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SUMMATION FORMULAS AND INTEGER POINTS UNDER SHIFTED GENERALIZED HYPERBOLAE

by

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Summation formulas and integer points under shifted generalized hyperbolae

Leonardo Colzani and Giacomo Gigante

Abstract

A hyperbolic domain is a domain below a convex function in the first quadrant of the plane. We prove an analog of a classical formula of Voronoï for the number of integer points in shifted hyperbolic domains.

1. *Introduction*

The problem of counting the average number of representations of a number as a sum of two squares leads to estimate the number of integer points inside a circle, the Gauss circle problem. If $r(n) = \#\left\{(x,y) \in \mathbb{Z}^2 : x^2 + y^2 = n\right\}$ and if $R(N) = \sum_{n<N} r(n)$, then

$$\frac{R(N-) + R(N+)}{2} = \pi N + \sqrt{N} \sum_{n=1}^{\infty} \frac{r(n)}{\sqrt{n}} J_1(2\pi\sqrt{nN}).$$

Here $J_1(z)$ is a Bessel function of first kind. The term πN is the area of the circle of radius $\sqrt{N}$ and it is much larger than the Fourier Bessel series as $N \rightsquigarrow +\infty$. This series can be considered as a remainder and a simple geometric argument shows that it is smaller than the perimeter $2\pi\sqrt{N}$, but the conjecture is that it is dominated by $CN^{1/4+\varepsilon}$ for every $\varepsilon > 0$.

Similarly, counting the average number of divisors of a number leads to estimate the number of integer points under a hyperbola, the Dirichlet divisor problem. If $d(n) = \#\left\{(x,y) \in \mathbb{Z}_+^2 : xy = n\right\}$ and if $D(N) = \sum_{n<N} d(n)$, then

$$\frac{D(N-) + D(N+)}{2} =$$
$$N \log(N) + (2\gamma - 1) N + 1/4 - \sqrt{N} \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} \left(Y_1(4\pi\sqrt{nN}) + \frac{2}{\pi} K_1(4\pi\sqrt{nN}) \right).$$

Again, $Y_1(z)$ is a Bessel function of second kind and $K_1(z)$ a Bessel function of imaginary argument. The term $N \log(N) + (2\gamma - 1) N$ is much larger than the Fourier Bessel series and a simple geometric argument shows that this remainder is smaller than $C\sqrt{N}$, but the conjecture is that it is dominated by $CN^{1/4+\varepsilon}$ for every $\varepsilon > 0$.

Finally, counting the numbers which are power of two primes p and q leads to estimate the number of integer points in $\{p^x q^y < N\}$, and by taking logarithms one recognizes a triangle.

The above formulas for circles and hyperbolae were stated by Voronoï in [**17**], [**18**] and proved by Hardy in [**5**], and the triangles were considered by Ramanujan, see [**6**, Chapter V]. More generally, Voronoï conjectured the existence of summation formulas also for weighted sums of arithmetic functions. In particular, if $g(t)$ is a smooth function and $0 < a < b$ are not

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integers, then

$$\sum_{a<n<b} g(n)\, r(n) = \pi \sum_{n=1}^{\infty} r(n) \int_a^b g(z)\, J_0(2\pi\sqrt{nz})dz,$$

$$\sum_{a<n<b} g(n)\, d(n) =$$

$$\int_a^b g(z)\,(\log(z) + 2\gamma)\, dz + \sum_{n=1}^{\infty} d(n) \int_a^b g(z)\left(4K_0(4\pi\sqrt{nz}) - 2\pi Y_0(4\pi\sqrt{nz})\right) dz.$$

If $g(t)$ is piecewise smooth with a discontinuity at an integer point, or if a or b are integers, then one has to consider a mean value of $g(t)$.

A formula for the number of integer points in a shifted disc is proved in [**8**], and a formula for the number of integer points in a shifted hyperbola is stated without proof in [**12**]. A general reference to these problems is [**9**], and a shorter self contained introduction, somehow related to what follows, is [**15**, Chapter 8]. Our goal is to generalize these formulas. The starting point is an observation by Kendall: If Ω is a bounded domain in $\mathbb{R}^d$, the number of integer points in a translated $\Omega - t$ is a $\mathbb{Z}^d$ periodic function of the translation t with Fourier expansion

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} \chi_{\Omega-t}(k) &= \sum_{j\in\mathbb{Z}^d} \left(\int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d} \chi_{\Omega-s}(k) \exp(-2\pi i j\cdot s) ds \right) \exp(2\pi i j\cdot t) \\
&= \sum_{j\in\mathbb{Z}^d} \left(\int_{\mathbb{R}^d} \chi_{\Omega}(s) \exp(-2\pi i j\cdot s) ds \right) \exp(2\pi i j\cdot t) \\
&= |\Omega| + \sum_{j\in\mathbb{Z}^d-\{0\}} \widehat{\chi}_{\Omega}(j) \exp(2\pi i j\cdot t).
\end{aligned}$$

Here $|\Omega|$ is the measure of the domain, $\widehat{G}(\xi) = \int_{\mathbb{R}^d} G(s)\exp(-2\pi i\xi\cdot s)ds$ is the Fourier transform in $\mathbb{R}^d$, and the above formula is nothing but the Poisson summation formula

$$\sum_{k\in\mathbb{Z}^d} G(k+t) = \sum_{j\in\mathbb{Z}^d} \widehat{G}(j) \exp(2\pi i j t).$$

It follows from Parseval's equality that the mean value of the number of integer points in $\Omega - t$ is the measure of the domain $|\Omega|$, and the variance is

$$\int_{\mathbb{T}^d} \left| \sum_{k\in\mathbb{Z}^d} \chi_{\Omega-t}(k) - |\Omega| \right|^2 dt = \sum_{j\in\mathbb{Z}^d-\{0\}} \left|\widehat{\chi}_{\Omega}(j)\right|^2.$$

In particular, the Fourier transform of a ball is a Bessel function,

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^d, |x|<R\}} \exp(-2\pi i\xi\cdot x)dx &= R^{d/2}\,|\xi|^{-d/2}\, J_{d/2}\left(2\pi R\,|\xi|\right) \\
&\approx \pi^{-1} R^{(d-1)/2}\,|\xi|^{-(d+1)/2} \cos\left(2\pi R\,|\xi| - (d+1)\,\pi/4\right).
\end{aligned}$$

Hence, disregarding any problems of convergence, one formally obtains the above mentioned formula for the number of integer points in a disc. This also implies that the standard deviation in the estimate of lattice points in a ball of radius R is of the order of $R^{(d-1)/2}$. The motivation of our work is to adapt the above Fourier approach from bounded to unbounded domains. The problem with this approach is that it seems to lose sense with domains of infinite measure, since in this case some non zero frequencies can be infinite. But also with bounded domains, some non zero frequencies can give large contributions to the Fourier expansions. A possible way to overcome this problem is the following. If $G(x,y)$ is integrable in $\mathbb{R}^2$ and $\widehat{G}(\xi,\eta)$ is its

Fourier transform then, by the two dimensional Poisson summation formula,

$$\begin{aligned}&\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha,n+\beta)\\ &= -\widehat{G}(0,0)+\sum_{m\in\mathbb{Z}}\widehat{G}(m,0)\exp(2\pi i m\alpha)+\sum_{n\in\mathbb{Z}}\widehat{G}(0,n)\exp(2\pi i n\beta)\\ &\quad+\sum\sum_{(m,n)\in\mathbb{Z}^2,\ mn\neq 0}\widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta)).\end{aligned}$$

Define

$$\lambda=-\int\int_{\mathbb{R}^2} G(x,y)\,dy,\quad h(x)=\int_{\mathbb{R}} G(x,y)\,dy,\quad k(y)=\int_{\mathbb{R}} G(x,y)\,dx.$$

Then, by Fubini's theorem, $h(x)$ and $k(y)$ are integrable in $\mathbb{R}$ and, by the one dimensional Poisson summation formula,

$$\sum_{m\in\mathbb{Z}}\widehat{G}(m,0)\exp(2\pi i m\alpha)=\sum_{m\in\mathbb{Z}} h(m+\alpha)=\varphi(\alpha),$$

$$\sum_{n\in\mathbb{Z}}\widehat{G}(0,n)\exp(2\pi i n\beta)=\sum_{n\in\mathbb{Z}} k(m+\alpha)=\psi(\beta).$$

Hence,

$$\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha,n+\beta)=\lambda+\varphi(\alpha)+\psi(\beta)+\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0}\widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta)).$$

The main point in what follows is that this last formula may make sense even when $G(x,y)$ is not integrable and some of the Fourier coefficients $\widehat{G}(m,0)$ and $\widehat{G}(0,n)$ are infinite. Indeed, our main result is a generalization of the formula of Voronoï:

Let $f(x)$ be a non negative decreasing convex function in $\{0<x<+\infty\}$ vanishing at infinity, and let $g(t)$ be a function with bounded variation in $\{-\infty<t<+\infty\}$ and compact support. Finally, if $0<t,x,y<+\infty$ let $t=\tau(x,y)$ be the solution to the equation $y/t=f(x/t)$, and let

$$G(x,y)=\begin{cases} g(\tau(x,y)) & \text{if } 0<x,y<+\infty,\\ 0 & \text{otherwise.}\end{cases}$$

Then, there exist a constant ΛG and two periodic functions $\Phi G(\alpha)$ and $\Psi G(\beta)$, such that

$$\begin{aligned}&\left|\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha,n+\beta)-\Lambda G-\Phi G(\alpha)-\Psi G(\beta)\right|\\ &\leq\int_0^{+\infty}\left(4+3tf(1/t)+3tf^{-1}(1/t)\right)|dg(t)|.\end{aligned}$$

Moreover, $\widehat{G}(\xi,\eta)$ is a continuous function in $\{0<|\xi|,|\eta|<+\infty\}$, and

$$\begin{aligned}&\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha,n+\beta)-\Lambda G-\Phi G(\alpha)-\Psi G(\beta)\\ &=\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0}\widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta)).\end{aligned}$$

In the above formula, $dg(t)$ denotes the signed measure associated to a function with bounded variation $g(t)$, and $|dg(t)|$ denotes the total variation of this measure. The above Fourier series converges in $\mathbb{L}^2(\mathbb{T}^2)$ and, under some mild extra assumptions, its square norm is dominated by

$$\left\{ \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \left|\widehat{G}\left(m,n\right)\right|^2 \right\}^{1/2}$$
$$\leq C \left(\int_0^{+\infty} \sqrt{1+t}\left|dg\left(t\right)\right| \right) \sqrt{1+\left(f'\right)^{-1}\left(-1\right)+\left(\left(f^{-1}\right)'\right)^{-1}\left(-1\right)}.$$

Moreover, the Fourier series is also pointwise spherically convergent at every point where the expanded function is smooth,

$$\lim_{R\rightsquigarrow+\infty} \left\{ \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0,\ \sqrt{m^2+n^2}<R} \widehat{G}\left(m,n\right)\exp\left(2\pi i\left(m\alpha+n\beta\right)\right) \right\}$$
$$= \sum\sum_{(m,n)\in\mathbb{Z}^2} G\left(m+\alpha,n+\beta\right) - \Lambda G - \Phi G\left(\alpha\right) - \Psi G\left(\beta\right).$$

In particular, if $g\left(t\right)=\chi_{\left(0,\sqrt{N}\right)}\left(t\right)$, then $G\left(x,y\right)$ is the characteristic function of a domain dilated by a factor $\sqrt{N}$ and $-dg\left(t\right)$ in $\left\{0<t<+\infty\right\}$ is a unit mass concentrated at $\sqrt{N}$. Hence the above square norm is dominated by $CN^{1/4}$ as $N\rightsquigarrow+\infty$, and this supports the conjectures in the circle and divisor problems. More generally, if $g\left(t\right)=h\left(t/\sqrt{N}\right)$ is the dilation of a function by a factor $\sqrt{N}\geq 1$, then

$$\int_0^{+\infty} \sqrt{1+t}\left|dg\left(t\right)\right| \leq N^{1/4}\int_0^{+\infty} \sqrt{1+t}\left|dh\left(t\right)\right|.$$

Hence the norm grows at most as $CN^{1/4}$, and indeed this growth can be attained. In specific examples, ΛG, $\Phi G\left(\alpha\right)$, $\Psi G\left(\beta\right)$, and $\widehat{G}\left(\xi,\eta\right)$, can be expressed in terms of known special functions. The Fourier transform of the hyperbola $\left\{0<y<N/x\right\}$ is a Bessel function, the functions $\Phi G\left(\alpha\right)$ and $\Psi G\left(\beta\right)$ are logarithmic derivatives of the Gamma function, and ΛG is a logarithm. If $\left\{0<y<N^{1/2+\delta/2}x^{-\delta}\right\}$ with $\delta\neq 1$ is a generalized hyperbola, then the Hurwitz Zeta function appears. The last example revisits the problem of counting integer points in a right angled triangle $\left\{x,y>0,\ \omega x+\omega' y<\sqrt{N}\right\}$, and here the Diophantine properties of the slope ω/ω' enter to play.

2. *Euler McLaurin and Poisson summation formulas*

The following is a mix between the Euler McLaurin and the Poisson summation formulas, and it is presented in order to introduce in a simple case the results in the next sections.

Definition. *Let $\sigma\left(t\right)$ be the saw tooth function, $\sigma\left(t\right)=t-\left[t\right]-1/2$. If $G\left(x,y\right)$ is a locally integrable function with support in the quadrant $\left\{0<x,y<+\infty\right\}$, let*

$$\Lambda G = \int_1^{+\infty}\int_1^{+\infty} G\left(x,y\right)dxdy - \int_0^1\int_0^1 G\left(x,y\right)dxdy.$$

Also, let $\Phi G\left(\alpha\right)$ and $\Psi G\left(\beta\right)$ be the $\mathbb{Z}$ periodic functions defined when $0<\alpha\leq 1$ and $0<\beta\leq 1$ by

$$\Phi G\left(\alpha\right) = \int_0^{+\infty} G\left(\alpha,y\right)dy + \left(1/2-\alpha\right)\int_0^{+\infty} G\left(1,y\right)dy + \int_1^{+\infty} \sigma\left(x-\alpha\right)\frac{\partial}{\partial x}\left(\int_0^{+\infty} G\left(x,y\right)dy\right)dx,$$

$$\Psi G\left(\beta\right) = \int_0^{+\infty} G\left(x,\beta\right)dx + \left(1/2-\beta\right)\int_0^{+\infty} G\left(x,1\right)dx + \int_1^{+\infty} \sigma\left(y-\beta\right)\frac{\partial}{\partial y}\left(\int_0^{+\infty} G\left(x,y\right)dx\right)dy.$$

Finally, let $\Sigma G(\alpha,\beta)$ be the $\mathbb{Z}^2$ periodic function defined by

$$\Sigma G(\alpha,\beta)=\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha,n+\beta)-\Lambda G-\Phi G(\alpha)-\Psi G(\beta).$$

For the functions $G(x,y)$ considered in what follows, ΛG, $\Phi G(\alpha)$, $\Psi G(\beta)$, and $\Sigma G(\alpha,\beta)$, will be always well defined and finite.

THEOREM 2.1. *If $G(x,y)$ is an integrable function, with integrable first derivatives, and with support in the quadrant $\{0<x,y<+\infty\}$, then $\Sigma G(\alpha,\beta)$ is a periodic locally integrable function, with Fourier expansion*

$$\Sigma G(\alpha,\beta)=\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta)).$$

Proof. By the two dimensional Poisson summation formula, if $G(x,y)$ is integrable then $\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha,n+\beta)$ is a periodic locally integrable function with Fourier expansion

$$\begin{aligned}\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha,n+\beta)&=\sum\sum_{(m,n)\in\mathbb{Z}^2}\widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta))\\ &=-\widehat{G}(0,0)+\sum_{m\in\mathbb{Z}}\widehat{G}(m,0)\exp(2\pi i m\alpha)+\sum_{n\in\mathbb{Z}}\widehat{G}(0,n)\exp(2\pi i n\beta)\\ &\quad+\sum\sum_{(m,n)\in\mathbb{Z}^2,\ mn\neq 0}\widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta)).\end{aligned}$$

The Fourier coefficient at the origin is

$$\widehat{G}(0,0)=\int_0^{+\infty}\int_0^{+\infty} G(x,y)\,dxdy.$$

By the one dimensional Poisson summation formula, $\sum_{m\in\mathbb{Z}}\widehat{G}(m,0)\exp(2\pi i m\alpha)$ is the Fourier expansion of a locally integrable function,

$$\sum_{m\in\mathbb{Z}}\widehat{G}(m,0)\exp(2\pi i m\alpha)=\sum_{m=0}^{+\infty}\int_0^{+\infty} G(m+\alpha,y)\,dy.$$

If $\partial G(x,y)/\partial x$ is integrable, the Euler McLaurin summation formula and an integration by parts give

$$\sum_{m\in\mathbb{Z}}\widehat{G}(m,0)\exp(2\pi i m\alpha)=\sum_{m=0}^{+\infty}\int_0^{+\infty} G(m+\alpha,y)\,dy$$

$$=\int_\alpha^{+\infty}\left(\int_0^{+\infty} G(x,y)\,dy\right)dx+\frac{1}{2}\int_0^{+\infty} G(\alpha,y)\,dy+\int_0^{+\infty}\sigma(x)\frac{\partial}{\partial x}\left(\int_0^{+\infty} G(x+\alpha,y)\,dy\right)dx$$

$$=\int_\alpha^{+\infty}\left(\int_0^{+\infty} G(x,y)\,dy\right)dx+\frac{1}{2}\int_0^{+\infty} G(\alpha,y)\,dy$$

$$+\int_\alpha^{1}\sigma(x-\alpha)\frac{\partial}{\partial x}\left(\int_0^{+\infty} G(x,y)\,dy\right)dx+\int_1^{+\infty}\sigma(x-\alpha)\frac{\partial}{\partial x}\left(\int_0^{+\infty} G(x,y)\,dy\right)dx$$

$$=\int_1^{+\infty}\left(\int_0^{+\infty} G(x,y)\,dy\right)dx$$

$$+\int_0^{+\infty} G(\alpha,y)\,dy+(1/2-\alpha)\int_0^{+\infty} G(1,y)\,dy+\int_1^{+\infty}\sigma(x-\alpha)\frac{\partial}{\partial x}\left(\int_0^{+\infty} G(x,y)\,dy\right)dx.$$

Similarly, if $\partial G(x,y)/\partial y$ is integrable,

$$\sum_{n\in\mathbb{Z}} \widehat{G}(0,n)\exp(2\pi i n\beta) = \sum_{n=0}^{+\infty}\int_0^{+\infty} G(x,n+\beta)\,dx$$
$$= \int_1^{+\infty}\left(\int_0^{+\infty} G(x,y)\,dx\right)dy$$
$$+\int_0^{+\infty} G(x,\beta)\,dx + (1/2-\beta)\int_0^{+\infty} G(x,1)\,dx + \int_1^{+\infty}\sigma(y-\beta)\frac{\partial}{\partial y}\left(\int_0^{+\infty} G(x,y)\,dx\right)dy.$$

Finally, the sum with appropriate signum of the double integrals gives

$$-\int_0^{+\infty}\int_0^{+\infty} G(x,y)\,dydx + \int_1^{+\infty}\left(\int_0^{+\infty} G(x,y)\,dy\right)dx + \int_1^{+\infty}\left(\int_0^{+\infty} G(x,y)\,dx\right)dy$$
$$= \int_1^{+\infty}\int_1^{+\infty} G(x,y)\,dydx - \int_0^1\int_0^1 G(x,y)\,dydx.$$

□

3. *Voronoï summation formula*

The assumptions in Theorem 2.1 can be relaxed. In particular, the main point in what follows is to prove an analog of the summation formula in the above theorem with $G(x,y)$ non necessarily integrable and some of the Fourier coefficients $\widehat{G}(m,0)$ and $\widehat{G}(0,n)$ infinite. The idea is simple; if a function is a suitable limit of functions to which the theorem applies, then the theorem applies also to this limit function. The following lemmas state that the Fourier transform of a hyperbolic domain is a continuous function away from the axes, and in particular $\widehat{G}(m,n)$ is well defined when $mn \neq 0$. In what follows, $f(x)$ will be a non negative decreasing convex function in $\{0<x<+\infty\}$ vanishing at infinity. One such function is invertible from its support onto its image and, with a small abuse of notation, $f^{-1}(y)=0$ for all positive y outside the image of $f(x)$. In other words, $f^{-1}(y)$ can be defined in $\{0<y<+\infty\}$ by $|\{0<x<+\infty,\ f(x)>y\}|$. In particular, this inverse function is defined even when $f(x)$ is bounded or when it has compact support.

Lemma 3.1. *Let $f(x)$ be a non negative decreasing convex function in $\{0<x<+\infty\}$ vanishing at infinity, and let Ω be the hyperbolic region between the axes $x=0$ and $y=0$ and the curve $y=f(x)$,*

$$\Omega = \{(x,y):\ 0<x,\ 0<y<f(x)\}.$$

Moreover, let $\Omega(\varepsilon)$ be the truncated hyperbolic domain

$$\Omega(\varepsilon) = \{(x,y):\ \varepsilon<x,\ \varepsilon<y<f(x)\}.$$

Then $\widehat{\chi}_{\Omega}(\xi,\eta)=\lim_{\varepsilon\rightsquigarrow 0+}\left\{\widehat{\chi}_{\Omega(\varepsilon)}(\xi,\eta)\right\}$*, the Fourier transform of the characteristic function of* Ω *in the sense of tempered distributions, is a continuous function in* $\{\xi\eta\neq 0\}$*. Moreover,*

$$\left|\widehat{\chi}_{\Omega}(\xi,\eta)-\widehat{\chi}_{\Omega(\varepsilon)}(\xi,\eta)\right|\leq\begin{cases}\dfrac{\varepsilon}{\pi|\xi|}+\dfrac{\varepsilon}{\pi|\eta|} & \text{for every } \xi\neq 0 \text{ and } \eta\neq 0,\\ \dfrac{\varepsilon}{\pi|\eta|}+\dfrac{f(1/|\xi|)}{\pi|\xi|}+\int_{f(1/|\xi|)}^{\varepsilon}f^{-1}(y)\,dy & \text{if } f(1/|\xi|)\leq\varepsilon,\\ \dfrac{\varepsilon}{\pi|\xi|}+\dfrac{f^{-1}(1/|\eta|)}{\pi|\eta|}+\int_{f^{-1}(1/|\eta|)}^{\varepsilon}f(x)\,dx & \text{if } f^{-1}(1/|\eta|)\leq\varepsilon,\\ \dfrac{f(1/|\xi|)}{\pi|\xi|}+\int_{f(1/|\xi|)}^{\varepsilon}f^{-1}(y)\,dy+\dfrac{f^{-1}(1/|\eta|)}{\pi|\eta|}+\int_{f^{-1}(1/|\eta|)}^{\varepsilon}f(x)\,dx & \\ \quad\text{if } f(1/|\xi|)\leq\varepsilon \text{ and } f^{-1}(1/|\eta|)\leq\varepsilon. & \end{cases}$$

In particular, if γ *is the solution to the equation* $f(x)=x$*, then* $\Omega(\gamma)$ *is empty and*

$$\left|\widehat{\chi}_{\Omega}(\xi,\eta)\right|\leq\frac{\gamma}{\pi|\xi|}+\frac{\gamma}{\pi|\eta|}.$$

Proof. Assume that $\varepsilon<f(\varepsilon)$, the other case is similar. Observe that

$$\left|\int_{a}^{b}\exp(-2\pi i\gamma z)\,dz\right|=\frac{|\sin(\pi\gamma(b-a))|}{\pi|\gamma|}\leq\min\{|b-a|,1/\pi|\gamma|\}.$$

If $0<\delta<\varepsilon$ then $\Omega(\varepsilon)\subseteq\Omega(\delta)$ and, decomposing $\Omega(\delta)$ into $\Omega(\varepsilon)$ and the "asymptotes" $\{\delta<x<\varepsilon,\ x<y<f(x)\}$ and $\left\{\delta<y<\varepsilon,\ y<x<f^{-1}(y)\right\}$, one obtains

$$\begin{aligned}&\widehat{\chi}_{\Omega(\delta)}(\xi,\eta)-\widehat{\chi}_{\Omega(\varepsilon)}(\xi,\eta)\\ &=\int_{\delta}^{\varepsilon}\left(\int_{x}^{f(x)}\exp(-2\pi i\eta y)\,dy\right)\exp(-2\pi i\xi x)\,dx\\ &+\int_{\delta}^{\varepsilon}\left(\int_{y}^{f^{-1}(y)}\exp(-2\pi i\xi x)\,dx\right)\exp(-2\pi i\eta y)\,dy\\ &=\int_{\delta}^{\varepsilon}\frac{\exp(-2\pi i\eta f(x))-\exp(-2\pi i\eta x)}{-2\pi i\eta}\exp(-2\pi i\xi x)\,dx\\ &+\int_{\delta}^{\varepsilon}\frac{\exp\left(-2\pi i\xi f^{-1}(y)\right)-\exp(-2\pi i\xi y)}{-2\pi i\xi}\exp(-2\pi i\eta y)\,dy.\end{aligned}$$

For every $\xi\neq 0$ and $\eta\neq 0$,

$$\left|\int_{\delta}^{\varepsilon}\frac{\exp(-2\pi i\eta f(x))-\exp(-2\pi i\eta x)}{-2\pi i\eta}\exp(-2\pi i\xi x)\,dx\right|\leq\frac{\varepsilon-\delta}{\pi|\eta|},$$

$$\left|\int_{\delta}^{\varepsilon}\frac{\exp\left(-2\pi i\xi f^{-1}(y)\right)-\exp(-2\pi i\xi y)}{-2\pi i\xi}\exp(-2\pi i\eta y)\,dy\right|\leq\frac{\varepsilon-\delta}{\pi|\xi|}.$$

And if $f(1/|\xi|)\leq\varepsilon$ or if $f^{-1}(1/|\eta|)\leq\varepsilon$, by the inequality $|\exp(i\alpha)-\exp(i\beta)|\leq\min\{2,|\alpha-\beta|\}$,

$$\begin{aligned}&\left|\int_{\delta}^{\varepsilon}\frac{\exp(-2\pi i\eta f(x))-\exp(-2\pi i\eta x)}{-2\pi i\eta}\exp(-2\pi i\xi x)\,dx\right|\\ &\leq\frac{f^{-1}(1/|\eta|)}{\pi|\eta|}+\int_{f^{-1}(1/|\eta|)}^{\varepsilon}(f(x)-x)\,dx,\end{aligned}$$

$$\left|\int_\delta^\varepsilon \frac{\exp\left(-2\pi i\xi f^{-1}(y)\right)-\exp(-2\pi i\xi y)}{-2\pi i\xi}\exp(-2\pi i\eta y)\,dy\right|$$
$$\leq \frac{f(1/|\xi|)}{\pi|\xi|}+\int_{f(1/|\xi|)}^\varepsilon\left(f^{-1}(y)-y\right)dy.$$

□

The domain where the Fourier transform is finite and continuous may be optimal, and $\widehat{\chi}_\Omega(\xi,\eta)$ can be infinite on the axes $\{\xi\eta=0\}$. Indeed,

$$\widehat{\chi}_\Omega(\xi,0)=\int_0^{1/4|\xi|} f(x)\cos(2\pi\xi x)\,dx+\int_{1/4|\xi|}^{+\infty} f(x)\cos(2\pi\xi x)\,dx-i\int_0^{+\infty} f(x)\sin(2\pi\xi x)\,dx.$$

If $f(x)$ is not integrable at the origin, the first integral is infinite, the second is finite, and the third is imaginary, hence $\widehat{\chi}_\Omega(\xi,0)$ is infinite. Similarly, if $f(x)$ is not integrable at infinity, then $\widehat{\chi}_\Omega(0,\eta)$ is infinite. In particular, if $\Omega=\{0<y<N/x\}$ then the above lemma and an homogeneity argument gives the estimate $|\widehat{\chi}_\Omega(\xi,\eta)|\leq CN\log(1/|\xi\eta|)$ when $|\xi\eta|\rightsquigarrow 0$. Indeed this Fourier transform can be expressed explicitly in terms of Bessel functions, and it has a logarithmic singularity along the axes. See Corollary 5.3 and the remarks that follow. The above result for characteristic functions can be easily generalized to functions whose level curves are the family of hyperbolae $y/t=f(x/t)$.

Lemma 3.2. *Let $f(x)$ be as in the previous lemma, and let $g(t)$ be a function of bounded variation in $\{-\infty<t<+\infty\}$ with compact support. Also, if $0<t,x,y<+\infty$ let $t=\tau(x,y)$ be the unique solution to the equation $y/t=f(x/t)$, and let*

$$G(x,y)=\begin{cases} g(\tau(x,y)) & \text{if } 0<x,y<+\infty,\\ 0 & \text{otherwise.}\end{cases}$$

Then $G(x,y)$ is a superposition of characteristic functions of the domains $t\Omega$,

$$G(x,y)=-\int_0^{+\infty}\chi_{t\Omega}(x,y)\,dg(t).$$

Similarly, $\widehat{G}(\xi,\eta)$, the Fourier transform in the sense of tempered distributions, in $\{\xi\eta\neq 0\}$ is a superposition of the Fourier transform of the domains $t\Omega$,

$$\widehat{G}(\xi,\eta)=-\int_0^{+\infty}t^2\widehat{\chi}_\Omega(t\xi,t\eta)\,dg(t).$$

This Fourier transform is a continuous function in $\{\xi\eta\neq 0\}$. Finally, if γ is the solution to the equation $f(x)=x$, then

$$\left|\widehat{G}(\xi,\eta)\right|\leq\left(\frac{\gamma}{\pi|\xi|}+\frac{\gamma}{\pi|\eta|}\right)\int_0^{+\infty}t\,|dg(t)|.$$

Proof. The representation of $G(x,y)$ as superposition of $\chi_{t\Omega}(x,y)$ follows from the definition of the measure $dg(t)$,

$$g(\tau(x,y))=-\int_{\tau(x,y)}^{+\infty}dg(t)=-\int_0^{+\infty}\chi_{(0,t)}(\tau(x,y))\,dg(t).$$

Observe that $\chi_{(0,t)}(\tau(x,y))=\chi_{t\Omega}(x,y)$. Let $\varepsilon>0$ and let

$$\Omega(\varepsilon,t)=\{\varepsilon<x,\ \varepsilon<y<tf(x/t)\}.$$

Also, let

$$G(\varepsilon, x, y) = -\int_0^{+\infty} \chi_{\Omega(\varepsilon,t)}(x, y)\, dg(t).$$

This function is integrable, and it has a Fourier transform

$$\widehat{G}(\varepsilon, \xi, \eta) = -\int_0^{+\infty} \widehat{\chi}_{\Omega(\varepsilon,t)}(\xi, \eta)\, dg(t).$$

Pointwise and in the topology of tempered distributions, $\lim_{\varepsilon \rightsquigarrow 0+} \{G(\varepsilon, x, y)\} = G(x, y)$. Hence, also $\lim_{\varepsilon \rightsquigarrow 0+} \left\{\widehat{G}(\varepsilon, \xi, \eta)\right\} = \widehat{G}(\xi, \eta)$ in the topology of tempered distributions. On the other hand, it follows from the estimate on the Fourier transform of $\Omega(\varepsilon, t)$ in Lemma 3.1 that $\left\{\widehat{G}(\varepsilon, \xi, \eta)\right\}$ also converges uniformly away from the axes $\{\xi\eta = 0\}$. Finally, the estimate on the size of $\widehat{G}(\xi, \eta)$ follows from the corresponding estimates on $\widehat{\chi}_{t\Omega}(\xi, \eta) = t^2 \widehat{\chi}_{\Omega}(t\xi, t\eta)$. □

The following is an analog of a classical formula of Voronoï for the number of integer points in shifted hyperbolic domains.

Theorem 3.3. *If $f(x)$ and $G(x, y)$ are as in the previous lemmas and if $\Sigma G(\alpha, \beta)$ is defined as in the previous section, then $\Sigma G(\alpha, \beta)$ is a bounded function,*

$$|\Sigma G(\alpha, \beta)| \le \int_0^{+\infty} \left(4 + 3tf(1/t) + 3tf^{-1}(1/t)\right) |dg(t)|.$$

Moreover, this bounded function has the Fourier expansion

$$\Sigma G(\alpha, \beta) = \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \widehat{G}(m, n) \exp(2\pi i (m\alpha + n\beta)).$$

Proof. It suffices to show that $G(x, y)$ is a suitable limit of functions to which the theorem applies. Let us split this proof into a number of steps. In particular, let us first prove the theorem when $G(x, y)$ is the characteristic function of the hyperbolic domain $\Omega = \{0 < y < f(x)\}$.

Step 1: *The functions $\sum\sum_{(m,n)\in\mathbb{Z}^2} \chi_\Omega(m + \alpha, n + \beta)$, $\Phi\chi_\Omega(\alpha)$, $\Psi\chi_\Omega(\beta)$, and $\Lambda\chi_\Omega$, are well defined and finite for every α and β. Moreover, an alternative definition for $\Lambda\chi_\Omega$, $\Phi\chi_\Omega(\alpha)$, $\Psi\chi_\Omega(\beta)$, if $0 < \alpha, \beta \le 1$, is the following:*

$$\Lambda\chi_\Omega = |\Omega \cap \{1 < x, y < +\infty\}| - |\Omega \cap \{0 < x, y < 1\}|,$$

$$\Phi\chi_\Omega(\alpha) = f(\alpha) + (1/2 - \alpha) f(1) + \int_1^{+\infty} \sigma(x - \alpha) \frac{\partial}{\partial x} f(x)\, dx,$$

$$\Psi\chi_\Omega(\beta) = f^{-1}(\beta) + (1/2 - \beta) f^{-1}(1) + \int_1^{+\infty} \sigma(y - \beta) \frac{\partial}{\partial y} f^{-1}(y)\, dy.$$

The assumption that Ω is open implies that the number of integer points in $\Omega - (\alpha, \beta)$ is always finite. The assumptions that $f(x)$ and $f^{-1}(y)$ vanish at infinity imply that $\Lambda\chi_\Omega$ is well defined and finite. Finally, the assumption $f(x)$ convex implies that $\frac{\partial}{\partial x} f(x)$ and $\frac{\partial}{\partial y} f^{-1}(y)$ are increasing, and that the integrals in the definitions of $\Phi\chi_\Omega(\alpha)$ and $\Psi\chi_\Omega(\beta)$ are well defined as generalized integrals.

Step 2: *If $0 < \alpha, \beta \le 1$, then*

$$\left|(1/2 - \alpha) f(1) + \int_1^{+\infty} \sigma(x - \alpha) \frac{\partial}{\partial x} f(x)\, dx\right| \le f(1),$$

$$\left|(1/2-\beta)\, f^{-1}(1)+\int_1^{+\infty}\sigma(y-\beta)\,\frac{\partial}{\partial y}f^{-1}(y)\,dy\right|\leq f^{-1}(1)\,.$$

Indeed, splitting the integral at $1+\alpha$ and integrating by parts,

$$\begin{aligned}&(1/2-\alpha)\, f(1)+\int_1^{+\infty}\sigma(x-\alpha)\,\frac{\partial}{\partial x}f(x)\,dx\\ &=-\int_1^{1+\alpha}f(x)\,dx+\frac{1}{2}f(1+\alpha)+\int_{1+\alpha}^{+\infty}\sigma(x-\alpha)\,\frac{\partial}{\partial x}f(x)\,dx.\end{aligned}$$

By the positivity and monotonicity of $f(x)$,

$$0\leq\int_1^{1+\alpha}f(x)\,dx\leq\alpha f(1)\,.$$

By the monotonicity of $\partial f(x)/\partial x$ and the periodicity and mean zero of $\sigma(x-\alpha)$,

$$\begin{aligned}0&\leq\frac{1}{2}f(1+\alpha)+\int_{1+\alpha}^{+\infty}\sigma(x-\alpha)\,\frac{\partial}{\partial x}f(x)\,dx\\ &\leq\frac{1}{2}f(1+\alpha)+\int_{1+\alpha}^{3/2+\alpha}\sigma(x-\alpha)\,\frac{\partial}{\partial x}f(x)\,dx\\ &\leq\frac{1}{2}f(1+\alpha)-\frac{1}{2}\int_{1+\alpha}^{3/2+\alpha}\frac{\partial}{\partial x}f(x)\,dx\\ &=f(1+\alpha)-\frac{1}{2}f(3/2+\alpha)\leq f(1)\,.\end{aligned}$$

Step 3: *If* $0<\alpha,\beta\leq1$, *then*

$$\begin{aligned}&\left|\sum\sum_{mn\neq0}\chi_{\Omega}(m+\alpha,n+\beta)-|\Omega\cap\{1<x,y<+\infty\}|\right|\\ &=\left|\sum\sum_{(m,n)\in\mathbb{Z}^2}\chi_{\Omega\cap\{1<x,y<+\infty\}}(m+\alpha,n+\beta)-|\Omega\cap\{1<x,y<+\infty\}|\right|\\ &\leq\max\{1,|\partial(\Omega\cap\{1<x,y<+\infty\})|\}\leq1+2f(1)+2f^{-1}(1)\,.\end{aligned}$$

This follows from a theorem of Jarnik: If the perimeter of a domain is larger than 1, then the number of integer points differs from the area by at most the perimeter. See [**16**].

Step 4: *Let* $\Omega(\varepsilon)=\{\varepsilon<x,\ \varepsilon<y<f(x)\}$, *with* $0<\varepsilon<1$, *and let*

$$W(\varepsilon,\alpha,\beta)=\begin{cases}\Lambda\chi_{\Omega}+\Phi\chi_{\Omega}(\alpha)+\Psi\chi_{\Omega}(\beta) & \text{if } \varepsilon<\alpha\leq1 \text{ and } \varepsilon<\beta\leq1,\\ \Lambda\chi_{\Omega}+\Phi\chi_{\Omega}(\alpha)+\Psi\chi_{\Omega}(\beta)-f(\alpha) & \text{if } 0<\alpha\leq\varepsilon \text{ and } \varepsilon<\beta\leq1,\\ \Lambda\chi_{\Omega}+\Phi\chi_{\Omega}(\alpha)+\Psi\chi_{\Omega}(\beta)-f^{-1}(\beta) & \text{if } \varepsilon<\alpha\leq1 \text{ and } 0<\beta\leq\varepsilon,\\ \Lambda\chi_{\Omega}+\Phi\chi_{\Omega}(\alpha)+\Psi\chi_{\Omega}(\beta)-f(\alpha)-f^{-1}(\beta) & \text{if } 0<\alpha\leq\varepsilon \text{ and } 0<\beta\leq\varepsilon.\end{cases}$$

Then, if $0<\alpha,\beta\leq1$,

$$\begin{aligned}&\sum\sum_{(m,n)\in\mathbb{Z}^2}\chi_{\Omega(\varepsilon)}(m+\alpha,n+\beta)-W(\varepsilon,\alpha,\beta)\\ &=\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq0}\widehat{\chi}_{\Omega(\varepsilon)}(m,n)\exp(2\pi i(m\alpha+n\beta))+\textit{Remainder}.\end{aligned}$$

The first term is a bounded function, the above is an equality in $\mathbb{L}^2(\mathbb{T}^2)$, *and the remainder is negligible when* $\varepsilon\rightsquigarrow0+$ *uniformly in* $0<a,\beta\leq1$,

$$|\textit{Remainder}|\leq6\varepsilon.$$

Assume that $0 \leq f(\varepsilon) \leq \varepsilon$. In this case $\Omega(\varepsilon)$ is empty and the remainder is $-W(\varepsilon,\alpha,\beta)$. If $\varepsilon < \alpha \leq 1$, by Step 2,

$$\begin{aligned} |\Phi\chi_\Omega(\alpha)| &\leq f(\alpha) + \left|(1/2-\alpha) f(1) + \int_1^{+\infty} \sigma(x-\alpha) \frac{\partial}{\partial x} f(x)\, dx\right| \\ &\leq f(\alpha) + f(1) \leq 2f(\varepsilon) \leq 2\varepsilon. \end{aligned}$$

If $0 < \alpha \leq \varepsilon$,

$$\begin{aligned} |\Phi\chi_\Omega(\alpha) - f(\alpha)| &= \left|(1/2-\alpha) f(1) + \int_1^{+\infty} \sigma(x-\alpha) \frac{\partial}{\partial x} f(x)\, dx\right| \\ &\leq f(1) \leq f(\varepsilon) \leq \varepsilon. \end{aligned}$$

Similar estimates hold for $\Psi\chi_\Omega(\beta)$. Finally, $\Omega \cap \{1 < x, y < +\infty\}$ is empty and

$$|\Lambda\chi_\Omega| = |\Omega \cap \{0 < x, y < 1\}| \leq 2\varepsilon.$$

Hence, if $0 \leq f(\varepsilon) \leq \varepsilon$ then

$$|Remainder| = |W(\varepsilon,\alpha,\beta)| \leq 6\varepsilon.$$

Assume that $f(\varepsilon) > \varepsilon$. By the two dimensional Poisson summation formula applied to the integrable function $\chi_{\Omega(\varepsilon)}(x,y)$, the number of integer points in $\Omega(\varepsilon) - (\alpha,\beta)$ is a bounded periodic function of the translation (α,β) with Fourier expansion

$$\sum\sum_{(m,n)\in\mathbb{Z}^2} \chi_{\Omega(\varepsilon)}(m+\alpha, n+\beta) = \sum\sum_{(m,n)\in\mathbb{Z}^2} \widehat{\chi}_{\Omega(\varepsilon)}(m,n) \exp(2\pi i (m\alpha + n\beta)).$$

The Fourier coefficient $\widehat{\chi}_{\Omega(\varepsilon)}(0,0)$ is the area of $\Omega(\varepsilon)$,

$$|\Omega(\varepsilon)| = \int_\varepsilon^{f^{-1}(\varepsilon)} f(x)\, dx - \varepsilon f^{-1}(\varepsilon) + \varepsilon^2.$$

In order to study the Fourier series along the axis $(m,0)$, define

$$h(x) = \int_{\mathbb{R}} \chi_{\Omega(\varepsilon)}(x,y)\, dy = \begin{cases} f(x) - \varepsilon & \text{if } \varepsilon < x < f^{-1}(\varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

Then, by the one-dimensional Poisson summation formula,

$$\sum_{m\in\mathbb{Z}} \widehat{\chi}_{\Omega(\varepsilon)}(m,0) \exp(2\pi i m\alpha) = \sum_{m\in\mathbb{Z}} h(m+\alpha).$$

This is an equality in $\mathbb{L}^2(\mathbb{T})$. Moreover if $\varepsilon < \alpha \leq 1$ and if M is the smallest integer with $M + \alpha \geq f^{-1}(\varepsilon)$, so that $f(x+\alpha) \leq \varepsilon$ if $x \geq M$, by the Euler McLaurin summation formula,

$$\begin{aligned} \sum_{m\in\mathbb{Z}} h(m+\alpha) &= -M\varepsilon - f(M+\alpha) + \sum_{m=0}^{M} f(m+\alpha) \\ = -M\varepsilon - f(M+\alpha) &+ \int_0^M f(x+\alpha)\, dx + \frac{1}{2} f(\alpha) + \frac{1}{2} f(M+\alpha) + \int_0^M \sigma(x) \frac{\partial}{\partial x} f(x+\alpha)\, dx \\ &= -M\varepsilon - \frac{1}{2} f(M+\alpha) + \int_\alpha^{M+\alpha} f(x)\, dx + \frac{1}{2} f(\alpha) \\ &+ \int_\alpha^1 \sigma(x-\alpha) \frac{\partial}{\partial x} f(x)\, dx + \int_1^{M+\alpha} \sigma(x-\alpha) \frac{\partial}{\partial x} f(x)\, dx \\ = -\varepsilon f^{-1}(\varepsilon) &+ \int_1^{f^{-1}(\varepsilon)} f(x)\, dx + f(\alpha) + (1/2-\alpha) f(1) + \int_1^{+\infty} \sigma(x-\alpha) \frac{\partial}{\partial x} f(x)\, dx \\ &+ Remainder. \end{aligned}$$

The remainder is

$$\varepsilon\left(f^{-1}(\varepsilon)-M\right)-\frac{1}{2}f(M+\alpha)+\int_{f^{-1}(\varepsilon)}^{M+\alpha}f(x)\,dx-\int_{M+\alpha}^{+\infty}\sigma(x-\alpha)\frac{\partial}{\partial x}f(x)\,dx.$$

This remainder can be bounded by 3ε, independently of $f(x)$ and α,

$$\left|\varepsilon\left(f^{-1}(\varepsilon)-M\right)\right|\leq\varepsilon,$$

$$\left|\frac{1}{2}f(M+\alpha)\right|\leq\varepsilon/2,$$

$$\left|\int_{f^{-1}(\varepsilon)}^{M+\alpha}f(x)\,dx\right|\leq\varepsilon\left(M+\alpha-f^{-1}(\varepsilon)\right)\leq\varepsilon,$$

$$\left|\int_{M+\alpha}^{+\infty}\sigma(x-\alpha)\frac{\partial}{\partial x}f(x)\,dx\right|\leq\frac{1}{2}\int_{M+\alpha}^{M+\alpha+1/2}\left|\frac{\partial}{\partial x}f(x)\right|dx\leq\frac{f(M+\alpha)}{4}\leq\frac{\varepsilon}{4}.$$

The last inequality follows from the alternating signum of $\sigma(x-\alpha)$ and the monotonicity of $\partial f(x)/\partial x$. Similarly, if $0<\alpha\leq\varepsilon$,

$$\sum_{m\in\mathbb{Z}}h(m+\alpha)=-(M-1)\varepsilon-f(M+\alpha)+\sum_{m=1}^{M}f(m+\alpha)$$

$$=-\varepsilon f^{-1}(\varepsilon)+\int_{1}^{f^{-1}(\varepsilon)}f(x)\,dx+(1/2-\alpha)f(1)+\int_{1}^{+\infty}\sigma(x-\alpha)\frac{\partial}{\partial x}f(x)\,dx$$

$$+\mathit{Remainder}.$$

And again this remainder has size at most 3ε. In order to study the Fourier series along the axis $(0,n)$, define

$$k(y)=\int_{\mathbb{R}}\chi_{\Omega(\varepsilon)}(x,y)\,dx=\begin{cases}f^{-1}(y)-\varepsilon & \text{if } \varepsilon<x<f(\varepsilon),\\ 0 & \text{otherwise.}\end{cases}$$

Then, by the one-dimensional Poisson summation formula,

$$\sum_{n\in\mathbb{Z}}\widehat{\chi}_{\Omega(\varepsilon)}(0,n)\exp(2\pi in\beta)=\sum_{n\in\mathbb{Z}}k(n+\beta).$$

Moreover if $\varepsilon<\beta\leq1$, by the Euler McLaurin summation formula,

$$\sum_{n\in\mathbb{Z}}k(n+\beta)$$

$$=-\varepsilon f(\varepsilon)+\int_{1}^{f(\varepsilon)}f^{-1}(y)\,dy+f^{-1}(\beta)+(1/2-\beta)f^{-1}(1)+\int_{1}^{+\infty}\sigma(y-\beta)\frac{\partial}{\partial y}f^{-1}(y)\,dy$$

$$+\mathit{Remainder}.$$

While if $0<\beta\leq\varepsilon$,

$$\sum_{n\in\mathbb{Z}}k(n+\beta)$$

$$=-\varepsilon f(\varepsilon)+\int_{1}^{f(\varepsilon)}f^{-1}(y)\,dy+(1/2-\beta)f^{-1}(1)+\int_{1}^{+\infty}\sigma(y-\beta)\frac{\partial}{\partial y}f^{-1}(y)\,dy$$

$$+\mathit{Remainder}.$$

And again the remainders have size at most 3ε. Hence, collecting all the terms,

$$\begin{aligned}
&\sum\sum_{(m,n)\in\mathbb{Z}^2} \chi_{\Omega(\varepsilon)}(m+\alpha,n+\beta) \\
&= -\varepsilon f(\varepsilon) - \int_\varepsilon^1 f(x)\,dx + \int_1^{f(\varepsilon)} f^{-1}(y)\,dy \\
&+ (1/2 \pm 1/2) f(\alpha) + (1/2-\alpha) f(1) + \int_1^{+\infty} \sigma(x-\alpha) \frac{\partial}{\partial x} f(x)\,dx \\
&+ (1/2 \pm 1/2) f^{-1}(\beta) + (1/2-\beta) f^{-1}(1) + \int_1^{+\infty} \sigma(y-\beta) \frac{\partial}{\partial y} f^{-1}(y)\,dy \\
&+ \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \widehat{\chi}_{\Omega(\varepsilon)}(m,n) \exp(2\pi i(m\alpha+n\beta)) + \textit{Remainder}.
\end{aligned}$$

Of course, $(1/2 \pm 1/2)$ is either 0 or 1, according to $\alpha, \beta \gtreqless \varepsilon$. Finally, the terms with ε give

$$\begin{aligned}
&-\varepsilon f(\varepsilon) - \int_\varepsilon^1 f(x)\,dx + \int_1^{f(\varepsilon)} f^{-1}(y)\,dy \\
&= |\Omega \cap \{1 < x, y < +\infty\}| - |\Omega \cap \{0 < x, y < 1\}|.
\end{aligned}$$

Step 5: *For every* $0 < \varepsilon < 2/3$ *and every* $0 < \alpha, \beta \leq 1$,

$$\left| \sum\sum_{(m,n)\in\mathbb{Z}^2} \chi_{\Omega(\varepsilon)}(m+\alpha,n+\beta) - W(\varepsilon,\alpha,\beta) \right| \leq 4 + 3f(1) + 3f^{-1}(1).$$

Assume $0 \leq f(\varepsilon) \leq \varepsilon$. Then $\Omega(\varepsilon)$ is empty and, by Step 4,

$$|W(\varepsilon,\alpha,\beta)| \leq 6\varepsilon < 4.$$

Assume $f(\varepsilon) > \varepsilon$. If $\varepsilon < \alpha, \beta \leq 1$, then

$$\begin{aligned}
&\sum\sum_{(m,n)\in\mathbb{Z}^2} \chi_{\Omega(\varepsilon)}(m+\alpha,n+\beta) - W(\varepsilon,\alpha,\beta) \\
&= \sum\sum_{(m,n)\in\mathbb{Z}^2} \chi_\Omega(m+\alpha,n+\beta) - \Lambda\chi_\Omega - \Phi\chi_\Omega(\alpha) - \Psi\chi_\Omega(\beta) \\
&= \left(\sum_{n\in\mathbb{Z}} \chi_\Omega(\alpha,n+\beta) - f(\alpha) \right) + \left(\sum_{m\in\mathbb{Z}} \chi_\Omega(m+\alpha,\beta) - f^{-1}(\beta) \right) \\
&- \chi_\Omega(\alpha,\beta) + |\Omega \cap \{0 < x, y < 1\}| \\
&+ \sum\sum_{mn\neq 0} \chi_\Omega(m+\alpha,n+\beta) - |\Omega \cap \{1 < x, y < +\infty\}| \\
&- (1/2-\alpha) f(1) - \int_1^{+\infty} \sigma(x-\alpha) \frac{\partial}{\partial x} f(x)\,dx \\
&- (1/2-\beta) f^{-1}(1) - \int_1^{+\infty} \sigma(y-\beta) \frac{\partial}{\partial y} f^{-1}(y)\,dy.
\end{aligned}$$

All these terms are uniformly bounded in $0<\alpha,\beta\leq 1$. More precisely,

$$\left|\sum_{n\in\mathbb{Z}}\chi_\Omega\left(\alpha,n+\beta\right)-f\left(\alpha\right)\right|\leq 1,$$
$$\left|\sum_{m\in\mathbb{Z}}\chi_\Omega\left(m+\alpha,\beta\right)-f^{-1}\left(\beta\right)\right|\leq 1,$$
$$\left|-\chi_\Omega\left(\alpha,\beta\right)+\left|\Omega\cap\{0<x,y<1\}\right|\right|\leq 1,$$
$$\left|-\left(1/2-\alpha\right)f\left(1\right)-\int_1^{+\infty}\sigma\left(x-\alpha\right)\frac{\partial}{\partial x}f\left(x\right)dx\right|\leq f\left(1\right),$$
$$\left|-\left(1/2-\beta\right)f^{-1}\left(1\right)-\int_1^{+\infty}\sigma\left(y-\beta\right)\frac{\partial}{\partial y}f^{-1}\left(y\right)dy\right|\leq f^{-1}\left(1\right),$$
$$\left|\sum\sum_{mn\neq 0}\chi_\Omega\left(m+\alpha,n+\beta\right)-\left|\Omega\cap\{1<x,y<+\infty\}\right|\right|\leq 1+2f\left(1\right)+2f^{-1}\left(1\right).$$

The first three inequalities are immediate, the fourth and fifth inequalities follow from Step 2, and the sixth from Step 3. If $0<\alpha\leq\varepsilon<\beta\leq 1$,

$$\begin{aligned}&\sum\sum_{(m,n)\in\mathbb{Z}^2}\chi_{\Omega(\varepsilon)}\left(m+\alpha,n+\beta\right)-W\left(\varepsilon,\alpha,\beta\right)\\
&=\left(\sum_{m\in\mathbb{Z}}\chi_\Omega\left(m+\alpha,\beta\right)-f^{-1}\left(\beta\right)\right)+\left|\Omega\cap\{0<x,y<1\}\right|\\
&\quad+\sum\sum_{mn\neq 0}\chi_\Omega\left(m+\alpha,n+\beta\right)-\left|\Omega\cap\{1<x,y<+\infty\}\right|\\
&\quad-\left(1/2-\alpha\right)f\left(1\right)-\int_1^{+\infty}\sigma\left(x-\alpha\right)\frac{\partial}{\partial x}f\left(x\right)dx\\
&\quad-\left(1/2-\beta\right)f^{-1}\left(1\right)-\int_1^{+\infty}\sigma\left(y-\beta\right)\frac{\partial}{\partial y}f^{-1}\left(y\right)dy.\end{aligned}$$

Again, all these terms are uniformly bounded in the square $0<\alpha,\beta\leq 1$. The cases $0<\beta\leq\varepsilon<\alpha$ and $0<\alpha,\beta\leq\varepsilon$ are similar.

Step 6: *Pointwise in $0<\alpha,\beta\leq 1$ and in the metric of $\mathbb{L}^2\left(\mathbb{T}^2\right)$,*

$$\Sigma G\left(\alpha,\beta\right)=\lim_{\varepsilon\rightsquigarrow 0+}\left\{\sum\sum_{(m,n)\in\mathbb{Z}^2}\chi_{\Omega(\varepsilon)-(\alpha,\beta)}\left(m,n\right)-W\left(\varepsilon,\alpha,\beta\right)\right\}.$$

The pointwise convergence follows from the definitions of $\Omega\left(\varepsilon\right)$ and $W\left(\varepsilon,\alpha,\beta\right)$, Step 5 implies that this convergence is dominated, and the square norm convergence follows from the dominated convergence.

Step 7: *In the metric of $\mathbb{L}^2\left(\mathbb{T}^2\right)$,*

$$\begin{aligned}&\lim_{\varepsilon\rightsquigarrow 0+}\left\{\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq 0}\widehat{\chi}_{\Omega(\varepsilon)}\left(m,n\right)\exp\left(2\pi i\left(m\alpha+n\beta\right)\right)\right\}\\
&=\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq 0}\widehat{\chi}_{\Omega}\left(m,n\right)\exp\left(2\pi i\left(m\alpha+n\beta\right)\right).\end{aligned}$$

By Step 4 and Step 6, $\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq 0}\widehat{\chi}_{\Omega(\varepsilon)}\left(m,n\right)\exp\left(2\pi i\left(m\alpha+n\beta\right)\right)$ converges in the metric of $\mathbb{L}^2\left(\mathbb{T}^2\right)$, and this implies the convergence of the Fourier coefficients

$\left\{\widehat{\chi}_{\Omega(\varepsilon)}(m,n)\right\}_{mn\neq 0}$ to the Fourier coefficients of the limit function. On the other hand, by Lemma 3.1, $\left\{\widehat{\chi}_{\Omega(\varepsilon)}(m,n)\right\}_{mn\neq 0}$ converges to $\{\widehat{\chi}_{\Omega}(m,n)\}_{mn\neq 0}$ uniformly.

By the previous steps, the theorem holds for characteristic functions. The extension to superposition of characteristic functions is easy.

Step 8: *The theorem holds for the function $G(x,y)$ defined by*

$$G(x,y)=-\int_0^{+\infty}\chi_{t\Omega}(x,y)\,dg(t).$$

For every $0<\alpha,\beta\leq 1$,

$$\sum\sum_{(m,n)\in\mathbb{Z}^2}G(m+\alpha,n+\beta)=-\int_0^{+\infty}\left(\sum\sum_{(m,n)\in\mathbb{Z}^2}\chi_{t\Omega}(m+\alpha,n+\beta)\right)dg(t).$$

Observe that if $g(t)$ has compact support then the above sums are finite and the integral is well defined. Also ΛG and $\Phi G(\alpha)$ and $\Psi G(\beta)$ are well defined and finite,

$$\Lambda G=-\int_0^{+\infty}\Lambda\chi_{t\Omega}dg(t),$$
$$\Phi G(\alpha)=-\int_0^{+\infty}\Phi\chi_{t\Omega}(\alpha)\,dg(t),$$
$$\Psi G(\beta)=-\int_0^{+\infty}\Psi\chi_{t\Omega}(\beta)\,dg(t).$$

Integrating the Voronoï formula for $t\Omega$ proved in the previous steps, one obtains

$$\Sigma G(\alpha,\beta)=-\int_0^{+\infty}\Sigma\chi_{t\Omega}(\alpha,\beta)\,dg(t).$$

This function $\Sigma G(\alpha,\beta)$ is bounded,

$$\int_0^{+\infty}|\Sigma\chi_{t\Omega}(\alpha,\beta)|\,|dg(t)|\leq\int_0^{+\infty}\left(4+3tf(1/t)+3tf^{-1}(1/t)\right)|dg(t)|.$$

The Fourier coefficients of this bounded function are

$$\begin{aligned}&\int_0^1\int_0^1\left(-\int_0^{+\infty}\Sigma\chi_{t\Omega}(\alpha,\beta)\,dg(t)\right)\exp(-2\pi i(m\alpha+n\beta))\,d\alpha d\beta\\&=-\int_0^{+\infty}\left(\int_0^1\int_0^1\Sigma\chi_{t\Omega}(\alpha,\beta)\exp(-2\pi i(m\alpha+n\beta))\,d\alpha d\beta\right)dg(t)\\&=\begin{cases}-\int_0^{+\infty}t^2\widehat{\chi}_{\Omega}(tm,tn)\,dg(t) & \text{if } mn\neq 0,\\ 0 \quad \text{if } mn\neq 0.\end{cases}\end{aligned}$$

Finally by Lemma 3.2, if $mn\neq 0$,

$$-\int_0^{+\infty}t^2\widehat{\chi}_{\Omega}(tm,tn)\,dg(t)=\widehat{G}(m,n).$$

□

Observe that the estimates in the statement of the above and following theorems do not depend explicitly on the support of $g(t)$. This suggests that the assumption of compact support can be relaxed, and replaced by a suitable decay of $dg(t)$ at infinity.

4. *Estimates of the Fourier expansions*

An immediate corollary of the pointwise estimate in Theorem 3.3 is that for every $1 \leq p \leq +\infty$,

$$\left\{ \int_0^1 \int_0^1 |\Sigma G(\alpha, \beta)|^p \, d\alpha d\beta \right\}^{1/p} \leq \int_0^{+\infty} \left(4 + 3tf(1/t) + 3tf^{-1}(1/t)\right) |dg(t)| .$$

On the other hand, the Hausdorff Young inequality when $2 \leq p < +\infty$ and $1/p + 1/q = 1$, or more simply the Parseval equality when $p = 2$, give

$$\left\{ \int_0^1 \int_0^1 \left| \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \widehat{G}(m,n) \exp(2\pi i (m\alpha + n\beta)) \right|^p d\alpha d\beta \right\}^{1/p}$$
$$\leq \left\{ \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \left| \widehat{G}(m,n) \right|^q \right\}^{1/q} .$$

In several cases the first estimate is far from optimal, and the second estimate is better than the first one. These inequalities hold under the sole assumption of convexity, but under some mild extra assumptions one can give more precise quantitative estimates.

Lemma 4.1. *Let $f(x)$ be a non negative decreasing convex function on $0 < x < +\infty$ vanishing at infinity and with both $f''(x)$ and $\left(f^{-1}\right)''(y)$ decreasing. Let $g(t)$ be a function of bounded variation with compact support. Let $t = \tau(x, y)$ be the unique solution to the equation $y/t = f(x/t)$ in $0 < t, x, y < +\infty$. Finally, let $G(x, y) = g(\tau(x, y))$ if $0 < x, y < +\infty$, and $G(x, y) = 0$ otherwise. If $\xi\eta < 0$, then*

$$\left|\widehat{G}(\xi,\eta)\right| \leq \frac{5}{4\pi^2 |\xi\eta|} \int_0^{+\infty} |dg(t)| .$$

If $\xi\eta > 0$, then

$$\left|\widehat{G}(\xi,\eta)\right| \leq \frac{1}{2\pi^2 |\xi\eta|} \int_0^{+\infty} |dg(t)|$$
$$+\sqrt{\frac{3}{2\pi^3}} \left|\eta^3 f''\left(\left(f'\right)^{-1}(-\xi/\eta)\right)\right|^{-1/2} \int_0^{+\infty} \sqrt{t}\, |dg(t)|$$
$$+\sqrt{\frac{3}{2\pi^3}} \left|\xi^3 \left(f^{-1}\right)''\left(\left(\left(f^{-1}\right)'\right)^{-1}(-\eta/\xi)\right)\right|^{-1/2} \int_0^{+\infty} \sqrt{t}\, |dg(t)| .$$

Proof. Recall that, by Lemma 3.2, if $\Omega = \{0 < x,\ 0 < y < f(x)\}$ then

$$\widehat{G}(\xi,\eta) = -\int_0^{+\infty} t^2 \widehat{\chi}_\Omega(t\xi, t\eta)\, dg(t) .$$

It then suffices to estimate $t^2\widehat{\chi}_\Omega(t\xi, t\eta)$. In particular, it suffices to estimate the Fourier transform of $\Omega(\varepsilon) = \{\varepsilon < x,\ \varepsilon < y < f(x)\}$, with $0 < \varepsilon < f(\varepsilon)$. The main tool is the lemma of van der Corput: If a phase $\varphi(z)$ is smooth real valued and if $\varphi'(z)$ is monotone in $a < z < b$, then

$$\left| \int_a^b \exp(i\varphi(z))\, dz \right| \leq \frac{3}{\inf_{a<z<b}\{|\varphi'(z)|\}} .$$

First assume that $\xi\eta < 0$. The Fourier transform of $\Omega(\varepsilon)$ is

$$\begin{aligned}\widehat{\chi}_{\Omega(\varepsilon)}(\xi,\eta) &= \int_{\varepsilon}^{f^{-1}(\varepsilon)}\left(\int_{\varepsilon}^{f(x)} \exp(-2\pi i\eta y)\,dy\right)\exp(-2\pi i\xi x)\,dx \\ &= \frac{\exp\left(-2\pi i\left(\xi f^{-1}(\varepsilon)+\eta\varepsilon\right)\right)-\exp(-2\pi i(\xi\varepsilon+\eta\varepsilon))}{4\pi^2\xi\eta} \\ &\quad -\frac{1}{2\pi i\eta}\int_{\varepsilon}^{f^{-1}(\varepsilon)}\exp(-2\pi i(\xi x+\eta f(x)))\,dx.\end{aligned}$$

Since $f'(x) \le 0$, if $\xi\eta < 0$ then

$$\left|\frac{\partial}{\partial x}(\xi x+\eta f(x))\right| = |\xi+\eta f'(x)| \ge |\xi|.$$

And, by the lemma of van der Corput,

$$\left|\frac{1}{2\pi i\eta}\int_{\varepsilon}^{f^{-1}(\varepsilon)}\exp(-2\pi i(\xi x+\eta f(x)))\,dx\right| \le \frac{3}{4\pi^2|\xi\eta|}.$$

Hence, if $\xi\eta < 0$,

$$\left|\widehat{\chi}_{\Omega(\varepsilon)}(\xi,\eta)\right| \le \frac{5}{4\pi^2|\xi\eta|}.$$

Now assume that $\xi\eta > 0$. By the assumptions, $f'(x)$ is invertible from the support of $f(x)$ onto the whole negative axis and, similarly, $\left(f^{-1}\right)'(y)$ is invertible from the image of $f(x)$ onto the whole negative axis. Let $p = (f')^{-1}(-\xi/\eta)$ and $q = \left(\left(f^{-1}\right)'\right)^{-1}(-\eta/\xi)$. Observe that $q = f(p)$, and that at this point (p,q) the tangent to the curve $y = f(x)$ is parallel to the level sets of the phase $\xi x+\eta y$ of the Fourier transform, hence it is a critical point of the oscillatory integral. If both p and q are larger than ε, then

$$\Omega(\varepsilon) = \{\varepsilon < x < p,\ \varepsilon < y < q\} \cup \{\varepsilon < x < p,\ q < y < f(x)\} \cup \left\{\varepsilon < y < q,\ p < x < f^{-1}(y)\right\}.$$

The Fourier transform of the rectangle $\{\varepsilon < x < p,\ \varepsilon < y < q\}$ is

$$\begin{aligned}&\int_{\varepsilon}^{p}\exp(-2\pi i\xi x)\,dx\int_{\varepsilon}^{q}\exp(-2\pi i\eta y)\,dy \\ &= \frac{\exp(-2\pi i(\xi p+\eta q))-\exp(-2\pi i(\xi p+\eta\varepsilon))-\exp(-2\pi i(\xi\varepsilon+\eta q))+\exp(-2\pi i(\xi\varepsilon+\eta\varepsilon))}{-4\pi^2\xi\eta}.\end{aligned}$$

The Fourier transform of the curvilinear triangle $\left\{\varepsilon < y < q,\ p < x < f^{-1}(y)\right\}$ is

$$\begin{aligned}&\int_{\varepsilon}^{p}\left(\int_{q}^{f(x)}\exp(-2\pi i\eta y)\,dy\right)\exp(-2\pi i\xi x)\,dx = \\ &\frac{\exp(-2\pi i(\xi p+\eta q))-\exp(-2\pi i(\xi\varepsilon+\eta q))}{4\pi^2\xi\eta}-\frac{1}{2\pi i\eta}\int_{\varepsilon}^{p}\exp(-2\pi i(\xi x+\eta f(x)))\,dx.\end{aligned}$$

The Fourier transform of the curvilinear triangle $\{\varepsilon < x < p,\ q < y < f(x)\}$ is

$$\begin{aligned}&\int_{\varepsilon}^{q}\left(\int_{p}^{f^{-1}(y)}\exp(-2\pi i\xi x)\,dx\right)\exp(-2\pi i\eta y)\,dy = \\ &\frac{\exp(-2\pi i(\xi p+\eta q))-\exp(-2\pi i(\xi p+\eta\varepsilon))}{4\pi^2\xi\eta}-\frac{1}{2\pi i\xi}\int_{\varepsilon}^{q}\exp\left(-2\pi i\left(\xi f^{-1}(y)+\eta y\right)\right)dy.\end{aligned}$$

Hence, the Fourier transform of $\Omega(\varepsilon)$ is

$$\widehat{\chi}_{\Omega(\varepsilon)}(\xi,\eta) = \frac{\exp(-2\pi i(\xi p+\eta q)) - \exp(-2\pi i(\xi\varepsilon+\eta\varepsilon))}{4\pi^2\xi\eta}$$
$$-\frac{1}{2\pi i\eta}\int_\varepsilon^p \exp(-2\pi i(\xi x+\eta f(x)))\,dx - \frac{1}{2\pi i\xi}\int_\varepsilon^q \exp\left(-2\pi i\left(\xi f^{-1}(y)+\eta y\right)\right)dy.$$

The sum of the two exponentials is bounded by $2^{-1}\pi^{-2}|\xi\eta|^{-1}$,

$$\left|\frac{\exp(-2\pi i(\xi p+\eta q)) - \exp(-2\pi i(\xi\varepsilon+\eta\varepsilon))}{4\pi^2\xi\eta}\right| \le \frac{1}{2\pi^2|\xi\eta|}.$$

The two integrals can be estimated by the lemma of van der Corput. The critical point of the phase $\xi x+\eta f(x)$ is $x=(f')^{-1}(-\xi/\eta)=p$ and, since $|f'(x)|$ is decreasing, for every $0<\delta\le p-\varepsilon$ one has

$$\left|\frac{1}{2\pi i\eta}\int_\varepsilon^p \exp(-2\pi i(\xi x+\eta f(x)))\,dx\right|$$
$$\le \left|\frac{1}{2\pi i\eta}\int_\varepsilon^{p-\delta} \exp(-2\pi i(\xi x+\eta f(x)))\,dx\right| + \frac{\delta}{2\pi|\eta|}$$
$$\le \frac{3}{4\pi^2|\eta||\xi+\eta f'(p-\delta)|} + \frac{\delta}{2\pi|\eta|}.$$

Under the assumption that $f''(x)$ is decreasing,

$$|\eta||\xi+\eta f'(p-\delta)| = |\eta|^2|f'(p-\delta)-f'(p)| \ge \delta|\eta|^2|f''(p)|.$$

Hence,

$$\left|\frac{1}{2\pi i\eta}\int_\varepsilon^p \exp(-2\pi i(\xi x+\eta f(x)))\,dx\right| \le \frac{3}{4\pi^2|\eta|^2|f''(p)|\delta} + \frac{\delta}{2\pi|\eta|}.$$

Then the choice $\delta=\sqrt{3/(2\pi|\eta||f''(p)|)}$ gives

$$\left|\frac{1}{2\pi i\eta}\int_\varepsilon^p \exp(-2\pi i(\xi x+\eta f(x)))\,dx\right| \le (3/2)^{1/2}\pi^{-3/2}|\eta|^{-3/2}|f''(p)|^{-1/2}.$$

Similarly, by symmetry,

$$\left|\frac{1}{2\pi i\xi}\int_\varepsilon^q \exp\left(-2\pi i\left(\xi f^{-1}(y)+\eta y\right)\right)dy\right| \le (3/2)^{1/2}\pi^{-3/2}|\xi|^{-3/2}\left|\left(f^{-1}\right)''(q)\right|^{-1/2}.$$

Finally, the cases $p\le\varepsilon$ or $q\le\varepsilon$ are similar, and even easier. □

Theorem 4.2. *There exists an absolute positive constant C with the following property: If $f(x)$ is non negative decreasing convex in $\{0<x<+\infty\}$ and vanishing at infinity, if also $f''(x)$ and $\left(f^{-1}\right)''(y)$ and $-f'(x)^3/f''(x)$ and $-\left(f^{-1}\right)'(y)^3/\left(f^{-1}\right)''(y)$ are decreasing in $\{0<x,y<+\infty\}$, if $g(t)$ has bounded variation and compact support, and if $G(x,y)$ is associated to $f(x)$ and $g(t)$ as in the above lemma, then*

$$\left\{\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn<0}\left|\widehat{G}(m,n)\right|^2\right\}^{1/2} \le C\left(\int_0^{+\infty}\sqrt{1+t}\,|dg(t)|\right)\sqrt{1+(f')^{-1}(-1)+\left(\left(f^{-1}\right)'\right)^{-1}(-1)}.$$

Proof. Let us prove the theorem in the particular case of $G(x,y)=\chi_\Omega(x,y)$. The general case is similar. By Lemma 4.1,

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn<0}\left|\widehat{\chi}_\Omega(m,n)\right|^2 \le C\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn<0}|mn|^{-2} \le C.$$

Moreover,

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn>0} \left|\widehat{\chi}_\Omega(m,n)\right|^2 \le C \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn>0} |mn|^{-2}$$
$$+C \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn>0} \left| |n|^3 f''\left((f')^{-1}(-m/n)\right)\right|^{-1}$$
$$+C \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn>0} \left| |m|^3 \left(f^{-1}\right)''\left(\left(\left(f^{-1}\right)'\right)^{-1}(-n/m)\right)\right|^{-1}.$$

The following functions are decreasing in the quadrant $\{\xi > 1/2,\ \eta > 1/2\}$:

$$\xi \rightsquigarrow \frac{1}{\left|\eta^3 f''\left((f')^{-1}(-\xi/\eta)\right)\right|},$$
$$\eta \rightsquigarrow \frac{1}{\left|\eta^3 f''\left((f')^{-1}(-\xi/\eta)\right)\right|} = \frac{|-\xi/\eta|^3}{|\xi|^3 f''\left((f')^{-1}(-\xi/\eta)\right)},$$
$$\xi \rightsquigarrow \frac{1}{\left|\xi^3 \left(f^{-1}\right)''\left(\left(\left(f^{-1}\right)'\right)^{-1}(-\eta/\xi)\right)\right|} = \frac{|-\eta/\xi|^3}{|\eta|^3 \left(f^{-1}\right)''\left(\left(\left(f^{-1}\right)'\right)^{-1}(-\eta/\xi)\right)},$$
$$\eta \rightsquigarrow \frac{1}{\left|\xi^3 \left(f^{-1}\right)''\left(\left(\left(f^{-1}\right)'\right)^{-1}(-\eta/\xi)\right)\right|}.$$

To see this it suffices to apply the assumptions with the change of variables $f'(x) = -\xi/\eta$ and $\left(f^{-1}\right)'(y) = -\eta/\xi$. In particular, $\left|\widehat{\chi}_\Omega(\xi,\eta)\right|$ has a majorant which is decreasing in both the variables ξ and η in the quadrant $\{\xi > 0,\ \eta > 0\}$, and this allows to substitute a sum with an integral,

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn<0} \left| |n|^3 f''\left((f')^{-1}(-m/n)\right)\right|^{-1} \le 4\int_{1/2}^{+\infty}\int_{1/2}^{+\infty} \frac{d\xi d\eta}{\eta^3 f''\left((f')^{-1}(-\xi/\eta)\right)}.$$

With the change of variables

$$\begin{cases} u = (f')^{-1}(-\xi\eta), \\ v = (f')^{-1}(-\xi/\eta), \end{cases} \qquad d\xi d\eta = \frac{f''(u)\, f''(v)}{-2f'(v)} du dv,$$

the integral becomes

$$\int_{1/2}^{+\infty}\int_{1/2}^{+\infty} \frac{d\xi d\eta}{\eta^3 f''\left((f')^{-1}(-\xi/\eta)\right)}$$
$$= \frac{1}{2}\int_{\{-1<f'(v)<0\}} (-f'(v))^{1/2} \left(\int_0^{(f')^{-1}\left(1/\left(4f'(v)\right)\right)} \frac{f''(u)}{(-f'(u))^{3/2}} du\right) dv$$
$$+\frac{1}{2}\int_{\{f'(v)<-1\}} (-f'(v))^{1/2} \left(\int_0^{(f')^{-1}\left(f'(v)/4\right)} \frac{f''(u)}{(-f'(u))^{3/2}} du\right) dv$$
$$= 2\,(f')^{-1}(-1) + 2f\left((f')^{-1}(-1)\right).$$

By symmetry,

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn>0} \left| |m|^3 \left(f^{-1}\right)'' \left(\left(\left(f^{-1}\right)'\right)^{-1}(-n/m)\right)\right|^{-1}$$
$$\leq 4\int_{1/2}^{+\infty}\int_{1/2}^{+\infty} \frac{d\xi d\eta}{\xi^3 \left(f^{-1}\right)'' \left(\left(\left(f^{-1}\right)'\right)^{-1}(-\eta/\xi)\right)}$$
$$= 8\left(\left(f^{-1}\right)'\right)^{-1}(-1) + 8f^{-1}\left(\left(\left(f^{-1}\right)'\right)^{-1}(-1)\right).$$

Then observe that

$$\left(\left(f^{-1}\right)'\right)^{-1}(-1) = f\left(\left(f'\right)^{-1}(-1)\right),$$
$$f^{-1}\left(\left(\left(f^{-1}\right)'\right)^{-1}(-1)\right) = \left(f'\right)^{-1}(-1).$$

□

The assumptions $-f'(x)^3/f''(x)$ and $-\left(f^{-1}\right)'(y)^3/\left(f^{-1}\right)''(y)$ decreasing are only used to substitute a sum with an integral, and they may be replaced by other assumptions. The estimates in the above lemma and theorem are essentially best possible. For example, when $\Omega = \left\{0 < y < x^{-\delta}\right\}$ with $0 < \delta < +\infty$, the lemma gives

$$|\widehat{\chi}_\Omega(\xi,\eta)| \leq \begin{cases} C\,|\xi\eta|^{-1} & \text{if } \xi\eta < 0, \\ C\,|\xi|^{-(2+\delta)/(2+2\delta)}\,|\eta|^{-(2+1/\delta)/(2+2/\delta)} & \text{if } \xi\eta > 0. \end{cases}$$

Indeed, a homogeneity argument shows that $\widehat{\chi}_\Omega(\xi,\eta) = |\xi|^{\delta-1}\,\widehat{\chi}_\Omega\left(\pm 1, |\xi|^\delta \eta\right)$, and it can be proved that the above estimates can be improved as $|\xi|^\delta |\eta| \rightsquigarrow 0$, but they are sharp as $|\xi|^\delta |\eta| \rightsquigarrow +\infty$.

The above Fourier expansions not only converge in norm, but also pointwise. By a result in [**11**], the partial sums of Fourier series in Sobolev spaces of positive order converge almost everywhere. In particular, if the Fourier coefficients are a bit more than square integrable, as suggested by Lemma 4.1, then the Fourier expansions converge almost everywhere. However, the following result is slightly more precise.

THEOREM 4.3. *If $f(x)$, $g(t)$, and $G(x,y)$ are as in the previous Theorem 4.2, then the Fourier series*

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \widehat{G}(m,n)\exp\left(2\pi i\left(m\alpha + n\beta\right)\right)$$

is spherically convergent at every point (α,β) where the expanded function $\Sigma G(\alpha,\beta)$ is smooth,

$$\lim_{R\rightsquigarrow+\infty}\left\{\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0,\, \sqrt{m^2+n^2}<R} \widehat{G}(m,n)\exp\left(2\pi i\left(m\alpha+n\beta\right)\right)\right\} = \Sigma G(\alpha,\beta).$$

Proof. In [**2**] it is proved that for spherical partial sums of Fourier series of a periodic integrable function $F(x,y)$ localization holds under the assumption

$$\lim_{R\rightsquigarrow+\infty}\left\{\sum\sum_{R-1<\sqrt{m^2+n^2}<R}\left|\widehat{F}(m,n)\right|\right\} = 0.$$

More precisely, under this assumption, for every (x,y) in an open set where $F(x,y)$ is smooth,

$$\lim_{R\rightsquigarrow+\infty}\left\{\sum\sum_{\sqrt{m^2+n^2}<R}\widehat{F}(m,n)\exp(2\pi i(m\alpha+n\beta))\right\}=F(x,y).$$

In particular, in order to prove the theorem it suffices to show that the Fourier coefficients $\left\{\widehat{G}(m,n)\right\}_{mn\neq0}$ satisfy the above decay condition. As in the proof of Theorem 4.2 one can substitute a sum with an integral,

$$\begin{aligned}&\sum\sum_{mn\neq0,\ R-1<\sqrt{m^2+n^2}<R}\left|\widehat{G}(m,n)\right|\\&\leq C\sum\sum_{m>0,\ n>0,\ R-1<\sqrt{m^2+n^2}<R}\frac{1}{mn}\\&+C\iint_{\left\{\xi>1/2,\ \eta>1/2,\ R-2<\sqrt{\xi^2+\eta^2}<R\right\}}\frac{d\xi d\eta}{\eta^{3/2}f''\left((f')^{-1}(-\xi/\eta)\right)^{1/2}}\\&+C\iint_{\left\{\xi>1/2,\ \eta>1/2,\ R-2<\sqrt{\xi^2+\eta^2}<R\right\}}\frac{d\xi d\eta}{\xi^{3/2}(f^{-1})''\left(((f^{-1})')^{-1}(-\eta/\xi)\right)^{1/2}}.\end{aligned}$$

The first sum converges to zero as $R\rightsquigarrow+\infty$,

$$\sum\sum_{m>0,\ n>0,\ R-1<\sqrt{m^2+n^2}<R}\frac{1}{mn}\leq C\frac{\log(R)}{R}.$$

The two integrals are similar, and it suffices to consider only the first one. In polar coordinates, if R is large,

$$\begin{aligned}&\iint_{\left\{\xi>1/2,\ \eta>1/2,\ R-2<\sqrt{\xi^2+\eta^2}<R\right\}}\frac{d\xi d\eta}{\eta^{3/2}f''\left((f')^{-1}(-\xi/\eta)\right)^{1/2}}\\&=\int_{R-2}^{R}\left(\int_{\arctan(1/2\rho)}^{\pi/2-\arctan(1/2\rho)}\frac{d\vartheta}{\sin(\vartheta)^{3/2}f''\left((f')^{-1}(-\cos(\vartheta)/\sin(\vartheta))\right)^{1/2}}\right)\rho^{-1/2}d\rho.\end{aligned}$$

With the change of variables

$$(f')^{-1}(-\cos(\vartheta)/\sin(\vartheta))=x,\quad d\vartheta=\frac{f''(x)}{1+f'(x)^2}dx,$$

the integral becomes

$$\int_{R-2}^{R}\left(\int_{(f')^{-1}(-2\rho)}^{(f')^{-1}(-1/2\rho)}\left(1+f'(x)^2\right)^{-1/4}f''(x)^{1/2}dx\right)\rho^{-1/2}d\rho.$$

Split the inner integral at $x = (f')^{-1}(-1)$. The integral below $(f')^{-1}(-1)$ is dominated by

$$\begin{aligned}
&\int_{R-2}^{R}\left(\int_{(f')^{-1}(-2\rho)}^{(f')^{-1}(-1)}\left(1+f'(x)^2\right)^{-1/4} f''(x)^{1/2}\,dx\right)\rho^{-1/2}d\rho\\
&\leq \left\{(f')^{-1}(-1)\right\}^{1/2}\int_{R-2}^{R}\left\{\int_{(f')^{-1}(-2\rho)}^{(f')^{-1}(-1)}|f'(x)|^{-1} f''(x)\,dx\right\}^{1/2}\rho^{-1/2}d\rho\\
&= \left\{(f')^{-1}(-1)\right\}^{1/2}\int_{R-2}^{R}\log^{1/2}(2\rho)\,\rho^{-1/2}d\rho\\
&\leq CR^{-1/2}\log^{1/2}(R).
\end{aligned}$$

Observe that if $g(x)$ is positive decreasing and convex in $0 < x < +\infty$, then

$$-g'(x) \leq -\frac{g(x/2)-g(x)}{x/2-x} < \frac{g(x/2)}{x/2}.$$

Applying this first to $-f'(x)$ and then to $f(x)$ gives

$$f''(x) \leq -\frac{f'(x/2)}{x/2} \leq 8\frac{f(x/4)}{x^2}.$$

Hence the integral above $(f')^{-1}(-1)$ is dominated by

$$\begin{aligned}
&= \int_{R-2}^{R}\left(\int_{(f')^{-1}(-1)}^{(f')^{-1}(-1/2\rho)}\left(1+f'(x)^2\right)^{-1/4} f''(x)^{1/2}\,dx\right)\rho^{-1/2}d\rho\\
&\leq \int_{R-2}^{R}\left(\int_{(f')^{-1}(-1)}^{(f')^{-1}(-1/2\rho)} f''(x)^{1/2}\,dx\right)\rho^{-1/2}d\rho\\
&\leq 2^{3/2}\int_{R-2}^{R}\left(\int_{(f')^{-1}(-1)}^{(f')^{-1}(-1/2\rho)}\frac{f(x/4)^{1/2}}{x}dx\right)\rho^{-1/2}d\rho\\
&\leq 2^{3/2} f\left((f')^{-1}(-1)/4\right)^{1/2}\int_{R-2}^{R}\left(\log\left((f')^{-1}(-1/2\rho)\right)-\log\left((f')^{-1}(-1)\right)\right)\rho^{-1/2}d\rho\\
&\leq CR^{-1/2}\log\left((f')^{-1}(-1/R)\right).
\end{aligned}$$

It then suffices to observe that

$$\lim_{R\rightsquigarrow+\infty}\left\{R^{-1/2}\log\left((f')^{-1}(-1/R)\right)\right\} = \lim_{x\rightsquigarrow+\infty}\left\{|f'(x)|^{1/2}\log(x)\right\} = 0.$$

On the other hand, by convexity, if $x \rightsquigarrow +\infty$ then

$$-f'(x)\log(x) \leq 2f(x/2)\frac{\log(x)}{x} \rightsquigarrow 0.$$

□

In particular, if $\mathbb{Z}^2 \cap \partial(\Omega - (\alpha,\beta)) = \varnothing$, then the function that counts the integer points is constant in a neighborhood of (α,β), while $\Lambda\chi_\Omega + \Phi\chi_\Omega(\alpha) + \Psi\chi_\Omega(\beta)$ has the same regularity as $f(x)$ and $f^{-1}(y)$. Hence the above theorem applies. A more refined analysis based on [**2**] and [**3**] shows that convergence of the above Fourier series may occur also at all points not on the axes. By the way, the argument is slightly more delicate than it appears, indeed localization for multiple Fourier series of piecewise smooth functions holds only in dimension 1 and 2, but not in dimension 3 or higher. Finally, it should be noticed that the classical Voronoï formula for the integer points in the hyperbola $\{xy < N\}$ mentioned in the introduction is summed not spherically, but hyperbolically.

5. *Examples*

The following lemma gives an expression of ΛG, $\Phi G(\alpha)$, $\Psi G(\beta)$, directly in terms of $g(t)$, $f(x)$, $f^{-1}(y)$.

Lemma 5.1. *If $f(x)$, $g(t)$, and $G(x,y)$ are as in Lemma 3.1 and Lemma 3.2, and if $H(x) = f(x) - xf'(x)$ and $K(y) = f^{-1}(y) - y\left(f^{-1}\right)'(y)$, then*

$$\Lambda G = \int_0^{+\infty} \left\{ t \int_{1/t}^{f^{-1}(1/t)} H(x)\,dx \right\} g(t)\,dt = \int_0^{+\infty} \left\{ t \int_{1/t}^{f(1/t)} K(x)\,dx \right\} g(t)\,dt,$$

$$\Phi G(\alpha) = \int_0^{+\infty} \left\{ H(\alpha/t) + (1/2-\alpha) H(1/t) + \int_1^{+\infty} \sigma(x-\alpha) \frac{\partial}{\partial x} (H(x/t))\,dx \right\} g(t)\,dt,$$

$$\Psi G(\beta) = \int_0^{+\infty} \left\{ K(\beta/t) + (1/2-\beta) K(1/t) + \int_1^{+\infty} \sigma(y-\beta) \frac{\partial}{\partial y} (K(y/t))\,dy \right\} g(t)\,dt.$$

Proof. If $t = \tau(x,y)$ is the solution to the equation $y/t = f(x/t)$, for a fixed x, by the implicit function theorem,

$$dt = \frac{\partial \tau}{\partial y}(x,y)\,dy$$
$$= -\frac{\partial/\partial y \{y/t - f(x/t)\}}{\partial/\partial t \{y/t - f(x/t)\}} dy = \frac{dy}{y/t - (x/t) f'(x/t)} = \frac{dy}{f(x/t) - (x/t) f'(x/t)}.$$

Therefore $dy = H(x/t)\,dt$ and

$$\int_0^{+\infty} G(x,y)\,dy = \int_0^{+\infty} g(t) H(x/t)\,dt.$$

Symmetrically, if y is fixed, then $dx = K(y/t)\,dt$ and

$$\int_0^{+\infty} G(x,y)\,dx = \int_0^{+\infty} g(t) K(y/t)\,dt.$$

This gives the expressions of $\Phi G(\alpha)$ and $\Psi G(\beta)$. Finally,

$$\Lambda G = \int_1^{+\infty} \int_1^{+\infty} g(\tau(x,y))\,dxdy - \int_0^1 \int_0^1 g(\tau(x,y))\,dxdy$$
$$= \int_1^{+\infty} \left(\int_{\tau(x,1)}^{+\infty} g(t) H(x/t)\,dt \right) dx - \int_0^1 \left(\int_0^{\tau(x,1)} g(t) H(x/t)\,dt \right) dx$$
$$= \int_{\tau(1,1)}^{+\infty} g(t) \left(\int_1^{tf^{-1}(1/t)} H(x/t)\,dx \right) dt - \int_0^{\tau(1,1)} g(t) \left(\int_{tf^{-1}(1/t)}^1 H(x/t)\,dx \right) dt$$
$$= \int_0^{+\infty} g(t) \left(t \int_{1/t}^{f^{-1}(1/t)} H(x)\,dx \right) dt.$$

□

It may be of some interest to estimate how the above formulas behave under the dilations $G_T(x,y) = G(x/T, y/T)$. The Fourier transform of a dilation is a dilation of the Fourier transform, $\widehat{G}_T(\xi,\eta) = T^2 \widehat{G}(T\xi, T\eta)$. The function $\Lambda G_T + \Phi G_T(\alpha) + \Psi G_T(\beta)$ does not seem to behave in a simple way under dilations, however if $G_T(x,y)$ is not identically zero and non negative then $\Lambda G_T + \Phi G_T(\alpha) + \Psi G_T(\beta)$ grows at least as T^2. This implies that the Fourier series is much smaller than $\Lambda G_T + \Phi G_T(\alpha) + \Psi G_T(\beta)$, and it can be considered as a remainder. In particular, the following corollary supports the conjectured estimates of the remainder in the circle and divisor problems.

Corollary 5.2. *(1) If $f(x)$, $g(t)$, and $G(x,y)$ are as in Lemma 3.1 and Lemma 3.2, if $G(x,y)$ is non negative and not identically zero, and if $G_T(x,y) = G(x/T, y/T)$, then there exists a positive constant C such that for every large T,*

$$\Lambda G_T + \Phi G_T(\alpha) + \Psi G_T(\beta) \geq CT^2.$$

(2) Under the assumptions in Theorem 4.2, there exists a constant C such that for every large T,

$$\left\{ \int_0^1 \int_0^1 \left| \Sigma G_T(\alpha,\beta) \right|^2 d\alpha d\beta \right\}^{1/2} \leq C\sqrt{T}.$$

Proof. (1) Indeed it can be proved that for every $0 < \alpha, \beta \leq 1$ and every $T > 1$, $\Phi G_T(\alpha) \geq 0$ and $\Psi G_T(\beta) \geq 0$, and ΛG_T grows at least as T^2 as $T \rightsquigarrow +\infty$.

(2) It suffices to observe that if $G(x,y)$ is associated to $g(t)$, then $G(x/T, y/T)$ is associated to $g(t/T)$ and, if $T \geq 1$,

$$\int_0^{+\infty} \sqrt{1+t}\, |dg(t/T)| = \int_0^{+\infty} \sqrt{1+Tt}\, |dg(t)| \leq \sqrt{T} \int_0^{+\infty} \sqrt{1+t}\, |dg(t)|.$$

□

The following corollaries specify the set Ω. In particular, the Fourier transform of an hyperbola $\{0 < y < T^2/x\}$ can be expressed in terms of Bessel functions, the functions $\Phi\chi_\Omega(\alpha)$ and $\Psi\chi_\Omega(\beta)$ in terms of logarithmic derivatives of the Gamma function, and $\Lambda\chi_\Omega$ is a logarithm. For the generalized hyperbolic domains $\{0 < y < T^{1+\delta} x^{-\delta}\}$ the Hurwitz Zeta function comes to play.

Corollary 5.3. *(1) If $0 < \delta < +\infty$, define*

$$\Phi^{(1)}(z) = (1+\delta) \int_0^{+\infty} t^{-\delta} \exp\left(-2\pi i z \left(t - t^{-\delta}\right)\right) dt,$$

$$\Phi^{(2)}(z) = (1+\delta) \int_0^{+\infty} t^{-\delta} \exp\left(-2\pi i z \left(t + t^{-\delta}\right)\right) dt.$$

Moreover, if $g(t)$ is a function of bounded variation in $\{-\infty < t < +\infty\}$ with compact support, define

$$G(x,y) = \begin{cases} g\left(x^{\delta/(\delta+1)} y^{1/(\delta+1)}\right) & \text{if } 0 < x, y < +\infty, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\widehat{G}(\xi,\eta) = \begin{cases} |\xi/\eta|^{(\delta-1)/(\delta+1)} \int_0^{+\infty} r g(r)\, \Phi^{(1)}\left(-|\xi|^{1/(1+1/\delta)} |\eta|^{1/(1+\delta)} r\right) dr & \text{if } \xi < 0 \text{ and } \eta > 0, \\ |\xi/\eta|^{(\delta-1)/(\delta+1)} \int_0^{+\infty} r g(r)\, \Phi^{(1)}\left(|\xi|^{1/(1+1/\delta)} |\eta|^{1/(1+\delta)} r\right) dr & \text{if } \xi > 0 \text{ and } \eta < 0, \\ |\xi/\eta|^{(\delta-1)/(\delta+1)} \int_0^{+\infty} r g(r)\, \Phi^{(2)}\left(-|\xi|^{1/(1+1/\delta)} |\eta|^{1/(1+\delta)} r\right) dr & \text{if } \xi < 0 \text{ and } \eta < 0, \\ |\xi/\eta|^{(\delta-1)/(\delta+1)} \int_0^{+\infty} r g(r)\, \Phi^{(2)}\left(|\xi|^{1/(1+1/\delta)} |\eta|^{1/(1+\delta)} r\right) dr & \text{if } \xi > 0 \text{ and } \eta > 0. \end{cases}$$

(2) If $\delta \neq 1$, if $\zeta(z,s) = \sum_{n=0}^{+\infty} (n+z)^{-s}$ is the Hurwitz Zeta function, and if $0 < \alpha, \beta \leq 1$, then

$$\begin{aligned}&\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha, n+\beta)\\ &= (1+\delta)\zeta(\alpha,\delta)\int_0^{+\infty} t^{\delta} g(t)\,dt + (1+1/\delta)\zeta(\beta,1/\delta)\int_0^{+\infty} t^{1/\delta} g(t)\,dt\\ &\quad + \sum\sum_{(m,n)\in\mathbb{Z}^2,\ mn\neq 0} \widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta)).\end{aligned}$$

(3) If $\delta = 1$, if $\Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Gamma function, and if $0 < \alpha, \beta \leq 1$, then

$$\begin{aligned}&\sum\sum_{(m,n)\in\mathbb{Z}^2} G(m+\alpha, n+\beta)\\ &= 4\int_0^{+\infty} t\log(t)\,g(t)\,dt - 2\left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \frac{\Gamma'(\beta)}{\Gamma(\beta)}\right)\int_0^{+\infty} tg(t)\,dt\\ &\quad + \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta)).\end{aligned}$$

(4) If $0 < \delta < +\infty$, $2 \leq p < \min\{2+2\delta, 2+2/\delta\}$, and $1/p + 1/q = 1$, then there exists a positive constant C such that

$$\begin{aligned}&\left\{\int_0^1\int_0^1 \left|\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} \widehat{G}(m,n)\exp(2\pi i(m\alpha+n\beta))\right|^p d\alpha d\beta\right\}^{1/p}\\ &\leq \left\{\sum\sum_{(m,n)\in\mathbb{Z}^2,\ mn\neq 0} \left|\widehat{G}(m,n)\right|^q\right\}^{1/q} \leq C\left(\int_0^{+\infty}\sqrt{1+t}\,|dg(t)|\right).\end{aligned}$$

Proof. (1) One can compute the Fourier transform with the hyperbolic coordinates

$$\begin{cases} x = rt, \\ y = \pm rt^{-\delta}, \end{cases} \quad -\infty < r < +\infty,\ 0 < t < +\infty,$$

$$\begin{cases} \xi = \rho\vartheta, \\ \eta = \pm\rho\vartheta^{-\delta}, \end{cases} \quad -\infty < \rho < +\infty,\ 0 < \vartheta < +\infty,$$

$$dxdy = (1+\delta)|r|\,t^{-\delta} drdt.$$

Since $G(x,y) = g(r)$ when $0 < x, y, r < +\infty$, and $G(x,y) = 0$ otherwise,

$$\begin{aligned}&\widehat{G}\left(\rho\vartheta, \pm\rho\vartheta^{-\delta}\right)\\ &= \int_0^{+\infty} rg(r)\left((1+\delta)\int_0^{+\infty} t^{-\delta}\exp\left(-2\pi i\rho r\left(\vartheta t \pm (\vartheta t)^{-\delta}\right)\right)dt\right)dr\\ &= \vartheta^{\delta-1}\int_0^{+\infty} rg(r)\left((1+\delta)\int_0^{+\infty} t^{-\delta}\exp\left(-2\pi i\rho r\left(t \pm t^{-\delta}\right)\right)dt\right)dr.\end{aligned}$$

Then observe that $\vartheta^{\delta-1} = |\xi/\eta|^{(\delta-1)/(\delta+1)}$ and $\rho = \pm|\xi|^{1/(1+1/\delta)}|\eta|^{1/(1+\delta)}$.

(2) If $f(x) = x^{-\delta}$ then $f(x) - xf'(x) = (1+\delta)x^{-\delta}$ and $f^{-1}(y) - y\left(f^{-1}\right)'(y) = (1+1/\delta)y^{-1/\delta}$ and, by Lemma 5.1, if $\delta \neq 1$ the constant ΛG is

$$\Lambda G = \int_0^{+\infty} \left\{ t \int_{1/t}^{t^{1/\delta}} (1+\delta)x^{-\delta} dx \right\} g(t)\,dt = \frac{1+\delta}{1-\delta} \int_0^{+\infty} \left(t^{1/\delta} - t^{\delta}\right) g(t)\,dt.$$

Again by Lemma 5.1, if $0 < \alpha \le 1$ the function $\Phi G(\alpha)$ is

$$\Phi G(\alpha) = (1+\delta)\left(\alpha^{-\delta} + 1/2 - \alpha - \delta \int_1^{+\infty} \sigma(x-\alpha)x^{-\delta-1}dx\right) \int_0^{+\infty} t^{\delta} g(t)\,dt.$$

By the Euler McLaurin summation formula, if $\delta > 1$,

$$\begin{aligned}\zeta(\alpha,\delta) &= \sum_{n=0}^{+\infty} (n+\alpha)^{-\delta} \\ &= \int_0^{+\infty} (x+\alpha)^{-\delta}\,dx + \alpha^{-\delta}/2 + \int_0^{+\infty} \sigma(x) \frac{\partial}{\partial x}(x+\alpha)^{-\delta}\,dx \\ &= 1/(\delta-1) + \alpha^{-\delta} + 1/2 - \alpha - \delta \int_1^{+\infty} \sigma(x-\alpha)x^{-\delta-1}dx.\end{aligned}$$

By analytic continuation, the same formula holds when $0 < \delta < 1$. Hence, if $\delta \neq 1$,

$$\Phi G(\alpha) = (1+\delta)\left(\zeta(\alpha,\delta) + 1/(1-\delta)\right) \int_0^{+\infty} t^{\delta} g(t)\,dt.$$

The formula for $\Psi G(\beta)$ is similar, with δ replaced by $1/\delta$,

$$\Psi G(\beta) = (1+1/\delta)\left(\zeta(\beta,1/\delta) + 1/(1-1/\delta)\right) \int_0^{+\infty} t^{1/\delta} g(t)\,dt.$$

Hence, if $\delta \neq 1$ and $0 < \alpha, \beta \le 1$,

$$\begin{aligned}&\Lambda G + \Phi G(\alpha) + \Psi G(\beta) \\ &= (1+\delta)\zeta(\alpha,\delta) \int_0^{+\infty} t^{\delta} g(t)\,dt + (1+1/\delta)\zeta(\beta,1/\delta) \int_0^{+\infty} t^{1/\delta} g(t)\,dt.\end{aligned}$$

(3) This follows from Euler definition of the Gamma function and the Euler McLaurin summation formula,

$$\Gamma(z) = \lim_{n \rightsquigarrow +\infty} \left\{ \frac{n! n^z}{z(z+1)(z+2)\ldots(z+n)} \right\},$$

$$\begin{aligned}\frac{\Gamma'(z)}{\Gamma(z)} &= \frac{d}{dz}\log(\Gamma(z)) = \lim_{n \rightsquigarrow +\infty} \left\{ \log(n) - \sum_{m=0}^{n} \frac{1}{z+m} \right\} \\ &= \lim_{n \rightsquigarrow +\infty} \left\{ \log(n) - \int_1^{n+z} t^{-1}dt - (n+z)^{-1}/2 - z^{-1} - 1/2 + z - \int_1^{n+z} \sigma(t-z)t^{-2}dt \right\} \\ &= -z^{-1} - 1/2 + z + \int_1^{+\infty} \sigma(t-z)t^{-2}dt.\end{aligned}$$

More simply, the case $\delta = 1$ is the limit of (2) when $\delta \rightsquigarrow 1$. First observe that

$$\Lambda G = \lim_{\delta \rightsquigarrow 1} \left\{ \frac{1+\delta}{1-\delta} \int_0^{+\infty} \left(t^{1/\delta} - t^{\delta}\right) g(t)\,dt \right\} = 4 \int_0^{+\infty} t \log(t) g(t)\,dt.$$

Then recall that the Hurwitz Zeta function is a meromorphic function with a simple pole in $\delta = 1$ and, see [**1**, 1.10.9],

$$\lim_{\delta \rightsquigarrow 1} \left\{ \zeta(z,\delta) - \frac{1}{\delta - 1} \right\} = -\frac{\Gamma'(z)}{\Gamma(z)}.$$

Thus, if $\delta = 1$ and $0 < \alpha, \beta \leq 1$,

$$\Lambda G + \Phi G(\alpha) + \Psi G(\beta)$$
$$= 4\int_0^{+\infty} t \log(t)\, g(t)\, dt - 2\left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \frac{\Gamma'(\beta)}{\Gamma(\beta)}\right)\int_0^{+\infty} tg(t)\, dt.$$

(4) The estimate of the norm of the discrepancy follows from the Hausdorff Young inequality and the estimate in Lemma 4.1,

$$\left|\widehat{G}(\xi,\eta)\right| \leq \begin{cases} C\left(\int_0^{+\infty} |dg(t)|\right) |\xi\eta|^{-1} & \text{if } \xi\eta < 0, \\ C\left(\int_0^{+\infty} \sqrt{1+t}\, |dg(t)|\right) |\xi|^{-(2+\delta)/(2+2\delta)}\, |\eta|^{-(2+1/\delta)/(2+2/\delta)} & \text{if } \xi\eta > 0. \end{cases}$$

A final remark: Up to a logarithmic transgression, the estimate of the norm of $\Sigma G(\alpha, \beta)$ holds also at the critical index $p = \min\{2 + 2\delta, 2 + 2/\delta\}$. For example, let $\delta = 1$ and let

$$\sup_{\alpha,\beta}\{|\Sigma G(\alpha,\beta)|\} = A,$$

$$\sup_{(\xi,\eta)\in\mathbb{Z}^2,\, \xi\eta\neq 0}\left\{|\xi\eta|^{3/4}\left|\widehat{G}(\xi,\eta)\right|\right\} = B.$$

Interpolating between $2 \leq p < 4$ and $+\infty$, one obtains

$$\left\{\int_0^1\int_0^1 |\Sigma G(\alpha,\beta)|^4\, d\alpha d\beta\right\}^{1/4}$$
$$\leq \left\{\sup_{\alpha,\beta}\left\{|\Sigma G(\alpha,\beta)|^{4-p}\right\}\int_0^1\int_0^1 |\Sigma G(\alpha,\beta)|^p\, d\alpha d\beta\right\}^{1/4}$$
$$\leq \sup_{\alpha,\beta}\{|\Sigma G(\alpha,\beta)|\}^{1-p/4}\left\{\sum\sum_{(m,n)\in\mathbb{Z}^2,\ m\cdot n\neq 0}\left|\widehat{G}(m,n)\right|^{p/(p-1)}\right\}^{(p-1)/4}$$
$$\leq A^{1-p/4}B^{p/4}\left\{2\sum_{k=1}^{+\infty} k^{-(3/4)p/(p-1)}\right\}^{(p-1)/2}$$
$$\leq CB\,(A/B)^{1-p/4}\,(4-p)^{(1-p)/2}.$$

Then the choice $p = 4 - 1/\log(A/B)$ gives

$$\left\{\int_0^1\int_0^1 |\Sigma G(\alpha,\beta)|^4\, d\alpha d\beta\right\}^{1/4} \leq CB\log^{3/2}(A/B).$$

□

When $\delta = 1$ then the above defined functions $\Phi^{(j)}(z)$ can be expressed in terms of Bessel function of imaginary argument $K_0(4\pi|z|)$, and Bessel functions of third kind $H_0^{(j)}(\pm 4\pi z)$,

$$\Phi^{(1)}(z) = 4K_0(4\pi|z|),$$

$$\Phi^{(2)}(z) = \begin{cases} 2\pi i H_0^{(1)}(-4\pi z) & \text{if } z < 0, \\ -2\pi i H_0^{(2)}(4\pi z) & \text{if } z > 0. \end{cases}$$

See [**1**, 7.3.27] or [**10**, pages 117 and 140]. In particular, the corollary with $\delta = 1$, $g(t) = \chi_{(0,\sqrt{N})}(t)$, and $\alpha = \beta = 1$, gives the classical Voronoï formula quoted in the introduction. Indeed, the case $\delta = 1$, $g(t) = \chi_{(0,\sqrt{N})}(t)$, $p = 2$, of the above corollary is stated in [**12**], and the cases $0 < \delta < +\infty$, $g(t) = \chi_{(0,\sqrt{N})}(t)$, $p = 2$, are also contained in [**13**]. As a last example, we want to present alternative proofs of known results on the number of integer points in triangles. See [**6**, Chapter V] for an interesting introduction to these problems.

Corollary 5.4. *(1) If* $0<\omega,\omega'<+\infty$ *and if* $\Omega=\{x,y>0,\ \omega x+\omega' y<T\}$ *is the triangle with vertices in* $(0,0)$, $(T/\omega,0)$, *and* $(0,T/\omega')$, *then*

$$\sum\sum_{(m,n)\in\mathbb{Z}^2}\chi_\Omega\left(m+\alpha,n+\beta\right)$$
$$=\frac{T^2}{2\omega\omega'}-\frac{T}{\omega'}\sigma\left(\alpha\right)-\frac{T}{\omega}\sigma\left(\beta\right)+\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq 0}\widehat{\chi}_\Omega\left(m,n\right)\exp\left(2\pi i\left(m\alpha+n\beta\right)\right)+O\left(1\right).$$

(2) If the slope ω/ω' *is irrational and if it satisfies the Diophantine condition* $|m\omega'-n\omega|>c\,|n|^{-\gamma}$, *for some* $c>0$, $1\le\gamma<+\infty$, *and for every integer* m *and* n, *then there exists a positive constant* C *such that for every* $T>2$,

$$\left\{\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq 0}\left|\widehat{\chi}_\Omega\left(m,n\right)\right|^2\right\}^{1/2}\le\begin{cases}C\sqrt{\log\left(T\right)} & \text{if } \gamma=1,\\ CT^{1-1/\gamma} & \text{if } \gamma>1.\end{cases}$$

Proof. (1) If $0<\alpha,\beta<1$, a straightforward computation gives

$$\Lambda\chi_\Omega+\Phi\chi_\Omega\left(\alpha\right)+\Psi\chi_\Omega\left(\beta\right)=\frac{T^2}{2\omega\omega'}-\frac{T}{\omega'}\left(\alpha-\frac{1}{2}\right)-\frac{T}{\omega}\left(\beta-\frac{1}{2}\right)+O\left(1\right),$$

(2) By the divergence theorem, if $\xi\neq 0$ and $\eta\neq 0$ and $\omega'\xi-\omega\eta\neq 0$, then

$$\widehat{\chi}_\Omega\left(\xi,\eta\right)$$
$$=\frac{\exp\left(-2\pi iT\eta/\omega'\right)-1}{4\pi^2\left(\xi^2+\eta^2\right)}\frac{\xi}{\eta}+\frac{\exp\left(-2\pi iT\xi/\omega\right)-1}{4\pi^2\left(\xi^2+\eta^2\right)}\frac{\eta}{\xi}$$
$$+\frac{\exp\left(-2\pi iT\eta/\omega'\right)-\exp\left(-2\pi iT\xi/\omega\right)}{4\pi^2\left(\xi^2+\eta^2\right)}\frac{\omega\xi+\omega'\eta}{\omega'\xi-\omega\eta}.$$

The square norms of the first two terms are uniformly bounded in T,

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq 0}\left|\frac{m}{n}\frac{\exp\left(-2\pi iTn/\omega'\right)-1}{4\pi^2\left(m^2+n^2\right)}\right|^2\le\pi^{-2}\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{m^2}{n^2\left(m^2+n^2\right)^2}\le C,$$

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq 0}\left|\frac{n}{m}\frac{\exp\left(-2\pi iTm/\omega\right)-1}{4\pi^2\left(m^2+n^2\right)}\right|^2\le\pi^{-2}\sum_{m=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{n^2}{m^2\left(m^2+n^2\right)^2}\le C.$$

The main contribution is the term

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\,mn\neq 0}\left|\frac{\omega m+\omega' n}{\omega' m-\omega n}\frac{\exp\left(-2\pi iTn/\omega'\right)-\exp\left(-2\pi iTm/\omega\right)}{4\pi^2\left(m^2+n^2\right)}\right|^2.$$

If $|\omega' m-\omega n|>c\,|n|^{-\gamma}$ then it can be proved that the above sum is dominated by $C\log\left(T\right)$ when $\gamma=1$, and by $CT^{2-2/\gamma}$ when $\gamma>1$. The case $\gamma=1$ is explicitly considered in [**4**], and the case $\gamma>1$ is similar. The case ω/ω' rational which is excluded in (2) is indeed a sort of limit for $\gamma\rightsquigarrow+\infty$ of the above Diophantine condition. In this case the sum over the frequencies on the line $\omega' m-\omega n=0$ gives a contribution of the order of CT^2, while the other frequencies give a bounded contribution. Indeed, when $\gamma\rightsquigarrow+\infty$ then $T^{1-1/\gamma}\rightsquigarrow T$, which corresponds to bounding the remainder with the perimeter of the domain. See [**16**]. □

The above corollary on the quadratic estimate of the discrepancy between area and integer points is essentially due to Davenport (see [**4**]), who considered only the case $\gamma = 1$, although the other cases are similar. By a result in [**14**] the quadratic estimate $C \log(T)$ for discrepancy cannot be improved. This should be also compared with [**7**], where it is proved that if the continued fraction expansion of ω/ω' has bounded quotients, then

$$\sum\sum_{(m,n)\in\mathbb{Z}^2} \chi_\Omega(m,n) = \frac{T^2}{2\omega\omega'} - \frac{T}{2\omega} - \frac{T}{2\omega'} + O(\log(T)).$$

Observe that this pointwise estimate $O(\log(T))$ is larger than the standard deviation $O\left(\sqrt{\log(T)}\right)$ that comes from the corollary.

The statement (1) in the above corollary holds for every slope ω/ω', but when applied to a rational ω/ω' it gives the somehow trivial estimate

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0} |\widehat{\chi}_\Omega(m,n)|^2 \leq CT^2.$$

Indeed, for ω/ω' rational, a more refined formula holds, with a remainder with a uniformly bounded square norm.

Corollary 5.5. *If $\Omega = \{x, y > 0,\ \omega x + \omega' y < T\}$ is the triangle with vertices in $(0,0)$, $(T/\omega, 0)$ and $(0, T/\omega')$, with ω and ω' relatively prime integers, then*

$$\begin{aligned} &\sum\sum_{(m,n)\in\mathbb{Z}^2} \chi_\Omega(m+\alpha, n+\beta) \\ &= \frac{T^2}{2\omega\omega'} - \frac{T}{\omega'}\sigma(\alpha) - \frac{T}{\omega}\sigma(\beta) + \frac{T}{\omega\omega'}\sigma(\omega\alpha + \omega'\beta - T) \\ &+ \sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0,\, |\omega' m - \omega n|\neq 0} \widehat{\chi}_\Omega(m,n)\exp(2\pi i(m\alpha + n\beta)) + O(1). \end{aligned}$$

Moreover, there exists a positive constant C such that for every $T > 1$,

$$\sum\sum_{(m,n)\in\mathbb{Z}^2,\, mn\neq 0,\ |\omega' m - \omega n|\neq 0} |\widehat{\chi}_\Omega(m,n)|^2 \leq C.$$

Proof. The only difference with the previous corollary is a non negligible contribution of the frequencies on the line $\omega' m = \omega n$. Indeed, for any integer $k \neq 0$,

$$\widehat{\chi}_\Omega(\omega k, \omega' k) = \frac{\exp(-2\pi i T k) - 1}{4\pi^2\omega\omega' k^2} - \frac{T\exp(-2\pi i T k)}{2\pi i \omega\omega' k}.$$

Therefore

$$\begin{aligned} &\sum\sum_{(m,n)\in\mathbb{Z}^2,\, \omega' m = \omega n \neq 0} \widehat{\chi}_\Omega(m,n)\exp(2\pi i(m\alpha + n\beta)) \\ &= \sum_{k\in\mathbb{Z},\, k\neq 0} \widehat{\chi}_\Omega(\omega k, \omega' k)\exp(2\pi i k(\omega\alpha + \omega'\beta)) \\ &= -\frac{T}{\omega\omega'}\sum_{k\in\mathbb{Z},\, k\neq 0} \frac{\exp(2\pi i k(\omega\alpha + \omega'\beta - T))}{2\pi i k} \\ &- \sum_{k\in\mathbb{Z},\, k\neq 0} \frac{\exp(2\pi i k(\omega\alpha + \omega'\beta))}{4\pi^2\omega\omega' k^2} + \sum_{k\in\mathbb{Z},\, k\neq 0} \frac{\exp(2\pi i k(\omega\alpha + \omega'\beta - T))}{4\pi^2\omega\omega' k^2} \\ &= \frac{T}{\omega\omega'}\sigma(\omega\alpha + \omega'\beta - T) + O(1). \end{aligned}$$

□

This corollary is a particular case of a more general result which will appear elsewhere. For an alternative formula, see [**6**, Chapter V].

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