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Bi-Hamiltonian Aspects of a Matrix Harry Dym Hierarchy

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Abstract

We study the Harry Dym hierarchy of nonlinear evolution equations from the bi-Hamiltonian view point. This is done by using the concept of an $\mathcal{S}$-hierarchy, which permits us to define a matrix Harry Dym hierarchy. We conclude by showing that the conserved densities of the matrix Harry Dym equation can be found by means of a Riccati-type equation.

1 Introduction

An intriguing equation known as the Harry Dym (HD) equation has attracted the attention of a number of researchers in integrable systems [4, 9, 12, 13, 14, 21, 26, 27, 28]. In one of its incarnations it can be written as

$$q_t = 2(1/\sqrt{(1+q)})_{xxx} \tag{1}$$

or equivalently

$$\rho_t = \rho^3 \rho_{xxx} \tag{2}$$

after the substitution $\rho = -(1+q)^{-1/2}$.

Equation (1) was discovered in an unpublished work by Harry Dym [15], and appeared in a more general form in works of P. C. Sabatier [29, 30, 31]. More recently, its relations with the Kadomtsev-Petviashvili (KP) and modified-KP hierarchy have been studied in detail by Oevel and Carillo [20].

In the present work we discuss the HD hierarchy from the bi-Hamiltonian point of view and show that it is amenable to the systematic treatment developed in [5, 10, 11, 18, 19]. The main result of this paper is the existence of a *matrix HD hierarchy*, giving rise to the usual (scalar) HD hierarchy after a projection. It is well known that a lot of integrable PDEs can be obtained as suitable reductions from (integrable) hierarchies living on loop-algebras, the main example being the Drinfeld-Sokolov reduction [8]. However, only recently it was realized that also the Camassa-Holm hierarchy has this important property [11]. The HD case is settled in this work, starting from the results in [17], where it has been shown that the bi-Hamiltonian structure of HD is the reduction of a suitable bi-Hamiltonian structure on the space $\mathcal{M} = C^\infty(S^1, \mathfrak{sl}(2))$ of C^∞ maps from the unit circle to $\mathfrak{sl}(2)$.

The plan of this article is the following:

In Section 2 we review the general definitions of Poisson geometry and bi-Hamiltonian theory. We review the important concept of an $\mathcal{S}$-hierarchy which was already used in [23] in connection with the Boussinesq equation.

Section 3 is devoted to endowing the loop-space on the Lie algebra of traceless 2×2 real matrices with a bi-Hamiltonian structure following a construction in [17].

Section 4 describes the construction of the matrix HD hierarchy, i.e., a hierarchy of commuting Hamiltonian flows in two fields that reduces to the Harry Dym equation (1) upon a suitable reduction. Two-component extensions of the HD equation have interested a number of researchers, see, e.g., [2, 3, 25]. We will see at the end of Section 5 that the hierarchy presented herein is different from those presented by these authors.

We conclude in Section 5 with a Riccati-type equation for the conserved quantities of the matrix HD hierarchy.

2 Bi-Hamiltonian preliminaries

This section collects a number of facts from bi-Hamiltonian geometry. More information could be found in [18].

A bi-Hamiltonian manifold is a triple $(\mathcal{M}, P_1, P_2)$ consisting of a manifold $\mathcal{M}$ and of two compatible Poisson tensors P_1 and P_2 on $\mathcal{M}$. In this context, we fix a symplectic leaf $\mathcal{S}$ of P_1 and consider the distribution $D = P_2(\mathrm{Ker} P_1)$ on $\mathcal{M}$. As it turns out, the distribution D is integrable. Furthermore, if $E = D \cap T\mathcal{S}$ is the distribution induced by D on $\mathcal{S}$ and the quotient space $\mathcal{N} = \mathcal{S}/E$ is a manifold, then it is a bi-Hamiltonian manifold. In situations where an explicit description of the quotient manifold $\mathcal{N}$ is not readily available, the following technique to compute the reduced bi-Hamiltonian structure is very useful [7]. Assume that $\mathcal{Q}$ is a submanifold of $\mathcal{S}$ that is transversal to the distribution E, in the sense that

$$T_p\mathcal{Q} \oplus E_p = T_p\mathcal{S} \qquad \text{for all } p \in \mathcal{Q} \ . \tag{3}$$

Then, $\mathcal{Q}$ is locally diffeomorphic to $\mathcal{N}$ and inherits a bi-Hamiltonian structure from $\mathcal{M}$. The reduced Poisson pair on $\mathcal{Q}$ is given by

$$\left(P_i^{\mathrm{rd}}\right)_p \alpha = \Pi_p\left((P_i)_p\tilde{\alpha}\right) \ , \qquad i = 1, 2 \ , \tag{4}$$

where $p \in \mathcal{Q}$, $\alpha \in T_p^*\mathcal{Q}$, the map $\Pi_p : T_p\mathcal{S} \to T_p\mathcal{Q}$ is the projection relative to (3), and $\tilde{\alpha} \in T_p^*\mathcal{M}$ satisfies

$$\tilde{\alpha}|_{D_p} = 0 \ , \qquad \tilde{\alpha}|_{T_p\mathcal{Q}} = \alpha \ . \tag{5}$$

Let us assume that $\{H_j\}_{j\geq 0}$ is a bi-Hamiltonian hierarchy on $\mathcal{M}$, that is, $P_2 dH_j = P_1 dH_{j+1}$ for all $j \geq 0$ and $P_1 dH_0 = 0$. In other words, $H(\lambda) = \sum_{j\geq 0} H_j \lambda^{-j}$ is a (formal) Casimir of the *Poisson pencil* $P_2 - \lambda P_1$. The bi-Hamiltonian vector fields associated with the hierarchy can be reduced on the quotient manifold $\mathcal{N}$ according to

Proposition 1 *The functions H_j restricted to $\mathcal{S}$ are constant along the distribution E. Thus, they give rise to functions on $\mathcal{N}$. Such functions form a bi-Hamiltonian hierarchy with respect to the reduced Poisson pair. The vector fields $X_j = P_2 dH_j = P_1 dH_{j+1}$ are tangent to $\mathcal{S}$ and project on $\mathcal{N}$. Their projections are the vector fields associated with the reduced hierarchy.*

In the sequel, we shall need a more general definition than that of a bi-Hamiltonian hierarchy. The point being that, once we have fixed a symplectic leaf $\mathcal{S}$ of P_1, it is not always possible to determine a hierarchy on $\mathcal{M}$ that is defined also on $\mathcal{S}$. In other words, there exist singular leaves for the hierarchies of a bi-Hamiltonian manifold. Nevertheless, it is sometimes possible to define hierarchies which are, in a certain sense "local" on $\mathcal{S}$.

Definition 2 *An* $\mathcal{S}$–hierarchy *is a sequence* $\{V_j\}_{j\geq 0}$ *of maps from* $\mathcal{S}$ *to* $T^*\mathcal{M}$*,*

$$V_j : s \mapsto V_j(s) \in T^*_s\mathcal{M} \ ,$$

with the following properties:

- V_j *restricted to* $T\mathcal{S}$ *is an exact* 1*-form, that is, there exist functions* H_j *on* $\mathcal{S}$ *such that* $V_j|_{T\mathcal{S}} = dH_j$*;*
- $P_2V_j = P_1V_{j+1}$ *for all* $j \geq 0$ *and* $P_1 dH_0 = 0$*.*

Obviously, every bi-Hamiltonian hierarchy on $(\mathcal{M}, P_1, P_2)$ defined in a neighborhood of $\mathcal{S}$ gives rise to an $\mathcal{S}$–hierarchy. In contradistinction, in this paper we will see an example of $\mathcal{S}$–hierarchy that does not come from any bi-Hamiltonian hierarchy. This is also the case of the Boussinesq hierarchy [23].

It is not difficult to extend Proposition 1 to the case of $\mathcal{S}$–hierarchies. In the sequel, whenever talking about $\mathcal{S}$–hierarchies and referring to such result, it shall be understood that we mean such straightforward extension.

3 A bi-Hamiltonian structure on a loop-algebra

In this section we recall from [17] that the bi-Hamiltonian structure of the (usual) HD hierarchy can be obtained by means of a reduction.

Let $\mathcal{M} = C^\infty(S^1, \mathfrak{sl}(2))$ be the loop-space on the Lie algebra of traceless 2×2 real matrices, i.e., the space of C^∞ functions from the unit circle S^1 to $\mathfrak{sl}(2)$. The tangent space $T_S\mathcal{M}$ at $S \in \mathcal{M}$ is identified with $\mathcal{M}$ itself, and we will assume that $T_S\mathcal{M} \simeq T^*_S\mathcal{M}$ by the non-degenerate form

$$\langle V_1, V_2 \rangle = \int \mathrm{tr}(V_1(x)V_2(x))\, \mathrm{d}x, \qquad V_1, V_2 \in \mathcal{M} \ ,$$

where the integral is taken (here and throughout this article) on S^1. It is well-known [16] that the manifold $\mathcal{M}$ has a 3-parameter family of compatible Poisson tensors. To wit,

$$P_{(\lambda_1,\lambda_2,\lambda_3)} = \lambda_1\partial_x + \lambda_2[\,\cdot\,, S] + \lambda_3[\,\cdot\,, A]\ , \tag{6}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, the matrix $A \in \mathfrak{sl}(2)$ is constant, and

$$S = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \in \mathcal{M}\ .$$

In this paper we focus on the pencil

$$P_\lambda = P_2 - \lambda P_1 = \partial_x + [\,\cdot\,,\ A + \lambda S] \tag{7}$$

with

$$A = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right)\ .$$

This means that

$$P_2 = \partial_x + [\,\cdot\,,\ A]\ , \qquad P_1 = [S,\ \cdot\,]\ . \tag{8}$$

Remark 1 We briefly recall from [18] the construction of the matrix KdV hierarchy, since what we will do in the following for the HD case is completely similar, even though technically more complicated. In the KdV case the Poisson pair is

$$P_2 = \partial_x + [\,\cdot\,,\ S]\ , \qquad P_1 = [A,\ \cdot\,]$$

and the symplectic leaf is chosen to be

$$\mathcal{S} = \left\{ \begin{pmatrix} p & q \\ 1 & -p \end{pmatrix} \mid p, q \in C^\infty(S^1, \mathbb{R}) \right\}\ .$$

The quotient space $\mathcal{N}$ can be identified with $C^\infty(S^1, \mathbb{R})$ and the projection from $\mathcal{S}$ to $\mathcal{N}$ is given by

$$(p, q) \mapsto u = p_x + p^2 + q\ . \tag{9}$$

The matrix KdV hierarchy is given by the flows

$$\frac{\partial S}{\partial t_j} = P_2 V_{j-1} = P_1 V_j$$

on the symplectic leaf $\mathcal{S}$, where the V_j are the coefficients of

$$V(\lambda) = \sum_{j\geq -1} V_j \lambda^{-j}$$

and $V(\lambda)$ is uniquely determined by the conditions $P_\lambda V(\lambda) = 0$ and $\operatorname{tr} V(\lambda)^2 = 2\lambda$. The conserved densities can be found also by solving the Riccati equation $h_x + h^2 = p_x + p^2 + q + \lambda$. The usual (scalar) KdV hierarchy lives on the quotient space $\mathcal{N}$ and can be obtained from the matrix one by applying the projection (9). In this case, this amounts to the well known Drinfeld-Sokolov reduction.

We close this remark by recalling that the Hamiltonian flows

$$\partial_t S = \partial_x V + [V, S] \ , \quad V = \mathrm{d}H \ ,$$

of the Poisson tensor $P_2 = \partial_x + [\,\cdot\,,\,S]$ admit the zero-curvature representation

$$[\partial_t + V, \partial_x + S] = 0.$$

Such a representation does not seem to exist in our (HD) case.

In [17] the bi-Hamiltonian reduction procedure was applied to the pair (P_1, P_2). In this case,

$$D_S = \left\{ \begin{pmatrix} (\mu p)_x + \mu q & (\mu q)_x \\ (\mu r)_x - 2\mu p & -(\mu p)_x - \mu q \end{pmatrix} \mid \mu \in C^\infty(S^1, \mathbb{R}) \right\} \ , \qquad S \in \mathcal{M}.$$

The distribution D is not tangent to the generic symplectic leaf of P_1. However, it is tangent to the symplectic leaf

$$\mathcal{S} = \left\{ \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \mid p^2 + qr = 0, (p, q, r) \neq (0, 0, 0) \right\} \ , \tag{10}$$

so that $E_p = D_p \cap T_p\mathcal{S}$ coincides with D_p for all $p \in \mathcal{S}$. It is not difficult to prove that the submanifold

$$\mathcal{Q} = \left\{ S(q) = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \mid q \in C^\infty(S^1, \mathbb{R}), q(x) \neq 0 \ \forall x \in S^1 \right\} \tag{11}$$

of $\mathcal{S}$ is transversal to the distribution E and that the projection $\Pi_{S(q)}$: $T_{S(q)}\mathcal{S} \to T_{S(q)}\mathcal{Q}$ is given by

$$\Pi_{S(q)} : (\dot{p}, \dot{q}) \mapsto (0, \dot{q} - \dot{p}_x) \ . \tag{12}$$

The reduced bi-Hamiltonian structure (4) coincides with the bi-Hamiltonian structure of the Harry Dym hierarchy (see [17] for details):

$$\left(P_1^{\mathrm{rd}}\right)_q = -(2q\partial_x + q_x)$$
$$\left(P_2^{\mathrm{rd}}\right)_q = -\frac{1}{2}\partial_x^3 .$$

Starting from the Casimir $\int \sqrt{q}\,\mathrm{d}x$ of P_1^{rd}, one constructs a bi-Hamiltonian hierarchy, which is called the HD hierarchy. We refer to [24] and the references therein for more details, and for a discussion about a "KP extension" of the HD hierarchy (see also [20]).

Remark 2 We take this opportunity for correcting a mistake in [24]. Equation (3.5) in that paper should be replaced with

$$K^{(2j+1)} = \lambda \left(-\frac{1}{2}(\lambda^j w)_{+,x} + k(\lambda^j w)_+ \right) . \tag{13}$$

The remaining of the paper is correct, up to minor changes.

We consider now the bi-Hamiltonian hierarchies of the Poisson pair (8). According to the bi-Hamiltonian theory, the computation of the flows of these hierarchies is divided in two steps.

1. First we have to look for a 1-form $V(\lambda) = \sum V_j \lambda^{-j}$ that belongs to the kernel of the pencil P_λ. By construction the coefficients V_j satisfy the relations
$$P_2 V_j = P_1 V_{j+1}.$$
By definition the flows of the hierarchy are
$$\frac{\partial S}{\partial t_j} = P_2 V_{j-1} = P_1 V_j. \tag{14}$$
In the case of $\mathcal{S}$–hierarchies the 1-form $V(\lambda)$ is defined only on a symplectic leaf $\mathcal{S}$.

2. Then, we have to verify that the 1-form is exact. If this is the case, the coefficients of the potential $H(\lambda) = \sum H_j \lambda^{-j}$, the so-called Casimir of the pencil, are the Hamiltonians of the flows (14). In the case of $\mathcal{S}$–hierarchies the Hamiltonians are the coefficients of the potential of the restriction of $V(\lambda)$ to $T\mathcal{S}$.

We apply now this procedure to the Poisson pencil (7). In particular, we shall study a 1-form $V(\lambda)$ defined only on the symplectic leaf $\mathcal{S}$ defined by (10) and the corresponding $\mathcal{S}$–hierarchy.

Let us suppose that

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

is a solution of $P_\lambda V = 0$, that is,

$$V_x + [V\,, A + \lambda S] = 0\ , \tag{15}$$

and let us write the previous equation in componentwise form

$$\begin{cases} \alpha_x + (\lambda r + 1)\,\beta - \lambda q\,\gamma = 0 \\ \beta_x + 2\lambda q\,\alpha - 2\lambda p\,\beta = 0 \\ \gamma_x + 2\lambda p\,\gamma - 2(\lambda r + 1)\,\alpha = 0 \end{cases} \tag{16}$$

Upon expressing α and γ in terms of β,

$$\begin{cases} \alpha = \dfrac{1}{2q}\big(-\dfrac{\beta_x}{\lambda} + 2\beta p\big) \\ \gamma = -\dfrac{\beta_{xx}}{2\lambda^2 q^2} + \dfrac{\beta_x}{\lambda q^2}\big(p + \dfrac{q_x}{2\lambda q}\big) + \beta\big(\dfrac{p_x}{\lambda q^2} - \dfrac{q_x p}{\lambda q^3} + \dfrac{r}{q} + \dfrac{1}{\lambda q}\big) \end{cases} \tag{17}$$

we find that β satisfies the equation

$$-\frac{\beta_{xxx}}{2q^2\lambda^2} + \frac{3q_x\,\beta_{xx}}{2q^3\lambda^2} + \left(\frac{2}{q\lambda} + \frac{2p_x}{q^2\lambda} + \frac{2r}{q} - \frac{3q_x^2}{2q^4\lambda^2} - \frac{2q_x p}{q^3\lambda} + \frac{q_{xx}}{2q^3\lambda^2} + \frac{2p^2}{q^2}\right)\beta_x +$$
$$+\left(\frac{r_x}{q} - \frac{q_x}{q^2\lambda} + \frac{p_{xx}}{q^2\lambda} - \frac{3q_x p_x}{q^3\lambda} + \frac{3q_x^2 p}{q^4\lambda} - \frac{q_{xx}p}{q^3\lambda} - \frac{q_x r}{q^2} + \frac{2pp_x}{q^2} - \frac{2p^2 q_x}{q^3}\right)\beta = 0$$

This equation can be rewritten as

$$\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}x}(\alpha^2 + \beta\gamma) = 0\ . \tag{18}$$

Indeed, it is a well-known consequence of equation (15) that the spectrum of V does not depend on x, so that $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{tr}\,V^2 = 0$. Let us set

$$\mathrm{tr}\,\frac{V^2}{2} = \alpha^2 + \beta\gamma = F(\lambda)\ , \tag{19}$$

where $F(\lambda)_x = 0$. Then, the equation for β becomes

$$2q\beta_{xx}\beta - q\beta_x^2 - 2q_x\beta\beta_x + 4(q_x p - qp_x - q^2)\beta^2\lambda - 4q(p^2+qr)\lambda^2\beta^2 + 4q^3F(\lambda)\lambda^2 = 0\,. \tag{20}$$

We now consider the possibility of finding a solution $\beta(\lambda)$ of (20) as a formal series expansion in (negative) powers of λ, that is, $\beta = \sum_{i=-1}^{\infty}\beta_i\lambda^{-i}$. In order to find the coefficients β_i recursively, we must equate the coefficients of the same degree in λ starting from the highest order one. We choose $F(\lambda)$ to be a power of λ. Let us suppose that $q(x) \neq 0$ for all x. Then, it turns out that we have to distinguish the two cases:

- If $p^2 + q\,r \neq 0$, then the degree of $F(\lambda)$ has to be even;
- If $p^2 + q\,r = 0$, then the degree of $F(\lambda)$ has to be odd.

We are interested in the latter case, in order to perform the reduction process described in Section 2. If the degree of $F(\lambda)$ is odd then there exists a solution β expanded in a formal Laurent series only on the symplectic leaf $\mathcal{S}$. Such a solution cannot be extended outside $\mathcal{S}$ because if $p^2 + q\,r \neq 0$, then the degree of $F(\lambda)$ has to be even. This means that in this article, we will study a $\mathcal{S}$-hierarchy on the symplectic leaf (10) that cannot be obtained from a bi-Hamiltonian hierarchy on the whole $\mathcal{M} = C^\infty(S^1, \mathfrak{sl}(2))$.

The bi-Hamiltonian hierarchy corresponding to the former case will not be considered here, although the proof of the exactness given in the next section can be easily adapted to this case.

4 The matrix HD hierarchy

In this section we will show that it is possible to find a solution

$$V = \sum_{i=-1}^{\infty} V_i\lambda^{-i} = \sum_{i=-1}^{\infty}\begin{pmatrix}\alpha_i & \beta_i\\ \gamma_i & -\alpha_i\end{pmatrix}\lambda^{-i} \tag{21}$$

of equation (15) at the points of the symplectic leaf $\mathcal{S}$, such that every V_i restricted to $T\mathcal{S}$ is an exact 1-form. This yields an $\mathcal{S}$-hierarchy, to be called the matrix HD hierarchy. We will see that it projects to the usual (scalar) HD hierarchy. In particular, its second vector field projects to the HD equation.

First of all, we restrict to the symplectic leaf $\mathcal{S}$ and we use the now classical dressing transformation method [32, 8, 6] to show that the matrix

$V(\lambda)$ whose entries are given by the solution of (20) and (17) defines an $\mathcal{S}$-hierarchy if $F(\lambda)$ does not depend on the point of $\mathcal{S}$. Indeed, equation (19) implies that there exists a nonsingular matrix K such that

$$V(\lambda) = K\Lambda K^{-1} \ ,$$

where [1]

$$\Lambda = \begin{pmatrix} 0 & 1 \\ F(\lambda) & 0 \end{pmatrix} \ .$$

Let us introduce

$$M = K^{-1}(S + \frac{A}{\lambda})K - \frac{1}{\lambda}K^{-1}K_x \ . \tag{22}$$

Thus, we have the following:

Proposition 3 *If $F(\lambda)$ does not depend on the point $S \in \mathcal{S}$, then $V(\lambda)$ restricted to $T\mathcal{S}$ is an exact 1-form. More precisely, if $H : \mathcal{S} \to \mathbb{R}$ is given by*

$$H(\lambda) = \int \mathrm{tr}\,(M\Lambda)\,\mathrm{d}x \ , \tag{23}$$

then $V|_{T\mathcal{S}} = \mathrm{d}H$.

Proof. If V is a solution of (15), then

$$\frac{1}{\lambda}K^{-1}V_xK + \frac{1}{\lambda}K^{-1}[V, A + \lambda S]K = 0 \ .$$

This in turn, implies that

$$\frac{1}{\lambda}\Lambda_x + [\Lambda, M] = 0 \ .$$

Since Λ does not depend on x, we have that Λ commutes with M. Therefore, for every tangent vector $\dot{S}$ to the symplectic leaf $\mathcal{S}$, we have

$$\begin{aligned} \langle \mathrm{d}H, \dot{S}\rangle &= \int \mathrm{tr}\,(\dot{M}\Lambda)\,\mathrm{d}x = \int \mathrm{tr}\,(K^{-1}\dot{S}K\Lambda) + \mathrm{tr}\,([M, K^{-1}\dot{K}]\Lambda)\,\mathrm{d}x \\ &= \int \mathrm{tr}\,(\dot{S}K\Lambda K^{-1})\,\mathrm{d}x = \int \mathrm{tr}\,(\dot{S}V)\,\mathrm{d}x = \langle V, \dot{S}\rangle \ , \end{aligned}$$

[1]Of course any other traceless matrix Λ depending only on λ and such that $\mathrm{tr}\,\frac{\Lambda^2}{2} = F(\lambda)$ is suitable for our purpose. Our choice simplifies the following computations.

since $\int \operatorname{tr}([M, K^{-1}\dot{K}]\Lambda)\,\mathrm{d}x = 0$. This completes the proof. ∎

Let us now compute explicitly H. A possible choice for K is

$$K = \begin{pmatrix} \beta^{\frac{1}{2}} & 0 \\ -\alpha\beta^{-\frac{1}{2}} & \beta^{-\frac{1}{2}} \end{pmatrix} .$$

Since M commutes with Λ and both matrices have distinct eigenvalues, it follows that M is a polynomial of Λ. However, since they are traceless and we are working with 2×2 matrices it follows that M is a multiple of Λ. This simplifies the computation of M, since it becomes

$$M = \frac{q}{\beta}\Lambda .$$

Thus, we have that

$$H(\lambda) = \int 2\frac{q}{\beta}F(\lambda)\,\mathrm{d}x . \tag{24}$$

We define the *matrix HD hierarchy* to be the $\mathcal{S}$-hierarchy corresponding to the choice

$$F(\lambda) = \lambda.$$

In order to find its first vector fields, let us substitute $p^2 + qr = 0$ and $F(\lambda) = \lambda$ in equation (20), to find

$$2q\beta_{xx}\beta - q\beta_x^2 - 2q_x\beta\beta_x + 4(q_x p - qp_x - q^2)\beta^2\lambda + 4q^3\lambda^3 = 0 . \tag{25}$$

From now on, we use the functions p and q to describe a point of $\mathcal{S}$. We know that it is possible to solve equation (25) recursively, starting from the highest power of λ:

$$\lambda^3 : \qquad 4(q_x p - qp_x - q^2)\beta_{-1}^2 = -4q^3 .$$

We choose the positive solution

$$\beta_{-1} = \sqrt{\frac{q^3}{q^2 - q_x p + qp_x}} . \tag{26}$$

Using the expressions (17) for α and γ we get a recursive formula for the matrices V_i. Indeed, we have that

$$\begin{cases} \alpha_{-1} = p\dfrac{\beta_{-1}}{q} \\ \gamma_{-1} = r\dfrac{\beta_{-1}}{q} \end{cases} \tag{27}$$

and

$$\begin{cases} \alpha_i = \dfrac{1}{2q}(-(\beta_{i-1})_x + 2\beta_i p) \\ \gamma_i = \dfrac{1}{q}((\alpha_{i-1})_x + \beta_i r + \beta_{i-1}) \end{cases} \tag{28}$$

for all $i \geq 0$. Therefore, we can compute the first 1-form

$$V_{-1} = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \varphi(x) \ ,$$

where

$$\varphi(x) := \sqrt{\frac{q}{q^2 - q_x p + q p_x}} \ .$$

We can verify immediately that V_{-1} indeed commutes with S, as expected.

Applying the Poisson tensor P_2 to V_{-1} we obtain the first vector field $X_0 := P_2(V_{-1}) = V_{-1_x} + [V_{-1}, A]$ of the hierarchy:

$$\begin{cases} \dot{p} = (p\varphi)_x + q\varphi \\ \dot{q} = (q\varphi)_x \end{cases} \tag{29}$$

We saw in Section 2 that every $\mathcal{S}$-hierarchy can be projected on the reduced bi-Hamiltonian manifold. We will show in Remark 3 that the projection of the matrix HD hierarchy is the (scalar) HD hierarchy. Now we compute the projections of the first vector fields of the hierarchy. Since V_{-1} belongs to the kernel of P_1, we have that $V_{-1}|_{T\mathcal{S}} = 0$ and that $P_2(V_{-1})$ belongs to the distribution D, so that the projection of X_0 vanishes. However, let us check it explicitly. We must evaluate X_0 at the points $p = 0$ of the transversal submanifold $\mathcal{Q}$, then we have to project this vector field according to the formula (12), thus obtaining the predicted result:

$$\frac{\partial q}{\partial t_0} = \dot{q} - \dot{p}_x = 0 \ .$$

More generally, let us observe that the formula for the vector field $X_i := P_2(V_{i-1})$ for $i \geq 0$ is

$$\begin{cases} \dot{p} = \alpha_{i-1_x} + \beta_{i-1} \\ \dot{q} = \beta_{i-1_x} \end{cases}$$

and its projection is $\dot{q} - \dot{p}_x = -\alpha_{i-1_{xx}}$ evaluated at $p = 0$.

The next step in the iteration is:

$$\lambda^2: \qquad 2q\beta_{-1}\beta_{-1_{xx}} - q(\beta_{-1_x})^2 - 2q_x\beta_{-1}\beta_{-1_x} = 8(q^2 - q_xp + qp_x)\beta_{-1}\beta_0\ .$$

Using also equation (26), we get that

$$\beta_0 = \frac{\varphi}{8\,q}\left(2q^2\varphi\,\varphi_{xx} + 2q\varphi^2 q_{xx} - {\varphi_x}^2 q^2 - 3\varphi^2{q_x}^2\right) \tag{30}$$

and then

$$V_0 = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & -\alpha_0 \end{pmatrix}\ ,$$

where

$$\begin{aligned}
\alpha_0 &= -\frac{\varphi_x}{2} - \frac{q_x\varphi}{2q} + \frac{p\varphi}{8\,q^2}\left(2q^2\varphi\varphi_{xx} + 2q\varphi^2 q_{xx} - {\varphi_x}^2 q^2 - 3\varphi^2{q_x}^2\right) \\
\gamma_0 &= \frac{1}{8\,q^3}\left(8\varphi_x pq^2 + 8\varphi p_x q^2 - 2\varphi^2p^2q^2\varphi_{xx} - 2\varphi^3p^2qq_{xx} + \varphi p^2{\varphi_x}^2q^2 + \right. \\
&\quad \left. + 3\varphi^3p^2{q_x}^2 + 8\varphi q^3\right)\ .
\end{aligned}$$

We can now determine the second vector field $X_1 := P_2(V_0) = V_{0x} + [V_0, A]$. It is given by

$$\left\{\begin{array}{l}
\dot{p} = -\dfrac{1}{2}\varphi_{xx} - \left(\dfrac{q_x}{2q}\varphi\right)_x + \left(\dfrac{p}{q}\beta_0\right)_x + \beta_0 \\
\dot{q} = \beta_{0x}
\end{array}\right. \tag{31}$$

Using equation (30), we can write out the above equation as follows:

$$\left\{\begin{array}{rl}
\dot{p} = & \left(\dfrac{\varphi}{8q} + \dfrac{p_x\varphi}{8q^2} - \dfrac{q_x\varphi}{8q^3}\right)(2q^2\varphi\,\varphi_{xx} + 2q\varphi^2q_{xx} - {\varphi_x}^2q^2 - 3\varphi^2{q_x}^2) + \\
& -\dfrac{1}{2}\varphi_{xx} - \left(\dfrac{q_x}{2q}\varphi\right)_x + \dfrac{p}{q}\left(\dfrac{\varphi}{8q}\,(2q^2\varphi\varphi_{xx} + 2q\varphi^2q_{xx} - {\varphi_x}^2q^2 - 3\varphi^2{q_x}^2)\right)_x \\
& \\
\dot{q} = & \left(\dfrac{\varphi}{8\,q}\,(2q^2\varphi\,\varphi_{xx} + 2q\varphi^2q_{xx} - {\varphi_x}^2q^2 - 3\varphi^2{q_x}^2)\right)_x
\end{array}\right. \tag{32}$$

Starting from (31), we calculate the reduced vector field, first evaluating X_1 at the points $p = 0$ of the transversal submanifold $\mathcal{Q}$,

$$\left\{ \begin{array}{l} \dot{p} = \dfrac{5\,{q_x}^2}{32\,q^{\frac{5}{2}}} - \dfrac{q_{xx}}{8\,q^{\frac{3}{2}}} \\ \\ \dot{q} = \left(-\dfrac{7\,{q_x}^2}{32\,q^{\frac{5}{2}}} + \dfrac{q_{xx}}{8\,q^{\frac{3}{2}}} \right)_x \end{array} \right. \tag{33}$$

and then projecting this vector field on the transversal submanifold. We thus obtain the HD equation (1)

$$\frac{\partial q}{\partial t_1} = \dot{q} - \dot{p}_x = -\frac{1}{2}\left(\frac{1}{\sqrt{q}}\right)_{xxx} \quad .$$

This equation is equivalent to (1) after the change of variables $q \mapsto (1+q)$ and $t_1 \mapsto -4t$.

5 A Riccati equation for the conserved densities

The goal of this final section is to point out that the conserved densities of the matrix HD hierarchy can also be found by means of a Riccati-type equation.

We recall that equation (24) gives, for $F(\lambda) = \lambda$, the expression of the potential H of the 1-form $V|_{T\mathcal{S}}$. The corresponding density is clearly defined up to a total x-derivative. This fact allows us to introduce

$$h = \frac{q\lambda^{\frac{3}{2}}}{\beta} + \frac{\beta_x}{2\beta} \ ,$$

which transforms the equation (20) in the Riccati-type equation

$$h_x + h^2 - \frac{q_x}{q}h = \left(p_x + q - \frac{q_x}{q}p\right)\lambda \ . \tag{34}$$

Its solution h yields

$$H(\lambda) = \frac{2}{\sqrt{\lambda}}\int h\mathrm{d}x \tag{35}$$

for the functional H. We set $z = \sqrt{\lambda}$, and substitute $h = \sum_{i=-1}^{\infty} h_i z^{-i}$ in the Riccati equation (34), which takes the form

$$\sum_{i=-1}^{\infty} \left(h_{ix} + \sum_{j=0}^{1}(h_{i-j}h_i)\right) z^{-i} - \frac{q_x}{q} \sum_{i=-1}^{\infty} h_i z^{-i} = \left(p_x + q - \frac{q_x}{q} p\right) z^2 \ . \tag{36}$$

Once again, this equation can be solved recursively, starting from the highest degree of z. The first step is

$$z^2 : \qquad h_{-1}^2 = p_x + q - \frac{q_x}{q} p \ \ ,$$

which gives, up to a sign,

$$h_{-1} = \sqrt{p_x + q - \frac{q_x}{q} p} \ .$$

Similarly, we have that

$$z^1 : \qquad h_{-1_x} + 2h_{-1}h_0 - \frac{q_x}{q} h_{-1} = 0$$

from which we obtain

$$h_0 = -\frac{h_{-1_x}}{2h_{-1}} + \frac{q_x}{2q} \ .$$

Let us notice that this is a total $x-$derivative. More generally, it is evident from (35) that every even density is a total $x-$derivative. Indeed, $H(\lambda) = \sum_{i \geq 0} H_i \lambda^{-i}$, with

$$H_i = 2 \int h_{2i-1} \mathrm{d}x \ . \tag{37}$$

In particular, $H_0 = 2 \int h_{-1} \mathrm{d}x$, and it can be checked that $\mathrm{d}H_0 = V_0|_{T\mathcal{S}}$, as claimed in Proposition 3.

The next equation is

$$z^0 : \qquad h_{0x} + 2h_{-1}h_1 + h_0^2 - \frac{q_x}{q} h_0 = 0$$

and the corresponding density is

$$h_1 = -\frac{1}{2h_{-1}} \left(h_{0x} + h_0^2 + \frac{q_x}{q} h_0\right) \ .$$

This leads to

$$H_1 = 2\int \left(\frac{h_{-1xx}}{4h_{-1}^2} - \frac{3h_{-1x}^{\ 2}}{8h_{-1}^3} - \frac{q_{xx}}{4h_{-1}q} + \frac{3q_x^2}{8h_{-1}q^2} \right) \mathrm{d}x \ .$$

Integrating by parts and substituting the expression for h_1, we find

$$H_1 = 2\int \left(\frac{\left(p_{xx} + q_x - (\frac{pq_x}{q})_x \right)^2}{32\left(p_x + q - \frac{pq_x}{q} \right)^{\frac{5}{2}}} - \frac{q_{xx}}{4q\sqrt{p_x + q - \frac{pq_x}{q}}} + \frac{3q_x^2}{8\,q^2\sqrt{p_x + q - \frac{pq_x}{q}}} \right) \mathrm{d}x \ .$$

This is the Hamiltonian (with respect to the symplectic structure obtained by restricting P_1 to its symplectic leaf $\mathcal{S}$) of the 2-component extension (32) of the HD equation.

Remark 3 In this paper we chose to express the equation (19) in terms of β, using the equations of the system (16). An analogous calculation can be performed in terms of γ, leading to the expression

$$\tilde{H}(\lambda) = \int 2\frac{\lambda r + 1}{\lambda\gamma} F(\lambda)\mathrm{d}x$$

for the functional on $\mathcal{S}$ such that $V|_{T\mathcal{S}} = \mathrm{d}\tilde{H}$ (see equation (24) and Proposition 3). The functional H defined by (24) could in principle differ from $\tilde{H}$ by an additive constant, but we will see that they coincide. In section 4 we set $F(\lambda) = \lambda$ and then in this section we proved that

$$H(\lambda) = \frac{2}{\sqrt{\lambda}} \int h\mathrm{d}x \ ,$$

where h satisfies the Riccati-type equation (34). In the same way we can prove that

$$\tilde{H}(\lambda) = \frac{2}{\sqrt{\lambda}} \int \tilde{h}\mathrm{d}x \ ,$$

where $\tilde{h}$ satisfies the following different Riccati-type equation,

$$(\tilde{h}_x + \tilde{h}^2)(\lambda r + 1) + \lambda r_x \tilde{h} = \lambda(q - p_x) + \lambda^2(r_x p - p_x r + p^2 + 2qr) \ , \quad (38)$$

where $p^2 + qr = 0$ since we are on the symplectic leaf $\mathcal{S}$. The two Riccati equations are both equivalent to equation (19), respectively expressed in terms of β and γ through the transformations

$$h = \frac{q\lambda^{\frac{3}{2}}}{\beta} + \frac{\beta_x}{2\beta} \ ,$$

and

$$\tilde{h} = \frac{\lambda r + 1}{\gamma}\lambda^{\frac{1}{2}} + \frac{\gamma_x}{2\gamma}.$$

It can be checked directly that the densities h and $\tilde{h}$ differ by a total x-derivative, so that H and $\tilde{H}$ actually coincide. Indeed, using (19) and the system (16), we obtain

$$\begin{aligned} h - \tilde{h} \ &= \ \frac{\sqrt{\lambda}\,\alpha_x}{\beta\gamma} + \frac{\beta_x}{2\beta} - \frac{\gamma_x}{2\gamma} \\ &= \ \frac{\sqrt{\lambda}\,\alpha_x}{\lambda - \alpha^2} + \frac{\beta_x}{2\beta} - \frac{\gamma_x}{2\gamma} \\ &= \ \frac{1}{2}\,\partial_x\left(\sqrt{\lambda}\,\log\left|\frac{\sqrt{\lambda}+\alpha}{\sqrt{\lambda}-\alpha}\right| + \log\left|\frac{\beta}{\gamma}\right|\right). \end{aligned}$$

The choice of dealing with the first Riccati equation (34) is due to the fact that equation (38) is more complicated to handle, in order to perform the iteration. But it is easy to see that (38), evaluated at the points of $\mathcal{Q}$, is the Riccati equation for the scalar HD hierarchy (see, e.g., [24]),

$$\tilde{h}_x + \tilde{h}^2 = q\lambda \ .$$

This shows that the matrix HD hierarchy projects on the usual HD hierarchy.

We close our paper with an explanation of the difference between our 2-component extension of the HD hierarchy and those already present in the literature. Ours is more precisely a *lifting*, since it gives rise to the usual HD hierarchy after a *projection*. On the contrary, those already known in the literature *restrict* to the HD hierarchy. For example, in [2] one has to put one of the two fields equal to zero. This is completely obvious if one looks at the corresponding second order linear problems (from our point of view, the Riccati equations): our equation (34) is similar to the one of the usual

HD hierarchy, while the one in [2, eq.(20)] is different, since a polynomial of degree 2 in λ appears. The same happens for the matrix KdV hierarchy (see Remark 1) and the N-component extensions discussed in [1]. Finally, we point out that there certainly exist coordinates in which the matrix HD hierarchy is triangular. Indeed, we showed that it projects on the usual HD hierarchy, and it is well known that every projectable vector field becomes triangular once written in coordinates which are adapted to the projection.

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