AN EXPLICIT CHEBOTAREV DENSITY THEOREM UNDER GRH

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Abstract. We prove an explicit version of the Chebotarev theorem for the density of prime ideals with fixed Artin symbol, under the assumption of the validity of the Riemann hypothesis for the Dedekind zeta functions. In appendix we also give some explicit formulas counting non-trivial zeros of Hecke's L-functions, in that case without assuming the truth of the Riemann hypothesis.

To Jurek Kaczorowski
for his 60th birthday

1. Introduction

In order to state the results we need to fix some notation. Thus, given a number field $\mathbb{K}$ we denote $n_{\mathbb{K}}$ its dimension and $r_1(\mathbb{K})$, $r_2(\mathbb{K})$ the number of its real, respectively imaginary places; the absolute value of its discriminant is denoted as $\Delta_{\mathbb{K}}$, $\mathfrak{p}$ always denotes a nonzero prime ideal of the integer ring $\mathcal{O}_{\mathbb{K}}$, and $\mathrm{N}\mathfrak{p}$ its absolute norm; $\Lambda_{\mathbb{K}}$ denotes the analogue of the von Mangoldt function, i.e. the function which is defined on the set of ideals of $\mathcal{O}_{\mathbb{K}}$ and whose value at an ideal $\mathfrak{I}$ is $\log \mathrm{N}\mathfrak{p}$ if $\mathfrak{I} = \mathfrak{p}^m$ for some $\mathfrak{p}$ and $m \geq 1$, and zero otherwise. Moreover, let $\mathbb{K} \subseteq \mathbb{L}$ be a Galois extension of number fields with relative discriminant $\Delta_{\mathbb{L}/\mathbb{K}}$, and let $\mathfrak{P}$ be a prime ideal of $\mathbb{L}$ above a non-ramified $\mathfrak{p}$ prime ideal of $\mathcal{O}_{\mathbb{K}}$. Then the Artin symbol $\left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right]$ denotes the conjugacy class of the Frobenius automorphism corresponding to $\mathfrak{P}/\mathfrak{p}$, and which is extended multiplicatively on the prime powers in $\mathcal{O}_{\mathbb{K}}$ coprime to $\Delta_{\mathbb{L}/\mathbb{K}}$. Let C be any conjugacy class in $G := \mathrm{Gal}(\mathbb{L}/\mathbb{K})$ and let ε_C be its characteristic function. Then the function π_C and the Chebyshev function ψ_C are defined as

$$
\begin{aligned}
\pi_C(x) &:= \sharp\big\{\mathfrak{p}\colon \mathfrak{p} \text{ non-ramified in } \mathbb{L}/\mathbb{K}, \mathrm{N}\mathfrak{p} \leq x, \left[\tfrac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right] = C\big\} \\
&= \sum_{\substack{\mathfrak{p} \\ \mathfrak{p} \text{ non-ram.} \\ \mathrm{N}\mathfrak{p} \leq x}} \varepsilon_C\big(\left[\tfrac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right]\big), \\
\psi_C(x) &:= \sum_{\substack{\mathfrak{I} \subset \mathcal{O}_{\mathbb{K}} \\ \mathfrak{I} \text{ non-ram.} \\ \mathrm{N}\mathfrak{I} \leq x}} \varepsilon_C\big(\left[\tfrac{\mathbb{L}/\mathbb{K}}{\mathfrak{I}}\right]\big)\Lambda_{\mathbb{K}}(\mathfrak{I}).
\end{aligned}
$$

The first function counts the number of non-ramified prime ideals with prescribed Artin symbol, while Chebyshev's function does the same but with a suitable logarithmic weight supported on prime powers. The celebrated Chebotarev density theorem states that $\pi_C(x) \sim \frac{|C|}{|G|}\frac{x}{\log x}$ when x diverges, a claim which can be stated equivalently by saying that $\psi_C(x) \sim \frac{|C|}{|G|}x$. We introduce also two other functions which are closely related to π_C and ψ_C but that are easier to deal with. They are built using an arithmetical function which comes from the theory of Artin L-functions and extends $\varepsilon_C\big(\left[\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right]\big)$ to ramifying prime ideals. To wit, for any prime ideal $\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{K}}$ (possibly ramified) let $\mathfrak{P}$ be any prime ideal dividing $\mathfrak{p}\mathcal{O}_{\mathbb{L}}$, let I be the

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inertia group of $\mathfrak{P}$ and τ be one of the $|I|$ Frobenius automorphisms corresponding to $\mathfrak{P}/\mathfrak{p}$. Let

$$\theta(C;\mathfrak{p}^m) := \frac{1}{|I|}\sum_{a\in I}\varepsilon_C(\tau^m a). \tag{1.1}$$

Notice that $\theta(C;\mathfrak{p}^m)\in[0,1]$, and that for non-ramified primes it is 1 if and only if τ^m belongs to C, and 0 otherwise. We define

$$\begin{aligned}\pi(C;x) &:= \sum_{\mathfrak{p}\colon\, \mathrm{N}\mathfrak{p}\leq x}\theta(C;\mathfrak{p})\log\mathrm{N}\mathfrak{p},\\ \psi(C;x) &:= \sum_{\substack{\mathfrak{I}\subset\mathcal{O}_{\mathbb{K}}\\ \mathrm{N}\mathfrak{I}\leq x}}\theta(C;\mathfrak{I})\Lambda_{\mathbb{K}}(\mathfrak{I}).\end{aligned}$$

Observe that $\psi_C(x)$ and $\psi(C;x)$ agree except on ramified-prime-powers ideals, being

$$\psi(C;x) = \psi_C(x)+\mathfrak{R}_C(x) \tag{1.2}$$

with

$$\mathfrak{R}_C(x) := \sum_{\mathfrak{p}|\Delta_{\mathbb{L}/\mathbb{K}}}\sum_{\substack{m\geq 1\\ \mathrm{N}\mathfrak{p}^m\leq x}}\theta(C;\mathfrak{p}^m)\log\mathrm{N}\mathfrak{p}. \tag{1.3}$$

In particular, $0\leq\psi_C(x)\leq\psi(C;x)$ for every x, so that every upper bound for $\psi(C;x)$ gives also a bound for $\psi_C(x)$, and a lower bound for $\psi_C(x)$ produces a lower bound for $\psi(C;x)$.

Jeffrey Lagarias and Andrew Odlyzko [12] provided versions of Chebotarev's theorem which are explicit in their dependence on the field $\mathbb{K}$ up to positive universal constants which however are not estimated, and Joseph Oesterlé [15] announced that

$$\Big|\frac{|G|}{|C|}\psi(C;x)-x\Big| \leq \sqrt{x}\Big[\Big(\frac{\log x}{\pi}+2\Big)\log\Delta_{\mathbb{L}}+\Big(\frac{\log^2 x}{2\pi}+2\Big)n_{\mathbb{L}}\Big] \qquad \forall\, x\geq 1 \tag{1.4}$$

under the assumption of the generalized Riemann hypothesis. On the other hand, Lowell Schoenfeld [22] proved that the Riemann hypothesis implies that

$$|\psi_{\mathbb{Q}}(x)-x| \leq \frac{1}{8\pi}\sqrt{x}\log^2 x \qquad \forall\, x\geq 59.$$

(He states this result for $x\geq 73.2$, but actually it is easy to check that the inequality holds also for $x\in[59,73.2]$). This result shows that it should be possible to improve the constants appearing in Oesterlé's result. Bruno Winckler [24, Th. 8.1] proved a result similar to (1.4), but with larger coefficients of logs in the $\log\Delta_{\mathbb{L}}$ and $n_{\mathbb{L}}$ parts.

In [7] we have proved an analogue of Schoenfeld's result for the easier case $\mathbb{K}=\mathbb{L}$, where all prime ideals are counted. In this paper we generalize this work to the full set of extensions and classes, as in Oesterlé's result, but with the improved constants. In fact, the following theorem is our main result.

Theorem 1.1. *Assume GRH holds. Then* $\forall x\geq 1$

$$\begin{aligned}\Big|\frac{|G|}{|C|}\psi(C;x)-x\Big| &\leq \sqrt{x}\Big[\Big(\frac{\log x}{2\pi}+2\Big)\log\Delta_{\mathbb{L}}+\Big(\frac{\log^2 x}{8\pi}+2\Big)n_{\mathbb{L}}\Big],\\ \Big|\frac{|G|}{|C|}\psi_C(x)-x\Big| &\leq \sqrt{x}\Big[\Big(\frac{\log x}{2\pi}+2\Big)\log\Delta_{\mathbb{L}}+\Big(\frac{\log^2 x}{8\pi}+2\Big)n_{\mathbb{L}}\Big].\end{aligned} \tag{1.5}$$

From the proof it will be clear that the constants +2 have nothing special and other values are possible. For instance, one can prove that

$$\Big|\frac{|G|}{|C|}\psi(C;x)-x\Big| \leq \sqrt{x}\Big[\Big(\frac{\log x}{2\pi}+2\Big)\log\Delta_{\mathbb{L}}+\frac{\log^2 x}{8\pi}n_{\mathbb{L}}\Big]+40,$$

again for all $x \geq 1$. Moreover, the +40 can be removed if $n_{\mathbb{L}} \geq 7$, and both +2, +40 can be removed if x is large enough. One can also prove a result of the form of [7, Corollary 1.3] where $\log x$ is substituted by $\log\big(\frac{cx}{\log^2 x}\big)$ for some constant c. All remarks apply also to $\psi_C(x)$.

By partial summation one deduces the following result.

Corollary 1.2. *Assume GRH holds. Then* $\forall x \geq 2$

$$\Big|\frac{|G|}{|C|}\pi(C;x)-\int_2^x \frac{\mathrm{d}u}{\log u}\Big| \leq\sqrt{x}\Big[\Big(\frac{1}{2\pi}+\frac{3}{\log x}\Big)\log\Delta_{\mathbb{L}}+\Big(\frac{\log x}{8\pi}+\frac{1}{4\pi}+\frac{6}{\log x}\Big)n_{\mathbb{L}}\Big],$$

$$\Big|\frac{|G|}{|C|}\pi_C(x)-\int_2^x \frac{\mathrm{d}u}{\log u}\Big| \leq\sqrt{x}\Big[\Big(\frac{1}{2\pi}+\frac{3}{\log x}\Big)\log\Delta_{\mathbb{L}}+\Big(\frac{\log x}{8\pi}+\frac{1}{4\pi}+\frac{6}{\log x}\Big)n_{\mathbb{L}}\Big].$$

This corollary also could be improved in the secondary terms as in [7, Corollary 1.4] which, unfortunately, was stated incorrectly and should read

Corollary ([7, Corollary 1.4])**.** *Assume GRH holds. Then* $\forall x \geq 2$

$$\Big|\pi_{\mathbb{K}}(x)-\int_2^x \frac{\mathrm{d}u}{\log u}\Big|$$
$$\leq \sqrt{x}\Big[\Big(\frac{1}{2\pi}-\frac{\log\log x}{\pi\log x}+\frac{5.8}{\log x}\Big)\log\Delta_{\mathbb{K}}+\Big(\frac{1}{8\pi}-\frac{\log\log x}{2\pi\log x}+\frac{3.6}{\log x}\Big)n_{\mathbb{K}}\log x+0.3+\frac{14}{\log x}\Big].$$

The general strategy for the proof is quite similar to the one of [12] and [6]. However, many estimations have to be done with special care, in order to reduce the range of fields $\mathbb{K}$, extensions $\mathbb{L}/\mathbb{K}$ and x where the claims have to be proved directly via explicit computations.

We have made available at the address:
`http://users.mat.unimi.it/users/molteni/research/chebotarev/chebotarev.gp`
the PARI/GP [17] code we have used to compute the constants in this paper.

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2. Facts

Let

$$\psi^{(1)}(C;x) := \int_0^x \psi(C;t)\,\mathrm{d}t.$$

As observed by Ingham [10, Ch. 2, Sec. 5], since $\psi(C;x)$ is non-decreasing as a function of x, one has the double inequality

$$\begin{aligned}\psi(C;x) &\leq \frac{\psi^{(1)}(C;x+h)-\psi^{(1)}(C;x)}{h} \qquad \text{if } h>0,\\ \psi(C;x) &\geq \frac{\psi^{(1)}(C;x+h)-\psi^{(1)}(C;x)}{h} \qquad \text{if } -x<h<0.\end{aligned} \tag{2.1}$$

We let, for $s>1$,

$$K(C;s) := \sum_{\mathfrak{I}\subseteq\mathcal{O}_{\mathbb{K}}} \theta(C;\mathfrak{I})\Lambda_{\mathbb{K}}(\mathfrak{I})(\mathrm{N}\mathfrak{I})^{-s}. \tag{2.2}$$

As in [10, Ch. IV Sec. 4, p. 73] and [12, Sec. 5], we have the integral representation

$$\psi^{(1)}(C;x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} K(C;s)\frac{x^{s+1}}{s(s+1)}\,\mathrm{d}s.$$

Let g be any element in C, then the orthogonality of the irreducible characters ϕ of G allows one to write

$$\theta(C;\mathfrak{p}^m) = \frac{|C|}{|G|}\sum_{\phi}\bar{\phi}(g)\phi_{\mathbb{K}}(\mathfrak{p}^m)$$

where

$$\phi_{\mathbb{K}}(\mathfrak{p}^m) := \frac{1}{|I|}\sum_{a\in I}\phi(\tau^m a).$$

The definitions of $\theta(C;\cdot)$ and $\phi_{\mathbb{K}}$ are modelled on the definition of the Artin L-functions $L(s,\phi,\mathbb{L}/\mathbb{K})$, giving the equality

$$K(C;s) = \sum_{\mathfrak{I}\subseteq\mathcal{O}_{\mathbb{K}}}\theta(C;\mathfrak{I})\Lambda_{\mathbb{K}}(\mathfrak{I})(\mathrm{N}\mathfrak{I})^{-s} = -\frac{|C|}{|G|}\sum_{\phi}\bar{\phi}(g)\frac{L'}{L}(s,\phi,\mathbb{L}/\mathbb{K})$$

for $\operatorname{Re}s > 1$.
Following an argument of Lagarias and Odlyzko (which comes from Deuring [5] and MacCluer [13]) we can modify the identity in order to use only Hecke L-functions, for which the continuation as holomorphic functions (apart at $s=1$) in $\mathbb{C}$ is proved: it is [12, Lemma 4.1], but a quick review can be useful.
As above, let g be any fixed element in C. Let H be the cyclic group generated by g and let $\mathbb{E} := \mathbb{L}^H = \mathbb{L}^g$, the subfield of $\mathbb{L}$ fixed by H. Let $f_g\colon H\to\mathbb{C}$ be the characteristic function of $\{g\}$. A direct computation shows that it induces on G the class function $\operatorname{Ind}_H^G f_g\colon G\to\mathbb{C}$ whose values are

$$(\operatorname{Ind}_H^G f_g)(y) = \frac{1}{|H|}\sum_{s\in G} f_g(s^{-1}ys) = \begin{cases}\frac{|G|}{|C||H|} & \text{if } y\in C\\ 0 & \text{otherwise.}\end{cases}$$

Thus, the characteristic function of C is $\frac{|C||H|}{|G|}\operatorname{Ind}_H^G f_g$. By orthogonality of characters χ of H one has

$$f_g = \frac{1}{|H|}\sum_{\chi}\bar{\chi}(g)\chi,$$

thus

$$(\operatorname{Ind}_H^G f_g)(y) = \frac{1}{|H|}\sum_{\chi}\bar{\chi}(g)(\operatorname{Ind}_H^G \chi)(y),$$

and the characteristic function of C is now written as $\frac{|C|}{|G|}\sum_{\chi}\bar{\chi}(g)\operatorname{Ind}_H^G\chi$. Using the definition of $\theta(C;\cdot)$, we find that

$$\theta(C;\mathfrak{p}^m) = \frac{|C|}{|G|}\sum_{\chi}\bar{\chi}(g)\chi_{\mathbb{K}}(\mathfrak{p}^m)$$

where

$$\chi_{\mathbb{K}}(\mathfrak{p}^m) := \frac{1}{|I|}\sum_{a\in I}(\operatorname{Ind}_H^G\chi)(\tau^m a).$$

In this way we get

$$K(C;s) = -\frac{|C|}{|G|}\sum_{\chi}\bar{\chi}(g)\frac{L'}{L}(s,\operatorname{Ind}_H^G\chi,\mathbb{L}/\mathbb{K}) = -\frac{|C|}{|G|}\sum_{\chi}\bar{\chi}(g)\frac{L'}{L}(s,\chi,\mathbb{L}/\mathbb{E}), \tag{2.3}$$

which means

$$\psi^{(1)}(C;x) = -\frac{|C|}{|G|}\sum_{\chi}\bar{\chi}(g)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s,\chi,\mathbb{L}/\mathbb{E})\frac{x^{s+1}}{s(s+1)}\,\mathrm{d}s, \tag{2.4}$$

where only abelian (i.e., Hecke, by class field theory) L-functions appear.

Thus, let $\mathbb{E}\subseteq\mathbb{L}$ be an abelian extension of fields and let χ be any irreducible character of $\mathrm{Gal}(\mathbb{L}/\mathbb{E})$. We will use $L(s,\chi)$ to denote $L(s,\chi,\mathbb{L}/\mathbb{E})$. Also, set $\delta_\chi = 1$ if χ is the trivial character, 0 otherwise.

We recall that for each χ there exist uniquely determined non-negative integers a_χ, b_χ such that

$$a_\chi + b_\chi = n_{\mathbb{E}}$$

and a positive integer $Q(\chi)$ such that if we define

$$\Gamma_\chi(s) := \Big[\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\Big]^{a_\chi}\Big[\pi^{-\frac{s+1}{2}}\Gamma\Big(\frac{s+1}{2}\Big)\Big]^{b_\chi} \tag{2.5}$$

and

$$\xi(s,\chi) := [s(s-1)]^{\delta_\chi}Q(\chi)^{s/2}\Gamma_\chi(s)L(s,\chi), \tag{2.6}$$

then $\xi(s,\chi)$ satisfies the functional equation

$$\xi(1-s,\bar{\chi}) = W(\chi)\xi(s,\chi), \tag{2.7}$$

where $W(\chi)$ is a certain constant of absolute value 1. For the trivial character χ, the Hecke L-function $L(s,\chi,\mathbb{L}/\mathbb{E})$ coincides with Dedekind's zeta function $\zeta_{\mathbb{E}}(s)$, and in this case $a_\chi = r_1(\mathbb{E})+r_2(\mathbb{E})$ and $b_\chi = r_2(\mathbb{E})$. Furthermore, $\xi(s,\chi)$ is an entire function (by class field theory) of order 1 and does not vanish at $s=0$, and hence by Hadamard's product theorem we have

$$\xi(s,\chi) = e^{A_\chi + B_\chi s}\prod_{\rho\in Z_\chi}\Big(1-\frac{s}{\rho}\Big)e^{s/\rho} \tag{2.8}$$

for some constants A_χ and B_χ, where Z_χ is the set of zeros (multiplicity included) of $\xi(s,\chi)$. They are precisely those zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$ for which $0<\beta<1$, the so-called "non-trivial zeros" of $L(s,\chi)$. From now on ρ will denote a non-trivial zero of $L(s,\chi)$.

Lastly, we introduce a special notation for the type of sum on characters as the one appearing in (2.4), and for any $f\colon \widehat{\mathrm{Gal}(\mathbb{L}/\mathbb{E})}\to\mathbb{C}$ we set

$$\mathcal{M}_C f := \sum_{\chi}\bar{\chi}(g)f(\chi)$$

where we recall that g is a fixed element of C.

3. Preliminary inequalities

3.1. **Reduction to Dedekind Zeta functions.** Differentiating (2.6) and (2.8) logarithmically we obtain the identity

$$\frac{L'}{L}(s,\chi) = B_\chi + \sum_{\rho\in Z_\chi}\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) - \frac{1}{2}\log Q(\chi) - \delta_\chi\Big(\frac{1}{s}+\frac{1}{s-1}\Big) - \frac{\Gamma'_\chi}{\Gamma_\chi}(s), \tag{3.1}$$

for all complex s. Using (2.5), (2.6) and (3.1) one sees that

$$\begin{aligned}\frac{L'}{L}(s,\chi) &= \frac{a_\chi-\delta_\chi}{s} + r_\chi + O(s) \quad \text{as } s\to 0,\\ \frac{L'}{L}(s,\chi) &= \frac{b_\chi}{s+1} + r'_\chi + O(s+1) \quad \text{as } s\to -1,\end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
r_\chi &= B_\chi+\delta_\chi-\frac{1}{2}\log\frac{Q(\chi)}{\pi^{n_{\mathbb{E}}}}-\frac{a_\chi}{2}\frac{\Gamma'}{\Gamma}(1)-\frac{b_\chi}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}\Big),\\
r'_\chi &= -\frac{L'}{L}(2,\bar\chi)-\log\frac{Q(\chi)}{\pi^{n_{\mathbb{E}}}}-\frac{n_{\mathbb{E}}}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{3}{2}\Big)-\frac{n_{\mathbb{E}}}{2}\frac{\Gamma'}{\Gamma}(1).
\end{aligned}
\tag{3.3}
$$

Comparing the previous formula for r_χ and (3.1), we get

$$
\begin{aligned}
r_\chi =& \frac{L'}{L}(s,\chi)-\sum_{\rho\in Z_\chi}\frac{s}{\rho(s-\rho)}+\delta_\chi\Big(\frac{1}{s}+\frac{1}{s-1}\Big)\\
&+\frac{a_\chi}{2}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)-\frac{\Gamma'}{\Gamma}(1)\Big)+\frac{b_\chi}{2}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{s+1}{2}\Big)-\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}\Big)\Big)
\end{aligned}
$$

for every $s\in\mathbb{C}$. Setting $s=2$ this formula simplifies to

$$
r_\chi = \frac{L'}{L}(2,\chi)-\sum_\rho\frac{2}{\rho(2-\rho)}+\frac{3}{2}\delta_\chi+b_\chi. \tag{3.4}
$$

We come back to the situation where $g\in C$ and $\mathbb{E}=\mathbb{L}^g$, so that $\mathbb{L}/\mathbb{E}$ is a cyclic extension for which g is a generator of $\mathrm{Gal}(\mathbb{L}/\mathbb{E})$. The following lemma computes the mean values of the parameters a_χ and b_χ appearing in (2.5). To simplify the formulas, we will write from now on r_1 and r_2 for $r_1(\mathbb{L})$ and $r_2(\mathbb{L})$.

Lemma 3.1. *Let*

$$
\mathbf{S} := \begin{cases} r_1+r_2 & \text{if } g \text{ has order } 1,\\ r_2-2r_2(\mathbb{E}) & \text{if } g \text{ has order } 2,\\ 0 & \text{otherwise,}\end{cases}
$$

and let δ_C defined to be 1 *if C is the trivial class and* 0 *otherwise. Then*

$$
\begin{aligned}
\mathcal{M}_C a_\chi &= \sum_\chi \bar\chi(g)a_\chi = \mathbf{S},\\
\mathcal{M}_C b_\chi &= \sum_\chi \bar\chi(g)b_\chi = \delta_C n_{\mathbb{E}}-\mathbf{S} = \delta_C n_{\mathbb{L}}-\mathbf{S}.
\end{aligned}
$$

Proof. If C is the trivial class, i.e. g has order 1, we have $\mathcal{M}_C a_\chi = \sum_\chi a_\chi = r_1+r_2 = \mathbf{S}$ because the extension $\mathbb{L}/\mathbb{E}$ is Galois, hence $\prod_\chi L(s,\chi)=\zeta_{\mathbb{L}}(s)$. We have as well $\mathcal{M}_C b_\chi = \sum b_\chi = r_2 = n_{\mathbb{L}}-\mathbf{S}$, hence the result is proved. We henceforth assume that g has order at least 2.

By duality, the set of characters of $\mathrm{Gal}(\mathbb{L}/\mathbb{E})$ is cyclic: let φ be a generator. The character χ corresponds to a Hecke character $\tilde\chi$ of the idèles of $\mathbb{E}$. For any real embedding of $\mathbb{E}$, let $p_\ell(\chi)$ be 1 if the local component of $\tilde\chi$ at ℓ is the sign character, and 0 otherwise. We furthermore denote s_χ the number of ℓ's for which $p_\ell(\chi)=1$. The construction of Hecke characters and L-functions shows that $a_\chi = r_1(\mathbb{E})+r_2(\mathbb{E})-s_\chi$, see [9]. In particular,

$$
\sum_\chi \bar\chi(g)a_\chi = -\sum_\chi \bar\chi(g)s_\chi.
$$

For every fixed real embedding ℓ one has $p_\ell(\chi\chi') = p_\ell(\chi)+p_\ell(\chi') \pmod 2$, thus $s_\chi=0$ when χ is an even power of φ, and $s_\chi=s_\varphi$ otherwise. This shows that if $|\mathrm{Gal}(\mathbb{L}/\mathbb{E})|$ is odd, then $s_\chi=0$ for every character, while when $|\mathrm{Gal}(\mathbb{L}/\mathbb{E})|$ is even one gets

$$
\sum_\chi \bar\chi(g)a_\chi = -s_\varphi\bar\varphi(g)\sum_{k=0}^{|\mathrm{Gal}(\mathbb{L}/\mathbb{E})|/2-1}(\bar\varphi^2)^k(g).
$$

This is the sum on the subgroup of the square characters, thus it is zero unless $\varphi^2(g) = 1$. This happens if and only $|\operatorname{Gal}(\mathbb{L}/\mathbb{E})| = 2$, because g is a generator, and in this case $\varphi(g) = -1$. Thus we get:

$$\sum_\chi \bar\chi(g)a_\chi = \begin{cases} s_\varphi & \text{if } |\operatorname{Gal}(\mathbb{L}/\mathbb{E})| = 2 \\ 0 & \text{otherwise.} \end{cases}$$

To conclude, we have $p_\ell(\varphi) = 1$ if and only if ℓ ramifies in $\mathbb{L}/\mathbb{E}$ hence $s_\varphi = r_2(\mathbb{L}) - 2r_2(\mathbb{E})$. This proves the lemma for the sum of the a_χ's. For the sum of the b_χ's it is sufficient to observe that

$$\sum_\chi \bar\chi(g)(a_\chi + b_\chi) = \sum_\chi \bar\chi(g) n_\mathbb{E} = 0. \qquad \square$$

Note that if g has order 1, then $\mathbf{S} = r_1 + r_2 = \frac{n_\mathbb{L}+r_1}{2}$. In the other cases we have $0 \le \mathbf{S} \le r_2 = \frac{n_\mathbb{L}-r_1}{2}$. Thus in all cases $0 \le \mathbf{S} \le \frac{n_\mathbb{L}-r_1}{2} + \delta_C r_1$.

Lemma 3.2. *Let $\mathbb{L}/\mathbb{E}$ be a cyclic extension and let Z be the multiset of non-trivial zeros of the Dedekind zeta function $\zeta_\mathbb{L}$. Let f be any complex function with $\sum_{\rho\in Z}|f(\rho)| < \infty$. Then*

$$\mathcal{M}_C \sum_{\rho\in Z_\chi} f(\rho) = \sum_{\rho\in Z} \epsilon(\rho) f(\rho)$$

where, for any $\rho \in Z$, $|\epsilon(\rho)| = 1$ and $\epsilon(\overline{\rho}) = \overline{\epsilon(\rho)}$.

Proof. Since $\zeta_\mathbb{L} = \prod_\chi L(s,\chi)$, the multiset Z is the disjoint union of the multisets Z_χ. Moreover, for each ρ in Z there is a well defined character χ such that $\rho \in Z_\chi$; for this ρ we set $\epsilon(\rho) := \overline{\chi(g)}$. This rule respects the formula $\epsilon(\overline{\rho}) = \overline{\epsilon(\rho)}$, because ρ belongs to Z_χ if and only if $\bar\rho$ belongs to $Z_{\bar\chi}$. Thus, we can write

$$\mathcal{M}_C \sum_{\rho\in Z_\chi} f(\rho) = \sum_\chi \sum_{\rho\in Z_\chi} \bar\chi(g) f(\rho) = \sum_{\rho\in Z} \epsilon(\rho) f(\rho).$$

The equality $|\epsilon(\rho)| = 1$ is obvious. $\square$

Lemma 3.3. *Let Z be the multiset of non-trivial zeros of the Dedekind zeta function $\zeta_\mathbb{L}$. Recall that $\mathbb{L}/\mathbb{E}$ is a cyclic extension and that $\mathbf{S}$ and $\epsilon(\rho)$ are defined in Lemmas 3.1 and 3.2, respectively. We have*

$$\mathcal{M}_C r_\chi = 2\sum_{\rho\in Z} \frac{\epsilon(\rho)}{\rho(2-\rho)} - \frac{n_\mathbb{L}}{n_\mathbb{K}|C|} \sum_{\mathfrak{I}\subseteq\mathcal{O}_\mathbb{K}} \theta(C;\mathfrak{I})\Lambda_\mathbb{K}(\mathfrak{I})(\mathrm{N}\mathfrak{I})^{-2} + n_\mathbb{L}\delta_C - \mathbf{S} + \frac{3}{2}.$$

Proof. By (3.4) and Lemmas 3.1 and 3.2 we get

$$\mathcal{M}_C r_\chi = 2\sum_{\rho\in Z} \frac{\epsilon(\rho)}{\rho(2-\rho)} + \mathcal{M}_C \frac{L'}{L}(2,\chi) + n_\mathbb{L}\delta_C - \mathbf{S} + \frac{3}{2}. \tag{3.5}$$

Moreover, by (2.3) we have

$$\mathcal{M}_C \frac{L'}{L}(2,\chi) = -\frac{|G|}{|C|} K(C;2)$$

hence by (2.2) we have

$$\mathcal{M}_C \frac{L'}{L}(2,\chi) = -\frac{n_\mathbb{L}}{n_\mathbb{K}|C|} \sum_{\mathfrak{I}\subseteq\mathcal{O}_\mathbb{K}} \theta(C;\mathfrak{I})\Lambda_\mathbb{K}(\mathfrak{I})(\mathrm{N}\mathfrak{I})^{-2}. \tag{3.6}$$

The result follows from (3.5) and (3.6). $\square$

Lemma 3.4. *We have*

$$|\mathcal{M}_C r'_\chi| \leq \log\Delta_{\mathbb{L}} + n_{\mathbb{L}}\Big|\frac{\zeta'}{\zeta}\Big|(2) + (\log 2\pi + \gamma - 1) n_{\mathbb{L}}\delta_C,$$

where $\gamma = 0.5772\dots$ is the Euler–Mascheroni constant.

Proof. By (3.3) we get

$$\mathcal{M}_C r'_\chi = -\mathcal{M}_C\frac{L'}{L}(2,\bar\chi) - \mathcal{M}_C\log Q(\chi) + n_{\mathbb{E}}\Big(\log\pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{3}{2}\Big) - \frac{1}{2}\frac{\Gamma'}{\Gamma}(1)\Big)\mathcal{M}_C 1.$$

Replacing C by $C_1 = [g^{-1}]$ and g by g^{-1} in (2.3) and conjugating, we get

$$\mathcal{M}_C\frac{L'}{L}(2,\bar\chi) = -\frac{|G|}{|C|}\overline{K(C^{-1};2)}$$

which by (2.2) is estimated by $\frac{n_{\mathbb{L}}}{|C|}\big|\frac{\zeta'}{\zeta}\big|(2)$ because $0 \leq \theta(C;\cdot) \leq 1$ by definition. Moreover,

$$|\sum_\chi \bar\chi(g)\log Q(\chi)| \leq \sum_\chi \log Q(\chi) = \log\Delta_{\mathbb{L}},$$

by the product formula for conductors.
The result follows because $\frac{\Gamma'}{\Gamma}\big(\frac{3}{2}\big) + \frac{\Gamma'}{\Gamma}(1) = 2 - \log 4 - 2\gamma$ and $n_{\mathbb{E}}\mathcal{M}_C 1 = n_{\mathbb{L}}\delta_C$. □

Lemma 3.5. *We define, for any $x > 1$ and any character χ,*

$$f_1(x) := \sum_{r=1}^{\infty}\frac{x^{1-2r}}{2r(2r-1)}, \qquad f_2(x) := \sum_{r=2}^{\infty}\frac{x^{2-2r}}{(2r-1)(2r-2)},$$
$$R_\chi(x) := -(a_\chi - \delta_\chi)(x\log x - x) + b_\chi(\log x + 1) - a_\chi f_1(x) - b_\chi f_2(x)$$

and

$$R_C(x) := \mathcal{M}_C R_\chi(x).$$

Then for any $x > 1$,

$$R_C(x) = \int_0^x \log u\,\mathrm{d}u - \mathbf{S}\int_1^{x+1}\log u\,\mathrm{d}u + \delta_C\frac{n_{\mathbb{L}}}{2}\Big[\log(x^2-1) + x\log\Big(\frac{x+1}{x-1}\Big)\Big],$$
$$R'_C(x) = \log x - \mathbf{S}\log(x+1) + \delta_C\frac{n_{\mathbb{L}}}{2}\log\Big(\frac{x+1}{x-1}\Big).$$

Proof. We have

$$f_1(x) = \frac{1}{2}\Big[x\log(1-x^{-2}) + \log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big)\Big],$$
$$f_2(x) = 1 - \frac{1}{2}\Big[\log(1-x^{-2}) + x\log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big)\Big].$$

Assume first that C is not the trivial class. By Lemma 3.1,

$$\begin{aligned}
R_C(x) &= -(\mathbf{S}-1)(x\log x - x) - \mathbf{S}(\log x + 1) - \mathbf{S}f_1(x) + \mathbf{S}f_2(x)\\
&= x\log x - x + \mathbf{S}\Big(-(x+1)\log x + x - \frac{x+1}{2}\Big(\log(1-x^{-2}) + \log\Big(\frac{1+x^{-1}}{1-x^{-1}}\Big)\Big)\Big)\\
&= x\log x - x + \mathbf{S}(-(x+1)\log x + x - (x+1)\log(1+x^{-1}))\\
&= x\log x - x + \mathbf{S}(x - (x+1)\log(x+1)),
\end{aligned}$$

which produces the formulas for R_C and R'_C stated in the lemma for a non-trivial class. For the trivial class we have to add $n_{\mathbb{L}}$ times

$$1+\log x-f_2(x)=\frac{1}{2}\Big[\log(x^2-1)+x\log\Big(\frac{x+1}{x-1}\Big)\Big]$$

to R_C and $\frac{1}{2}\log\big(\frac{x+1}{x-1}\big)$ to its derivative. □

3.2. Bounds for the ramification term.

Lemma 3.6. *Let $x\geq 1$. Then*

$$\mathfrak{R}_C(x)\leq \min\Big(\frac{|C|}{p},1\Big)\mathfrak{n}\log x$$

where p is the smallest prime dividing $|G|$, and $\mathfrak{n}:=\sum_{\mathfrak{p}|\Delta_{\mathbb{L}/\mathbb{K}}}1$ is the number of prime ideals of $\mathbb{K}$ dividing $\Delta_{\mathbb{L}/\mathbb{K}}$.

Proof. From its definition (1.3) we have

$$\begin{aligned}\mathfrak{R}_C(x)&\leq \max_{\substack{\mathfrak{p}|\Delta_{\mathbb{L}/\mathbb{K}}\\ m\geq 1}}(\theta(C;\mathfrak{p}^m))\sum_{\mathfrak{p}|\Delta_{\mathbb{L}/\mathbb{K}}}\log \mathrm{N}\mathfrak{p}\sum_{\substack{m\geq 1\\ \mathrm{N}\mathfrak{p}^m\leq x}}1=\max_{\substack{\mathfrak{p}|\Delta_{\mathbb{L}/\mathbb{K}}\\ m\geq 1}}(\theta(C;\mathfrak{p}^m))\sum_{\mathfrak{p}|\Delta_{\mathbb{L}/\mathbb{K}}}\log \mathrm{N}\mathfrak{p}\Big\lfloor\frac{\log x}{\log \mathrm{N}\mathfrak{p}}\Big\rfloor\\ &\leq \max_{\substack{\mathfrak{p}|\Delta_{\mathbb{L}/\mathbb{K}}\\ m\geq 1}}(\theta(C;\mathfrak{p}^m))\mathfrak{n}\log x,\end{aligned}$$

and (1.1) immediately shows that $\theta(C;\mathfrak{p}^m)\leq\min(|C|/|I|,1)$. The proof concludes because the order of the inertia group is at least p for ramified primes. □

Lemma 3.7. *Let $\mathfrak{n}=\sum_{\mathfrak{p}|\Delta_{\mathbb{L}/\mathbb{K}}}1$ as in Lemma 3.6. We have the following bounds:*

i. If $\mathbb{L}\neq\mathbb{Q}[\sqrt{\pm 3}]$ and $\mathbb{L}\neq\mathbb{Q}[\sqrt{\pm 15}]$ then $\mathfrak{n}\leq\frac{\log\Delta_{\mathbb{L}}}{\log 4}$.
ii. If $n_{\mathbb{L}}=3$, the bound improves to $\mathfrak{n}\leq\frac{\log\Delta_{\mathbb{L}}}{\log 49}$.
iii. If $n_{\mathbb{L}}/n_{\mathbb{K}}$ is not prime, the bound improves to $\mathfrak{n}\leq\frac{\log\Delta_{\mathbb{L}}}{\log 22}$ except for the quartic fields of discriminant in $\{144,225,400,441,3600,7056,176400\}$ (twenty five fields in total).
iv. If $\log\Delta_{\mathbb{L}}>e^{1.1714}\,n_{\mathbb{K}}$, then

$$\mathfrak{n}\leq\frac{\log\Delta_{\mathbb{L}}}{\log\log\Delta_{\mathbb{L}}-\log n_{\mathbb{K}}-1.1714}.$$

The proof will make clear that Item iv is valid even when $\mathbb{L}/\mathbb{K}$ is not Galois. Moreover, the inequality $\log\Delta_{\mathbb{L}}>e^{1.1714}\,n_{\mathbb{K}}$ holds except for just a few fields when $\mathbb{L}\neq\mathbb{K}$. Precisely, the only exceptions for $n_{\mathbb{K}}=1$ are the fields $\mathbb{L}$ with $\Delta_{\mathbb{L}}\leq 25$ (i.e., the cubic field of discriminant -23 and seventeen quadratic fields), for $n_{\mathbb{K}}=2$ they are the twenty four quartic fields with $\Delta_{\mathbb{L}}\leq 634$, for $n_{\mathbb{K}}=3$ the four sextic fields with $\Delta_{\mathbb{L}}\leq 15986$. There are no exceptions with $n_{\mathbb{K}}\geq 4$.

Proof. We can assume $|G|\geq 2$ otherwise $\mathfrak{n}=0$.
ITEM *i.*
Suppose $\mathbb{K}\neq\mathbb{Q}$. We split the set of primes dividing $\Delta_{\mathbb{L}/\mathbb{K}}$ into three (possibly empty) sets: $\{\mathfrak{p}_i\}_{i=1}^a$, $\{\mathfrak{q}_j\}_{j=1}^b$ and $\{\mathfrak{s}_\ell\}_{\ell=1}^c$, which are the set of primes whose norm is 2, 3 and ≥ 4, respectively. Note that a, $b\leq n_{\mathbb{K}}$. Then

$$\Delta_{\mathbb{L}}=\Delta_{\mathbb{K}}^{[\mathbb{L}:\mathbb{K}]}\mathrm{N}(\Delta_{\mathbb{L}/\mathbb{K}})=\Delta_{\mathbb{K}}^{n_{\mathbb{L}}/n_{\mathbb{K}}}\mathrm{N}\Big(\prod_i\mathfrak{p}_i\prod_j\mathfrak{q}_j\prod_\ell\mathfrak{s}_\ell\Big)\geq\Delta_{\mathbb{K}}^2 2^a3^b4^c.$$

Moreover by Minkowski's bound we know that $\Delta_{\mathbb{K}}^{1/n_{\mathbb{K}}} \geq \sqrt{3}$, i.e. $\Delta_{\mathbb{K}}^2 \geq 3^{n_{\mathbb{K}}}$. Thus we get

$$\Delta_{\mathbb{L}} \geq 3^{n_{\mathbb{K}}}2^a3^b4^c = 2^{n_{\mathbb{K}}}\big(\tfrac{3}{2}\big)^{n_{\mathbb{K}}}2^a3^b4^c \geq 2^a\big(\tfrac{3}{2}\big)^b2^a3^b4^c = 4^a\big(\tfrac{9}{2}\big)^b4^c \geq 4^{a+b+c} = 4^{\mathfrak{n}}$$

as claimed.
Suppose $\mathbb{K} = \mathbb{Q}$. Then $\mathfrak{n} = \omega(\Delta_{\mathbb{L}})$. Let p_j, $j = 2, 3, \ldots$ be the sequence of primes. Note that if $\Delta_{\mathbb{L}} \in [\prod_{k\leq j} p_k, \prod_{k\leq j+1} p_k)$ then

$$\frac{\mathfrak{n}}{\log \Delta_{\mathbb{L}}} = \frac{\omega(\Delta_{\mathbb{L}})}{\log \Delta_{\mathbb{L}}} \leq \frac{j}{\log(\prod_{k\leq j} p_k)} = \frac{j}{\vartheta(p_j)}.$$

The sequence $\vartheta(p_j)/j$ is strictly increasing because it is the sequence of mean values of the increasing sequence $\log p_j$. Since $\frac{j}{\vartheta(p_j)} \leq 1/\log 4$ for $j = 4$, and since $\prod_{k\leq 4} p_k = 210$, the previous remark shows that $\mathfrak{n} \leq \log \Delta_{\mathbb{L}}/\log 4$ as soon as $\Delta_{\mathbb{L}} \geq 210$. Moreover, $\omega(\Delta_{\mathbb{L}}) \leq 3$ when $\Delta_{\mathbb{L}} \in [30, 210)$. Thus in this range $\mathfrak{n}/\log \Delta_{\mathbb{L}} \leq 3/\log \Delta_{\mathbb{L}}$ so that it is $\leq 1/\log 4$ as soon as $\Delta_{\mathbb{L}} \geq 4^3 = 64$. There are only 21 + 19 (resp. 4 + 1) quadratic (resp. cubic) fields with $\Delta_{\mathbb{L}} < 64$; for all of them the inequality $\mathfrak{n} \leq \log \Delta_{\mathbb{L}}/\log 4$ holds but for $\mathbb{Q}[\sqrt{\pm 3}]$ and for $\mathbb{Q}[\sqrt{\pm 15}]$.
Item *ii.*
Since $\mathbb{L}$ has to be a non-trivial Galois extension of $\mathbb{K}$, we must have $\mathbb{K} = \mathbb{Q}$ and G cyclic of order 3. We thus know that the discriminant of $\mathbb{L}$ (hence $\Delta_{\mathbb{L}}$) is the square of an integer. By [8] or [3, Th. 6.4.11, p. 341], the only primes that can divide $\Delta_{\mathbb{L}}$ are 3 and the primes congruent to 1 modulo 3 and, if $3 \mid \Delta_{\mathbb{L}}$ then $81 \mid \Delta_{\mathbb{L}}$. This proves that $\Delta_{\mathbb{L}} \geq 49^{\mathfrak{n}}$, as needed.

Item *iii.*
We prove that $\mathfrak{p}^2 \mid \Delta_{\mathbb{L}/\mathbb{K}}$ for each prime ideal $\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{K}}$ ramifying in $\mathbb{L}$. In fact, we are assuming that $|G|$ is not a prime, thus G has a proper subgroup and by Galois duality there is a proper intermediate field $\mathbb{F}$, so that $\mathbb{Q} \subseteq \mathbb{K} \subset \mathbb{F} \subset \mathbb{L}$. Thus

$$\Delta_{\mathbb{L}/\mathbb{K}} = \Delta_{\mathbb{F}/\mathbb{K}}^{[\mathbb{L}:\mathbb{F}]} \mathrm{N}_{\mathbb{F}/\mathbb{K}}(\Delta_{\mathbb{L}/\mathbb{F}})$$

Let $\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{K}}$ be a prime ideal ramifying in $\mathbb{L}$. If $\mathfrak{p}$ ramifies in $\mathbb{F}$, then $\mathfrak{p}^{[\mathbb{L}:\mathbb{F}]} \mid \Delta_{\mathbb{L}/\mathbb{K}}$, hence $\mathfrak{p}^2 \mid \Delta_{\mathbb{L}/\mathbb{K}}$.
Suppose now that $\mathfrak{p}$ does not ramify in $\mathbb{F}$. Let $\mathfrak{P} \subseteq \mathcal{O}_{\mathbb{L}}$ be a prime above $\mathfrak{p}$. As $\mathbb{L}/\mathbb{K}$ is Galois, it follows that $\mathfrak{q} := \mathfrak{P} \cap \mathbb{F}$ ramifies in $\mathbb{L}/\mathbb{F}$. Thus $\mathfrak{q} \mid \Delta_{\mathbb{L}/\mathbb{F}}$. This proves that $\prod_{\mathfrak{q}|\mathfrak{p}\mathcal{O}_{\mathbb{F}}} \mathfrak{q} \mid \Delta_{\mathbb{L}/\mathbb{F}}$. Hence $\mathfrak{p}\mathcal{O}_{\mathbb{F}} \mid \Delta_{\mathbb{L}/\mathbb{F}}$, because $\mathfrak{p}\mathcal{O}_{\mathbb{F}} = \prod_{\mathfrak{q}|\mathfrak{p}\mathcal{O}_{\mathbb{F}}} \mathfrak{q}$ (because $\mathfrak{p}$ does not ramify in $\mathbb{F}$, by hypothesis). Therefore $\mathfrak{p}^{[\mathbb{F}:\mathbb{K}]} = \mathrm{N}_{\mathbb{F}/\mathbb{K}}(\mathfrak{p}\mathcal{O}_{\mathbb{F}}) \mid \Delta_{\mathbb{L}/\mathbb{K}}$. In particular $\mathfrak{p}^2 \mid \Delta_{\mathbb{L}/\mathbb{K}}$ also in this case.

Suppose $\mathbb{K} = \mathbb{Q}$. The previous computation shows that there exist integers A and B such that $\Delta_{\mathbb{L}} = A^2B$ with B squarefree and $B \mid A$. As a consequence

$$\frac{\mathfrak{n}}{\log \Delta_{\mathbb{K}}} = \frac{\omega(A^2B)}{\log(A^2B)} \leq \frac{\omega(A)}{2\log A}$$

and if $A \in [\prod_{k\leq j} p_k, \prod_{k\leq j+1} p_k)$ then

$$\frac{\mathfrak{n}}{\log \Delta_{\mathbb{K}}} \leq \frac{j}{2\vartheta(p_j)}.$$

Since $\frac{j}{2\vartheta(p_j)} \leq 1/\log 22$ for $j = 5$, and since $\prod_{k\leq 5} p_k = 2310$, the previous remark shows that $\mathfrak{n} \leq \log \Delta_{\mathbb{L}}/\log 22$ as soon as $A \geq 2310$. Moreover, $\omega(A) \leq 4$ when $A < 2310$. Thus in this case $\mathfrak{n}/\log \Delta_{\mathbb{L}} \leq 4/\log \Delta_{\mathbb{L}}$ which is $\leq 1/\log 22$ as soon as $\Delta_{\mathbb{L}} \geq 22^4 = 234256$. Odlyzko's Table 3 shows that $\Delta_{\mathbb{L}} \leq 234256$ is possible only for degrees $n_{\mathbb{L}} \leq 7$, and, given our hypothesis, it remains to test only $n_{\mathbb{L}} = 4$ and $n_{\mathbb{L}} = 6$. All quartic and sextics fields with absolute discriminant up to 234256 appear in megrez table: exploring the table we found that there

are only twenty five quartic fields which are Galois extensions of $\mathbb{Q}$ and which do not satisfy the bound (they are the fields with discriminant in $\{144, 225, 400, 441, 3600, 7056, 176400\}$), and no sextic fields.
Suppose $\mathbb{K} \neq \mathbb{Q}$. We will prove that $\mathfrak{n} \leq \frac{\log \Delta_{\mathbb{L}}}{\log 24}$. For $n = 2$, 3, 4 let S_n be the set of prime ideals dividing $\Delta_{\mathbb{L}/\mathbb{K}}$ and whose norm is n and let S_5 be the set of prime ideals dividing $\Delta_{\mathbb{L}/\mathbb{K}}$ and whose norm is ≥ 5. For all $2 \leq n \leq 5$, let a_n be the cardinality of S_n. Then

$$\Delta_{\mathbb{L}} = \Delta_{\mathbb{K}}{}^{[\mathbb{L}:\mathbb{K}]}\mathrm{N}(\Delta_{\mathbb{L}/\mathbb{K}}) \geq \Delta_{\mathbb{K}}{}^{n_{\mathbb{L}}/n_{\mathbb{K}}}(\mathrm{N}(\prod_{n=2}^{5}\prod_{\mathfrak{p}\in S_n}\mathfrak{p}))^2 \geq \Delta_{\mathbb{K}}{}^{n_{\mathbb{L}}/n_{\mathbb{K}}}(2^{a_2}3^{a_3}4^{a_4}5^{a_5})^2.$$

Hence

$$\log \Delta_{\mathbb{L}} \geq \frac{n_{\mathbb{L}}}{n_{\mathbb{K}}}\log \Delta_{\mathbb{K}}+2\sum_{n=2}^{5} a_n \log n.$$

The number appearing on the right-hand side is larger than $(\log 24)\sum_n a_n$ as soon as

$$\frac{n_{\mathbb{L}}}{n_{\mathbb{K}}}\log \Delta_{\mathbb{K}} \geq \sum_{n=2}^{5} a_n \log(24/n^2). \tag{3.7}$$

Note that $a_3 \leq n_{\mathbb{K}}$ and that $a_2+2a_4 \leq n_{\mathbb{K}}$ (because these primes factorize $2\mathcal{O}_{\mathbb{K}}$). As $n_{\mathbb{L}}/n_{\mathbb{K}} \geq 4$, Inequality (3.7) holds for sure when

$$\log(\Delta_{\mathbb{K}}{}^{1/n_{\mathbb{K}}}) \geq \frac{1}{4}\log\Big(\frac{24^2}{2^2{\cdot}3^2}\Big) = \log 2,$$

i.e. $\Delta_{\mathbb{K}}{}^{1/n_{\mathbb{K}}} \geq 2$. The root discriminant of $\mathbb{K}$ satisfies this inequality for $n_{\mathbb{K}} \geq 3$, as one can see from line $b = 1$ in Odlyzko's Table 3. For $n_{\mathbb{K}} = 2$ this is true for $\Delta_{\mathbb{K}} \geq 4$, thus $\mathbb{K} = \mathbb{Q}[\sqrt{-3}]$ is the unique exception to this argument. However, in this case S_2 is empty and a_3, $a_4 \leq 1$, thus the claim is true anyway.

Item *iv.*
Set $p_0 := 1$ and let $A\colon [0, +\infty) \to \mathbb{R}$ be the function such that

$$\forall j \geq 0, \forall x \in [\vartheta(p_j), \vartheta(p_{j+1})), \quad A(x) := \frac{x-\vartheta(p_j)}{\log p_{j+1}}+j,$$

i.e., the continuous and piecewise affine map satisfying $A(\vartheta(p_j)) = j$ for every j. It is an increasing and concave map.
We also introduce on $(e^{1.1714}, +\infty)$ the function $R(x) := \frac{x}{\log x-1.1714}$. It is increasing for $x \geq x_R := e^{2.1714}$, convex for $x \leq ex_R$ and concave for $x \geq ex_R$.
Guy Robin [19] proved that $\omega(n) \leq R(\log n)$ for all $n \geq 26$. As a consequence,

$$\forall x > e^{1.1714}, \quad A(x) \leq R(x).$$

Indeed, $A(\vartheta(p_j)) = j = \omega(\prod_{k=1}^{j} p_k) \leq R(\vartheta(p_j))$ when $j \geq 4$ by Robin's result, and $A(ex_R) \leq R(ex_R)$, by explicit computation. Thus, $A(x) \leq R(x)$ for $x \geq ex_R$ because A is piecewise affine and R is concave in this range. On $(e^{1.1714}, ex_R)$ the inequality still holds because R is convex here and the tangent to its graph in ex_R stays above the graph of A.
Let $j_0 := \lfloor \mathfrak{n}/n_{\mathbb{K}} \rfloor$ and $x_0 := \vartheta(p_{j_0})+(\mathfrak{n}/n_{\mathbb{K}}-j_0)\log p_{j_0+1}$, so that $\mathfrak{n} = A(x_0)n_{\mathbb{K}}$.
Let $\mathfrak{p}_j$, $j = 1, \ldots, \mathfrak{n}$ be the primes ramifying in $\mathbb{L}/\mathbb{K}$. For each j let p_{k_j} be the prime integer below $\mathfrak{p}_j$ and f_j be such that $\mathrm{N}(\mathfrak{p}_j) = p_{k_j}^{f_j}$. We suppose that the ideals are ordered such that the sequence p_{k_j} is non-decreasing. We have

$$\mathrm{N}(\Delta_{\mathbb{L}/\mathbb{K}}) = \prod_{j=1}^{\mathfrak{n}} \mathrm{N}\mathfrak{p}_j = \prod_{j=1}^{\mathfrak{n}} p_{k_j}^{f_j} \geq \prod_{j=1}^{\mathfrak{n}} p_{k_j}.$$

For a given p_k, there are at most $n_{\mathbb{K}}$ values of j such that $p_k = p_{k_j}$, thus we get

$$\Delta_{\mathbb{L}} \geq \mathrm{N}(\Delta_{\mathbb{L}/\mathbb{K}}) \geq \Big(\prod_{k=1}^{j_0} p_k\Big)^{n_{\mathbb{K}}} p_{j_0+1}^{\mathfrak{n}-j_0 n_{\mathbb{K}}},$$

so that $\log \Delta_{\mathbb{L}} \geq x_0 n_{\mathbb{K}}$. Hence

$$\frac{\mathfrak{n}}{n_{\mathbb{K}}} = A(x_0) \leq A\Big(\frac{\log \Delta_{\mathbb{L}}}{n_{\mathbb{K}}}\Big) \leq R\Big(\frac{\log \Delta_{\mathbb{L}}}{n_{\mathbb{K}}}\Big) = \frac{1}{n_{\mathbb{K}}} \frac{\log \Delta_{\mathbb{L}}}{\log(\Delta_{\mathbb{L}}/n_{\mathbb{K}}) - 1.1714}$$

when $\log \Delta_{\mathbb{L}} > e^{1.1714} n_{\mathbb{K}}$. □

Lemma 3.8. *For every integer n, let $\tilde{\Lambda}_{\mathbb{L}}(n) := \sum_{\mathrm{N}\mathfrak{I}=n} \Lambda_{\mathbb{L}}(\mathfrak{I})$. Then for any $\ell \geq 1$ and any prime p we have*

$$\sum_{r=1}^{n_{\mathbb{L}}} \tilde{\Lambda}_{\mathbb{L}}(p^{\ell n_{\mathbb{L}}+r}) \geq n_{\mathbb{L}} \log p.$$

Proof. From the definition of $\tilde{\Lambda}_{\mathbb{L}}$, we have

$$\sum_{r=1}^{n_{\mathbb{L}}} \tilde{\Lambda}_{\mathbb{L}}(p^{\ell n_{\mathbb{L}}+r}) = \sum_{r=1}^{n_{\mathbb{L}}} \sum_{\mathfrak{p}|p} \sum_{\substack{m:\\ \mathrm{N}\mathfrak{p}^m = p^{\ell n_{\mathbb{L}}+r}}} \log(\mathrm{N}\mathfrak{p}) = \sum_{\mathfrak{p}|p} f_{\mathfrak{p}} \Big(\sum_{r=1}^{n_{\mathbb{L}}} \sum_{\substack{m:\\ m f_{\mathfrak{p}} = \ell n_{\mathbb{L}}+r}} 1 \Big) \log p,$$

where $f_{\mathfrak{p}}$ is the inertia degree of $\mathfrak{p}$ in the extension $\mathbb{Q} \subseteq \mathbb{L}$. To conclude, it is sufficient to prove that

$$\sum_{r=1}^{n_{\mathbb{L}}} \sum_{\substack{m:\\ m f_{\mathfrak{p}} = \ell n_{\mathbb{L}}+r}} 1 \geq e_{\mathfrak{p}},$$

where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$, because $\sum_{\mathfrak{p}|p} f_{\mathfrak{p}} e_{\mathfrak{p}} = n_{\mathbb{L}}$. To prove this inequality, we pick $r \in \{1, ..., f_{\mathfrak{p}}\}$ such that $\ell n_{\mathbb{L}} + r = 0 \pmod{f_{\mathfrak{p}}}$. We then set $m = (\ell n_{\mathbb{L}} + r)/f_{\mathfrak{p}}$, and this contributes by 1 to the inner sum on m. We repeat this procedure in the first $e_{\mathfrak{p}}$ blocks of length $f_{\mathfrak{p}}$: the claim follows since $e_{\mathfrak{p}} f_{\mathfrak{p}} \leq n_{\mathbb{L}}$. □

3.3. **Bounds for sums on zeros of Dedekind Zeta functions.**

Lemma 3.9. *Assume GRH. Then we have*

$$\sum_{|\gamma| \leq 2\pi} \frac{1}{|\rho|} + \sum_{|\gamma| > 2\pi} \frac{|1/2 + 2\pi i|}{|\rho|^2} \leq 1.348 \log \Delta_{\mathbb{L}} - 1.557 n_{\mathbb{L}} + 7.786 - 0.406 r_1 - e_{n_{\mathbb{L}}}, \tag{3.8}$$

where the sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta_{\mathbb{L}}$. Here $e_{n_{\mathbb{L}}}$ is positive, with $e_1 \geq 5.529$, $e_2 \geq 0.751$ and $e_3 \geq 0.313$.

Proof. We prove this lemma with the same method of [7, Lemma 3.1]. Thus, let

$$g(\gamma) := \begin{cases} \frac{2}{(1+4\gamma^2)^{1/2}} & \text{if } |\gamma| \leq 2\pi \\ \frac{2|1+4\pi i|}{1+4\gamma^2} & \text{otherwise} \end{cases}$$

so that

$$\sum_{|\gamma| \leq 2\pi} \frac{1}{|\rho|} + \sum_{|\gamma| > 2\pi} \frac{|1/2 + 2\pi i|}{|\rho|^2} = \sum_{\gamma} g(\gamma).$$

We observe that g is continuous in $\mathbb{R}$. Moreover, let $f(s,\gamma) := 4(2s-1)/((2s-1)^2+4\gamma^2)$ and $f_{\mathbb{L}}(s) := \sum_\gamma f(s,\gamma)$. We look for a finite linear combination of $f(s,\gamma)$ at suitable points s_j such that

$$g(\gamma) \le F(\gamma) := \sum_j a_j f(s_j,\gamma) \qquad \forall \gamma\in\mathbb{R}, \tag{3.9}$$

so that

$$\sum_{|\gamma|\le 2\pi} \frac{1}{|\rho|} + \sum_{|\gamma|>2\pi} \frac{|1/2+2\pi i|}{|\rho|^2} \le \sum_j a_j f_{\mathbb{L}}(s_j). \tag{3.10}$$

Once (3.10) is proved, we recover a bound for the sum on zeros recalling the identity

$$f_{\mathbb{L}}(s) = 2\,\mathrm{Re}\,\frac{\zeta_{\mathbb{L}}'}{\zeta_{\mathbb{L}}}(s) + \log\frac{\Delta_{\mathbb{L}}}{\pi^{n_{\mathbb{L}}}} + \mathrm{Re}\Big(\frac{2}{s}+\frac{2}{s-1}\Big) + (r_1+r_2)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) + r_2\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{s+1}{2}\Big). \tag{3.11}$$

To determine a convenient set of constants a_j's we set $s_j = 1+j/2$ with $j=1,\dots,23$,

$$\Upsilon := \{0.62, 1, 1.6, 2.1, 2.8, 3.8, 4.6, 5.8, 7.5, 9.3, 12.9, 14, 16, 17, 18, 19, 20, 30, 40, 50, 10^2, 10^3, 10^4\},$$

and we require:

(1) $F(\gamma) = g(\gamma)$ for all $\gamma\in\Upsilon\cup\{0,2\pi\}$,
(2) $F'(\gamma) = g'(\gamma)$ for all $\gamma\in\Upsilon$,
(3) $\lim_{\gamma\to\infty}\gamma^2 F(\gamma) = \lim_{\gamma\to\infty}\gamma^2 g(\gamma) = |1/2+2\pi i|$.

This produces a set of 49 linear equations for the 49 constants a_j's ensuring (3.9), at least for $\gamma\in\Upsilon$. With an abuse of notation we take for a_j's the solution of the system, rounded above to 10^{-7}: this produces the numbers in Table 2. Then, using Sturm's algorithm, we prove that the values found actually give an upper bound for g, so that (3.9) holds with such a_j's. These constants verify

$$\begin{aligned} &\sum_{j=1}^{49} a_j = 1.3479\dots, && \sum_{j=1}^{49} a_j\Big(\frac{2}{s_j}+\frac{2}{s_j-1}\Big) \le 7.786,\\ &\sum_{j=1}^{49} a_j\frac{\Gamma'}{\Gamma}\Big(\frac{s_j}{2}\Big) \le -0.421, && \sum_{j=1}^{49} a_j\frac{\Gamma'}{\Gamma}\Big(\frac{s_j+1}{2}\Big) \le 0.392. \end{aligned} \tag{3.12}$$

This suffices to manage all terms in (3.10) coming from all terms in (3.11) but the first one. However, we observe that $a_1>0$, $a_2>0$ and the signs of the a_j's alternate for $2\le j\le 49$. We write $\sum_{j=1}^{49} a_j\frac{\zeta_{\mathbb{L}}'}{\zeta_{\mathbb{L}}}(s_j)$ as

$$-\sum_n \tilde\Lambda_{\mathbb{L}}(n)S(n) \quad\text{with}\quad S(n) := \sum_{j=1}^{49}\frac{a_j}{n^{s_j}}.$$

We isolate the first three terms in $S(n)$, and group the other ones by consecutive pairs

$$S(n) = \Big(\frac{a_1}{n^{3/2}}+\frac{a_2}{n^2}+\frac{a_3}{n^{5/2}}\Big)+\Big(\frac{a_4}{n^3}+\frac{a_5}{n^{7/2}}\Big)+\Big(\frac{a_6}{n^4}+\frac{a_7}{n^{9/2}}\Big)+\dots+\Big(\frac{a_{48}}{n^{25}}+\frac{a_{49}}{n^{51/2}}\Big).$$

It is easy to verify that each group decreases for $n\ge 85597$, and that hence the same holds for $S(n)$. A direct computation shows that $S(n+1)<S(n)$ holds also for $n\le 85597$. Thus S is a decreasing sequence. Since $a_1>0$ we know that $S(n)>0$ definitively and hence always. Thus, we can deduce that $-e_{n_{\mathbb{L}}} := 2\sum_{j=1}^{49} a_j\frac{\zeta_{\mathbb{L}}'}{\zeta_{\mathbb{L}}}(s_j) = -2\sum_{n\ge1}\tilde\Lambda_{\mathbb{L}}(n)S(n)\le 0$ which suffices to prove the claim for a generic $n_{\mathbb{L}}$, via (3.10–3.12).

With the help of Lemma 3.8 we can produce a better upper bound for $-e_{n_{\mathbb{L}}}$, at least when $n_{\mathbb{L}}$ is small. In fact S is decreasing, so that

$$\begin{aligned}\sum_{j=1}^{49} a_j \frac{\zeta'_{\mathbb{L}}}{\zeta_{\mathbb{L}}}(s_j) &= -\sum_p \sum_m \tilde{\Lambda}_{\mathbb{L}}(p^m) S(p^m) = -\sum_p \sum_{\ell=0}^{\infty} \sum_{r=1}^{n_{\mathbb{L}}} \tilde{\Lambda}_{\mathbb{L}}(p^{\ell n_{\mathbb{L}}+r}) S(p^{\ell n_{\mathbb{L}}+r}) \\ &\leq -\sum_p \sum_{\ell=0}^{\infty} \sum_{r=1}^{n_{\mathbb{L}}} \tilde{\Lambda}_{\mathbb{L}}(p^{\ell n_{\mathbb{L}}+r}) S(p^{(\ell+1) n_{\mathbb{L}}}).\end{aligned}$$

From Lemma 3.8 and since $S \geq 0$, this is

$$\leq -n_{\mathbb{L}} \sum_p \sum_{\ell=1}^{\infty} (\log p) S(p^{\ell n_{\mathbb{L}}}) = -n_{\mathbb{L}} \sum_p \sum_{\ell=1}^{\infty} \Lambda(p^\ell) S(p^{\ell n_{\mathbb{L}}}) = n_{\mathbb{L}} \sum_{j=1}^{49} a_j \frac{\zeta'}{\zeta}(s_j n_{\mathbb{L}}).$$

Hence

$$-e_{n_{\mathbb{L}}} = 2 \sum_{j=1}^{49} a_j \frac{\zeta'_{\mathbb{L}}}{\zeta_{\mathbb{L}}}(s_j) \leq 2 n_{\mathbb{L}} \sum_{j=1}^{49} a_j \frac{\zeta'}{\zeta}(s_j n_{\mathbb{L}})$$

whose value for $n_{\mathbb{L}} = 1$ is lower than -5.529, for $n_{\mathbb{L}} = 2$ is lower than -0.751 and for $n_{\mathbb{L}} = 3$ is lower than -0.313 (the gain unfortunately decreases quickly: it is -0.149 for $n_{\mathbb{L}} = 4$ and only -0.074 for $n_{\mathbb{L}} = 5$). □

Lemma 3.10. *Assume GRH. Then one has*

$$\sum_\rho \frac{1}{|\rho(\rho+1)|} \leq 0.5375 \log \Delta_{\mathbb{L}} - 1.0355 n_{\mathbb{L}} + 5.3879 - 0.2635 r_1,$$

where the sums run over the non-trivial zeros ρ of $\zeta_{\mathbb{L}}$.

Proof. This claim is [6, Lemma 4.1], but now we repeat the computations keeping the extra term which is proportional to r_1. Since

$$\sum_j a_j = 0.53747\ldots, \qquad \sum_j a_j \Big(\frac{2}{s_j} + \frac{2}{s_j - 1}\Big) \leq 5.3879,$$

$$\sum_j a_j \frac{\Gamma'}{\Gamma}\Big(\frac{s_j}{2}\Big) \leq -0.6838, \qquad \sum_j a_j \frac{\Gamma'}{\Gamma}\Big(\frac{s_j+1}{2}\Big) \leq -0.1567,$$

the claim follows. □

We rewrite Theorem A.1 for $\mathbb{E} = \mathbb{L}$ and trivial character as

$$\Big| N_{\mathbb{L}}(T) - \frac{T}{\pi} \log\Big(\Big(\frac{T}{2\pi e}\Big)^{n_{\mathbb{L}}} \Delta_{\mathbb{L}}\Big) - 2 + \frac{1}{4} r_1 \Big| \leq c_1 W_{\mathbb{L}}(T) + c_2 n_{\mathbb{L}} + c_3 \tag{3.13}$$

for every $T \geq T_0 \geq 1$, where $W_{\mathbb{L}}(T) := \log \Delta_{\mathbb{L}} + n_{\mathbb{L}} \log(T/2\pi)$, $c_1 = D_1$, $c_2 = D'_2 + D_1 \log 2\pi$ and $c_3 = D'_3$. With $T_0 = 2\pi$, the last line of Table 1 provides (3.13) with the constants

$$c_1 = 0.460, \quad c_2 = 2.491, \quad c_3 = 0.593.$$

Other and smaller values for c_1 are available in Table 1, but we need also a small value for c_2 and c_3: this choice is adequate to our purpose. This proves

Lemma 3.11. *For all $T \geq 2\pi$ one has*

$$\sum_{|\gamma| \leq T} 1 = N_{\mathbb{L}}(T) \leq T\Big(\frac{1}{\pi} + \frac{0.460}{T}\Big) W_{\mathbb{L}}(T) - T\Big(\frac{1}{\pi} - \frac{2.491}{T}\Big) n_{\mathbb{L}} + 2.593 - \frac{r_1}{4}. \tag{3.14}$$

As in [7, Second sum], one has

Lemma 3.12. *For all $T \geq 2\pi$ one has*

$$\sum_{|\gamma|\geq T} \frac{1}{|\rho|^2} \leq \Big(\frac{1}{\pi}+\frac{0.920}{T}\Big)\frac{W_{\mathbb{L}}(T)}{T}+\Big(\frac{1}{\pi}+\frac{5.220}{T}\Big)\frac{n_{\mathbb{L}}}{T}+\frac{1.186}{T^2}. \tag{3.15}$$

Proof. The proof remains the same in spite of the difference between the structure of (3.13) and Trudgian's formula we used in [7] for this purpose, because the term $-1+r_1/4$ disappears in integrations. This provides the upper bound

$$\sum_{|\gamma|\geq T} \frac{1}{|\rho|^2} \leq \Big(\frac{1}{\pi}+\frac{2c_1}{T}\Big)\frac{W_{\mathbb{L}}(T)}{T}+\Big(\frac{1}{\pi}+\frac{\log 2\pi}{12T^2}\Big)\frac{n_{\mathbb{L}}}{T}+\Big(2c_2+\frac{c_1}{2}\Big)\frac{n_{\mathbb{L}}}{T^2}+\frac{2c_3}{T^2},$$

and the claim follows from the selected values of c_j's. $\square$

Note that the formula improves upon the one in [7] because now c_1, c_2 and c_3 are smaller.

Lemma 3.13. *For all $T \geq 2\pi$ one has*

$$\begin{aligned}\sum_{|\gamma|\leq T}\frac{1}{|\rho|}+\sum_{|\gamma|>T}\frac{|1+2\pi i|}{|\rho|^2} \leq &\Big(\frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big)+1.067+\frac{2}{T}\Big)\log\Delta_{\mathbb{L}}\\ &+\Big(\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)+\frac{2}{T}\log\Big(\frac{eT}{2\pi}\Big)-1.633-\frac{0.460}{T}+\frac{1.446}{T^2}\Big)n_{\mathbb{L}}\\ &+7.834-0.406r_1-e_{n_{\mathbb{L}}}.\end{aligned} \tag{3.16}$$

Proof. Let (3.13) be written as $|N_{\mathbb{L}}(T)-A(T)| \leq R(T)$, with $A(T)$ representing the main term and $R(T)$ the bound for the remainder term. To ease notations, we set $\ell := |1/2+2\pi i|$. We write

$$\begin{aligned}\sum_{|\gamma|\leq T}\frac{1}{|\rho|}+\sum_{|\gamma|>T}\frac{\ell}{|\rho|^2} &= \sum_{|\gamma|\leq 2\pi}\frac{1}{|\rho|}+\sum_{|\gamma|>2\pi}\frac{\ell}{|\rho|^2}+\sum_{2\pi<|\gamma|\leq T}\Big(\frac{1}{|\rho|}-\frac{\ell}{|\rho|^2}\Big)\\ &\leq \sum_{|\gamma|\leq 2\pi}\frac{1}{|\rho|}+\sum_{|\gamma|>2\pi}\frac{\ell}{|\rho|^2}+\sum_{2\pi<|\gamma|\leq T}\Big(\frac{1}{|\gamma|}-\frac{2\pi}{\gamma^2}\Big),\end{aligned}$$

where the last step follows by the general inequality $\frac{1}{|1/2+i\gamma|}-\frac{\ell}{|1/2+i\gamma|^2} \leq \frac{1}{|\gamma|}-\frac{2\pi}{\gamma^2}$. By partial summation we get

$$\sum_{2\pi<|\gamma|\leq T}\Big(\frac{1}{|\gamma|}-\frac{2\pi}{\gamma^2}\Big) \leq \int_{2\pi}^{T}\Big(\frac{1}{\gamma}-\frac{2\pi}{\gamma^2}\Big)\,\mathrm{d}A(\gamma)+\frac{R(4\pi)}{4\pi}-\int_{2\pi}^{4\pi}\Big(\frac{1}{\gamma}-\frac{2\pi}{\gamma^2}\Big)R'(\gamma)\,\mathrm{d}\gamma+\int_{4\pi}^{T}\Big(\frac{1}{\gamma}-\frac{2\pi}{\gamma^2}\Big)R'(\gamma)\,\mathrm{d}\gamma$$

because $\frac{1}{\gamma}-\frac{2\pi}{\gamma^2}$ has a maximum at 4π. Since $R'(\gamma) = c_1 n_{\mathbb{L}}/\gamma$ this produces the bound

$$\sum_{2\pi<|\gamma|\leq T}\Big(\frac{1}{|\gamma|}-\frac{2\pi}{\gamma^2}\Big) \leq \int_{2\pi}^{T}\Big(\frac{1}{\gamma}-\frac{2\pi}{\gamma^2}\Big)\,\mathrm{d}A(\gamma)+\frac{R(4\pi)}{4\pi}+c_1\Big(\frac{1}{8\pi}-\frac{1}{T}+\frac{\pi}{T^2}\Big)n_{\mathbb{L}}.$$

The claim follows from this bound, the equality

$$\int_{2\pi}^{T}\Big(\frac{1}{\gamma}-\frac{2\pi}{\gamma^2}\Big)\,\mathrm{d}A(\gamma) = \Big(\frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big)-\frac{1}{\pi}+\frac{2}{T}\Big)\log\Delta_{\mathbb{L}}+\Big(\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)+\frac{2}{T}\log\Big(\frac{eT}{2\pi}\Big)-\frac{1}{\pi}\Big)n_{\mathbb{L}},$$

the result in (3.8) and the chosen values for the c_j's constants. $\square$

4. A parametric result

Theorem 4.1. *(GRH) For every $x \geq 4$ and $T \geq 2\pi$ we have:*

$$\frac{|G|}{|C|}\psi(C;x)-x \leq L_a(x,T,n_{\mathbb{L}},\log\Delta_{\mathbb{K}}), \tag{4.1}$$

$$-\Big(\frac{|G|}{|C|}\psi_C(x)-x\Big) \leq L_a(x,T,n_{\mathbb{L}},\log\Delta_{\mathbb{K}})+D(x,T,n_{\mathbb{L}},\log\Delta_{\mathbb{K}})+\frac{|G|}{|C|}\mathfrak{R}_C(x), \tag{4.2}$$

with

$$\begin{aligned}
L_a(x,T,n,\mathcal{L}) &:= F(x,T)\mathcal{L}+G(x,T)n+H(x,T,n),\\
F(x,T) &:= \sqrt{x}\Big[\frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big)+1.704+\frac{1.858}{T}\Big]+1.075,\\
G(x,T) &:= \sqrt{x}\Big[\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(\frac{2}{\pi}+\frac{1.858}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.633+\frac{7.729}{T}\Big]-1.501,\\
H(x,T,n) &:= H_1(x,T)+H_2(x,T,n),\\
H_1(x,T) &:= \frac{x+2}{T}+\sqrt{x}\Big(7.834+\frac{3.779}{T}\Big)+9.276,\\
H_2(x,T,n) &:= -\sqrt{x}\Big(\Big(0.406+\frac{1}{4T}\Big)r_1+e_n\Big)+(1-\mathbf{S})\log x+\mathbf{S}-0.744n\delta_C-0.527r_1,\\
D(x,T,n,\mathcal{L}) &:= 2(\mathbf{S}-1)(\log x-1)+1-0.445n+2n\delta_C\\
&\qquad -\frac{\sqrt{x}}{T}\big(1.167+0.743\tfrac{\mathcal{L}}{n}+0.743\log\big(\tfrac{T}{2\pi}\big)\big)n.
\end{aligned}$$

Proof. Following (2.4), we consider for a character χ of $\mathrm{Gal}(\mathbb{L}/\mathbb{E})$ the integral

$$I_\chi(x) := -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s,\chi)\frac{x^{s+1}}{s(s+1)}\,\mathrm{d}s.$$

Shifting the axis of integration arbitrarily far to the left, one gets for every $x > 1$ the identity

$$I_\chi(x) = \delta_\chi\frac{x^2}{2}-\sum_{\rho\in Z_\chi}\frac{x^{\rho+1}}{\rho(\rho+1)}-xr_\chi+r'_\chi+R_\chi(x)$$

where $R_\chi(x)$ is defined in Lemma 3.5 and r_χ and r'_χ are defined in (3.2). The shift is done in a way similar to [12, § 6], further simplified by the fact that the integral is absolutely convergent on vertical lines. By (2.4), Lemma 3.2 and using R_C as defined in Lemma 3.5, this gives

$$\frac{|G|}{|C|}\psi^{(1)}(C;x) = \mathcal{M}_C I_\chi(x) = \frac{x^2}{2}-\sum_{\rho\in Z}\epsilon(\rho)\frac{x^{\rho+1}}{\rho(\rho+1)}-x\mathcal{M}_C r_\chi+\mathcal{M}_C r'_\chi+R_C(x) \tag{4.3}$$

so that for any $h\neq 0$, one has

$$\frac{|G|}{|C|}\frac{\psi^{(1)}(C;x+h)-\psi^{(1)}(C;x)}{h} = x+\frac{h}{2}-\sum_{\rho\in Z}\epsilon(\rho)\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}-\mathcal{M}_C r_\chi+R'_C(\eta)$$

for a suitable η in the interval between x and $x+h$. By (2.1) we deduce for $h > 0$:

$$\frac{|G|}{|C|}\psi(C;x)-x \leq \frac{h}{2}+\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|-\mathcal{M}_C r_\chi+R'_C(\eta) \tag{4.4}$$

and for $h<0$

$$-\Big[\frac{|G|}{|C|}\psi(C;x)-x\Big] \le -\frac{h}{2}+\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|+\mathcal{M}_C r_\chi - R'_C(\eta). \tag{4.5}$$

To get an upper bound for the sum of zeros we split its contribution into two parts: above and below T. Moreover, in the lower range we isolate the contribution of $\sum_{|\gamma|\le T} x^\rho/\rho$, which will produce the main term. Thus,

$$\begin{aligned}\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big| &\le \sum_{|\gamma|\le T}\Big|\frac{x^\rho}{\rho}\Big|+\sum_{|\gamma|\le T}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}-h(\rho+1)x^\rho}{h\rho(\rho+1)}\Big|+\sum_{|\gamma|>T}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big| \\ &\le \sum_{|\gamma|\le T}\frac{\sqrt{x}}{|\rho|}+\frac{|h|}{\sqrt{x}}\sum_{|\gamma|\le T}|w_\rho|+\frac{x^{3/2}}{|h|}\Big(\Big(1+\frac{h}{x}\Big)^{3/2}+1\Big)\sum_{|\gamma|>T}\frac{1}{|\rho|^2} \end{aligned} \tag{4.6}$$

with

$$w_\rho := \frac{\big(1+\frac{h}{x}\big)^{\rho+1}-1-(\rho+1)\frac{h}{x}}{\rho(\rho+1)\big(\frac{h}{x}\big)^2}.$$

The technique we apply to bound (4.4) and (4.5) changes in some details. We thus proceed separately for the two cases.
To prove (4.1) we bound the right hand side of (4.4). Let $h>0$, then $|w_\rho|\le\frac{1}{2}$ from [7, Lemma 2.1], and (4.6) gives

$$\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big| \le \sqrt{x}\sum_{|\gamma|\le T}\frac{1}{|\rho|}+\frac{h}{2\sqrt{x}}N_{\mathbb{L}}(T)+\frac{x^{3/2}}{h}\Big(\Big(1+\frac{h}{x}\Big)^{3/2}+1\Big)\sum_{|\gamma|>T}\frac{1}{|\rho|^2}.$$

By (3.14) we know that $N_{\mathbb{L}}(T)$ has order $TW_{\mathbb{L}}(T)$, by (3.15) that $\sum_{|\gamma|>T}\frac{1}{|\rho|^2}$ has order $W_{\mathbb{L}}(T)/T$, and by (3.16) that $\sum_{|\gamma|\le T}\frac{1}{|\rho|}$ has order $(\log T)W_{\mathbb{L}}(T)$. The comparison of the second and the last term, hence, suggests to take $h\approx x/T$. We set $h=2x/T$. In this way we get:

$$\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big| \le \sqrt{x}\sum_{|\gamma|\le T}\frac{1}{|\rho|}+\frac{\sqrt{x}}{T}N_{\mathbb{L}}(T)+\frac{T\sqrt{x}}{2}\Big(\Big(1+\frac{2}{T}\Big)^{3/2}+1\Big)\sum_{|\gamma|>T}\frac{1}{|\rho|^2}.$$

Since $(1+\frac{2}{T})^{3/2}+1\le 2+\frac{3}{T}+\frac{3}{2T^2}$ we conclude

$$\frac{1}{\sqrt{x}}\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big| \le \Big(\sum_{|\gamma|\le T}\frac{1}{|\rho|}+\sum_{|\gamma|>T}\frac{2\pi}{|\rho|^2}\Big)+\frac{N_{\mathbb{L}}(T)}{T}+\Big(1+\frac{3}{2T}+\frac{3}{4T^2}-\frac{2\pi}{T}\Big)\sum_{|\gamma|>T}\frac{T}{|\rho|^2}.$$

Substituting (3.14), (3.15) and (3.16) in this equation, after some rearrangements we get:

$$\begin{aligned}\frac{1}{\sqrt{x}}\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big| &\le \Big[\frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big)+1.704+\frac{1.858}{T}\Big]\log\Delta_{\mathbb{L}} \\ &\quad+\Big[\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(\frac{2}{\pi}+\frac{1.858}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.633+\frac{7.729}{T}\Big]n_{\mathbb{L}} \\ &\quad+7.834+\frac{3.779}{T}-\Big(0.406+\frac{1}{4T}\Big)r_1-e_{n_{\mathbb{L}}}. \end{aligned}\tag{4.7}$$

The explicit formula for R'_C in Lemma 3.5 gives

$$R'_C(\eta)\le\log\eta-\mathbf{S}\log(\eta+1)+0.256n_{\mathbb{L}}\delta_C$$

under the assumption that $x\geq 4$. Using that and Lemma 3.3,

$$-\mathcal{M}_C r_\chi + R'_C(\eta) \leq \sum_{\rho\in Z}\frac{2}{|\rho(2-\rho)|} - \frac{\zeta'(2)}{\zeta(2)}n_{\mathbb{L}} - n_{\mathbb{L}}\delta_C + \mathbf{S} - \frac{3}{2} + (1-\mathbf{S})\log x + 0.256 n_{\mathbb{L}}\delta_C + \frac{2}{T}. \tag{4.8}$$

Following (4.4), we sum (4.7) and (4.8), to get:

$$\begin{aligned}\frac{|G|}{|C|}\psi(C;x)-x \leq{}& \frac{x}{T}+\sqrt{x}\Big[\frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big)+1.704+\frac{1.858}{T}\Big]\log\Delta_{\mathbb{L}}\\ &+\sqrt{x}\Big[\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(\frac{2}{\pi}+\frac{1.858}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.633+\frac{7.729}{T}\Big]n_{\mathbb{L}}\\ &+\sqrt{x}\Big[7.834+\frac{3.779}{T}-\Big(0.406+\frac{1}{4T}\Big)r_1-e_{n_{\mathbb{L}}}\Big]\\ &+\sum_{\rho\in Z}\frac{2}{|\rho(2-\rho)|}-\frac{\zeta'(2)}{\zeta(2)}n_{\mathbb{L}}-n_{\mathbb{L}}\delta_C+\mathbf{S}-\frac{3}{2}+(1-\mathbf{S})\log x+0.256n_{\mathbb{L}}\delta_C+\frac{2}{T}.\end{aligned} \tag{4.9}$$

Moreover, $|2-\rho|=|\rho+1|$ since we are assuming GRH. Thus, by Lemma 3.10

$$\sum_{\rho\in Z}\frac{2}{|\rho(2-\rho)|}\leq 1.075\log\Delta_{\mathbb{L}}-2.071n_{\mathbb{L}}+10.776-0.527r_1.$$

The upper bound in (4.9) thus gives

$$\begin{aligned}\frac{|G|}{|C|}\psi(C;x)-x \leq{}& \sqrt{x}\Big[\frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big)+1.704+\frac{1.858}{T}\Big]\log\Delta_{\mathbb{L}}\\ &+\sqrt{x}\Big[\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(\frac{2}{\pi}+\frac{1.858}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.633+\frac{7.729}{T}\Big]n_{\mathbb{L}}\\ &+\sqrt{x}\Big[7.834+\frac{3.779}{T}-\Big(0.406+\frac{1}{4T}\Big)r_1-e_{n_{\mathbb{L}}}\Big]+1.075\log\Delta_{\mathbb{L}}-2.071n_{\mathbb{L}}+10.776-0.527r_1\\ &+0.570n_{\mathbb{L}}-0.744n_{\mathbb{L}}\delta_C+\mathbf{S}-\frac{3}{2}+(1-\mathbf{S})\log x+\frac{x+2}{T}.\end{aligned}$$

This is the bound in (4.1), once the definition of L_a is considered.
To prove (4.2) we first bound the right hand side of (4.5). In this case $h<0$, thus $|w_\rho|\leq\frac{1}{2}+\frac{|h|}{6x}$ from [7, Lemma 2.1], so that (4.6) gives

$$\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|\leq\sqrt{x}\sum_{|\gamma|\leq T}\frac{1}{|\rho|}+\frac{|h|}{\sqrt{x}}\Big(\frac{1}{2}+\frac{|h|}{6x}\Big)N_{\mathbb{L}}(T)+\frac{x^{3/2}}{|h|}\Big(\Big(1+\frac{h}{x}\Big)^{3/2}+1\Big)\sum_{|\gamma|>T}\frac{1}{|\rho|^2}.$$

Setting $h=-\frac{2x}{T}$, and estimating $(1+\frac{h}{x})^{3/2}+1=(1-\frac{2}{T})^{3/2}+1\leq 2-\frac{3}{T}+\frac{20}{T^2}$ (valid as soon as $T\geq 2$), we get

$$\begin{aligned}\frac{1}{\sqrt{x}}\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|\leq{}&\Big(\sum_{|\gamma|\leq T}\frac{1}{|\rho|}+\sum_{|\gamma|>T}\frac{2\pi}{|\rho|^2}\Big)+\Big(1+\frac{2}{3T}\Big)\frac{N_{\mathbb{L}}(T)}{T}\\ &+\Big(1-\frac{3}{2T}+\frac{10}{T^2}-\frac{2\pi}{T}\Big)\sum_{|\gamma|>T}\frac{T}{|\rho|^2},\end{aligned}$$

which with (3.14), (3.15) (which can be used because $1-\frac{3}{2T}+\frac{10}{T^2}-\frac{2\pi}{T}$ is positive for $T\geq 2\pi$) and (3.16) produces

$$\frac{1}{\sqrt{x}}\sum_{\rho\in Z}\Big|\frac{(x+h)^{\rho+1}-x^{\rho+1}}{h\rho(\rho+1)}\Big|\leq\Big[\frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big)+1.704+\frac{1.115}{T}\Big]\log\Delta_{\mathbb{L}}$$

$$+\Big[\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(\frac{2}{\pi}+\frac{1.115}{T}-\frac{2.206}{T^2}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.633+\frac{6.562}{T}\Big]n_{\mathbb{L}}$$

$$+7.834+\frac{3.779}{T}-\frac{5.614}{T^2}-\Big(0.406+\frac{1}{4T}\Big)r_1-e_{n_{\mathbb{L}}}. \tag{4.10}$$

Then $-R'_C(\eta)\le-\log\eta+\mathbf{S}\log(\eta+1)$ hence, using Lemma 3.3,

$$\mathcal{M}_C r_\chi-R'_C(\eta)\le\sum_{\rho\in Z}\frac{2}{|\rho(2-\rho)|}+n_{\mathbb{L}}\delta_C-\mathbf{S}+\frac{3}{2}+\mathbf{S}\log(x+1)-\log\Big(x-\frac{2x}{T}\Big). \tag{4.11}$$

Summing (4.10) and (4.11), we get from (4.5):

$$\begin{aligned}-\Big(\frac{|G|}{|C|}\psi(C;x)-x\Big)\le&\frac{x}{T}+\sqrt{x}\Big[\frac{1}{\pi}\log\Big(\frac{T}{2\pi}\Big)+1.704+\frac{1.115}{T}\Big]\log\Delta_{\mathbb{L}}\\&+\sqrt{x}\Big[\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)+\Big(\frac{2}{\pi}+\frac{1.115}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)-1.633+\frac{6.562}{T}\Big]n_{\mathbb{L}}\\&+\sqrt{x}\Big[7.834+\frac{3.779}{T}-\frac{5.614}{T^2}-\Big(0.406+\frac{1}{4T}\Big)r_1-e_{n_{\mathbb{L}}}\Big]\\&+\sum_{\rho\in Z}\frac{2}{|\rho(2-\rho)|}+n_{\mathbb{L}}\delta_C-\mathbf{S}+\frac{3}{2}+(\mathbf{S}-1)\log x+\frac{\mathbf{S}}{x}-\log\Big(1-\frac{2}{T}\Big).\end{aligned}$$

Reorganizing as above we get

$$-\Big(\frac{|G|}{|C|}\psi(C;x)-x\Big)\le L_a(x,T,n_{\mathbb{L}},\log\Delta_{\mathbb{K}})+\mathcal{A} \tag{4.12}$$

with

$$\begin{aligned}\mathcal{A}:=&2(\mathbf{S}-1)\log x+\frac{\mathbf{S}}{x}-2\mathbf{S}+1.744n_{\mathbb{L}}\delta_C+3-0.570n_{\mathbb{L}}-\log\Big(1-\frac{2}{T}\Big)-\frac{2}{T}\\&-\frac{\sqrt{x}}{T}\Big[0.743W_{\mathbb{L}}(T)+1.167n_{\mathbb{L}}+\frac{5.614}{T}\Big].\end{aligned}$$

We observe that, for $T\ge2\pi$, we have $-\log(1-2/T)-2/T\le2.561/T^2\le5.614\sqrt{x}/T^2$, and that $\mathbf{S}/x\le0.256n_{\mathbb{L}}\delta_C+0.125n_{\mathbb{L}}$, under the assumption $x\ge4$. We then get

$$\begin{aligned}\mathcal{A}&\le2(\mathbf{S}-1)(\log x-1)-0.445n_{\mathbb{L}}+1+2n_{\mathbb{L}}\delta_C-\frac{\sqrt{x}}{T}(0.743W_{\mathbb{L}}(T)+1.167n_{\mathbb{L}})\\&=D(x,T,n_{\mathbb{L}},\log\Delta_{\mathbb{L}}).\end{aligned} \tag{4.13}$$

By (1.2), we have (4.2) from (4.12) and (4.13). □

5. Proof of Theorem 1.1

For $\mathbb{L}=\mathbb{Q}$, the theorem is weaker than Lowell Schoenfeld's result for $x\ge59$, and true in the range $[1,59]$ by explicit computation. We assume henceforth that $\mathbb{L}\ne\mathbb{Q}$, i.e. $n_{\mathbb{L}}\ge2$. Since $\psi(C;x)\ge\psi_C(x)$, for the proof of the theorem it is sufficient to show that

$$\frac{|G|}{|C|}\psi(C;x)-x\le\sqrt{x}\Big[\Big(\frac{\log x}{2\pi}+2\Big)\log\Delta_{\mathbb{L}}+\Big(\frac{\log^2 x}{8\pi}+2\Big)n_{\mathbb{L}}\Big], \tag{5.1}$$

$$-\Big(\frac{|G|}{|C|}\psi_C(x)-x\Big)\le\sqrt{x}\Big[\Big(\frac{\log x}{2\pi}+2\Big)\log\Delta_{\mathbb{L}}+\Big(\frac{\log^2 x}{8\pi}+2\Big)n_{\mathbb{L}}\Big] \tag{5.2}$$

hold $\forall x\ge1$. Let then

$$B_a(x,T,n,\mathcal{L}):=\frac{L_a(x,T,n,\mathcal{L})}{n\sqrt{x}}-\Big(\frac{\log x}{2\pi}+2\Big)\frac{\mathcal{L}}{n}-\Big(\frac{\log^2 x}{8\pi}+2\Big),$$

$$B_b(x,T,n,\mathcal{L},g) := B_a(x,T,n,\mathcal{L})+\frac{D(x,T,n,\mathcal{L})}{n\sqrt{x}}+\frac{g}{p}\frac{\mathfrak{N}(\mathcal{L})}{n}\frac{\log x}{\sqrt{x}},$$

where g is an integer, p is the smallest prime divisor of g and $\mathfrak{N}(\log\Delta_{\mathbb{L}})$ is an upper bound for $\mathfrak{n}$, as given by Lemma 3.7, that will be made explicit later. To prove (5.1) it is sufficient to show that there is an $\bar{x}^+ \geq 4$ such that it is trivial for $x \in [1,\bar{x}^+]$ and that when $x \geq \bar{x}^+$, by (4.1), there exists a value of $T \geq 2\pi$ such that $B_a(x,T,n_{\mathbb{L}},\log\Delta_{\mathbb{L}}) \leq 0$. To prove (5.2) it is sufficient to show that there is an $\bar{x}^- \geq 4$ such that it is trivial for $x \in [1,\bar{x}^-]$ and that when $x \geq \bar{x}^-$, by (4.2) and Lemma 3.6, there exists a value of $T \geq 2\pi$ such that $B_b(x,T,n_{\mathbb{L}},\log\Delta_{\mathbb{L}}) \leq 0$.
We assume, from now on, that $T = T(x) := c\sqrt{x}/\log x$ with $c := 5.2$. This ensures in particular that $T \geq 2\pi$ for any $x > 1$.

5.1. **Upper bound.** We first prove (5.1).
Step 1: trivial bound. We notice that $\psi(C;x) \leq \psi_{\mathbb{K}}(x) \leq \psi_{\mathbb{Q}}(x)n_{\mathbb{K}}$. Hence, given that $n_{\mathbb{L}} = |G|n_{\mathbb{K}}$, the bound (5.1) is true if

$$\sqrt{x}\Big[\Big(\frac{\log x}{2\pi}+2\Big)\frac{\log\Delta_{\mathbb{L}}}{n_{\mathbb{L}}}+\Big(\frac{\log^2 x}{8\pi}+2\Big)\Big] \geq \psi_{\mathbb{Q}}(x)-\frac{x}{n_{\mathbb{L}}}.$$

We will call this bound the trivial bound. We observe that $\psi_{\mathbb{Q}}$ is constant on the intervals $[p^m,q^n)$ where p^m and q^n are consecutive prime powers, hence if the trivial bound is true in p^m it is true in the whole interval $[p^m,q^n)$. We check that the bound is true for $x < 61$ if $n_{\mathbb{L}} = 4$ and for $x < 71$ for any other value of $n_{\mathbb{L}} \in [2,13]$ using the explicit lower bounds for $\log\Delta_{\mathbb{L}}$ in [16] and [14, Table 3]. For $n_{\mathbb{L}} \geq 14$, $\frac{\log\Delta_{\mathbb{L}}}{n_{\mathbb{L}}} \geq 2.12$ as follows from entry $b = 2.1$ in [14, Table 3]. We this lower bound, we check that the stronger bound without the $x/n_{\mathbb{L}}$ term is true for $x < 71$. This ensures that it is true for $x < 71$ and $n_{\mathbb{L}} \geq 14$.
Hence (5.1) is a consequence of the trivial bound if either $n_{\mathbb{L}} = 4$ and $x < 61$ or $n_{\mathbb{L}} \neq 4$ and $x < 71$.
Step 2: function B_a is decreasing in $\mathcal{L}$. We have

$$\begin{aligned}B_a(x,T(x),n_{\mathbb{L}},\mathcal{L}) =&\Big[\frac{1}{\pi}\log\Big(\frac{c/(2\pi)}{\log x}\Big)-0.296+\frac{1.858}{T}+\frac{1.075}{\sqrt{x}}\Big]\frac{\mathcal{L}}{n_{\mathbb{L}}}\\&+\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)-\frac{1}{8\pi}\log^2 x+\Big(\frac{2}{\pi}+\frac{1.858}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)-3.633+\frac{7.729}{T}-\frac{1.501}{\sqrt{x}}\\&+\frac{1}{n_{\mathbb{L}}\sqrt{x}}\Big[\frac{x+2}{T}+(1-\mathbf{S})\log x+\mathbf{S}+9.276-0.744n_{\mathbb{L}}\delta_C-0.527r_1\Big]\\&+\frac{1}{n_{\mathbb{L}}}\Big[7.834+\frac{3.779}{T}-\Big(0.406+\frac{1}{4T}\Big)r_1-e_{n_{\mathbb{L}}}\Big].\end{aligned}$$

Since $T(x)$ is an increasing function of $x \geq e^2$, $\frac{\partial B_a}{\partial\mathcal{L}}$ is decreasing with x. As $\frac{\partial B_a}{\partial\mathcal{L}}(61,T(61)) \leq 0$, we have that $\frac{\partial B_a}{\partial\mathcal{L}} \leq 0$ for any $x \geq 61$.
Step 3: function B_a is decreasing in x. We have

$$\begin{aligned}\frac{\partial B_a}{\partial x}(x,T(x),n_{\mathbb{L}},\mathcal{L}) \leq&\frac{-\log 3}{2}\Big[\frac{1}{\pi x\log x}+\frac{1.858T'}{T^2}+\frac{1.075}{2x\sqrt{x}}\Big]+\frac{T'}{T}\Big(\frac{1}{\pi}-\frac{1.858}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)\\&-\frac{\log x}{4\pi x}+\frac{2T'}{\pi T}-\frac{5.621T'}{T^2}+\frac{\log x+0.772}{2x\sqrt{x}}+\frac{1}{cn_{\mathbb{L}}x}-\frac{4.638}{n_{\mathbb{L}}x^{3/2}}\end{aligned}$$

where we have removed a few terms whose decreasing behaviour is evident, and used the facts that $\mathcal{L}/n_{\mathbb{L}} \geq \frac{1}{2}\log 3$, $\delta_C \leq 1$, $\mathbf{S} \leq n_{\mathbb{L}}$ and $r_1 \leq n_{\mathbb{L}}$. Since $n_{\mathbb{L}} \geq 2$, we bound the last two terms by $\max(0,1/(cx)-4.638x^{-3/2})/2$ and the resulting function is an elementary one variable function which is negative for $x \geq 61$.

Step 4: estimates for $n_{\mathbb{L}} \geq 4$. For $n_{\mathbb{L}} \geq 4$, we have $\log \Delta_{\mathbb{L}} \geq n_{\mathbb{L}}$ (this is true for all number fields except $\mathbb{Q}$ and the four quadratic fields with $\Delta_{\mathbb{L}} \leq 7$). Given that B_a is a decreasing function of $\mathcal{L}$ for $x \geq 61$, we have

$$B_a(x, T(x), n_{\mathbb{L}}, \log \Delta_{\mathbb{L}}) \leq B_a(x, T(x), n_{\mathbb{L}}, n_{\mathbb{L}})$$

as soon as $n_{\mathbb{L}} \geq 4$ and $x \geq 61$.
Since $\delta_C \geq 0$, $r_1 \geq 0$, $\mathbf{S} \geq 0$ and $e_{n_{\mathbb{L}}} \geq 0$, we have

$$\begin{aligned} B_a(x, T(x), n_{\mathbb{L}}, n_{\mathbb{L}}) \leq & \frac{1}{\pi} \log\Big(\frac{c/(2\pi)}{\log x}\Big) - 0.296 + \frac{1.858}{T} + \frac{1.075}{\sqrt{x}} \\ & + \frac{1}{2\pi} \log^2\Big(\frac{T}{2\pi}\Big) - \frac{1}{8\pi} \log^2 x + \Big(\frac{2}{\pi} + \frac{1.858}{T}\Big) \log\Big(\frac{T}{2\pi}\Big) - 3.633 + \frac{7.729}{T} - \frac{1.501}{\sqrt{x}} \\ & + \frac{1}{n_{\mathbb{L}}\sqrt{x}}\Big[\frac{x+2}{T} + \log x + 9.276\Big] + \frac{1}{n_{\mathbb{L}}}\Big[7.834 + \frac{3.779}{T}\Big]. \end{aligned}$$

This upper bound is decreasing in $n_{\mathbb{L}}$ because $n_{\mathbb{L}}$ only appears as the denominator of a fraction with positive numerator. Since $B_a(61, T(61), 4, 4) < 0$, the decreasing behaviour of B_a in x, n and $\mathcal{L}$ proves that $B_a(x, T(x), n_{\mathbb{L}}, \log \Delta_{\mathbb{L}}) < 0$ if $n_{\mathbb{L}} \geq 4$ and $x \geq 61$. With the trivial bound in Step 1, we see that $B_a(x, T(x), n_{\mathbb{L}}, \log \Delta_{\mathbb{L}}) < 0$ if $n_{\mathbb{L}} \geq 4$ and $x \geq 1$.
Step 5: estimates for $n_{\mathbb{L}} = 3$, $r_1 = 3$. In this case $\Delta_{\mathbb{L}} \geq 49$ and $B_a(71, T(71), 3, \log 49) < 0$ (where we use, as above, that $\delta_C \geq 0$ and $\mathbf{S} \geq 0$) which, including the trivial bound, concludes the proof.
Step 6: estimates for $n_{\mathbb{L}} = 3$, $r_1 = 1$. In this case $\Delta_{\mathbb{L}} \geq 23$ and we necessarily have $\mathbb{L} = \mathbb{K}$, hence $\delta_C = 1$ and $\mathbf{S} = (n_{\mathbb{L}} + r_1)/2 = 2$. Since $B_a(71, T(71), 3, \log 23) < 0$, the proof is complete for $n_{\mathbb{L}} = 3$.
Step 7: estimates for $n_{\mathbb{L}} = 2$, large $\Delta_{\mathbb{L}}$ or large x. We observe that the trivial bound extends to $x < 607$ when $\Delta_{\mathbb{L}} \geq 300$. As above the worst case is for $\delta_C = 0$ and $r_1 = 0$ and in that case $\mathbf{S} = 1$. We have $B_a(607, T(607), 2, \log 300) < 0$, which means that the case where $n_{\mathbb{L}} = 2$, $\Delta_{\mathbb{L}} \geq 300$ is proved.
Besides, we observe that also $B_a(10^5, T(10^5), 2, \log 3) < 0$, keeping the worst case $\delta_C = 0$, $r_1 = 0$ and $\mathbf{S} = 1$, hence (5.1) for $n_{\mathbb{L}} = 2$ is proved also for $x \geq 10^5$. Hence (5.1) is proved for $n_{\mathbb{L}} = 2$ if either $\Delta_{\mathbb{L}} \geq 300$ or $x \geq 10^5$.
Step 8: estimates for $n_{\mathbb{L}} = 2$, small $\Delta_{\mathbb{L}}$ and small x. For the remaining quadratic fields $\mathbb{L}$ the proof will be made together with the lower bound.

5.2. **Lower bound.** We now turn to (5.2).
Lemma 3.7(iv) shows that $\mathfrak{n} \leq \log \Delta_{\mathbb{L}}/(\log\log \Delta_{\mathbb{L}} - \log n_{\mathbb{K}} - 1.1714)$ when $\log \Delta_{\mathbb{L}} > e^{1.1714}\, n_{\mathbb{K}}$. To get an easier estimate we use line $b = 4.1$ of Table 3 in [14], producing the lower bound

$$\begin{aligned} \log\log \Delta_{\mathbb{L}} - \log n_{\mathbb{K}} - 1.1714 &\geq \log(n_{\mathbb{L}} \log 25.585 - 28.36) - \log n_{\mathbb{K}} - 1.1714 \\ &= \log\Big(|G| \log 25.585 - \frac{28.36}{n_{\mathbb{K}}}\Big) - 1.1714 \geq \log(|G| - 8.79). \end{aligned}$$

Moreover, Lemma 3.7(iii) implies that $\mathfrak{n} \leq 0.4 + \log \Delta_{\mathbb{L}}/\log 22$ if $|G|$ is not prime – where the 0.4 has been added to handle the exceptions. We thus define

$$\mathfrak{N}(\mathcal{L}) := \begin{cases} 0 & \text{if } |G| = 1, \\ \mathcal{L}/\log(|G| - 8.79) & \text{if } |G| \geq 32, \\ \mathcal{L}/\log 4 & \text{if } |G| \text{ is a prime} \leq 31 \text{ and} \neq 3, \\ \mathcal{L}/\log 49 & \text{if } |G| = 3, \\ 0.4 + \mathcal{L}/\log 22 & \text{otherwise.} \end{cases}$$

In this way, from Lemma 3.7 we have $\mathfrak{n} \le \mathfrak{N}(\log \Delta_{\mathbb{L}})$.
Before starting the proof, we observe that if $\mathbb{K} = \mathbb{L}$, then $\mathfrak{N}(\mathcal{L}) = 0$. Thus, when we are able to prove that $B_b \le 0$ for suitable x, T (and a certain value for the parameters r_1 and $\mathbf{S}$) under the assumption that $\mathbb{K} \ne \mathbb{L}$, then with the same values for x and T, we have $B_b \le 0$ also for $\mathbb{K} = \mathbb{L}$ (and the same value for r_1 and $\mathbf{S}$).
Step 1: trivial bound. Bound (5.2) is satisfied if

$$\Big(\frac{\log x}{2\pi}+2\Big)\log \Delta_{\mathbb{L}}+\Big(\frac{\log^2 x}{8\pi}+2\Big)n_{\mathbb{L}} \ge \sqrt{x}$$

because in this case it is weaker than the trivial bound $\psi_C(x) \ge 0$. Since for $n_{\mathbb{L}} \ge 3$ we have $\log \Delta_{\mathbb{L}} \ge n_{\mathbb{L}}$ we see that this is true if $x \le 16 n_{\mathbb{L}}^2$. This extends to $n_{\mathbb{L}} = 2$ by direct computation.
For the end of this subsection, we will assume $x \ge 16 n_{\mathbb{L}}^2$ (and hence $x \ge 16|G|^2$ and $x \ge 64$).
Step 2: function B_b is decreasing in $\mathcal{L}$. We have

$$\begin{aligned}
B_b(x,T(x),n_{\mathbb{L}},\mathcal{L},|G|)=&\Big[\frac{1}{\pi}\log\Big(\frac{c/(2\pi)}{\log x}\Big)-0.296+\frac{1.115}{T}+\frac{1.075}{\sqrt{x}}\Big]\frac{\mathcal{L}}{n_{\mathbb{L}}}+\frac{|G|}{p}\frac{\mathfrak{N}(\mathcal{L})}{n_{\mathbb{L}}}\frac{\log x}{\sqrt{x}}\\
&+\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)-\frac{1}{8\pi}\log^2 x+\Big(\frac{2}{\pi}+\frac{1.115}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)\\
&-3.633+\frac{6.562}{T}-\frac{1.946}{\sqrt{x}}+\frac{1}{n_{\mathbb{L}}}\Big[7.834-\Big(0.406+\frac{1}{4T}\Big)r_1-e_{n_{\mathbb{L}}}+\frac{3.779}{T}\Big]\\
&+\frac{1}{n_{\mathbb{L}}\sqrt{x}}\Big[\frac{x+2}{T}+(\mathbf{S}-1)\log x-\mathbf{S}+12.276+1.256 n_{\mathbb{L}}\delta_C-0.527 r_1\Big].
\end{aligned}$$

We observe that the derivative $\mathfrak{N}'$ is a constant depending only on $|G|$. Moreover, since $x \ge 16 n_{\mathbb{L}}^2 \ge 16|G|^2$,

$$\frac{\partial}{\partial \mathcal{L}}\Big[\frac{|G|}{p}\mathfrak{N}(\mathcal{L})\frac{\log x}{\sqrt{x}}\Big] = \frac{|G|\mathfrak{N}'\log x}{p\sqrt{x}} \le \frac{\mathfrak{N}'\log(4|G|)}{2p}.$$

By computing the values for $2 \le |G| \le 32$, and using the lower bound $x \ge 16|G|^2$, we observe that

$$\frac{1.075}{\sqrt{x}}+\frac{\mathfrak{N}'\log(4|G|)}{2p} \le 0.51.$$

The conclusion holds also for any $|G| > 32$ because

$$\frac{\mathfrak{N}'\log(4|G|)}{2p} \le \frac{\log(4|G|)}{4\log(|G|-8.79)}$$

which decreases in $|G|$. We thus get

$$n_{\mathbb{L}}\frac{\partial B_b}{\partial \mathcal{L}} \le \frac{1}{\pi}\log\Big(\frac{c/(2\pi)}{\log x}\Big)-0.296+\frac{1.115}{T}+0.51$$

which is negative because $x \ge 64$ hence $T \ge 10$.
Step 3: function B_b is decreasing in x. We have

$$\begin{aligned}
\frac{\partial B_b}{\partial x}(x,T(x),n_{\mathbb{L}},\mathcal{L},|G|) \le& \frac{-\log 3}{2}\Big[\frac{1}{\pi x\log x}+\frac{1.115T'}{T}+\frac{1.075}{2x\sqrt{x}}\Big]+\frac{T'}{T}\Big(\frac{1}{\pi}-\frac{1.115}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)\\
&-\frac{\log x}{4\pi x}+\frac{2T'}{\pi T}-\frac{5.197T'}{T^2}+\frac{2.473}{2x\sqrt{x}}+\frac{1}{c n_{\mathbb{L}} x}+\frac{\log x-2}{2n_{\mathbb{L}}x\sqrt{x}}
\end{aligned}$$

which is negative as well for $x \ge 64$.

Step 4: estimates for $n_{\mathbb{L}} \geq 4$. We have $\log \Delta_{\mathbb{L}} \geq n_{\mathbb{L}}$. Given that B_b is a decreasing function of $\mathcal{L}$ for $x \geq 64$, we have

$$B_b(x, T(x), n_{\mathbb{L}}, \log \Delta_{\mathbb{L}}, |G|) \leq B_b(x, T(x), n_{\mathbb{L}}, n_{\mathbb{L}}, |G|)$$

as soon as $x \geq 64$. We know that $\mathbf{S} \leq (n_{\mathbb{L}}+r_1)/2$; introducing this bound in B_b, the term depending on r_1 in B_b becomes

$$\frac{r_1}{n_{\mathbb{L}}\sqrt{x}}\Big(\frac{1}{2}(\log x-1)-0.527-\Big(0.406+\frac{1}{4T}\Big)\sqrt{x}\Big)$$

which is ≤ 0 for every x. Its larger value is therefore reached for $r_1 = 0$. Once the bound $\delta_C \leq 1$ is also considered, we get the upper bound

$$\begin{aligned}B_b(x, T, n_{\mathbb{L}}, n_{\mathbb{L}}, |G|) \leq &\frac{1}{\pi}\log\Big(\frac{c/(2\pi)}{\log x}\Big)-0.296+\frac{1.115}{T}+\frac{1.075}{\sqrt{x}}+\frac{|G|}{p}\frac{\mathfrak{N}(n_{\mathbb{L}})}{n_{\mathbb{L}}}\frac{\log x}{\sqrt{x}}\\ &+\frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big)-\frac{1}{8\pi}\log^2 x+\Big(\frac{2}{\pi}+\frac{1.115}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)\\ &-3.633+\frac{6.562}{T}+\frac{\log x-2.380}{2\sqrt{x}}+\frac{1}{n_{\mathbb{L}}\sqrt{x}}\Big[\frac{x+2}{T}-\log x+12.276\Big]\\ &+\frac{1}{n_{\mathbb{L}}}\Big[7.834+\frac{3.779}{T}\Big].\end{aligned}$$

Once again this is decreasing in $n_{\mathbb{L}}$, as long as $|G|/p$ remains constant and $\mathfrak{N}$ does not change form, since $7.834\sqrt{x}-\log x > 0$. We check that B_b is negative in the proper range of its arguments by checking that this upper bound is negative, too. Doing this, we can restrict the test to the cases with $|G| \geq 2$: in fact, $\frac{|G|}{p}\frac{\mathfrak{N}(n_{\mathbb{L}})}{n_{\mathbb{L}}}\frac{\log x}{\sqrt{x}}$ is the unique term depending on $|G|$ appearing there, and it is zero when $|G| = 1$. Moreover, for each $|G|$, we only need to check whether the right hand side with $x = 16n_{\mathbb{L}}^2$, $T = T(16n_{\mathbb{L}}^2)$ is negative when $n_{\mathbb{L}} = |G|$ (if $|G| \geq 4$) or when $n_{\mathbb{L}} = 2|G|$ (if $|G| = 2$ or 3).
If $|G| \geq 32$, then $n_{\mathbb{L}} = |G|$ and

$$\frac{|G|}{p}\frac{\mathfrak{N}(n_{\mathbb{L}})}{n_{\mathbb{L}}}\frac{\log(16n_{\mathbb{L}}^2)}{\sqrt{16n_{\mathbb{L}}^2}} = \frac{\log(4|G|)}{2p\log(|G|-8.79)} \leq \frac{\log(4|G|)}{4\log(|G|-8.79)}$$

which is decreasing in $|G|$, so, we just need to test the value for $n_{\mathbb{L}} = |G| = 32$.
If $|G| \leq 31$ is not prime, we need to check for $|G|/p \in \{2, \dots, 15\}$, but from the decreasing argument (now in p with fixed $|G|/p$) we only need to check the case $p = 2$, i.e. $|G|$ even in $[4, 30]$.
If $|G| \leq 31$ is prime (but different from 3) we have

$$\frac{|G|}{p}\frac{\mathfrak{N}(n_{\mathbb{L}})}{n_{\mathbb{L}}}\frac{\log(16n_{\mathbb{L}}^2)}{\sqrt{16n_{\mathbb{L}}^2}} = \frac{\log(4n_{\mathbb{L}})}{2n_{\mathbb{L}}\log 4},$$

which decreases in $n_{\mathbb{L}}$. Thus we just need to check the case $n_{\mathbb{L}} = 4$, and hence $|G| = 2$.
If $|G| = 3$, then $n_{\mathbb{L}} = 6$ and

$$\frac{|G|}{p}\frac{\mathfrak{N}(n_{\mathbb{L}})}{n_{\mathbb{L}}}\frac{\log(16n_{\mathbb{L}}^2)}{\sqrt{16n_{\mathbb{L}}^2}} = \frac{\log(4n_{\mathbb{L}})}{2n_{\mathbb{L}}\log 49},$$

which is smaller than what we got previously for the case $|G| = 6$.
In total we have sixteen cases: $n_{\mathbb{L}} = |G| = 32$, $n_{\mathbb{L}} = |G|$ even in $[4, 30]$ and $n_{\mathbb{L}} = 4$ with $|G| = 2$. All sixteen values are negative. We have covered all cases for $|G|/p$ and $\mathfrak{N}$ hence, together with the trivial bound, this proves the lower bound for $n_{\mathbb{L}} \geq 4$.

Step 5: estimates for $n_{\mathbb{L}} = 3$. We have $\Delta_{\mathbb{L}} \geq 23$, $\delta_C \leq 1$. As for the previous case, we estimate $\mathbf{S}$ with $(n_{\mathbb{L}}+r_1)/2$ and the emerging term depending on r_1 with its largest value, which now corresponds to $r_1 = 1$ (because for $n_{\mathbb{L}} = 3$ the unique admissible values for r_1 are 1 and 3). This produces the bound

$$\begin{aligned}B_b(x, T(x), 3, \log 23, |G|) \leq & \Big[\frac{1}{\pi}\log\Big(\frac{c/(2\pi)}{\log x}\Big) - 0.296 + \frac{1.115}{T} + \frac{1.075}{\sqrt{x}}\Big]\frac{\log 23}{3} + \frac{\log 23}{3\log 49}\frac{\log x}{\sqrt{x}}\\ & + \frac{1}{2\pi}\log^2\Big(\frac{T}{2\pi}\Big) - \frac{1}{8\pi}\log^2 x + \Big(\frac{2}{\pi} + \frac{1.115}{T}\Big)\log\Big(\frac{T}{2\pi}\Big)\\ & - 3.633 + \frac{6.562}{T} - \frac{1.946}{\sqrt{x}} + \frac{1}{3}\Big[7.115 + \frac{3.529}{T}\Big] + \frac{1}{3\sqrt{x}}\Big[\frac{x+2}{T} + \log x + 13.517\Big],\end{aligned}$$

which is negative for $x = 16n_{\mathbb{L}}^2 = 16{\cdot}9$ and $T = T(16{\cdot}9)$. This completes the proof of the claim for $n_{\mathbb{L}} = 3$.

Step 6: estimates for $n_{\mathbb{L}} = 2$, large $\Delta_{\mathbb{L}}$ or large x. The worst case happens when $\delta_C = 1$, $|G| = 2$, $\mathbf{S} = 1+r_1/2$ and $r_1 = 0$. For $\Delta_{\mathbb{L}} \geq 300$, we observe that the trivial bound extends to $x \leq 598$ and that $B_b(598, T(598), 2, \log 300, 2) < 0$ if $r_1 = 0$. This means that the case where $\Delta_{\mathbb{L}} \geq 300$ is proved. We observe that $B_b(10^5, T(10^5), 2, \log 3, 2) < 0$, hence the claim is proved for $x \geq 10^5$.

Step 7: estimates for $n_{\mathbb{L}} = 2$, small $\Delta_{\mathbb{L}}$ and small x. For the remaining fields $\mathbb{L}$, which are quadratic with $\Delta_{\mathbb{L}} < 300$, let $x_1(\mathbb{L}) \geq 61$ be such that $B_a(x_1(\mathbb{L}), T(x_1(\mathbb{L})), n_{\mathbb{L}}, \log \Delta_{\mathbb{L}}) < 0$ (with $\delta_C = 0$) and $B_b(x_1(\mathbb{L}), T(x_1(\mathbb{L})), n_{\mathbb{L}}, \log \Delta_{\mathbb{L}}, 2) < 0$ (with $\delta_C = 1$), where we use the true value of $\mathfrak{n}$. As we have seen, for all fields $x_1(\mathbb{L}) \leq 10^5$. To complete the proof of Theorem 1.1 we have built a program that checks for each integer $x \in [1, x_1(\mathbb{L})]$ that

$$\begin{aligned}&-\mathcal{B}+1 \leq \psi_{\mathbb{L}}(x) - x \leq \mathcal{B},\\ &-\mathcal{B}+1 \leq 2\psi_C(x) - x \leq 2\psi(C;x) - x \leq \mathcal{B},\end{aligned}$$

where

$$\mathcal{B} := \sqrt{x}\Big[\Big(\frac{\log x}{2\pi} + 2\Big)\log \Delta_{\mathbb{L}} + 2\Big(\frac{\log^2 x}{8\pi} + 2\Big)\Big].$$

6. Proof of Corollary 1.2

The bounds stated in the corollary are certainly true as soon as

$$\sqrt{x}\Big[\Big(\frac{1}{2\pi} + \frac{3}{\log x}\Big)\frac{\log \Delta_{\mathbb{L}}}{n_{\mathbb{L}}} + \Big(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x}\Big)\Big] \geq \max\Big(\int_2^x \frac{\mathrm{d}u}{\log u}, \pi(x) - \frac{1}{n_{\mathbb{L}}}\int_2^x \frac{\mathrm{d}u}{\log u}\Big),$$

because in this case the conclusion is weaker than the elementary bound $0 \leq \pi_C(x) \leq \pi(C;x) \leq \pi(x)n_{\mathbb{K}}$. The first inequality holds when $x \in [2, 193)$, because $\frac{1}{n_{\mathbb{L}}}\log \Delta_{\mathbb{L}} \geq \frac{1}{2}\log 3$, and

$$\sqrt{x}\Big[\Big(\frac{1}{2\pi} + \frac{3}{\log x}\Big)\frac{1}{2}\log 3 + \Big(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x}\Big)\Big] \geq \int_2^x \frac{\mathrm{d}u}{\log u}$$

holds in this range. The second inequality

$$\sqrt{x}\Big[\Big(\frac{1}{2\pi} + \frac{3}{\log x}\Big)\frac{\log \Delta_{\mathbb{L}}}{n_{\mathbb{L}}} + \Big(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x}\Big)\Big] \geq \pi(x) - \frac{1}{n_{\mathbb{L}}}\int_2^x \frac{\mathrm{d}u}{\log u}$$

is checked for $x \in [2, 193)$ by testing it for each $n_{\mathbb{L}} \leq 20$ (using the lower bound for $\log \Delta_{\mathbb{L}}$ as follows from Odlyzko's tables for each degree). The case $n_{\mathbb{L}} = 20$ is checked in the stronger version where $-\frac{1}{n_{\mathbb{L}}}\int_2^x \frac{\mathrm{d}u}{\log u}$ is removed, so that its validity implies the validity also for all

$n_{\mathbb{L}} \geq 20$. In this way the corollary is fully proved up to 193.
Let

$$\vartheta(C;x) := \sum_{\substack{\mathfrak{p} \\ \mathrm{N}\mathfrak{p}\leq x}} \theta(C;\mathfrak{p}) \log \mathrm{N}\mathfrak{p}.$$

Then by partial summation

$$\begin{aligned}\frac{|G|}{|C|}\pi(C;x) - \int_2^x \frac{\mathrm{d}u}{\log u} = \frac{|G|}{|C|}\pi(C;73) - \frac{\frac{|G|}{|C|}\vartheta(C;73)}{\log 73} + \frac{73}{\log 73} - \int_2^{73} \frac{\mathrm{d}u}{\log u} \\ + \frac{\frac{|G|}{|C|}\vartheta(C;x) - x}{\log x} + \int_{73}^x \frac{\frac{|G|}{|C|}\vartheta(C;u) - u}{u \log^2 u}\,\mathrm{d}u.\end{aligned}$$

Assuming $x \geq 193$, we have

$$0 \leq \pi(C;73) - \frac{\vartheta(C;73)}{\log 73} \leq \sum_{\mathrm{N}\mathfrak{p}\leq 73}\Big(1 - \frac{\log \mathrm{N}\mathfrak{p}}{\log 73}\Big) \leq \sum_{\substack{p\leq 73 \\ p \text{ prime}}}\Big(1 - \frac{\log p}{\log 73}\Big) n_{\mathbb{K}} \leq 5.65 n_{\mathbb{K}} \leq 2.15\frac{\sqrt{x}}{\log x} n_{\mathbb{K}},$$

$$0 \leq \int_2^{73} \frac{\mathrm{d}u}{\log u} - \frac{73}{\log 73} \leq 6.1 \leq 1.16\frac{\sqrt{x}}{\log x} n_{\mathbb{L}},$$

and

$$\forall x \geq 1, \qquad 0 \leq \psi(C;x) - \vartheta(C;x) \leq \psi_{\mathbb{K}}(x) - \vartheta_{\mathbb{K}}(x) \leq 1.43\sqrt{x} n_{\mathbb{K}}$$

by [20, Th. 13]. We deduce that

$$\begin{aligned}\Big|\frac{|G|}{|C|}\pi(C;x) - \int_2^x \frac{\mathrm{d}u}{\log u}\Big| &\leq \frac{\big|\frac{|G|}{|C|}\psi(C;x) - x\big| + 2.59\sqrt{x} n_{\mathbb{L}}}{\log x} + \int_2^x \frac{\big|\frac{|G|}{|C|}\psi(C;u) - u\big| + 1.43\sqrt{u} n_{\mathbb{L}}}{u \log^2 u}\,\mathrm{d}u \\ &\leq \sqrt{x}\Big[\Big(\frac{1}{2\pi} + \frac{2}{\log x}\Big)\log \Delta_{\mathbb{L}} + \Big(\frac{\log x}{8\pi} + \frac{4.59}{\log x}\Big) n_{\mathbb{L}}\Big] \\ &\quad + \int_{73}^x \frac{\big(\frac{\log u}{2\pi} + 2\big)\log \Delta_{\mathbb{L}} + \big(\frac{\log^2 u}{8\pi} + 3.43\big) n_{\mathbb{L}}}{\sqrt{u}\log^2 u}\,\mathrm{d}u.\end{aligned}$$

Since $\int_{73}^x \frac{\frac{\log u}{2\pi}+2}{\sqrt{u}\log^2 u}\,\mathrm{d}u \leq \frac{\sqrt{x}}{\log x}$, and $\int_{73}^x \frac{\mathrm{d}u}{\sqrt{u}\log^2 u} \leq 0.33\frac{\sqrt{x}}{\log x}$ (for $x \geq 193$), we get

$$\leq \sqrt{x}\Big[\Big(\frac{1}{2\pi} + \frac{3}{\log x}\Big)\log \Delta_{\mathbb{L}} + \Big(\frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x}\Big) n_{\mathbb{L}}\Big],$$

which concludes the proof of the claim for $\pi(C;x)$. For $\pi_C(x)$ the argument is the same.

Appendix A. Number of zeros

Trudgian [23] showed how to take advantage of both Backlund's and Rosser's approaches to produce good explicit bounds for the function $N(T)$ counting non-trivial zeros ρ with $|\operatorname{Im}\rho| \leq T$ for Dirichlet and Dedekind L-functions. Note that, contrary to the rest of this paper, Trudgian's approach doest not require to assume any form of the Riemann Hypothesis. Studying his paper we have found some possible improvements in the way some terms are bounded. We have also noted that the original paper does not isolate the role of a special constant (the analogue of the constant $-7/8$ appearing for Riemann's zeta in [4, Ch. 15, (1)]). However, isolating this term allows to formulate the bound with smaller constants, and this is very useful when sums on zeros of type $\sum_{|\operatorname{Im}\rho|\geq a} f(\rho)$ with $a > 0$ are estimated via partial summation, because in this case that term does not contribute and only the smaller constants appear. This is very important for our application, since we need to take advantage of every possible method to improve the constants, in order to reduce the set of explicit computations

which are needed to prove Theorem 1.1.
Moreover, we have also noticed that essentially the same strategy can be applied to study the zeros of all Hecke's L-functions of finite order Größencharakter, thus we have formulated the results for this more general set, for possible future reference.
We stress once again that the main strategy for this computation has to be credited to Trudgian, our contribution being limited to the points cited above.

Let $\mathbb{E}$ be a number field. Let χ be a Hecke Größencharakter of $\mathbb{E}$ which is primitive and of finite order. Let $\mathfrak{f}(\chi)$ denote the conductor of χ and set $Q(\chi) = \Delta_{\mathbb{E}} \mathrm{N}_{\mathbb{E}/\mathbb{Q}}(\mathfrak{f}(\chi))$. Let δ_χ be 1 if χ is trivial and 0 otherwise. Let $N(T,\chi)$ be the number (multiplicity included) of non-trivial zeros ρ (i.e. with $\operatorname{Re}\rho \in (0,1)$) with $|\operatorname{Im}\rho| \leq T$ for $L(s,\chi)$.

Theorem A.1. *Unconditionally,*

$$\Big|N(T,\chi) - \frac{T}{\pi}\log\Big[Q(\chi)\Big(\frac{T}{2\pi e}\Big)^{n_{\mathbb{E}}}\Big] - 2\delta_\chi + \frac{a_\chi - b_\chi}{4}\Big| \leq D_1(\log Q(\chi) + n_{\mathbb{E}}\log T) + D_2' n_{\mathbb{E}} + \delta_\chi D_3'$$

when $T \geq T_0$, for T_0, D_1, D_2' and D_3' as in Table 1.

If χ is the trivial character, then $\mathbb{E} = \mathbb{L}$ and $N(T,\chi) = N_{\mathbb{L}}(T)$ is the number of non-trivial zeros of $\zeta_{\mathbb{L}}$ with imaginary part in $[-T,T]$. In that case $Q(\chi) = \Delta_{\mathbb{L}}$ and $a_\chi - b_\chi = r_1$. If one want to compare this result with the analogue contained in [23, Theorem 2] one has to take note of the extra term $-2+\frac{1}{4}r_1$ that we have put in evidence (as for Riemann's zeta in [4, Ch. 15, (1)]).

Table 1: Parameters for Theorem A.1

	$T_0 = 1$		$T_0 = 2\pi$		$T_0 = 10$	
D_1	D_2'	D_3'	D_2'	D_3'	D_2'	D_3'
0.230	16.577	1.330	16.032	0.033	16.004	0.014
0.247	8.180	1.435	7.614	0.083	7.585	0.062
0.265	6.416	1.515	5.834	0.150	5.805	0.129
0.282	5.409	1.598	4.812	0.213	4.783	0.192
0.299	4.696	1.699	4.083	0.275	4.053	0.254
0.316	4.158	1.814	3.526	0.335	3.495	0.313
0.333	3.735	1.961	3.082	0.400	3.050	0.371
0.350	3.425	2.185	2.731	0.429	2.698	0.402
0.367	3.206	2.426	2.467	0.453	2.432	0.423
0.384	3.043	2.687	2.257	0.478	2.221	0.444
0.401	2.918	2.966	2.083	0.503	2.044	0.465
0.460	2.666	4.082	1.645	0.593	1.598	0.540

Proof. We first suppose that χ is non-trivial. Let $\sigma_1 \in (1,2)$ and let $\mathcal{R}$ be the rectangle with vertices $\sigma_1 \pm iT$ and $1-\sigma_1 \pm iT$, positively oriented. We furthermore assume that T is not the imaginary part of any zero of $L(s,\chi)$. The conclusion for the missing T's follows because $N(T,\chi)$ is upper-continuous and all other functions are continuous. Cauchy's argument principle shows that

$$2\pi N(T,\chi) = \Delta_{\mathcal{R}} \arg\xi(s,\chi),$$

where $\Delta_{\mathcal{R}} \arg\xi(s,\chi)$ is the variation of the argument of $\xi(s,\chi)$ along $\mathcal{R}$. The functional equation shows that the variation of the argument we have in the left half-rectangle equals the variation in the right half-rectangle. Hence

$$\pi N(T,\chi) = \Delta_{\mathcal{C}} \arg\xi(s,\chi)$$

where $\mathcal{C}$ is the path $1/2-iT \to \sigma_1-iT \to \sigma_1+iT \to 1/2+iT$ and $\Delta_{\mathcal{C}}$ is the variation along $\mathcal{C}$. Hence

$$\begin{aligned}\pi N(T,\chi) &= \Delta_{\mathcal{C}} \arg(Q(\chi)^{\frac{s}{2}})+\Delta_{\mathcal{C}} \arg\Gamma_\chi(s)+\Delta_{\mathcal{C}} \arg L(s,\chi)\\ &= \Delta_{\mathcal{C}} \arg(Q(\chi)^{\frac{s}{2}})+a_\chi\Delta_{\mathcal{C}} \arg\Big(\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\Big)+b_\chi\Delta_{\mathcal{C}} \arg\Big(\pi^{-\frac{s+1}{2}}\Gamma\Big(\frac{s+1}{2}\Big)\Big)+\Delta_{\mathcal{C}} \arg L(s,\chi).\end{aligned}$$

Letting $q(\chi) := Q(\chi)^{1/n_{\mathbb{E}}}$ it becomes:

$$\begin{aligned}&= \Delta_{\mathcal{C}} \arg\Big(\Big(\frac{q(\chi)}{\pi}\Big)^{sn_{\mathbb{E}}/2}\Big)+a_\chi\Delta_{\mathcal{C}} \arg\Gamma\Big(\frac{s}{2}\Big)+b_\chi\Delta_{\mathcal{C}} \arg\Gamma\Big(\frac{s+1}{2}\Big)+\Delta_{\mathcal{C}} \arg L(s,\chi)\\ &= n_{\mathbb{E}}T\log\Big(\frac{q(\chi)}{\pi}\Big)+2a_\chi \operatorname{Im}\log\Gamma\Big(\frac{1}{4}+\frac{iT}{2}\Big)+2b_\chi \operatorname{Im}\log\Gamma\Big(\frac{3}{4}+\frac{iT}{2}\Big)+\Delta_{\mathcal{C}} \arg L(s,\chi).\end{aligned}$$

We define the function $g(\alpha,T)$ by

$$\operatorname{Im}\log\Gamma\Big(\frac{1+2\alpha}{4}+\frac{iT}{2}\Big) =: \frac{T}{2}\log\frac{T}{2e}+(2\alpha-1)\frac{\pi}{8}+g(\alpha,T) \tag{A.1}$$

for $T>0$, and by Stirling's formula we know that $g(\alpha,T)=O(1/T)$ as $T\to+\infty$. Thus, in terms of $g(\alpha,T)$ we get

$$\pi N(T,\chi) = n_{\mathbb{E}}T\log\Big(\frac{q(\chi)T}{2\pi e}\Big)+\frac{\pi}{4}(b_\chi-a_\chi)+2a_\chi g(0,T)+2b_\chi g(1,T)+\Delta_{\mathcal{C}} \arg L(s,\chi).$$

We first show that $g(1,T)\le g(0,T)$ for every $T\ge 0$. In fact, setting $z:=\frac{1}{4}+\frac{iT}{2}$, by Euler's reflection formula

$$\frac{\Gamma(\frac{1}{4}+\frac{iT}{2})}{\Gamma(\frac{3}{4}+\frac{iT}{2})} = \frac{\Gamma(z)}{\Gamma(1-z)} = \frac{|\Gamma(z)|^2}{\pi\sqrt{2}}\Big(\cosh\Big(\frac{\pi T}{2}\Big)-i\sinh\Big(\frac{\pi T}{2}\Big)\Big).$$

Since this fraction is in the fourth quadrant, this equality implies that

$$g(0,T)-g(1,T) = \frac{\pi}{4}+\arg\Big(\frac{\Gamma(\frac{1}{4}+\frac{iT}{2})}{\Gamma(\frac{3}{4}+\frac{iT}{2})}\Big) = \frac{\pi}{4}-\operatorname{atan}\Big(\tanh\Big(\frac{\pi T}{2}\Big)\Big) > 0.$$

For $g(\alpha,T)$ we have the equalities:

$$g(\alpha,T) = -\frac{2\alpha-1}{4}\operatorname{atan}\Big(\frac{2\alpha+1}{2T}\Big)+\frac{T}{4}\log\Big(1+\frac{(2\alpha+1)^2}{4T^2}\Big)-\frac{T}{6|\frac{1}{2}+\alpha+iT|^2}+\frac{3\theta}{40|\frac{1}{2}+\alpha+iT|^3} \tag{A.2}$$

for some $\theta\in[-1,1]$ (see [1, Th. 1.4.2], with $m=2$), and

$$\begin{aligned}g(\alpha,T) = -\frac{2\alpha-1}{4}\operatorname{atan}\Big(\frac{2\alpha+1}{2T}\Big)+\frac{T}{4}\log\Big(1+\frac{(2\alpha+1)^2}{4T^2}\Big)&\\ +\int_0^{+\infty}\Big(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\Big)\frac{e^{-(2\alpha+1)t/4}}{t}\sin\Big(\frac{tT}{2}\Big)\,\mathrm{d}t&\end{aligned}$$

(see [1, Th. 1.6.3 (i)]) when $2\alpha+1>0$. The first formula is strong enough to prove that $g(1,T)>0$ for $T\ge 1.5$ (but an explicit computation shows that this holds also for $T\in[1,1.5]$). The second one (with some tedious but elementary work) shows that $g(0,T)$ decreases for $T\ge 1$. Therefore

$$\Big|N(T,\chi)-\frac{n_{\mathbb{E}}T}{\pi}\log\Big(\frac{q(\chi)T}{2\pi e}\Big)+\frac{a_\chi-b_\chi}{4}\Big| \le \frac{2n_{\mathbb{E}}}{\pi}g(0,T_0)+\frac{1}{\pi}|\Delta_{\mathcal{C}} \arg L(s,\chi)| \tag{A.3}$$

for every $T\ge T_0\ge 1$.

To bound $\Delta_{\mathcal{C}} \arg L(s,\chi)$ we split $\mathcal{C}$ in three segments $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ where $\mathcal{C}_2$ is the vertical one. We have

$$|\Delta_{\mathcal{C}_2} \arg L(s,\chi)| \le 2|\log\zeta_{\mathbb{E}}(\sigma_1)| \le 2n_{\mathbb{E}}\log\zeta(\sigma_1). \tag{A.4}$$

To bound $\Delta_{\mathcal{C}_1}\arg L(s,\chi)$ and $\Delta_{\mathcal{C}_3}\arg L(s,\chi)$ we apply Backlund's argument [2], in the version given by Trudgian [23]. Let

$$f(s) := \frac{1}{2}\big(L(s+iT,\chi)^N + L(s-iT,\bar\chi)^N\big) \tag{A.5}$$

for some positive integer N. Suppose that there are n distinct zeros of $f(\sigma)=\mathrm{Re}(L(\sigma+iT,\chi)^N)$ for $\sigma\in[\frac{1}{2},\sigma_1]$. These zeros partition the segment into $n+1$ intervals. On each interval $\arg(L(\sigma+iT,\chi)^N)$ can vary by at most π. Thus

$$|\Delta_{\mathcal{C}_3}\arg L(s,\chi)| = \frac{1}{N}|\Delta_{\mathcal{C}_3}\arg L(s,\chi)^N| \leq \frac{(n+1)\pi}{N}.$$

By symmetry the same bound applies on $\mathcal{C}_1$, thus (A.3) becomes

$$\Big|N(T,\chi) - \frac{n_{\mathbb{E}}T}{\pi}\log\Big(\frac{q(\chi)T}{2\pi e}\Big) + \frac{a_\chi - b_\chi}{4}\Big| \leq \frac{2n_{\mathbb{E}}}{\pi}(g(0,T)+\log\zeta(\sigma_1)) + \frac{2(n+1)}{N}. \tag{A.6}$$

In order to bound n we apply Jensen's formula, see [11, (8)] or [21, Th. 15.18 p. 307],

$$\log\frac{R^m}{|a_1a_2\cdots a_m|} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(a+Re^{i\phi})|\,\mathrm{d}\phi - \log|f(a)|$$

where f is any function which is holomorphic in the disc centred in a and radius R, $f(a)$ is assumed to be not zero, and a_j for $j=1,\dots,m$ is the list of all zeros of f in the disc (further assuming that there are no zeros on the boundary). We set $a=1+\eta$ with $\eta\in(0,1]$, $R=r(\frac{1}{2}+\eta)$, $r>0$ and apply Jensen's formula to the function in (A.5). Assuming for the moment that $f(1+\eta)\neq 0$, [23, Lemma 2] (a special realization of Backlund's trick) shows that if $\sigma_1=\frac{1}{2}+\sqrt{2}(\frac{1}{2}+\eta)$ and $1-\sigma_1>a-R$ (which corresponds to $r>1+\sqrt{2}$) there are $n'\geq n-2-\frac{NE}{\pi}$ real zeros in the circle and smaller than $1/2$ which coupled with the n zeros allow one to prove that

$$\log\frac{R^m}{|a_1a_2\cdots a_m|} \geq \log\frac{R^{n+n'}}{|a_1a_2\cdots a_{n+n'}|} \geq (n+n')\log r \geq 2\Big(n-1-\frac{NE}{2\pi}\Big)\log r,$$

where E is any upper bound for

$$|\Delta_+\arg L(s,\chi) + \Delta_-\arg L(s,\chi)|, \tag{A.7}$$

where $\Delta_\pm\arg$ denotes the change of the argument between the points $\frac{1}{2}\pm\delta+iT$, with $\delta:=\sigma_1-\frac{1}{2}$, and the point $\frac{1}{2}+iT$, proviso that

$$|\Delta_{\mathcal{C}_3}\arg L(s,\chi)^N| \geq 3\pi + NE. \tag{A.8}$$

An argument of Heath-Brown [23, Subsection 3.1] shows that the same conclusion holds also if $\sigma_1<a+R$ but without the assumption $1-\sigma_1>a-R$. As a consequence, for n (the number of zeros of $f(\sigma)$ in $[\frac{1}{2},\sigma_1]$) we have the bound

$$n \leq 1 + \frac{NE}{2\pi} + \frac{1}{4\pi\log r}\int_0^{2\pi}\log|f(a+Re^{i\phi})|\,\mathrm{d}\phi - \frac{1}{2\log r}\log|f(a)|, \tag{A.9}$$

when (A.8) holds. To bound the integral, we first use the inequality $|f(s)|\leq|L(s,\chi)|^N$. For $\phi\in[-\pi/2,\pi/2]$, we bound $L(s,\chi)$ with what we get from its representation as Dirichlet series on the half-circle $a+Re^{i\phi}$. Thus,

$$\begin{aligned}\frac{1}{N}\int_{-\pi/2}^{\pi/2}\log|f(a+Re^{i\phi})|\,\mathrm{d}\phi &\leq \frac{1}{N}\int_{-\pi/2}^{\pi/2}\log|L(a+iT+Re^{i\phi},\chi)^N|\,\mathrm{d}\phi \\ &\leq \int_{-\pi/2}^{\pi/2}\log(\zeta_{\mathbb{E}}(a+R\cos\phi))\,\mathrm{d}\phi \leq n_{\mathbb{E}}\int_{-\pi/2}^{\pi/2}\log(\zeta(a+R\cos\phi))\,\mathrm{d}\phi. \end{aligned}\tag{A.10}$$

For the remaining part of the domain, following [23, Subsection 4.1], we use Lindelöf's convexity bound [18] on the strip $p \le \sigma \le a$, where the negative parameter p has to satisfy both $p \ge -1/2$ to use [18], and $p \le a-R$ so that the left half-circle is included in the strip. In fact, by (2.5), (2.6), (2.7) and [18, Lemmas 1, 2] we get

$$|L(s,\chi)| = \Big(\frac{Q(\chi)}{\pi^{n_{\mathbb{E}}}}\Big)^{\frac12-\sigma}\Big|\frac{\Gamma\big(\frac{1-s}{2}\big)}{\Gamma\big(\frac{s}{2}\big)}\Big|^{a_\chi}\Big|\frac{\Gamma\big(\frac{2-s}{2}\big)}{\Gamma\big(\frac{1+s}{2}\big)}\Big|^{b_\chi}|L(1-s,\chi)|$$
$$\le \Big(\frac{Q(\chi)}{(2\pi)^{n_{\mathbb{E}}}}\Big)^{\frac12-\sigma}|1+s|^{(\frac12-\sigma)n_{\mathbb{E}}}|L(1-s,\chi)|$$

for $\sigma \in [-\frac12,\frac12]$. In particular, for $p \in [-\frac12, 0)$

$$|L(p+it,\chi)| \le \Big(\frac{q(\chi)|1+p+it|}{2\pi}\Big)^{(\frac12-p)n_{\mathbb{E}}}\zeta(1-p)^{n_{\mathbb{E}}}$$

and by [18, Th. 2] we conclude

$$|L(s,\chi)| \le \Big\{\Big(\frac{q(\chi)|1+s|}{2\pi}\Big)^{(1/2-p)(1+\eta-\sigma)}\zeta(1-p)^{1+\eta-\sigma}\zeta(1+\eta)^{\sigma-p}\Big\}^{n_{\mathbb{E}}/(1+\eta-p)},$$

valid for $p \le \sigma \le 1+\eta$ where $-\frac12 \le p < 0 < \eta \le \frac12$. We thus have

$$\frac{1}{N}\int_{\pi/2}^{3\pi/2}\log|f(a+Re^{i\phi})|\,\mathrm{d}\phi \le \frac{1}{N}\int_{\pi/2}^{3\pi/2}\log|L(a+iT+Re^{i\phi},\chi)^N|\,\mathrm{d}\phi$$
$$\le \frac{1-2p}{1+\eta-p}Rn_{\mathbb{E}}\log\Big(\frac{q(\chi)T}{2\pi}\Big)+\pi n_{\mathbb{E}}\log\zeta(1+\eta)+\frac{2Rn_{\mathbb{E}}}{1+\eta-p}\log\Big(\frac{\zeta(1-p)}{\zeta(1+\eta)}\Big)$$
$$+\frac{1/2-p}{1+\eta-p}Rn_{\mathbb{E}}\int_{\pi/2}^{3\pi/2}(-\cos\phi)\log\big(w(T,\phi,\eta,R)\big)\,\mathrm{d}\phi \tag{A.11}$$

where, as in [23, (4.8)] (but using R instead of r as the last argument of w)

$$w(T,\phi,\eta,R)^2 = 1+\frac{2R\sin\phi}{T}+\frac{R^2+(2+\eta)^2+2R(2+\eta)\cos\phi}{T^2}.$$

To bound this integral we use the elementary inequality $\log x \le \frac{x^2-1}{2}$, which applied to w produces a function which can be explicitly integrated. The resulting function is decreasing in T, so that it can be bounded with its value at T_0. With this method from (A.11) we get

$$\frac{1}{N}\int_{\pi/2}^{3\pi/2}\log|f(a+Re^{i\phi})|\,\mathrm{d}\phi \le \frac{1-2p}{1+\eta-p}Rn_{\mathbb{E}}\log\Big(\frac{q(\chi)T}{2\pi}\Big)+\pi n_{\mathbb{E}}\log\zeta(1+\eta)$$
$$+\frac{2R}{1+\eta-p}n_{\mathbb{E}}\log\Big(\frac{\zeta(1-p)}{\zeta(1+\eta)}\Big)+\frac{1/2-p}{1+\eta-p}Rn_{\mathbb{E}}\frac{2R^2+2(2+\eta)^2-\pi R(2+\eta)}{2T_0^2} \tag{A.12}$$

valid for all $T \ge T_0 \ge 1$, as long as $-1/2 \le p < 0 < \eta \le 1/2$, $p \le a-R$ and $\sigma_1 < a+R$. We still have to bound $-\log|f(a)|$ and for that we let N diverge along a sequence such that $N\arg L(a+iT,\chi)$ tends to 0 modulo 2π. In the limit we get $\lim\frac{1}{N}\log|f(a)| = \log|L(a+iT,\chi)|$. We use

$$\log|L(a+iT,\chi)| = \Big|\prod_{\mathfrak{p}}\big(1-\chi(p)\mathrm{N}\mathfrak{p}^{-a-iT}\big)^{-1}\Big| \ge \prod_{\mathfrak{p}}\big(1+\mathrm{N}\mathfrak{p}^{-a}\big)^{-1}$$
$$= \prod_p\prod_{j=1}^{g_p}\big(1+p^{-af_j}\big)^{-1} \ge \prod_p\big(1+p^{-a}\big)^{-n_{\mathbb{E}}} = \Big(\frac{\zeta(2a)}{\zeta(a)}\Big)^{n_{\mathbb{E}}}. \tag{A.13}$$

In order to compute a convenient bound for E in (A.7), we notice that the functional equation (2.7) shows that $\Delta_-\arg\xi(s,\chi)=-\Delta_+\arg\xi(s,\chi)$, and that $\Delta_\pm\arg(Q(\chi)\pi^{-n_{\mathbb{E}}})^{s/2}=0$, thus (A.7) equals

$$|\Delta_+\arg\Gamma_\chi(s)+\Delta_-\arg\Gamma_\chi(s)|.$$

Recalling the definition of Γ_χ and the bound in (A.1)–(A.2), this may be estimated by

$$a_\chi G(0,\delta,T)+b_\chi G(1,\delta,T)\le n_{\mathbb{E}}G(0,\delta,T)$$

where

$$\begin{aligned}G(\alpha,\delta,T):=&\frac{1}{2}\Big(\alpha-\frac{1}{2}+\delta\Big)\operatorname{atan}\Big(\frac{\alpha+\frac{1}{2}+\delta}{T}\Big)+\frac{1}{2}\Big(\alpha-\frac{1}{2}-\delta\Big)\operatorname{atan}\Big(\frac{\alpha+\frac{1}{2}-\delta}{T}\Big)\\&-\Big(\alpha-\frac{1}{2}\Big)\operatorname{atan}\Big(\frac{\alpha+\frac{1}{2}}{T}\Big)-\frac{T}{4}\log\Big(1+\frac{2\delta^2(T^2-(\frac{1}{2}+\alpha)^2)+\delta^4}{(T^2+(\frac{1}{2}+\alpha)^2)^2}\Big)\\&+\frac{1}{4}\Big(\frac{1}{|\frac{1}{2}+\delta+\alpha+iT|}+\frac{1}{|\frac{1}{2}-\delta+\alpha+iT|}+\frac{2}{|\frac{1}{2}+\alpha+iT|}\Big)\end{aligned}$$

and we have used the inequalities $0<G(1,\delta,T)\le G(0,\delta,T)$. Observing that $G(0,\delta,T)$ is decreasing in T for $T\ge1$, we have

$$|\Delta_+\arg L(s,\chi)+\Delta_-\arg L(s,\chi)|\le n_{\mathbb{E}}G(0,\delta,T_0) \tag{A.14}$$

for $T\ge T_0\ge1$. We thus let $E:=n_{\mathbb{E}}G(0,\delta,T_0)$.

In the final inequality (A.15) the coefficient of $\log(q(\chi)T)$ is $\frac{(1/2-p)R}{2\pi(1+\eta-p)\log r}$. It is minimal for $r=\frac{1+\eta-p}{1/2+\eta}$, hence this is the choice we make. We then have $R=1+\eta-p$, hence $a-R=p$ and $a+R=2+2\eta-p>\frac{1}{2}+\sqrt{2}(\frac{1}{2}+\eta)=\sigma_1$. From (A.6), (A.9), (A.10), (A.12), (A.13) and (A.14) we have, recalling that $r=\frac{1+\eta-p}{1/2+\eta}$,

$$\Big|\frac{N(T,\chi)}{n_{\mathbb{E}}}-\frac{T}{\pi}\log\Big(\frac{q(\chi)T}{2\pi e}\Big)+\frac{a_\chi-b_\chi}{4n_{\mathbb{E}}}\Big|\le C_1\log(q(\chi)T)+C_2' \tag{A.15}$$

with

$$C_1:=\frac{1/2-p}{\pi\log r} \tag{A.16}$$

and

$$\begin{aligned}C_2':=&\frac{2}{\pi}\Big(g(0,T_0)+\log\zeta\Big(\frac{1}{2}+\sqrt{2}\Big(\frac{1}{2}+\eta\Big)\Big)+\frac{1}{2}G\big(0,\sqrt{2}(\frac{1}{2}+\eta),T_0\big)\Big)\\&+\frac{1}{2\pi\log r}\int_{-\pi/2}^{\pi/2}\log(\zeta(a+(1+\eta-p)\cos\phi))\,\mathrm{d}\phi\\&+\frac{1/2-p}{4\pi T_0^2\log r}[2(1+\eta-p)^2+2(2+\eta)^2-\pi(1+\eta-p)(2+\eta)]\\&-\frac{1/2-p}{\pi\log r}\log(2\pi)+\frac{\log\zeta(1+\eta)}{2\log r}+\frac{\log\big(\frac{\zeta(1-p)}{\zeta(1+\eta)}\big)}{\pi\log r}+\frac{1}{\log r}\log\Big(\frac{\zeta(1+\eta)}{\zeta(2(1+\eta))}\Big)\end{aligned} \tag{A.17}$$

valid for $-1/2\le p<0<\eta\le1/2$ and $T\ge T_0\ge1$, and proviso that (A.8) holds. In case (A.8) is false, by (A.3), (A.4) and (the opposite of) (A.8) we still get (A.15) but with

$$C_1:=0, \tag{A.18}$$

$$C_2':=\frac{2}{\pi}\Big(g(0,T_0)+\log\zeta\Big(\frac{1}{2}+\sqrt{2}\Big(\frac{1}{2}+\eta\Big)\Big)\Big)+G\big(0,\sqrt{2}(\frac{1}{2}+\eta),T_0\big)\Big). \tag{A.19}$$

To obtain the values in Table 1, we observe that by (A.16) we have

$$\eta = \frac{1/2-p}{\exp((1/2-p)/(\pi C_1))-1} - \frac{1}{2}$$

for every given C_1 and $p \in [-\frac{1}{2}, 0)$.

Coming to the case where χ is trivial, we follow the proof of [23, Theorem 2] with the modifications we have made above, and we observe that $\Delta_{\mathcal{C}} s(s-1) = 2\pi$, which accounts for the $-2\delta_\chi$ in the main term of $N(T,\chi) = N_{\mathbb{L}}(T)$.

For the remaining terms, we observe that $g(T) := \operatorname{Im}\log\Gamma(1/2+iT) - T\log(T/e)$ and $g(0,T)$ both decrease to 0 as $T \to \infty$, and that $g(T) \leq g(0,T)$, hence we can use $D_1 := C_1$ and $D_2' := C_2'$.

Moreover using that $\log x \leq (x^2-1)/2$ to bound the integrals in the expression of D_3 of [23, (5.12)], we can use

$$D_3' := \frac{1}{\pi\log r}\log\Big(\frac{1-p}{1+p}\Big) + \frac{1}{\pi}F\big(\sqrt{2}(\tfrac{1}{2}+\eta), T_0\big) + \frac{\pi r^2(\frac{1}{2}+\eta)^2 - 4r(\frac{1}{2}+\eta) + \pi\eta^2 + 2\pi\eta + 2\pi}{2\pi T_0^2\log r}$$

where $F(\delta, T) := 2\operatorname{atan}\frac{1}{2T} - \operatorname{atan}\frac{1/2+\delta}{T} - \operatorname{atan}\frac{1/2-\delta}{T}$.

We use the formula given above for η in terms of $C_1 = D_1$ and p, we compute the values of $D_2' = C_2'$ for a suitable choice of p as given by (A.17) and we test that it is greater than the value produced by (A.19); an upper bound for D_2', a rounding of the computed value of η and the chosen value of p are indicated in the table below (the sequences of values of D_1 are the same in the three subtables and are those indicated in [23, Table 2], plus the two extremal values 0.230 and 0.460).

	$T_0 = 1$				$T_0 = 2\pi$			
D_1	D_2'	D_3'	η	p	D_2'	D_3'	η	p
0.230	16.577	1.330	0.00090	−0.00070	16.032	0.033	0.00090	−0.00070
0.247	8.180	1.435	0.03058	−0.05681	7.614	0.083	0.03111	−0.05542
0.265	6.416	1.515	0.05175	−0.14367	5.834	0.150	0.05390	−0.13792
0.282	5.409	1.598	0.06920	−0.23355	4.812	0.213	0.07236	−0.22490
0.299	4.696	1.699	0.08646	−0.32500	4.083	0.275	0.09004	−0.31500
0.316	4.158	1.814	0.10280	−0.42000	3.526	0.335	0.10982	−0.40000
0.333	3.735	1.961	0.12462	−0.50000	3.082	0.400	0.12808	−0.49000
0.350	3.425	2.185	0.17432	−0.50000	2.731	0.429	0.17432	−0.50000
0.367	3.206	2.426	0.22435	−0.50000	2.467	0.453	0.22435	−0.50000
0.384	3.043	2.687	0.27467	−0.50000	2.257	0.478	0.27467	−0.50000
0.401	2.918	2.966	0.32520	−0.50000	2.083	0.503	0.32520	−0.50000
0.460	2.666	4.082	0.50000	−0.50000	1.645	0.593	0.50000	−0.50000

	$T_0 = 10$			
D_1	D_2'	D_3'	η	p
0.230	16.004	0.014	0.00091	−0.00067
0.247	7.585	0.062	0.03164	−0.05404
0.265	5.805	0.129	0.05390	−0.13792
0.282	4.783	0.192	0.07236	−0.22490
0.299	4.053	0.254	0.09004	−0.31500
0.316	3.495	0.313	0.10982	−0.40000
0.333	3.050	0.371	0.13156	−0.48000
0.350	2.698	0.402	0.17432	−0.50000
0.367	2.432	0.423	0.22435	−0.50000
0.384	2.221	0.444	0.27467	−0.50000
0.401	2.044	0.465	0.32520	−0.50000
0.460	1.598	0.540	0.50000	−0.50000

□

TABLE 2: Constants for Lemma 3.9.

j	$a_j \cdot 10^7$	j	$a_j \cdot 10^7$
1	67441107	26	4711532246020032770961059850536842961
2	129064216397	27	−9979971210677326363399566081587309621
3	−33671827706277	28	19147233119732562826091118305794764779
4	4159437592468632	29	−33274047709559371113992775342599269485
5	−315432926321374242	30	52358220195286687433763798635287630555
6	16370077474919646336	31	−74548381119823637972378393085833994786
7	−620228745134606597597	32	95937426238030011573589993986867291432
8	17934517713943067903261	33	−111421834266414109909340554112526772452
9	−408973952667945326004549	34	116550516507798376160362309501875288819
10	7542955862267902755091933	35	−109525478172827789046963052436691334874
11	−114797714164799489558618807	36	92171825266689255311105390157515626975
12	1465278757842284478556905563	37	−69194087310394938615774929447065136471
13	−15896327170655789866055422304	38	46115594804031958286535245249055216023
14	148210358380111290581087608810	39	−27125577798003271571724298417346235682
15	−1198675077750183343628567667972	40	13979915122173412958783020578998059040
16	8475563352018452380288345356252	41	−6255803435136676900876694551147848415
17	−52742543205461653283881602845090	42	2402825607446165955037188420531530836
18	290485125582627204720553754700530	43	−780487429206171024872362699598861667
19	−1422762853575378758435389963062636	44	210196127819906522271561747433766713
20	6222222002869884071289885659404750	45	−45668115875651680795706979313599659
21	−24380706266315957556815280817594915	46	7690167072902618888205802917980935
22	85837343646704274150965262557412097	47	−941636162712117945732981144066824
23	−272183051338763525712916125735803989	48	74577580991057238830195411510057
24	778809192744980652056346184699871878	49	−2867250294949111291564065810976
25	−2013896299428527154913597515037117583		

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