# Second-order stochastic dominance for decomposable multiparametric families with applications to order statistics

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**Abstract**

We provide a simple method for deriving second-order stochastic dominance between multiparametric families which can be decomposed into a functional composition of two cumulative distributions and a quantile function. The method is applied to stochastic comparisons of order statistics.

*Keywords:* stochastic dominance, order statistics, generalized distribution, beta family

## 1. Introduction

The second order stochastic dominance (SSD) is probably the most widely used stochastic order in areas such as economics, finance, decision science and management. Yet, investigating dominance relations within multiparametric families of distributions is often complicated, owing to the many parameters or non-closed functional forms (Wilfling, 1996a,b; Kleiber, 1999; Sarabia et al., 2002; Belzunce et al., 2013; Ortobelli et al., 2016). To solve

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this problem, we observe that a wide class of multiparametric families can be decomposed into the functional composition of two cumulative distributions (CDFs) and a quantile function (QF). This approach is the inverse procedure of the T-X method (Alzaatreh et al., 2013; Aljarrah et al., 2014; Lee et al., 2013). We show that sufficient SSD conditions for a pair of decomposable multiparametric distributions can be derived straightforwardly by checking dominance conditions of the more manageable distributions that compose the models. We apply our method to the beta-generated (BG) family of Jones (2004), which may generate the generalized betas of the first and second kinds (GB1 and GB2) of McDonald (1984) (see also McDonald and Xu (1995)) —which are the main distributions for modelling size phenomena on bounded or unbounded support, respectively (Kleiber and Kotz, 2003)— and many others. Interestingly, the distribution of an order statistic of an i.i.d. sample from any underlying random variable (RV) belongs to the BG family. Hence, we use our results to derive SSD relations between order statistics of i.i.d. samples from the same or from different RVs.

## 2. Preliminaries

### *2.1. Stochastic orders*

In this paper, we consider absolutely continuous RVs with finite means. Let $U$ be an RV with CDF $F_U$ and probability density function (PDF) $f_U$. We recall the basic definitions of first order stochastic dominance (FSD) and SSD.

**Definition 1.** We say that $U_1$ dominates $U_2$ w.r.t. FSD and we write $U_1 \geq_1 U_2$ iff $F_{U_1}(u) \leq F_{U_2}(u), \forall u \in \mathbb{R}$.

**Definition 2.** We say that $U_1$ dominates $U_2$ w.r.t. SSD and we write $U_1 \geq_2 U_2$ iff $\int_{-\infty}^{u} F_{U_1}(t)dt \leq \int_{-\infty}^{u} F_{U_2}(t)dt, \forall u \in \mathbb{R}$.

When the integral condition of Definition 2 is difficult to verify, we may derive the SSD by checking whether the CDFs cross (at most) once (Hanoch and Levy, 1969, Theorem 3) or the PDFs cross (at most) twice (Shaked, 1982, Theorem 2.2) (see also Ramos et al. (2000, Theorem 2.2)). However, crossing verification is an issue for most multiparametric distributions, whose CDFs and PDFs are not easily tractable from a mathematical point of view.

### *2.2. The T-X family*

The T-X method, which was introduced by Alzaatreh et al. (2013), is based on the composition of the CDFs of two RVs, namely, $X$ and $T$, with a differentiable function, which we denote as $w$, that fulfils specified requirements (Lee et al., 2013). Aljarrah et al. (2014) define $w$ more practically as the QF of a third RV, namely, $Y$. This method, which is denoted as $T\text{-}X\{Y\}$, can be outlined as follows: given three RVs, namely, $X$, $Y$ and $T$, where the support of $T$ is included in that of $Y$, a new RV, namely, $Z$, is defined via the CDF

$$F_Z = F_T \circ Q_Y \circ F_X, \tag{1}$$

where $Q_Y$ is the QF of $Y$. In this formula, $F_T$ plays the role of a generator distribution and $F_X$ represents a baseline distribution. The support of $Z$ is included in that of $X$, whereas if $T$ and $Y$ have the same support, then $X$ and $Z$ have the same support. The composite function $h = F_T \circ Q_Y$ is a distortion function —which is defined as a non-decreasing function $h$ such that $h(0) = 0$ and $h(1) = 1$— of the baseline CDF $F_X$. Also, it is interesting to note that $T$ is a transformation of the RV $Z$, namely $T = Q_Y \circ F_X(Z)$. With the notation that was introduced by Aljarrah et al. (2014), we can specify and highlight the roles of the distributions within the composition: for example, gamma-normal{exponential(1)} represents that $T$ is a gamma distribution, $X$ is a normal distribution and $Y$ is a unit exponential distribution.

It should be stressed that a baseline CDF, $F_X$, can be simply transformed into a new CDF, $F_Z$, via the probability transformation $F_Z = F_V \circ F_X$, where $V$ is any RV defined on $[0, 1]$ (Jones, 2015, Family 4). The T-X family, obtained for $F_V = F_T \circ Q_Y$, can be considered as a rather enigmatic generalization of such an approach. In particular, different compositions in (1) may yield the same family, which is arguably a backward step when simply using the method to generate distributions. However, for technical reasons that will become clear in the next section, the T-X method is helpful in our context, as it may be used to decompose in an alternative way existing models of practical relevance, such as the BG family of Jones (2004), which includes the GB1, the GB2 and the generalized gamma of McDonald and Xu (1995).

## 3. Main result

The objective of this paper is to establish dominance relations between pairs of multiparametric distributions, namely, $F_{Z_1}$ and $F_{Z_2}$, which can be decomposed according to (1). For the sake of simplicity, we assume $Q_{Y_1} = Q_{Y_2} = Q_Y$ (in most applications in the literature, $Q_Y$ is not parametrised).

FSD can be derived straightforwardly: Let $F_{Z_i} = F_{T_i} \circ Q_Y \circ F_{X_i}$ for $i = 1, 2$. If $X_1 \geq_1 X_2$ and $T_1 \geq_1 T_2$, then $Z_1 \geq_1 Z_2$.

Regarding SSD, the following result holds:

**Theorem 1.** *Let $F_{Z_i} = F_{T_i} \circ Q_Y \circ F_{X_i}$, for $i = 1, 2$, where $Q_Y \circ F_{X_2}$ is* *convex. Let i) $X_1 \geq_1 X_2$ or ii) $X_1 \geq_2 X_2$ and $X_1, X_2$ belong to a location-scale family with support $\mathbb{R}$. Then $T_1 \geq_2 T_2$ implies $Z_1 \geq_2 Z_2$.*

*Proof.* The proof is based on the following argument. Let $\varepsilon_1, \ldots, \varepsilon_n$ and $c_1, \ldots, c_n$, where $c_1 \geq \cdots \geq c_n \geq 0$, be two sequences of real numbers. If $s_k = \sum_{j=1}^{k} \varepsilon_j \geq 0$, $\forall k = 1, \ldots, n$, then

$$\sum_{j=1}^{n} c_j \varepsilon_j = s_n c_n + \sum_{j=1}^{n-1} s_j \left(c_j - c_{j+1}\right) \geq 0. \tag{2}$$

i) $X_1 \geq_1 X_2$ implies $\int_{-\infty}^{z} F_{T_1} \circ Q_Y \circ F_{X_1}(t)\, dt \leq \int_{-\infty}^{z} F_{T_1} \circ Q_Y \circ F_{X_2}(t)\, dt, \forall z \in \mathbb{R}$ since $F_{T_1} \circ Q_Y$ is increasing. It is sufficient to prove that $W \geq_2 Z_2$, where $W$ is the RV with CDF $F_{T_1} \circ Q_Y \circ F_{X_2}$. Via a change of variables, we must show

$$\int_{-\infty}^{z} F_{T_1}(t)\, g(t)\, dt \leq \int_{-\infty}^{z} F_{T_2}(t)\, g(t)\, dt, \forall z \in \mathbb{R}, \tag{3}$$

where $g = \left(Q_{X_2} \circ F_Y\right)'$ is a decreasing function, by the assumed convexity of $Q_Y \circ F_{X_2}$. Let $g$ be a decreasing step function, that is, $g(t) = c_j$ for $a_{j-1} < t < a_j$, with a decreasing sequence of $c_j$ and an increasing sequence of $a_j$, $j = 1, \ldots, n$, $a_0 = -\infty$, $a_n = z$. By setting $\varepsilon_j = \int_{a_{j-1}}^{a_j} \left(F_{T_2}(t) - F_{T_1}(t)\right) dt$ in (2), we obtain $\sum_{j=1}^{n} c_j \varepsilon_j = \int_{-\infty}^{z} \left(F_{T_2}(t) - F_{T_1}(t)\right) h(t)\, dt \geq 0, \forall z \in \mathbb{R}$. Then, (3) holds for every decreasing $g$, because all decreasing functions can be approximated by decreasing step functions.

ii) $X_i$, $i = 1, 2$, have location and scale parameters $\mu_i$ and $\sigma_i$ such that $F_{X_i}(z) = F\left(\frac{z - \mu_i}{\sigma_i}\right), z \in \mathbb{R}$, where $F$ is a given CDF. For location-scale families, $X_1 \geq_1 X_2$ iff $E(X_1) \geq E(X_2)$ and $\sigma_1 = \sigma_2$, whereas $X_1 \geq_2 X_2$ iff $E(X_1) \geq E(X_2)$ and $\sigma_1 \leq \sigma_2$. Then, SSD follows from condition i) by setting $\sigma_1 = \sigma_2$. Nevertheless, $\mu_i$ and $\sigma_i$ are location and scale parameters also for $Z_i$ (namely, $F_{Z_i}(z; a_i, b_i, \mu_i, \sigma_i) = F_{Zi}(z\sigma_i + \mu_i; a_i, b_i, 0, 1)$). Thus, the condition $\sigma_1 = \sigma_2$ can be replaced by $\sigma_1 \leq \sigma_2$, which yields $X_1 \geq_2 X_2$

($E(X_1) \geq E(X_2)$ and $\sigma_1 \leq \sigma_2$ imply $X_1 \geq_2 X_2$ for Hanoch and Levy (1969, Theorem 3)). $\square$

Theorem 1 states that i) FSD (in the general case) or ii) SSD (for real-valued location-scale families) among baseline RVs and SSD among generators imply SSD for the generated model. As for ii), SSD conditions within a location-scale family (defined on $\mathbb{R}$) are especially simple. Let $X_i$, for $i = 1, 2$, have location and scale parameters $\mu_i = E(X_i)$ and $\sigma_i$, respectively. Then $\mu_1 \geq \mu_2$ and $\sigma_2 \geq \sigma_1$ imply $X_1 \geq_2 X_2$ for Hanoch and Levy (1969, Theorem 3).

Theorem 1 is useful since it is typically far simpler to compare the pairs $(F_{X_1}, F_{X_2})$ and $(F_{T_1}, F_{T_2})$ than the pair $(F_{Z_1}, F_{Z_2})$. The requirement of the convexity of $Q_Y \circ F_{X_2}$ depends on the choice of $Q_Y$. To simplify the notation, let $F_{X_2} = F$. If $Q_Y(p) = -\ln(1-p)$ is the QF of a unit exponential RV (namely, the T-X{exponential(1)} family of Alzaatreh et al. (2013), which generates many models) then we require convexity of $-\ln(1-F)$, which is equivalent to having an increasing failure rate (IFR); many well-known distributions satisfy this condition. Note that IFR distributions are of great interest in reliability theory (Shaked and Shanthikumar, 2007; Kochar, 2012). In the next section, we choose $Q_Y(p) = p/(1-p)$. With such decomposition we require convexity of $F/(1-F)$. Such a condition holds for all IFR distributions plus others, since convexity of $-\ln(1-F)$ implies convexity of $F/(1-F)$ (for instance, the Pareto and the log-logistic distributions are not IFR, but they satisfy convexity of $F/(1-F)$ iff they have finite means).

## 4. Comparisons of order statistics: an application

In this section, we derive SSD within the BG family of Jones (2004), which is obtained via the composition of the beta distribution with any CDF $F$. Such a family generates many relevant multiparametric models: for instance, we can obtain GB1 and GB2 by taking $F$ to be a power function or a log-logistic CDF, respectively. In particular, the distributions of order statistics of i.i.d. samples from any underlying distribution $F$ belong to the BG class. SSD conditions for the BG family can be derived easily from Theorem 1. This enables the comparison of order statistics in terms of SSD in various sampling scenarios. In reliability theory, stochastic comparisons of order statistics are particularly relevant (Shaked and Shanthikumar, 2007; Kochar, 2012; Kundu and Chowdhury, 2016). Order statistics may represent the waiting time until fewer than $k$ components remain functioning in a system of $n$ components. Thus, engineering is concerned with maximizing the mean life while also reducing the variability since predictable life length is desirable. In the literature, several works deal with this issue using the Lorenz order (LO) (Arnold and Villaseñor, 1991; Wilfling, 1996b; Kochar, 2006, 2012); however, we argue that this scenario is even more suitable for SSD, which considers both the variability and the size (the LO is a size-independent version of SSD for non-negative RVs).

### *4.1. Beta-generated family*

Let $beta(p, q)$ denote the beta distribution with shape parameters $p, q > 0$. Starting from a baseline RV, $X$, and a generator RV, $B \sim beta(p, q)$, $Z$ has a BG distribution if its CDF can be expressed as $F_Z = F_B \circ F_X$. Via our approach, the BG model can be decomposed trivially by using the beta distribution as the generator $T$ and the uniform distribution on $[0, 1]$ for the QF, thereby giving rise to the beta–X {uniform[0,1]} family. However, since the QF of the uniform distribution is the identity function, to apply Theorem 1 we would require the convexity of $Q_Y{}^\circ F_X = F_X$, which is a highly restrictive condition. Thus, in Theorem 2 below we use an alternative T–X{Y} decomposition with $Q_Y(p) = p/(1-p)$, thereby rendering it possible to consider the BG as a B2–X{log-logistic(1,1)}, where B2 denotes the beta distribution of the second kind (Kleiber and Kotz, 2003), defined via the CDF $F_T = F_B \circ F_Y$ $(T = \frac{B}{1-B})$. We show that SSD conditions can be derived easily if $Q_Y{}^\circ F_X = \frac{F_X}{1-F_X}$, namely, the odds (in favour) of $F_X$, is convex.

**Theorem 2.** *For $i = 1,2$, let $F_{Z_i} = F_{B_i} \circ F_{X_i}$, where $B_i \sim beta(p_i, q_i)$ $(q_i > 1)$. Let $F_{X_2}/(1-F_{X_2})$ be convex. Let i) $X_1 {\geq_1} X_2$ or ii) $X_1, X_2$ belong to a location-scale family with support $\mathbb{R}$ and $X_1 {\geq_2} X_2$. Then $p_1 {\geq} p_2$ and $\frac{p_1}{q_1-1} {\geq} \frac{p_2}{q_2-1}$ imply $Z_1 {\geq_2} Z_2$.*

*Proof.* For $i = 1,2$, $F_{Z_i}$ can be decomposed via the T-X method as expressed in (1), where $T_i$ is a B2 with parameters $p_i$ and $q_i$ and PDF $f_{T_i}(t; p_i, q_i) = B(p_i, q_i)^{-1} t^{p_i - 1}(1+t)^{-p_i - q_i}$ $(t, p_i, q_i > 0)$ and $Y$ has QF $Q_Y(u) = u/(1-u)$. According to Ramos et al. (2000), it is sufficient to study the function $r(t) = \left(\frac{f_{T_1}(t)}{f_{T_2}(t)}\right)' = a(t)(c + dt)$, where $a(t) > 0$ for every $t, p_i, q_i > 0$, $i = 1,2$, $c = p_1 - p_2$ and $d = q_2 - q_1$. If $cd \geq 0$, then $r$ is a (strictly) monotone function, whereas if $cd < 0$, then $r$ is unimodal. If $p_1 \geq p_2$ and $q_2 \geq q_1$, then $T_1 \geq_1 T_2$ (implying $T_1 \geq_2 T_2$), because $r$ is increasing. If $p_1 > p_2$ and $q_1 > q_2$, then the mode is a maximum and we can apply Theorem 2.2 of Ramos et al. (2000), recalling that $E(X_i) = \frac{p_i}{q_i - 1}$, if $q_i > 1$. Finally we obtain:

$$p_1 \geq p_2 \text{ and } \frac{p_1}{q_1-1} \geq \frac{p_2}{q_2-1} \text{ (with } q_1, q_2 > 1) \text{ implies } T_1 \geq_2 T_2.$$

Then, the thesis follows from Theorem 1. □

Table 1: SSD sufficient conditions for various BG families satisfying $p_1 \geq p_2$ and $\frac{p_1}{q_1-1} \geq \frac{p_2}{q_2-1}$.

| **BG family** | ***X*** | $\boldsymbol{F_X}$ | $\boldsymbol{F_X/(1-F_X)}$ | **Conditions** |
|---|---|---|---|---|
| GB1 | power function | $(x/b)^a, a, b > 0, x \in [0, b]$ | convex, $\forall a, b$ | $a_1 \geq a_2,\ b_1 \geq b_2$ |
| GB2 | log-logistic | $\frac{1}{1+(x/b)^{-a}}, a, b, x > 0$ | convex, $\forall a > 1, b$ | $a_2 \geq a_1 > 1,\ b_1 \geq b_2$ |
| BG-Pareto | Pareto | $1-\left(\frac{b}{x}\right)^a, a, b > 0, x > b$ | convex, $\forall a > 1, b$ | $a_2 \geq a_1 > 1, b_1 \geq b_2$ |
| BG-uniform | uniform | $\frac{x-a}{b-a},\ b > a, x \in [a, b]$ | convex, $\forall a, b$ | $a_1 \geq a_2,\ b_1 \geq b_2$ |
| BG-normal | normal | $\frac{1}{2}\text{erf}\left((\mu - x)/\sqrt{2}\sigma\right)$, $\sigma > 0,\ x \in \mathbb{R}$ | convex, $\forall \mu, \sigma$ | $\mu_1 \geq \mu_2, \sigma_1 \leq \sigma_2$ |
| BG-logistic | logistic | $\frac{1}{1+\exp\left(\frac{\mu-x}{\sigma}\right)}$, $\sigma > 0, x \in \mathbb{R}$ | convex, $\forall \mu, \sigma$ | $\mu_1 \geq \mu_2, \sigma_1 \leq \sigma_2$ |

In Table 1, we describe several BG families according to the baseline CDF, namely, $F_X$, and we specify the conditions under which $F_X/(1-F_X)$ is convex. SSD conditions for such families can be obtained by combining items in the "conditions" column with those for the B2 generator distributions, namely, $p_1 \geq p_2$ and $\frac{p_1}{q_1-1} \geq \frac{p_2}{q_2-1}$.

*4.2. Relations among order statistics: sampling from different populations*

Let $X_1, \ldots, X_n$ denote a sample of i.i.d. RVs from an RV $X$ and let $Y_1, \ldots, Y_m$ denote a sample of i.i.d. RVs from another RV $Y$. Then, the CDFs of $X_{i:n}$ and $Y_{j:m}$ are $F_{X_{i:n}} = F_{B_1} \circ F_X$ and $F_{X_{j:m}} = F_{B_2} \circ F_Y$, where $B_1 \sim beta(i, n-i+1)$ and $B_2 \sim beta(j, m-j+1)$. The following theorem enables the determination of the sample sizes $n$ and $m$ and the ranks $i$ and $j$

such that $X_{i:n} \geq_2 Y_{j:m}$.

**Theorem 3.** *Let* $X$ *and* $Y$ *be RVs such that* $F_Y/(1-F_Y)$ *is convex. Let i)* $X \geq_1 Y$ *or ii)* $X, Y$ *belong to a location-scale family with support* $\mathbb{R}$ *and* $X \geq_2 Y$*. Then* $i \geq j$ *and* $\frac{i}{n} \geq \frac{j}{m}$ *imply* $X_{i:n} \geq_2 Y_{j:m}$.

*Proof.* The thesis follows from Theorem 2 with $p_1 = i, p_2 = j, q_1 = n-i+1$ and $q_2 = m-j+1$. The system of inequalities $i \geq j$ and $\frac{i}{n-i} \geq \frac{j}{m-j}$ can be reduced to $i \geq j$ and $\frac{i}{n} \geq \frac{j}{m}$. □

*Sampling from normal distributions: an example*

Let $X \sim N(\mu_1, \sigma_1)$ and $Y \sim N(\mu_2, \sigma_2)$. Then, $X_{i:n}$ and $Y_{j:m}$ have BG-normal distributions. According to Table 1, if $\mu_1 \geq \mu_2$ and $\sigma_1 \leq \sigma_2$, we can apply Theorem 3. For instance, let $n = 35$, $m = 30$ and $j = 20$. The minimum rank $i$ such that $X_{i:35} \geq_2 Y_{20:30}$ is given by $i = \left\lceil \max\{j, \frac{nj}{m}\} \right\rceil = 24$, where $\lceil \bullet \rceil$ denotes the ceiling function ($X_{i:35} \geq_2 Y_{20:30}$ for $i \geq 24$).

### *4.3. Relations among order statistics: sampling from the same population*

Since the condition $X \geq_1 X$ always holds, interesting properties can be derived easily as a corollary of Theorem 3.

**Corollary 1.** *Let* $X$ *be an RV such that the odds function* $F_X/(1-F_X)$ *is convex. Then:*

1. $X_{i+1:n} \geq_2 X_{i:n}, \forall i, n$ $(i = 1, \ldots, n-1)$.
2. $X_{i:n} \geq_2 X_{i:n+1}, \forall i, n$.

The results of Corollary 1 extend to the SSD case and to a larger class of distributions those that were obtained for the LO by Arnold and Villaseñor (1991); Wilfling (1996b) for uniform, power function and Pareto distributions. The interpretation of Corollary 1 is as follows: 1) larger order statistics in a sample dominate smaller ones within the same sample, 2) order statistics from larger samples dominate order statistics (with the same rank) from smaller samples.

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