
Discrepancy for convex bodies with isolated flat points

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Abstract. We consider the discrepancy of the integer lattice with respect to the collection of all translated copies of a dilated convex body having a finite number of flat, possibly non-smooth, points in its boundary. We estimate the L^p norm of the discrepancy with respect to the translation variable, as the dilation parameter goes to infinity. If there is a single flat point with normal in a rational direction we obtain, for certain values of p , an asymptotic expansion for this norm. Anomalies may appear when two flat points have opposite normals. Our proofs depend on careful estimates for the Fourier transform of the characteristic function of the convex body.

1. Introduction

Let B be a convex body in \mathbb{R}^d , that is a convex bounded set with nonempty interior, and for every $R > 1$ and $\mathbf{z} \in \mathbb{R}^d$ let

$$D_R(\mathbf{z}) = -R^d |B| + \sum_{\mathbf{m} \in \mathbb{Z}^d} \chi_{RB}(\mathbf{z} + \mathbf{m})$$

be the discrepancy between the number of integer points inside a dilated and translated copy of B and its volume. The function $\mathbf{z} \mapsto D_R(\mathbf{z})$ is periodic and a straightforward computation shows that it has the Fourier expansion

$$(1.1) \quad \sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} R^d \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}}$$

where $\widehat{\chi}_B(\boldsymbol{\zeta})$ denotes the Fourier transform of $\chi_B(\mathbf{z})$, that is

$$\widehat{\chi}_B(\boldsymbol{\zeta}) = \int_B e^{-2\pi i \boldsymbol{\zeta} \cdot \mathbf{z}} d\mathbf{z}.$$

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The size of $D_R(\mathbf{z})$ as $R \rightarrow +\infty$ is therefore closely connected to the decay of $\widehat{\chi}_B(\boldsymbol{\zeta})$ as $|\boldsymbol{\zeta}| \rightarrow +\infty$. For example, if the boundary of B is smooth and has everywhere positive Gaussian curvature then $\widehat{\chi}_B(\boldsymbol{\zeta})$ has the decay

$$(1.2) \quad |\widehat{\chi}_B(\boldsymbol{\zeta})| \leq c |\boldsymbol{\zeta}|^{-\frac{d+1}{2}}$$

(see [29, Chapter 8]), and it can be shown that this rate of decay is optimal. Under the assumption (1.2), in [2, Corollary 3] the authors proved the following estimates for the L^p norm of the discrepancy function

$$(1.3) \quad \left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} \leq \begin{cases} cR^{\frac{d-1}{2}} & 1 \leq p < 2d/(d-1), \\ cR^{\frac{d-1}{2}} \log^{\frac{d-1}{2d}}(R) & p = 2d/(d-1), \\ cR^{\frac{d(d-1)}{d+1} \left(1 - \frac{1}{p}\right)} & p > 2d/(d-1). \end{cases}$$

In [2, Theorem 5] it has also been shown that the above estimates are sharp in the range $1 \leq p < 2d/(d-1)$. More precisely, using the asymptotic expansion for $\widehat{\chi}_B(\boldsymbol{\zeta})$, it has been proved that when B is not symmetric about a point or $d \not\equiv 1 \pmod{4}$ one has, for every $p \geq 1$,

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} \geq cR^{\frac{d-1}{2}}.$$

On the other hand, when B is symmetric about a point and $d \equiv 1 \pmod{4}$,

$$\begin{aligned} \limsup_{R \rightarrow +\infty} R^{-\frac{d-1}{2}} \left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} &> 0 \quad \text{for every } p \geq 1, \\ \liminf_{R \rightarrow +\infty} R^{-\frac{d-1}{2}} \left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} &= 0 \quad \text{for every } p < \frac{2d}{d-1}. \end{aligned}$$

Up to now we have considered the case of positive Gaussian curvature.

When the Gaussian curvature of the boundary of B vanishes at some point the estimate (1.2) fails and the rate of decay depends on the direction.

More precisely the decay of the Fourier transform (1.2) holds in a given direction Θ if the Gaussian curvature does not vanish at the points on the boundary of B where the normal is $\pm\Theta$. When the curvature vanishes the rate of decay of $\widehat{\chi}_B(\rho\Theta)$ can be significantly smaller. We will see that in this case the behavior of the L^p norms of the discrepancy function may differ from the case of positive Gaussian curvature.

To the authors' knowledge the discrepancy for convex bodies with vanishing Gaussian curvature has been considered only for specific classes of convex bodies and only for L^∞ and L^2 estimates. See e.g. [9], [14], [15], [16], [21], [22], [23], [27], [28]. See for example [22] for sharp estimate of the L^2 discrepancy associated to the curve $x^2 + y^4 = 1$. See also [7] for a lower bound in terms of irregularities of distribution.

Throughout the paper we will use bold symbols only for d -dimensional points and non-bold symbol for lower dimensional points. Moreover when we write a point $\mathbf{z} = (x, t)$ or $\zeta = (\xi, s)$ we agree that $x, \xi \in \mathbb{R}^{d-1}$ and $t, s \in \mathbb{R}$.

We are happy to thank Gabriele Bianchi for some interesting remarks on the geometric properties of the convex bodies considered in this paper (see [1]).

2. Statements of the results

In this paper we study the L^p norms of the discrepancy function associated to a convex body whose boundary has a finite number of isolated flat points. The relevant example is a convex body B such that ∂B has everywhere positive Gaussian curvature except at the origin and such that, in a neighborhood of the origin, ∂B is the graph of the function $t = |x|^\gamma$, with $x \in \mathbb{R}^{d-1}$ and some $\gamma > 2$. This function is smooth at the origin only when γ is a positive even integer, and the geometric control of the Fourier transform in [8] does not apply directly.

We are actually interested in a larger class of convex bodies and this is why we introduce the following definition.

Definition 1. Let U be a bounded open neighborhood of the origin in \mathbb{R}^{d-1} , let $\Phi \in C^\infty(U \setminus \{0\})$ and let $\gamma > 1$. For every $x \in U \setminus \{0\}$ let $\mu_1(x), \dots, \mu_{d-1}(x)$ be the eigenvalues of the Hessian matrix of Φ . We say that $\Phi \in S_\gamma(U)$ if for $j = 1, \dots, d-1$,

$$(2.1) \quad 0 < \inf_{x \in U \setminus \{0\}} |x|^{2-\gamma} \mu_j(x)$$

and, for every multi-index α ,

$$(2.2) \quad \sup_{x \in U \setminus \{0\}} |x|^{|\alpha|-\gamma} \left| \frac{\partial^{|\alpha|} \Phi}{\partial x^\alpha}(x) \right| < +\infty.$$

Observe that if $\Phi \in S_\gamma(U)$ then for some $c_1, c_2 > 0$,

$$c_1 |x|^{\gamma-2} \leq \mu_j(x) \leq c_2 |x|^{\gamma-2}.$$

Moreover, since $\gamma > 1$, we have $\Phi \in C^1(U)$.

Definition 2. Let B be a convex body in \mathbb{R}^d and let $\mathbf{z} \in \partial B$ and let $\gamma > 1$. We say that \mathbf{z} is an isolated flat point of order γ if, in a neighborhood of \mathbf{z} and in a suitable Cartesian coordinate system with the origin in \mathbf{z} , ∂B is the graph of a function $\Phi \in S_\gamma(U)$, as in the previous definition.

Convex bodies with flat points can be easily constructed by taking powers of strictly convex functions.

Proposition 3. Let U be a bounded open neighborhood of the origin in \mathbb{R}^{d-1} , let $H \in C^\infty(U)$ such that $H(0) = 0$, $\nabla H(x) = 0$ and assume that its Hessian matrix is positive definite at the origin. Let $\gamma > 1$. Then the function $\Phi(x) = [H(x)]^{\gamma/2} \in S_\gamma(U)$.

We have already observed that some of the results in this paper for the singularity $|x|^\gamma$ with γ even integer are not new. However observe that $|x|^{2n}$ is analytic, while the above definition does not imply that the boundary is smooth. For example, in dimension 2 consider a singularity of the kind $\Phi''(x) = 2 + \sin(\log(|x|))$.

Interestingly, in the following Proposition 11 concerning the decay of the Fourier transform of a convex body with a flat point of order γ , the case $\gamma = 2$ with non smooth flat points requires some extra care.

The discrepancy for convex bodies with flat points in the above class is described by the following theorem.

Theorem 4. *Let B be a bounded convex body in \mathbb{R}^d . Assume that ∂B is smooth with everywhere positive Gaussian curvature except for a finite number of isolated flat points of order at most γ .*

1) For $1 < \gamma \leq 2$ we have

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} \leq \begin{cases} cR^{\frac{d-1}{2}} & 1 \leq p < 2d/(d-1), \\ cR^{\frac{d-1}{2}} \log^{\frac{d-1}{2d}}(R) & p = 2d/(d-1), \\ cR^{\frac{d(d-1)}{d+1}(1-\frac{1}{p})} & p > 2d/(d-1). \end{cases}$$

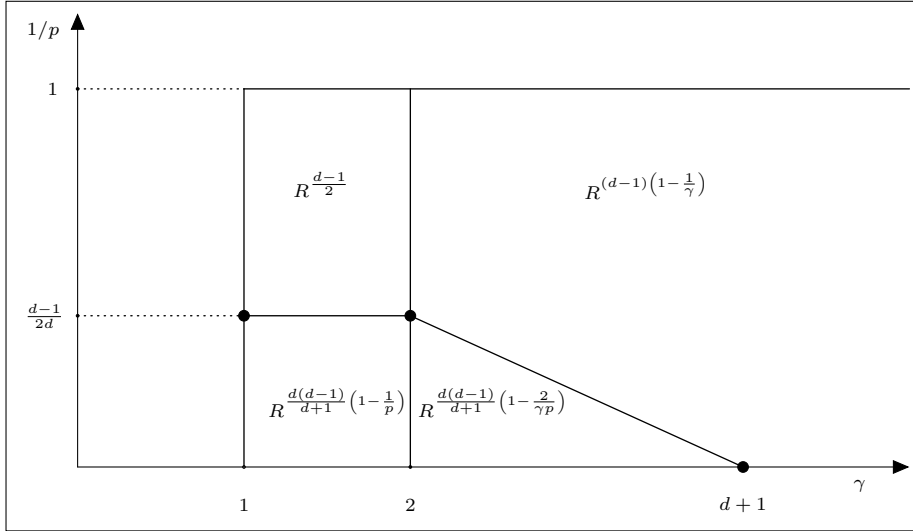
2) For $2 < \gamma \leq d+1$ we have

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} \leq \begin{cases} cR^{(d-1)(1-\frac{1}{\gamma})} & 1 \leq p \leq (2d)/(d+1-\gamma) \\ cR^{\frac{d(d-1)}{d+1}(1-\frac{2}{\gamma p})} & p > (2d)/(d+1-\gamma) \end{cases}$$

3) For $\gamma > d+1$ and every $p \geq 1$ we have

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} \leq cR^{(d-1)(1-\frac{1}{\gamma})}.$$

The picture summarizes our estimates for the discrepancy.



The estimates in point 1) are the same as in [2] for the case of positive Gaussian curvature and are independent of γ . On the contrary, as we will see from the proof, in the cases 2) and 3) the flat points give the main contribution. In the case 2) if $p = +\infty$ then

$$\frac{d(d-1)}{d+1} \left(1 - \frac{2}{p\gamma}\right) = \frac{d(d-1)}{d+1}.$$

Hence, when $\gamma \leq d+1$ the estimates for the L^∞ discrepancy of B match Landau's estimates for the L^∞ discrepancy of the ball. See e.g. [25]. For $p \geq 2d/(d-1)$ the above result extends a theorem of Colin de Verdiere [9].

The proof of the above theorem is inspired by the classical works of Kendall [22] and Hlakwa [18], and relies on the study of the Fourier series of the discrepancy which, by the Poisson summation formula, is related to the Fourier transform $\widehat{\chi}_B(\zeta)$. The decay of $\widehat{\chi}_B(\zeta)$ is of order of $|\zeta|^{-(d+1)/2}$ along generic directions, but it is of order of $|\zeta|^{-1-(d-1)/\gamma}$ along singular directions normal to flat points. See Proposition 11 below. One can split the Fourier series of the discrepancy into two series $S = S_{generic} + S_{singular}$ according to these two different decays. Roughly speaking $S_{generic}$ has many terms, but these terms are small, while $S_{singular}$ has few terms, but these terms are larger. For every p and γ the contribution of $S_{singular}$ is of order $R^{(d-1)(1-\frac{1}{\gamma})}$, and this is sharp. The contribution of $S_{generic}$ depends on p and γ and we are able to show that our estimates are sharp only in a restricted range of p . In particular, when $2 < \gamma \leq d+1$ and $1 \leq p \leq (2d)/(d+1-\gamma)$ or $\gamma > d+1$ and $p \geq 1$, the contribution of the singular part is larger than the contribution of the generic part and we obtain sharp estimates of the norm of the discrepancy. On the other hand for convex bodies with smooth boundary of positive Gaussian curvature, sharp estimates for the supremum of the discrepancy are not known, except for specific bodies and dimensions.

In the next theorem we consider convex bodies with a flat point with normal pointing in a rational direction. In this case, some of the previous estimates can be improved to asymptotic estimates.

Theorem 5. *Let B be a bounded convex body in \mathbb{R}^d . Assume that ∂B is smooth with everywhere positive Gaussian curvature except at most at two points P and Q with outward unit normals $-\Theta$ and Θ which are flat of order γ_P and γ_Q respectively. Let*

$$S(t) = |\{z \in B : z \cdot \Theta = t\}|$$

be $(d-1)$ -dimensional measure of the slices of B that are orthogonal to Θ . The function $S(t)$ is supported in $P \cdot \Theta \leq t \leq Q \cdot \Theta$ and is smooth in $P \cdot \Theta < t < Q \cdot \Theta$. Assume that there exist two smooth functions $G_P(r)$ and $G_Q(r)$ with $G_P(0) \neq 0$ and $G_Q(0) \neq 0$ such that, for $u \geq 0$ sufficiently small

$$S(P \cdot \Theta + u) = u^{\frac{d-1}{\gamma_P}} G_P(u^{1/\gamma_P})$$

and

$$S(Q \cdot \Theta - u) = u^{\frac{d-1}{\gamma_Q}} G_Q(u^{1/\gamma_Q}).$$

Finally, assume that the direction Θ is rational, that is $\alpha\Theta \in \mathbb{Z}^d$ for some α , and denote by \mathbf{m}_0 the first non-zero integer point in the direction Θ . Define

$$A_P(\mathbf{z}) = \frac{2G_P(0)\Gamma\left(\frac{d-1}{\gamma_P}+1\right)}{(2\pi|\mathbf{m}_0|)^{\frac{d-1}{\gamma_P}+1}} \sum_{k=1}^{+\infty} k^{-1-\frac{d-1}{\gamma_P}} \sin\left(2\pi k\mathbf{m}_0 \cdot \mathbf{z} - \frac{\pi d-1}{2\gamma_P}\right),$$

and

$$A_Q(\mathbf{z}) = -\frac{2G_Q(0)\Gamma\left(\frac{d-1}{\gamma_Q}+1\right)}{(2\pi|\mathbf{m}_0|)^{\frac{d-1}{\gamma_Q}+1}} \sum_{k=1}^{+\infty} k^{-1-\frac{d-1}{\gamma_Q}} \sin\left(2\pi k\mathbf{m}_0 \cdot \mathbf{z} + \frac{\pi d-1}{2\gamma_Q}\right).$$

1) Let $\gamma_P > \gamma_Q \geq 2$ and assume that one of the two alternatives holds:

$$2 < \gamma_P \leq d+1 \text{ and } p < (2d)/(d+1-\gamma_P),$$

or

$$\gamma_P > d+1 \text{ and } p \leq +\infty.$$

Then there exist constants $\delta > 0$ and $c > 0$ such that for every $R \geq 1$,

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z}) - R^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z}-RP)|^p d\mathbf{z}\right)^{1/p} \leq cR^{(d-1)(1-1/\gamma_P)-\delta}.$$

In particular, as $R \rightarrow +\infty$, we have the following asymptotic

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z}\right)^{1/p} \sim R^{(d-1)(1-1/\gamma_P)} \left(\int_{\mathbb{T}^d} |A_P(\mathbf{z})|^p d\mathbf{z}\right)^{1/p}.$$

2) Let $\gamma_P = \gamma_Q = \gamma$ and assume that one of the two alternatives holds:

$$2 < \gamma \leq d+1 \text{ and } p < (2d)/(d+1-\gamma),$$

or

$$\gamma > d+1 \text{ and } p \leq +\infty.$$

Then there exist constants $\delta > 0$ and $c > 0$ such that for every $R \geq 1$,

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z}) - R^{(d-1)(1-1/\gamma)} (A_P(\mathbf{z}-RP) + A_Q(\mathbf{z}-RQ))|^p d\mathbf{z}\right)^{1/p} \leq cR^{(d-1)(1-1/\gamma)-\delta}.$$

Note that the series that define $A_P(\mathbf{z})$ and $A_Q(\mathbf{z})$ converge uniformly and absolutely. In particular these functions are bounded and continuous.

Observe that the asymptotic estimate of point 1) includes the case of a single flat point, that is $\gamma_P > \gamma_Q = 2$. In point 2) it is not excluded that for particular values of $P, Q, G_P(0), G_Q(0)$ and R , the terms $A_P(\mathbf{z}-RP)$ and $A_Q(\mathbf{z}-RQ)$ may cancel each other and the discrepancy gets smaller.

Corollary 6. *Under the assumptions in point 2 in the previous theorem assume furthermore that $G_P(0) = G_Q(0)$ and that $(d-1)/\gamma$ is an even integer. Then for every R such that $R\mathbf{m}_0 \cdot (P-Q)$ is an integer we have*

$$A_P(\mathbf{z}-RP) + A_Q(\mathbf{z}-RQ) = 0.$$

In particular with this choice of the parameters

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} \leq cR^{(d-1)(1-1/\gamma)-\delta}.$$

The case $\gamma = 2$ is not covered by the above corollary, however observe that for $\gamma = (d-1)/(2k) = 2$, that is $d \equiv 1 \pmod{4}$, and $p < 2d/(d-1)$, one formally would obtain

$$\liminf_{R \rightarrow +\infty} \left\{ R^{\frac{d-1}{2}} \left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p d\mathbf{z} \right)^{1/p} \right\} = 0.$$

Actually this is true, even if the proof is more delicate. The case of a ball and $p = 2$ has been proved by L. Parnowski and A. Sobolev in [24]. Moreover, in [2] it is shown that this phenomenon also occurs for convex smooth domains with positive Gaussian curvature and $p < 2d/(d-1)$ if and only if the domains are symmetric and $d \equiv 1 \pmod{4}$.

As remarked by Kendall the above L^p estimates for the discrepancy can be turned into almost everywhere pointwise estimates using a Borel-Cantelli type argument. See [22, §3] for the proof.

Proposition 7. *Assume that for some $\beta > 0$*

$$\left(\int_{\mathbb{T}^d} |D_R(\mathbf{z})|^p \right)^{1/p} \leq \kappa R^\beta,$$

let $\lambda(t)$ be an increasing function and let $R_n \rightarrow +\infty$ such that

$$\sum_{n=1}^{+\infty} \lambda(R_n)^{-p} < +\infty.$$

Then for almost every $\mathbf{z} \in \mathbb{T}^d$ there exists $c > 0$ such that

$$|D_{R_n}(\mathbf{z})| < cR_n^\beta \lambda(R_n).$$

If the flat points on the boundary of the domain B have “irrational” normals then the discrepancy can be smaller than the one described in the above theorems. In particular, we have the following result that applies to every convex body, without curvature or smoothness assumption.

Theorem 8. *Let B be a bounded convex body in \mathbb{R}^d and for $\sigma \in SO(d)$ denote by $D_{R,\sigma}$ the discrepancy associated to the rotated body σB . Then we have the following mixed norm inequalities.*

1) If $1 \leq p \leq 2$, we have

$$\left(\int_{SO(d)} \left(\int_{\mathbb{T}^d} |D_{R,\sigma}(\mathbf{z})|^p d\mathbf{z} \right)^{2/p} d\sigma \right)^{1/2} \leq cR^{\frac{d-1}{2}}.$$

2) If $2 \leq p < 2d/(d-1)$, we have

$$\left(\int_{SO(d)} \left(\int_{\mathbb{T}^d} |D_{R,\sigma}(\mathbf{z})|^p d\mathbf{z} \right)^{1/(p-1)} d\sigma \right)^{\frac{p-1}{p}} \leq cR^{\frac{d-1}{2}}.$$

For the planar case $d = 2$ we can state a slightly more precise result.

Theorem 9. *Let B be a bounded convex body in \mathbb{R}^2 . Assume that ∂B is smooth with everywhere positive curvature except a single flat point of order $\gamma > 2$. Let (α, β) be the unit outward normal at the flat point and assume the Diophantine property that for some $\delta < 2/(\gamma - 2)$ there exists $c > 0$ such that for every $n \in \mathbb{Z}$*

$$\left\| n \frac{\alpha}{\beta} \right\| \geq \frac{c}{|n|^{1+\delta}}.$$

Here $\|x\|$ denotes the distance of x from the closest integer. Then

$$\left(\int_{\mathbb{T}^2} |D_R(\mathbf{z})|^2 d\mathbf{z} \right)^{1/2} \leq cR^{\frac{1}{2}}.$$

By a classical result of Jarnik (see e.g. [12, §10.3]) the set of real numbers ω that are $(2 + \delta)$ -well approximable, that is

$$\|n\omega\| \leq n^{1-(2+\delta)} = n^{-1-\delta}$$

for infinitely many n , has Hausdorff dimension $\frac{2}{2+\delta}$. In particular the exceptional set in the above theorem, where the discrepancy may be larger than $R^{1/2}$ has Hausdorff dimension at most $\frac{\gamma-2}{\gamma-1}$.

3. Estimates for the Fourier transforms

The main ingredient in the proof of our results on the discrepancy comes from suitable estimates of the decay of $\widehat{\chi}_B(\zeta)$. We start studying a family of oscillatory integrals.

As usual we write d -dimensional points through the notation $\mathbf{z} = (x, t)$ and $\zeta = (\xi, s)$ (see the Introduction).

Lemma 10. *Let $U \subset \mathbb{R}^{d-1}$ be an open ball about the origin of radius b , let $\Phi \in S_\gamma(U)$ for some $\gamma > 1$, let ψ be a smooth function supported in $\{0 < a \leq |x| \leq b\}$, for every positive integer k let*

$$\Phi_k(x) = 2^{k\gamma} \Phi(2^{-k}x)$$

and let

$$I_k(\xi, s) = \int_{\mathbb{R}^{d-1}} (\nabla \Phi(2^{-k}x), -1) e^{-2\pi i(\xi, s) \cdot (x, \Phi_k(x))} \psi(x) dx.$$

Then there exist constants $c, c_1, c_2 > 0$ and, for every $M > 0$, a constant c_M such that for every $k \geq 0$

$$|I_k(\xi, s)| \leq \begin{cases} c(1 + |s| + |\xi|)^{-\frac{d-1}{2}} & \text{for every } (\xi, s) \\ c_M(1 + |s|)^{-M} & \text{if } |\xi| \leq c_1 |s|, \\ c_M(1 + |\xi|)^{-M} & \text{if } c_2 |s| \leq |\xi|. \end{cases}$$

Proof. The behaviour of the oscillatory integral $I_k(\xi, s)$ depends on the points where the amplitude $\psi(x)$ is not zero and the phase $(\xi, s) \cdot (x, \Phi_k(x))$ is stationary. This happens only when $|\xi| \approx |s|$ and in this case, since the phase is non degenerate, one obtains the classical estimate $c|(\xi, s)|^{-(d-1)/2}$. In all other directions the oscillatory integral has a fast decay. In particular, when $|\xi| \leq c_1 |s|$ one obtains the decay $c_M(1 + |s|)^{-M}$, and when $|\xi| > c_2 |s|$, one obtains the decay $c_M(1 + |\xi|)^{-M}$. For the sake of completeness we include the full details of the proof.

By the definition of the class S_γ we have

$$(3.1) \quad \left| \frac{\partial^{|\alpha|} \Phi_k}{\partial x^\alpha}(x) \right| = 2^{k\gamma} 2^{-k|\alpha|} \left| \frac{\partial^{|\alpha|} \Phi}{\partial x^\alpha}(2^{-k}x) \right| \leq 2^{k\gamma} 2^{-k|\alpha|} c_\alpha |2^{-k}x|^{\gamma-|\alpha|} \\ \leq c_\alpha |x|^{\gamma-|\alpha|}.$$

In particular when x belongs to the support of $\psi(x)$ we have

$$\left| \frac{\partial^{|\alpha|} \Phi_k}{\partial x^\alpha}(x) \right| \leq c_\alpha.$$

Moreover, the Hessian matrix of $\Phi_k(x)$ satisfies

$$\text{Hess } \Phi_k(x) = 2^{k(\gamma-2)} \text{Hess } \Phi(2^{-k}x)$$

and it follows that the eigenvalues $\mu_j^{(k)}(x)$ of $\text{Hess } \Phi_k(x)$ are related to the eigenvalues $\mu_j(x)$ of $\text{Hess } \Phi(x)$ by the identity

$$\mu_j^{(k)}(x) = 2^{k(\gamma-2)} \mu_j(2^{-k}x).$$

By (2.1)

$$(3.2) \quad \mu_j^{(k)}(x) = \frac{\mu_j(2^{-k}x)}{|2^{-k}x|^{\gamma-2}} |x|^{\gamma-2} \geq c |x|^{\gamma-2}.$$

If x belongs to the support of $\psi(x)$ we have

$$\mu_j^{(k)}(x) \geq c > 0.$$

Since

$$\nabla \Phi (2^{-k}x) = 2^{-k(\gamma-1)} \nabla \Phi_k (x)$$

by (3.1) all the derivatives of $\nabla \Phi (2^{-k}x)$ are uniformly bounded.

The phase in the integral $I_k (\xi, s)$ is stationary when

$$\nabla (\xi \cdot x + s\Phi_k (x)) = \xi + s\nabla \Phi_k (x) = 0.$$

By (3.1) there exists $c_2 > 0$ such that $|\nabla \Phi_k (x)| \leq \frac{c_2}{2}$ for every k . It follows that for $|\xi| \geq c_2 |s|$ we have

$$|\xi + s\nabla \Phi_k (x)| \geq |\xi| - |s| |\nabla \Phi_k (x)| \geq \frac{1}{2} |\xi|.$$

Integrating by parts M times gives (see e.g. Proposition 4, p. 341, in [29])

$$|I_k (\xi, s)| \leq c_M (1 + |\xi|)^{-M}.$$

Let now $|\xi| \leq c_1 |s|$ where c_1 is a constant which will be determined later on. Let us consider the function

$$F (t) = \nabla \Phi_k (tx) \cdot x$$

with $t \in [0, 1]$. Then, for $t \in (0, 1]$

$$F' (t) = x^T \text{Hess } \Phi_k (tx) x$$

and by (3.2) the eigenvalues of $\text{Hess } \Phi_k (tx)$ are bounded from below by $t^{\gamma-2} |x|^{\gamma-2}$. Then

$$F (1) \geq \int_0^1 x^T \text{Hess } \Phi_k (tx) x dt \geq c \int_0^1 t^{\gamma-2} |x|^{\gamma-2} |x|^2 dt \geq c |x|^\gamma$$

and therefore

$$|\nabla \Phi_k (x)| \geq \frac{\nabla \Phi_k (x) \cdot x}{|x|} \geq c |x|^{\gamma-1} \geq 2c_1 > 0.$$

It follows that

$$|\xi + s\nabla \Phi_k (x)| \geq |s| |\nabla \Phi_k (x)| - |\xi| \geq c_1 |s|.$$

Integrating by parts M times gives

$$|I_k (\xi, s)| \leq c_M (1 + |s|)^{-M}.$$

Finally, for every (ξ, s) , Theorem 1, p. 348, in [29] gives

$$|I_k (\xi, s)| \leq c (1 + |\xi| + |s|)^{-\frac{d-1}{2}}.$$

□

Proposition 11. *Let $\gamma > 1$ and let B be a bounded convex body in \mathbb{R}^d with everywhere positive Gaussian curvature with the exception of a single flat point of order γ . Let Θ be the outward unit normal to ∂B at the flat point and for every $\zeta \in \mathbb{R}^d$ write $\zeta = \xi + s\Theta$, with $s = \zeta \cdot \Theta$ and $\xi \cdot \Theta = 0$. Then, if $1 < \gamma \leq 2$*

$$(3.3) \quad |\widehat{\chi}_B(\zeta)| \leq c |\zeta|^{-\frac{d+1}{2}}.$$

If $\gamma > 2$ the following three upper bounds hold:

$$(3.4) \quad |\widehat{\chi}_B(\zeta)| \leq \begin{cases} c |s|^{-1-\frac{d-1}{\gamma}}, \\ c |\xi|^{-(d-1)\frac{\gamma-2}{2(\gamma-1)}} |s|^{-\frac{d-1}{2(\gamma-1)}-1}, \\ c |\xi|^{-\frac{d+1}{2}}. \end{cases}$$

The particular case where the boundary in a neighborhood of the flat point has equation $t = |x|^\gamma$ with $\gamma > 2$ and $d = 2$ has been already considered in [6]. The same case with $\gamma > 2$ and $d > 2$ has been considered in [4], but we acknowledge that the proof of the rate of decay in the horizontal directions was not correctly justified.

More precise estimates for the Fourier transform of the characteristic functions of a convex body with boundary having flat points in which the principal curvatures may vanish of different orders are contained in [21, Proposition 1.2].

Proof. Choose a smooth function $\eta(\mathbf{z})$ supported in a neighborhood of the flat point and such that $\eta(\mathbf{z}) = 1$ in a smaller neighborhood. For every $\mathbf{z} \in \partial B$ let $\nu(\mathbf{z})$ be its outward unit normal. Applying the divergence theorem we decompose the Fourier transform as

$$(3.5) \quad \begin{aligned} \widehat{\chi}_B(\zeta) &= \int_B e^{-2\pi i \zeta \cdot \mathbf{z}} d\mathbf{z} = \frac{-1}{4\pi^2 |\zeta|^2} \int_{\partial B} \nabla (e^{-2\pi i \zeta \cdot \mathbf{z}}) \cdot \nu(\mathbf{z}) d\sigma(\mathbf{z}) \\ &= \frac{-1}{2\pi i |\zeta|^2} \int_{\partial B} \zeta \cdot \nu(\mathbf{z}) e^{-2\pi i \zeta \cdot \mathbf{z}} d\sigma(\mathbf{z}) \\ &= \frac{-1}{2\pi i |\zeta|^2} \int_{\partial B} \zeta \cdot \nu(\mathbf{z}) e^{-2\pi i \zeta \cdot \mathbf{z}} \eta(\mathbf{z}) d\sigma(\mathbf{z}) \\ &\quad + \frac{-1}{2\pi i |\zeta|^2} \int_{\partial B} \zeta \cdot \nu(\mathbf{z}) e^{-2\pi i \zeta \cdot \mathbf{z}} [1 - \eta(\mathbf{z})] d\sigma(\mathbf{z}) \\ &= K_1(\zeta) + K_2(\zeta). \end{aligned}$$

Since in the support of the function $1 - \eta(\mathbf{z})$ the Gaussian curvature is bounded away from zero, the method of stationary phase gives the classical estimate (see Theorem 1, p. 348, in [29])

$$(3.6) \quad |K_2(\zeta)| \leq c |\zeta|^{-\frac{d+1}{2}}.$$

By a suitable choice of coordinates we can assume that $\mathbf{z} = (x, t)$, the flat point is the point $(0, 0)$, its outward normal is $(0, -1)$ and that the relevant part of the

surface ∂B is described by the equation $t = \Phi(x)$ with $\Phi \in S_\gamma$. Hence

$$\nu(x, \Phi(x)) = \frac{(\nabla\Phi(x), -1)}{\sqrt{1 + |\nabla\Phi(x)|^2}}.$$

Write $\varphi(x) = \eta(x, \Phi(x))$ and $\psi(x) = \varphi(x) - \varphi(2x)$ so that for every $x \neq 0$

$$\varphi(x) = \sum_{k=0}^{+\infty} \psi(2^k x).$$

Observe that $\varphi(x)$ is smooth and a suitable choice of $\eta(\mathbf{z})$ guarantees $\varphi(x) = 1$ if $|x| \leq \varepsilon/2$ and $\varphi(x) = 0$ if $|x| \geq \varepsilon$ for some $\varepsilon > 0$. With the above choice of coordinate we can also write $\zeta = (\xi, -s)$ so that, following the notation of the previous lemma, we have

$$\begin{aligned} & K_1(\zeta) \\ &= \frac{-1}{2\pi i |\zeta|^2} \int_{\mathbb{R}^{d-1}} \zeta \cdot \frac{(\nabla\Phi(x), -1)}{\sqrt{1 + |\nabla\Phi(x)|^2}} e^{-2\pi i \zeta \cdot (x, \Phi(x))} \varphi(x) \sqrt{1 + |\nabla\Phi(x)|^2} dx \\ (3.7) \quad &= \frac{-\zeta}{2\pi i |\zeta|^2} \cdot \int_{\mathbb{R}^{d-1}} (\nabla\Phi(x), -1) e^{-2\pi i \zeta \cdot (x, \Phi(x))} \varphi(x) dx \\ &= \frac{-\zeta}{2\pi i |\zeta|^2} \cdot \sum_{k=0}^{+\infty} \int_{\mathbb{R}^{d-1}} (\nabla\Phi(x), -1) e^{-2\pi i \zeta \cdot (x, \Phi(x))} \psi(2^k x) dx \\ &= \sum_{k=0}^{+\infty} \frac{-2^{-k(d-1)} \zeta}{2\pi i |\zeta|^2} \\ &\quad \cdot \int_{\mathbb{R}^{d-1}} (\nabla\Phi(2^{-k}y), -1) e^{-2\pi i (2^{-k}\xi, -2^{-k}\gamma s) \cdot (y, 2^{k\gamma} \Phi(2^{-k}y))} \psi(y) dy \\ &= \sum_{k=0}^{+\infty} 2^{-k(d-1)} \frac{-\zeta}{2\pi i |\zeta|^2} \cdot I_k(2^{-k}\xi, -2^{-k}\gamma s). \end{aligned}$$

By the previous lemma

$$|I_k(2^{-k}\xi, -2^{-k}\gamma s)| \leq c(1 + |2^{-k}\gamma s| + |2^{-k}\xi|)^{-\frac{d-1}{2}}.$$

Hence,

$$|K_1(\zeta)| \leq \frac{c}{|(\xi, s)|} \sum_{k=0}^{+\infty} 2^{-k(d-1)} (1 + |2^{-k}\gamma s| + |2^{-k}\xi|)^{-\frac{d-1}{2}}.$$

In particular, for every $(\xi, s) \in \mathbb{R}^d$ we have

$$|K_1(\zeta)| \leq \frac{c}{|\xi|} \sum_{k=0}^{+\infty} 2^{-k(d-1)} |2^{-k}\xi|^{-\frac{d-1}{2}} \leq c|\xi|^{-\frac{d+1}{2}}.$$

Assume now $\gamma \neq 2$. Our second estimate for $K_1(\zeta)$ is as follows. For every $(\xi, s) \in \mathbb{R}^d$ we have

$$\begin{aligned}
|K_1(\zeta)| &\leq \frac{c}{|s|} \sum_{k=0}^{+\infty} 2^{-k(d-1)} (1 + |2^{-k\gamma}s|)^{-\frac{d-1}{2}} \\
&\leq \frac{c}{|s|} \sum_{2^k \geq |s|^{1/\gamma}} 2^{-k(d-1)} + \frac{c}{|s|} \sum_{2^k < |s|^{1/\gamma}} 2^{-k(d-1)} |2^{-k\gamma}s|^{-\frac{d-1}{2}} \\
&\leq c|s|^{-1-\frac{d-1}{\gamma}} + c|s|^{-1-\frac{d-1}{2}} \sum_{2^k < |s|^{1/\gamma}} 2^{k(d-1)(\frac{\gamma}{2}-1)} \\
&\leq \begin{cases} c|s|^{-\frac{d+1}{2}} & \gamma < 2, \\ c|s|^{-1-\frac{d-1}{\gamma}} & \gamma > 2. \end{cases}
\end{aligned}$$

Note that when $\gamma = 2$ the previous computation gives $c|s|^{-\frac{d+1}{2}} \log(2 + |s|)$. However when $\Phi(y)$ is smooth it is well known that the correct estimate is $c|s|^{-\frac{d+1}{2}}$. With a more careful analysis we show that this is the case also in our setting, even if we do not assume smoothness at the flat point. Indeed notice that condition (2.2) allows higher derivatives to blow up at the flat point. Let $\gamma = 2$. Then

$$(3.8) \quad |K_1(\zeta)| \leq \frac{c}{|(\xi, s)|} \sum_{k=0}^{+\infty} 2^{-k(d-1)} |I_k(2^{-k}\xi, -2^{-k}\gamma s)|$$

By the previous lemma we have

$$|I_k(2^{-k}\xi, -2^{-2k}s)| \leq \begin{cases} c(1 + |2^{-2k}s| + |2^{-k}\xi|)^{-\frac{d-1}{2}} & \text{for every } (\xi, s) \\ c_M(1 + |2^{-2k}s|)^{-M} & \text{if } 2^k \leq c_1 \frac{|s|}{|\xi|}, \\ c_M(1 + |2^{-k}\xi|)^{-M} & \text{if } c_2 \frac{|s|}{|\xi|} \leq 2^k. \end{cases}$$

so that

$$\begin{aligned}
&\sum_{k=0}^{+\infty} 2^{-k(d-1)} |I_k(2^{-k}\xi, -2^{-2k}s)| \\
&\leq c \sum_{2^k \leq c_1 |s|/|\xi|} 2^{-k(d-1)} (1 + |2^{-2k}s|)^{-M} + c \sum_{c_2 |s|/|\xi| \leq 2^k} 2^{-k(d-1)} (1 + |2^{-k}\xi|)^{-M} \\
&+ c \sum_{c_1 |s|/|\xi| < 2^k < c_2 |s|/|\xi|} 2^{-k(d-1)} (1 + |2^{-2k}s| + |2^{-k}\xi|)^{-\frac{d-1}{2}} \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

We have

$$S_1 \leq c \sum_{k=0}^{+\infty} 2^{-k(d-1)} (1 + |2^{-2k}s|)^{-M}$$

$$\begin{aligned}
&\leq c \sum_{|s|^{1/2} \leq 2^k} 2^{-k(d-1)} + c |s|^{-M} \sum_{|s|^{1/2} > 2^k} 2^{2kM} 2^{-k(d-1)} \\
&\leq c |s|^{-\frac{d-1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
S_2 &\leq c \sum_{c_2 |s|/|\xi| \leq 2^k} 2^{-k(d-1)} (1 + |2^{-k} \xi|)^{-M} \\
&\leq c \sum_{\max(c_2 |s|/|\xi|, |\xi|) \leq 2^k} 2^{-k(d-1)} + c |\xi|^{-M} \sum_{c_2 |s|/|\xi| \leq 2^k < |\xi|} 2^{Mk-k(d-1)}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{\max(c_2 |s|/|\xi|, |\xi|) \leq 2^k} 2^{-k(d-1)} &\leq c \min \left(\left(\frac{|\xi|}{|s|} \right)^{d-1}, |\xi|^{-(d-1)} \right) \\
&\leq c |s|^{-\frac{d-1}{2}} \min \left(\frac{|\xi|^{d-1}}{|s|^{\frac{d-1}{2}}}, \frac{|s|^{\frac{d-1}{2}}}{|\xi|^{d-1}} \right) \leq c |s|^{-\frac{d-1}{2}}.
\end{aligned}$$

Also, the series

$$|\xi|^{-M} \sum_{c_2 |s|/|\xi| \leq 2^k < |\xi|} 2^{Mk-k(d-1)}$$

is non void only if $c_2 \frac{|s|}{|\xi|} < |\xi|$, that is $c |s|^{1/2} < |\xi|$. In this case we have

$$\begin{aligned}
c |\xi|^{-M} \sum_{c_2 |s|/|\xi| \leq 2^k < |\xi|} 2^{Mk-k(d-1)} &\leq c |\xi|^{-M} \sum_{|\xi| > 2^k} 2^{Mk-k(d-1)} \leq c |\xi|^{-(d-1)} \\
&\leq c |s|^{-\frac{d-1}{2}}.
\end{aligned}$$

We now turn to S_3 that contains a sum with a finite number of terms. We have,

$$S_3 \leq c |s|^{-\frac{d-1}{2}} \sum_{c_1 |s|/|\xi| < 2^k < c_2 |s|/|\xi|} 1 \leq c |s|^{-\frac{d-1}{2}}.$$

Substituting into (3.8) gives the estimate

$$|\widehat{\chi}_B(\zeta)| \leq c |s|^{-\frac{d+1}{2}}$$

when $\gamma = 2$. It remains to prove the second row in (3.4). We have

$$\begin{aligned}
&|K_1(\zeta)| \\
&\leq \frac{c}{|s|} \sum_{k=0}^{+\infty} 2^{-k(d-1)} (|2^{-k\gamma} s| + |2^{-k} \xi|)^{-\frac{d-1}{2}}
\end{aligned}$$

$$\begin{aligned} &\leq \frac{c}{|s|} \left(|s|^{-\frac{d-1}{2}} \sum_{2^k \leq (|s|/|\xi|)^{\frac{1}{\gamma-1}}} 2^{k(d-1)\left(\frac{\gamma}{2}-1\right)} + |\xi|^{-\frac{d-1}{2}} \sum_{2^k > (|s|/|\xi|)^{\frac{1}{\gamma-1}}} 2^{-k\frac{d-1}{2}} \right) \\ &\leq c |\xi|^{-\frac{(d-1)(\gamma-2)}{2(\gamma-1)}} |s|^{-\frac{d-1}{2(\gamma-1)}-1}. \end{aligned}$$

□

Remark 12. Let $\mathbf{z} \in \partial B$, let $T_{\mathbf{z}}$ be the tangent hyperplane to ∂B in \mathbf{z} and let

$$S(\mathbf{z}, \delta) = \{\mathbf{w} \in \partial B : \text{dist}(\mathbf{w}, T_{\mathbf{z}}) < \delta\}.$$

In [8] it is proved that when the boundary of B is smooth and of finite type (every one dimensional tangent line to ∂B makes finite order of contact with ∂B), then

$$|\widehat{\chi}_B(\zeta)| \leq c |\zeta|^{-1} \left[\sigma \left(S(\mathbf{z}^+, |\zeta|^{-1}) \right) + \sigma \left(S(\mathbf{z}^-, |\zeta|^{-1}) \right) \right]$$

where \mathbf{z}^+ and \mathbf{z}^- are the two points on ∂B with outer normal parallel to ζ and σ is the surface measure. In our case ∂B is not necessarily smooth, but the above result in fact holds. Indeed, let B as in Proposition 11, and choose coordinates such that $\mathbf{z} = (x, t)$, the flat point is the point $(0, 0)$, its outward normal is $(0, -1)$ and that the relevant part of the surface ∂B is described by the equation $t = \Phi(x)$ with $\Phi \in S_\gamma$. Fix $\mathbf{z} = (x, t) \in \partial B$. Elementary geometric observations lead to

$$(3.9) \quad \sigma(S(\mathbf{z}, \delta)) \geq c \sigma(S(\mathbf{0}, c_1 \delta)) \approx \left(\delta^{1/\gamma} \right)^{d-1} \quad \text{for } \delta \geq c|x|^\gamma,$$

$$(3.10) \quad \sigma(S(\mathbf{z}, \delta)) \geq c \left(\delta \left(|x|^{\gamma-2} \right)^{-1} \right)^{\frac{d-1}{2}} \quad \text{for } \delta \leq c|x|^\gamma.$$

The unit normal to ∂B in $(x, \Phi(x))$ is $\frac{(\nabla \Phi(x), -1)}{\sqrt{|\nabla \Phi(x)|^2 + 1}}$. It follows that for a given $\zeta = (\xi, s)$ the point $(x, \Phi(x))$ in ∂B with normal in the direction ζ satisfies $|\xi|/|s| = |\nabla \Phi(x)| \approx |x|^{\gamma-1}$.

a) If $|s| \geq |\xi|^\gamma$ then $|s|^{1-\gamma} \geq (|\xi|/|s|)^\gamma \approx |x|^{(\gamma-1)\gamma}$. Hence

$$\frac{1}{|\zeta|} \approx \frac{1}{|s|} \geq c|x|^\gamma$$

so that, by (3.9),

$$\sigma \left(S(\mathbf{z}, |\zeta|^{-1}) \right) \geq c \sigma \left(S(\mathbf{0}, c_1 |\zeta|^{-1}) \right) \approx |\zeta|^{-(d-1)/\gamma} \approx |s|^{-(d-1)/\gamma}.$$

b) If $|\xi| \leq |s| \leq |\xi|^\gamma$, then as before

$$\frac{1}{|\zeta|} \approx \frac{1}{|s|} \leq c|x|^\gamma$$

and by (3.10)

$$\sigma\left(S\left(\mathbf{z}, |\zeta|^{-1}\right)\right) \geq c \left(\frac{1}{|s|} \left(\frac{|\xi|}{|s|}\right)^{\frac{2-\gamma}{\gamma-1}}\right)^{\frac{d-1}{2}} = c |\xi|^{-(d-1)\frac{\gamma-2}{2(\gamma-1)}} |s|^{-\frac{d-1}{2(\gamma-1)}}$$

c) If $|\xi| \geq |s|$, then $|x| \approx 1$ and by (3.10)

$$\sigma\left(S\left(\mathbf{z}, |\zeta|^{-1}\right)\right) \geq |\xi|^{-\frac{d-1}{2}}.$$

The geometric estimate

$$|\widehat{\chi}_B(\zeta)| \leq c |\zeta|^{-1} \sigma\left(S\left(\mathbf{z}, |\zeta|^{-1}\right)\right).$$

now follows from Proposition 11.

As said before, the above proposition is the main ingredient in the estimate of the discrepancy associated to the convex body B . In particular it follows that the directions where the Fourier transform has the slowest rate of decay play a relevant role in the estimates of the discrepancy.

Actually the Fourier transform in a given direction depends on the two points in ∂B have normals in that direction. The interplay between the contribution of these points is exploited in the following proposition.

Proposition 13. *Let B be a bounded convex body in \mathbb{R}^d . Assume that ∂B is smooth with everywhere positive Gaussian curvature except at most at two points P and Q which are flat of order γ_P and γ_Q respectively and have outward unit normals $-\Theta$ and Θ . Let*

$$S(t) = |\{\mathbf{z} \in B : \mathbf{z} \cdot \Theta = t\}|$$

be $(d-1)$ -dimensional measures of the slices of B that are orthogonal to Θ . The function $S(t)$ is supported in $P \cdot \Theta \leq t \leq Q \cdot \Theta$ and is smooth in $P \cdot \Theta < t < Q \cdot \Theta$. Assume that there exist two smooth functions $G_P(r)$ and $G_Q(r)$ with $G_P(0) \neq 0$ and $G_Q(0) \neq 0$ such that, for $u \geq 0$ sufficiently small

$$S(P \cdot \Theta + u) = u^{\frac{d-1}{\gamma_P}} G_P\left(u^{1/\gamma_P}\right)$$

and

$$S(Q \cdot \Theta - u) = u^{\frac{d-1}{\gamma_Q}} G_Q\left(u^{1/\gamma_Q}\right).$$

Then, as $|s| \rightarrow +\infty$,

$$\begin{aligned} \widehat{\chi}_B(s\Theta) &= e^{-2\pi i s \Theta \cdot P} G_P(0) \frac{\Gamma\left(\frac{d-1}{\gamma_P} + 1\right)}{(2\pi)^{\frac{d-1}{\gamma_P} + 1}} e^{-i\frac{\pi}{2}\left(\frac{d-1}{\gamma_P} + 1\right) \operatorname{sgn}(s)} |s|^{-1 - \frac{d-1}{\gamma_P}} \\ &+ e^{-2\pi i s \Theta \cdot Q} G_Q(0) \frac{\Gamma\left(\frac{d-1}{\gamma_Q} + 1\right)}{(2\pi)^{\frac{d-1}{\gamma_Q} + 1}} e^{i\frac{\pi}{2}\left(\frac{d-1}{\gamma_Q} + 1\right) \operatorname{sgn}(s)} |s|^{-1 - \frac{d-1}{\gamma_Q}} \\ &+ O\left(|s|^{-1 - \frac{d}{\max(\gamma_P, \gamma_Q)}}\right). \end{aligned}$$

Observe that when in a neighborhood of the points P and Q the boundary of B is smooth with positive Gaussian curvature $K(P)$ and $K(Q)$ then we have $\gamma_P = \gamma_Q = \gamma = 2$, $G_P(0) = \frac{(2\pi)^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} K^{-1/2}(P)$, and $G_Q(0) = \frac{(2\pi)^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} K^{-1/2}(Q)$. Hence we obtain the classical formula

$$\begin{aligned} \widehat{\chi}_B(s\Theta) &= \frac{1}{2\pi} e^{-2\pi i s \Theta \cdot P} K^{-1/2}(P) e^{-i\frac{\pi}{2}(\frac{d-1}{2}+1) \operatorname{sgn}(s)} |s|^{-\frac{d+1}{2}} \\ &\quad + \frac{1}{2\pi} e^{-2\pi i s \Theta \cdot Q} K^{-1/2}(Q) e^{i\frac{\pi}{2}(\frac{d-1}{2}+1) \operatorname{sgn}(s)} |s|^{-\frac{d+1}{2}} \\ &\quad + O\left(|s|^{-\frac{d+2}{2}}\right). \end{aligned}$$

See [17] and [18]. See also [19, Corollary 7.7.15].

Proof. The d -dimensional Fourier transform is the one dimensional Fourier transform of a Radon transform. In [1, Lemma 4.3] it is proved that the Radon transform $S(t)$ is smooth inside $P \cdot \Theta < t < Q \cdot \Theta$. It follows that the asymptotic behaviour of the Fourier transform of $S(t)$ depends only on $S(t)$ in a neighborhood of the endpoints $t = P \cdot \Theta$ and $t = Q \cdot \Theta$. Let $\eta(t)$ be a smooth cutoff function with $\eta(t) = 1$ if $|t| \leq \varepsilon$ and $\eta(t) = 0$ if $|t| \geq 2\varepsilon$ with ε small. For every $N > 0$ we have

$$\begin{aligned} \widehat{\chi}_B(s\Theta) &= \int_{\mathbb{R}^d} \chi_B(\mathbf{z}) e^{-2\pi i \mathbf{z} \cdot s\Theta} d\mathbf{z} = \int_{-\infty}^{+\infty} \left(\int_{\{\mathbf{z} \cdot \Theta = t\}} \chi_B(\mathbf{z}) d\mathbf{z} \right) e^{-2\pi i s t} dt \\ &= \int_{P \cdot \Theta}^{Q \cdot \Theta} S(t) e^{-2\pi i s t} dt \\ &= \int_0^{+\infty} \eta(u) S(P \cdot \Theta + u) e^{-2\pi i s (P \cdot \Theta + u)} du \\ &\quad + \int_0^{+\infty} \eta(u) S(Q \cdot \Theta - u) e^{-2\pi i s (Q \cdot \Theta - u)} du + O\left(|s|^{-N}\right) \\ &= e^{-2\pi i s P \cdot \Theta} \int_0^{+\infty} u^{\frac{d-1}{\gamma_P}} G_P\left(u^{1/\gamma_P}\right) \eta(u) e^{-2\pi i s u} du \\ &\quad + e^{-2\pi i s Q \cdot \Theta} \int_0^{+\infty} u^{\frac{d-1}{\gamma_Q}} G_Q\left(u^{1/\gamma_Q}\right) \eta(u) e^{2\pi i s u} du + O\left(|s|^{-N}\right). \end{aligned}$$

It is enough to consider

$$K(s) = \int_0^{+\infty} u^{\frac{d-1}{\gamma}} G\left(u^{1/\gamma}\right) \eta(u) e^{-2\pi i s u} du.$$

Since $G(r)$ is smooth, for every $N > 0$ we can write the Taylor expansion

$$\begin{aligned} K(s) &= \sum_{k=0}^{N-1} \frac{G^{(k)}(0)}{k!} \int_0^{+\infty} u^{\frac{d-1+k}{\gamma}} \eta(u) e^{-2\pi i s u} du \\ &\quad + \int_0^{+\infty} u^{\frac{d-1+N}{\gamma}} G_N\left(u^{1/\gamma}\right) \eta(u) e^{-2\pi i s u} du. \end{aligned}$$

For N large enough, the function $u^{\frac{d-1+N}{\gamma}} G_N(u^{1/\gamma}) \eta(u)$ has enough bounded derivatives so that a repeated integration by parts gives

$$\left| \int_0^{+\infty} \eta(u) G_N(u^{1/\gamma}) u^{\frac{d-1+N}{\gamma}} e^{-2\pi i s u} du \right| \leq c |s|^{-1-\frac{d}{\gamma}}.$$

Finally, all other terms in the above sum have the form

$$\int_0^{+\infty} \eta(u) u^\alpha e^{-2\pi i s u} du$$

and can be estimated by the following lemma. □

Lemma 14. *If η is as above then, for every $\alpha > -1$ and $s \neq 0$, we have*

$$\int_0^{+\infty} t^\alpha e^{-2\pi i s t} \eta(t) dt = \frac{\Gamma(\alpha+1)}{(2\pi |s|)^{\alpha+1}} e^{-i\frac{\pi}{2}(\alpha+1)\operatorname{sgn}(s)} + O(|s|^{-N}).$$

The above result is not surprising since, in the sense of distributions,

$$\int_0^{+\infty} t^\alpha e^{-2\pi i s t} dt = \frac{\Gamma(\alpha+1)}{(2\pi |s|)^{\alpha+1}} e^{-i\frac{\pi}{2}(\alpha+1)\operatorname{sgn}(s)}.$$

See e.g. [13]. The following is a direct proof.

Proof. Assume first $s > 0$. An integration by parts gives

$$\begin{aligned} \int_0^{+\infty} t^\alpha e^{-2\pi i s t} \eta(t) dt &= \frac{1}{2\pi i s} \int_0^{+\infty} e^{-2\pi i s t} \frac{d}{dt} [t^\alpha \eta(t)] dt \\ &= \frac{\alpha}{2\pi i s} \int_0^{+\infty} e^{-2\pi i s t} t^{\alpha-1} \eta(t) dt + \frac{1}{2\pi i s} \int_0^{+\infty} e^{-2\pi i s t} t^\alpha \eta'(t) dt. \end{aligned}$$

Since $\operatorname{supp} \eta' \subset (\varepsilon, 2\varepsilon)$ the term $t^\alpha \eta'(t)$ is smooth so that

$$\frac{1}{2\pi i s} \int_0^{+\infty} e^{-2\pi i s t} t^\alpha \eta'(t) dt = O(|s|^{-N}).$$

Repeating the integration by parts k times, with $k \leq \alpha$ gives

$$\begin{aligned} \int_0^{+\infty} t^\alpha e^{-2\pi i s t} \eta(t) dt \\ = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{(2\pi i s)^k} \int_0^{+\infty} e^{-2\pi i s t} t^{\alpha-k} \eta(t) dt + O(|s|^{-N}). \end{aligned}$$

Assume first that α is an integer and take $k = \alpha$. Then

$$\int_0^{+\infty} t^\alpha e^{-2\pi i s t} \eta(t) dt = \frac{\alpha!}{(2\pi i s)^\alpha} \int_0^{+\infty} e^{-2\pi i s t} \eta(t) dt + O(|s|^{-N})$$

$$\begin{aligned}
&= \frac{\alpha!}{(2\pi is)^{\alpha+1}} + \frac{\alpha!}{(2\pi is)^{\alpha+1}} \int_0^{+\infty} e^{-2\pi ist} \eta'(t) dt \\
&= \frac{\alpha!}{(2\pi is)^{\alpha+1}} + O(|s|^{-N}).
\end{aligned}$$

If α is not an integer we take $k = [\alpha] + 1$. Then

$$\begin{aligned}
&\int_0^{+\infty} t^\alpha e^{-2\pi ist} \eta(t) dt \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-[\alpha])}{(2\pi is)^{[\alpha]+1}} \int_0^{+\infty} e^{-2\pi ist} t^{\lambda-1} \eta(t) dt + O(|s|^{-N}),
\end{aligned}$$

where $\lambda = \alpha - [\alpha]$. By [11, (4) pag. 48] we have

$$\begin{aligned}
&\int_0^{+\infty} e^{-2\pi ist} t^{\lambda-1} \eta(t) dt = \overline{\int_0^{+\infty} e^{2\pi ist} t^{\lambda-1} \eta(t) dt} \\
&= - \sum_{n=0}^{N-1} \frac{\Gamma(n+\lambda)}{n!} e^{-\pi i(n+\lambda-2)/2} \eta^{(n)}(0) (2\pi s)^{-n-\lambda} + O(s^{-N})
\end{aligned}$$

and since $\eta^{(n)}(0) = 0$ for every $n > 0$ and $\eta(0) = 1$ we obtain

$$\begin{aligned}
&\int_0^{+\infty} t^\alpha e^{-2\pi ist} \eta(t) dt \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-[\alpha])}{(2\pi is)^{[\alpha]+1}} \int_0^{+\infty} e^{-2\pi ist} t^{\lambda-1} \eta(t) dt + O(|s|^{-N}) \\
&= \frac{\Gamma(\alpha+1)}{(2\pi s)^{\alpha+1}} e^{-\frac{\pi}{2}i(\alpha+1)} + O(s^{-N})
\end{aligned}$$

also in this case.

Let now $s < 0$. Then

$$\begin{aligned}
&\int_0^{+\infty} t^\alpha e^{-2\pi ist} \eta(t) dt = \overline{\int_0^{+\infty} t^\alpha e^{-2\pi i(-s)t} \eta(t) dt} \\
&= \frac{\Gamma(\alpha+1)}{(2\pi |s|)^{\alpha+1}} e^{i\frac{\pi}{2}(\alpha+1)} + O(|s|^{-N})
\end{aligned}$$

□

In the next proposition we show that assumptions of Proposition 13 are satisfied when the flat points are as in Proposition 3.

Proposition 15. *Let $\gamma > 1$ and let B be a bounded convex body in \mathbb{R}^d . Let U be a bounded open neighborhood of the origin in \mathbb{R}^{d-1} and let $H(x) \in C^\infty(U)$ such that $H(0) = 0$, $\nabla H(0) = 0$ and $\text{Hess } H(0)$ positive definite (see Proposition 3).*

Assume there exists a neighborhood of the origin $W \subset \mathbb{R}^d$ such that, in suitable coordinates,

$$\partial B \cap W = \left\{ (x, t) \in \mathbb{R}^d : t = (H(x))^{\gamma/2} \right\} \cap W.$$

As before, let

$$S(t) = |\{x \in \mathbb{R}^{d-1} : (x, t) \in B\}|.$$

Then, there exists a smooth function $G(r)$ such that for $t > 0$ sufficiently small we have

$$S(t) = t^{\frac{d-1}{\gamma}} G\left(t^{1/\gamma}\right)$$

with $G(0)$ equal to the $(d-1)$ -dimensional measure of the ellipsoid

$$\left\{ x \in \mathbb{R}^{d-1} : \frac{1}{2} \sum_{j,k=1}^{d-1} \frac{\partial^2 H}{\partial x_j \partial x_k}(0) x_j x_k \leq 1 \right\}.$$

Proof. For t small enough we have

$$S(t) = \int_{\{x \in \mathbb{R}^{d-1} : t \geq (H(x))^{\gamma/2}\}} dx.$$

By Morse's lemma (see [29, p. 346]), there exists a diffeomorphism $\Psi(y)$ between two small neighborhoods of the origin in \mathbb{R}^{d-1} such that

$$H(\Psi(y)) = |y|^2.$$

Then,

$$\begin{aligned} S(t) &= \int_{\{x \in \mathbb{R}^{d-1} : t \geq (H(x))^{\gamma/2}\}} dx = \int_{\{|y| \leq t^{1/\gamma}\}} J_{\Psi}(y) dy \\ &= t^{\frac{d-1}{\gamma}} \int_{\{|u| \leq 1\}} J_{\Phi}(t^{1/\gamma}u) du = t^{\frac{d-1}{\gamma}} G\left(t^{1/\gamma}\right) \end{aligned}$$

where

$$G(r) = \int_{\{|z| \leq 1\}} J_{\Phi}(ru) du.$$

Finally observe that

$$\begin{aligned} G(0) &= \lim_{r \rightarrow 0} \int_{\{|u| \leq 1\}} J_{\Phi}(ru) du = \lim_{r \rightarrow 0} \frac{1}{r^{d-1}} \int_{\{|w| \leq r\}} J_{\Phi}(w) dw \\ &= \lim_{r \rightarrow 0} \frac{1}{r^{d-1}} \int_{\{|w|^2 \leq r^2\}} J_{\Phi}(w) dw \\ &= \lim_{r \rightarrow 0} \frac{1}{r^{d-1}} \int_{\{H(y) \leq r^2\}} dy = \lim_{r \rightarrow 0} \int_{\{\frac{H(rx)}{r^2} \leq 1\}} dx \\ &= \left| \left\{ x \in \mathbb{R}^{d-1} : \frac{1}{2} \sum_{j,k=1}^{d-1} \frac{\partial^2 H}{\partial x_j \partial x_k}(0) x_j x_k \leq 1 \right\} \right|. \end{aligned}$$

□

4. Proofs of the results

Proof of Proposition 3. Let α be a multi-index. It is not difficult to prove by induction on $|\alpha|$ that

$$\frac{\partial^{|\alpha|}\Phi}{\partial x^\alpha}(x)$$

is a finite sum of terms of the form

$$c [H(x)]^{\gamma/2-k} \frac{\partial^{|\beta_1|}H}{\partial x_{\beta_1}}(x) \times \cdots \times \frac{\partial^{|\beta_k|}H}{\partial x_{\beta_k}}(x)$$

with $k \leq |\alpha|$ and multi-indices β_1, \dots, β_k such that $|\beta_1| + \cdots + |\beta_k| = |\alpha|$. Since $\text{Hess } H(0)$ is positive definite there are positive constants c_1 and c_2 such that in a neighborhood of the origin

$$c_1 |x|^2 \leq H(x) \leq c_2 |x|^2$$

and

$$\left| \frac{\partial H}{\partial x_j}(x) \right| \leq c_2 |x|.$$

Moreover, since $H(x)$ is smooth

$$\left| \frac{\partial^{|\beta_j|}H}{\partial x_{\beta_j}}(x) \right| \leq c |x|^{\max(2-|\beta_j|, 0)} \leq c |x|^{2-|\beta_j|}.$$

It follows that

$$\begin{aligned} & \left| [H(x)]^{\gamma/2-k} \frac{\partial^{|\beta_1|}H}{\partial x_{\beta_1}}(x) \times \cdots \times \frac{\partial^{|\beta_k|}H}{\partial x_{\beta_k}}(x) \right| \\ & \leq c \left(|x|^2 \right)^{\gamma/2-k} |x|^{2-|\beta_1|} \times \cdots \times |x|^{2-|\beta_k|} \leq c |x|^{\gamma-2k} |x|^{2k-(|\beta_1|+\cdots+|\beta_k|)} \\ & \leq c |x|^{\gamma-|\alpha|}. \end{aligned}$$

This proves (2.2). To prove (2.1) let us write

$$\begin{aligned} & \frac{\partial^2 \Phi}{\partial x_j \partial x_k}(x) \\ & = \frac{\gamma}{2} (\gamma/2 - 1) [H(x)]^{\gamma/2-2} \frac{\partial H}{\partial x_j}(x) \frac{\partial H}{\partial x_k}(x) + \frac{\gamma}{2} [H(x)]^{\gamma/2-1} \frac{\partial^2 H}{\partial x_j \partial x_k}(x) \\ & = \frac{\gamma}{2} [H(x)]^{\gamma/2-1} \left(\frac{\partial^2 H}{\partial x_j \partial x_k}(x) + (\gamma/2 - 1) \frac{\frac{\partial H}{\partial x_j}(x) \frac{\partial H}{\partial x_k}(x)}{H(x)} \right) \end{aligned}$$

so that

$$\text{Hess } \Phi(x) = \frac{\gamma}{2} [H(x)]^{\gamma/2-1} \text{Hess } M(x)$$

where $M(x)$ is the matrix with entries

$$\frac{\partial^2 H}{\partial x_j \partial x_k}(x) + (\gamma/2 - 1) \frac{\frac{\partial H}{\partial x_j}(x) \frac{\partial H}{\partial x_k}(x)}{H(x)}.$$

Let $A = \text{Hess } H(0)$, since

$$\begin{aligned} H(x) &= \frac{1}{2} x^T A x + O(|x|^3), \\ \nabla H(x) &= A x + O(|x|^2) \\ \text{Hess } H(x) &= A + O(|x|) \end{aligned}$$

we have

$$\begin{aligned} M(x) &= A + O(|x|) + (\gamma/2 - 1) \frac{\left(Ax + O(|x|^2)\right) \left(Ax + O(|x|^2)\right)^T}{\frac{1}{2} x^T A x + O(|x|^3)} \\ &= A + (\gamma - 2) \frac{Ax(Ax)^T}{x^T A x} + O(|x|). \end{aligned}$$

Let us show that the matrix

$$A + (\gamma - 2) \frac{Ax(Ax)^T}{x^T A x}$$

is positive definite. Indeed, for all $y \in \mathbb{R}^{d-1}$ we have

$$y^T \left(A + (\gamma - 2) \frac{Ax(Ax)^T}{x^T A x} \right) y = y^T A y + (\gamma - 2) \frac{(y^T A x)^2}{x^T A x}.$$

When $\gamma \geq 2$ we easily obtain

$$y^T \left(A + (\gamma - 2) \frac{Ax(Ax)^T}{x^T A x} \right) y \geq y^T A y \geq \lambda_1 |y|^2$$

where λ_1 is the smallest eigenvalue of A . For $1 < \gamma < 2$, by Cauchy-Schwarz inequality for the inner product defined by $\langle y, x \rangle = y^T A x$ we have

$$\frac{(y^T A x)^2}{x^T A x} \leq \frac{(y^T A y)(x^T A x)}{x^T A x} = y^T A y.$$

Hence

$$\begin{aligned} y^T \left(A + (\gamma - 2) \frac{Ax(Ax)^T}{x^T A x} \right) y &\geq y^T A y + (\gamma - 2) y^T A y \\ &= (\gamma - 1) y^T A y \geq (\gamma - 1) \lambda_1 |y|^2. \end{aligned}$$

Let $\mu_1(x)$ be the smallest eigenvalue of $\text{Hess } \Phi(x)$. We want to show that $\mu_1(x) \geq c|x|^{\gamma-2}$. This is equivalent to show that

$$y^T \text{Hess } \Phi(x) y \geq c|x|^{\gamma-2}|y|^2.$$

We have

$$\begin{aligned} y^T \text{Hess } \Phi(x) y &= \frac{\gamma}{2} [H(x)]^{\gamma/2-1} y^T \left(A + (\gamma-2) \frac{Ax(Ax)^T}{x^T Ax} \right) y \\ &\quad + [H(x)]^{\gamma/2-1} y^T O(|x|) y \\ &\geq c_1 \left(|x|^2 \right)^{\gamma/2-1} |y|^2 - c_2 \left(|x|^2 \right)^{\gamma/2-1} |x| |y|^2 \\ &= c_1 |x|^{\gamma-2} |y|^2 - c_2 |x|^{\gamma-2} |x| |y|^2 \geq c|x|^{\gamma-2}|y|^2 \end{aligned}$$

for $|x|$ small enough. \square

To prove the theorems and the corollary it is convenient to introduce a mollified discrepancy. In the next lemma we assume that the origin is in the interior of the body. Since we will apply this lemma to prove estimates that are invariant under translations of the body we can always reduce to this case.

Lemma 16. *Assume that the origin is an interior point of B and let $\varphi(\mathbf{z})$ be a nonnegative compactly supported smooth function in \mathbb{R}^d with integral 1. Then, if the support of $\varphi(\mathbf{z})$ is sufficiently small, for every $0 < \varepsilon < 1$ and $R > 1$ we have*

$$\varepsilon^{-d} \varphi(\varepsilon^{-1} \cdot) * \chi_{(R-\varepsilon)B}(\mathbf{z}) \leq \chi_{RB}(\mathbf{z}) \leq \varepsilon^{-d} \varphi(\varepsilon^{-1} \cdot) * \chi_{(R+\varepsilon)B}(\mathbf{z}).$$

In particular,

$$|B| \left((R-\varepsilon)^d - R^d \right) + D_{\varepsilon, R-\varepsilon}(\mathbf{z}) \leq D_R(\mathbf{z}) \leq |B| \left((R+\varepsilon)^d - R^d \right) + D_{\varepsilon, R+\varepsilon}(\mathbf{z}),$$

where

$$D_{\varepsilon, R}(\mathbf{z}) = R^d \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R \mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}}.$$

The above lemma is well known. See e.g. [3, pag. 195] for a proof.

Also the following result is well known, the following is elementary proof.

Lemma 17. *For every integer $M > 0$ and every neighborhood U of the origin in \mathbb{R}^d there exists a smooth function $\varphi(\mathbf{z})$ supported in U such that*

$$\widehat{\varphi}(\mathbf{0}) = 1$$

and for every multi-index α , with $0 < |\alpha| \leq M$

$$\frac{\partial^{|\alpha|} \widehat{\varphi}}{\partial \zeta^\alpha}(\mathbf{0}) = 0.$$

Proof. Let $\psi(\mathbf{z})$ be a smooth function supported in U such that

$$\int_{\mathbb{R}^d} \psi(\mathbf{z}) d\mathbf{z} = 1.$$

We want to find constants c_0, c_1, \dots, c_M such that the function

$$\varphi(\mathbf{z}) = \sum_{k=0}^M 2^{kd} c_k \psi(2^k \mathbf{z})$$

satisfies the lemma. We have

$$\widehat{\varphi}(\zeta) = \sum_{k=0}^M c_k \widehat{\psi}(2^{-k} \zeta)$$

so that

$$\widehat{\varphi}(0) = \sum_{k=0}^M c_k \widehat{\psi}(0) = \sum_{k=0}^M c_k$$

and for every multi-index α

$$\frac{\partial^{|\alpha|} \widehat{\varphi}}{\partial \zeta^\alpha}(0) = \sum_{k=0}^M c_k 2^{-k|\alpha|} \frac{\partial^{|\alpha|} \widehat{\psi}}{\partial \zeta^\alpha}(0).$$

Hence the coefficients c_k are the solution of the non singular linear system

$$\begin{cases} c_0 + c_2 + \dots + c_M = 1 \\ (2^{-1})^0 c_0 + (2^{-1})^1 c_1 + \dots + (2^{-1})^M c_M = 0 \\ \vdots \\ (2^{-M})^0 c_0 + (2^{-M})^1 c_1 + \dots + (2^{-M})^M c_M = 0 \end{cases}$$

□

The following lemma collects the main estimates that we will use later.

Lemma 18. *Assume the inequalities*

$$|\widehat{\chi}_B(\zeta)| \leq \begin{cases} c |s|^{-1 - \frac{d-1}{\gamma}}, \\ c |\xi|^{-(d-1)\frac{\gamma-2}{2(\gamma-1)}} |s|^{-\frac{d-1}{2(\gamma-1)} - 1}, \\ c |\xi|^{-\frac{d+1}{2}}. \end{cases}$$

proved in Proposition 11, where $\zeta = \xi + s\Theta$, with $s = \zeta \cdot \Theta$ and $\xi \cdot \Theta = 0$ for some $\Theta \in \mathbb{R}^d$ with $|\Theta| = 1$ and $\gamma > 2$ and let $\varphi(\mathbf{z})$ as in the previous lemma.

1) *For every $\tau > 0$ and $p > 2d/(d-1)$ there exists c such that for every $\varepsilon > 0$ and $R > 1$,*

$$\left(\int_{\mathbb{T}^d} \left| R^d \sum_{|\mathbf{m} - (\mathbf{m} \cdot \Theta)\Theta| \geq \tau} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \right|^p d\mathbf{z} \right)^{1/p} \leq c R^{\frac{d-1}{2}} \varepsilon^{-\frac{d-1}{2} + \frac{d}{p}}.$$

2) For every $\tau > 0$ there exists c such that for every $\varepsilon > 0$, $R > 1$ and $\mathbf{z} \in \mathbb{T}^d$,

$$\left| R^d \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d, |\mathbf{m} - (\mathbf{m} \cdot \Theta)\Theta| < \tau} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \right| \leq c R^{(d-1)(1-\frac{1}{\gamma})}.$$

Proof. Let us prove 1). For every $\mathbf{m} \in \mathbb{Z}^d$, write $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$ with $\mathbf{m}_1 = \mathbf{m} - (\mathbf{m} \cdot \Theta)\Theta$ and $\mathbf{m}_2 = (\mathbf{m} \cdot \Theta)\Theta$. Also observe that for every $M > 0$,

$$|\widehat{\varphi}(\zeta)| \leq c_M (1 + |\zeta|)^{-M}.$$

Since $p \geq 2$, by the Hausdorff-Young inequality with $1/p + 1/q = 1$ and the assumption on $\widehat{\chi}_B(\zeta)$ we have

$$\begin{aligned} & \left(\int_{\mathbb{T}^d} \left| R^d \sum_{|\mathbf{m}_1| \geq \tau} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \right|^p d\mathbf{z} \right)^{q/p} \\ & \leq R^{dq} \sum_{|\mathbf{m}_1| \geq \tau} |\widehat{\varphi}(\varepsilon \mathbf{m})|^q |\widehat{\chi}_B(R\mathbf{m})|^q \\ & \leq cR^{q\frac{d-1}{2}} \sum_{\tau \leq |\mathbf{m}_1| \leq |\mathbf{m}_2|} (1 + \varepsilon |\mathbf{m}_2|)^{-M} |\mathbf{m}_1|^{-q(d-1)\frac{\gamma-2}{2(\gamma-1)}} |\mathbf{m}_2|^{-q\frac{d-1}{2(\gamma-1)}-q} \\ & + cR^{q\frac{d-1}{2}} \sum_{|\mathbf{m}_1| > \max(|\mathbf{m}_2|, \tau)} (1 + \varepsilon |\mathbf{m}_1|)^{-M} |\mathbf{m}_1|^{-\frac{d+1}{2}q} \\ & = A + B. \end{aligned}$$

Since in the series in A the quantities $|\mathbf{m}_1|$ and $|\mathbf{m}_2|$ are bounded away from zero we can control the series with an integral,

$$\begin{aligned} & \sum_{\tau < |\mathbf{m}_1| \leq |\mathbf{m}_2|} (1 + \varepsilon |\mathbf{m}_2|)^{-M} |\mathbf{m}_1|^{-q(d-1)\frac{\gamma-2}{2(\gamma-1)}} |\mathbf{m}_2|^{-q\frac{d-1}{2(\gamma-1)}-q} \\ & \leq c \iint_{\{\tau < |\xi| \leq |s|\}} (1 + \varepsilon |s|)^{-M} |\xi|^{-q(d-1)\frac{\gamma-2}{2(\gamma-1)}} |s|^{-q\frac{d-1}{2(\gamma-1)}-q} d\xi ds \\ & \leq c \int_{\tau}^{+\infty} (1 + \varepsilon |s|)^{-M} |s|^{-q\frac{d-1}{2(\gamma-1)}-q} \left[\int_{\{|\xi| \leq |s|\}} |\xi|^{-q(d-1)\frac{\gamma-2}{2(\gamma-1)}} d\xi \right] ds \\ & \leq c \int_{\tau}^{+\infty} (1 + \varepsilon |s|)^{-M} |s|^{d-1-q\frac{d+1}{2}} ds \leq c\varepsilon^{-(d-q\frac{d+1}{2})} \end{aligned}$$

(note that since $p > 2d/(d-1)$ we have $q < 2d/(d+1)$). Similarly, for the series in B ,

$$\begin{aligned} & \sum_{|\mathbf{m}_1| > \max(|\mathbf{m}_2|, \tau)} (1 + \varepsilon |\mathbf{m}_1|)^{-M} |\mathbf{m}_1|^{-\frac{d+1}{2}q} \\ & \leq c \iint_{\{|\xi| > |s|\}} (1 + \varepsilon |\xi|)^{-M} |\xi|^{-\frac{d+1}{2}q} d\xi ds \end{aligned}$$

$$= c\varepsilon^{\frac{d+1}{2}q-d} \int_{\mathbb{R}^{d-1}} (1+|\xi|)^{-M} |\xi|^{1-\frac{d+1}{2}q} d\xi = c\varepsilon^{\frac{d+1}{2}q-d}.$$

This proves point 1) in the statement. Similarly, to prove point 2) observe that, by the assumption on $\widehat{\chi}_B(\zeta)$, we have

$$\begin{aligned} \left| R^d \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d, |\mathbf{m}_1| < \tau} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \right| &\leq R^d \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d, |\mathbf{m}_1| < \tau} |\widehat{\chi}_B(R\mathbf{m})| \\ &\leq cR^{(d-1)(1-\frac{1}{\gamma})} \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d, |\mathbf{m}_1| < \tau} |\mathbf{m}_2|^{-1-\frac{d-1}{\gamma}}. \end{aligned}$$

Note that the last series is essentially one dimensional and it is convergent. \square

Proof of Theorem 4. Without loss of generality we can assume that the origin is an interior point of B . The discrepancy will be estimated using the size of $\widehat{\chi}_B(\zeta)$. Since the main contribution to the size of this Fourier transform comes from the flat points on ∂B and since with a suitable partition of unity we can isolate such flat points, without loss of generality we can assume the existence of only a single flat point of order γ .

The case $1 < \gamma \leq 2$ follows from the argument used in [2] for the smooth case. This essentially reduces to the Hausdorff-Young inequality and follows from the estimate

$$|\widehat{\chi}_B(\zeta)| \leq c|\zeta|^{-\frac{d+1}{2}}$$

that holds true also in our case by (3.3). Let us now prove point 2) and point 3) in the theorem. To prove point 2) we observe that the case $p \leq 2d/(d+1-\gamma)$ follows from the case $p = 2d/(d+1-\gamma)$, and the case $2d/(d+1-\gamma) \leq p \leq +\infty$ follows by interpolation between $p = 2d/(d+1-\gamma)$ and $p = +\infty$. Hence to prove point 2) it suffices to consider only the cases $p = 2d/(d+1-\gamma)$ and $p = +\infty$. Similarly to prove point 3) it suffices to consider only the case $p = +\infty$. Observe that since $\gamma > 2$, all these values of p are greater than $2d/(d-1)$.

By Lemma 16 we have

$$\begin{aligned} \|D_R\|_{L^p(\mathbb{T}^d)} &\leq |B| \max_{\pm} \left| (R \pm \varepsilon)^d - R^d \right| + \max_{\pm} \|D_{\varepsilon, R \pm \varepsilon}\|_{L^p(\mathbb{T}^d)} \\ &\leq cR^{d-1}\varepsilon + \max_{\pm} \|D_{\varepsilon, R \pm \varepsilon}\|_{L^p(\mathbb{T}^d)}. \end{aligned}$$

Replacing $R \pm \varepsilon$ with R for simplicity, Lemma 18, with a fixed $\tau > 0$, gives

$$\begin{aligned} \|D_{\varepsilon, R}\|_{L^p(\mathbb{T}^d)} &= \left(\int_{\mathbb{T}^d} \left| R^d \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \right|^p d\mathbf{z} \right)^{1/p} \\ &\leq \left(\int_{\mathbb{T}^d} \left| R^d \sum_{|\mathbf{m} - (\mathbf{m} \cdot \Theta)\Theta| \geq \tau} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \right|^p d\mathbf{z} \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
& + \left| R^d \sum_{\substack{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d, \\ |\mathbf{m} - (\mathbf{m} \cdot \Theta)\Theta| < \tau}} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \right| \\
& \leq cR^{\frac{d-1}{2}} \varepsilon^{-\frac{d-1}{2} + \frac{d}{p}} + cR^{(d-1)(1-\frac{1}{\gamma})}
\end{aligned}$$

The choice $\varepsilon = R^{-\frac{d-1}{d+1-2d/p}}$ then gives

$$\|D_R\|_{L^p(\mathbb{T}^d)} \leq cR^{d(d-1)\frac{p-2}{p-2d+dp}} + cR^{(d-1)(1-\frac{1}{\gamma})}.$$

For $p = 2d/(d+1-\gamma)$ and $2 < \gamma \leq d+1$, or $p = +\infty$ and $\gamma > d+1$ we obtain

$$\|D_R\|_{L^p(\mathbb{T}^d)} \leq cR^{(d-1)(1-\frac{1}{\gamma})}.$$

For $2 < \gamma < d+1$ and $p = +\infty$ we obtain

$$\|D_R\|_{L^\infty(\mathbb{T}^d)} \leq cR^{\frac{d(d-1)}{d+1}}.$$

□

Proof of Theorem 5. Once again we can assume that the origin is an interior point of B . The smoothness of $S(t)$ is proved in [1, Lemma 4.3]. For every $\mathbf{m} \in \mathbb{Z}^d$, write $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$ with $\mathbf{m}_1 = \mathbf{m} - (\mathbf{m} \cdot \Theta)\Theta$ and $\mathbf{m}_2 = (\mathbf{m} \cdot \Theta)\Theta$, and split the Fourier expansion of the discrepancy as

$$(4.1) \quad D_R(\mathbf{z}) = R^d \sum_{\mathbf{m}_1=\mathbf{0}, \mathbf{m}_2 \neq \mathbf{0}} \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} + R^d \sum_{\mathbf{m}_1 \neq \mathbf{0}} \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}}.$$

We will see that the main term is the first one and it follows from Proposition 13 that

$$\begin{aligned}
& R^d \sum_{\mathbf{m}_1=\mathbf{0}, \mathbf{m}_2 \neq \mathbf{0}} \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \\
& \sim R^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z}-RP) + R^{(d-1)(1-1/\gamma_Q)} A_Q(\mathbf{z}-RQ).
\end{aligned}$$

The details are as follows. Let $D_{\varepsilon,R}(\mathbf{z})$ be the mollified discrepancy as in the proof of Theorem 4 and let

$$Y(\mathbf{z},R) = R^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z}-RP) + R^{(d-1)(1-1/\gamma_Q)} A_Q(\mathbf{z}-RQ).$$

From Lemma 16 with a cut-off function as in Lemma 17 we have

$$\begin{aligned}
(4.2) \quad & |D_R(\mathbf{z}) - Y(\mathbf{z},R)| \\
& \leq |B| \max_{\pm} \left| (R \pm \varepsilon)^d - R^d \right| + \max_{\pm} |D_{\varepsilon,R \pm \varepsilon}(\mathbf{z}) - Y(\mathbf{z},R \pm \varepsilon)| \\
& + |Y(\mathbf{z},R \pm \varepsilon) - Y(\mathbf{z},R)|
\end{aligned}$$

The first term in the right-hand side is bounded by $cR^{d-1}\varepsilon$. For the third term we have

$$\begin{aligned} & |Y(\mathbf{z}, R \pm \varepsilon) - Y(\mathbf{z}, R)| \\ & \leq \left| (R \pm \varepsilon)^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z} - (R \pm \varepsilon)P) - R^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z} - RP) \right| \\ & + \left| (R \pm \varepsilon)^{(d-1)(1-1/\gamma_Q)} A_Q(\mathbf{z} - (R \pm \varepsilon)Q) - R^{(d-1)(1-1/\gamma_Q)} A_Q(\mathbf{z} - RQ) \right|. \end{aligned}$$

The two terms are similar, let us consider only the first one. Then

$$\begin{aligned} & \left| (R \pm \varepsilon)^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z} - (R \pm \varepsilon)P) - R^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z} - RP) \right| \\ & \leq (R \pm \varepsilon)^{(d-1)(1-1/\gamma_P)} |A_P(\mathbf{z} - (R \pm \varepsilon)P) - A_P(\mathbf{z} - RP)| \\ & + \left| (R \pm \varepsilon)^{(d-1)(1-1/\gamma_P)} - R^{(d-1)(1-1/\gamma_P)} \right| |A_P(\mathbf{z} - RP)| \end{aligned}$$

Since

$$\begin{aligned} & |A_P(\mathbf{z} - (R \pm \varepsilon)P) - A_P(\mathbf{z} - RP)| \\ & \leq c \sum_{k=1}^{+\infty} k^{-1-\frac{d-1}{\gamma_P}} \\ & \times \left| \sin\left(2\pi k \mathbf{m}_0 \cdot (\mathbf{z} - (R \pm \varepsilon)P) - \frac{\pi}{2} \frac{d-1}{\gamma_P}\right) - \sin\left(2\pi k \mathbf{m}_0 \cdot (\mathbf{z} - RP) - \frac{\pi}{2} \frac{d-1}{\gamma_P}\right) \right| \\ & \leq c \sum_{k=1}^{+\infty} k^{-1-\frac{d-1}{\gamma_P}} |\sin(\varepsilon \pi k \mathbf{m}_0 \cdot P)| \\ & \leq c \varepsilon^{\frac{d-1}{\gamma_P}}. \end{aligned}$$

and $|A_P(\mathbf{z} - RP)| \leq c$ we have

$$\begin{aligned} & \left| (R \pm \varepsilon)^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z} - (R \pm \varepsilon)P) - R^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z} - RP) \right| \\ & \leq c R^{(d-1)(1-1/\gamma_P)} \varepsilon^{\frac{d-1}{\gamma_P}} + R^{(d-1)(1-1/\gamma_P)-1} \varepsilon. \end{aligned}$$

It remains to estimate the second term in (4.2) and for simplicity we replace $R \pm \varepsilon$ with R . We have

$$\begin{aligned} (4.3) \quad & D_{\varepsilon, R}(\mathbf{z}) - Y(\mathbf{z}, R) \\ & = \left(R^d \sum_{\mathbf{m}_1=\mathbf{0}, \mathbf{m}_2 \neq \mathbf{0}} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R \mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} - Y(\mathbf{z}, R) \right) \\ & + \left(R^d \sum_{\mathbf{m}_1 \neq \mathbf{0}} \widehat{\varphi}(\varepsilon \mathbf{m}) \widehat{\chi}_B(R \mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} \right) \\ & = I(\mathbf{z}) + II(\mathbf{z}). \end{aligned}$$

For $I(\mathbf{z})$ we have a pointwise estimate,

$$|I(\mathbf{z})| \leq R^d \sum_{s \neq 0} |\widehat{\varphi}(\varepsilon s \mathbf{m}_0) - 1| |\widehat{\chi}_B(Rs \mathbf{m}_0)| \\ + \left| R^d \sum_{s \neq 0} \widehat{\chi}_B(Rs \mathbf{m}_0) e^{2\pi i s \mathbf{m}_0 \cdot \mathbf{z}} - Y(\mathbf{z}, R) \right|.$$

Our choice of the function $\varphi(\mathbf{z})$ yields

$$\frac{d^{|\alpha|} \widehat{\varphi}}{d\zeta^\alpha}(\mathbf{0}) = 0$$

for every multi-index α with $|\alpha| \leq M$. Hence

$$|\widehat{\varphi}(\zeta) - 1| \leq c_M |\zeta|^M,$$

and by Proposition 11 (recall that $\gamma_P \geq \gamma_Q$)

$$R^d \sum_{s \neq 0} |\widehat{\varphi}(\varepsilon s \mathbf{m}_0) - 1| |\widehat{\chi}_B(Rs \mathbf{m}_0)| \\ \leq cR^{(d-1)(1-\frac{1}{\gamma_P})} \sum_{s \neq 0} \min(\varepsilon^M |s|^M, 1) |s|^{-1-\frac{d-1}{\gamma_P}} \\ \leq cR^{(d-1)(1-\frac{1}{\gamma_P})} \sum_{\varepsilon |s| \leq 1} \varepsilon^M |s|^M |s|^{-1-\frac{d-1}{\gamma_P}} + cR^{(d-1)(1-\frac{1}{\gamma_P})} \sum_{\varepsilon |s| > 1} |s|^{-1-\frac{d-1}{\gamma_P}} \\ \leq cR^{(d-1)(1-\frac{1}{\gamma_P})} \varepsilon^{\frac{d-1}{\gamma_P}}.$$

By our assumption on the direction Θ and by Proposition 13 a long but direct computation gives

$$R^d \sum_{\mathbf{m}_1 = \mathbf{0}, \mathbf{m} \neq \mathbf{0}} \widehat{\chi}_B(R\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{z}} = R^d \sum_{s \neq 0} \widehat{\chi}_B(Rs \mathbf{m}_0) e^{2\pi i s \mathbf{m}_0 \cdot \mathbf{z}} \\ = R^{(d-1)(1-1/\gamma_P)} A_P(\mathbf{z} - RP) + R^{(d-1)(1-1/\gamma_Q)} A_Q(\mathbf{z} - RQ) \\ + O\left(R^{d-1-\frac{d}{\gamma_P}}\right) \\ = Y(\mathbf{z}, R) + O\left(R^{d-1-\frac{d}{\gamma_P}}\right)$$

Hence, we have the pointwise estimate

$$(4.4) \quad |I(\mathbf{z})| \leq cR^{(d-1)(1-\frac{1}{\gamma_P})} \varepsilon^{\frac{d-1}{\gamma_P}} + R^{d-1-\frac{d}{\gamma_P}}.$$

The assumption that $\alpha\Theta \in \mathbb{Z}^d$ for some α implies that the requirement $\mathbf{m}_1 \neq \mathbf{0}$ is equivalent to $|\mathbf{m}_1| \geq \tau$ for some $\tau > 0$. By Lemma 18 we therefore have

$$(4.5) \quad \|II\|_p \leq cR^{\frac{d-1}{2}} \varepsilon^{-\frac{d-1}{2} + \frac{d}{p}}.$$

Collecting the estimates (4.3), (4.4) and (4.5) we have

$$\begin{aligned} & \|D_R(\mathbf{z}) - Y(\mathbf{z}, R)\|_{L^p(\mathbb{T}^d)} \\ & \leq cR^{(d-1)(1-1/\gamma_P)} \varepsilon^{\frac{d-1}{\gamma_P}} + cR^{d-1} \varepsilon + cR^{d-1-\frac{d}{\gamma_P}} + cR^{\frac{d-1}{2}} \varepsilon^{-\frac{d-1}{2}+d/p}. \end{aligned}$$

The choice $\varepsilon = R^{-\frac{d-1}{d+1-2d/p}}$ gives

$$\begin{aligned} & \|D_R(\mathbf{z}) - Y(\mathbf{z}, R)\|_{L^p(\mathbb{T}^d)} \\ & \leq cR^{(d-1)\left(1-\frac{1}{\gamma_P}\right)} \left(R^{-\frac{d-1}{d+1-2d/p} \frac{d-1}{\gamma_P}} + R^{\frac{d-1}{\gamma_P} - \frac{d-1}{d+1-2d/p}} + R^{-\frac{1}{\gamma_P}} \right). \end{aligned}$$

Since our assumption implies $1/p > (d+1-\gamma_P)/(2d)$, all the exponents of R in the parenthesis are negative and therefore

$$\|D_R(\mathbf{z}) - Y(\mathbf{z}, R)\|_{L^p(\mathbb{T}^d)} \leq cR^{(d-1)\left(1-\frac{1}{\gamma_P}\right)-\delta}$$

for some $\delta > 0$. This proves immediately point 2). It also prove point 1) as long as one notices that if $\gamma_P > \gamma_Q$

$$\left\| R^{(d-1)(1-1/\gamma_Q)} A_Q(\mathbf{z}-RQ) \right\|_{L^p(\mathbb{T}^d)} \leq cR^{(d-1)\left(1-\frac{1}{\gamma_P}\right)-\delta}$$

for a suitable $\delta > 0$. □

Proof of Corollary 6. Because of our assumptions the constants in front of the two series that define $A_P(\mathbf{z}-RP)$ and $A_Q(\mathbf{z}-RQ)$ are the same. A simple computation gives

$$\begin{aligned} & A_P(\mathbf{z}-RP) + A_Q(\mathbf{z}-RQ) \\ & = \frac{4G_P(0)\Gamma\left(\frac{d-1}{\gamma}+1\right)}{(2\pi|\mathbf{m}_0|)^{\frac{d-1}{\gamma}+1}} \sum_{k=1}^{+\infty} k^{-1-\frac{d-1}{\gamma}} \sin\left(\pi\left(k\mathbf{m}_0 \cdot R(Q-P) - \frac{d-1}{2\gamma}\right)\right) \\ & \quad \times \cos\left(2\pi k\mathbf{m}_0 \cdot \left(\mathbf{z}-R\frac{P+Q}{2}\right)\right). \end{aligned}$$

and

$$A_P(\mathbf{z}-RP) + A_Q(\mathbf{z}-RQ) = 0$$

since $\mathbf{m}_0 \cdot R(Q-P)$ and $\frac{d-1}{2\gamma}$ are integers. □

Proof of Theorem 8. By Theorem 1.1 in [5] we have the following estimate for the L^2 average decay of the Fourier transform

$$\left(\int_{SO(d)} |\widehat{\chi}_B(R\sigma\mathbf{m})|^2 d\sigma \right)^{1/2} \leq c(R|\mathbf{m}|)^{-\frac{d+1}{2}}.$$

Hence, applying the Hausdorff-Young inequality to (1.1) with $2 \leq p < 2d/(d-1)$ and $1/p + 1/q = 1$, we obtain

$$\begin{aligned} \int_{SO(d)} \left(\int_{\mathbb{T}^d} |D_{R,\sigma}(\mathbf{z})|^p d\mathbf{z} \right)^{q/p} d\sigma &\leq R^{dq} \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d} \int_{SO(d)} |\widehat{\chi}_B(R\sigma\mathbf{m})|^q d\sigma \\ &\leq R^{dq} \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d} \left(\int_{SO(d)} |\widehat{\chi}_B(R\sigma\mathbf{m})|^2 d\sigma \right)^{q/2} \\ &\leq cR^{q\frac{d-1}{2}} \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d} |\mathbf{m}|^{-q\frac{d+1}{2}} \leq cR^{q\frac{d-1}{2}}. \end{aligned}$$

In a similar way if $1 \leq p \leq 2$ we have,

$$\begin{aligned} \int_{SO(d)} \left(\int_{\mathbb{T}^d} |D_{R,\sigma}(\mathbf{z})|^p d\mathbf{z} \right)^{2/p} d\sigma &\leq \int_{SO(d)} \int_{\mathbb{T}^d} |D_{R,\sigma}(\mathbf{z})|^2 d\mathbf{z} d\sigma \\ &= R^{2d} \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d} \int_{SO(d)} |\widehat{\chi}_B(R\sigma\mathbf{m})|^2 d\sigma \\ &\leq cR^{d-1} \sum_{\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d} |\mathbf{m}|^{-(d+1)} \leq cR^{d-1}. \end{aligned}$$

□

Proof of Theorem 9. Without loss of generality we can assume $|\alpha| \leq |\beta|$. By Proposition 11 we have

$$|\widehat{\chi}_B(m, n)| \leq \begin{cases} c|\alpha m + \beta n|^{-1-\frac{1}{\gamma}}, \\ c|-\beta m + \alpha n|^{-\frac{\gamma-2}{2(\gamma-1)}} |\alpha m + \beta n|^{-\frac{1}{2(\gamma-1)}-1}, \\ c|-\beta m + \alpha n|^{-\frac{3}{2}}. \end{cases}$$

We have

$$\begin{aligned} \int_{\mathbb{T}^d} |D_R(\mathbf{z})|^2 d\mathbf{z} &= R^4 \sum_{(m,n) \neq (0,0)} |\widehat{\chi}_B(Rm, Rn)|^2 \\ &\leq R^4 \sum_{0 < |-\beta m + \alpha n| < 1/2} |\widehat{\chi}_B(Rm, Rn)|^2 \end{aligned}$$

$$\begin{aligned}
& + R^4 \sum_{1/2 \leq |-\beta m + \alpha n| \leq |\alpha m + \beta n|} |\widehat{\chi}_B(Rm, Rn)|^2 \\
& + R^4 \sum_{0 < |\alpha m + \beta n| < |-\beta m + \alpha n|} |\widehat{\chi}_B(Rm, Rn)|^2 \\
& = I + II + III.
\end{aligned}$$

Using the above estimate for $\widehat{\chi}_B(Rm, Rn)$ we have

$$III \leq cR \sum_{0 < |\alpha m + \beta n| < |-\beta m + \alpha n|} |(m, n)|^{-3} \leq cR \sum_{(m, n) \neq (0, 0)} |(m, n)|^{-3} \leq cR.$$

In the term II the quantity $|-\beta m + \alpha n|$ and $|\alpha m + \beta n|$ are bounded away from zero so that, arguing as in the proof of Lemma 18, we can replace the series with the corresponding integral,

$$\begin{aligned}
II & \leq cR \sum_{1/2 \leq |-\beta m + \alpha n| < |\alpha m + \beta n|} |-\beta m + \alpha n|^{-\frac{\gamma-2}{\gamma-1}} |\alpha m + \beta n|^{-\frac{1}{\gamma-1}-2} \\
& \leq cR \int_{\{1/2 \leq |\xi| < |s|\}} |\xi|^{-\frac{\gamma-2}{\gamma-1}} |s|^{-\frac{1}{\gamma-1}-2} ds d\xi \leq cR.
\end{aligned}$$

In the term I observe that $|-\beta m + \alpha n| < 1/2$ implies $|\alpha m + \beta n| \approx |n|$. Then

$$\begin{aligned}
I & \leq cR \sum_{|-\beta m + \alpha n| < 1/2} |-\beta m + \alpha n|^{-\frac{\gamma-2}{\gamma-1}} |\alpha m + \beta n|^{-\frac{1}{\gamma-1}-2} \\
& \leq cR \sum_{|-\beta m + \alpha n| < 1/2} \left\| \frac{\alpha}{\beta} n \right\|^{-\frac{\gamma-2}{\gamma-1}} |n|^{-\frac{1}{\gamma-1}-2} \\
& \leq cR \sum_{|-\beta m + \alpha n| < 1/2} \left(|n|^{-1-\delta} \right)^{-\frac{\gamma-2}{\gamma-1}} |n|^{-\frac{1}{\gamma-1}-2} \\
& \leq cR \sum_{n=1}^{+\infty} n^{(1+\delta)\frac{\gamma-2}{\gamma-1}} n^{-\frac{1}{\gamma-1}-2} \leq cR.
\end{aligned}$$

In the last inequality we used the assumption $\delta < 2/(\gamma - 2)$. \square

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