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Critical velocities of a beam on nonlinear elastic foundation under harmonic moving load

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Abstract

The present paper is concerned with the numerical FEM analysis of the dynamic response of a simply-supported Euler-Bernoulli elastic beam resting on a spatially homogeneous nonlinear cubic elastic Winkler foundation subjected to a concentrated moving load. The load moves at a constant velocity along the beam, displaying a harmonic-varying magnitude in time, defined in terms of mean value, amplitude and frequency of oscillation. Parametric analyses are performed to investigate the influence of the moving load parameters on the so-called *critical velocities*, leading to large displacements, possibly harmful for the structural system. The relationship between critical velocities and moving load parameters is portrayed in appropriate analytical curves, derived from the obtained numerical results, by proposed analytical functional dependencies, with calibrated coefficients. The ensuing outcomes shall reveal practical implications in the description and control of track vibrations induced by high-speed trains, within contemporary railway engineering scenarios, where the critical velocity onset may be lowered down by the effect of the magnitude-varying moving load.

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1. Introduction

Moving load dynamic problems are very common in various engineering fields and particularly in the context of railway tracks, where an accurate prediction of the railway response is stimulated by the continuous development of fast transportation systems, nowadays at increasingly high vehicle velocities. Thereby, it appears fundamental to investigate how the interaction between the support structures and the moving mechanical system influences the amplitudes of the track displacements and in which manner they may depend on the velocity of the moving vehicle and on the nature of the acting forces. In the literature, elastic beams and various types of foundation models

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have been adopted to represent railway tracks, i.e. flexible structures that are mainly one-dimensional in geometry. Comprehensive literature reviews about moving load problems concerning beams and plates may be found in [13].

The simplest modelization of a track foundation may be based on the well-known Winkler model, where a uniform layer of infinitely closely spaced springs push up/pull down the beam with a force linearly related to the beam deflection [5,6]. Nevertheless, the support structure usually shows up to be highly nonlinear, due to the mechanical characteristics of the ballast. The nonlinear elastic foundation model with linear plus cubic stiffness has been widely recognized as one of the most reliable and convenient modelizations for the dynamic analysis of railway tracks [1].

Concerning the analysis of beams on a (visco)elastic foundation, under moving load, different approaches have been adopted so far. In case of linear foundations, classical ones considered the steady-state response of infinite beams subjected to a constant-magnitude moving load (see e.g. [8,11]), as well as to a moving load with harmonically varying amplitude [3,12], or the transient response of finite beams [4]. Besides analytical methods, FEM modelizations were also applied in case of nonlinear foundations [1,2], even for moving oscillator problems [14] and for taut strings [10]. The outcomes of many research works revealed that structural vibrations induced by moving objects may become rather high, when their velocity attains a characteristic value, referred to as *critical velocity*, which for an infinite beam is the minimum phase velocity of the bending waves propagating within the beam-foundation system [11], while for a finite beam corresponds to the lowest resonant velocity among those of the natural modes of the beam [3,4].

The effect of the frequency of a harmonic moving load for both infinite and finite beams had been already studied for linear elastic foundations [3,12]. Bifurcation curves were derived, coupling the critical velocity and the frequency of oscillation of the load magnitude. To the authors' knowledge, however, bifurcation curves for nonlinear foundations have been obtained only recently in [7], where the single effect of the frequency variation was investigated.

This paper is concerned with the transient dynamic response of a simply-supported Euler-Bernoulli elastic beam resting on a spatially homogeneous nonlinear Winkler elastic foundation under the action of a transverse concentrated load with harmonic-varying magnitude, moving at a constant velocity along the beam. The subgrade reaction of the foundation depends on the displacements of the beam according to a nonlinear cubic constitutive law. The dynamic response is obtained numerically by an autonomous FEM implementation and a HHT- α algorithm for the direct time integration, as presented in [7,9].

The aim of the present paper is to extend the analyses in [7], by also exploring the dependency of the critical velocities on the mean value of the harmonic moving load, together with its frequency of oscillation, outlining here new results for the considered nonlinear foundation case. Through extensive numerical simulations, the value and number of the critical velocities is shown to depend on the characteristic parameters of the harmonic moving load. *The manifestation that three different critical velocities may appear is revealed.* Then, three-branched curves are reproduced by appropriate analytical fitting proposals with calibrated coefficients, providing effective formulas to represent the critical velocities/load frequency dependencies, as reported in [7] for a zero mean moving load magnitude.

The paper is organized as follows. Section 2 presents the governing boundary value problem for the nonlinear partial differential equation of motion of a simply-supported elastic beam resting on a Winkler elastic foundation, and its FEM discretization. In Section 3 maximum displacements and critical velocities are numerically determined as a function of the mean value of the moving load and of its frequency of oscillation, and the dependency of the critical velocities is depicted in appropriate fitted curves. Finally, main conclusions are outlined in closing Section 4.

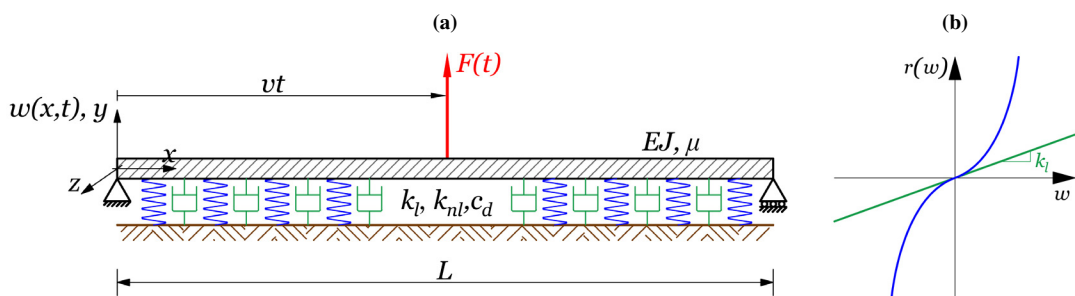


Fig. 1: Finite simply-supported Euler-Bernoulli elastic beam lying on an elastic nonlinear foundation subjected to a moving load with variable magnitude (a); nonlinear cubic constitutive law of the foundation (b).

2. Model equations and finite element formulation

Consider the idealized system shown in Fig. 1, consisting of a finite simply-supported Euler-Bernoulli elastic beam with Young's modulus E , moment of inertia J , length L , mass per unit length μ , lying on a nonlinear cubic Winkler elastic foundation and subjected to a concentrated load of magnitude $F(t)$, varying in time, moving with constant velocity v along the beam. The load is assumed to be positive if upward and described by a variable harmonic magnitude $F(t) = F_0 + F \sin(\Omega t)$, where $F_0 = \alpha F$ is the mean value, F the reference amplitude and Ω the angular frequency of the harmonic magnitude variation in time. From these assumptions, the equation of motion describing the transverse deflection of the beam, possibly accounting also for smeared structural damping c_d , is:

$$EJ \frac{\partial^4 w(x, t)}{\partial x^4} + \mu \frac{\partial^2 w(x, t)}{\partial t^2} + c_d \frac{\partial w(x, t)}{\partial t} + r(w(x)) = F(t) \delta(x - vt); \quad 0 < x < L, \quad t > 0; \quad (1)$$

$$r(w(x)) = k_l w(x) + k_{nl} w(x)^3; \quad (2)$$

where x is the axial coordinate, with the origin fixed on the left end of the beam, t is the time variable, with the origin at the instant on which the force starts its motion from the left end of the beam ($x = 0$) and $w(x, t)$ is the vertical deflection of the beam (positive if upward). In foundation constitutive Eq. (2), k_l is a classical linear Winkler coefficient and k_{nl} defines an additional nonlinear stiffness coefficient attached to the cubic term. The right hand side of Eq. (1) represents, by means of Dirac delta function δ , with relative position $x - vt$ as argument, the effect of a unit concentrated moving load acting at time t at position vt .

By applying a classical displacement-based FEM modeling, differential equation of motion (1) may be rewritten for an arbitrary finite element in semi-discretized form [1]. Then, by assembling all finite element contributions and imposing the boundary conditions of zero transverse displacements at the two extreme nodes of the beam (simply-supported boundary conditions), the global discretized equations of motion are obtained as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}_d\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{Q}_{nl}(\mathbf{q}) = F(t)\mathbf{\Psi}(vt); \quad (3)$$

where \mathbf{M} , \mathbf{C}_d and \mathbf{K} are the global structural mass, damping and stiffness matrices, \mathbf{Q}_{nl} is the global vector of the nonlinear forces of the foundation, $\mathbf{\Psi}(vt)$ is the global vector of shape functions (nodal generalized forces equivalent to the Dirac delta function), evaluated at instantaneous load location vt , \mathbf{q} , $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are the global vectors of generalized displacements, velocities and accelerations, respectively. Both mass and stiffness matrices are symmetric and positive definite. The previously defined matrices and vectors can be derived according to the formulation in [1]. Mass-proportional damping may be assumed, according to [4].

Assuming here standard homogeneous initial conditions $\mathbf{q}(0) = \mathbf{0}$; $\dot{\mathbf{q}}(0) = \mathbf{0}$, the numerical solution of Eq. (3) can be achieved through a HHT- α implementation, as in [7]. The main methodology and computational details about the whole FEM implementation are reported in [8].

3. Results and discussion

The analyzed beam is a UIC60 rail (see Fig. 2a), a most diffused steel profile in railway tracks (mechanical properties reported in Fig. 2c). A beam length of $L = 200$ m has been selected, in order to reasonably represent the case of a long beam and approach that of an infinite beam. The reference amplitude is taken as $F = 83.4$ kN [1]. The number of adopted finite elements is 200 (mesh size $h = 1$ m). The time span taken throughout the integration process corresponds to the interval of time along which the moving load is acting along the beam, that is $\tau = L/v$, plus one fifth of τ for free vibrations. The adopted time step is $\Delta t = \min\{10^{-3}\text{s}, h/(5v)\}$ in order to warrant a sufficient accuracy.

By assuming constant values of elastic coefficients of the foundation [1], for each load frequency Ω and ratio $\alpha = F_0/F$, computations are performed for a moving load velocity varying between 10 m/s and 300 m/s, with a step variation of 1 m/s, recording, for each simulation, the maximum upward w_{max}^+ (positive) and downward w_{max}^- (negative) displacements of the beam. The frequency variation is chosen according to the stiffness coefficients of the foundation and the mechanical characteristics of the rail profile, and ranges from 0 to 30 rad/s, with intervals of 10 rad/s. Parameter α is varied in order to obtain the harmonic moving load magnitude trends depicted in Fig. 2b. Finally, since damping has been observed to display a negligible influence on the scrutinized critical velocities, it has been neglected for the results reported here ($c_d = 0$), as it was done in [1].

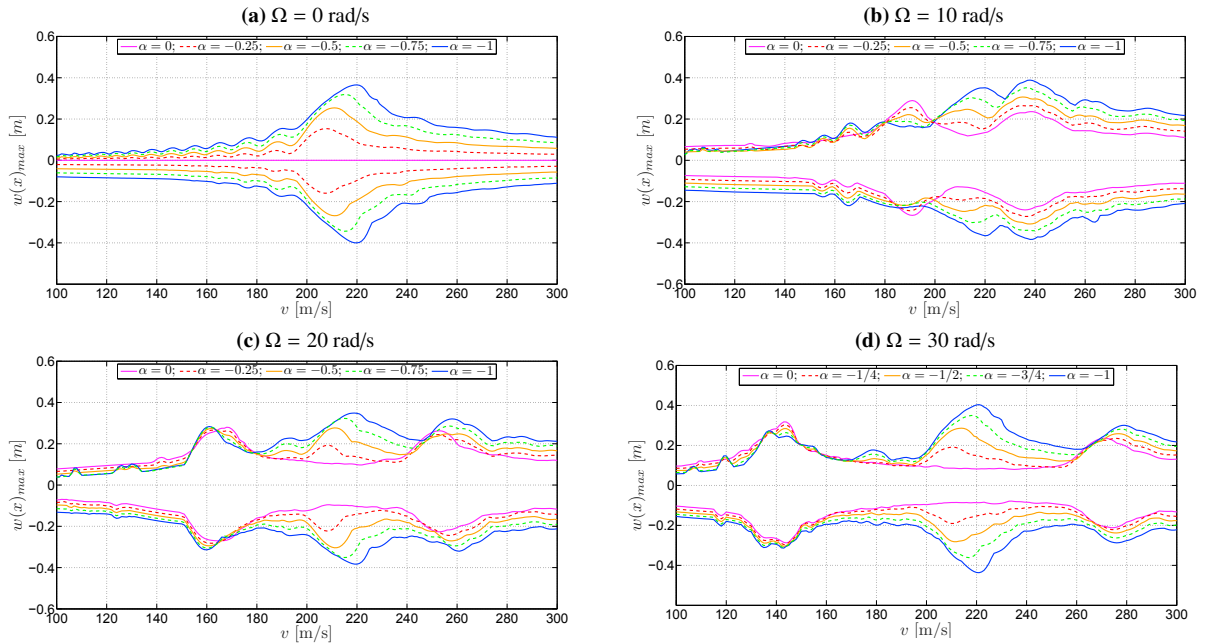


Fig. 4: Representation of beam maximum displacements as a function of load velocity v and load ratio $\alpha = F_0/F$ for an undamped nonlinear cubic elastic foundation with stiffnesses $k_l = 2.5 \times 10^2 \text{ kN/m}^2$ and $k_{nl} = 2.5 \times 10^3 \text{ kN/m}^4$. Load magnitude frequency Ω ranges from 0 rad/s to 30 rad/s.

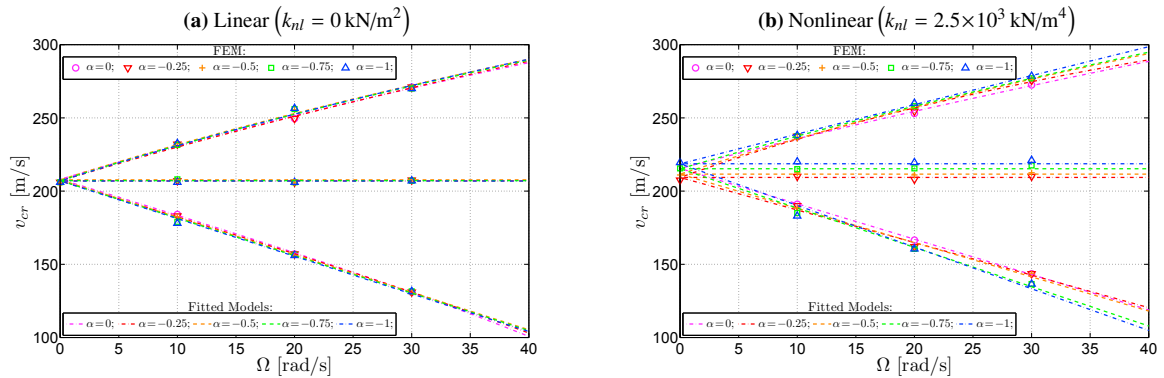


Fig. 5: Critical velocities and load frequency three-branched curves for undamped linear or nonlinear cubic elastic foundations ($k_l = 2.5 \times 10^2 \text{ kN/m}^2$). Results extracted from FEM results in Figs. 3-4 and fitted curves from predictions in Eq. (4).

Table 1: Values of fitted a_{ij} coefficients according to Eq. (4).

α	Linear model				Nonlinear model				
	$a_{01}=a_{02}=a_{03}$ [m ² /s ²]	$a_{11}=-a_{13}$ [m ² /(rad s)]	a_{21} [m ² /rad ²]	a_{23} [m ² /rad ²]	$a_{01}=a_{02}=a_{03}$ [m ² /s ²]	a_{11} [m ² /(rad s)]	a_{13} [m ² /(rad s)]	a_{21} [m ² /rad ²]	a_{23} [m ² /rad ²]
0.00	$4.326 \cdot 10^4$	$-1.006 \cdot 10^3$	4.451	$-4.588 \cdot 10^{-1}$	$4.641 \cdot 10^4$	$-1.041 \cdot 10^3$	5.833	$9.101 \cdot 10^3$	$2.910 \cdot 10^1$
-0.25	$4.283 \cdot 10^4$	$-1.008 \cdot 10^3$	5.054	$1.291 \cdot 10^{-1}$	$4.380 \cdot 10^4$	$-9.305 \cdot 10^2$	4.942	$1.213 \cdot 10^3$	-5.272
-0.50	$4.312 \cdot 10^4$	$-1.063 \cdot 10^3$	6.552	$-9.597 \cdot 10^{-1}$	$4.475 \cdot 10^4$	$-9.890 \cdot 10^2$	5.465	$1.100 \cdot 10^3$	-1.501
-0.75	$4.290 \cdot 10^4$	$-1.060 \cdot 10^3$	6.553	$-9.513 \cdot 10^{-1}$	$4.632 \cdot 10^4$	$-1.156 \cdot 10^3$	7.207	$1.007 \cdot 10^3$	$2.157 \cdot 10^1$
-1.00	$4.280 \cdot 10^4$	$-1.063 \cdot 10^3$	6.605	$-7.281 \cdot 10^{-1}$	$4.780 \cdot 10^4$	$-1.244 \cdot 10^3$	8.093	$8.939 \cdot 10^3$	3.497

The extrapolated relationship between frequency Ω and critical velocities $v_{cr,j}$, $j = 1, 2, 3$ is explicitly depicted in Fig. 5. Such threefold curves have been derived by appropriate analytical fitting proposals on the data collected from the numerical simulations. The following fitting expressions have been adopted for both considered types of

foundation, linear and nonlinear cubic:

$$v_{cr,j} = \sqrt{a_{0j} + a_{1j}\Omega + a_{2j}\Omega^2}, \quad j = 1, 3; \quad v_{cr,2} = \sqrt{a_{02}}; \quad (4)$$

where each a_{ij} coefficient is implicitly depending on load ratio α . The associated fitting procedure, with results in Table 1, has been performed by imposing additional constraints on the a_{ij} coefficients, as reported in Table 1 itself.

4. Conclusions

In this work, the dynamic transient response of a simply-supported beam, of a finite, though wide, length, thus approaching the configuration of an infinite beam, lying on a linear or a nonlinear cubic elastic Winkler foundation subjected to a harmonic load moving with constant velocity is numerically analyzed by a FEM implementation. Several numerical simulations have been performed by varying the moving load velocity, magnitude mean value and frequency, providing the following interesting outcomes in terms of the mechanical behavior of high-speed rail tracks.

If the moving load magnitude is constant, only one critical velocity appears, while two critical velocities are observed for a harmonic moving load with zero mean magnitude, one increasing, the other decreasing, as the moving load frequency increases. Three critical velocities are instead detected for a harmonic moving load with non-zero mean magnitude, which also depend on the moving load mean magnitude, in the nonlinear case. The relationship between the critical velocities and the moving load frequency has been portrayed in appropriate three-branch curves (Fig. 5). These curves have been achieved by fitting the values of the critical velocities computed at different load frequencies through effective analytical formulas (4), which may be assessed for practical treatments.

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