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This research proposes several applications of stochastic orderings to portfolio selection problems. In the first chapter, an analysis of the relationship between second order stochastic dominance efficient set and the mean variance efficient frontier is proposed. In the second chapter, extensions of the classic definition of stochastic dominance efficiency linked to behavioral finance are given. The research continues by introducing risk diversification measures, along with the concept of mean risk diversification efficiency.

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UNIVERSITÀ DEGLI STUDI DI BERGAMO

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Introduction

Portfolio selection is the process of finding the optimal allocation of wealth across risky assets. Optimal allocation is better understood in terms of efficiency. The seminal work of Markowitz inaugurated this field of study (see Markowitz, 1952a). The mean-variance efficient frontier yields efficiency based on the trade-off between expected return and risk.

Consider a market with N assets. Call $Z = [Z_1, \dots, Z_N]'$ the stochastic vector of asset returns and Σ their covariance matrix. Under mean-variance efficiency, investors seek the portfolio with the lowest possible variance for a desired level of mean μ. Portfolios satisfying this condition are called mean-variance efficient (Markowitz, 1952a). The original problem formulation is:

$$\begin{aligned} \min_{w \in \mathbb{R}^N} \quad & w'\Sigma w \\ & w'e = 1 \\ & w'\mathbb{E}[Z] \geq \mu \\ & 0 \leq w_i \leq 1 \;\; i = 1, \dots, N \end{aligned}$$

where $e = [1, \dots, 1]'$. Let $w^*(\mu)$ be the solution; then the mean-variance efficient frontier is given by the set of points satisfying $(w^*(\mu)'\mathbb{E}[Z], w^*(\mu)'\Sigma w^*(\mu))$ for all admissible levels of μ.

The Sharpe ratio, defined as

$$SR(w) = \frac{w'\mathbb{E}[Z]}{\sqrt{w'\Sigma w}}$$

then provides an additional criteria for investor decision making (Sharpe, 1964). So, all investors with a mean-variance type of preference would prefer, among all mean-variance efficient portfolios, the one that maximizes the Sharpe ratio, i.e., the portfolio solution of the following problem (see Black, 1972; Stoyanov et al., 2007):

$$\max_{w^*(\mu)} \frac{w'\mathbb{E}[Z]}{\sqrt{w'\Sigma w}}$$

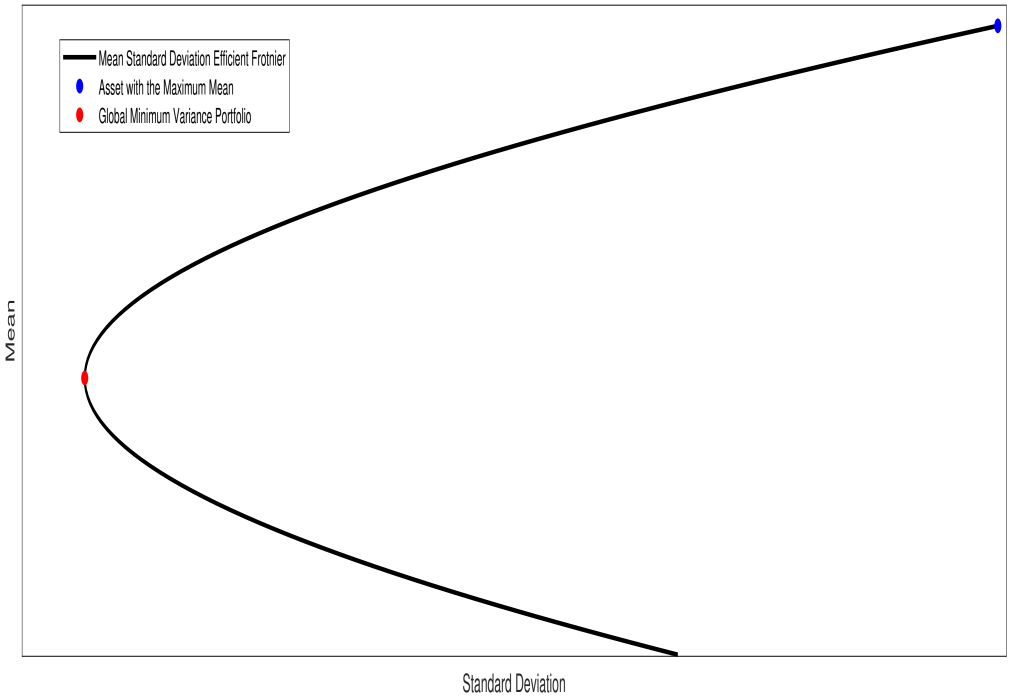


Figure 1: Example of mean-variance efficient frontier. The red and blue dots represent the projection of the global minimum-variance portfolio and the asset with the maximum mean onto the standard deviation mean plane. According to the theory of mean-variance efficiency, portfolios whose projection lies on the efficient frontiers present an optimal combination of expected return and risk.

The portfolio solution of the last problem is called the Markowitz market portfolio, or tangent portfolio (see Black, 1972; Ingersoll, 1987; Stoyanov et al., 2007).

Following Markowitz's analysis, Sharpe (1964), Lintner (1964), Mossin (1966), and Black (1972) developed one of the most famous asset-pricing models: the capital asset pricing model (CAPM). Under a series of strong assumptions, the CAPM established a relationship between any asset's return and the market portfolio (see Harris, 1972; Ingersoll, 1987). In particular, the expected excess return of any asset, over the risk-free rate, is proportional to the expected excess return of the market portfolio over the risk-free rate (see Ingersoll, 1987). Formally:

$$\mathbb{E}[Z_i] - r_f = \beta_{i,m} \left(\mathbb{E}[Z_m] - rf\right)$$

where r_f is the risk-free rate, Z_m is the return of the market portfolio, and $\beta_m = \frac{cov(Z_i, Z_m)}{var(Z_m)}$. $\beta_{m,i}$ is usually interpreted as the marginal contribution of the asset i to the market portfolio.

Despite being fascinating in its simplicity, mean-variance efficiency suffers from limitations in its theoretical validity. First, mean-variance preferences imply that investors have increasing risk aversion, a hypothesis that has been rejected in many empirical studies (see Bawa, 1975; Ingersoll, 1987; Levy, 1992; Levy and Levy, 2002). Furthermore, mean-variance efficiency requires the underlying asset-return distribution to be elliptical (see, for example, Bawa, 1975; Chamberlain, 1983). Expected utility is an alternative approach in portfolio optimization, overcoming such criticisms. Expected utility is based on an axiomatic description of investor preferences (see Von Neumann and Morgenstern, 2007). Typically, investors' preferences are represented via expected utility functions. An agent with a utility function u prefers a portfolio P_1 over a portfolio P_2 if $\mathbb{E}[u(P_1)] \geq \mathbb{E}[u(P_2)]$. The optimal allocation for such an investor would then be the solution of the following optimization problem:

$$\max_{w} \ \mathbb{E}[u(w'Z)]$$

Nevertheless, except in obvious circumstances, the true form of investors' utility functions is not known. Typically, instead of looking for the optimal allocation for a given utility function, we classify investors according to their attitude toward risk, and then express an ordering relationship for such categories. Stochastic dominance, and in general stochastic ordering, serve this purpose (see, for example, Quirk and Saposnik, 1962; Hadar and Russell, 1969; Hanoch and Levy, 1969; Bawa, 1975; Fishburn, 1976; Dybvig and Ross, 1982; Dybvig, 1988; Mosler and Scarsini, 1991; Levy, 1992; Post, 2003; Kuosmanen, 2004; De Giorgi, 2005; Baucells and Heukamp, 2006; Dentcheva and Ruszczyński, 2006; De Giorgi and Post, 2008; Kopa and Chovanec, 2008; Eeckhoudt et al., 2009; Ma and Wong, 2010; Hodder et al., 2014).

Stochastic dominance is a partial ordering in the space of distribution functions. Let X and Y be two random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution functions $F_X(t) = \mathbb{P}\left[X \leq t\right]$ and $F_Y(t) = \mathbb{P}\left[Y \leq t\right]$ respectively. The classic definition of stochastic dominance is given in terms of conditions imposed on iterative integrals of the distribution function. X dominates Y in the first order of stochastic dominance (FSD), if $F_X(t) \leq F_Y(t)$ $\forall t \in \mathbb{R}$; X dominates Y in the second order of stochastic dominance (SSD), if $\int_t^\infty F_X(u)du \leq \int_t^\infty F_Y(u)du$ $\forall t \in \mathbb{R}$. It can be shown that FSD is coherent with the choices of non-satiable investors, i.e., investors with a non-decreasing utility function ($u' > 0$), and SSD is coherent with the choices of non-satiable and risk-averse investors, i.e., investors with non-decreasing and concave utility functions ($u' > 0$ and $u'' \leq 0$).

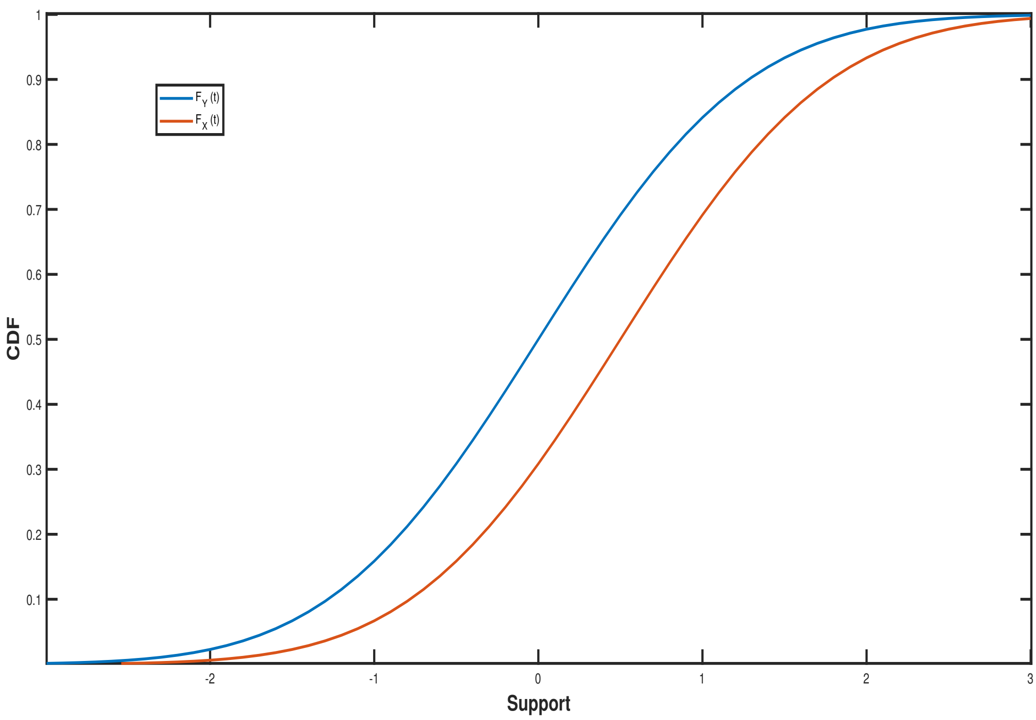


Figure 2: Cumulative distribution functions for X and Y. Since F_X always lies below F_Y, X dominates in the sense of the first order of stochastic dominance Y. Moreover, all non-satiable investors prefer X over Y.

Expected utility and stochastic dominance also provide a general definition of risk. In this context, risk can be defined as a property of random outcomes linked to investors' utility functions, i.e., risk is what risk-averse investors dislike (see, among others, Ingersoll, 1987). A general and widely accepted definition is given in terms of the mean-preserving spread, which implies that a random variable is riskier if it allocates more weight to the tails of the distribution. This very same concept is equivalent to the Rothschild-Stiglitz order of stochastic dominance: X dominates Y in the sense of Rothschild-Stiglitz order if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\int_t^\infty F_X(u)du \leq \int_t^\infty F_Y(u)du \; \forall t \in \mathbb{R}$ (Rothschild and Stiglitz, 1970, 1971; Landsberger and Meilijson, 1993).[1]

Expected utility and stochastic dominance have different definitions of efficiency with respect to mean variance. A portfolio is said to be efficient, with respect to a given stochastic ordering, if no

[1]Under the assumption of normally distributed returns, the variance is consistent with the Rothschild-Stiglitz order.

other portfolio is able to dominate it with respect to the given stochastic ordering. Efficient allocation with respect to a given stochastic ordering are then optimal for the corresponding category of investors. Stochastic orderings and mean-variance theory are rival approaches in portfolio selection. In particular, there is a considerable stream of literature suggesting that mean-variance efficiency is not consistent with SSD efficiency, implying that only under very rare circumstances would a non-satiable and risk-averse investor be a mean-variance optimizer (see, for example, Borch, 1969; Bawa, 1975; Markowitz, 2014; Loistl, 2015).

Based on this fact, Chapter 1 (Pareto optimal choices versus mean-variance optimal choices: A paradigm of portfolio theory) of this thesis serves as a justification for the use of stochastic orderings in portfolio selection. It considers a market composed of assets belonging to the Dow Jones Industrial Average (DJIA) index that contains 30 stocks of large, publicly owned United States–based companies (see Dybvig and Ross, 1982; Roman et al., 2006). The analysis is performed on monthly observations of stocks belonging to the DJIA from May 1997 to October 2017. Chapter 1 aims to compare mean-variance and SSD-efficient sets (see Dybvig and Ross, 1982; Roman et al., 2006). The two portfolio sets are indeed quite different in terms of portfolio composition and moments. In fact, the only portfolio belonging to both sets is the one composed of only the asset with the maximum mean. An interesting result is also the link to diversification: the diversification levels in the two sets are quite different. Portfolios belonging to the SSD-efficient set are, in general, less diversified than those in the mean-variance efficient frontier. It is well known that high risk aversion should imply higher diversification. However, the second order of stochastic dominance is consistent with diversification only under strong assumptions (see Wong, 2007; Egozcue and Wong, 2010). When applied to real markets, SSD efficiency doesn't select highly diversified portfolios (see, for example, Mansini et al., 2007). This last observation might serve as an explanation of the Statman diversification puzzle, i.e., that the diversification level observed in real markets is lower than those predicted by mean-variance theory (see Statman, 2004). Non-satiable and risk-averse investors prefer portfolios that are more concentrated than mean-variance efficient ones. The main results, however, are related to the different definitions of efficiency in the two approaches. In fact, it turns out that portfolios belonging to the mean-variance frontier where the mean is "low" are second-order stochastically dominated. This area also encompasses the global minimum-variance portfolio. The second part of the chapter illustrates the non-efficiency of the global minimum-variance (GMV) portfolio, constructs dominating strategies that outperform the GMV portfolio in terms of wealth and other performance measures, such as Sharpe ratio, maximum drawdown, and Rachev ratio (see Young, 1998; Pedersen and Satchell, 2002; Biglova et al., 2004; Deng et al., 2005; Ruttiens, 2013;

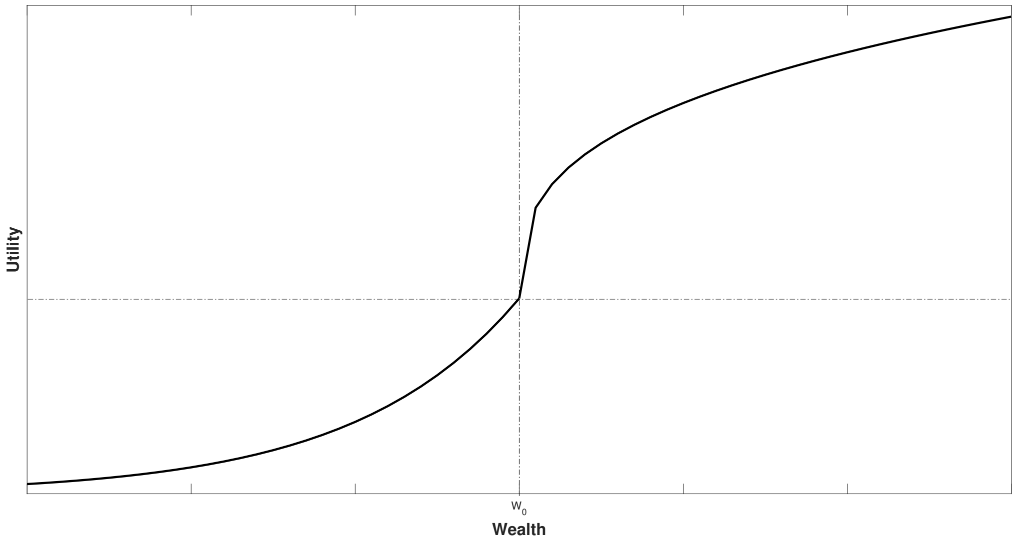
Ortobelli et al., 2013).

According to recent findings in the literature, the risk-averse assumption might be too strong to describe investors' behavior. Typically, investors prefer more to less, i.e., are non-satiable, and are neither risk-averse nor risk-seeking (see Markowitz, 1952b; Kahneman and Tversky, 1979; Tversky and Kahneman, 1992; Barberis and Thaler, 2003). Prospect theory aims to describe this category of investors. It relies on four pillars: (1) investors' decisions are based on relative changes in wealth, rather than total or final wealth; (2) investors are risk-averse for gains and risk-seeking for losses, i.e., investors avoid risk to protect gains but seek risk to recoup losses and their preferences are represented by an S-shaped utility function; (3) investors look at subjective rather than objective probabilities, and suffer from a framing effect (see Kahneman and Tversky, 1979; Tversky and Kahneman, 1992). On a similar approach, Markowitz in 1952 proposed that investors' utility functions have a fourfold behavior: convex–concave–convex–concave (see Markowitz, 1952b). Investors, then, are risk-averse for losses and risk-seeking for gains, while in the case of extreme events, they are risk-seeking for losses and risk-averse for gains. Such a typology of utility function is called inverse S-shaped utility.

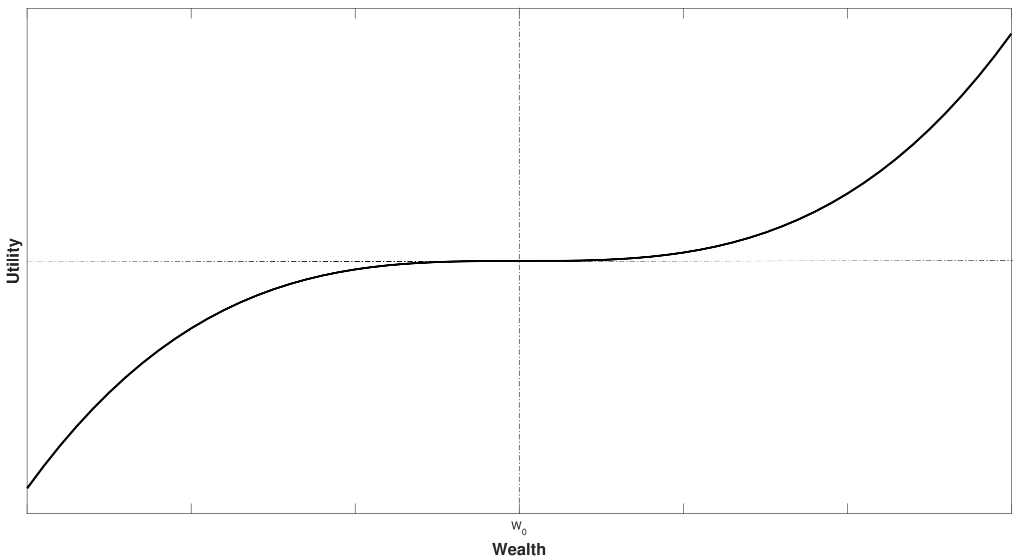
Similar to the case of expected utility theory, it is possible to define stochastic orderings even for the S-shaped and inverse S-shaped utility functions. In particular, X dominates Y in Markowitz stochastic dominance (MSD) if $\int_{-\infty}^{y} F_Y(u) - F_X(u)du \geq 0 \; \forall y \leq 0$ and $\int_{x}^{\infty} F_Y(u) - F_X(u)du \geq 0$ $\forall x \geq 0$. An MSD-efficient allocation corresponds to an optimal allocation for all investors with inverse S-shaped utility functions (see Levy and Levy, 2002; Baucells and Heukamp, 2006). Similarly, X dominates Y in the sense of prospect stochastic dominance (PSD) if $\int_{y}^{o} F_Y(u) - F_X(u)du \geq 0$ $\forall y \leq 0$ and $\int_{0}^{x} F_Y(u) - F_X(u)du \geq 0 \; \forall x \geq 0$.

One of the main ideas of behavioral finance is that, depending on market conditions, investors are sometimes risk-averse, and sometimes risk-seeking. Classic definitions of stochastic dominance, then, are not flexible enough to describe what efficiency is. Moreover, aggressive-coherent functionals are consistent with the preferences of such a category of investors (see Biglova et al., 2004; Stoyanov et al., 2007; Ortobelli et al., 2009). To bridge this gap and to develop an ordering consistent with the preferences of investors that are non-satiable and neither risk-averse nor risk-seeking, Chapter 2 (Testing for parametric ordering efficiency) considers a particular family of distribution (Ortobelli, 2001).

Distributions belonging to this family depend on reward and risk measures, and other distributional parameters. This family can be seen as an extension of the elliptical family, widely used in finance and portfolio theory, see, for example, Owen and Rabinovitch (1983); Adcock (2010), where



(a) Prospect theory utility function



(b) Markowitz's utility of wealth

Figure 3: Examples of utility functions in behavioral finance. The prospect theory utility function describes behavior that is risk-averse for gains and risk-seeking for losses, while Markowitz's utility describes the behavior of an investor that is risk-averse for losses and risk-seeking for gains investor.

reward and risk measures serve as location and scale parameters. Under minimal assumptions, stochastic dominance conditions are extended for the case of general risk and reward measures. Firstly, in the case where the mean is assumed to be the reward measure, to guarantee SSD efficiency it is not necessary for the risk measure to be convex. Convexity of risk measures guarantees that diversification does not increase risk (Artzner et al., 1999; Rachev et al., 2008). This in some sense confirms the empirical findings of Chapter 1.

Diversification is not a necessary condition for SSD efficiency, and thus, for non-satiable risk-averse investors. Secondly, in the case where a reward measure different than the mean is considered, it turns out that the behavior of non-satiable and neither risk-averse nor risk-seeking investors changes according to market conditions. In particular, agents behave as non-satiable and risk-averse when the reward measure is lower than the mean, and as non-satiable and risk-seeking when the reward is higher than the expected return. This allows us to state that efficiency for non-satiable and neither risk-averse nor risk-seeking investors corresponds to that of non-satiable, risk-averse investors in a market where the reward is lower than the expected return, and to that of non-satiable, risk-seeking investors when the reward is higher than the mean. The last part of Chapter 2 combines these stochastic dominance relations with estimation function theory to develop a hypothesis-testing methodology for portfolio efficiency (see also Godambe and Thompson, 1989). Such methodology is then applied on monthly observations of the Fama and French market portfolio and the six Fama and French benchmark portfolios formed on the basis of size and book-to-market equity ratio from July 1963 to October 2001 and, on daily observations of assets belonging to the Nasdaq and New York Stock Exchange (NYSE) from December 1995 to May 2017 (see Fama and French, 1993).

As an extension of Chapter 2, Chapter 3 (On the efficiency of portfolio choices) presents an application on different data and a different functional defining the ordering. It considers assets belonging to the Standard and Poor's 500 (S&P500) index from January 2000 to June 2017. The risk functional is based on a linear combination of tail Gini measures, and corresponds to a weighted difference between a given percentage of worst and a given percentage of best outcomes. Results of this ordering suggest that the market portfolio is almost never efficient, even if in some situations, depending on the configuration of the functional, it can be hard to find a dominating portfolio.

Chapters 1 and 2 both show applications of stochastic orderings in portfolio theory. In particular, they discuss different definitions of efficient choice, and what an optimal allocation is from the point of view of various investors. Starting from slightly different assumptions about investors' behavior, they end with similar conclusions on diversification. Diversification is not necessary for an efficient portfolio choice. In ordering theory, diversification is understood in terms of majorization ordering.

A portfolio is said to be more diversified than another if the ordered weights of the second majorizes those of the first. In other words, diversification considers the number of assets in which a positive proportion of wealth is invested, rather than *how* it is invested (Marshall et al., 1943; Wong, 2007; Egozcue and Wong, 2010). Nevertheless, diversification ordering and stochastic dominance are consistent under strong assumptions that are almost never satisfied by real data (see Samuelson, 1967; Ortobelli et al., 2018).

These last observations motivate Chapter 4 (Risk diversification), which proposes a new approach in introducing *risk diversification*. Risk diversification can be defined as the way idiosyncratic risk is diversified among a portfolio's components. Some diversification measures already present in the literature can be seen as special cases of risk diversification measures (RDMs), see, e.g., Choueifaty and Coignard (2008); Vermorken et al. (2012); Clarke et al. (2013); Flores et al. (2017).

The empirical analysis firstly introduces the mean-risk-diversification frontier. Similar to the mean-variance efficient frontier, the mean-risk diversification efficient frontier establishes a definition of efficiency. A portfolio is said to be mean-risk diversification efficient if its projection onto the mean-risk diversification plane belongs to the frontier. First, the chapter illustrates that different risk measures have different corresponding mean-risk diversification efficiencies. Second, the analysis shows how strategies based on portfolio risk diversification perform under periods of financial distress. The results suggest that the higher the risk aversion, the higher the concentration of a portfolio controlled for risk diversification. This result might seem counterintuitive, but it is in agreement with the results of a high degree of co-movement between asset returns, see, for example Dhaene et al. (2012); Linders et al. (2015).

Chapter 1

Pareto optimal choices versus mean-variance optimal choices: A paradigm of portfolio theory

Summary

In this chapter, we compare two of the main paradigms of portfolio theory: mean-variance analysis and expected utility. In particular, we show empirically that mean-variance efficient portfolios are typically suboptimal for non-satiable and risk-averse investors. We illustrate that the second-order stochastic dominance (SSD) efficient set is the solution of a multi-objective optimization problem. We further show that the market portfolio is not necessarily a solution to this optimization problem.

We also conduct empirical analysis to examine the ex ante and ex post performance of SSD and mean-variance efficient portfolios. In the ex ante analysis, we compare empirical moments, the level of diversification, and set distances of mean-variance and SSD-efficient sets. We also show that the global minimum-variance (GMV) portfolio and the part of the mean-variance efficient frontier composed of highly diversified portfolios is second-order stochastically dominated. This result also provides a possible alternative explanation for the diversification puzzle. In the ex post analysis, we construct second-order stochastic dominating strategies that outperform the GMV portfolio in terms of wealth and various other performance measures.

1.1 Introduction

Many authors have developed well-known studies based on the seminal work of Markowitz (see, for example, Markowitz, 1952a; Sharpe, 1964; Lintner, 1964; Mossin, 1966; Black, 1972). Significant effort has been applied to the estimation side. Expected-value estimation seldom yields reliable estimates, and the literature suggests applying specific estimation techniques, such as shrinkage, on the variance–covariance matrix, and focusing on the global minimum-variance portfolio (GMV) only (see Chopra and Ziemba, 1993; Broadie, 1993; Jagannathan and Ma, 2003; Ledoit and Wolf, 2004). This particular portfolio has proven to be not very sensitive to risk, and also to be able to outperform, in the out-of-sample analysis, most if not all of the other mean-variance efficient portfolios (see DeMiguel et al., 2009; Clarke et al., 2011).

A different definition of efficiency, and thus optimal allocation, is related to expected utility and decision theory under uncertainty. Typically, investors are classified according to their attitude toward risk (i.e., according to the shape of their utility function), and it is then possible to express the preference of an entire investor category via stochastic dominance. In particular, the second order of stochastic dominance (SSD) is coherent with the preferences of non-satiable and risk-averse investors, i.e., those with non-decreasing and concave utility functions (see Bawa, 1975; Müller and Stoyan, 2002). A portfolio is said to be SSD efficient if there does not exist another portfolio able to dominate it in terms of SSD order (see Post, 2003; Kuosmanen, 2004). An efficient portfolio is then optimal for all non-satiable risk-averse investors. To construct the SSD-efficient set, one may choose a coherent risk measure consistent with the SSD (see Ogryczak and Ruszczyński, 2002; Roman et al., 2006; Mansini et al., 2007; Eeckhoudt et al., 2009; Fábián et al., 2011; Hodder et al., 2014).[1] A considerable amount of literature has also developed hypothesis-testing procedures to verify whether a portfolio is SSD efficient (see, for example, Post, 2003; Kuosmanen, 2004; Kopa and Chovanec, 2008; Kopa and Post, 2015; Bruni et al., 2017).

The mean-variance efficient frontier (MVEF) and expected utility lack mutual coherence and are different in their implications and drawbacks. First of all, mean-variance efficient portfolios are consistent with the choice of risk-averse investors. However, mean-variance efficient portfolios are consistent with the preferences of non-satiable risk-averse investors only under the assumption of elliptically distributed returns, or when investors optimize a quadratic utility function (see Borch, 1969; Feldstein, 1969; Hakansson, 1971; Porter et al., 1973; Bawa, 1975). On the one hand, the MVEF

[1]See Ogryczak and Ruszczyński (1999), De Giorgi (2005), and De Giorgi and Post (2008) for sets of conditions that a risk measure must satisfy in order to achieve consistency with the SSD.

provides a justification for diversification and, when short sales are allowed, implies the convexity of the optimal choices set (efficient set).

Whereas diversification is of the most interest for practical purposes, convexity, from an economic point of view, is a desirable property for such sets; in fact, it allows the construction of the two-fund separation theorem, in which each optimal portfolio is a linear combination of the risk-free asset and the market portfolio (see Sharpe, 1964; Lizyayev and Ruszczyński, 2012). On the other hand, variance is not a coherent risk measure, and many authors, even Markowitz himself, have questioned the use of variance as a measure of risk and proposed different risk measures of semi-variance or semi-deviation (see, for example, Mao, 1970; Artzner et al., 1999).

Typically, MVEF works as a dimensionality-reduction technique. After the MVEF has been derived, the careful choice of a portfolio on the frontier would approximate the optimal choices of a great number of non-satiable and risk-averse investors (see Markowitz, 2014; Loistl, 2015). In virtue of this approximation, many authors believe that a lack of consistency with expected utility is not a major concern from the portfolio-management perspective. This statement is often supported by the fact that models based on expected utility often behave better in in-sample analysis, but are outperformed by MVEF-based models in out-of-sample studies (e.g., Simaan, 2014). The expected utility and stochastic dominance approaches are theoretically based, well established in the literature, and provide general criteria valid for entire categories of agents that share the same attitude toward risk (see Müller and Stoyan, 2002). However, SSD-coherent models generally show neither the convex efficient set nor diversification. A possible explanation is that SSD accounts for all the levels of risk aversion, though a considerable number of these might be unrealistic (see Dybvig and Ross, 1982; Ogryczak and Ruszczyński, 2002; Mansini et al., 2007; Lizyayev and Ruszczyński, 2012).

In this paper, we exploit stochastic dominance and multi-objective optimization to challenge the MVEF paradigm. Our aim is twofold: to prove that mean-variance efficiency is suboptimal from the perspective of non-satiable and risk-averse investors and evaluating and comparing the impact of mean-variance efficient choice and SSD-optimal choice in the market. To do so, we construct and compare two efficient sets. The mean-variance efficient set is composed by portfolios belonging to the MVEF. The SSD-efficient set is composed by average value at risk (AVaR) Pareto-optimal portfolios (see Pflug, 2000; Rockafellar and Uryasev, 2000, 2002; Roman et al., 2006).[2]

To find $AVaR$ Pareto optimal portfolios, we resort to multi-objective optimization (see, among

[2]The $AVaR$ is also known as conditional value at risk (CVaR) or expected shortfall (ES). Under the assumption of continuity of the return distributions, $AVaR$ is a coherent risk measure (see Artzner et al., 1999).

others, Roman et al., 2006; Mansini et al., 2007; Miettinen, 2012). We perform two types of analysis: ex ante and ex post. In the ex ante empirical analysis, we construct and compare the two efficient sets in terms of central moments, level of diversification, and set distances. It was found that GMV and portfolios belonging to the part of the MVEF with a low mean are SSD-dominated, implying that no investors would hold any of those portfolios. In other words, highly diversified portfolios are not preferable for non-satiable and risk-averse investors.

Moreover, we propose an alternative efficient frontier based on three parameters: expected return, $AVaR$, and confidence level, where we are able to interpret the confidence level as a risk-tolerance parameter. We exploit the non-efficiency of GMV in the ex post empirical analysis, constructing $AVaR$-based strategies able to dominate (ex ante) the GMV portfolio. This analysis confirms the ex post dominance of the proposed strategies.

In section 1.2, we present the construction of the efficient set of portfolios for non-satiable and risk-averse agents, and the related mathematical problem and its properties. In section 1.3, we discuss the empirical result with in- and out-of-sample analyses.

1.2 Efficient choices for non-satiable risk-averse investors

In this section, we define and describe the portfolio efficient sets for non-satiable investors and non-satiable risk-averse investors. Consider a market with N assets and denote $R = [R_1, \dots, R_N]'$ as the random vector of returns.[3]

Let $x \in \mathbb{R}^N$ denote the portfolio weights, and assume that all the portfolios $X = R'x$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume also that no short sales are allowed, i.e., $x \in \Delta = \{x \in \mathbb{R}^N : \sum_{j=1}^{N} x_j = 1, : x_j \geq 0 : j = 1, \dots, N\}$.

Generally, to describe investors' optimal choices, we must know the ordering consistent with their preferences. According to the preferences of non-satiable investors and non-satiable risk-averse investors, we recall the following classic definitions of stochastic dominance orderings (see Bawa, 1975; Mosler and Scarsini, 1991; Müller and Stoyan, 2002).

Definition 1. Let X and Y be two portfolios of returns with distribution functions $F_X(t) = \mathbb{P}[X \leq t]$ and $F_Y(t) = \mathbb{P}[Y \leq t]$ respectively.

[3] We call the return of the i-th asset at time t over the period $[t, t+1]$ has a value of $R_{t+1,i} = p_{t+1,i}/p_{t,i} - 1$, where $p_{t,i}$ is the stock value of asset i at time t. We sometimes consider gross returns $r_{t+1,i} = p_{t+1,i}/p_{t,i} = R_{t+1,i} + 1$, which are positive random variables, or log returns $z_{t+1,i} = \ln(r_{t+1,i})$, which are unbounded random variables. However, the results are presented in terms of returns, i.e., $R_{t,i}$, for uniformity with relevant literature in the field.

- All non-satiable investors prefer X to Y (i.e., $E(u(X)) \geq E(u(Y))$ for all non-decreasing utility functions u) or, equivalently, X dominates Y with respect to the first order of stochastic dominance ($X\ FSD\ Y$) if and only if $F_X(t) \leq F_Y(t)$, $\forall t \in \mathbb{R}$.

- All non-satiable risk-averse investors prefer X to Y (i.e., $E(u(X)) \geq E(u(Y))$ for all non-decreasing and concave utility functions u) or, equivalently, X dominates Y with respect to the second order of stochastic dominance ($XSSDY$) if and only if $\int_{-\infty}^{t} F_X(u)du \leq \int_{-\infty}^{t} F_Y(u)du$, $\forall t \in \mathbb{R}$.

The above inequalities are strict for at least one $t \in \mathbb{R}$ or a utility function u.

FSD and SSD relationships can be expressed in terms of well-known risk measures, i.e., X $FSD\ Y$ if and only if $VaR_\alpha(X) \leq VaR_\alpha(Y)$ $\forall \alpha \in [0,1]$ with at least one strict inequality (for some α), and $X\ SSD\ Y$ if and only if $AVaR_\alpha(X) \leq AVaR_\alpha(Y)$, $\forall \alpha \in [0,1]$ with at least one strict inequality, where $VaR_\alpha(X) = -F_X^{-1}(\alpha) = -\inf\{u : F_X(u) \geq \alpha\}$ is the value at risk and $AVaR_\alpha(X) = \frac{-1}{\alpha}\int_0^\alpha F_X^{-1}(u)du$ is the average value at risk (see, for example, Ogryczak and Ruszczyński, 2002; Kopa and Chovanec, 2008). Typically, stochastic dominance relationships are implemented to find efficient choices for whole categories of investors. Following Kuosmanen (2004), a portfolio is SSD (or FSD) efficient if there is no other admissible portfolio that is able to dominate it in the SSD (or FSD) sense. We recall these definitions in terms of VaR and $AVaR$ (see Roman et al., 2006).

Definition 2. A portfolio X^* is FSD efficient if and only if it is not FSD dominated by other portfolios, i.e., there does not exist another portfolio $X = R'x$ s.t. $VaR_\alpha(X) \leq VaR_\alpha(X^*)$, $\forall \alpha \in [0,1]$, and $VaR_\alpha(X) < VaR_\alpha(X^*)$ for at least one $\alpha \in [0,1]$. A portfolio X^* is SSD efficient if and only if it is not SSD dominated by other portfolios, i.e., there does not exist another portfolio $X = R'x$ s.t. $AVaR_\alpha(X) \leq AVaR_\alpha(X^*)$ $\forall \alpha \in [0,1]$ and $AVaR_\alpha(X) < AVaR_\alpha(X^*)$ for at least one $\alpha \in [0,1]$. We call this the efficient set for non-satiable investors, when all portfolios are FSD efficient, and similarly, we call this the efficient set for non-satiable and risk-averse investors if all portfolios are SSD efficient.

Still, this definition is based on infinitely many conditions. Stochastic dominance rules indeed provide selection criteria based on continuous constraints which, in practice, are not feasible most of the time, because they translate into an infinite number of constraints (see Bawa, 1975). However, under the assumption of a finite probability space Ω with T elements with uniform probability, then, according to Kuosmanen (2004) and Kopa and Chovanec (2008), we can consider only T confidence levels $\alpha_i = \frac{i}{T}$ ($i = 1, \ldots, T$) for ordering portfolios with respect to FSD and SSD. Typically, we identify the (FSD, SSD) optimal choices considering T "equiprobable" historical observations and thus, SSD- (FSD-) efficient portfolios satisfy the following "empirical" condition: X^* is said to be SSD (FSD) efficient if and only if there does not exist $X = R'x$ s.t. $AVaR_{\alpha_i}(X) \leq AVaR_{\alpha_i}(X^*)$ with $\alpha_i = \frac{i}{T}$ for every $i = 1, \ldots, T$ and $AVaR_{\alpha_j}(X) < AVaR_{\alpha_j}(X^*)$ for at least one j ($\nexists\ X = R'x$ s.t. $VaR_{\alpha_i}(X) \leq VaR_{\alpha_i}(X^*)$ with $\alpha_i = \frac{i}{T}$ for every $i = 1, \ldots, T$ and $VaR_{\alpha_j}(X) < VaR_{\alpha_j}(X^*)$ for at least one j.

In this paper, we assume a finite probability space with T elements and with uniform probability. Such a typology of sets is called Pareto optimal, and is typically the solution set of a multi-objective optimization problem (see Roman et al., 2006; Miettinen, 2012; Luc, 2016). As a matter of fact, the solutions $x^* \in \Delta$ of multi-object problems $\min_{x \in \Delta} f(x)$ where the function: $f : \Delta \to \mathbb{R}^T$ s.t. $f(x) = (f_1(x), ..., f_T(x))$ are called Pareto optimal because $\nexists\ x \in \Delta$ s.t $f_i(x) \leq f_i(x^*)$ $\forall i$ and $f_j(x) < f_j(x^*)$ for at least j. In our case, f_i is either VaR_{α_i} or $AVaR_{\alpha_i}$, for every $i = 1, ..., T$, and x is the vector of portfolio weights belonging to the convex set $\Delta \subset \mathbb{R}^N$. The literature on finding the Pareto optimal of a multi-objective problem is wide and broad. In particular, when the multivalued function f is continuously differentiable, we know that $x^* \in \Delta$ is Pareto optimal for f if and only if it solves the following optimization problem for any $i \in \{1, ..., T\}$:

$$\begin{aligned} &\min_{x \in \Delta} \quad f_i(x) \\ &s.t. \quad f_j(x) - m_j \leq 0 \qquad j = 1, ..., T;\ j \neq i \end{aligned}$$

where $m_j = f_j(x^*)$. This transformation of the multi-object problem in a minimization problem of a real function $f_i : \Delta \to \mathbb{R}$ with constraints is also known as the ϵ-constrained method. Since the minimization of VaR could give more than one local optimum, determining the efficient set of all non-satiable investors is not a simple problem and will be the object of future research, following the seminal work of Bawa (1976) in a mean-variance framework. For this reason, in this chapter, we essentially discuss and study the characteristics of the portfolio SSD-efficient set that is a fundamental subset of the FSD-efficient set. Thus, according to the ϵ-constrained method for the SSD-efficient set, a necessary and sufficient condition for $X^* = R'x^*$ to be a Pareto optimal portfolio and, consequently, the SSD-efficient portfolio is that x which solves the following optimization problem for all $i \in \{1, ..., T\}$:

$$\begin{aligned} \min_{x \in \Delta} \quad & AVaR_{\alpha_i}(X) \\ s.t. \quad & AVaR_{\alpha_k}(X) \leq m_k \qquad \forall k \neq i \end{aligned} \tag{1.1}$$

and some values m_k.

Remark 1. *According to Roman et al. (2006) and Luc (2016), determination of the SSD-efficient portfolios can be rearranged as a multi-objective linear program. Then, the efficient set of a multi-objective optimization consists of faces of the feasible set. Moreover, if a relative interior point of a face is Pareto optimal, every point of the face is Pareto optimal. These properties should enlighten the discussion of the non-convexity of the Pareto set (see Dybvig and Ross, 1982). One of the reasons behind the success of the MVEF is the convexity of the optimal portfolio weights. In fact, this property opened the door for the two-funds separation theorem and the capital asset pricing model (Mossin, 1966; Lintner, 1964; Sharpe, 1964). However, as noted by Dybvig and Ross (1982) and Lizyayev and Ruszczyński (2012), the efficient set for non-satiable risk-averse investors is a finite union of convex sets and, therefore, it is generally not convex (except in a few particular cases). Thus, it could be that the market portfolio is not SSD efficient. Recall that it is possible to derive SSD-efficiency tests applying different optimality conditions for multi-objective linear problems (see Kopa and Post, 2015, and reference therein).*

Recall that the function $AVaR_{\alpha_i}(R'x)$ is a convex linearizable function that, under some regularity conditions (see, among others, Mangasarian, 1979) admits a unique global minimum for any $i \in \{1, ..., T\}$. Clearly, when for some i and m_k a unique portfolio solution to problem (1.1) exists, then that portfolio is efficient. Unicity of the minimization problem solutions also guarantees the avoidance of redundant returns and, even for this reason, this hypothesis is sometimes required to solve practical problems. In the following proposition, requiring that there exists a unique solution to the global minimum $AVaR_{\alpha_i}$ lets us identify a property of the SSD-dominated portfolios (the inefficient ones).

Proposition 1. *Assume that, for any $i \in \{1, ..., T\}$, there exists a unique solution $x \in \Delta$ to the global minimum $AVaR_{\alpha_i}(R'x)$. If a portfolio $X = R'x$ is second-order stochastically dominated, then for any $i \in \{1, ..., T\}$, there exists a portfolio $Y_{(i)}$ such that*

$$AVaR_{\alpha_i}(Y_{(i)}) < AVaR_{\alpha_i}(X). \tag{1.2}$$

Proof. Assume that for a given i, there does not exist a portfolio $Y_{(i)}$ s.t. $AVaR_{\alpha_i}(Y_{(i)}) < AVaR_{\alpha_i}(X)$. Then, X is the unique global minimum $AVaR_{\alpha_i}$, and is thus a solution of problem (1.1) and is SSD efficient, which contradicts our hypothesis. □

An economic interpretation of this result is that agents prefer outcomes where the allocation of undertaken risk is optimal. In other words, if a non-satiable risk-averse investor can undertake less risk, they will. However, the main impact of this result is in its application. In the empirical analysis, we will apply proposition (1) to the portfolios belonging to the MVEF to verify their SSD efficiency. We know that MVEF portfolios are efficient for risk-averse investors because if all risk-averse investors prefer portfolio X to portfolio Y, then portfolio X should have lower variance than portfolio Y. However, it could be that optimal MVEF portfolios are not efficient for non-satiable risk-averse investors because returns are not generally Gaussian (elliptically) distributed. In practice, we know that the GMV portfolio is generally FSD- and SSD-dominated even if it is often considered a potential optimal portfolio by practitioners.

In fact, portfolio returns (or portfolios of returns) are random variables bounded from below when we assume that no short sales, and limited liabilities, are allowed. Thus, we cannot assume that the portfolio of returns is Gaussian (or elliptically) distributed. In addition, establishing dominance among returns also implies dominance among log returns (even if the converse is not necessarily true—see, for example Fishburn, 1964). Therefore, according to Ortobelli (2001), many of the MVEF portfolios (which the GMV portfolio is among) are not optimal for non-satiable risk-averse investors. Then, to verify if a mean-variance efficient portfolio $X_{mv} = R'x_{mv}$ is SSD efficient, we solve the following optimization problem:

$$\begin{aligned} \min_{x\in\Delta} \quad & AVaR_{\alpha_i}(X) \\ s.t. \quad & AVaR_{\alpha_k}(X) \leq AVaR_{\alpha_k}(X_{mv}) \qquad \forall k = 1, ..., T \end{aligned} \tag{1.3}$$

for some i. Clearly, if we can find at least one portfolio solution of problem (1.3) that is different from the mean-variance solution, X_{mv}, then portfolio X_{mv} is SSD dominated. Moreover, if X_{mv} is SSD dominated, then by proposition (1), the feasible set of problem (1.3) is non-empty for any $i \in \{1, ..., T\}$.

Let us give a simple practical example based on some real historical observations.

Example 1. *Consider the return series of two assets (Celgene and Schlumberger) with 10 years of yearly* *observations (from June 2006 to July 2016) and compute the MVEF between them.*[4] *Let X and Y be portfolios of returns and arrange them in ascending order, i.e., $X_{[1]} \leq ... \leq X_{[T]}$ and $Y_{[1]} \leq ... \leq Y_{[T]}$. Then, X second-order stochastically dominates Y if and only if*

$$\frac{-1}{k}\sum_{i=1}^{k}\left(X_{[i]} - Y_{[i]}\right) \leq 0 \qquad \forall k = 1, ..., T. \tag{1.4}$$

Observe that formula (1.4) represents the differences in all $AVaR_{\alpha_k}$ levels. Figure 1.1 shows the MVEF and $AVaR_{\alpha_k}$ of the two assets and the global minimum-variance (GMV) portfolio. As seen in panel (b), Celgene's $AVaRs$ are lower than those for the GMV portfolio, implying that they second-order stochastically dominate the GMV portfolio. In addition, the difference is strict for all $AVaR_{\alpha_i}$ levels according to proposition (1).

According to Ortobelli (2001), example 1, and proposition 1, we expect that the GMV portfolio is SSD dominated by the portfolio solution of problem (1.3) for any $i \in \{1, ..., T\}$. Moreover, many other portfolios of the mean-variance efficient frontier could be SSD dominated. In the next section, we compare the $AVaR$ Pareto optimal set and the mean-variance efficient frontier using portfolios composed of assets belonging to the DJIA.

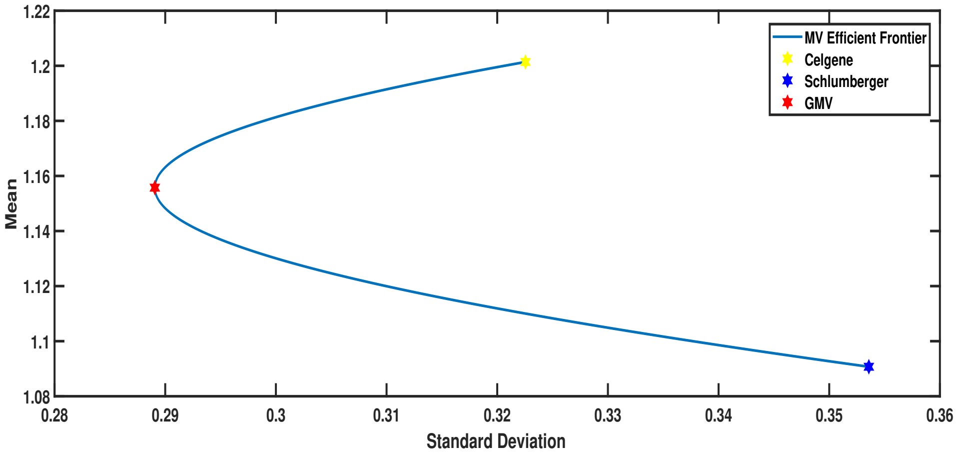
1.3 The second-order stochastic dominance efficient set in practice

In this section, we propose ex ante and ex post empirical analyses. In the ex ante empirical analysis, we examine the differences between the mean-variance efficient frontier and the efficient set for non-satiable risk-averse investors. In the ex post empirical analysis, we evaluate the impact of these differences. We consider monthly observations of stocks belonging to the DJIA index from 18 March 1997 to 14 October 2017. The dataset is composed of assets belonging to the DJIA on 14 October 2017. For the ex ante analysis, the total number of assets used is 28. The two missing stocks are

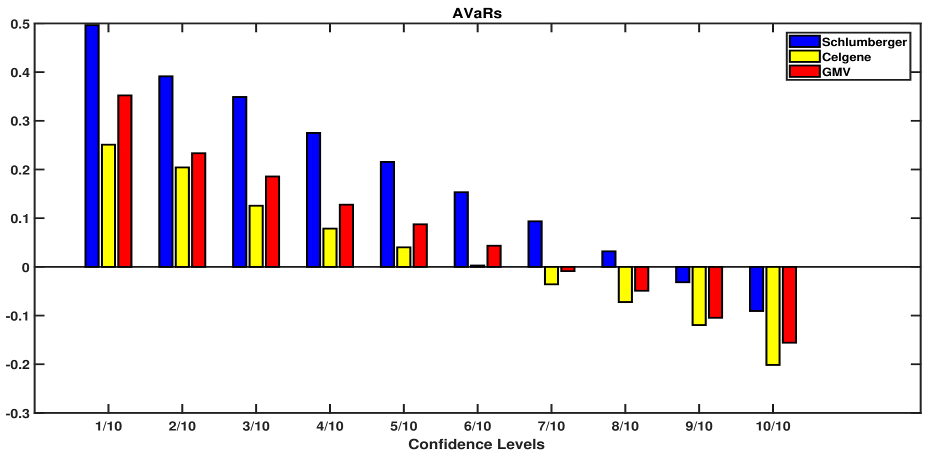
[4]Celgene's adjusted return series, rounded to the third decimal place, is $R_C = [0.268,\ 0.114,\ -0.251,\ 0.062,\ 0.184,\ 0.032,\ 0.938,\ 0.498,\ 0.327,\ -0.157]'$.

Schlumberger's adjusted gross return series, rounded to the third decimal place, is $R_S = [0.399, 0.265, -0.496, 0.023, 0.539,\ -0.264,\ 0.158,\ 0.621,\ -0.286,\ -0.053]'$.

The GMV portfolio gross return series, rounded to the third decimal place, is $R_{GMV} = [0.322, 0.176, -0.352, 0.046, 0.330,\ -0.091,\ 0.616,\ 0.549,\ 0.074,\ -0.115]'$.



(a) MVEF between Schlumberger and Celgene. The GMV portfolio weights are 0.412 and 0.588



(b) AVaRs of Schlumberger, Celgene, and GMV portfolio, with $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 07, 0.8, 0.9., 1$. Celgene cumulative AVaRs are smaller than GMV ones for each confidence level α, and hence, satisfy condition (1.4)

Figure 1.1: Example with only two assets

Visa, which went public in 2008, and Cisco Systems. In the ex post analysis, Visa is then included in the dataset after 20 March 2008.

1.3.1 Ex ante analysis

For ex ante analysis, we consider 10 years of monthly observations of assets belonging to the dataset, from 13 October 2007 to 14 October 2017.

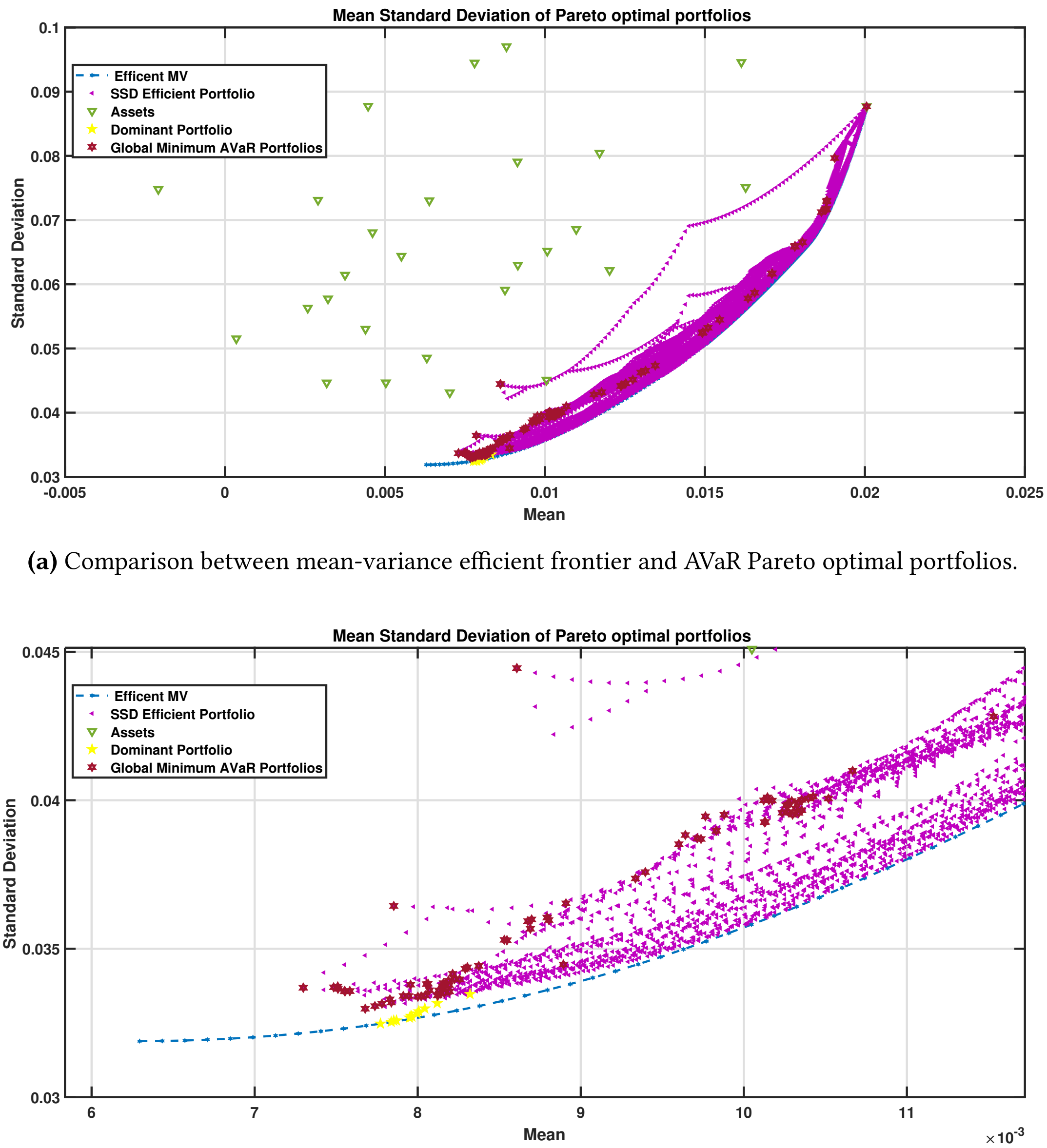
The aim of this section is to compare MVEF and SSD-efficient sets empirically, and to show that the two sets differ and that part of the MVEF is second-order stochastically dominated. First, we solve the quadratic program associated with the mean-variance efficient set and solve problem (1.1) to construct the SSD-efficient set. Second, for each mean-variance efficient portfolio, according to proposition (1), we solve problem (1.3).

If we cannot find, for each level of risk aversion α_i, less risky portfolios $R'z$, it means that the tested mean-variance efficient portfolio is also efficient for non-satiable and risk-averse investors. The MVEF is composed of 100 portfolios, whereas we consider approximately 12,000 SSD-efficient portfolios. Then, we project these onto the expected value, standard deviation plane. Figure 1.2 shows the result: (a) depicts the entire projection, whereas (b) is an enlargement of the first part.

These graphs provide interesting insights into the relationship between mean-variance efficiency and $AVaR$ Pareto optimal portfolios. The line with blue triangles is the classical MVEF, the purple triangles are the SSD-efficient portfolios, the green triangles are the 28 assets, the yellow stars are the $AVaR$ Pareto optimal portfolios that second-order stochastically dominate MVEF portfolios, and the red stars are the 120 global-minimum $AVaR$ portfolios. As expected, the asset with the maximum mean over the period belongs to both efficient sets, whereas none of the others belong either to the mean-variance or the SSD-efficient sets. From panel (a), which shows the entire picture, we see that the majority of SSD-efficient portfolios lie near the MVEF, but there is still a conspicuous number with higher standard deviations.

Their projection on the mean–standard deviation plane also presents two “kinks” around points (0.015,0.07) and (0.015,0.055). The inner kink might suggest that, in the blank zone in the SSD-efficient portfolio projection, there lies some SSD-efficient portfolio that we cannot construct due to the discretization in problem (1.1). The other kink is probably due to the statistics of the assets belonging to the DJIA index.

The only asset that belongs to the SSD efficient set is the one with the highest mean; therefore, the mean and standard deviation of the other assets affect the shape of the SSD-efficient projection. From (b), we see that the first part of the MVEF is second-order stochastically dominated, approximately the first 12%, confirming the theoretical results in Ortobelli (2001), coherently with Roman et al. (2006) and Mansini et al. (2007).



(a) Comparison between mean-variance efficient frontier and AVaR Pareto optimal portfolios.

(b) Enlargement of the first part.

Figure 1.2: Comparison between mean-variance efficient frontier and AVaR Pareto optimal portfolios. The asset universe is composed of monthly observations of assets belonging to the DJIA index, from 18 March 1997 to 13 May 2007. The dashed blue line is the MVEF, purple triangles represent the mean and the standard deviation of the SSD-efficient portfolios, and green triangles represent the mean and the standard deviation of the stand-alone assets. The yellow stars depict the mean and standard deviation of portfolios dominating the GMV, i.e., portfolio solutions of problem 1.3, whereas red stars are the 120 portfolios of global-minimum AVaR. Panel (a) shows the entire mean–standard deviation plan. As we can see, there is an area near the mean-variance efficient frontier formed by SSD-efficient portfolios. Panel (b) shows an enlargement of the first part. The yellow stars are portfolios able to dominate a mean-variance efficient portfolio w.r.t. SSD.

MVEF selection criteria are based on the comparison between the first two central distributional moments, while stochastic dominance, in general, compares whole distributions. In the literature, the first approach is also referred to in terms of primary probability functionals, while the second is in terms of simple probability functionals (see, among others, Ortobelli et al., 2009; Rachev et al., 2011). Therefore, to understand the differences between the two efficient sets more fully, we first report some portfolio statistics, which concerns the first typology of orderings, and then study some distributional distances linked to FSD and SSD. Because we are dealing with a large number of portfolios, we divide the area of the mean–standard deviation plane into 10 parts, equally spaced along the mean axis.

In the first case, we compute the average of the following statistics of portfolios belonging to each area: expected return, standard deviation, skewness, kurtosis, the sum of the portfolio's squared weights, and the number of assets that present a non-zero weight.

Table 1.1 reports the results; the first row of each column refers to the mean-variance efficient set, while the second row refers to the $AVaR$ Pareto optimal set. As expected, the $AVaR$ Pareto optimal set presents higher average standard deviations, whereas the mean is approximately the same across the efficient sets, except for the classes around the GMV portfolio, where the mean of SSD-efficient portfolios is slightly higher. The most interesting area is the first, where the GMV lies, where portfolios belonging to the $AVaR$ Pareto optimal set present less negative skewness and lower kurtosis, on average. Moreover, in all other areas, SSD-efficient portfolios present more negative skewness and higher kurtosis than mean-variance efficient portfolios.

The relationship with diversification can be seen in the last two columns; the portfolio diversification level is similar, as explained by the sum of squared weights, but the number of invested assets is higher in the $AVaR$ Pareto optimal set. This implies that portfolios in the $AVaR$ Pareto optimal set present a higher number of invested assets, but only a few of them have a considerable invested amount. In other words, these portfolios are more concentrated than those in the mean-variance efficient set. Combining this result with panel (b) of figure 1.2, we see that the part of the MVEF where diversification is higher is also second-order stochastically dominated. In other words, a portion of the mean-variance efficient portfolios are not optimal for non-satiable and risk-averse investors. Because this part of the efficient frontier is composed of highly diversified portfolios, this result might, in some sense, provide an explanation for the diversification puzzle in portfolio theory (see, for example, Statman, 2004; Egozcue et al., 2011). Agents do not really seek high diversification because, according to these results, highly diversified portfolios appear to be second-order stochastically dominated and therefore suboptimal for non-satiable and risk-averse investors.

Table 1.1: Average of mean, standard deviation, skewness, kurtosis, sum of squared weights, and number of invested assets of portfolios belonging to each of the 10 groups.

Mean range		Mean	St. dev.	Skewness	Kurtosis	$\sum_i x_i^2$	♯ Assets
$[0.0063, 0.0077)$	MV	0.0069	0.0320	-1.6861	10.869	0.128	11
	SSD	0.0075	0.0338	-1.3034	9.091	0.166	16.7
$[0.0077, 0.009)$	MV	0.0083	0.0330	-1.4574	9.867	0.171	9.3
	SSD	0.0085	0.0345	-1.5937	11.393	0.181	19.6
$[0.009, 0.0104)$	MV	0.0096	0.0351	-1.3924	9.882	0.198	8.2
	SSD	0.0097	0.0371	-1.6562	12.239	0.186	20.1
$[0.0104, 0.0118)$	MV	0.0110	0.0382	-1.4003	10.412	0.207	8.1
	SSD	0.0111	0.0406	-1.7076	13.213	0.189	20.2
$[0.0118, 0.0132)$	MV	0.0124	0.0420	-1.3616	10.490	0.214	7.1
	SSD	0.0124	0.0446	-1.6037	12.906	0.206	19.7
$[0.0132, 0.0145]$	MV	0.0138	0.0465	-1.2889	10.250	0.237	6.2
	SSD	0.0138	0.0490	-1.5095	12.451	0.220	18.7
$[0.145, 0.0159)$	MV	0.0152	0.0517	-1.3077	10.754	0.234	5
	SSD	0.015	0.0541	-1.4584	12.218	0.236	18.3
$[0.0159, 0.0173)$	MV	0.0166	0.0580	-1.2955	10.960	0.274	5
	SSD	0.0166	0.0600	-1.4283	12.092	0.282	17.6
$[0.0173, 0.0187)$	MV	0.0180	0.0653	-1.2302	10.563	0.393	4
	SSD	0.0180	0.0662	-1.3672	11.294	0.389	16.5
$[0.0187, 0.0201]$	MV	0.0193	0.0776	-0.8590	7.666	0.704	3
	SSD	0.0193	0.0786	-0.9546	7.828	0.726	14.1

Having shown the differences between the two sets in terms of mean statistics, we proceed to consider differences in terms of portfolio weights and distributions. We consider three different distances. Let $X = R'x$ and $Y = R'y$ be two portfolios with weights x and y, respectively. Define $d_1(X, Y)$ as the following distance:

$$d_1(X, Y) = \sum_{i=1}^{N} |x_i - y_i|.$$

$d_1(X, Y)$ defines a distance between the weights of two chosen portfolios, which takes a value

Table 1.2: Minimum, mean, and maximum values for all distances between portfolios belonging to the SSD-efficient set and MVEF.

Mean Range	d_1 mean	d_1 min	d_1 max	d_2 mean	d_2 min	d_2 max	d_3 mean	d_3 min	d_3 max
$[0.0063, 0.0077)$	0.620	0.529	0.737	0.012	0.011	0.0141	0.099	0.076	0.135
$[0.0077, 0.009)$	0.275	0.191	0.361	0.008	0.006	0.0109	0.036	0.026	0.048
$[0.009, 0.0104)$	0.122	0.099	0.162	0.005	0.003	0.0070	0.014	0.010	0.022
$[0.0104, 0.0118)$	0.114	0.093	0.145	0.005	0.003	0.0085	0.016	0.011	0.021
$[0.0118, 0.0132)$	0.208	0.191	0.221	0.007	0.006	0.0084	0.020	0.016	0.028
$[0.0132, 0.0145]$	0.094	0.063	0.148	0.005	0.003	0.0086	0.015	0.009	0.022
$[0.145, 0.0159)$	0.042	0.022	0.062	0.003	0.001	0.0058	0.012	0.004	0.020
$[0.0159, 0.0173)$	0.093	0.032	0.137	0.007	0.003	0.0116	0.029	0.008	0.048
$[0.0173, 0.0187)$	0.069	0.006	0.101	0.006	0.0007	0.0088	0.025	0.002	0.040
$[0.0187, 0.0201]$	0.013	0	0.027	0.002	0	0.0041	0.005	0	0.013

in the interval $[0, 2]$, where 0, by definition, implies that the two portfolios are equal and 2 implies that the two portfolios are completely different in that they do not invest in the same set of assets. The second distance we consider is a particular type of Levy quasi-semidistance (see Rachev, 1991; Rachev et al., 2011). It can be defined as

$$d_2(X, Y) = \max_t |F_X^{-1}(t) - F_Y^{-1}(t)|.$$

$d_2(X, Y)$ takes values in $[0, \infty]$, can be interpreted as a measure of closeness of distribution graphs, and can be considered as a distance in terms of value at risk. Moreover, d_2 is related to FSD, in the sense that it metrizes the preference relation induced by FSD.[5] Similarly, a distance in terms of $AVaR$ and linked with the SSD can be defined as follows:

$$d_3(X, Y) = \max_t \left| \int_0^t F_X^{-1}(u) - F_Y^{-1}(u) du \right|.$$

In each of the 10 areas, we compute the distances between the mean-variance efficient and SSD-efficient portfolios. Table 1.2 reports minimum, mean, and maximum values.

Distances d_2 and d_3 are set distances: when the minimum is equal to 0, it implies that the intersection between the two sets is non-empty; when the maximum distance is 0, it implies that the mean-variance efficient set is a subset of the SSD one. In other words, when the minimum of d_2

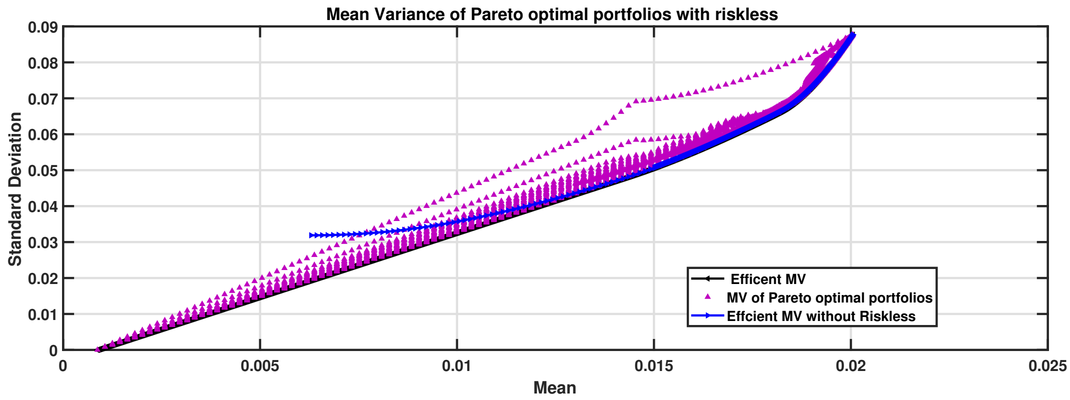
[5] See Rachev et al. (2011) for a complete and detailed discussion on this.

(or d_3) is equal to 0, then at least a mean-variance efficient portfolio belongs to the SSD-efficient set, while when the maximum of d_2 is equal to 0, all the mean-variance efficient portfolios in that area are SSD-efficient. The economic interpretation of this, is that when $d_2 = 0$, there exists at least one mean-variance efficient portfolio that is at least as *preferable* as the SSD-efficient one, according to all non-satiable investors' preferences. When $d_3 = 0$, the same is valid, but for all non-satiable and risk-averse investors (see Rachev et al., 2011).

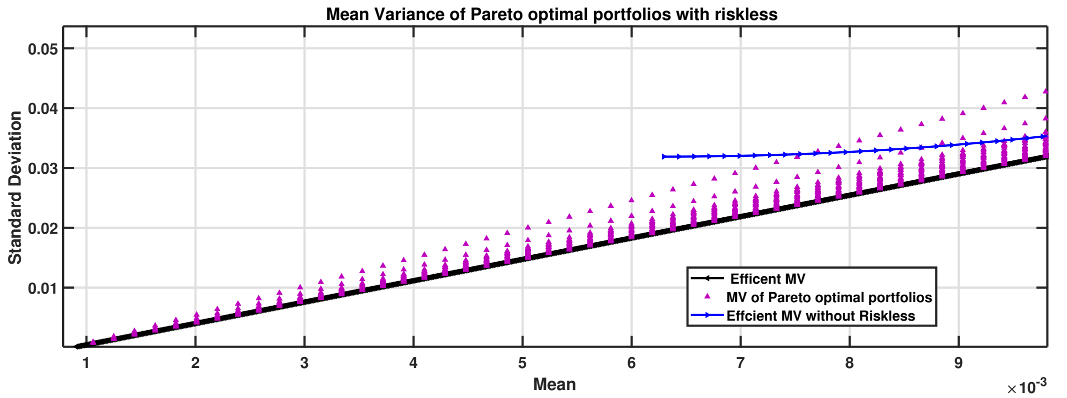
The first three columns of table 1.2 show that portfolio dissimilarity is higher in the area near the GMV than elsewhere; more than 25% of the portfolio composition is different, and this tends to decrease as the mean increases. Note that the minimum value for area 10 is zero for all distances, because the asset with the maximum mean belongs to both efficient sets. Further, set distances remain remarkable, even if they tend to decrease as the mean increases. As we see from the results for d_2 and d_3, none of the mean-variance efficient portfolios belong to the SSD-efficient sets rather than the maximum mean asset. This does not imply that all MVEFs are second-order stochastically dominated, but more reasonably, that the discretization, due to computational feasibility, affects the results in some sense.

We perform the same empirical analysis now also including the risk-free rate, which is taken from the three-month treasury bill at 17 October 2017, and is equal to 0.0105. Graphical inspection of the result is reported in figure 1.3. The blue line represents, as before, the MVEF without the risk-free rate, while the black line is the one with the risk-free rate. The behavior so far is exactly standard. The purple triangles are Pareto optimal portfolios. The structure in panel (a) resembles that in figure 1.2 panel (a), in the sense that all the Pareto optimal portfolios lie in an area near the MVEF. In panel (b), we focus on the first part. The enlargement, (c), tells an even more interesting story. There is a point after which the SSD-efficient set converges with the MVEF. A possible explanation of this is the presence of the maximum mean portfolio on the MVEF. Since this portfolio is also not second-order stochastically dominated, such convergence has to occur at a certain point.

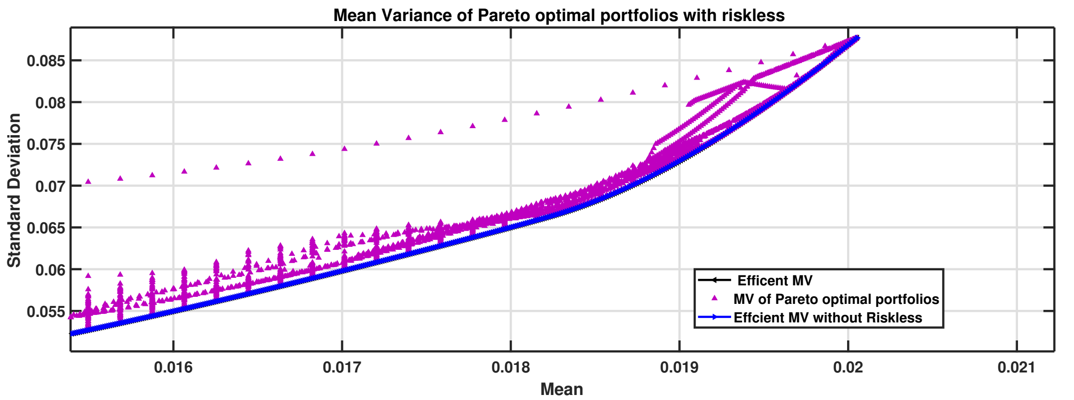
Before moving to ex post analysis, we investigate one last features of the results. According to section 1.2, the efficient set for non-satiable risk-averse investors can be completely described only in a T-dimensional space where T points out the number of all possible $AVaR_{\alpha_k}$ (in our case, T is equal to the number of observations, which is 120). However, we can always summarize some non-satiable risk-averse investors' choices, reducing the number of dimensions to check. Figure 1.4 shows a particular three-dimensional efficient frontier for monthly returns. We have on the two horizontal axes the $AVaR$s and the relative confidence level, while the vertical axis considers the expected portfolio return (corresponding to -$AVaR_{\alpha_T}$). Thus, for each confidence



(a) Comparison between mean-variance efficient frontier and AVaR Pareto optimal portfolios considering a risk-free rate.



(b) Enlargement of the first part.



(c) Enlargement of the last part.

Figure 1.3: Comparison between mean-variance efficient frontier and AVaR Pareto optimal portfolios without risk. The asset universe is composed by monthly observation of the assets belonging to the DJIA index, from 18 March 1997 to 13 May 2007, and a risk-free rate equal to 0.0105. The dashed blue and black lines are the MVEF. The purple triangles represent the mean and the standard deviation of the SSD-efficient portfolios. Panel (a) shows the entire mean–standard deviation plan. As we can see, there is an area nearby the mean-variance efficient frontier formed by SSD-efficient portfolios. Panel (b) shows an enlargement of the first part.

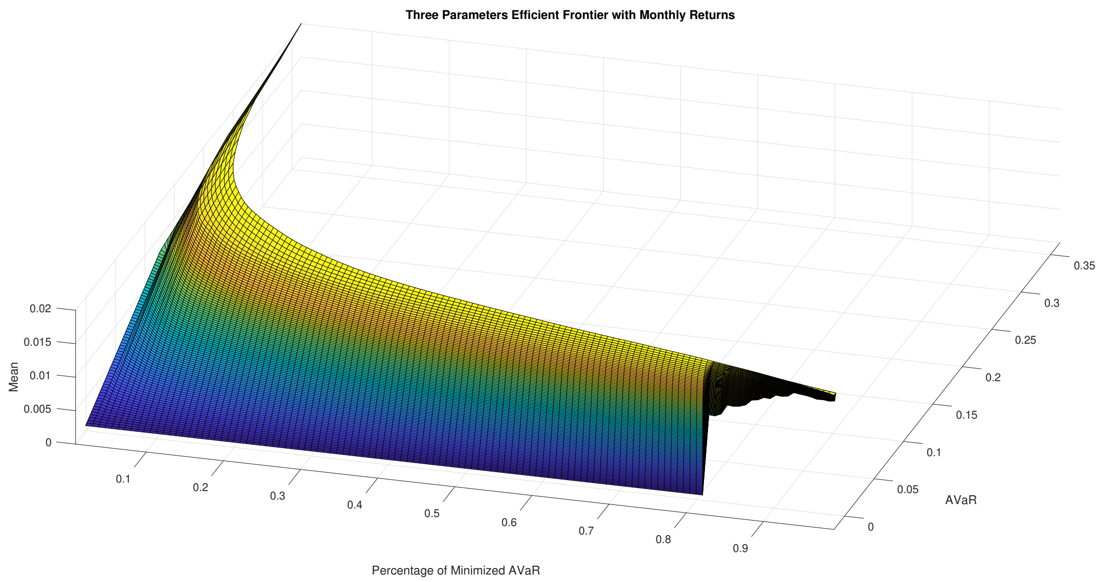


Figure 1.4: Three-dimensional AVaR efficient frontier. This graph shows an efficient-frontier alternative to the MVEF. For each confidence level $\frac{i}{120}$ ($i = 1, ..., 120$), and for a fixed expected portfolio return, we compute the minimum-AVaR portfolio.

level $\frac{i}{120}$ ($i = 1, ..., 120$), and for a fixed expected portfolio return, we compute the minimum-$AVaR$ portfolio.

Figure 1.4 shows some kind of convexity with respect to confidence level $\frac{i}{120}$, which lets us interpret it as the opposite of a risk-aversion (risk-tolerance) coefficient: a less risk-averse investor prefers an $AVaR$ at a higher confidence level. In other words, in this figure we have $AVaR$, a coherent risk measure, the confidence level as a risk-tolerance coefficient, and expected return as a reward measure, meaning that if an investor seeks a revenue level, given their risk-aversion coefficient, they must undertake at least the quantity of risk given by the global minimum $AVaR$. For each portfolio, is possible to determine aggressive and defensive securities among its components.

1.3.2 Ex post analysis

In this section, we present an ex post analysis where according to problem (1.3) we determine 120 portfolios that dominate the GMV one (corresponding to 120 different confidence levels α_k of $AVaR_{\alpha_k}$). In particular, we perform a rolling window analysis, with a time window of 10 years of monthly observations (the first window considers monthly returns from 18 March 1997 to 13 May

2007) where we solve problem (1.3) and recalibrate the portfolio monthly (each 21 trading days) considering a proportional transaction cost of 0.2%. Finally, we compare the ex post performance of the resulting $AVaR$ strategies in terms of wealth and other performance measures.

Figure 1.5 shows the ex post wealth of the GMV portfolio and of 120 $AVaR$ strategies. At the end of the period, the GMV portfolio reaches a level of wealth of 1.8. It shows a downward trend during the subprime crisis of 2008, and as expected, it is not so volatile. The $AVaR$ strategies instead seem to reach a higher level of wealth, but with higher volatility. This aspect is further confirmed by figure 1.6, where the ex post wealth of some selected strategies outperform the GMV (up to 150%).[6] The dense red line is the ex post behavior of the GMV portfolio, while the blue dense line is the behavior of the asset with the maximum mean. All other dashed lines represent the ex post wealth of other $AVaR$ strategies.

We also consider five well-known performance measures: Sharpe ratio, MinMax ratio, maximum drawdown, Ruttiens ratio, Sortino and Satchell ratio, and Rachev ratio. Let $X = R'x$ be a portfolio; then the Sharpe ratio can be defined as:

$$SR(X) = \frac{\mathbb{E}[X]}{\sqrt{\mathbb{V}(X)}}$$

The MinMax ratio is defined as the ratio between expected and portfolio values in the worst possible scenario. It is also consistent with the behavior of the most risk-averse investor. It is defined as follows:

$$MM(X) = \frac{\mathbb{E}[X]}{-\min_s X_s}$$

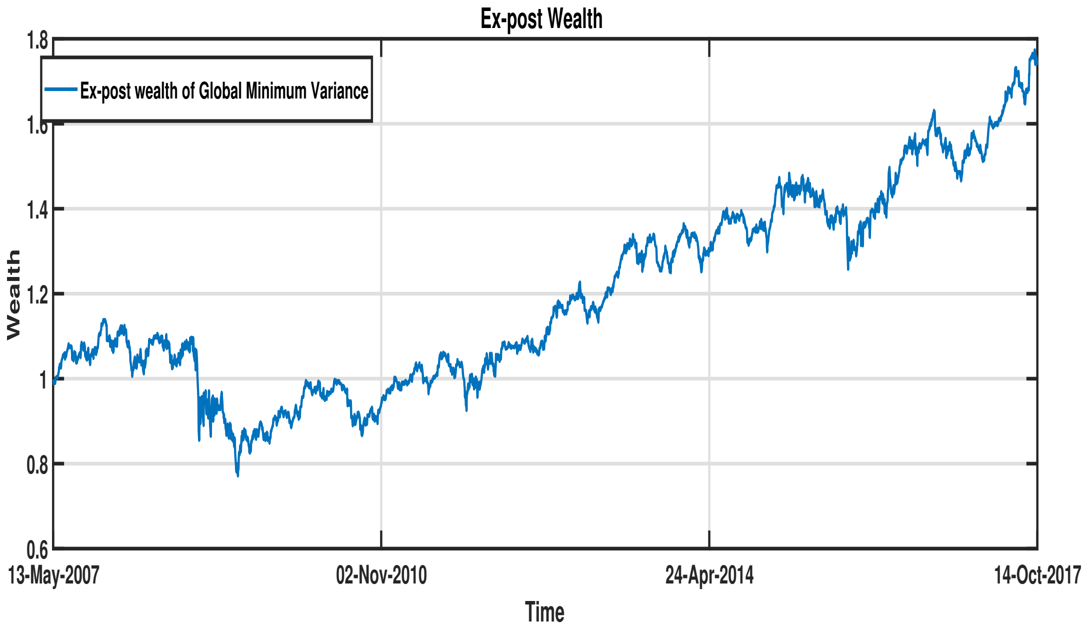
where $X_{(s)}$ is the portfolio value in the state of the world s (i.e., in our case, $-\min_s X_{(s)} = AVaR_{\alpha_1}$ with $\alpha_1 = 1/120$)) (see Young, 1998; Deng et al., 2005). The maximum drawdown is the maximum of the drawdown function, defined in terms of the maximum of the portfolio up to time t and the portfolio at time t:

$$DD(X,t) = \frac{\max_{0\leq\tau\leq t}(X_\tau - X_t)}{\max_{0\leq\tau\leq t} X_\tau}$$

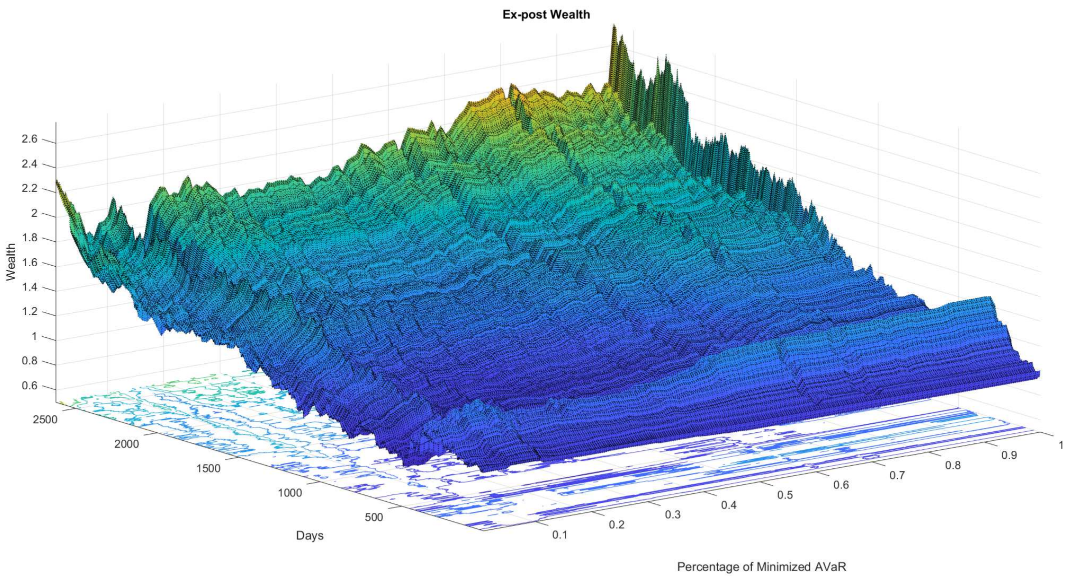
Then, the maximum drawdown of a portfolio X over the period $[0, T]$ is:

$$MDD(X) = max_{0\leq t\leq T} DD(X,t)$$

[6]The selected strategies are portfolios solving problem (1.3) with objective functions: $AVaR_{5\%}$, $AVaR_{10\%}$, $AVaR_{30\%}$, $AVaR_{50\%}$, $AVaR_{75\%}$, $AVaR_{90\%}$, $AVaR_{95\%}$, $AVaR_{100\%}$



(a) Ex post wealth of the minimum variance portfolio.



(b) Ex post wealth of the 120 AVaR Pareto optimal portfolios

Figure 1.5: Ex post strategies

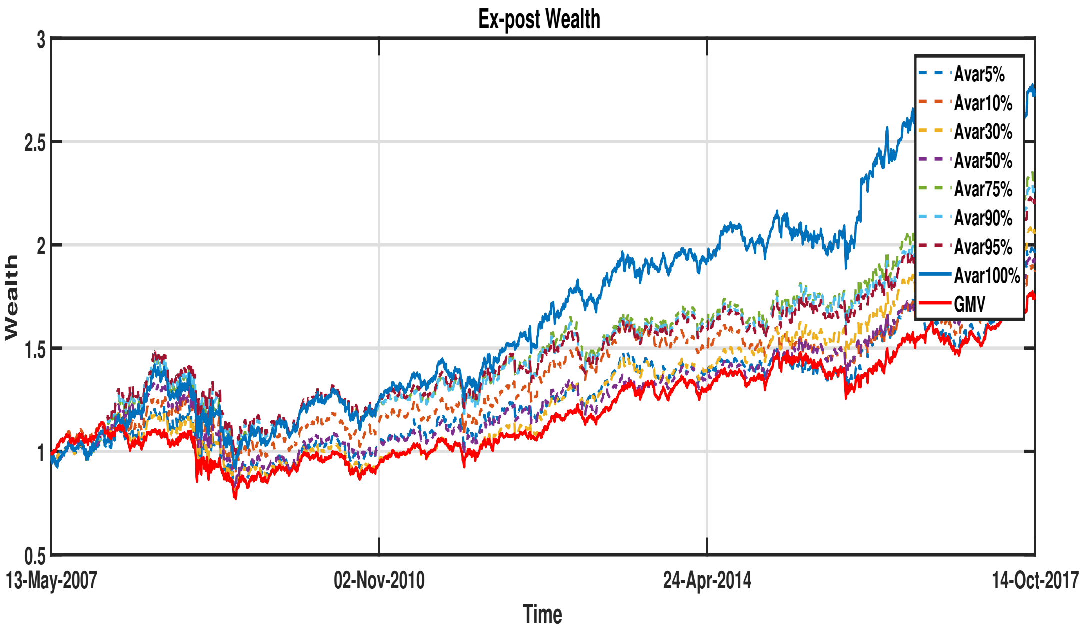


Figure 1.6: Ex post wealth of selected strategies from May 2007 to October 2017. The selected strategies correspond to $AVaR_{5\%}$, $AVaR_{10\%}$, $AVaR_{30\%}$, $AVaR_{50\%}$, $AVaR_{75\%}$, $AVaR_{90\%}$, $AVaR_{95\%}$, $AVaR_{100\%}$.

The Ruttiens ratio is a dynamic performance ratio, in the sense that it considers how returns change over time and is based on portfolio wealth. The risk measure is taken as the standard deviation of the quantity $C_t = W_t - (\frac{t}{T})(W_T - W_0)$, where W_t is the wealth of the portfolio at time t, i.e., (see Ruttiens, 2013):

$$Risk = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(C_t - \mathbb{E}[C])^2}$$

Then, the Ruttiens ratio is defined as the excess wealth, over the period, over a discount rate proportional to the $Risk$ (see Ortobelli et al., 2013):

$$Ruttiens(W_T) = \frac{W_T - 1}{1 + kRisk}$$

Finally, the Sortino and Satchell ratio is the ratio of expected value to semi-standard deviation, and is suitable in case of asymmetric return distributions (see Pedersen and Satchell, 2002):

$$SS(X) = \frac{\mathbb{E}[X]}{\sigma(h)}$$

where $\sigma(h) = \frac{1}{T}\sum_{k=1}^{T}(h - X_k)_+$ and $(a)_+ = \max(a, 0)$.

The Rachev ratio is a performance measure specifically designed to consider the tail behavior of portfolio distribution, and is defined as a ratio of $AVaR$ at different confidence levels (see Biglova et al., 2004):

$$RR_{\alpha,\beta}(X) = \frac{AVaR_\alpha(-X)}{AVaR_\beta(X)}$$

Table 1.3 reports the ex post performance measure values for the GMV, the selected $AVaR$ strategies, a maximum Sharpe ratio strategy (MSR), and the equally weighted portfolio (naive).[7] The $AVaR$ strategies reach an ex post expected return of up to 180% of that of the GMV portfolio. Even in the ex post analysis, the GMV presents the lowest standard deviation among all the selected strategies, and the MSR and naive strategies. Nevertheless, all other performance measures seem to advocate for the other selected strategies.

First of all, the GMV portfolio presents, ex post, a lower value for all performance ratios than the $AVaR$ strategies, Sharpe ratio included. Moreover, it is also possible to infer some decision-making rules based on ex post results. An agent with preferences consistent with the Sharpe ratio or the Rachev ratio, would prefer the portfolio solution of problem (1.3) with $AVaR_{100\%}$ over all the selected strategies. The MinMax ratio suggests that the $AVaR$ strategies seem to respond better to the worst possible scenario, especially those with higher confidence levels. Ruttiens ratio ex post values advocate for the $AVaR_{100\%}$, implying that the ex post dynamic wealth behavior of this strategy is favorable. As per the Sortino and Satchell ratio, portfolio solutions to problem (1.3) with $AVaR$ as the objective function at a confidence level at 100% seem to be preferable for investors with semi-standard deviation or expected value–type preferences.

Moreover, all the $AVaR$ strategies perform better, even considering the maximum drawdown. Summarizing these results, all the selected $AVaR$ strategies are in general less risky and perform better ex post than the GMV portfolio, even considering distribution tails, the worst-case scenario, dynamic point of view, asymmetry in returns distribution, and drawdown.

Table 1.3 also reports on, as references, the MSR strategy and the naive portfolio. The MSR

[7] We compute the Rachev ratio with different parameters, $RR_{1\%,1\%}(X)$, $RR_{5\%,5\%}(X)$, and $RR_{50\%,10\%}(X)$, and the Ruttiens ratio with $k = 1$.

strategy reaches the highest expected return among all analyzed strategies, but it is also riskier than the $AVaR$ strategies, as it has a higher standard deviation, a lower Rachev ratio for all parameterizations, the lowest MinMax ratio, and a higher maximum drawdown. The MSR seems to perform better in terms of expected returns and ex post wealth, but does not allocate the risk efficiently. The naive strategy is usually difficult to beat (see DeMiguel et al., 2009). Even in our ex post analysis, the naive strategy performs quite well in comparison with the $AVaR$ strategies. It reaches a similar level of expected return and has a favorable dynamic of wealth. However, it seems to be more exposed to drawdown and the worst-case scenario.

Table 1.3: Ex post performance of the GMV portfolio and selected strategies. This table reports the ex post value of some performance ratios. The GMV portfolio still presents the lowest standard deviation, but performs poorly according to all other chosen criteria.

Strategies	Mean	St. dev.	$SR(X)$	$MM(X)$	$Ruttiens(W_T)$	$SS(X)$	$MDD(X)$	$RR_{\alpha,\beta}$		
								$RR_{1\%,1\%}(X)$	$RR_{5\%,5\%}(X)$	$RR_{50\%,10\%}(X)$
GMV	0.0053	0.0327	0.1616	0.0584	0.6757	0.5031	2.727	0.8708	0.9203	0.5609
$AVaR_{5\%}$	0.0072	0.0373	0.1928	0.0566	0.8682	0.6336	2.927	0.9280	1.2161	0.6683
$AVaR_{10\%}$	0.0066	0.0364	0.1813	0.0523	0.7971	0.5749	2.533	0.9618	1.1162	0.6438
$AVaR_{30\%}$	0.0069	0.0360	0.1935	0.0646	0.9336	0.6334	2.543	1.0369	1.0625	0.5987
$AVaR_{50\%}$	0.0070	0.0369	0.1895	0.0644	0.8178	0.6174	2.213	1.0508	1.1162	0.5929
$AVaR_{75\%}$	0.0081	0.0383	0.2119	0.0638	1.1667	0.7242	2.268	0.9532	1.1337	0.6205
$AVaR_{90\%}$	0.0081	0.0382	0.2114	0.0633	1.1171	0.7274	2.268	0.9532	1.1225	0.6261
$AVaR_{95\%}$	0.0079	0.0381	0.2066	0.0620	1.0677	0.7042	2.268	0.9532	1.1289	0.6308
$AVaR_{100\%}$	0.0100	0.0392	0.2545	0.0784	1.5241	0.9484	2.268	1.1866	1.3923	0.7244
MSR	0.0130	0.0759	0.1682	0.7336	1.2980	0.6201	3.759	0.9665	0.5153	0.0458
naive	0.0079	0.0446	0.1761	1.0120	1.2112	0.6293	4.044	0.8634	0.4710	0.0539

To confirm this analysis, we verify whether we can identify some type of stochastic dominance among the ex post portfolio returns of the $AVaR$ strategies and the GMV portfolio. Thus, we evaluate and test (with 95% confidence level) the dominance for all non-satiable investors (FSD), all non-satiable risk-averse investors (SSD), all non-satiable risk-seeking investors (increasing convex order [ICX]) (see, among others, Müller and Stoyan, 2002; Davidson and Duclos, 2000). The ICX ordering (generally less used in financial decision-making problems) accounts for the choice of non-satiable risk-seeking investors (i.e., $X\ ICX\ Y$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$, for all non-decreasing and convex functions u).

These tests suggest that the ex post $AVaR$ strategies fail to dominate the GMV portfolio in the FSD and SSD senses.[8] However, all the ex post $AVaR$ returns ICX dominate those of the GMV portfolio. Therefore, we deduce that even if the GMV portfolio is always ex ante SSD dominated, it maintains a strong conservative stability ex post, where it can be dominated only in the ICX sense.

1.4 Conclusion

Portfolio selection deals with investors making decisions under uncertainty. In this chapter, we have analyzed two of the main approaches to optimal portfolio choice proposed in the literature: mean-variance analysis and expected utility. Following the seminal work of Markowitz (1952a), many studies have further developed and extended modern portfolio theory and examined properties of the MVEF (see, for example, Sharpe, 1964; Lintner, 1964; Mossin, 1966). The properties of the mean-variance efficient set determine the main reasons behind its success. For example, the convexity of the efficient set, also implying the efficiency of the market portfolio, has opened the door for the two-fund separation theorem and the capital asset pricing model. In mean-variance analysis, the concept of diversification also plays a major role and has become a key concept for investors and risk-management professionals who must fulfill regulatory requirements or are particularly concerned with reducing risk or achieving more consistent returns over time.

On the other hand, the concepts of expected utility and stochastic dominance yield general criteria for optimal asset allocation for entire classes of agents. In particular, SSD provides a decision-making rule valid for all non-satiable and risk-averse investors. Here, the efficient set is not necessarily convex, and non-satiable risk-averse investors might prefer more concentrated portfolios over well-diversified ones. It is important to note that mean-variance analysis and expected utility

[8] On the one hand, this result is not a surprise, given the stability of GMV characteristics. On the other hand, it is well known that the in-sample and out-of-sample types of analysis could differ substantially (see Roll, 1976, 1978).

are consistent only if returns are assumed to follow an elliptical distribution or if agents have mean-variance type preferences.

In this study, we have used multi-objective optimization to construct portfolio efficient sets for all non-satiable and risk-averse investors, and then compared the performance of these portfolios with mean-variance efficient ones. In particular, we have thoroughly discussed the definition of the SSD rule, and how a multi-objective minimization problem, having as its objective vector the AVaR for all admissible levels, can be used to compute the efficient set. Since AVaR, under the hypothesis of equiprobable scenarios, can be easily linearized, the market portfolio will not necessarily belong to the efficient set for all non-satiable and risk-averse investors. We have also shown that, as a consequence of the transferable assumption of the preference relation for non-satiable and risk-averse investors, agents will not hold a portfolio for which, at any level of risk aversion, there exists a less risky portfolio.

In the conducted empirical analysis, we have found that the global minimum-variance portfolio and approximately 12% of the portfolios belonging to the MVEF are second-order stochastically dominated. These portfolios typically belong to the part of the MVEF where diversification is higher. This result might be a possible explanation for the *diversification puzzle* (Statman, 2004), since neither the market portfolio nor highly diversified portfolios are sought by non-satiable and risk-averse investors.

Portfolios belonging to either of the two efficient sets present, on average, different distributional moments. In particular, for lower levels of expected return, SSD-efficient portfolios seem to have less negative skewness and lower kurtosis, and tend to be more concentrated than those belonging to the MVEF. Moreover, MVEF-efficient portfolios and SSD-efficient portfolios are remarkably different from a distributional perspective.

We have provided empirical evidence that the confidence level of AVaR can be directly interpreted as a risk-tolerance coefficient, and proposed a three-parameter efficient frontier.

In an out-of-sample analysis, we have compared the performances of GMV and 120 $AVaR$ strategies that stochastically dominate GMV. Some of these strategies are able to outperform GMV with respect to the mean return, generating a terminal wealth that is up to 150% higher than that of the GMV portfolio. The strategies typically also perform better for various additional performance criteria, including the Sharpe, Rachev, MinMax, Ruttiens, and Sortino and Satchell ratios. These results suggest that, in general, $AVaR$ strategies reach higher levels of expected return without excessively increasing the standard deviation and risk of the portfolio. A strategy based on maximizing the Sharpe ratio, in our case, will gain more in terms of expected return, but at the same time also substantially increases the risk. Therefore, our analysis suggests that $AVaR$ strategies are more favorable than an MSR strategy with regard to the considered risk-adjusted performance measures.

Overall, our study provides valuable insights into the relationship between mean-variance efficient and SSD-efficient investment strategies. We have also illustrated that the latter typically outperform the GMV portfolio in terms of terminal wealth, as well as with respect to various performance measures.

Chapter 2

Testing for parametric ordering efficiency

Summary

In this chapter, we develop and empirically compare semi-parametric tests to evaluate the efficiency of a benchmark portfolio with respect to different stochastic orderings. Firstly, we classify investors' choices when returns depend on a finite number of parameters: a reward measure, a risk measure, and other parameters. We extend stochastic dominance theory under minimal assumptions about reward and risk measures. We prove that, when choices depend on a finite number of parameters and the reward measure is isotonic with investors' preferences, agents behave as non-satiable and risk-averse when the reward measure is lower than the mean. Similarly, when the reward measure is higher than the mean, we prove that investors behave as non-satiable and risk-seeking when the reward measure is higher than the mean. Secondly, we introduce a new class of stochastic ordering consistent with the choices of investors that are non-satiable and neither risk-averse nor risk-seeking. Then, we propose a methodology to test the efficiency of a portfolio when the return distribution is uniquely identified by four parameters. Finally, we empirically test whether the Fama and French market portfolio, as well as the NYSE and the Nasdaq indexes, are efficient with respect to alternative stochastic orderings.

2.1 Introduction

Several studies in the theory of decision-making under uncertainty have categorized investors according to their utility functions. Typically, investors are considered to be non-satiable and risk-averse, i.e., with non-decreasing and concave utility functions, or non-satiable and risk-seeking, i.e., with non-decreasing and convex utility functions. Nevertheless, it is very hard to know a priori the exact functional shape of agents' utility functions. For these reasons, many hypothesis tests assessing the efficiency of a portfolio are based on stochastic dominance.

Stochastic dominance (SD) ranks prospects based on general regularity conditions for decision-making under risk (Quirk and Saposnik, 1962; Hadar and Russell, 1969; Hanoch and Levy, 1969; Rothschild and Stiglitz, 1970). SD can be seen as a model-free alternative to mean-variance (MV) dominance. The MV criterion is consistent with expected utility for elliptical distributions such as the normal distribution (Chamberlain, 1983; Owen and Rabinovitch, 1983), but has limited economic meaning when the probability distribution cannot be characterized completely by its location and scale.

Simaan (1993), de Athayde and Flôres (2004), and Mencia and Sentana (2009) develop a mean-variance skewness framework based on generalizations of elliptical distributions that are fully characterized by their first three moments. SD presents a further generalization that accounts for all moments of the return distributions without assuming a particular family of distributions.

SD is traditionally applied for comparing a pair of given prospects, for example, two income distributions or two medical treatments. Davidson and Duclos (2000), Barrett and Donald (2003) and Linton et al. (2005), among others, develop statistical tests for such pairwise comparisons.

A more general, multivariate problem is that of testing whether a given prospect is stochastically efficient relative to all mixtures of a discrete set of alternatives (Bawa et al., 1985; Shalit and Yitzhaki, 1994; Post, 2003; Kuosmanen, 2004; Roman et al., 2006). This problem arises naturally in applications of portfolio theory and asset-pricing theory, where the mixtures are portfolios of financial securities. Post and Kopa (2013) and Kopa and Post (2015) developed a linear problem formulation which can be implemented in testing for any n-th order of stochastic dominance (see also Kopa and Post, 2015, and references therein). Post and Versijp (2007), Scaillet and Topaloglou (2010), Linton et al. (2014), and Post and Potì (2017) address this problem using various statistical methods. Their stochastic efficiency tests can be seen as model-free alternatives to tests for MV efficiency, such as the Shanken test (without a riskless asset) (Shanken, 1985, 1986) and the Gibbons Ross Shanken (GRS) test (with a riskless asset) (Gibbons et al., 1989).

Several studies in behavioral finance suggest that investors prefer more to less, and are neither risk-averse nor risk-seeking (see, for example, Markowitz, 1952a; Kahneman and Tversky, 1979; Tversky and Kahneman, 1992; Levy, 1992; Barberis and Thaler, 2003).[1] Thus, it is not clear whether

[1]An efficiency test for prospect stochastic dominance and Markovitz stochastic dominance can be found in Arvanitis

investors' preferences vary with market conditions, and in particular in periods of financial distress. Moreover, it is well known that asset returns exhibit skewness and excess kurtosis (see, for example, Rachev et al., 2011). For these reasons, we propose a new approach able to test for the efficiency of a portfolio from the perspective of non-satiable and neither risk-averse nor risk-seeking investors, and to consider also higher moments of return distributions.

The aims of this chapter are twofold. Firstly, we categorize the preferences of non-satiable investors, under the assumption that return distributions depend on a finite number of parameters. We extend stochastic dominance conditions for FSD, ICV, and ICX, when return distributions depend on a positive-homogeneous and translation-equivariant reward measure, a positive-homogeneous and translation-invariant risk measure, and other distributional parameters.

We show that investors' risk attitudes change according to market conditions. In particular, in a market where the reward measure is higher than the expected return, investors behave as non-satiable and risk-seeking, while when the reward measure is lower than the expected return, they behave as non-satiable and risk-averse. Then, we define a new class of stochastic ordering, coherent with the preferences of investors that are non-satiable and neither risk-averse nor risk-seeking, which we call λ-Rachev orderings, identified by a reward measure based on a linear combination of conditional value at risk (CVaR) (see Biglova et al., 2004; Ortobelli et al., 2009, 2013; Pflug, 2000; Rockafellar and Uryasev, 2000).

Secondly, exploiting estimation function theory, we propose a methodology to test whether a given portfolio is efficient with respect to ICV, ICX, and λ-Rachev ordering (see Godambe and Thompson, 1989). Finally, we propose an empirical analysis by testing whether the Fama and French market portfolio (see Fama and French, 1993) can be considered efficient according to the proposed semi-parametric tests. Starting with Banz (1981), small-cap portfolios are of particular interest in behavioral finance, since they earn a return that defies rational expectation.[2] To apply our methodology to a large-scale problem, we also test whether the NYSE and the Nasdaq market indexes are efficient during the period June 2006 to May 2017.

This paper is organized into four sections. In section 2.2, we propose some ordering criteria when the portfolios distributions are uniquely determined by a reward measure, a deviation measure and a finite number of other parameters. In section 2.3, we propose semi-parametric tests based on the estimating function theory. In section 2.4, we perform the empirical analysis. Section 2.5 briefly summarizes the main results.

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and Topaloglou (2017).

[2]Similar conclusions on small-cap stocks can be found in Post and Kopa (2013) and Arvanitis and Topaloglou (2017).

2.2 Optimal choices depending on a finite number of parameters

In this section, we classify choices for different categories of investors when return distributions depend on a finite number of parameters. We focus on the ordering of parametric choices consistent with investors' preferences. Such a typology of orderings can be placed into the class of FORS orderings (see Ortobelli et al., 2009, 2013). We consider the first order of stochastic dominance (FSD), concave order (also called Rothschild-Stiglitz order [RS or CV]) (see Rothschild and Stiglitz, 1971), increasing and concave order, also called the second order of stochastic dominance (ICV or SSD) and increasing and convex order (ICX), which are respectively consistent with the preferences of non-satiable investors, risk-averse investors, non-satiable risk-averse investors, and non-satiable risk-seeking investors.[3] Recall the classical definitions of different orderings of stochastic dominance.

Definition 3. Given a pair of random variables W and Y, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution functions F_W and F_Y respectively, we say:

- W dominates Y in the sense of the first order of stochastic dominance (i.e., W FSD Y) if and only if $F_W(\lambda) \leq F_Y(\lambda)\ \forall \lambda \in \mathbb{R}$, or equivalently W FSD Y if and only if $E(u(W)) \geq E(u(Y))$ for all non-decreasing functions u.

- W dominates Y in the sense of the second order of stochastic dominance (i.e., W ICV Y) if and only if $\int_{-\infty}^{\lambda} F_W(t)dt \leq \int_{-\infty}^{\lambda} F_Y(t)dt\ \forall\, \lambda \in \mathbb{R}$, or equivalently W ICV Y if and only if $E(u(W)) \geq E(u(Y))$ for all non-decreasing and concave functions u.

- W dominates Y in the sense of the concave order of stochastic dominance (i.e., W CV Y) if and only if W ICV Y and $E(W) = E(Y)$, or equivalently W CV Y if and only if $E(u(W)) \geq E(u(Y))$ for all concave functions u.

- W dominates Y in the sense of the increasing and convex order of stochastic dominance (i.e., W ICX Y) if and only if $\int_{\lambda}^{\infty} 1 - F_W(t)dt \geq \int_{\lambda}^{\infty} 1 - F_Y(t)dt\ \forall\, \lambda \in \mathbb{R}$, or equivalently W ICX Y if and only if $E(u(W)) \geq E(u(Y))$ for all non-decreasing and convex functions u.

[3] Classic stochastic dominance order can be placed into the class of FORS orderings (see a series of papers by Ortobelli, Ortobelli et al., 2009, 2013). The FORS ordering class also contains behavioral orderings often used in the recent financial literature (see, among others, Levy, 1992), the Markowitz ordering (Markowitz, 1952a), and prospect behavioral–type ordering (Tversky and Kahneman, 1992).

- W dominates Y in the sense of the convex order of stochastic dominance (i.e., W CX Y) if and only if W ICX Y and $E(W) = E(Y)$, or equivalently W CX Y if and only if $E(u(W)) \geq E(u(Y))$ for all convex functions u (or equivalently if and only if Y CV W).

where all the above inequalities are strict for at least a real λ and for at least one utility function u.

Let us now consider the optimal portfolio problem. Call $Z = [Z_1, \dots, Z_N]'$ a vector of asset gross returns and, $w = [w_1, \dots, w_N]$ a portfolio weights vector.[4] Let any portfolio $W = w'Z$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Under institutional constraints, such as no short sales allowed (i.e., $w_i \geq 0, i = 1, \dots, N$) and limited liabilities ($Z_i \geq 0, i = 1, \dots, N$), we can assume that all portfolio gross returns are positive random variables and belong to a scale-invariant family, with positive translation properties (see Ortobelli, 2001). We assume that all portfolio gross returns belong to a scale-invariant family $\sigma\tau_q^+(\bar{a})$, with parameters $(\mu(W), \rho(W), a_1(W), \dots, a_{q-2}(W))$, satisfying the following assumptions:

1. Every distribution function $F \in \sigma_q^+(\bar{a})$ is weakly determined by the set of parameters $(\mu(X), \rho(X), a_1(X), \dots, a_{q-2}(X))$, i.e., $F_W, F_Y \in \sigma_q^+(\bar{a})$; then

$$(\mu(W), \rho(W), a_1(W), \dots, a_{q-2}(W)) = (\mu(Y), \rho(Y), a_1(Y) \dots, a_{q-2}(Y))$$

   implies $W \stackrel{d}{=} Y$, but the converse is not necessarily true.

2. $\mu(X)$ is a reward measure that is translation equivariant, i.e., for $t \geq 0$ $\mu(X + t) = \mu(X) + t$, and positive homogeneous, i.e., $\forall\ h \in \mathbb{R}$, $\mu(hX) = h\mu(X)$; and $\rho(X)$ is a risk measure that is consistent with the additive shift, i.e., $\rho(X + t) \leq \rho(X)\ \forall t \geq 0$, and positive homogeneous, while parameters $a_1(X), \dots a_{q-2}(X)$ are translation- and scalar-invariant, i.e., $a_i(X + t) = a_i(X)$ and $a_i(bX) = a_i(X)$ $i = 1, \dots, q - 2$.[5]

Assumption 1 above guarantees that every admissible set of parameters corresponds to a portfolio distribution, and implies that instead of looking at all the distributions, it is sufficient to consider only certain parameters. Assumption 2 has a interesting economic interpretation. Translation

[4] We call the gross return of the i-th asset at time t over the period $[t, t + 1]$ the value $R_{t+1,i} = p_{t+1,i}/p_{t,i}$, where $p_{t,i}$ is the stock value of the asset i at time t.

[5] In this paper we adopt the definition of translation invariance and translation equivariance proposed by Gaivoronski and Pflug (2005), which slightly differs from the one proposed by Artzner et al. (1999).

invariance and consistency with additive shift imply that in a market where there exists a sure gain, i.e., a risk-free rate, the reward measure shifts by the same sure amount, while riskiness does not increase. Positive homogeneity implies that position size linearly affects both riskiness and reward. The first q moments of a distribution are admissible parameters for the $\sigma\tau_q^+(\bar{a})$ family, but the family could admit more general parametrization. In the following, we establish and extend dominance conditions when the $\sigma\tau_q^+(\bar{a})$ family depends on general reward and risk measures.

Theorem 1. *Assume all random admissible portfolios of gross returns belong to a $\sigma\tau_q^+(\bar{a})$ class. Let $W = w'Z$ and $Y = y'Z$ be the random returns of two portfolios determined by the parameters:*

$$(\mu(W), \rho(W), a_1(W), ..., a_{q-2}(W)) \textit{ and } (\mu(Y), \rho(Y), a_1(Y), ..., a_{q-2}(Y))$$

where, $a_i(W) = a_i(Y)$, $i = 1, \dots, q-2$, w and y are the portfolio weights vectors and Z the vector of gross returns. Then, the following implications hold:

1. *$\frac{\mu(W)}{\rho(W)} \geq \frac{\mu(Y)}{\rho(Y)}$ and $\rho(W) \geq \rho(Y)$ (with at least one strict inequality) implies $W\ FSD\ Y$.*
2. *Suppose $\dfrac{\mu(W)}{\rho(W)} = \dfrac{\mu(Y)}{\rho(Y)}$. $\rho(W) > \rho(Y)$ if and only if $W\ FSD\ Y$.*
3. *Suppose the risk measure ρ is translation invariant (i.e., $\rho(X+t) = \rho(X), \forall t \geq 0$) and $\rho(W) = \rho(Y)$. Then W FSD Y, if and only if $\mu(W) > \mu(Y)$.*

Theorem 1 extends some results on the FSD when the distributions depend on general reward and risk measures. The relationship between FSD, μ, and ρ is similar to the case where the reward and risk measures are the first two moments (see, for example, Ortobelli, 2001).

In general, admissible reward measures for the $\sigma\tau_q^+(\bar{a})$ are isotonic with FSD, while risk measure are isotonic with FSD only when certain conditions on the risk reward ratio are met[6]. When conditions on risk reward ratio, of theorem 1 (point 2) are matched, all non-satiable investors will prefer a portfolio with an higher risk measure over one with a lower. Moreover, when the risk measure is translation invariant, if two portfolios have the same risk measure, all non-satiable investors will prefer the one with the higher reward measure.

Having extended FSD conditions in cases of different risk and reward measures, we now discuss conditions for ICV and ICX orderings. We first establish stochastic dominance relations when

[6]A reward measure μ is said to be isotonic with a preferences order $\succ$ when, give two random variables, X and Y, if $X \succ Y$ then $\mu(X) \geq \mu(Y)$.

the $\sigma\tau_q^+(\bar{a})$ depends on the mean and on a positive-homogeneous and translation-invariant risk measure. Then, we extend those conditions also for the other reward measure (different from the mean). We recall that the mean (used as reward measure) is always isotonic, with either risk-averse or risk-seeking behavior, because it can be seen as the expected utility of a linear positive function that is both convex and concave.

Remark 2. *Suppose every admissible portfolio of gross returns belongs to a family $\sigma\tau_q^+(\bar{a})$ as uniquely characterized by the mean μ, a translation-invariant risk measure $\rho(X)$ (i.e., $\rho(X+t) = \rho(X), \forall t \geq 0$) and other $q-2$ scalar- and translation-invariant parameters $a_1, ..., a_{q-2}$. Let $W = w'Z$ and $Y = y'Z$ be two portfolios determined by the parameters $(E(W), \rho(W), a_1(W), ..., a_{q-2}(W))$ and $(E(Y), \rho(Y), a_1(Y), ..., a_{q-2}(Y))$, where $a_i(W) = a_i(Y)$, $i = 1, \dots, q-2$, w and y are the portfolio weights vectors and Z is the vector of gross returns. Suppose W and Y have different distributions and W does not FSD Y. Then, according to Ortobelli (2001), assuming fixed $a_1, ..., a_{q-2}$, we get*

$$\begin{aligned} W\ ICV\ Y\ &\Leftrightarrow \mathbb{E}[W] \geq \mathbb{E}[Y] \text{ and } \rho(W) < \rho(Y) \\ W\ CV\ Y\ &\Leftrightarrow \mathbb{E}[W] = \mathbb{E}[Y] \text{ and } \rho(W) < \rho(Y) \end{aligned} \tag{2.1}$$

As a consequence of this remark, we obtain the following corollary, presenting the ICX-dominance conditions.

Corollary 1. *Suppose every admissible portfolio of gross returns belongs to a family $\sigma\tau_q^+(\bar{a})$ as uniquely characterized by the mean μ, a translation-invariant risk measure $\rho(X)$ (i.e., $\rho(X+t) = \rho(X), \forall t \geq 0$) and other $q-2$ scalar- and translation-invariant parameters $a_1, ..., a_{q-2}$. Let $W = w'Z$ and $Y = y'Z$ be a couple of portfolios determined by the parameters $(E(W), \rho(W), a_1(W), ..., a_{q-2}(W))$ and $(E(Y), \rho(Y), a_1(Y), ..., a_{q-2}(Y))$, where $a_i(W) = a_i(Y)$, $i = 1, \dots, q-2$, w, and y are the portfolio weights vectors and Z is the vector of gross returns.*

Suppose W and Y have different distributions and W does not FSD Y. Then, we get:

$$\begin{aligned} W\ ICX\ Y\ &\Leftrightarrow \mathbb{E}[W] \geq \mathbb{E}[Y] \text{ and } \rho(W) > \rho(Y) \\ W\ CX\ Y\ &\Leftrightarrow \mathbb{E}[W] = \mathbb{E}[Y] \text{ and } \rho(W) > \rho(Y) \end{aligned} \tag{2.2}$$

Remark 2 and corollary 1 establish necessary and sufficient conditions for ICV and ICX dominance when return distributions depend on the mean and a risk measure ρ. When all gross returns belong to the $\sigma\tau_q^+(\bar{a})$ family, dominance conditions for ICX and ICV resemble those when the family is weakly determined by the first q moments. Notice that the risk measure does not necessarily have to be convex. Furthermore, the risk measure discriminates between risk-averse and risk-seeking behavior.

A portfolio is said to be ICV (ICX) efficient if a dominant portfolio in the sense of ICV (ICX) does not exist. As consequence of remark 2 and corollary 1, we state ICV and ICX efficiency conditions for portfolios whose random gross returns belong to the $\sigma\tau_q^+(\bar{a})$ family.

Efficiency Condition 1. *Suppose every admissible portfolio of gross returns belongs to a family $\sigma\tau_q^+(\bar{a})$ uniquely characterized by the mean, a translation-invariant risk measure $\rho(X)$ (i.e., $\rho(X+t) = \rho(X), \forall t \geq 0$) and other $q-2$ scalar- and translation-invariant parameters $a_1, ..., a_{q-2}$ assumed to be fixed. Then, a portfolio $W = w'Z$ is said to be:*

1. *ICV efficient* $\Leftrightarrow \nexists Y = y'Z$ *such that* $\mathbb{E}[Y] \geq \mathbb{E}[W]$ *and* $\rho(Y) < \rho(W)$
2. *ICX efficient* $\Leftrightarrow \nexists Y = y'Z$ *such that* $\mathbb{E}[Y] \geq \mathbb{E}[W]$ *and* $\rho(Y) > \rho(W)$

The ICV efficiency condition is based on (2.1) and the ICX condition is based on (2.2). We will use the above efficiency conditions in the testing methodology in the next section. Nevertheless, remark 2 and corollary 1 do not describe investors' behavior when return distributions depend on a particular reward measure (different than the mean). First, we need to state the following proposition.

Proposition 2. *Suppose that a $\sigma\tau_2^+(\bar{a})$ distribution family admits two possible parametrizations, $(\mu_1(X), \rho(X))$ and $(\mu_2(X), \rho(X))$, with the same risk measure $\rho(X)$ that is translation-invariant (i.e., $\rho(X+t) = \rho(X)$ for any real t). If there exists a random variable $X \in \sigma\tau_2^+(\bar{a})$ such that $\mu_1(X) > \mu_2(X)$, then $\mu_1(Y) > \mu_2(Y)$ $\forall Y \in \sigma\tau_2^+(\bar{a})$ and $(\mu_1(X) - \mu_2(X))/\rho(X)$ is a constant for all random variables $X \in \sigma\tau_2^+(\bar{a})$.*

Proposition 2 is based on the assumption that distributions belonging to the $\sigma\tau_q^+(\bar{a})$ admit two possible parameterizations with different reward measures. In particular, when the risk measure is translation invariant, the ratio between the difference of two admissible reward measures and risk measures is constant, and the order relation between reward measures is the same for all the random variables belonging to the family. The same is also true for the ordering relation expressed in terms of risk–reward ratio.[7]

[7]Positivity is not necessary for proposition 2 to hold. In fact, it holds also for scale and translation-invariant families

Thanks to proposition 2, the following theorem allows us to classify the choices of investors that are non-satiable and neither risk-averse nor risk-seeking, when the $\sigma\tau_q^+(\bar{a})$ family depends on a positive -homogeneous and translation-equivariant reward measure.

Theorem 2. *Let every random admissible portfolio of gross returns be in a $\sigma\tau_q^+(\bar{a})$ family admitting two possible parametrizations with different reward measures. Assuming that the risk measure is translation invariant (i.e., $\rho(X+t)=\rho(X)$) and the mean is one of the two possible reward measures, we define $W=w'Z$ and $Y=y'Z$ random return of two portfolios determined by the parameters $(\mu(W),\rho(W),a_1(W),...,a_{q-2}(W))$ and $(\mu(Y),\rho(Y),a_1(Y),...,a_{q-2}(Y))$ (or $(E(W),\rho(W),a_1(W),...,a_{q-2}(W))$ and $(E(Y),\rho(Y),a_1(Y),...,a_{q-2}(Y))$), where $a_i(W)=a_i(Y)$, $i=1,\ldots,q-2$. Then*

1. *Whenever $\mu(W)>E(W)$, the following implications hold:*
   - 1a) $\frac{\mu(W)}{\rho(W)}\geq\frac{\mu(Y)}{\rho(Y)}$ *and $\mu(W)\geq\mu(Y)$ (with at least one strict inequality) implies W ICV Y.*
   - 1b) *$\mu(W)\geq\mu(Y)$ and $\rho(W)\leq\rho(Y)$ (with at least one strict inequality) implies W ICV Y.*
   - 1c) *$W\ CX\ Y$ (i.e., Y dominates W in the Rothschild-Stiglitz sense, $Y\ CV\ W$) implies $\mu(W)>\mu(Y)$ and $\rho(W)>\rho(Y)$.*
   - 1d) *$W\ ICX\ Y$, but W does not first-order dominate Y, implying that $\mu(W)\geq\mu(Y)$ and $\rho(W)>\rho(Y)$.*
2. *Else, $\mu(W)<E(W)$, and we have:*
   - 2a) *W ICV Y, implying $\mu(W)\geq\mu(Y)$.*
   - 2b) *W ICV Y, but W does not first-order dominate Y (this assumption includes the case W CV Y), implying $\mu(W)\geq\mu(Y)$ and $\rho(W)<\rho(Y)$.*
   - 2c) *$\mu(W)\geq\mu(Y)$ and $\rho(W)\geq\rho(Y)$ (with at least one strict inequality), implying $W\ ICX\ Y$.*

Theorem 2 distinguishes two cases. When $\mu(W)>\mathbb{E}[W]$, points $1a$) and $1b$) describe ICV dominance conditions, while according to point $1c$) and $1d$), reward measures are isotonic with respect to ICX. When $\mu(W)<\mathbb{E}[W]$, according to points $2a$) and $2b$), reward measures are isotonic

$\sigma\tau_2(\bar{a})$ (see Ortobelli, 2001). Notice that every $\sigma\tau_k^+(\bar{a})$ family can be seen as the union of scale-invariant families weakly determined by the same reward and risk measures, and each having a fixed vector of other distributional parameters. Therefore, proposition 2 can be applied to each component of the union. Nevertheless, it is not necessarily true that the ratio remains constant for all the distributions of the family.

with ICV, while point $1c$) is an ICX dominance condition. Thanks to conditions $1d$) and $2b$) in theorem 2, we can state ICV and ICX efficiency conditions when the $\sigma\tau_q^+(\bar{a})$ family depends on a general reward measure.

These conditions are essentially the same as efficiency condition 1, where instead of the mean, we have the reward measure μ.

Theorem 2 also implies that admissible reward measures for the $\sigma\tau_q^+(\bar{a})$ family are isotonic with investors' prevalent behavior (i.e., risk-seeking or risk-averse). In particular, the theorem classifies reward measures with respect to ICV and ICX orders:

1. those greater than the mean (for some fixed distributional parameters) are isotonic with risk-seeking prevalent behavior;

2. those lower than the mean (for some fixed distributional parameters) are isotonic with risk-averse prevalent behavior.

In particular, consider the conditional value at risk (CVaR), defined as $CVaR_\alpha(W) = \frac{1}{\alpha}\int_0^\alpha F_W^{-1}(d)du$, where F_W^{-1} is the left inverse of the cumulative distribution function, i.e., $F_W^{-1}(u) = \inf\{x : \mathbb{P}[W \leq x] = F_W(x) \geq u\}$. When all portfolio gross returns belong to the $\sigma\tau_q^+(\bar{a})$ family, for any portfolio W, $\mu(W) = -CVaR_\alpha(W)$ is an admissible reward measure and always isotonic with ICV, since $CVaR_\alpha$ is a coherent risk measure and, in this case, μ is always lower than the mean. Similarly, $\mu(W) = -CVaR_\alpha(-W)$ is an admissible reward measure and always isotonic with ICX. Combining together $CVaR_\alpha(W)$ and $CVaR_\alpha(-W)$, it is possible to construct a functional consistent with the behavior of investors that are non-satiable and neither risk-averse nor risk-seeking (see, for example, Biglova et al., 2004; Ortobelli et al., 2009, 2013). Therefore, we propose the following functional. For any $\alpha, \beta, \lambda \in [0, 1]$:

$$\gamma_{\alpha,\beta,\lambda}(X) = \lambda CVaR_\alpha(X) - (1-\lambda)CVaR_\beta(-X) \tag{2.3}$$

We call $\gamma_{\alpha,\beta,\lambda}$ the Rachev risk measure. The functional in equation (2.3) (varying α, β and keeping λ fixed) identifies the distribution of X and is consistent with FSD. By the properties of the coherent risk measure, $\mu(X) = -\gamma_{\alpha,\beta,\lambda}(X)$ is an admissible reward measure for the $\sigma\tau_q^+(\bar{a})$ family, and when $\lambda = 0$, it is isotonic with ICX, while when $\lambda = 1$, it is isotonic with ICV (see Artzner et al., 1999; Rachev et al., 2008; Ortobelli et al., 2009, 2013). For all the other values of λ, $\gamma_{\alpha,\beta,\lambda}$ matches the conditions in theorem 2, and therefore also identifies the choices of non-satiable, neither risk-averse nor risk-seeking investors.

Remark 3. *All the aggressive-coherent utility functionals of the form*

$$\mu(X) = \lambda\nu_1(-X) - (1-\lambda)\nu_2(X) \tag{2.4}$$

where ν_1 and ν_2 are coherent risk measures, are admissible reward measures for the $\sigma\tau_q^+(\bar{a})$ family, if they are positive for all admissible portfolios (see Artzner et al., 1999; Biglova et al., 2004; Ortobelli et al., 2013). Similarly, classical deviation measures satisfy the properties of the $\sigma\tau_q^+(\bar{a})$ family, and typical scalar- and translation-invariant parameters are skewness and kurtosis.

We can now introduce a new stochastic ordering, consistent with the preferences of investors that are non-satiable and neither risk-averse nor risk-seeking, based on $\gamma_{\alpha,\beta,\lambda}$.

Definition 4. Given two real-valued random variables X, Y, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, we say that X dominates Y in the sense of λ-Rachev ordering with parameters $\alpha, \beta, \lambda \in [0,1]$ (i.e., $X \geq^R_{\alpha,\beta,\lambda} Y$) if and only if $\gamma_{\alpha,\beta,\lambda}(X) \leq \gamma_{\alpha,\beta,\lambda}(Y)$, $\alpha, \beta \in [0,1]$.[8]

This class of orderings generalizes ICV and ICX orderings that we obtain with $\lambda = 1$ and $\lambda = 0$ respectively.

Remark 4. *The λ-Rachev ordering is a simple risk ordering for any fixed value of λ belonging to $[0,1]$.*

Thanks to theorem 2, investors with preferences coherent with the λ-Rachev ordering behave as non-satiable and risk-averse, when $\gamma_{\alpha,\beta,\lambda}$ is lower than the mean, while they behave as non-satiable and risk-seeking when $\gamma_{\alpha,\beta,\lambda}$ is higher than the mean. Therefore, it's possible to find a portfolio that dominates a given benchmark in the λ-Rachev ordering sense, exploiting ICV and ICX dominance conditions. The following example clarifies this point.

Example 2. *Let us assume that all portfolios of gross return distributions belong to the $\sigma\tau_4^+(\bar{a})$ family, weakly determined by parameters $(\mu(X), \rho(X), a_1(X), a_2(X))$. Suppose there is a portfolio with vector weights $x^{(P)} = [x_1^P, ..., x_n^P]'$ and parameters given by $\mu(P), \rho(P), s, k$, where the reward measure μ is translation equivariant and positive-homogeneous, the risk measure ρ is translation-invariant and positive-homogeneous, and s and k are other scalar- and translation-invariant parameters. If the portfolio P is not FSD dominated, then, according to corollary (1), remark (2) and theorem (2), all*

[8] Similarly, we can define other orderings (for the preferences of investors that are neither risk-seeking nor risk-averse), using the functional as in (2.4), based on parametric coherent spectral measures (see Ortobelli et al., 2009, 2013).

non-satiable risk-seeking investors prefer portfolios solving the following optimization problem:

$$\begin{aligned} \max_{x_i, i=1,\ldots,N} \; & \mu(x'Z) \\ & x'\mathbb{E}[Z] \geq x^{(P)'}\mathbb{E}[Z] \\ & \rho(x'Z) \geq \rho(x^{(P)'}Z) \\ & a_1(x'Z) = s, \; a_2(x'Z) = k \\ & \sum_{i=1}^{N} x_i = 1, \; x_i \geq 0, \; i = 1, \ldots, N \end{aligned} \tag{2.5}$$

over portfolio P. Similarly, all non-satiable risk-averse investors prefer the portfolio solving the following optimization problems:

$$\begin{aligned} \max_{x_i, i=1,\ldots,N} \; & \mu(x'Z) \\ & x'\mathbb{E}[Z] \geq x^{(P)'}\mathbb{E}[Z] \\ & \rho(x'Z) \leq \rho(x^{(P)'}Z) \\ & a_1(x'Z) = s, \; a_2(x'Z) = k \\ & \sum_{i=1}^{N} x_i = 1, \; x_i \geq 0, \; i = 1, \ldots, N \end{aligned} \tag{2.6}$$

over portfolio P.

Feasible regions for problems (2.5) and (2.6) coincide with the dominance conditions for ICX and ICV in remark 2 and corollary 1. With problem 2.5, when the feasible set is non-empty, we select, among the portfolios that dominate P in the ICX sense, the one that has the maximum reward measure μ. Similarly, in the case of problem 2.6, we select the portfolio with the maximum reward measure among those dominating P in the ICV sense. In the next section, we combine these results with estimation function theory to develop semi-parametric tests for ICV, ICX and λ-Rachev ordering.

2.3 Testing choices depending on a finite number of parameters

In this section, we combine estimation function theory with the results of previous sections to develop a hypothesis-testing procedure for Rachev ordering and ICV and ICX efficiency. In the literature, several tests based on the Kolmogorov-Smirnov statistic have been proposed to compare stochastic ordering preferences.[9] We discuss a semi-parametric statistic obtained by estimation function theory.

Let $R = (R_1, ..., R_T)$ be a random vector on a probability space, and the distribution family of this vector parametrized by $\xi = (\xi_1, ..., \xi_p)$. An estimating function (EF) $h(R_t, \xi)$ is called unbiased if $\mathbb{E}\left[h(R_t, \xi)\right] = 0$. Generally, the number of EFs is set equal to the number of parameters ξ_q ($q = 1, ..., p$) through the linear combinations of unbiased EFs $l_{\xi,q} = \sum_{t=1}^{T}\sum_{i=1}^{n}\delta_{q,i,t}h_i(R_t, \xi)$, with $q = 1, ..., p$ and $i = 1, ..., n$. These unbiased EFs $h_i(R_t, \xi)$ are also mutually orthogonal, i.e., for every $i \neq j$, $i, j = 1, \dots, n$, $\mathbb{E}\left[h_i(R_t, \xi)h_j(R_t, \xi)\right] = 0$. In particular, among every linear combination $l_{\xi,q} = \sum_{t=1}^{T}\sum_{i=1}^{n}\delta_{q,i,t}h_i(R_t, \xi)$ of unbiased mutually orthogonal EFs, we consider $l^*_{\xi,q} = \sum_{t=1}^{T}\sum_{i=1}^{n}\frac{\mathbb{E}\left[\frac{\partial h_i(R_t,\xi)}{\partial \xi_q}\right]}{\mathbb{E}\left[h_i^2(R_t, \xi)\right]}h_i(R_t, \xi)$ $q = 1, ..., p$ as the optimal EFs.[10] Then, an estimate $\hat{\xi}$ of ξ is obtained by solving the system of equations $l^*_{\xi,q} = 0$, $q = 1, ..., p$. According to the estimating function theory, the optimal EFs obtained as consistent solutions[11] of equations $l^*_{\xi,q} = 0$, have the property

$$\sqrt{T}\left(\hat{\xi} - \xi\right) \rightarrow N(0, V_{EF}^{-1})$$

where N is a normal distribution with a zero mean vector and variance matrix V_{EF}^{-1}, with $V_{EF} = [v_{i,j}]_{i,j=1,\dots,p}$ and $v_{i,j} = \mathbb{E}l\left[\frac{\partial l^*_{\xi,i}}{\partial \xi_j}\right]$ $i, j = 1, ...p$. Although in some cases we can easily obtain optimal solutions, it is common to introduce convergent methods to compute the roots of equations, starting with an approximate solution to get to the optimal estimates (Crowder, 1986). Typical examples of optimal estimating functions are those proposed by Godambe and Thompson (1989) based on the first four central moments of a given statistic. They propose a model with two unbiased and

[9] See Beach and Davidson (1983), Anderson (1996), Davidson and Duclos (2000), and Scaillet and Topaloglou (2010).

[10] These estimation functions are called optimal since they present the lowest variance among all the $l_{\xi,q}$.

[11] The concept of consistent solutions is given in Crowder (1986).

mutually orthogonal estimating functions:

$$h_1(R_t, \xi) = f(R_t) - \theta_1(\xi) \tag{2.7}$$

$$h_2(R_t, \xi) = (f(R_t) - \theta_1(\xi))^2 - \theta_2^2(\xi) - \theta_3(\xi)\theta_2(\xi)\,(f(R_t) - \theta_1(\xi)) \tag{2.8}$$

where f is a measurable real function $\mathbb{E}\left[f(R_t)\right] = \theta_1(\xi)$, $\theta_2^2(\xi) = \mathbb{E}\left[(f(R_t) - \theta_1(\xi))^2\right]$, and $\theta_3(\xi) = \dfrac{\mathbb{E}\left[(f(R_t) - \theta_1(\xi))^3\right]}{\theta_2^3(\xi)}$. Therefore, the optimal estimating functions are given by

$$l^*_{\xi,q} = \sum_{t=1}^{T} \left(d_{q,t} h_{1,q}(R_t, \xi) + b_{q,t} h_{2,q}(R_t, \xi)\right)$$

where $d_{q,t} = \dfrac{\mathbb{E}\left[\frac{\partial h_1(R_t,\xi)}{\partial \xi_q}\right]}{\mathbb{E}\left[h_1^2(R_t, \xi)\right]}$ and $b_{q,t} = \dfrac{\mathbb{E}\left[\frac{\partial h_2(R_t,\xi)}{\partial \xi_q}\right]}{\mathbb{E}\left[h_2^2(R_t, \xi)\right]}$ $q = 1, ..., p$. Under regularity assumptions, the following proposition determines the class of consistent estimators for $\theta_1(\xi)$.

Proposition 3. *Suppose we have a sample $R = (R_1, ..., R_T)$ of independent and identically distributed (i.i.d.) observations. Consistent estimators of $\widehat{\theta}_1(\xi)$ are given by the solutions of equations $l^*_{\xi,q} = 0$ for $q = 1, \ldots, p$,:*

$$\widehat{\theta}_1(\xi) = \begin{cases} \dfrac{1}{T}\sum\limits_{t=1}^{T} f\left(R_t\right) + c_q - \left(c_q^2 - \dfrac{1}{T}\sum\limits_{t=1}^{T}\left(f\left(R_t\right) - \dfrac{1}{T}\sum\limits_{t=1}^{T} f\left(R_t\right)\right)^2 + \theta_2^2(\xi)\right)^{1/2} & \textit{if}\, c_q > 0 \\ \dfrac{1}{T}\sum\limits_{t=1}^{T} f\left(R_t\right) + c_q + \left(c_q^2 - \dfrac{1}{T}\sum\limits_{t=1}^{T}\left(f\left(R_t\right) - \dfrac{1}{T}\sum\limits_{t=1}^{T} f\left(R_t\right)\right)^2 + \theta_2^2(\xi)\right)^{1/2} & \textit{if}\, c_q \leq 0 \end{cases} \tag{2.9}$$

where $c_q = \frac{d_q - b_q a_1(\xi)\rho(\xi)}{2b_q}$, $d_q = \dfrac{E\left(\frac{\partial h_1(R_t,\xi)}{\partial \xi_q}\right)}{E(h_1^2(R_t, \xi))}$ and $b_q = \frac{E\left(\frac{\partial h_2(R_t,\xi)}{\partial \xi_q}\right)}{E(h_2^2(R_t,\xi))}$. Moreover, we get optimal estimators $\hat{\xi}_q$ when the regularity conditions of the implicit function theorem are satisfied and we can determine the estimates in all of its whole domain.

We combine optimal estimators and their limiting distributions with the theoretical results of the previous section to derive a methodology to test for the efficiency of a given portfolio with respect to another, in the sense of λ-Rachev, ICV and ICX orderings.

2.3.1 Tests for ICV and ICX

Assume that all gross portfolio returns belong to the scale-invariant family $\sigma\tau_q^+(\bar{a})$, weakly determined by the first four moments, $(\mathbb{E}[X], \rho(X), a_1(X), a_2(X))$, where $\rho(X) = \mathbb{E}\left[(X - E(X))^2\right]^{1/2}$ $a_1 = \frac{\mathbb{E}\left[X - E(X))^3\right]}{\mathbb{E}[(X-E(X))^2]^{3/2}}$ and $a_2 = \frac{\mathbb{E}\left[(X-E(X))^4\right]}{\mathbb{E}[(X-E(X))^2]^2}$. Let P be the benchmark portfolio. Consider the ICX efficiency case first. We propose a two-step procedure that involves solving an optimization problem, and then a hypothesis test. According to efficiency condition 1, portfolio P is ICX efficient if there is not another portfolio with a greater or equal mean and greater $\rho(W)$. We consider the following optimization problem:

$$\max_{x_i, i=1,\ldots,N} x'\mathbb{E}\left[Z\right] \tag{2.10}$$
$$\rho(x'Z) \geq \rho(P)$$
$$a_1(x'Z) = s,\ a_2(x'Z) = k$$
$$\sum_{i=1}^{N} x_i = 1,\ x_i \geq 0, i = 1, \ldots, N$$

where s and k are the skewness and kurtosis of portfolio P. In problem (2.10), among the portfolios with the same level of skewness and kurtosis, and at least the same level of standard deviation as portfolio P, we select the one with the maximum mean. Call x^{icx} the solution vector of problem 2.10. Then, according to corollary 1, perform the following hypothesis test:

$$H_0^1 : \text{P is not ICX dominated by } x^{icx'}Z \qquad H_1^1 : \mathbb{E}\left[x^{icx'}Z\right] - \mathbb{E}\left[P\right] > 0 \tag{2.11}$$
$$\rho\left(x^{icx'}Z\right) - \rho\left(P\right) > 0$$

According to corollary 1 and efficiency condition 1, under the null hypothesis P is ICX efficient. The alternative hypothesis implies that $x^{icx'}Z\ ICX\ P$. This is because, with problem (2.10), we seek a portfolio able to dominate P in the ICX sense. If the mean of the candidate dominating portfolio is lower than the mean of P, it means that there is not a portfolio able to dominate it in the ICX sense.[12] The test statistic for the hypothesis test in 2.11 can be found using estimation function theory. In particular, let $R_t^{icx} = [R_1^{icx}, \ldots, R_t^{icx}]$ be historical observations of portfolio gross returns

[12] In case the hypothesis test in (2.11) is performed using the historical observations of gross returns of a portfolio not necessarily solving problem (2.5), the test simply verifies whether the chosen portfolio dominates P in the sense of ICX.

of $x^{icx'}Z$. Let us define the following:

$$\begin{aligned} \theta_1 &= \mathbb{E}\left[f(R_t)\right] \\ \theta_2^2 &= \mathbb{E}\left[\left(f(R_t)-\theta_1\right)^2\right] \\ \theta_3 &= \frac{\mathbb{E}\left[\left(f(R_t)-\theta_1\right)^3\right]}{\theta_2^2} \\ \theta_4 &= \frac{\mathbb{E}\left[\left(f(R_t)-\theta_1\right)^4\right]}{\theta_2^4} \end{aligned} \tag{2.12}$$

where $f(R_t) = R_t$. The optimal estimators for θ_1, $\widehat{\theta}_1$, and θ_2, $\widehat{\theta}_2$ are the roots of the following optimal EFs:

$$\begin{aligned} l_{\theta_1} &= \sum_{t=1}^{T} \frac{-1}{\theta_2^2}\left(f(R_t)-\theta_1\right) + \frac{\theta_3}{\theta_2^3\left(\theta_4-1-\theta_3^2\right)}\left(\left(f(R_t)-\theta_1\right)^2-\theta_2^2-\theta_3\theta_2\left(f(R_t)-\theta_1\right)\right) \\ l_{\theta_2} &= \sum_{t=1}^{T} \frac{-2}{\theta_2^3(\theta_4-1-\theta_3^2)}\left(\left(f(R_t)-\theta_1\right)^2-\theta_2^2-\theta_3\theta_2\left(f(R_t)-\theta_1\right)\right) \end{aligned} \tag{2.13}$$

The distribution of the optimal estimator then satisfies the following:

$$\sqrt{T}\left(\begin{bmatrix}\widehat{\theta}_1 \\ \widehat{\theta}_2\end{bmatrix} - \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}\right) \to^d N\left(0, V^{-1}\right) \tag{2.14}$$

where $V^{-1} = \frac{\theta_2^2}{4}\begin{bmatrix} 4 & 2\theta_3 \\ 2\theta_3 & \theta_4 - 1\end{bmatrix}$.

In the worst-case scenario, $x^{icx'}Z$ has the same distribution as portfolio P, and under the assumption that the parameters identifying the distribution of the benchmark portfolio are known, i.e., we know the true values of $\mathbb{E}[P]$ and $\rho(P)$, the asymptotic density of the vector $\left[\sqrt{T}\left(\widehat{\theta}_1 - \mathbb{E}[P]\right); \sqrt{T}\left(\widehat{\theta}_2 - \rho(P)\right)\right]$ is given by the density $\phi(x, y)$ of the bivariate normal distribution $N(0, V^{-1})$. Then the rejection region for the hypothesis test in (2.11) is given by $C^1 = \{\sqrt{T}\left(\widehat{\theta}_1 - \mathbb{E}[P]\right) > c_1^1, \sqrt{T}\left(\widehat{\theta}_2 - \rho(P)\right) > c_2^1\}$, where c_1^1 and c_2^1 are non-negative real numbers. This rejection region implies that we reject the null hypothesis when the differences between the parameters are positive and large enough. For a given test size, $\alpha \leq 0.5$, c_1^1 and c_2^1 are chosen such that $\int_{c_1^1}^{\infty} \int_{c_2^1}^{\infty} \phi(x,y)dxdy = \alpha$, so that:

$$\lim_{T\to\infty} \mathbb{P}[\text{reject } H_0^1 | H_0^1 \text{ is true}] \leq \int_{c_1^1}^{\infty} \int_{c_2^1}^{\infty} \phi(x,y)dxdy = \alpha$$

Similarly, we can define a testing methodology for ICV efficiency. In this case, we propose to first solve the following optimization problem:

$$\begin{aligned} \max_{x_i, i=1,\dots,N} \; & x'\mathbb{E}[Z] \\ & \rho(x'^{(P)'}Z) \\ & a_1(x'Z) = s, \; a_2(x'Z) = k \\ & \sum_{i=1}^{N} x_i = 1, \; x_i \geq 0, i = 1, \dots, N \end{aligned} \tag{2.15}$$

where in this case we select the portfolio with the maximum mean among those that have the same values of skewness and kurtosis as, and lower standard deviation than, portfolio P. Then call x^{icv} the solution of problem (2.15), and perform the following hypothesis test:

$$H_0^2 : \text{P is not ICV dominated by } x^{icv'}Z \qquad \begin{aligned} H_1^2 : \mathbb{E}\left[x^{icv'}Z\right] - \mathbb{E}[P] > 0 \\ \rho\left(x^{icv'}Z\right) - \rho(P) < 0 \end{aligned} \tag{2.16}$$

Also, in this case, under the null hypothesis, P is ICV efficient, while under the alternative, $x^{icv'}Z \; ICV \; P$. The optimal estimators are the roots of the optimal EF in (2.13). The limiting distribution is equivalent to the ICX case, while the rejection region is now $C^2 = \{\sqrt{T}\left(\widehat{\theta}_1 - \mathbb{E}[P]\right) > c_1^2, \sqrt{T}\left(\widehat{\theta}_2 - \rho(P)\right) < c_2^2\}$, where c_1^2 is a non-negative real number and c_2^2 is a non-positive real number such that $\int_{c_1^2}^{\infty} \int_{-\infty}^{c_2^2} \phi(x,y)dxdy = \alpha$. Hypothesis tests in (2.11) and (2.16), based on the rejection region of the form of C, are asymptotically efficient and unbiased, as shown in the following proposition.

Proposition 4. *When the portfolios belong to a family* $\sigma\tau_4^+(\bar{a})$ *uniquely determined by* $(\mu(X), \rho(X), a_1(X), a_2(X)$ *we can guarantee the existence of opportune values* c_1^i, c_2^i, $i = 1, 2$, *s.t.*

$$\lim_{T\to\infty} P\left(reject\ H_0^i \mid H_0^i\ is\ true\right) \leq \alpha \text{ and } \lim_{T\to\infty} P\left(reject\ H_0^i \mid H_0^i\ is\ false\right) = 1.$$

2.3.2 Test for λ-Rachev ordering

From definition (4), we know that $X \geq_{\alpha,\beta,\lambda} P$ if and only if $\gamma_{\alpha,\beta,\lambda}(X) \leq \gamma_{\alpha,\beta,\lambda}(P)$ for all $\alpha, \beta \in [0,1]$. Then, any investor consistent with the λ-Rachev ordering prefers X to P if and only if $\inf_{\alpha,\beta}\left(\gamma_{\alpha,\beta,\lambda}(P) - \gamma_{\alpha,\beta,\lambda}(X)\right) \geq 0$. Therefore, we say that a portfolio P is λ-Rachev efficient if a dominant portfolio X in the sense of λ-Rachev ordering does not exist, i.e., $\nexists X = x'Z$ such that $\inf_{\alpha,\beta}\left(\gamma_{\alpha,\beta,\lambda}(P) - \gamma_{\alpha,\beta,\lambda}(X)\right) \geq 0$. Thus, we can define a testing methodology for λ-Rachev efficiency. In this case, we propose to solve first the following optimization problem:

$$\max_{x} \min_{i,j=1,\dots,T} \gamma_{\alpha_i,\beta_j,\lambda}(P) - \gamma_{\alpha_i,\beta_j,\lambda}(x'Z) \qquad (2.17)$$
$$\sum_{k=1}^{N} x_k = 1,\ x_k \geq 0,\ k = 1, \dots, N$$

where T is the number of available observations, $\alpha_i = \frac{i}{T}$ and $\beta_j = \frac{j}{T}$ with $i, j = 1, \dots, T$. Thus, call x^R, i^*, and j^* the portfolio and indexes solving problem (2.17). Then, for testing the λ-Rachev efficiency, we should value and test the reward measure $\mu(X) = -\gamma_{\alpha_{i^*},\beta_{j^*},\lambda}(X)$. In particular, assume that all gross portfolio returns belong to the scale-invariant family $\sigma\tau_q^+(\bar{a})$, of positive random variables, with parameters $(\mu(X), \rho(X), a_1(X), a_2(X))$, where:

$$\begin{aligned}
\mu(X) &= E\left(f(X)\right) \\
\rho(X) &= E\left((f(X) - \mu(X))^2\right)^{0.5} \\
a_1(X) &= \frac{E\left((f(X) - \mu(X))^3\right)}{\rho^3(X)} \\
a_2(X) &= \frac{E\left((f(X) - \mu(X))^4\right)}{\rho^4(X)}
\end{aligned} \qquad (2.18)$$

with $f(X) = \lambda \dfrac{I_{[X \geq t_{\beta_{j^*}}(X)]}X}{(1-\beta_{j^*})} + (1-\lambda)\dfrac{I_{[X \leq t_{\alpha_{i^*}}(X)]}X}{\alpha_{i^*}}$ and $t_{\alpha_{i^*}}(X) = F_X^{-1}(\alpha_{i^*})$. Then, the test for λ-Rachev ordering efficiency can be formulated as:

$$H_0^3 : \text{P is not } \lambda - Rachev \text{ dominated by } x^{R'}Z \qquad H_1^3 : \mu(x^{R'}Z) - \mu(P) > 0 \qquad (2.19)$$

Similarly to the previous cases, under the null hypothesis, P is efficient in the λ-Rachev ordering sense, while under the alternative, $x^{R'}Z \geq_{\alpha,\beta,\lambda} P$. Even in this last case, the hypothesis test in (2.19) based on the rejection region of the form $C^3 = \{\sqrt{T}\left(\widehat{\theta}_1 - \mu(P)\right) > c^3\}$ is asymptotically efficient and unbiased. Observe that for $\lambda = 1$ and $\lambda = 0$, we are proposing alternative tests for ICX and ICV orderings respectively.

Proposition 5. *When the portfolios belong to a family* $\sigma\tau_4^+(\bar{a})$ *uniquely determined by* $(\mu(X), \rho(X), a_1(X), a_2(X)$ *we can guarantee the existence of opportune values* c^3*, such that*

$$\lim_{T\to\infty} P\left(reject\ H_0^3 \,\middle|\, H_0^3\ is\ true\right) \leq \alpha \textit{ and } \lim_{T\to\infty} P\left(reject\ H_0^3 \,\middle|\, H_0^3\ is\ false\right) = 1$$

2.4 Empirical applications

In this section, we discuss the results of two empirical applications. Illustrating the potential of the proposed semi-parametric approach, we test whether different reward–risk measures rationalize the market portfolio. Firstly, we test the efficiency of the Fama and French benchmark portfolios to confirm some classical results in portfolio literature (see, among others, Scaillet and Topaloglou, 2010; Post and Kopa, 2013; Arvanitis and Topaloglou, 2017). Secondly, we examine the efficiency of some stock indexes in large-scale markets (the NYSE and Nasdaq).

We deal with three different datasets. In the first case, we consider the Fama and French benchmark portfolios covering the period from July 1963 to October 2001.[13] In the second case, we test the efficiency of the Nasdaq and NYSE composite indexes. We use 876 assets in the NYSE market and 386 in the Nasdaq from 28 December 1995 to 12 May 2017.

2.4.1 Case I. The Fama and French market

This analysis considers the six Fama and French benchmark portfolios as a set of risky assets. They are constructed at the end of each June from the intersections of two size portfolios and three portfolios created according to the ratio between book equity (BE) and market equity (ME). The tested portfolio is the Fama and French market portfolio, which is the value-weighted average of all non-financial common stocks listed on the NYSE, AMEX, and Nasdaq, covered by CRSP and Compustat. A preliminary statistical analysis is reported in table 2.1, showing the first four moments of the Fama and French market portfolio and the six benchmark portfolios.

[13] 460 monthly observations obtained from the homepage of Kenneth French, http://mba.tuck.dartmouth.edu/pages/faculty/ken.French/.

Descriptive statistics				
Benchmark portfolios	Mean	Std. dev.	Skewness	Kurtosis
1	0.316	7.070	-0.337	-1.033
2	0.726	5.378	-0.512	0.570
3	0.885	5.385	-0.298	1.628
4	0.323	4.812	-0.291	-1.135
5	0.399	4.269	-0.247	-0.706
6	0.581	4.382	-0.069	-0.929
Fama and French market portfolio	0.462	4.461	-0.498	2.176

Table 2.1: Descriptive statistics of percentage monthly returns from July 1963 to October 2001 of the Fama and French market portfolio and the six Fama and French benchmark portfolios, formed on size and book-to-market equity ratio. Portfolio 1 has low BE/ME and small size, portfolio 2 has medium BE/ME and small size, etc.

An interesting feature is the comparison between the behavior of the Fama and French market portfolio and the benchmarks. A mean-variance analysis illustrates the inefficiency of the market portfolio considering these two measures. In fact, it is clearly dominated by the benchmark portfolio 6, and the presence of other benchmark portfolios with higher mean or lower standard deviation suggests the possibility of creating a Markowitz efficient frontier above the Fama and French portfolio. Thus, the market portfolio is mean-standard deviation inefficient. Analyzing the efficiency of the portfolio selection problem with a semi-parametric test, we want to verify whether the market portfolio is stochastically dominated in the sense of ICX, ICV, and λ-Rachev ordering (with $\lambda = 0, 0.1, 0.2, ..., 0.9, 1$). We parametrically test (confidence level 95%) the efficiency of the Fama and French portfolio in the λ-Rachev ordering, ICX, and ICV. When we try to solve problems (2.10), (2.15), and (2.17) for 11 levels of λ (i.e., $\lambda = 0, 0.1, 0.2, ..., 0.9, 1$), we find infeasible solutions for problems (2.10) and (2.15), while for all the other cases we did not find dominance, partially confirming the results from Post and Kopa (2013) and Arvanitis and Topaloglou (2017). Moreover, all the tests concord in their results. In fact, the λ-Rachev and ICX and ICV efficiency tests all agree in assessing the efficiency of the Fama and French Portfolio.

2.4.2 Case II. The NYSE and Nasdaq markets

In the second case, we analyze the efficiency of the NYSE and Nasdaq composite indexes, considering the four parameters. In particular, we test the possibility of building a portfolio from the set of the stock components that dominates the index in the ICX, ICV, and λ-Rachev sense (with $\lambda = 0, 0.1, 0.2, ..., 0.9, 1$). The efficiency of the market portfolio changes according to size and time frames, as do the risk and the preferences of investors. To control for those, we consider a rolling-window type of analysis with a window of 10 years. We first test FSD, using the methodology proposed by Scaillet and Topaloglou (2010), and find that only in two cases are the indexes FSD dominated.

When we test for ICX or ICV, we assume that all portfolio gross returns belong to the $\sigma\tau_4^+(\bar{a})$ family, weakly determined by mean, standard deviation, skewness, and kurtosis. We solve problems (2.10) or (2.15) every 21 trading days. In total, we solve 140 optimization problems per test. Then we perform the hypothesis test in (2.11) and (2.16) at a 95% confidence level. When we test λ-Rachev ordering (with $\lambda = 0, 0.1, 0.2, ..., 0.9, 1$) we assume that the portfolios are uniquely determined by the set of parameters in equations (2.18) and then test for λ-Rachev ordering for 11 levels of $\lambda = 0, 0.1, 0.2, ..., 0.9, 1$. Therefore, for any level of λ, we solve 140 optimization problems (2.17) per test.

From the structure of the objective functions, it is clear that these optimization problems may have multiple local optima. To overcome this limitation, we use as a starting point the optimal solution obtained with the heuristic algorithm proposed by Angelelli and Ortobelli (2009), and then improve this solution by applying the heuristic function "pattern search" implemented in Matlab 2015. Then, we perform the hypothesis test in (2.19) on each solution of each problem.

Table 2.2 reports, for each test, the percentage of times the benchmark portfolio is not efficient. We observe that most of the time, in up to 89% of cases, the Nasdaq and NYSE market portfolios are not efficient for non-satiable risk-averse and non-satiable risk-seeking investors. Nevertheless, ICX efficiency is easier to obtain than ICV efficiency, because market portfolios are well diversified and non-satiable risk-seeking investors prefer more concentrated portfolios over well-diversified ones (see, for example, Egozcue and Wong, 2010; Ortobelli et al., 2018). These facts also confirm the results from Kopa and Post (2015) on the efficiency of the market portfolios. Similarly, in the case of λ-Rachev ordering efficiency, we find that the percentage of times the stock indexes are dominated changes with respect to the parameter λ. In particular, for both indexes we see that these percentages are at maximum $\lambda = 0$ or $\lambda = 1$, corresponding to ICV and ICX orderings respectively. However, the λ-Rachev tests suggest that Nasdaq and NYSE market portfolios are most of the time (i.e., in more than 52% of cases) not efficient for investors that are non-satiable and neither risk-averse nor risk-seeking.

To analyze the power of our hypothesis-testing procedure for ICV, ICX, and λ-Rachev efficiency,

Portfolio problem	Ordering	Size	%-dominance NYSE	%-dominance Nasdaq
2.10	ICX	95%	89.286%	82.142%
2.15	ICV	95%	87.857%	77.857%
2.17	0-Rachev	95%	90.714%	83.571%
2.17	0.1-Rachev	95%	79.286%	76.429%
2.17	0.2-Rachev	95%	64.286%	62.143%
2.17	0.3-Rachev	95%	54.286%	56.429%
2.17	0.4-Rachev	95%	59.286%	52.857%
2.17	0.5-Rachev	95%	66.429%	57.857%
2.17	0.6-Rachev	95%	75%	65.714%
2.17	0.7-Rachev	95%	80.714%	74.286%
2.17	0.8-Rachev	95%	86.429%	80%
2.17	0.9-Rachev	95%	90.714%	84.286%
2.17	1-Rachev	95%	93.571%	87.143%

Table 2.2: Percentage of times we identify dominance from June 2006 till May 2017, for a total of 140 optimizations.

we follow a block bootstrap approach proposed by Scaillet and Topaloglou (2010) and Arvanitis and Topaloglou (2017). The block bootstrap resamples blocks in which data are divided, rather than the data themselves, considering the original dependence structure. Our proposed testing methodology presents a higher power than the test proposed by Scaillet and Topaloglou (2010) approximately 67% of the time.

2.5 Conclusion

Stochastic dominance efficiency tests have been developed under the assumption that investors are non-satiable and risk-averse (Post, 2003; Kuosmanen, 2004; Kopa and Post, 2015; Post and Kopa, 2016). Nevertheless, studies based on risk-averse investors have proven to be too restrictive in many circumstances (see, among others, Markowitz, 1952a; Levy and Levy, 2002; Kahneman and Tversky, 1979; Barberis and Thaler, 2003). In this paper, we develop a methodology to assess the efficiency of a given portfolio able to consider a wider class of investors. We extend classic dominance conditions, when the return distributions depend on a reward measure, a risk measure, and other distributional parameters, satisfying a minimal set of assumptions. In doing so, we also establish that investors that are non-satiable and neither risk-averse nor risk-seeking adjust their risk attitude according to market conditions. In a market where the reward measure is higher than the expected return, they are risk-seeking, while when the reward measure is lower than the expected return, they are risk-averse. Then, we define a new class of stochastic orderings called λ-Rachev, which are coherent with the preferences of investors that are non-satiable and neither risk-averse nor risk-seeking.

We employ these results to develop efficiency tests for ICV, ICX, and the newly introduced stochastic orderings. The test statistics are based on estimation function theory and its asymptotic properties. Finally, we illustrate the potential of the proposed statistic tests in two empirical applications. In the case of the Fama and French market portfolio, we find that this portfolio is efficient during the whole period and for all the tests, which is coherent with related literature assessing the efficiency of the Fama and French market portfolio (see, for example, Post and Kopa, 2013; Arvanitis and Topaloglou, 2017). Then, we apply the test statistics to the NYSE and Nasdaq composite indexes. Empirical results indicate that the two stock indexes are often dominated for the λ-Rachev, ICX and ICV orderings. The proposed methodology is general and can be applied to any stochastic ordering defined by a functional, satisfying positive homogeneity and translation equivariance.

Appendix A: Proofs

Theorem 1. *Implication 1:* As a consequence of the assumptions, it follows that

$$h = \mu(W) - \mu(Y) \geq 0 \text{ and } \rho(W) \geq \rho(Y) \geq \rho(Y+t)$$

for every $t \geq 0$. Moreover, for every $t \geq 0$, the function $g(t) \equiv \dfrac{\mu(Y)+t}{\rho(Y+t)}$ is an increasing continuous positive function that tends to infinity for large values of t. As a consequence of the definition of the $\sigma\tau_q^+(\bar{a})$ family, there exists $t \leq h$ such that the random variable $\frac{W}{\rho(W)}$ has the same parameters of $\frac{Y+t}{\rho(Y+t)}$, and hence $\frac{W}{\rho(W)} \stackrel{d}{=} \frac{Y+t}{\rho(Y+t)}$. Then, for every $\lambda \geq 0$:

$$\mathbb{P}\left[W \leq \lambda\right] \leq \mathbb{P}\left[\frac{Y+t}{\rho(Y+t)} \leq \frac{\lambda}{\rho(Y+t)}\right] \leq \mathbb{P}\left[Y \leq \lambda\right]$$

Observe that at least one of the two inequalities $\rho(W) \geq \rho(Y)$ and $h \geq 0$ is strict by hypothesis. Then, at least one of the previous inequalities is strict for some real $\lambda \geq 0$. Therefore, W FSD Y.

Implication 2: According to the definition of the $\sigma\tau_q^+(\bar{a})$ family, it follows that

$$\frac{W}{\rho(W)} \stackrel{d}{=} \frac{Y}{\rho(Y)}$$

because the two random variables have the same parameters. If $\rho(W) > \rho(Y)$, then for every $t \geq 0$

$$\mathbb{P}[W \leq t] = \mathbb{P}\left[\frac{W}{\rho(W)} \leq \frac{t}{\rho(W)}\right] \leq \mathbb{P}\left[\frac{W}{\rho(W)} \leq \frac{t}{\rho(Y)}\right] = \mathbb{P}[Y \leq t]$$

and the above inequality is strict for some t. Conversely, if W FSD Y, then according to Skorokhod's representation theorem, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two random variables X and Y defined on this space such that $X \geq Y$ and X, Y have the same distributions of W and Y. Since $\mu(X)$ is a simple reward measure, $\mu(W) \geq \mu(Y)$ and must be $\rho(W) > \rho(Y)$ (Skorokhod, 1977).

Implication 3: If W FSD Y, then $\mu(W) \geq \mu(Y)$ because any reward measure is isotonic with FSD order. If $\mu(W) = \mu(Y)$, then $W \stackrel{d}{=} Y$, thus $\mu(W) > \mu(Y)$. Conversely, if $\mu(W) > \mu(Y)$, then $W \stackrel{d}{=} Z = Y + \mu(W) - \mu(Y)$ and W FSD Y. □

Corollary 1. Recall that W CV Y if and only if Y CX W (or equivalently, Y ICX W and $\mathbb{E}[W] = \mathbb{E}[Y]$) i.e., every risk-seeking investor prefers Y to W. Thus, W CX Y $\Leftrightarrow$ $\mathbb{E}[W] = \mathbb{E}[Y]$ and $\rho(W) > \rho(Y)$ as consequence of remark 2. When $\mathbb{E}[W] \geq \mathbb{E}[Y]$ and $\rho(W) > \rho(Y)$, then $Z = Y + \mathbb{E}[W] - \mathbb{E}[Y]$ belongs to $\sigma\tau_q^+(\bar{a})$, and from remark 2, W CX Z and Z FSD Y if $\mathbb{E}[W] > \mathbb{E}[Y]$.

Thus, W ICX Y for the transitive property of the ICX ordering (considering that FSD implies ICX). Conversely, suppose W ICX Y, then $\mathbb{E}[W] \geq \mathbb{E}[Y]$. Since the case W ICX Y and $\mathbb{E}[W] = \mathbb{E}[Y]$ has been previously studied, we assume $\mathbb{E}[W] > \mathbb{E}[Y]$. Suppose W and Y have the same risk (i.e., $\rho(W) = \rho(Y)$); then, $Z = Y + \mathbb{E}[W] - \mathbb{E}[Y]$ has the same distribution of W (because Z has the same parameters as W) and W FSD Y counter to the hypothesis. Suppose W presents lower risk than Y (i.e., $\rho(W) < \rho(Y)$). Then $\frac{\mathbb{E}[W]}{\rho(W)} > \frac{\mathbb{E}[Y]}{\rho(Y)}$. Therefore, there exists a positive value $t > \mathbb{E}[W] - \mathbb{E}[Y] > 0$ such that $\frac{\mathbb{E}[W]}{\rho(W)} = \frac{\mathbb{E}[Y]+t}{\rho(Y)}$ and thus, $\frac{W}{\rho(W)} \stackrel{d}{=} \frac{Y+t}{\rho(Y)}$ because they have the same parameters. Then, the distributions of W and Y intersect themselves in the point $M = \frac{t}{\frac{\rho(Y)}{\rho(W)}-1} > 0$, and for every $\lambda \leq M$, $F_W(\lambda) \leq F_Y(\lambda)$, and for every $\lambda > M$, $F_W(\lambda) \geq F_Y(\lambda)$. However, a value $\lambda > M$ such that $F_W(\lambda) > F_Y(\lambda)$ cannot exist, because distribution functions are right-continuous and $\int_M^\infty (1 - F_W(u))\, du < \int_M^\infty (1 - F_Y(u))\, du$, against the assumption $w'Z$ ICX $y'Z$. Thus, if W ICX Y and $\rho(W) \leq \rho(Y)$, it implies W FSD Y, counter to the hypothesis. Hence, it must be that

$\rho(W) > \rho(Y)$. □

Proposition 1. Let $(\mu_1(X), \rho(X))$ and $(\mu_2(X), \rho(X))$ be two parametrizations of the $\sigma\tau_2$ family . Observe that for every distribution function $F_U, F_Y \in \sigma\tau_2^+(a)$, $F_{V_1} := F_{\frac{U-\mu_1(U)}{\rho(U)}} = F_{\frac{Y-\mu_1(Y)}{\rho(Y)}}$ and $F_{V_2} := F_{\frac{U-\mu_2(U)}{\rho(U)}} = F_{\frac{Y-\mu_2(Y)}{\rho(Y)}}$. Then, for every $F_X \in \sigma\tau_2^+(\bar{a})$ identified by the parameters $(\mu_i(X), \rho(X))$, $i = 1, 2$, we get

$$F_X = F_{\rho(X)V_1+\mu_1(X)} = F_{\rho(X)V_2+\mu_2(X)}.$$

Thus, $V_1 + (\mu_1(X) - \mu_2(X))/\rho(X) \overset{d}{=} V_2$, and $(\mu_1(X) - \mu_2(X))/\rho(X)$ is constant for every $F_X \in \sigma\tau_2^+(\bar{a})$. □

Proof of theorem 2. *Case 1:* Suppose $\frac{\mu(W)}{\rho(W)} \geq \frac{\mu(Y)}{\rho(Y)}$ and $\mu(W) \geq \mu(Y)$. From theorem 1, if $\rho(W) \geq \rho(Y)$ implies W FSD Y, that implies W SSD Y. Thus, assume $\rho(W) < \rho(Y)$. From proposition (2), we know that $\dfrac{\mu(W)}{\rho(W)} - \dfrac{\mu(Y)}{\rho(Y)} = \dfrac{\mathbb{E}[W]}{\rho(W)} - \dfrac{\mathbb{E}[Y]}{\rho(Y)} \geq 0$. Since $\mu(W) > E(W)$, then $\dfrac{\mu(W) - \mathbb{E}[W]}{\rho(W)} = c > 0$, which implies $0 \leq \mu(W) - \mu(Y) = \mathbb{E}[W] - \mathbb{E}[Y] + c\,(\rho(W) - \rho(Y))$, i.e., $0 < c\,(\rho(Y) - \rho(W)) \leq \mathbb{E}[W] - \mathbb{E}[Y]$. Then, from Ortobelli (2001), we know that W SSD Y. Similar considerations follow when we assume $\mu(W) \geq \mu(Y)$ and $\rho(W) \leq \rho(Y)$. Moreover, if W CX Y, it is not possible that $\rho(W) \leq \rho(Y)$, since W ICX Y implies W FSD Y, counter to the hypothesis $E(W) = E(Y)$. Thus, it should be $\rho(W) > \rho(Y)$.

From proposition 1:

$$\mu(W) - \mu(Y) = \mathbb{E}[W] - \mathbb{E}[Y] + c\,(\rho(W) - \rho(Y)) = c\,(\rho(W) - \rho(Y)) > 0.$$

Similarly, for $1d)$ from corollary 1, we deduce $\mathbb{E}[W] \geq \mathbb{E}[Y]$ and $\rho(W) > \rho(Y)$. Then $c = \dfrac{\mu(W) - \mathbb{E}[W]}{\rho(W)} > 0$, which implies $\mu(W) - \mu(Y) = \mathbb{E}[W] - \mathbb{E}[Y] + c\,(\rho(W) - \rho(Y)) > 0$, i.e., $\mu(W) > \mu(Y)$.

Case 2: We know that W SSD Y implies that $\mathbb{E}[W] - \mathbb{E}[Y] \geq 0$. If W SSD Y and $\rho(W) \geq \rho(Y)$, from Ortobelli (2001), that implies W FSD Y, which implies $\mu(W) \geq \mu(Y)$ for theorem (1). If W SSD Y and $\rho(W) < \rho(Y)$, then $\dfrac{\mu(W) - \mathbb{E}[W]}{\rho(W)} = c < 0$, because $\mu(W) < \mathbb{E}[W]$. Then, using the same arguments as case 1, we find $0 \leq \mu(W) - \mu(Y) = \mathbb{E}[W] - \mathbb{E}[Y] + c\,(\rho(W) - \rho(Y))$, which explains case 2a), i.e., W SSD Y implies $\mu(W) \geq \mu(Y)$ such that $\mu(W) \geq \mu(Y)$. If W SSD Y and $\rho(W) \geq \rho(Y)$, we know from Ortobelli (2001) that it implies W FSD Y. Thus, it must be that $\rho(W) < \rho(Y)$. To get 2c), assume $\mu(W) \geq \mu(Y)$ and $\rho(W) \geq \rho(Y)$ (with at least one strict inequality) implies that $0 < \mu(W) - \mu(Y) - c(\rho(W) - \rho(Y)) = \mathbb{E}[W] - \mathbb{E}[Y]$, because $c = \dfrac{\mu(W) - \mathbb{E}[W]}{\rho(W)} < 0$. If $\rho(W) = \rho(Y)$, then W has the same distribution of $Z = Y + \mathbb{E}[W] - \mathbb{E}[Y]$, and thus W FSD Y, which implies W ICX Y. If $\rho(W) > \rho(Y)$, then W CX $Z = Y + \mathbb{E}[W] - \mathbb{E}[Y]$ and Z FSD Y, thus W ICX Y. □

Remark 3. Consider two random variables, X, Y, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. According to Ortobelli et al. (2009), a functional $\gamma_X : [0,1] \times [0,1] \to \mathbb{R}$ identifies a simple risk ordering if satisfies the following properties:

a) *Identity:* $X =^d Y$ if and only if $\gamma_X = \gamma_Y$.

b) *Consistency:* $\gamma_X(\alpha, \beta)$ is a risk measure, i.e., it is consistent with FSD, for any values of $\alpha, \beta \in [0,1]$.

We get $a)$ observing that, for $\beta = 1$ we have:

$$\begin{aligned}\gamma_{\alpha,\beta,\lambda}(X) &= \lambda CVaR_\alpha(X) - (1-\lambda)\mathbb{E}[X] \\ &= \lambda CVaR_\alpha\left(\bar{X} - \mathbb{E}[X]\right) - (1-\lambda)\mathbb{E}[X] \\ &= \lambda CVaR_\alpha(\bar{X}) - \mathbb{E}[X]\end{aligned}$$

where $\bar{X}$ is the centered random variable. Thus, if $\gamma_{\alpha,\beta,\lambda}(X) = \gamma_{\alpha,\beta,\lambda}(Y)\ \forall \alpha, \beta \in [0,1]$, it implies that:

$$\lambda\left(CVaR_\alpha(\bar{X}) - CVaR_\alpha(\bar{Y})\right) = \mathbb{E}[X] - \mathbb{E}[Y]\ \forall \alpha \in [0,1].$$

When $\alpha = 1$, we get $CVaR_1(\bar{X}) = CVaR_1(\bar{Y}) = 0$ and $\mathbb{E}[X] = \mathbb{E}[Y]$. Thus $CVaR_\alpha(\bar{X}) = CVaR_\alpha(\bar{Y})\ \forall \alpha \in [0,1]$. Hence, $X =^d Y$. □

Proposition 2. If we solve the estimating equations $l^*_{\xi,k} = 0$ in terms of $\theta_1(\xi)$, we obtain the two solutions:

$$\theta_1(\xi) = \frac{1}{T}\sum_{t=1}^{T} f(R_t) + c_q \pm \left(c_q^2 - \frac{1}{T}\sum_{t=1}^{T}\left(f(R_t) - \frac{1}{T}\sum_{t=1}^{T} f(R_t)\right)^2 + \theta_2^2(\xi)\right)^{1/2}.$$

However, when $T \to +\infty$, the unique consistent admissible equations are:

$\theta_1(\xi) = \frac{1}{T}\sum_{t=1}^{T} f(R_t) + c_q - \left(c_q^2 - \frac{1}{T}\sum_{t=1}^{T}\left(f(R_t) - \frac{1}{T}\sum_{t=1}^{T} f(R_t)\right)^2 + \theta_2^2(\xi)\right)^{1/2}$ if $c_q > 0$ and $\theta_1(\xi) = \frac{1}{T}\sum_{t=1}^{T} f(R_t) + c_q + \left(c_q^2 - \frac{1}{T}\sum_{t=1}^{T}\left(f(R_t) - \frac{1}{T}\sum_{t=1}^{T} f(R_t)\right)^2 + \theta_2^2(\xi)\right)^{1/2}$ if $c_q \leq 0$. □

Proposition 3. Observe that the following equalities hold:

$$\begin{aligned}\sqrt{T}\left(\widehat{\theta}_1 - \mathbb{E}[P]\right) &= \sqrt{T}\left(\widehat{\theta}_1 - \theta_1\right) + \sqrt{T}\left(\theta_1 - \mathbb{E}[P]\right) \\ \sqrt{T}\left(\widehat{\theta}_2 - \rho(P)\right) &= \sqrt{T}\left(\widehat{\theta}_2 - \theta_2\right) + \sqrt{T}\left(\theta_2 - \rho(P)\right)\end{aligned} \tag{2.20}$$

Consider the case of $H_0^1 \; vs \; H_1^1$.

When the H_0^1 hypothesis is true, then portfolio P is not ICX dominated, and one or both inequalities $\mathbb{E}\left(x^{icx'}Z\right) - \mathbb{E}[P] \leq 0$, $\rho\left(x^{icx'}Z\right) - \rho(P) \leq 0$ hold.

In the case both hold with equality, $\left(\mathbb{E}\left[x^{icx'}Z\right] = E[P]\right)$ and $\left(\rho\left(x^{icx'}Z\right) = \rho(P)\right)$, we can always choose opportunely c_1^1 and $c_2^2 > 0$, such that using the limit Gaussian distribution $\lim_{T\to\infty} P\left(reject\ H_0^1 \mid H_0^1\ is\ true\right) = \alpha$.

When both hold but at least one is strict, according to (2.20), we have

$$\begin{aligned}&\lim_{T\to\infty} P\left(reject\ H_0^1 \mid H_0^1\ is\ true\right) = \\ &= \lim_{T\to\infty} P\left(\sqrt{T}\left(\widehat{\theta}_1 - \mathbb{E}[P]\right) \geq c_1^1, \sqrt{T}\left(\widehat{\theta}_2 - \rho(P)\right) \geq c_2^1 \mid H_0^1\ is\ true\right) = 0\end{aligned}$$

When H_0^1 is false, portfolio $x^{icx'}Z$ ICX P. Thus, according to remark (2) and corollary (1), when the portfolios are uniquely determined by the first four moments and the portfolios present the same skewness and kurtosis, we have $\left(\mathbb{E}\left[x^{icx'}Z\right] - \mathbb{E}[P]\right) > 0$, $\left(\rho\left(x^{icx'}Z\right) - \rho(P)\right) > 0$. Then, for any c_1^1 and $c_2^1 > 0$ as a consequence of (2.20), we have

$$\begin{aligned}&\lim_{T\to\infty} P\left(reject\ H_0^1 \mid H_0^1\ is\ false\right) = \\ &= \lim_{T\to\infty} P\left(\sqrt{T}\left(\widehat{\theta}_1 - \mathbb{E}[P]\right) \geq c_1^1, \sqrt{T}\left(\widehat{\theta}_2 - \rho(P)\right) \geq c_2^1 \mid H_0^1\ is\ false\right) = 1.\end{aligned}$$

The case of $H_0^2 \; vs \; H_1^2$ follows analogously. □

Proposition 4. Notice that the following equality holds:

$$\begin{aligned}\mathbb{P}\left[\text{reject } H_0^3\right] &= \mathbb{P}\left[\sqrt{T}\left(\widehat{\theta}_1 - \mu(P)\right) > c^3\right] \\ &= \mathbb{P}\left[\sqrt{T}\left(\widehat{\theta}_1 - \mu(x^{R'}Z)\right) > c^3 + \sqrt{T}\left(\mu(P) - \mu(x^{R'}Z)\right)\right]\end{aligned}$$

So, under H_0^3 we have $\mu(P) - \mu(x^{R'}Z) \geq 0$. Thus, we can always find c^3 such that:

$$\lim_{T\to\infty} \mathbb{P}\left[\text{reject } H_0^3 | H_0^3 \text{ is true}\right] \leq \alpha$$

In addition, when H_0^3 is false, then $\mu(P) - \mu(x^{R'}Z) < 0$, so:

$$\lim_{T\to\infty} \mathbb{P}\left[\text{reject } H_0^3 | H_0^3 \text{ is false}\right] = 1$$

□

Chapter 3

On the efficiency of portfolio choices

Summary

This chapter proposes semi-parametric tests to evaluate the efficiency of a given portfolio with respect to a new stochastic ordering, called λ-Gini order, coherent with the preferences of investors that are non-satiable and neither risk-seeking nor risk-averse. Firstly, the λ-Gini ordering is introduced and its relation with risk aversion is discussed. Secondly, a testing methodology based on associated optimization problems is proposed. The solutions of such problems are designed specifically to dominate the tested portfolio in the λ-Gini order.

Finally, an empirical investigation of the efficiency of the S&P 500 index is proposed.

3.1 Introduction

The concept of expected utility is one of the cornerstones of modern portfolio theory. It relies on a finite number of axioms which characterize investors' preferences. In particular, under the assumptions of rationality of preferences, an agent prefers an investment X over Y if and only if $\mathbb{E}\left[u(X)\right] \geq \mathbb{E}\left[u(Y)\right]$, where u is a given utility function (Von Neumann and Morgenstern, 2007). Typically, optimal choices for different categories of investor can be distinguished using stochastic dominance. Each stochastic ordering identifies a specific investor behavior. Efficient allocations with respect to a given stochastic ordering are also optimal for the corresponding category of agents. The first order of stochastic dominance is coherent with the behavior of non-satiable investors. The second order of stochastic dominance is coherent with the preferences of non-satiable risk-averse investors, while efficient choices for non-satiable risk-seeking investors are coherent with the increasing and convex order of stochastic dominance (see, for example, Fishburn, 1976; Levy, 1992). The literature on testing stochastic dominance efficiency can be split into parametric and non-parametric. For parametric tests, see, among others, Post (2003), Kuosmanen (2004), and Kopa and Post (2015), and for non-parametric tests, see Gibbons et al. (1989), Linton et al. (2005), and Scaillet and Topaloglou (2010).

However, several studies have shown that investors prefer more to less and are neither risk-seeking nor risk-averse (see, among others, Markowitz, 1952b; Kahneman and Tversky, 1979; Levy and Levy, 2002). Therefore, the classic definitions of stochastic dominance appear to be too restrictive to fully characterize investors' preferences.

This chapter proposes semi-parametric tests for portfolio efficiency w.r.t. a new stochastic ordering coherent with the preferences of investors that are non-satiable and neither risk-seeking nor risk-averse, extending the results from chapter 2 to a new stochastic ordering. The new stochastic ordering, called λ-Gini order, is defined in terms of *aggressive-coherent* functionals and specifically designed to take into account the tail behavior of random variables (Biglova et al., 2004; Ortobelli et al., 2013). It is strictly linked to the Gini tail measure, which is an extension of the Gini index, widely used in economics and other social sciences, in particular in measuring disparities (Gini, 1921; Shalit and Yitzhaki, 2005; Ceriani and Verme, 2012, see, for example).

The functional defining the stochastic orderings depends on two parameters, to which correspond different levels of risk aversion.

Following chapter 2, the testing methodology is a two-step procedure.

The chapter is organized as follows: section 3.2 introduces the λ-Gini order, section 3.3 describes the testing methodology. Finally, section 3.4 presents an empirical application to the S&P 500.

3.2 Efficient choices

Classic definitions of stochastic dominance rules are given in terms of iterative integral conditions. For instance, first-order stochastic dominance is based on pairwise comparison of distribution functions. Second-order stochastic dominance is based on pairwise comparison of the integrals of distribution functions, and so on. Inverse stochastic dominance offers an alternative representation based on the left inverse of cumulative distribution functions (see, among others, Dybvig, 1988; Ogryczak and Ruszczyński, 2002; Kopa and Chovanec, 2008). Let X and Y be real random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution functions $F_X(t) = \mathbb{P}[X \leq t]$

and $F_Y(t) = \mathbb{P}[Y \leq t], \forall\, t \in \mathbb{R}$, respectively. Call $F_X^{-1}(p) = \inf\{t : F_X(t) \geq p\}\ \forall p \in [0,1]$ the left inverse of F, then:

- X dominates Y in the sense of the first order of stochastic dominance (i.e., W FSD Y) if and only if $F_X^{-1}(t) \geq F_Y^{-1}(t)\ \forall\, t \in [0,1]$, or equivalently, W FSD Y if and only if $E(u(X)) \geq E(u(Y))$ for all non-decreasing utility functions u.

- X dominates Y in the sense of the second order of stochastic dominance (i.e., X SSD Y) if and only if $\int_0^t F_X^{-1}(\lambda)d\lambda \geq \int_0^t F_Y^{-1}(\lambda)d\lambda\ \forall\, t \in \mathbb{R}$, or equivalently X SSD Y if and only if $E(u(X)) \geq E(u(Y))$ for all non-decreasing and concave utility functions u.

Both FSD and SSD can be interpreted in terms of important financial quantities widely used in recent developments in the literature: X FSD Y if and only if $V@R_\alpha(X) \leq V@R_\alpha(Y)$, and X SSD Y if and only if $CVaR_\alpha(X) \leq CVaR_\alpha(Y)$ for all confidence levels α, where $V@R_\alpha(X) = -F_X^{-1}(\alpha)$ is the value at risk and $CVaR_\alpha = \frac{-1}{\alpha}\int_0^\alpha F_X^{-1}(p)dp$ is the conditional value at risk (see Ogryczak and Ruszczyński, 2002; Kopa and Chovanec, 2008).

Typically, stochastic dominance relations identifies optimal choices for the whole category of investors. A portfolio is said to be optimal, for a given stochastic ordering, if another portfolio able to dominates it does not exist. Moreover, when a portfolio is non-dominated w.r.t. FSD, it is also optimal for all non-satiable investors (i.e., those with increasing utility functions), and when it is SSD non-dominated, it is also optimal for all non-satiable risk-averse investors (i.e., those with increasing and concave utility functions).

Nevertheless, many studies have shown that investors prefer “more” to “less” and are neither risk-seeking nor risk-averse (see, for example, Markowitz, 1952b; Kahneman and Tversky, 1979; Levy and Levy, 2002). Therefore, the aforementioned stochastic orderings appear to be too restrictive. To relax them and construct a stochastic ordering coherent with investors that are non-satiable and neither risk-seeking nor risk-averse, consider the definition of coherent risk measures (see Artzner et al., 1999).

A coherent risk measure is a map $\nu : \Lambda \subset L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, satisfying the following axioms:

- monotonicity: $\forall X, Y \in \Lambda$ such that $X \leq Y$, $\nu(X) \geq \nu(Y)$

- positive homogeneity: $\forall h \in \mathbb{R}$ and $X \in \Lambda$, $\nu(hX) = h\nu(X)$

- translation invariance: $\forall \alpha \in R$ and $X \in \Lambda$, $\nu(X + \alpha) = \nu(X) - \alpha$

- sub-additivity: $\forall X_1, X_2 \in \Lambda$, $\nu(X_1 + X_2) \leq \nu(X_1) + \nu(X_2)$

Sub-additivity and positive homogeneity also imply convexity. Convexity of coherent risk measures is of most importance from a practical point of view, because it means that diversification should not increase the amount of risk.[1]

Combining two coherent risk measures, it is possible to construct functionals consistent with the choices of investors that are non-satiable and neither risk-averse nor risk-seeking. Following Biglova et al. (2004), the resulting functionals are called *aggressive-coherent* and are defined as follows $\forall \lambda \in [0, 1]$:

$$\eta(X) = \lambda\nu_1(X) - (1 - \lambda)\nu_1(-X) \tag{3.1}$$

where ν_1 and ν_2 are coherent risk measures.

The functional in (3.1) is consistent with the preferences of investors that are non-satiable and neither risk-seeking nor risk-averse (see Biglova et al., 2004; Rachev et al., 2008). The next proposition shows some properties of the aggressive-coherent functional ν.

Proposition 6. *Let $\eta(X)$ be an aggressive-coherent functional defined as in (3.1). Then the following properties hold.*

- *translation invariance: $\forall \alpha \in \mathbb{R}$ and $X \in \Lambda$, $\eta(X + \alpha) = \eta(X) - \alpha$*
- *positive homogeneity: $\forall h \in \mathbb{R}$ and $X \in \Lambda$, $\eta(hX) = h\eta(X)$*
- *monotonicity: $\forall X, Y \in \Lambda$, such that $X \geq Y$ $\eta(X) \leq \eta(Y)$*

Proof. Translation invariance:

$$\begin{aligned}
\eta(X + t) &= \lambda\nu_1(X + \alpha) - (1 - \lambda)\,\nu_2(-X - \alpha) \\
&= \lambda\,(\nu_1(X) - \alpha) - (1 - \lambda)\,(\nu_2(-X) + \alpha) \\
&= \lambda\nu_1(X) - (1 - \lambda)\nu_2(-X) - \alpha = \eta(X) - \alpha
\end{aligned}$$

Positive homogeneity:

$$\begin{aligned}
\eta(hX) &= \lambda\nu_1(hX) - (1 - \lambda)\nu_2(-hX) \\
&= h\,[\lambda\nu_1(X) - (1 - \lambda)\nu_2(-X)] = h\eta(X)
\end{aligned}$$

[1]Note that some authors consider the axioms of positive homogeneity and sub-additivity too restrictive and, instead, assume convexity directly (see Acerbi, 2002).

Monotonicity: note that by the monotonicity of coherent risk measures, $\nu(-X) \geq \nu(-Y)$, therefore:

$$\lambda\left(\nu_1(X) - \nu_2(Y)\right) - (1-\lambda)\left(\nu_2(-X) - \nu_2(-Y)\right) \leq 0$$

then, $\eta(X) \leq \eta(Y)$. □

The properties of ν are strictly linked to those of coherent risk measures, but sub-additivity, and hence convexity, for the aggressive-coherent risk measure don't hold. By monotonicity, any aggressive-coherent functional is consistent with FSD, and as reported by Biglova et al. (2004) and Rachev et al. (2008), it is consistent with the preferences of investors who are non-satiable and neither risk-averse nor risk-seeking.

Given the general form of ν, it is possible to derive different functionals by specifying the coherent risk measures ν_1 and ν_2. In particular, consider the Gini type of risk measure. Based on extensions of the Gini index, widely used in measuring social disparities, Gini-type measures have been proposed as an alternative way of measuring prospect variabilities (Konno and Yamazaki, 1991; Shalit and Yitzhaki, 2005). The tail Gini measure (tGM), is a Gini-type measure that focuses on the tail behavior of random variables and also allows specific consideration in terms of risk aversion. It is defined as the integral of the Lorenz curve:

$$G_{\beta,\delta}(X) = -\frac{\delta(\delta-1)}{\beta^{\delta}} \int_0^{\beta} (\beta-u)^{\delta-2} L_X(u)du \tag{3.2}$$

where $L_X(p) = \int_0^p F_X^{-1}(p)$ is the Lorenz curve and δ ($\delta > 1$) governs the weight assigned to the lower β ($\beta \in [0,1]$) percentage of the portfolio distribution. There exists a different formulation for the tGM; in particular, for $\delta = 1$, it is possible to show that tGM is consistent with SSD (see Shalit and Yitzhaki, 1994, 2005; Ortobelli et al., 2013).

When $\delta = 2$, $G_{\beta,2}(X) = G_{\beta}(X)$, in the discrete case corresponding to the cumulative sum of the $\beta\%$ worst possible outcome. Furthermore, $G_{\beta}(X)$ is also consistent with dilation order (see Fagiuoli et al., 1999).

Therefore, following chapter 2, functionals coherent with the preferences of investors that are non-satiable and neither risk-seeking nor risk-averse can be defined as the difference of tGMs, i.e., $\forall, \alpha, \beta \in [0,1]$ and $\lambda \in [0,1]$:

$$\gamma_{\alpha,\beta,\lambda}(X) = \lambda G_{\alpha}(X) - (1-\lambda)G_{\beta}(-X) \tag{3.3}$$

For any fixed λ, and varying the parameters α, β, the functional in (3.3) identifies the distribution of X, and according to proposition (6) is consistent with the monotonic order, and so is a FORS risk measure (see Ortobelli et al., 2009). Moreover, in the discrete settings, it corresponds to a weighted difference between the $\alpha\%$ worst and $\beta\%$ best outcomes. It is possible to define the following new stochastic ordering, called λ-Gini order, consistent with the preferences of an investor agent that is non-satiable and neither risk-seeking nor risk-averse.

Definition 5. Given X and Y random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, X dominates Y in the λ-Gini ordering with parameters $\alpha, \beta, \lambda \in [0, 1]$ (i.e., $X \geq^{G}_{\alpha,\beta\gamma} Y$), if a and only if $\gamma_{\alpha,\beta,\lambda}(X) \leq \gamma_{\alpha,\beta,\lambda}(Y)$ for $\alpha, \beta \in [0, 1]$.

In the next section, we propose semi-parametric tests for the λ-Gini efficiency of a given portfolio.

3.3 Optimal choices depending on a finite number of parameters

Consider now a market with n assets. Denote with $Z = [Z_1, \ldots, Z_n]'$ and $x = [x_1, \ldots, x_n]'$ vectors of gross returns and portfolio weight respectively. Under the assumptions of no short sales allowed ($x_i \geq 0$, $i = 0, \ldots, n$ and $\sum_{i=0}^{n} x_i = 1$) and limited liabilities ($Z_i \geq 0$, $i = 1, \ldots, n$), all portfolios $X = x'Z$ are positive random variables. Following Ortobelli (2001), assume that all portfolio gross return distributions belong to a scale-invariant family $\sigma_4^+(\bar{a})$, with parameters $(\mu(X), \rho(X), a_1(X), a_2(X))$ and the following properties:

1. Every distribution function $F \in \sigma_4^+(\bar{a})$ is weakly determined by the set of parameters $(\mu(X), \rho(X), a_1(X), a_2(X))$, i.e., $F, G \in \sigma_4^+(\bar{a})$, then

$$(\mu(X), \rho(X), a_1(X), a_2(X)) = (\mu(Y), \rho(Y), a_1(Y), a_2(Y))$$

implies $X \overset{d}{=} Y$, but the converse is not necessarily true.

2. $\mu(X)$ is a reward measure translation invariant, i.e., $\mu(X + t) = \mu(X) + t$ for all admissible t and positive homogeneous, and $\rho(X)$ is a risk measure consistent with the additive shift, i.e., $\rho(X + t) \leq \rho(X)$ $\forall t \leq 0$ and positive homogeneous.[2]

[2]For the definitions of translation invariance and translation equivariance see Gaivoronski and Pflug (2005).

Assumption 2 is technical and guarantees that to every set of admissible parameters corresponds a portfolio and that, concerning optimal choices, it is sufficient to look directly at the parameters.

Assumption 3, instead, has interesting economic interpretations. Translation invariance and consistency with additive shift imply that in a market where there exists a sure gain, i.e., a risk-free rate, the reward measure shifts by the same sure amount while riskiness does not increase. Positive homogeneity implies that position size affects both riskiness and reward linearly. According to proposition (6), $\mu(X) = -\gamma_{\delta,\beta,\lambda}(X)$ is an admissible reward measure for the class $\sigma_4^+(\bar{a})$. Moreover, since $\gamma_{\delta,\beta,\lambda}(X)$ is consistent with the preferences of investors that are non-satiable and neither risk-seeking nor risk-averse, $\mu(X)$ is isotonic with the same preference relation.

Let $R = [R_1, \ldots, R_T]'$ be a vector of historical observations of portfolio $X = x'Z$. Then a set of distributional parameters admissible for the $\sigma_q^+(\bar{a})$ is:

$$
\begin{aligned}
\mu(X) &= \mathbb{E}\left[f(R_t)\right] \\
\rho(X) &= \sqrt{\mathbb{E}\left[(f(R_t) - \mu(X))^2\right]} \\
a_1(X) &= \frac{\mathbb{E}\left[(f(R_t) - \mu(X))^3\right]}{\rho(X)^3} \\
a_2(X) &= \frac{\mathbb{E}\left[(f(R_t) - \mu(X))^4\right]}{\rho(X)^4}
\end{aligned}
\tag{3.4}
$$

with

$$
f(R_t) = \lambda \left(\frac{2}{\alpha^2} \sum_{k=1}^{t} R^a_{k:T} \mathbf{1}_{t<[T\alpha]} \right) - (1-\lambda) \left(\frac{2}{\beta^2} \sum_{k}^{t} R^d_{k:T} \mathbf{1}_{t<[T\beta]} \right)
$$

where $R^a_{k:T}$ and $R^d_{k:T}$ are the k-th component of R sorted in ascending and descending order respectively, and $\mathbf{1}_{t<[T\alpha]}$ is the indicator function equal to 1 when t is lower than $[T\alpha]$, which indicates the closest integer number to $T\alpha$. As established in chapter 2, reward measures of the family $\sigma_4^+(\bar{a})$ are linked with SSD and ICX.[3] In particular, when a reward measure is greater than the mean of the distribution, it is also isotonic with the prevalent risk-seeking behavior, while when it is lower than the mean, it is isotonic with the prevalent risk-averse behavior. For example, consider figure 3.1, which depicts the values of mean and $\gamma_{0.05,0.1,0.5}$ and $\gamma_{0.1,0.1,0.5}$ for daily observations of the S&P 500 index with a rolling window of one year. As shown in the graph after 15 September 2008, which corresponds to the Lehman Brothers bankruptcy, $\gamma_{0.05,0.1,0.5}$ assumes lower values than

[3]ICX stands for increasing and convex order, defined as X ICX Y if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all non-satiable and risk-seeking investors.

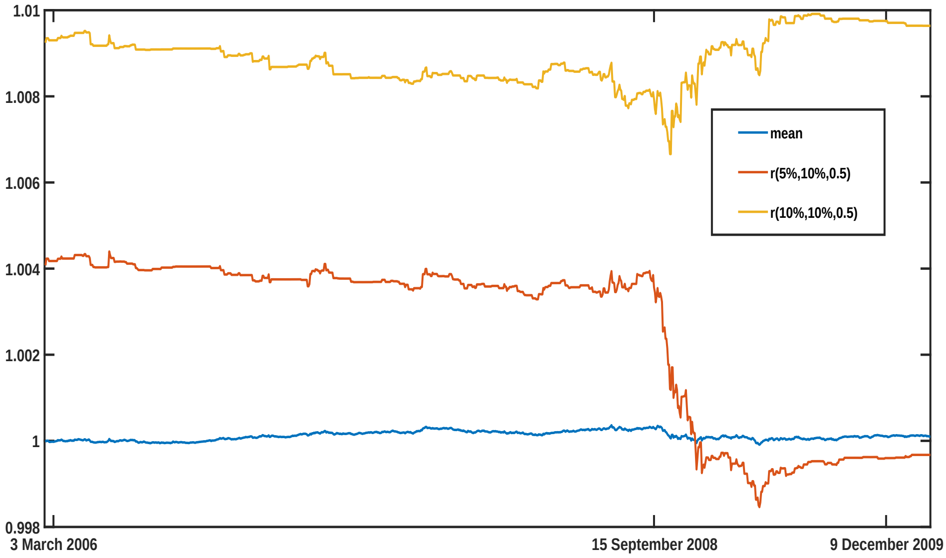


Figure 3.1: Comparison between mean and reward measures of the S&P 500 index computed with a rolling window of one year and step size of one day, from 2006 to 2009. Until 15 September 2008, the day of the Lehman Brothers bankruptcy, both the reward measures are greater than the mean and therefore consistent with the behavior of a non-satiable and risk-seeking investor. After that date, the reward measure $\gamma_{5\%,10\%,0.5}$ slowly becomes consistent with the preferences of non-satiable and risk-averse investors.

the mean, meaning that if before that date the functional was isotonic with risk-seeking behavior, after that date it was isotonic with risk-averse behavior.

Now let $P = p'Z$ be a portfolio, with parameters $(\mu(P), \rho(P), s, k)$, where s and k are skewness and kurtosis parameters respectively. According to chapter 2, it is possible to devise a semi-parametric test for efficiency of portfolio P, with respect to the λ-Gini order, following a two-step procedure.

From definition (5), we know that $X \geq^G_{\alpha,\beta,\lambda} P$ if and only if $\gamma_{\alpha,\beta,\lambda}(X) \leq \gamma_{\alpha,\beta,\lambda}(P)$ for all $\alpha, \beta \in [0,1]$, then any investor consistent with the λ-Gini ordering prefers X to P if and only if $\inf_{\alpha,\beta} \left(\gamma_{\alpha,\beta,\lambda}(P) - \gamma_{\alpha,\beta,\lambda}(X)\right) \geq 0$. Similarly to the case of λ-Rachev ordering, we solve first the following optimization problem:

$$\max_{x} \min_{i,j=1,\ldots,T} \gamma_{\alpha_i,\beta_j,\lambda}(P) - \gamma_{\alpha_i,\beta_j,\lambda}(x'Z) \tag{3.5}$$
$$\sum_{k=1}^{N} x_k = 1,\ x_k \geq 0,\ k = 1,\ldots,N$$

where T is the number of available observations, $\alpha_i = \frac{i}{T}$ and $\beta_j = \frac{j}{T}$ with $i,j = 1,\ldots,T$. Then call the solution of problem (3.5) x_G, $i*$, and j^*. Secondly, perform the following hypothesis test:

H_0 : P is not dominated in the sense of λ-Gini order $\qquad H_1 : \mu(x'_G Z) - \mu(P) > 0$

where $\mu(X) = \mathbb{E}[f^*(X)]$ and

$$f^*(X) = \lambda \left(\frac{2}{\alpha_{i^*}^2} \sum_{k=1}^{t} X_{k:T}^{a} \mathbf{1}_{t<[T\alpha_{i^*}]} \right) - (1-\lambda) \left(\frac{2}{\beta_{j^*}^2} \sum_{k}^{t} X_{k:T}^{d} \mathbf{1}_{t<[T\beta_{j^*}]} \right)$$

Under the null hypothesis, portfolio P is efficient with respect to λ-Gini ordering, while under the alternative, $x'_G Z \geq_{\alpha,\beta\lambda}^{G} P$. This test is based on a rejection region of the form $C = \{\sqrt{T}\,(\widehat{\mu}(x'_G Z) - \mu(P)) > c\}$, where c is a non-negative real number and $\widehat{\mu}(x'_G Z)$ is the optimal consistent estimator for $\mu(x'_G Z)$. According to chapter 2, such an estimator is the root of a linear combination of unbiased and mutually orthogonal estimation functions (see Crowder, 1986; Godambe and Thompson, 1989). Moreover, the limiting distribution of the test statistic is a Gaussian distribution with mean $\sqrt{T}\mu(x^G Z)$ and variance given by a function depending on the second, third, and fourth moment of $\mu(x^G Z)$ (see Crowder, 1986; Godambe and Thompson, 1989).

3.4 Empirical study

The methodology presented in the previous sections can be used to test whether a given portfolio is efficient in the λ-Gini ordering. As described before, the λ-Gini ordering is coherent with the preferences of investors that are non-satiable and neither risk-seeking nor risk-averse. This section proposes an empirical analysis of the efficiency of the S&P 500 index. The dataset is composed of daily observations, from January 2000 to June 2017, of 386 stocks. Since investors' preferences may change over time, the analysis is performed with a rolling window of approximately four years (1,000 daily observations), where portfolio problems are solved every 21 trading days.

According to the previous section, the efficiency can be tested as following a two-step procedure:

1. Assume that the distribution of S&P 500 index is weakly determined by the parameters in (3.4), and then test whether is not FSD dominated.
2. Solve the optimization problem in (3.5) and then perform the hypothesis test $H_0 \; vs \; H_1$.

We consider 11 values of λ ($\lambda = 0, 0.1, 0.2 \ldots, 1$) and the corresponding λ-Gini orderings:

Portfolio problem	Ordering	Confidence level	$\%$-dominance
3.5	0-Gini	95%	89.595
3.5	0.1-Gini	95%	83.815
3.5	0.2-Gini	95%	75.144
3.5	0.3-Gini	95%	68.208
3.5	0.4-Gini	95%	70.520
3.5	0.5-Gini	95%	72.254
3.5	0.6-Gini	95%	75.722
3.5	0.7-Gini	95%	78.034
3.5	0.8-Gini	95%	84.971
3.5	0.9-Gini	95%	91.329
3.5	1-Gini	95%	92.485

Table 3.1: Percentage of times we identify dominance from January 2000 to June 2017, for a total of 173 optimizations

Table (3.1) reports a summary of the testing procedure. Note that the portfolio solutions of each problem in general differ, due to the fact that each problem characterizes different investors' preferences. The last column presents the percentage of times the null hypothesis is rejected. In most cases, the null hypothesis, and hence the efficiency of the S&P 500 index, is rejected. Dominance is easier to obtain in the cases of 0-Gini and 1-Gini orderings than the others. As λ increases (or decreases) to a mid-range value, the dominance percentage of dominance decreases. This is because a mid-range value of λ corresponds to more complex investor behavior. In some sense, λ jeopardizes the risk-averse and risk-seeking preferences. Overall, these results confirm previous findings of chapter 2, and Kopa and Post (2015) and Arvanitis and Topaloglou (2017), on the efficiency of market portfolios.

3.5 Conclusions

This chapter proposes a methodology for testing for portfolio efficiency w.r.t. a stochastic ordering coherent with the preferences of investors that are non-satiable and neither risk-seeking nor risk-averse. A new stochastic ordering has been defined in terms of an aggressive-coherent functional linked to the Gini tail measure. The proposed testing methodology is a two-step procedure. Firstly, assume that gross return distributions belong to a scale-invariant family weakly determined by four parameters. Distributional parameters are chosen coherently with the λ-Gini ordering. Secondly, solve the proposed optimization problem, whose solution should dominate the tested portfolio w.r.t λ-Gini ordering. Finally, perform the associated hypothesis tests to assess the efficiency of the tested portfolio.

The last part of the chapter applies the proposed methodology to a market composed of assets belonging to the S&P 500. It turns out that during the analyzed period (2000–2017), the null hypothesis is rejected almost all the time, implying that the S&P 500 index is dominated in the λ-Gini ordering. This proposed procedure offers a way to test for efficiency w.r.t. λ-Gini ordering, and when the tested portfolio result dominates, it also provides dominating portfolios.

Chapter 4

Risk diversification

Summary

In this chapter, we propose a new approach for quantifying portfolio diversification. The proposed framework defined diversification with respect to a risk measure, and benefits can be interpreted as arising from risk diversification. When satisfying the axioms of coherency, the derived functional is called a coherent risk diversification measure, and it quantifies the percentage of idiosyncratic risk diversified in a portfolio. We also show that under the assumption of elliptically distributed returns, all coherent risk measures depend only on the first two moments of asset returns and portfolio distributions. Finally, we test the proposed risk diversification measures in empirical applications, taking into account various levels of risk aversion. We discuss the concept of mean-risk-diversification efficient frontiers and illustrate how risk-diversified portfolios perform during periods of financial distress. We further examine the ability of portfolio strategies based on risk diversification to outperform given tangent portfolios.

4.1 Introduction

The seminal work of Markowitz provides the foundational understanding of what diversification is (Markowitz, 1952a). In an efficient market, and when all investors are mean-variance optimizers, at equilibrium only non-diversifiable risk is priced, see, e.g., Sharpe (1964); Mossin (1966).

Since then, different approaches have been proposed in the literature. Starting from the empirical evidence that diversification benefits decrease when correlation increases, several studies have related diversification and correlation, see, for example, Levy and Sarnat (1970); Silvapulle and Granger (2001); Dopfel (2003); Skintzi and Refenes (2005). However, while correlation may be an indicator of diversification benefits, it simply pairwise relates asset returns. Thus, much effort has been made to better quantify diversification benefits. For example, Statman (2004) considers the return gap as a measure of diversification benefits. The return gap is based on both correlation and standard deviation: when the benefit of diversification is low, the return gap is low, while when standard deviation is high, the return gap is high. The idea of considering not only correlation but also asset-return standard deviations is also present in the diversification ratio (Choueifaty and Coignard, 2008; Clarke et al., 2013), which is based on the ratio between the weighted average of asset and portfolio standard deviations. The portfolio that maximizes the diversification ratio is called the most diversified portfolio and can be interpreted as the portfolio with the same level of correlation with each of its components Choueifaty and Coignard (2008); Clarke et al. (2013).

Empirical evidence suggests that asset returns exhibit excess kurtosis and skewness, see for example Rachev et al. (2011). Therefore, quantifying diversification only by the first two moments of a portfolio and asset returns might lead to incorrect decisions. For this reason, Vermorken et al. (2012) introduce a diversification measure based on the Shannon entropy, which measures the uncertainty of the entire portfolio distribution rather than only the first two moments. The diversification delta is then based on the ratio between the portfolio's entropy and a weighted average of the entropy of its components. Flores et al. (2017) propose a modified version of diversification delta, called revised diversification delta, which is left bounded. The measures quantifies the diversification of idiosyncratic risk in the portfolio and values the change in size in assets and portfolio in the same way. Using principal component analysis, Rudin and Morgan (2006) developed the so-called portfolio diversification index (PDI), based on the number of independent components of a portfolio. Meucci (2009) extend the PDI and introduce the diversification distribution.

In a different stream of the literature, portfolio diversification is understood as the number of assets with non-zero weights, rather than how the risk is allocated among these assets. Typically, in this literature portfolio diversification is compared by using diversification ordering (Marshall et al., 1943; Wong, 2007; Egozcue and Wong, 2010). Diversification ordering can be expressed as a comparison between the order statistics of two portfolio weight vectors (Wong, 2007; Egozcue and Wong, 2010; Ortobelli et al., 2018). Generally, the most used diversification measures, such as the Herfindahl index, and any Schur-convex function defined on portfolio weights, are consistent with diversification ordering.

In this chapter, we propose a new class of functionals, which we call risk diversification measures

(RDMs). RDMs are defined in terms of a given risk measure, and can be interpreted as the risk-reduction benefits arising from risk diversification. With a positive and convex risk measure, the corresponding value of the RDM can be interpreted as the percentage of idiosyncratic risk diversified among the portfolio components. When the risk measure satisfies all the axioms of coherency, the value of the RDM changes with the size of the assets or portfolio, and it is left bounded. In this case, we call the measure a *coherent* risk diversification measure (CRDM), and it represents the percentage of capital-requirement reduction arising from risk diversification.

We prove that, under the assumption of elliptically distributed returns, any CRDM depends on the mean and standard deviation of asset-returns as well as on the portfolio standard deviation. Moreover, any centered portfolio has a CRDM equivalent to the diversification ratio. On one hand, RDMs generalize diversification measures such as the diversification delta and the diversification ratio (Choueifaty and Coignard, 2008; Flores et al., 2017). On the other hand, the functionals belonging to the class are in general not consistent with diversification ordering, in the sense that there can exist two portfolios with the same order statistics, but with a different RDM, see, among others, Wong (2007); Egozcue and Wong (2010).

We also provide three empirical applications of RDMs and CRDMs. We consider a market composed of stocks belonging to the Dow Jones Industrial Average index (DJIA) from January 3, 2005, to October 14, 2017, and perform a static and a dynamic analysis. The DJIA represents a good candidate to test risk diversification measures: it shows increasing correlation during periods of financial distress, the index itself represents a well-diversified portfolio, and stocks belonging to the index are highly traded (Silvapulle and Granger, 2001; Skintzi and Refenes, 2005; Preis et al., 2012).

In the static analysis, we introduce the mean-risk-diversification frontier. Similarly to the mean-variance efficient frontier, it is formed by portfolios with the highest level of expected return for a desired level of risk diversification. We estimate and compare four different mean-risk-diversification frontiers, using the standard deviation and conditional value at risk with 90%, 95%, and 99% levels of confidence as risk measures (see Pflug, 2000; Rockafellar and Uryasev, 2000; Schulmerich and Trautmann, 2003). We observe that all the mean-risk-diversification frontiers are somewhat concave in the risk diversification measure, and as risk diversification increases, the mean decreases. Comparing the average statistics of portfolios belonging to the efficient frontiers, we observe that risk diversification measures based on conditional value at risk with a high confidence parameter seem to better allocate risk diversification (see Rockafellar and Uryasev, 2000; Pflug, 2000). Moreover, since the confidence level can be seen as the risk-aversion parameter, a higher risk aversion corresponds, on average, to more concentrated portfolios.

In the dynamic analysis, we test the adaptability of risk diversification measures based on the standard deviation and conditional value at risk, at confidence levels of 90%, 95%, and 99%. In paticular we consider periods of financial distress and the performance of RDM-oprimal portfolios in comparison to a given tangent portfolio. During the year after the Lehman Brothers bankruptcy, all strategies based on risk diversification measures produce positive returns. In the second dynamic analysis, we show that using RDMs, constructed portfolios can outperform tangent portfolios in terms of expected return, risk allocation and wealth by up to 150%.

The results of this study are of importance for both practitioners and academics, since RDMs and CRDMs offer alternative investment strategies while formalizing and generalizing the concept of risk diversification.

The remainder of this chapter is organized as follows: section 2 introduces the new class of risk diversification measures, section 3 shows the results of the conducted static and dynamic analysis. Finally, section 4 concludes.

4.2 Coherent Diversification Measures

This section describes the new class of diversification measures and their properties. Functionals belonging to the new class of diversification measures quantify the amount of risk diversified among the assets in the portfolio.

Let $R = [R_1, \ldots, R_N]'$ be a random vector of returns and $w \in \mathbb{R}^N$ be a vector of portfolio weights. Let $P = w'R$ be a portfolio defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and assume no short sales are allowed, i.e., $w \in \Delta$, where:

$$\Delta = \left\{ w \in \mathbb{R}^N : \sum_{i=1}^{N} w_i = 1 \text{ and } w_i \in [0,1],\ i = 1, \ldots, N \right\}.$$

Let us recall the definition of a *coherent risk measure*. A coherent risk measure is a map ν that assigns to each “risky” asset a real value, i.e., ν: $\Lambda \subset L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, satisfying the following properties (Artzner et al., 1999):

- monotonicity: $\forall X, Y \in \Lambda$ such that $X \leq Y$, $\nu(X) \geq \nu(Y)$
- positive homogeneity: for $h \geq 0$ and $X \in \Lambda$, $\nu(hX) = h\nu(X)$
- translation invariance: $\forall \alpha \in \mathbb{R}$ and $X \in \Lambda$, $\nu(X + \alpha) = \nu(X) - \alpha$

- sub-additivity: $\forall X_1, X_2 \in \Lambda$, $\nu(X_1 + X_2) \leq \nu(X_1) + \nu(X_2)$

A positive-homogeneous risk measure detects changes in asset and portfolio size in the same way. Translation invariance implies that the presence of a sure gain in the market, e.g., a risk-free rate, simply shifts the portfolio riskiness by the same sure amount leftward. Positive homogeneity and sub-additivity imply convexity. Convexity implies that diversification should not increase risk.

Coherent risk measures play an important role in the following definition.

Definition 6. Let $X = [X_1, \ldots, X_n]'$ be a vector of returns,[1] $w = [w_1, \ldots, w_n]'$ be a vector of portfolio weights, and $\nu : \Lambda \in L(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a risk measure. We call the risk diversification measure (RDM) for a portfolio $P = w'X$ a functional of the form:

$$D_\nu(P) = 1 - \frac{\nu(P)}{\sum_{i=1}^{n} w_i \nu(X_i)}.$$

A portfolio P_1 presents higher risk diversification with respect to the risk measure ν than a portfolio P_2, if $D_\nu(P_1) \geq D_\nu(P_2)$.

When ν is a coherent risk measure, then D_ν is called a coherent risk diversification measure (CRDM).

RDMs are defined as the ratio between the portfolio risk and the average risk of the portfolio's components. Given their structure, RDMs can be interpreted as a risk-reduction benefit arising from risk diversification. To any risk measure corresponds a different definition of risk, and therefore a different definition of risk diversification. When the risk measure ν satisfies some of the axioms of coherency, the corresponding RDM exhibits the following properties.

Proposition 7. *Let $X = [X_1, \ldots, X_n]$ be a return vector, $w = [w_1, \ldots, w_n]$ be a weights vector of portfolio $P = w'X$, and ν a risk measure. Then the following statements hold.*

1. *If ν is a convex risk measure (and generally consistent with the preference of risk-averse investors), and it assumes the same sign for all admissible portfolios P, then $D_\nu \in [0, 1]$.*

2. *If ν is positive-homogeneous and translation-invariant, and when $X_i = a_i Z + b_i$, with $a_i \geq 0$ for $i = 1, \ldots, n$, then $D(P) = 0$.*

[1] We denote the return of the i-th asset at time t over the period $[t, t+1]$ as $R_{t+1,i} = p_{t+1,i}/p_{t,i} - 1$, where $p_{t,i}$ is the stock value of the asset i at time t.

Proof. Point 1 above is a direct consequence of the definition of a convex function, since when ν is convex, for all admissible portfolios we have:

$$\sum_{i=1}^{n} w_i \nu(X_i) - \nu\left(\sum_{i=1}^{n} w_i X_i\right) \geq 0.$$

To establish point 2, consider $X = [X_1, \ldots, X_n]$ with $X_i = a_i X + b_i$. Then,

$$\begin{aligned} D(P) &= 1 - \frac{\nu\left(\sum_{i=1}^{n} w_i X_i\right)}{\sum_{i=1}^{n} w_i \nu(X_i)} \\ &= 1 - \frac{\nu\left(\sum_{i=1}^{n} w_i (a_i X + b_i)\right)}{\sum_{i=1}^{n} w_i \nu(a_i X + b_i)} \\ & 1 - \frac{\sum_{i=1}^{n} a_i w_i \nu(X) - \sum_{i=1}^{n} w_i b_i}{\sum_{i=1}^{n} a_i w_i \nu(X) - \sum_{i=1}^{n} w_i b_i} = 0. \end{aligned}$$

□

When the risk measure is convex and assumes the same sign for all the assets, D_ν takes values between 0 and 1. In this case, RDM can be interpreted as the percentage of diversified idiosyncratic risk. A portfolio P with $D_\nu(P) = 0$ has the same level of risk diversification with respect to ν as a stand-alone asset. At the otherend of the spectrum, i.e. for $D_\nu(P) = 1$ all the risk has been diversified. Nevertheless, under institutional constraints, such as where no short sales are allowed, this level of risk diversification cannot be reached and, as long as the weight vector $w \in \Delta$, any RDM is bounded away from 1. In particular, the maximum of any RDM is attained when the ratio between portfolio risk and the weighted risk of its components reaches its minimum. A portfolio reaching that level then has the maximum risk diversification with respect to the risk measure ν. When the risk measure is positive-homogeneous and translation-invariant, no diversification benefit can be earned from investments in linearly dependent assets. This is an important property for any RDM, because it ensures that the risk diversification is detected correctly (see for example Vermorken et al., 2012; Flores et al., 2017). Moreover, any RDM based on a positive-homogeneous risk measure is not affected by changes in assets and portfolio size.

Coherent risk measures are convex, positive-homogeneous, and translation-invariant, and describe the capital requirement to regulate the risk assumed by market participants, see, e.g., Artzner et al. (1999); Tasche (2006). So, any CRDM can be interpreted as the percentage of capital-requirement reduction arising from risk diversification.

Consider the conditional value at risk of portfolio P, defined as (Pflug, 2000; Rockafellar and Uryasev, 2000; Schulmerich and Trautmann, 2003):

$$CV@R_\alpha(P) = \inf\left\{a + \frac{1}{1-\alpha}\mathbb{E}\left[(-P-a)^+\right]\right\}.$$

Under this formulation, the $CV@R_\alpha$ corresponds to the mean in the $(1-\alpha)\%$ worst scenarios. Then, we call the diversification conditional value at risk, i.e., $D_{CV@R_\alpha}$, the following CRDM:

$$D_{CV@R_\alpha}(P) = 1 - \frac{CV@R_\alpha(P)}{\sum_{i=1}^N w_i CV@R_\alpha(X_i)}$$

The $D_{CV@R_\alpha}$ is defined as the ratio between the $CV@R_\alpha$ of a portfolio P and the weighted $CV@R_\alpha$ of its components. Different levels of α correspond to a different $CV@R_\alpha$ and, hence, a different CRDM.

Similar approaches to risk diversification have been proposed in the literature by two series of papers that considered similar functionals to D_ν: the diversification ratio (DR) and the revised diversification delta (DD^*). DR is a functional based on the ratio of the weighted average of assets' standard deviation and portfolio standard deviation (Choueifaty and Coignard, 2008; Clarke et al., 2013). DD^* is based on the ratio between the exponential entropy of the portfolio and the weighted average of the exponential entropy of each asset (Vermorken et al., 2012; Flores et al., 2017). Under the assumption of elliptically distributed returns, all the CRDMs are linked to both DD^* and DR.

Remark 5. *Let the return vectors of the individual assets be from an elliptical distribution with mean vector* $\mu_X = [\mu_1, \dots, \mu_n]'$ *and covariance matrix* Σ_X. *Let* $w = [w_1, \dots, w_n]'$ *be a vector of weights and* ν *a coherent risk measure satisfying the following identity property: if* $X \overset{d}{=} Y$ *then* $\nu(X) = \nu(Y)$.

Then, the D_ν *for portfolio* $P = w'X$ *is given by:*

$$D_\nu(P) = 1 - \frac{\sqrt{w'\Sigma_X w}\,\nu(Z) - w'\mu_X}{w'\sigma\nu(Z) - w'\mu_X} \tag{4.1}$$

where Z *is a random variable elliptically distributed with zero and mean and variance equal to 1, and* $sigma = [\sigma_1, \dots, \sigma_n]'$, *where* σ_i *is the standard deviation of asset* i.

Proof. Since $X \sim Ell(\mu_x, \Sigma_X)$, then $P \sim Ell(w'\mu_X, w'\Sigma_X w)$. So by the properties of elliptically distributed random variables, $P =^d w'\mu_X + \sqrt{w'\Sigma_X w}Z$, where $Z \sim Ell(0,1)$. Then:

$$
\begin{aligned}
D_\nu(P) &= 1 - \frac{\nu(P)}{\sum_{i=1}^n w_i \nu(X_i)} \\
&= 1 - \frac{\nu(w'\mu_X + \sqrt{w'\Sigma_X w}Z)}{\sum_{i=1}^n w_i \nu(\mu_i + \sigma_i Z)} \\
&= 1 - \frac{\sqrt{w'\Sigma_X w}\nu(Z) - w'\mu_X}{w'\sigma_i \nu(Z) - w'\mu_X}
\end{aligned}
$$

□

This result can be extended to any family of distributions, weakly determined by a finite number of parameters (see Ortobelli, 2001). Under the assumption of elliptical distribution, any CRDM depends on the standard deviation of each asset, the portfolio mean and standard deviation, and the risk measure evaluated using the standardized random variable as an argument. In the case of a centered portfolio, D_ν is equivalent to DD^* and DR, up to a one-to-one transformation (see Choueifaty and Coignard, 2008; Flores et al., 2017).

CRDMs and RDMs, in general, are not proper diversification measures. Typically, diversification measures are consistent with majorization ordering. We recall the definition of diversification ordering in terms of majorization ordering, see, for example, Wong (2007); Egozcue and Wong (2010); Ortobelli et al. (2018).

Definition 7. Let $X = [X_1, \dots, X_n]'$ be a vector of returns and $\alpha, \beta \in \Delta$ where $\Delta = \{w \in \mathbb{R}^n, w_i \in [0,1], \sum_{i=1}^n w_i = 1\}$, be portfolio weight vectors. Then $\alpha' X$ is more diversified, in the sense of the first order of majorization, than $y'X$, if y dominates in the sense of the first order of majorization w, i.e., $y \succ_M w$ if $F_y(k) = \sum_{i=1}^k y_i^o \geq F_w(k)$ for $k = 1, \dots, n$, where y_i^o is the i-th element of the vector β sorted in ascending order.

According to the definition, a portfolio is more diversified than another if the former majorizes the latter, which is if the ordered weights of the first portfolio are greater than or equal to the ordered weights of the second portfolio. Examples of functions consistent with diversification ordering are the Herfindahl-Hirschman (HH) index, weights vector moments, and in general all Schur-convex functions applied on the space of weights (see Fastrich et al., 2014; Ortobelli et al., 2018). Diversification per se is related to the number of assets in which the initial wealth is invested, rather than *how* the initial wealth is invested. Therefore, there can exist portfolios with the same ordered weights, that is, portfolios with the same diversification according to the literature, but with different risk diversification with respect to a risk measure ν. The following example clarifies this point.

Example 3. *Consider a market with three assets and three possible states of the world, described by the following matrix:*

$$X = \begin{bmatrix} 0.35 & 0.2 & 0.9 \\ 0.1 & -0.1 & 0.2 \\ 0.1 & 0.5 & -0.05 \end{bmatrix}$$

Take two portfolio weights vectors, $w = [0.6, 0.25, 0.15]'$ *and* $y = [0.15, 0.25, 0.6]'$*. Consider two risk diversification measures:* $D_{CV@R_{67\%}}$ *and* D_σ*, where* σ *is the standard deviation. While having the same ordered weights, portfolios* Xw *and* Xy *have* $D_{CV@R_{67\%}}(Xw) = 0.16$ *and* $D_{CV@R_{67\%}}(Xy) = 0.1$*, and* $D_\sigma(Xw) = 0.29$ *and* $D_\sigma(Xy) = 0.22$*.*

This example shows that even if the two portfolios have the same ordered weights (i.e., are equivalent with regard to diversification ordering), they differ in risk diversification.

Having discussed the basic properties of RDMs and their relationship with diversification in terms of majorization ordering, it is fundamental to investigate for which investor category RDMs are designed. Several studies have shown that investors can be classified according to their attitude toward risk (see among others Fishburn, 1980; Rothschild and Stiglitz, 1971; Levy, 1992; Levy and Levy, 2002). Investors prefer "more to less" and can be risk-averse, risk-seeking, or neither risk-averse nor risk-seeking. Typically, the optimal choice for different categories of investors can be distinguished using stochastic dominance. In particular, the choice of non-satiable investors, i.e., with non-decreasing utility functions, is implied by the first order of stochastic dominance.

Definition 8. Let X and Y be random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution functions $F_X(t) = \mathbb{P}[X \leq t]$ and $F_Y(t) = \mathbb{P}[Y \leq t]$ respectively. All non-satiable investors prefer X to Y (i.e., $E(u(X)) \geq E(u(Y))$ for all non-decreasing utility functions u), or, equivalently, X dominates Y with respect to the first order of stochastic dominance ($X\, FSD\, Y$), if and only if $F_X(t) \leq F_Y(t)$, $\forall t \in \mathbb{R}$.

Optimizing a risk measure consistent with FSD gives optimal choices for some non-satiable investors.[2] Therefore, FSD consistency is a desirable property for any risk measure. The next proposition shows under which conditions it is possible to guarantee the FSD consistency of RDMs.

[2]A risk measure ν is consistent with FSD if, given two random variables X and Y, $\nu(X) \leq \nu(Y)$ whenever $X\, FSD\, Y$.

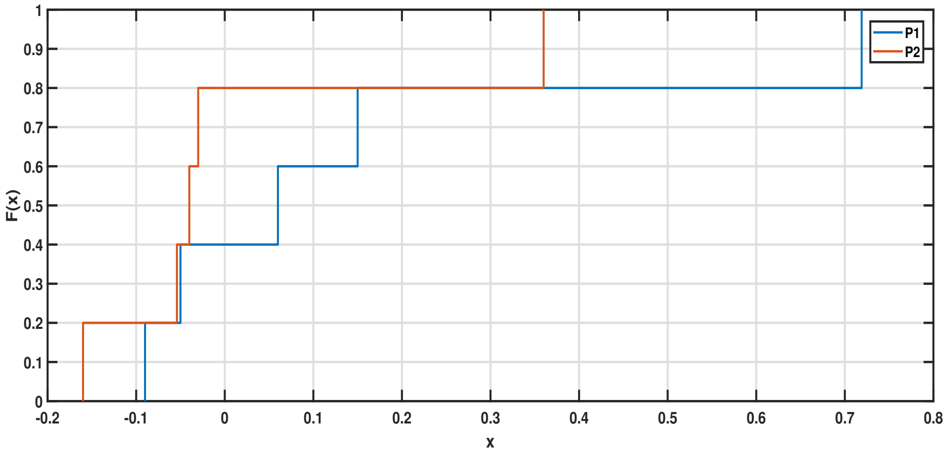


Figure 4.1: Cumulative distribution functions (CDFs) of portfolios P_1 and P_2. The condition for FSD in definition 8 can be applied by a graphical inspection of the CDFs. Blue and red lines are CDFs of P_1 and P_2 respectively. Since F_1 lies beneath F_2, P_1 FSD P_2.

Proposition 8. *Let $X = [X_1, \ldots, X_n]'$ be a vector of returns, $w = [w_1, \ldots, w_n]'$ and $y = [y_1, \ldots, y_n]'$ be portfolio weights, and ν be a risk measure consistent with FSD. Assume that all the asset returns present the same risk (i.e., $\nu(X_i) = \nu(X_j)$ for and i,j). Then, if $w'X$ FSD $y'X$ then $D_\nu(w'X) \geq D_\nu(y'X)$.*

The condition in proposition 8 is quite strong. In real applications, a market where all the assets' returns share the same risk measure hardly exists. This condition applies when the returns are independent identically distrbuted as supposed by Wong (2007); Egozcue and Wong (2010). Nevertheless, the following counterexample shows that when such an assumption is not satisfied, D_ν is not consistent with FSD.

Example 4. *Consider a market with three assets and five states of the world described by the following matrix:*

$$X = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.1 & -0.45 & 0.25 \\ 0.91 & 0 & -0.09 \\ -0.05 & 0.5 & -0.6 \\ -0.2 & 0.3 & 0.4 \end{bmatrix}$$

Take two portfolio weights vectors $w = [0.8, 0.1, 0.1]'$ and $y = [0, 0.4, 0.6]'$. Denote $P_1 = Xw$ and $P_2 = Xy$ with cumulative density functions F_1 and F_2 respectively. As shown in Figure 4.1, P_1 FSD P_2, *but* $D_\sigma(P_1) = 0.21$ *and* $D_\sigma(P_2) = 0.47$, *and* $D_{CV@R_{60\%}}(P_1) = 0.57$ *and* $D_{CV@R_{60\%}}(P_2) = 0.66$.

The last example shows that in general, RDMs are not consistent with FSD. Therefore, maximizing an RDM might give a solution inconsistent with FSD and therefore suboptimal for all non-satiable investors.[3] We propose instead to maximize the mean return of a portfolio for given values of an RDM. In particular, we consider the following optimization problem:

$$\max_w \ \mathbb{E}\left[w'X\right] \tag{4.2}$$

$$s.t. \ \ D_\nu(w'X) \geq \bar{d} \tag{4.3}$$

$$\sum_{i=1}^{n} w_i = 1 \ \ , \ \ 0 \leq w_i \leq 1, \ i = 1, \ldots, n$$

where $\bar{d}$ is a desired level of risk diversification. The solution then depends on the values $\bar{d}$ and on the choice of the risk measure ν.

In the next section, we solve the problem in (4.2) in three different empirical applications. Firstly, we compute and compare mean-risk-diversification efficient frontiers, using $CV@R_\alpha$ with α = 90%, 95%, and 99%, as well as the standard deviation $\sigma(X) = \left(\mathbb{E}\left[(X - \mathbb{E}[X])^2\right]\right)^{1/2}$ as risk measures. Secondly, we test the ability of RDM optimal portfolios to deal with periods of financial distress. Finally, we exploit the constraint in (4.3) to construct portfolios with the same level of risk-diversification as given tangent portfolios. We then examine dynamically generated returns of these portfolios with respect to various performance measures.

4.3 Empirical Analysis

In this section, we propose an empirical analysis of the newly introduced RDMs both in a static and dynamic setting. We consider a market composed of assets belonging to the DJIA from January 3, 2005, to October 13, 2017. [4] Assets belonging to the DJIA are well traded, so we avoid any non-synchronous trading problems, and the index itself can be considered to be similar to a well-

[3]Note that many authors have ignored this aspect, and proposed portfolio strategies that optimize an RDM (see Choueifaty and Coignard, 2008; Clarke et al., 2013; Vermorken et al., 2012). Nevertheless, the lack of consistency of RDM implies that no non-satiable investors would optimize any RDM.

[4]Note that we consider only the assets present in the index for the whole period. The excluded assets are Visa Inc. (NYSE: V), which went public in 2008, and Cisco Systems (NASDAQ: CSCO).

diversified market portfolio (Silvapulle and Granger, 2001; Skintzi and Refenes, 2005). Moreover, it exhibits increasing correlation under periods of financial distress, implying that the diversification benefit decreases when it is needed the most, see, for example, Silvapulle and Granger (2001); Skintzi and Refenes (2005); Preis et al. (2012).

The objective of this section is threefold. In the static analysis, we investigate the mean-risk-diversification efficient frontiers' composition, comparing different RDMs. In the dynamic analysis, we study the ability of RDMs to face periods of financial distress, and, finally, their ability to outperform given tangent portfolios.

4.3.1 Efficient Frontiers of RDMs

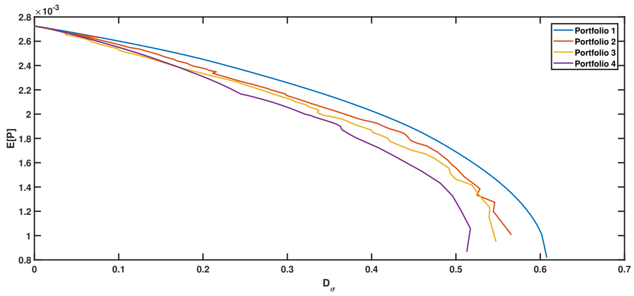
In this subsection, we introduce and describe efficient frontiers for $D_{CV@R_\alpha}$ and D_σ.[5] We consider one year of daily observations from October 13, 2016, to October 13, 2017 for stocks in the DJIA. Similar to the classic mean-variance efficient frontier, for a given risk measure ν, the solution of the problem in (4.2) depends on the value $\bar{d}$. Solving for all admissible levels of $\bar{d}$ then yields the mean-risk-diversification efficient frontier with respect to the risk measure ν. In the following, we consider four different risk measures: $CV@R_\alpha$, with α = 90%, 95%, and 99% as well as the standard deviation σ and for each of these risk measures, we adapt the constraint in (4.3) accordingly.

To estimate the four mean-risk-diversification efficient frontiers, we need to identify the "admissible levels" for $\bar{d}$. Since each of the RDMs measures risk diversification differently, we can expect to have different admissible levels $\bar{d}$. For each problem, the admissible levels $\bar{d}$ belong to the admissible interval $[0, \max_{w\in\Delta} D_\nu(P)]$. Thus, we select 100 equally spaced points in the admissible interval and then solve problem (4.2) for each RDM 100 times.

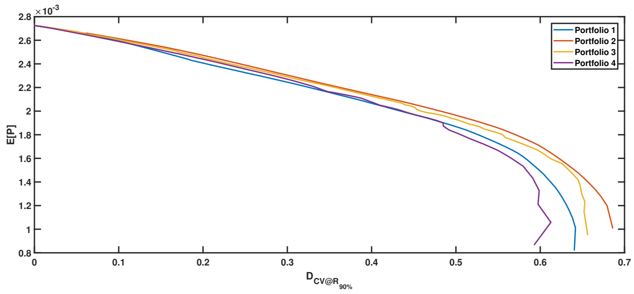
Figure 4.2 shows the four mean-risk-diversification efficient frontiers. Each panel depicts the mean and RDM of all solutions to the problem in (4.2) with different constraints (4.3). To be more precise, *ES 1*, *ES 2*, *ES 3*, and *ES 4* are efficient frontiers containing mean-D_σ, mean-$D_{CV@R_{90\%}}$, mean-$D_{CV@R_{95\%}}$, and mean-$D_{CV@R_{99\%}}$, respectively, efficient sets of portfolios. Thus, Panel (a) depicts the mean-D_σ efficient frontier (ES 1), and the values of mean and D_σ for the portfolios belonging to the other mean-risk-diversification frontiers, i.e. mean-$D_{CV@R_{90\%}}$, mean-$D_{CV@R_{95\%}}$, and mean-$D_{CV@R_{99\%}}$.[6] Panels (b), (c), and (d) are then constructed in a similar manner. Since $D_\nu(P) = 0$ when the portfolio is formed by a stand-alone asset, the asset with the maximum mean belongs to

[5]Note that D_σ can be represented as a one-to-one transformation of the DR in Choueifaty and Coignard (2008) and Clarke et al. (2013). We consider the RDM version of it to assure internal consistency in our empirical analysis.

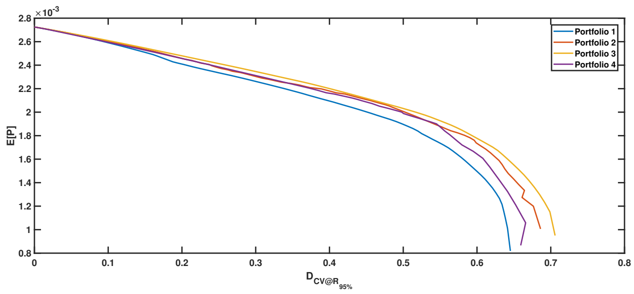
[6]By construction the latter portfolios will typically not be efficient in the mean-D_σ plane.



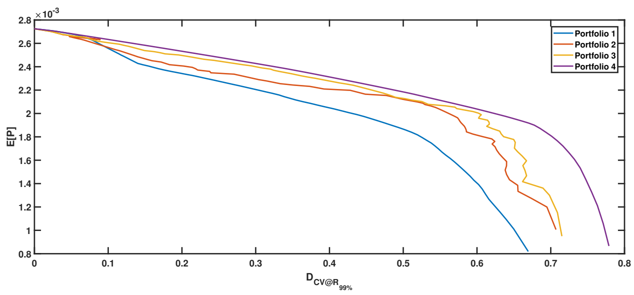
(a) Mean-D_σ efficient frontiers



(b) Mean-$D_{CV@R_{90\%}}$ efficient frontiers



(c) Mean-$D_{CV@R_{95\%}}$ efficient frontiers



(d) Mean-$D_{CV@R_{99\%}}$ efficient frontiers

Figure 4.2: Mean-risk-diversification efficient frontiers. These graphs report the mean-risk-diversification efficient frontiers of a market composed by the 30 assets belonging to the DJIA index between October 13, 2016, and October 13, 2017. We select 100 equally spaced points in each admissible interval and then solve the optimization problem in (4.2) 100 times for each RDM.

all the efficient frontiers. From the graphs, we also note that as risk diversification increases, the mean decreases, and, moreover, the magnitude of the decrease in the expected return varies across the RDMs. The shape of all efficient frontiers is somewhat concave, while the projection of each efficient frontier on different planes is mostly irregular. With regard to the maximum achievable level of risk diversification, Figure 4.2 suggests that this is typically higher for the mean-$D_{CV@R}$ efficient set than for the mean-D_σ efficient set. Our results also suggest that the achivable level of risk diversification seems to increases with the risk-aversion parameter α, i.e. for the considered portfolios it is the highest in the mean-$D_{CV@R_{95\%}}$ plane.

To better describe the features of mean-risk-diversification efficient frontiers, Table 4.1 provides descriptive statistics for various portfolios on the different mean-risk-diversification efficient frontiers. Since we are dealing with 400 portfolios, (one optimal portfolio for each level of $\bar{d}$ and each D_ν), we divide the efficient frontiers into 10 areas, equally spaced along the risk-diversification axis according to the range of $D_{CV@R_{99\%}}$, because it reaches the highest values.

In each of the 10 groups, we then compute the average of the portfolios' mean, standard deviation, skewness, and kurtosis, the average of the sum of the portfolios' squared weights (i.e., the inverse of the HH index), and the average number of assets with positive weights.

From Table 4.1, we see that on average, expected return, standard deviation, skewness, and kurtosis decrease as risk diversification increases. The evidence on the relationship between the proposed RDMs and standard measures of diversification is mixed, though, as can be seen from the last two columns of the table. For "low" values of risk diversification, diversification measured by the sum of squared portfolio weights or the number of invested assets typically increases with risk diversification. However, this is not true for "high" values of risk diversification. As shown in example 3, typically RDMs are not consistent with majorization ordering and, therefore, the relationship with diversification remains unclear.

Except for the first area, where risk diversification is the lowest, portfolios belonging to the mean-D_σ efficient set have on average the lowest expected return, standard deviation, skewness, and kurtosis. $D_{CV@R_{99\%}}$ has on average the highest expected return and standard deviation in all areas. More interesting is the relationship between diversification and different RDMs. Diversification decreases as the risk-aversion parameter, i.e., the confidence level α of the $D_{CV@R_\alpha}$, increases. This can be seen by the last two columns of table 4.1, where on average, the sum of squared weights and the average number of assets with a positive portion of wealth invested are the lowest for $D_{CV@R_{99\%}}$. This seems counterintuitive, but note that as shown by example 3, diversification and risk diversification are generally not consistent. As a general comment, portfolios belonging to the

Table 4.1: Average of mean, standard deviation, skewness, kurtosis, sum of squared weights, and number of invested assets, of portfolios belonging to each of the 10 groups. Portfolios are grouped together according to the range of $D_{CV@R_{99\%}}$.

Range	RDM	Mean	St. dev.	Skewness	Kurtosis	$\sum_i w_i^2$	$\sharp$ Assets
$[0, 0.078)$	D_σ	0.00268	0.0108	2.651	22.543	0.9186	2.166
	$D_{CV@R_{90\%}}$	0.00267	0.0109	2.667	22.782	0.9264	2
	$D_{CV@R_{95\%}}$	0.00268	0.0109	2.720	23.302	0.9345	2.636
	$D_{CV@R_{99\%}}$	0.00269	0.0109	2.726	23.307	0.9454	2.666
$[0.078, 0.156)$	D_σ	0.00257	0.0099	2.352	19.701	0.7382	3
	$D_{CV@R_{90\%}}$	0.00259	0.0104	2.462	20.852	0.8011	4
	$D_{CV@R_{95\%}}$	0.00258	0.0104	2.659	22.657	0.8262	3.727
	$D_{CV@R_{99\%}}$	0.00262	0.0104	2.640	22.321	0.8409	3
$[0.156, 0.234)$	D_σ	0.00246	0.0091	1.900	15.375	0.5760	3.25
	$D_{CV@R_{90\%}}$	0.00247	0.0097	2.252	19.014	0.6686	4.833
	$D_{CV@R_{95\%}}$	0.00248	0.0097	2.629	22.245	0.7165	4.272
	$D_{CV@R_{99\%}}$	0.00254	0.0099	2.530	20.953	0.7453	3
$[0.234, 0.311)$	D_σ	0.00231	0.0080	1.513	11.689	0.4233	5.5
	$D_{CV@R_{90\%}}$	0.00234	0.0088	2.180	18.198	0.5463	7.454
	$D_{CV@R_{95\%}}$	0.00238	0.0089	2.479	20.209	0.6045	4.909
	$D_{CV@R_{99\%}}$	0.00246	0.0094	2.388	19.093	0.6649	2.9
$[0.311, 0.389)$	D_σ	0.00214	0.0069	1.182	8.7405	0.3031	6.33
	$D_{CV@R_{90\%}}$	0.00222	0.0078	2.078	16.404	0.4401	7
	$D_{CV@R_{95\%}}$	0.00227	0.0082	2.217	17.109	0.4952	4.818
	$D_{CV@R_{99\%}}$	0.00237	0.0089	2.207	16.699	0.5972	2.3
$[0.389, 0.467)$	D_σ	0.00194	0.0059	0.897	6.2706	0.2125	7.8
	$D_{CV@R_{90\%}}$	0.00209	0.0070	1.769	12.213	0.3550	6.666
	$D_{CV@R_{95\%}}$	0.00214	0.0075	1.909	13.408	0.4051	5.363
	$D_{CV@R_{99\%}}$	0.00228	0.0085	2.016	14.149	0.5352	3
$[0.467, 0.545)$	D_σ	0.00165	0.0049	0.666	4.8942	0.1381	13.19
	$D_{CV@R_{90\%}}$	0.00195	0.0062	1.449	9.0999	0.2709	6.545
	$D_{CV@R_{95\%}}$	0.00201	0.0068	1.517	8.8859	0.3373	5.363
	$D_{CV@R_{99\%}}$	0.00205	0.0072	1.505	8.4170	0.3986	3.6
$[0.545, 0.623)$	D_σ	0.00124	0.0040	0.469	4.8144	0.0842	18.59
	$D_{CV@R_{90\%}}$	0.00175	0.0054	1.097	5.8251	0.1917	9.363
	$D_{CV@R_{95\%}}$	0.00182	0.0060	1.272	6.3980	0.2482	7.000
	$D_{CV@R_{99\%}}$	0.00217	0.0079	1.800	11.408	0.4768	3.363
$[0.623, 0.701)$	D_σ						
	$D_{CV@R_{90\%}}$	0.00137	0.0046	0.917	4.7324	0.1253	14
	$D_{CV@R_{95\%}}$	0.00144	0.0048	1.032	5.0948	0.1571	9.454
	$D_{CV@R_{99\%}}$	0.00191	0.0065	1.370	7.0917	0.3249	5.2
$[0.701, 0.778]$	D_σ						
	$D_{CV@R_{90\%}}$						
	$D_{CV@R_{95\%}}$	0.00095	0.0043	0.959	5.0430	0.1193	13
	$D_{CV@R_{99\%}}$	0.00148	0.0053	1.146	5.5559	0.2005	9

mean-$D_{CV@R_{99\%}}$ efficient frontier seem to have better average moments, for the same level of risk diversification, than portfolios lying on the other mean-risk diversification efficient frontiers.

4.3.2 Dynamic Analysis

In this section, we investigate the performance of RDM-optimal portfolios in a dynamic setting, where the portfolios are recalibrated on a monthly basis. We also examine the performance of the different portfolios during periods of financial distress. In particular, we focus on the period surrounding the financial crisis of 2008. We perform a rolling window analysis, using a one-year window of asset returns to calculate the portfolio that maximizes the corresponding RDM at each time step. Our sample covers the period from January 3, 2005, to October 7, 2014, and we recalibrate the portfolios every 21 trading days with transaction costs equal to 0.2% of the traded volume. At each time step, we solve problem 4.2, where $\bar{d} = \max D_\nu(P)$ for the four different risk measures, i.e. we calculate the portfolios that provide the maximum level of risk diversification for each measure and then evaluate the return of these portfolios over the next month. As reference points for the performance of the RDM-optimal portfolios, at each time step we also compute the global minimum variance portfolio (GMV), the equally weighted portfolio (EW), and the maximum Sharpe ratio portfolio (MSR) and evaluate the return of these portfolios over the next month.

Figure 4.3 shows the ex post wealth evolution of the four portfolios, solving the problem in 4.2. For comparison the wealth evolution is also reported for the strategies involving the GMV, EW, and MSR portfolios. The figure illustrates that during the financial crisis of 2008 all strategies experience a drop in the ex post wealth. Looking at the graph, we also observe that the portfolio controlled for $D_{CV@R_{99\%}}$ performs better during the crisis period, and is able to reach a wealth level comparable to the pre-crisis period much quicker than the othr strategies. Thus, at the end of the crisis period and for approximately 24 months after, the ex post wealth of the portfolio based on optimizing $D_{CV@R_{99\%}}$ is significantly higher than for the other strategies. Interestingly, when the entire sample period is considered, the portfolio based on maximizing D_σ reaches the highest wealth level, while the equally weighted portfolio also performs quite well outside the crisis period. Returns created by the GMV and maximizing $D_{CV@R_{95\%}}$ are less volatile but the created ex post wealth at the end of the sample period remains at a much lower level than for the other strategies.

We also evaluate the ex post performance of the constructed portfolios in terms of annualized expected return (μ_{ann}), annualized standard deviation (σ_{ann}), and Sharpe ratio, $CV@R_\alpha$, with

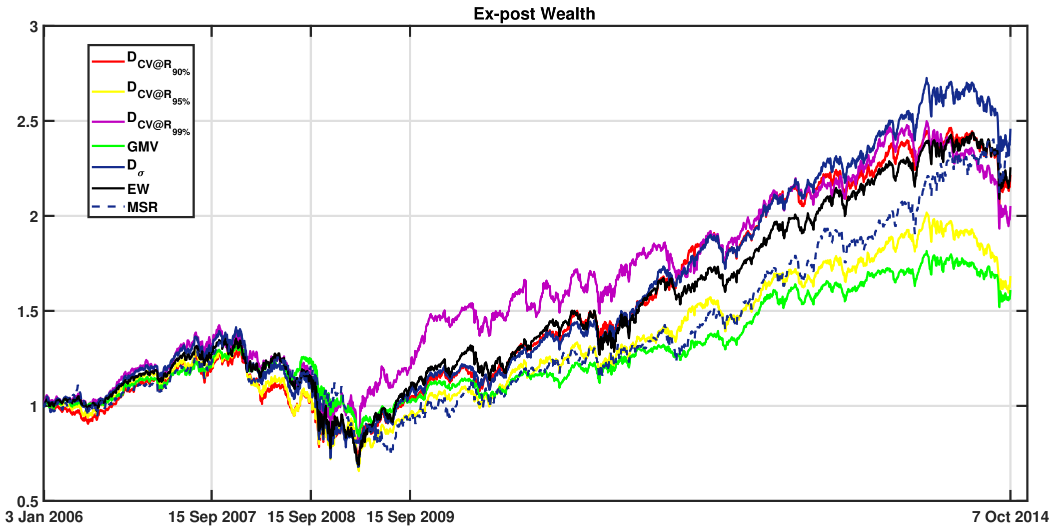


Figure 4.3: The figure shows the ex post evolution of the $DCV@R_{90\%}$, $DCV@R_{95\%}$, $DCV@R_{99\%}$ D_{σ}, global minimum variance, equally weighted, and maximum Sharpe ratio portfolios. We compute the ex post wealth, using a rolling window of one year and recalibrating the portfolios every 21 trading days, for the sample period January 3, 2005 - October 7, 2014. We also consider transaction costs equal to 0.2% of the traded volume. The picture illustrates that during the 2008-2009 crisis period, $D_{CV@R_{99\%}}$ recovers much faster than the other depicted strategies.

$\alpha = 90\%, 95\%$, and 99%[7]. To better understand the relationship between diversification and risk diversification, we also compute the number of assets with non-zero weights for each strategy and, finally, average transaction costs. Results for generated returns by each strategy during the period January 3, 2006, to October 7, 2014 are shown in Table 4.2.

We observe that MSR reaches the highest level of annualized expected return, while the GMV shows the lowest level of annualized standard deviation. Nevertheless, D_σ presents the highest ex post Sharpe ratio. In terms of tail risk, GMV has the lowest ex post $CV@R$s at 90%, 95%, and 99%, and looking at the average number of assets, EW and GMV dominate the other strategies with respect to diversification ordering. Among the risk diversification strategies, D_σ performs best with respect to expected return, but $D_{CVA@R_{90\%}}$ yields an equally high Sharpe ratio and lower ex post tail risk statistics.

We also consider five well-known performance ratios: MinMax ratio (MinMax), Ruttiens alternative risk measure, Sortino and Satchell ratio (Sortino), maximum drawdown (MDD), and Rachev ratio ($RR_{\alpha,\beta}$), see among others, Young (1998); Pedersen and Satchell (2002); Biglova et al. (2004); Ruttiens (2013).[8] Results for the conducted analysis are summarized in Table 4.3.

[7]Following the relevant literature in the field, we report annualized return and standard deviation, while performance and risk measures such as Sharpe ratio and CV@R are computed on daily returns (see, among the others Biglova et al., 2004; Kempf et al., 2015; Post and Kopa, 2016).

[8]We recall the definition of the Sortino and Satchell ratio: $SS(X) = \frac{\mathbb{E}[X]}{s(h)}$, where $s(h) = \frac{1}{T}\sum_{i=1}^{T}(h - X_i)_+$ and $(a)_+ = \max(0, a)$, and the definition of the Ruttiens ratio: $Ruttiens(W_T) = \frac{W_T - 1}{1 + kRisk}$, where $Risk = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(C_t - \mathbb{E}[C])^2}$, $c_t = W_t - \frac{t}{T}(W_T - W_0)$, where W_t is the ex post wealth at time t.

Table 4.2: Ex post performances of maximum $DCV@R_{90\%}$, $DCV@R_{95\%}$, $DCV@R_{99\%}$ D_σ, global minimum variance, equally weighted, and maximum Sharpe ratio portfolios. The considered ex post period starts on January 3, 2006 and ends on October 7, 2014. We perform a rolling window of one year (252 trading days), where we recalibrate the portfolio every 21 trading days. In total, we solve 117 optimization problems for each strategy. Among the maximum RDM strategies, D_σ has the highest Sharpe ratio.

Strategies	μ_{ann}	σ_{ann}	Sharpe ratio	$CV@R_\alpha$			$\sharp$ Assets	Transaction cost
				$CV@R_{90\%}$	$CV@R_{95\%}$	$CV@R_{99\%}$		
$D_{CV@R_{90\%}}$	0.11612	0.18083	0.03827	0.02002	0.02642	0.04490	11.26495	0.11291
$D_{CV@R_{95\%}}$	0.08841	0.18973	0.02813	0.02100	0.02812	0.04962	9.324786	0.11961
$D_{CV@R_{99\%}}$	0.11616	0.19402	0.03568	0.02155	0.02840	0.04819	6.641025	0.13457
D_σ	0.12472	0.19014	0.03894	0.02092	0.02814	0.04976	14.38461	0.05417
GMV	0.06687	0.14283	0.02855	0.01582	0.02106	0.03641	28	0.04523
EW	0.11059	0.19655	0.03362	0.02242	0.02971	0.05116	28	0
MSR	0.13742	0.20893	0.03883	0.02353	0.03036	0.04730	5.26495	0.14776

Table 4.3: Performances of maximum $DCV@R_{90\%}$, $DCV@R_{95\%}$, $DCV@R_{99\%}$ D_σ, global minimum variance, equally weighted, and maximum Sharpe ratio portfolios. The ex post period starts January 3, 2006 and ends on October 7, 2014. We compute the Ruttiens ratio with $k = 1$ and Sortino and Satchell ratio with $h = 0$. D_σ performs better in terms of the considered performance ratios than the other strategies.

Strategies	MinMax	Ruttiens	Sortino	MDD	$RR_{\alpha,\beta}$		
					$RR_{99\%,99\%}$	$RR_{95\%,95\%}$	$RR_{90\%,90\%}$
$D_{CV@R_{90\%}}$	0.00521	0.98209	0.12587	3.13489	1.57492	2.74542	1.12464
$D_{CV@R_{95\%}}$	0.00397	0.57409	0.09128	3.03121	1.55556	2.36867	1.08533
$D_{CV@R_{99\%}}$	0.00476	0.87951	0.11446	3.33045	1.04423	1.61416	1.12107
D_σ	0.00538	1.15112	0.13237	3.34127	1.54517	2.38751	1.10927
GMV	0.00339	0.53649	0.09079	2.91665	1.47771	2.59542	1.02465
EW	0.00506	1.04066	0.11157	2.77021	1.38271	1.99773	1.05237
MSR	0.00633	1.13696	0.12006	2.70091	1.34219	1.95733	1.15193

According to Table 4.3, the risk-adjusted performance ratios seem to advocate in favor of D_σ. Such a strategy responds better to the worst-case scenario, has a favorable ex post wealth evolution, and is preferable for investors with semi-deviation– or expected-return–type preferences. It is worth noting that also EW performs quite well in terms of the Ruttiens ratio and maximum drawdown.

The number of assets with non-zero weights confirms the preliminary results from the static analysis: diversification as defined by classical measures decreases as α increases. In other words, as the risk-aversion parameter increases, diversification decreases. This result might serve as an explanation of the Statman diversification puzzle (Statman, 2004; Goetzmann and Kumar, 2008). Investors controlling for risk diversification tend to hold a much more concentrated portfolio than mean-variance optimizers. Moreover, holding a more concentrated portfolio than GMV, during periods of financial distress, could lead to better ex post performance as long as the risk is efficiently diversified. This result is in agreement with a series of papers on the herd behavior index, suggesting that during periods of financial distress, the high degree of co-movement between asset returns nullifies the benefits of holding a well-diversified position (Dhaene et al., 2012; Linders et al., 2015). Nevertheless, risk diversification comes with higher transaction costs. In particular, portfolios based on $D_{CV@R_\alpha}$ have much higher transaction costs than those based on GMV or D_σ. Moreover, transaction costs increase with α, implying that higher risk aversion demands, on the one hand, a more concentrated portfolio, and on the other, more active investment strategies.

Since we want to examine more deeply the performance of RDM-optimal portfolios during periods of financial distress, we compute the ex post statistics and performance measures of Tables 4.2 and 4.3 for two different subsamples of the ex post period.

Table 4.4: Performances of maximum $DCV@R_{90\%}$, $DCV@R_{95\%}$, $DCV@R_{99\%}$ D_σ, global minimum variance, equally weighted, and maximum Sharpe ratio portfolios. The crisis period starts on September 15, 2008, and finishes on September 15, 2009, while the studied post crisis period ends on October 7, 2014. According to the results summarized in the table, $D_{CV@R_{99\%}}$ performs better during the period of financial distress than the other strategies. It also exhibits higher concentration and transaction costs.

Strategies	μ_{ann}	σ_{ann}	Sharpe ratio	$CV@R_\alpha$			$\sharp$ Assets	Transaction cost
				$CV@R_{90\%}$	$CV@R_{95\%}$	$CV@R_{99\%}$		
Crisis period								
$D_{CV@R_{90\%}}$	0.10068	0.39168	0.0154	0.0160	0.0202	0.0270	9.416	0.0009
$D_{CV@R_{95\%}}$	0.02101	0.41737	0.0031	0.0155	0.0199	0.0259	10	0.0010
$D_{CV@R_{99\%}}$	0.15317	0.40740	0.0220	0.0199	0.0239	0.0277	7.416	0.0013
D_σ	0.07286	0.43777	0.0101	0.0188	0.0238	0.0343	15.08	0.0005
GMV	-0.13380	0.30257	-0.0299	0.0124	0.0154	0.0198	28	0.0003
EW	0.07024	0.43057	0.0099	0.0215	0.0250	0.0346	28	0
MSR	-0.13733	0.40622	-0.0229	0.0352	0.0441	0.0634	5.08	0.0003
Post crisis								
$D_{CV@R_{90\%}}$	0.15598	0.13124	0.0695	0.0151	0.0192	0.0293	10.930	0.0009
$D_{CV@R_{95\%}}$	0.12196	0.13669	0.0530	0.0156	0.0198	0.0313	8.662	0.0010
$D_{CV@R_{99\%}}$	0.12235	0.14816	0.0491	0.0168	0.0212	0.0325	5.825	0.0011
D_σ	0.16513	0.13234	0.0727	0.0152	0.0194	0.0295	13.80	0.0004
GMV	0.08984	0.11142	0.0486	0.0128	0.0162	0.0251	28	0.0004
EW	0.13829	0.14628	0.0558	0.0170	0.0219	0.0344	28	0
MSR	0.20563	0.17260	0.0683	0.0196	0.0248	0.0363	4.848	0.0004

The upper panel of Table 4.4 illustrates the ex post performance of the constructed portfolios from September 15, 2008, to September 15, 2009. Interestingly, all strategies based on the RDMs show a positive annualized return during the twelve months after the collapse of Lehman brothers, while both the GMV and MSR portfolio generate a negative annualized return. Among the considered strategies, $D_{CV@R_{99\%}}$ yields the highest annualized men return and also the highest Sharpe ratio. Moreover, $D_{CV@R_{90\%}}$, $D_{CV@R_{95\%}}$, and $D_{CV@R_{99\%}}$ generate returns with a lower annualized standard deviation than D_σ, while $D_{CV@R_{90\%}}$ and $D_{CV@R_{99\%}}$ yield a higher annualized return. Overall, our results indicate a superior performance of RDM-optimal portfolios in comparison to the GMV and MSR portfolio. While the equally weighted portfolio performs better than GMV and MSR, it is still outperformed based on several of the considered criteria.

The lower panel of Table 4.4 shows the ex post performance of the constructed portfolios after the global financial crisis until the end of the sample period on October 7, 2014. All strategies yield positive annualized returns, and D_σ outperforms the other strategies in terms of Sharpe ratio and risk, at the same time offering lower transaction costs than the strategies based on optimizing $CV@R_\alpha$. Among the $D_{CV@R}$ strategies, $D_{CV@R_{90\%}}$ performs best, yielding the second-highest annualized return and second-highest Sharpe ratio. We find that for the period after the crisis, MSR yields the highest annualized return, but the strategy also creates returns with higher standard deviation and tail risk. Again, also the equally weighted portfolio performs quite well, yielding an annualized return, standard deviation, Sharpe ratio, $CVA@R_{95\%}$, and $CVA@R_{99\%}$ relatively similar to those of the D_σ-optimal portfolio, while offering transaction costs of zero. This is in agreement with the literature, since the equally weighted portfolio has been proven to perform quite well in many empirical settings and is typically hard to beat based on strategies involving a mean-variance approach, see, e.g., DeMiguel et al. (2009).

Table 4.5 reports the ex post performance ratios in the two subsamples.

Table 4.5: Performances of maximum $DCV@R_{90\%}$, $DCV@R_{95\%}$, $DCV@R_{99\%}$ D_{σ}, global minimum variance, equally weighted, and maximum Sharpe ratio portfolios. The crisis period starts on September 15, 2008, and finishes on September 15, 2009, while the studied post crisis period ends on October 7, 2014.

Strategies	MinMax	Ruttiens	Sortino	MDD		$RR_{\alpha,\beta}$	
					$RR_{99\%,99\%}$	$RR_{95\%,95\%}$	$RR_{90\%,90\%}$
Crisis period							
$D_{CV@R_{90\%}}$	0.00455	0.03809	0.04726	3.81245	1.57492	2.74867	1.08060
$D_{CV@R_{95\%}}$	0.00097	-0.04006	0.00937	3.74602	1.55556	2.38891	1.04024
$D_{CV@R_{99\%}}$	0.00618	0.18905	0.06509	3.95219	1.04423	1.67616	1.14163
D_{σ}	0.00322	0.05767	0.02987	3.66112	1.54517	2.40173	1.08215
GMV	-0.0075	0.01808	-0.0866	5.35443	1.47771	2.62005	1.00741
EW	0.00327	0.08727	0.02864	4.22226	1.38271	2.02429	1.04771
MSR	-0.00726	-0.07308	-0.06270	3.13291	1.34219	1.23457	1.11151
Post crisis							
$D_{CV@R_{90\%}}$	0.01292	1.06925	0.21182	2.75325	0.97087	0.98227	0.99651
$D_{CV@R_{95\%}}$	0.00881	0.61491	0.15816	2.83087	0.95659	0.97589	0.99193
$D_{CV@R_{99\%}}$	0.00790	0.93074	0.14608	4.87818	1.00850	1.01566	1.01239
D_{σ}	0.01235	1.27103	0.22459	2.54787	1.00110	0.97401	0.99433
GMV	0.00953	0.55908	0.14136	2.68860	0.91897	0.93515	0.97302
EW	0.00909	1.13558	0.17255	2.47181	0.94448	0.95225	0.96640
MSR	0.01612	1.29060	0.20735	2.20028	1.05774	1.02704	1.02161

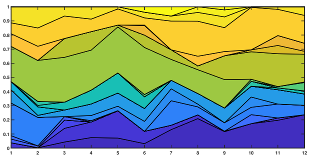
During the crisis period, the examined ex post performance ratios confirm the results of Table 4.4. In particular, $D_{CV@R_{99\%}}$ reaches the highest values for the MinMax ratio, Ruttiens ratio, Sortino and Satchell ratio, and Rachev ratio with parameters 90%,90%, while $D_{CV@R_{90\%}}$ exhibits the highest values for the Rachev ratio with parameters 99%,99% and 95%,95%. Among the competing strategies, MSR yields the lowest maximum drawdown, implying that even during the financial crisis, MSR optimal portfolios would have still been preferable for some non-satiable and risk-averse investors, see e.g. (Hodder et al., 2014).

Before moving to the second empirical exercise, we take a closer look at risk-diversification optimal portolfios and examine their composition during the height of the crisis, i.e. the year after the Lehman Brothers bankruptcy. Figure 4.4 shows the calculated portfolio weights in the rolling window rebalancing exercise for the four RDM strategies [panels (a), (b), (c) and (d)], GMV [panel (e)] and MSR [panel (f)]. We find that there is a significant difference in the portfolio weight evolutions between the different $D_{CV@R}$-optimal strategies and the D_σ strategy. In particular, we observe that, while the behavior of D_σ is quite similar to the GMV, as the confidence level increases, the weight allocation of $D_{CV@R_\alpha}$ becomes more variable. Our observations confirm the higher transaction cost in Table 4.2 for the strategies based on $D_{CV@R}$. The differences in portfolio weight evolutions are due to the different definitions of risk embedded in the RDMs, while the standard deviation measures the “spread” of a random variable, $CV@R_\alpha$ measures the tail risk.

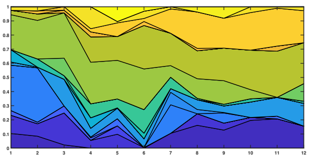
Risk diversification aims to reduce the idiosyncratic risk of a position. For this reason, in the following we use the risk diversification measures to construct portfolios that povide the same level of risk diversification as a chosen tangent portfolio. Since we are considering the standard deviation and $CV@R_\alpha$ with $\alpha = 90\%,\ 95\%,\ 99\%$ as risk measures, we first compute the relative tangent portfolios (i.e., Sharpe ratio and mean-$CV@R$ ratio), solving the following optimization problem for each risk measure:[9]

$$\begin{aligned} \max_w \ & \frac{\mathbb{E}[w'X]}{\nu(w'X)} \\ & \sum_{i=1}^n w_i = 1 \\ & 0 \le w_i \le 1,\ i = 1, \dots, n \end{aligned}$$

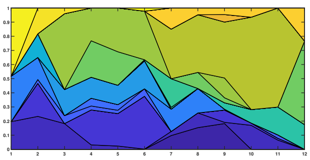
[9]It is well known that the portfolio solution of a maximization of the ratio of expected return and a risk measure is the market portfolio with respect to that risk measure, see, for example Stoyanov et al. (2007).



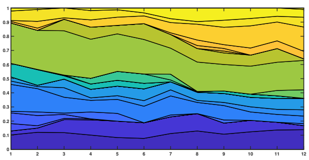
(a) Portfolio weights evolution of $D_{CV@R_{90\%}}$



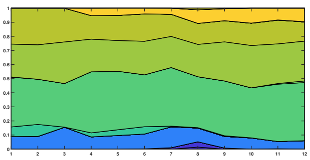
(b) Portfolio weights evolution of $D_{CV@R_{95\%}}$



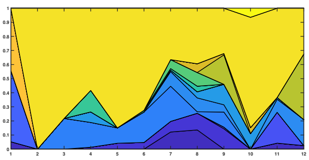
(c) Portfolio weights evolution of $D_{CV@R_{99\%}}$



(d) Portfolio weights evolution of D_{σ}



(e) Portfolio weights evolution of GMV



(f) Portfolio weights evolution of MSR

Figure 4.4: These pictures show the weights evolutions of the maximum $DCV@R_{90\%}$, $DCV@R_{95\%}$, $DCV@R_{99\%}$ D_{σ}, global minimum variance, and maximum Sharpe ratio portfolios during the period September 15, 2008, to September 15, 2009. The difference in these graphs is explained by the different definitions of risk embedded in each RDM.

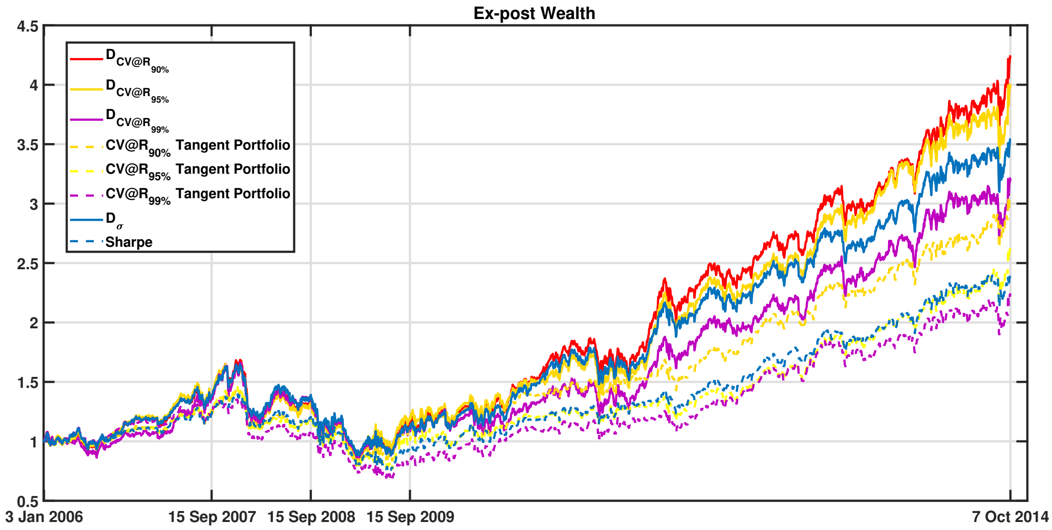


Figure 4.5: The figure shows the ex post evolution of dominating strategies and their corresponding tangent portfolios. We compute the ex post wealth, using a rolling window of one year and recalibrating the portfolios every 21 trading days, starting at January 3, 2005, and ending on October 7, 2014. We also consider transaction costs equal to 0.2% of the traded volume. All the dominating strategies show a higher ex post wealth than their corresponding tangent portfolios.

We denote the solution to this optimization problem as vector w_m. Then, we solve Problem (4.2) with $\bar{d} = D_\nu(w_m' X)$. In other words, we consider the portfolio that maximizes the mean with the same level of risk diversification as the market portfolio for each risk measure (henceforth referred to as dominating strategies).

Figure 4.5 shows the ex post wealth evolution of the dominating strategies and the corresponding tangent portfolios. In comparison to the evolution of wealth in Figure 4.3, we find that the ex post wealth appears to be less volatile during the crisis, while typically a higher level of wealth is reached at the end of the sample period. We also find that all dominating strategies clearly outperform the respective tangent portfolios in terms of final wealth. Surprisingly, the higher level of wealth creation for the dominating strategies in comparison to the tangent portfolios is quite substantial, reaching levels of up to 150%.

Table 4.6: Performances of dominating strategies and their corresponding tangent portfolios. The ex post period starts January 3, 2006 and ends on October 7, 2014. All the dominating strategies outperform their corresponding tangent portfolios.

Strategies	μ_{ann}	σ_{ann}	Sharpe ratio		$CV@R_{\alpha}$		$\sharp$ Assets	Transaction cost
				$CV@R_{90\%}$	$CV@R_{95\%}$	$CV@R_{99\%}$		
$D_{CV@R_{90\%}}$	0.21336	0.23796	0.05121	0.026954	0.035065	0.05626	5.01709	0.15512
$D_{CV@R_{95\%}}$	0.20812	0.24416	0.04879	0.027477	0.035472	0.05633	4.85470	0.15755
$D_{CV@R_{99\%}}$	0.18234	0.25032	0.04216	0.028166	0.036805	0.05863	3.89743	0.16772
$\frac{\mathbb{E}[P]}{CV@R_{90\%}}$	0.16934	0.22093	0.04462	0.024755	0.032544	0.05056	4.62393	0.15830
$\frac{\mathbb{E}[P]}{CV@R_{95\%}}$	0.15101	0.21939	0.04039	0.024644	0.031939	0.05113	4.35042	0.15837
$\frac{\mathbb{E}[P]}{CV@R_{99\%}}$	0.13417	0.22164	0.03579	0.024898	0.032414	0.05101	4.11111	0.17545
D_{σ}	0.18727	0.23361	0.04630	0.026543	0.034386	0.05515	5.29059	0.12869
MSR	0.13742	0.20893	0.03883	0.023532	0.030364	0.04730	5.26495	0.14776

Table 4.7: Performances of dominating strategies and their corresponding tangent portfolios. The ex post period starts January 3, 2006 and ends on October 7, 2014.

Strategies	MinMax	Ruttiens	Sortino	MDD		$RR_{\alpha,\beta}$	
					$RR_{99\%,99\%}$	$RR_{95\%,95\%}$	$RR_{90\%,90\%}$
$D_{CV@R_{90\%}}$	0.00953	2.30802	0.16067	2.20938	1.34299	1.67958	1.08774
$D_{CV@R_{95\%}}$	0.00939	2.14425	0.15304	2.28397	1.35623	1.68404	1.15221
$D_{CV@R_{99\%}}$	0.00824	1.67313	0.12891	2.36823	1.34403	1.64450	1.07754
$\frac{\mathbb{E}[P]}{CV@R_{90\%}}$	0.00770	1.63379	0.14140	2.69385	1.33523	1.84264	1.17386
$\frac{\mathbb{E}[P]}{CV@R_{95\%}}$	0.00659	1.26963	0.12628	2.70570	1.20621	1.79003	1.14349
$\frac{\mathbb{E}[P]}{CV@R_{99\%}}$	0.00619	0.97967	0.11101	3.45921	1.34403	1.85494	1.15857
D_{σ}	0.00847	1.91493	0.14336	2.24080	1.34583	1.71941	1.06354
MSR	0.00633	1.13696	0.12006	2.70091	1.34219	1.95733	1.15193

Table 4.6 shows ex post statistics and performance measures for the RDM-based strategies and the corresponding tangent portfolios. Overall, all RDM-based strategies have higher annualized returns, higher annualized standard deviations, higher Sharpe ratios, and higher $CV@R$ at 90%, 95%, and 99%, with respect to their corresponding tangent portfolio. At the same time, they are less concentrated and requiring less active investment, i.e. having lower transaction costs. Table 4.7 summarizes ex post results for the constructed portfolios with respect to the considered performance ratios. We find that also for these measures, the dominating strategies beat their tangent portfolio counterparts with regard to the MinMax ratio, Ruttiens risk measure, Sortino and Satchell ratio, maximum drawdown, and Rachev ratio with parameters 99%,99%. The only measures advocating in favor of tangent portfolios are the Rachev ratios with parameters 95%,95% and 90%,90%.

Similar to Table 4.5, in the upper panel of Table 4.8 we report some statistics for the performance during the year after the outbreak of the financial crisis, i.e. for the period from September 15, 2008 to September 15, 2009.

Table 4.8: Performances of dominating strategies and their corresponding tangent portfolios. The crisis period begins September 15, 2008, and finishes September 15, 2009, while the studied post crisis period ends on October 7, 2014.

Strategies	μ_{ann}	σ_{ann}	Sharpe ratio		$CV@R_\alpha$		$\sharp$ Assets	Transaction cost
				$CV@R_{90\%}$	$CV@R_{95\%}$	$CV@R_{99\%}$		
Crisis period								
$D_{CV@R_{90\%}}$	-0.03367	0.44882	-0.0048	0.0436	0.0557	0.0763	5.83	0.0014
$D_{CV@R_{95\%}}$	0.020357	0.45866	0.0028	0.0431	0.0542	0.0745	5.92	0.0014
$D_{CV@R_{99\%}}$	-0.09796	0.44395	-0.0146	0.0424	0.0544	0.0754	5	0.0011
$\frac{\mathbb{E}[P]}{CV@R_{90\%}}$	0.037641	0.44374	0.0053	0.0427	0.0506	0.0673	4.25	0.0015
$\frac{\mathbb{E}[P]}{CV@R_{95\%}}$	-0.06154	0.41422	-0.0097	0.0402	0.0505	0.0738	4.25	0.0017
$\frac{\mathbb{E}[P]}{CV@R_{99\%}}$	-0.14031	0.41796	-0.0228	0.0386	0.0484	0.0668	4.25	0.0016
D_σ	-0.09023	0.44119	-0.0135	0.0421	0.0537	0.0734	15.08	0.0009
MSR	-0.13733	0.40622	-0.0229	0.0352	0.0441	0.0634	5.08	0.0003
Post crisis								
$D_{CV@R_{90\%}}$	0.28697	0.19988	0.0796	0.0224	0.0285	0.0426	4.453	0.0014
$D_{CV@R_{95\%}}$	0.26687	0.20534	0.0726	0.0227	0.0289	0.0446	4.244	0.0014
$D_{CV@R_{99\%}}$	0.23963	0.20991	0.0645	0.0232	0.0294	0.0459	3.569	0.0014
$\frac{\mathbb{E}[P]}{CV@R_{90\%}}$	0.22292	0.18279	0.0694	0.0205	0.0262	0.0397	4.197	0.0013
$\frac{\mathbb{E}[P]}{CV@R_{95\%}}$	0.21013	0.19149	0.0628	0.0215	0.0275	0.0437	3.767	0.0013
$\frac{\mathbb{E}[P]}{CV@R_{99\%}}$	0.21878	0.19158	0.0651	0.0213	0.0275	0.0427	3.605	0.0015
D_σ	0.25001	0.19448	0.0723	0.0221	0.0279	0.0425	13.80	0.0012
MSR	0.20563	0.17260	0.0683	0.0196	0.0248	0.0363	4.848	0.0004

We find that only two of the strategies, namely $D_{CV@R_{95\%}}$ and the $CV@R_{90\%}$ tangent portfolio yield a positive annualized return durig the crisis period. In fact, the $CV@R_{90\%}$ market portfolio is the best-performing strategy in this case, in terms of risk and annualized return. Nevertheless, all the other dominating strategies outperform their equivalent tangent portfolio in terms of expected return and Sharpe ratio.

All of the strategies seem to be quite concentrated during 2008 except that based on D_σ, which shows higher diversification, including on average approximately 15 assets in the constructed portfolios. The strategies based on $D_{CV@R}$ are more concentrated in comparison to Table 4.2, while D_σ shows the same level of concentration. We also find that transaction costs have increased for all the strategies, but still the one based on D_σ is the least expensive among the dominating strategies.

The lower panel of Table 4.8 illustrates the performances of th constructed portfolios after the crisis, i.e. the period from September 15, 2009 until October 7, 2014. We find that the dominating strategies create higher annualized returns and Sharpe ratios, are more concentrated, and have higher average transaction costs in comparison to the portfolios illustrated in Table 4.2. This result suggests that risk diversification measures are a flexible tool and can be implemented in various settings. On the one hand, an agent facing a period of financial distress can adopt a strategy maximizing the level of risk diversification, using problem 4.2. On the other hand, portfolios maximizing the expected return, while having the the same level of risk diversification as a tangent portfolio, are suitable for a fund manger with a return driven target performance.

Table 4.9 summarizes the ex post performance of the constructed portfolios with regard to the considered performance ratios during (upper panel) and after (lower panel) the crisis period.

Table 4.9: Performances of dominating strategies and their corresponding tangent portfolios. The crisis period begins September 15, 2008, and finishes September 15, 2009, while the studied post crisis period ends on October 7, 2014.

Strategies	MinMax	Ruttiens	Sortino	MDD	$RR_{\alpha,\beta}$		
					$RR_{99\%,99\%}$	$RR_{95\%,95\%}$	$RR_{90\%,90\%}$
Crisis period							
$D_{CV@R_{90\%}}$	-0.00170	0.12218	-0.01306	3.58201	1.35844	1.12948	1.07471
$D_{CV@R_{95\%}}$	0.00100	0.16375	0.00767	3.60661	1.35623	1.18082	1.12604
$D_{CV@R_{99\%}}$	-0.00507	0.07694	-0.03949	3.92511	1.34403	1.13635	1.05773
$\frac{\mathbb{E}[P]}{CV@R_{90\%}}$	0.00182	0.08397	0.01439	3.08202	1.33523	1.21482	1.13515
$\frac{\mathbb{E}[P]}{CV@R_{95\%}}$	-0.00312	0.00482	-0.02602	3.15711	1.28207	1.21754	1.14682
$\frac{\mathbb{E}[P]}{CV@R_{99\%}}$	-0.00743	-0.16440	-0.06141	3.01855	1.34403	1.20593	1.11582
D_{σ}	-0.00466	0.10014	-0.03652	3.75527	1.34582	1.14495	1.06317
MSR	-0.00726	-0.07308	-0.06270	3.13291	1.34219	1.23457	1.11151
Post crisis							
$D_{CV@R_{90\%}}$	0.01244	2.82752	0.24990	2.71696	1.10226	1.04314	1.04203
$D_{CV@R_{95\%}}$	0.01264	2.56764	0.22657	2.47942	1.11031	1.06357	1.04903
$D_{CV@R_{99\%}}$	0.01075	1.93454	0.19635	2.73244	1.06584	1.03335	1.04274
$\frac{\mathbb{E}[P]}{CV@R_{90\%}}$	0.01428	1.83381	0.21575	2.29729	1.12489	1.04493	1.03586
$\frac{\mathbb{E}[P]}{CV@R_{95\%}}$	0.00894	1.41868	0.19663	2.80628	1.06912	1.03270	1.01198
$\frac{\mathbb{E}[P]}{CV@R_{99\%}}$	0.01256	1.12451	0.20240	2.38945	1.10708	1.03581	1.03584
D_{σ}	0.01219	2.29320	0.22300	2.54982	1.01436	1.01937	1.02290
MSR	0.01612	1.29060	0.20735	2.20028	1.05774	1.02704	1.02161

Overall, the results reported in Table 4.9 confirm our previous findings. During the crisis period, all dominating strategies perform better than their corresponding tangent portfolio counterparts with respect to MinMax ratio, Ruttiens ratio, Sortino and Satchell ratio, and Rachev ratio with parameters 99%,99% (except the $D_{CV@R_{90\%}}$, because the $CV@R_{90\%}$ tangent portfolio seems to perform quite well during the crisis period). Nonetheless, the maximum drawdown values are lower for the tangent portfolios than for the corresponding RDM strategies, confirming the previous result on the MSR. Thus, we conclude that these tangent portfolios are preferable for some non-satiable and risk-averse investors.

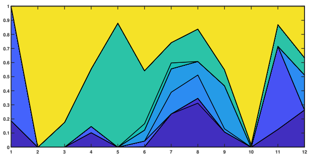
After the crisis period, all strategies yield positive annualized return, and overall the RDM strategies outperform the relative tangent portfolios, even if some non-satiable and risk-averse investors might still prefer the tangent portfolios.

Finally, similarly to Figure 4.4, in Figure 4.6 we report the portfolio weights evolution of the tangent portfolios and the dominating strategies during the year after Lehmann Brothers collapsed. We find that typically the portfolio allocations are quite similar in variability and concentration for all strategies, confirming the results of Table 4.6 on the average number of assets in a portfolio and the transaction costs.

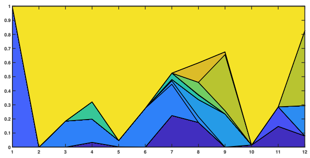
4.4 Conclusion

Diversification has been of utmost importance in portfolio theory since the seminal work of Markowitz (1952a). In this chapter, we propose a new class of diversification measures called risk diversification measures. The class extends some diversification measures already established in the literature such as the diversification ratio or the diversification delta (Choueifaty and Coignard, 2008; Vermorken et al., 2012; Clarke et al., 2013; Flores et al., 2017) that have gained some prominence among investors. The new measures depend on a given risk measure, and are defined as the ratio between a portfolio's risk and the average risk of its components. Under appropriate assumptions, the constructed RDMs can then be interpreted as the percentage of idiosyncratic risk diversified in the portfolio.

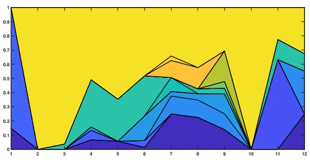
In our empirical analysis, we apply four different RDMs: next to the diversification ratio D_σ we also consider $D_{CV@R_\alpha}$ at various levels of α, allowing us to take into account different levels of risk aversion among investors. For each of the RDMs, we propose three empirical applications on a market composed by stocks belonging to the DJIA. In the first case, we introduce the mean-risk-diversification efficient frontier. Similarly to the mean-variance efficient frontier, this is an empirical



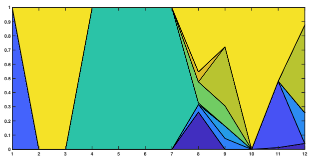
(a) Portfolio weights evolution of $D_{CV@R_{90\%}}$ dominating strategy



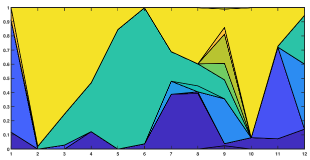
(b) Portfolio weights evolution of $CV@R_{90}\%$ tangent portfolio



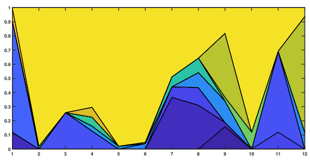
(c) Portfolio weights evolution of $D_{CV@R_{95\%}}$ dominating strategy



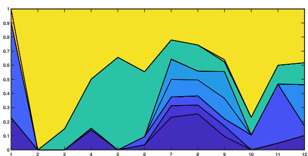
(d) Portfolio weights evolution of $CV@R_{95\%}$ tangent portfolio



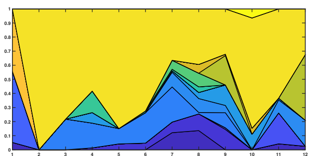
(e) Portfolio weights evolution of $D_{CV@R_{99\%}}$ dominating strategy



(f) Portfolio weights evolution of $CV@R_{99\%}$ tangent portfolio



(g) Portfolio weights evolution of D_σ dominating strategy



(h) Portfolio weights evolution of MSR

Figure 4.6: These charts show the portfolio weights evolution of the dominating strategies and their corresponding tangent portfolios during the period September 15, 2008 to September 15, 2009. In this case, all the portfolios show a comparable level of concentration.

tool that analyzes portfolios' efficiency in terms of expected return and risk diversification. Each of the efficient frontiers exhibits concavity with regard to risk diversification and the set of risk diversification efficient portfolios illustrates that as risk diversification increases, expected return decreases. Moreover, our results suggest that the level of risk diversification increases with risk aversion, while concentration decreases.

We also examine the performance of portfolios constructed by optimizing D_σ and $D_{CV@R_\alpha}$ with α = 90%, α = 95%, and 99% during periods of financial distress. We focus our analysis on the year after the collapse of Lehman Brothers and observe that all strategies based on the developed RDMs generate positive expected returns, illustrating a better performance in terms of risk–reward ratio than the global minimum variance or the tangency portfolio. The price to pay for this superior performance is typically the requirement of a more active investment strategies, i.e. more rebalancing, thus involving higher transaction cost.

Finally, we show how the developed risk diversification measures and coherent risk diversification measures can be used to outperform given tangent portfolios.

Our empirical results strongly support the application of risk diversification measurs for portfolio construction. The new measures could be employed by investors to craft alternative investment strategies, based on a new concept of diversification that takes into account the relationship between the risk of a portfolio and the weighted risk of a portfolio's individual components. For researchers, this study provides the first formalization and generalization of risk diversification, and encourages additional work to better understand the properties and empirical performance of portfolios that are constructed using risk diversification measures.

Chapter 5

Conclusion

Portfolio selection deals with decision-making under uncertainty, and aims to find the best wealth allocation among risky assets. To define what "best allocation" means, it is fundamental to have a definition of efficiency. Markowitz's mean-variance efficient frontier represents a milestone in modern portfolio theory, and established the first definition of efficiency (Markowitz, 1952a).

Even though, thanks to mean-variance efficiency, many important results in portfolio selection theory have been established, the concept appears to be insufficient to describe investors' behavior (Lintner, 1964; Sharpe, 1964; Mossin, 1966; Bawa, 1975; Levy, 1992). Here is where the concepts of expected utility and stochastic dominance are of help. Expected utility allows to classify investors according to their attitude toward risk. Then, with stochastic dominance, and generally with stochastic orderings, it is possible to define efficiency for an entire category of investors. In particular, an efficient allocation with respect to a stochastic ordering is also optimal for all investors whose preferences are coherent with the stochastic ordering.

This thesis discusses several applications of stochastic ordering to portfolio selection problems. In particular, it studies efficiency from the point of view of investors with different attitudes to risk.

Chapter 1 compares mean-variance efficiency with second-order stochastic dominance efficiency. In particular, the chapter shows that the second-order stochastic dominance efficient set is composed of a huge number of portfolios and is not necessarily convex. Portfolios belonging to the set are on average more concentrated than those belonging to the mean-variance efficient frontier. Moreover, the GMV portfolio and other portfolios belonging to the mean-variance efficient frontier with a low expected return are second-order stochastically dominated. These results question the validity of mean-variance efficiency for non-satiable risk-averse investors. The last part of the chapter exploits the non-efficiency of GMV to construct dominating strategies able to outperform the GMV portfolio in terms of wealth and other performance measures. These results confirm those of previous studies on the relationship between mean variance and expected utility (Bawa, 1975; Ingersoll, 1987; Levy, 1992; Dentcheva and Ruszczyński, 2006; Kopa and Chovanec, 2008; Lizyayev and Ruszczyński, 2012).

Based on recent findings in the literature, Chapter 2 considers efficiency for investors that are non-satiable and neither risk-averse nor risk-seeking. This particular category of investors is usually the subject of behavioral finance studies (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992; Barberis and Thaler, 2003). Chapter 2 assumes that asset-return distributions belong to a family uniquely determined by a finite number of parameters. This family of distributions can be seen as an extension of the elliptical family, which is a class widely used in portfolio selection and generally in financial economics. To make a link between distribution and investors' preferences, the location parameter is understood as a positive-homogeneous and translation-invariant reward measure, the scale parameter as a positive-homogeneous and translation-invariant risk measure, while the other parameters can be chosen to be other distributional parameters, such as moments (Ortobelli, 2001).

Under these assumptions, the first part of the chapter extends classic stochastic dominance conditions. In particular, these conditions do not necessitate a convex risk measure. Moreover, the behavior of investors that are non-satiable and neither risk-averse nor risk-seeking changes according to market conditions. When the expected value is higher than the reward measure, investors behave as non-satiable and risk-averse, while when the expected value is lower than the reward measure, they behave as non-satiable and risk-seeking. Thanks to these results, it is possible to define several orderings consistent with the preferences of investors that are non-satiable and neither risk-averse nor risk-seeking, of which the introduced λ-Rachev ordering is an example, see, among others, Biglova et al. (2004).

The last part of the chapter provides a methodology to test whether a given portfolio is efficient with respect to classic stochastic dominance orderings and the λ-Rachev ordering, using estimation function theory (see Godambe and Thompson, 1989; Crowder, 1986). Finally, this methodology is applied to test the efficiency of the Fama and French market portfolio as well as the NYSE and Nasdaq market portfolios. Our results are in line with those from other studies, confirming the efficiency of the Fama and French market portfolio, while the NYSE and Nasdaq market portfolio are not efficient (see, among others, Scaillet and Topaloglou, 2010; Kopa and Post, 2015; Arvanitis and Topaloglou, 2017).

The newly introduced stochastic ordering describes the behavior of non-satiable risk-averse investors differently from prospect theory or Markowitz's utility of wealth theory (Markowitz, 1952b; Kahneman and Tversky, 1979; Tversky and Kahneman, 1992). Both in prospect theory and under Markowitz utility, the behavior of investors is altered with changes in wealth. The results on the λ-Rachev ordering suggest that the change in behavior due to relative wealth is in reality endogenous to the market. Investors' behavior change according to the relationship between reward measure and expected return.

Chapter 3 applies theoretical results from Chapter 2 in defining a new ordering called λ-Gini ordering. The functional defining the ordering can be interpreted as a weighted difference between given percentages of worst and best outcomes. Results of this ordering suggest that a market portfolio is almost never efficient for investors that are non-satiable and neither risk-averse nor risk-seeking, even if in some situations, depending on the configuration of the functional, it can be hard to find a dominating portfolio.

Chapters 1 and 2 apply definitions of efficiency that are valid for different categories of investors. Assuming different investor behavior nevertheless leads to the same conclusion on diversification. Given the results in Chapter 1, optimal portfolios for non-satiable risk-averse investors have a lower level of diversification than mean-variance efficient ones. Chapter 2 attests that the convexity of a risk measure is not a necessary condition for optimality, even in the case of investors that are non-satiable and neither risk-averse nor risk-seeking.

In ordering theory, diversification is defined in terms of majorization ordering. In particular, diversification considers the number of assets in which a positive proportion of wealth is invested, rather than how it is invested, see, e.g. Marshall et al. (1943); Wong (2007); Egozcue and Wong (2010). These observations motivate Chapter 4.

As an alternative to classic diversifications, Chapter 4 introduces the definition of risk diversification measures which quantify the idiosyncratic risk diversified among portfolio components. When the risk measure satisfies the axioms of coherency, the resulting risk diversification measure can be seen as the percentage reduction in capital requirements due to the portfolio composition (see Artzner et al., 1999; Tasche, 2006).

The second part of the chapter then establishes the mean-risk diversification efficient frontier. Similar to the mean-variance efficient frontier, the mean-risk diversification frontier carries a new definition of portfolio efficiency. In particular, any risk measure corresponds to a different portfolio efficiency. Observing the results from different frontiers, a higher level of risk aversion corresponds to a higher level of risk diversification and a lower level of diversification in the classical sense. In conjunction with the results of Chapters 1 and 2, this might serve as an explanation of Statman's diversification puzzle, i.e., the observation that the level of observed diversification in real markets is typically lower than the one predicted by mean-variance theory (Statman, 2004). Non-satiable risk-averse investors, or investors controlling for risk diversification, do not aim for a highly diversified portfolio, even if the risk measure is convex.

The last part of the Chapter 4 presents an empirical application of risk diversification with a specific focus on periods of financial distress. The findings suggest that in particular during the Global Financial Crisis, strategies based on risk diversification would have performed better in terms of a risk–reward ratio than competing strategies. At the same time, constructed portfolios based on the risk diversification measures are typically more concentrated on a small number of assets and require more rebalancing, what also involves higher transaction costs.

Bibliography

Acerbi, C. (2002). Spectral measures of risk: a coherent representation of subjective risk aversion. *Journal of Banking & Finance 26*(7), 1505–1518.

Adcock, C. J. (2010). Asset pricing and portfolio selection based on the multivariate extended skew-student-t distribution. *Annals of Operations Research 176*(1), 221–234.

Anderson, G. (1996). Nonparametric tests of stochastic dominance in income distributions. *Econometrica 64*(5), 1183–1193.

Angelelli, E. and S. L. Ortobelli (2009). American and european portfolio selection strategies: the markovian approach. In P. Catlere (Ed.), *Financial Hedging*, Volume 5, pp. 119–152. Nova Science Publisher.

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1999). Coherent measures of risk. *Mathematical Finance 9*(3), 203–228.

Arvanitis, S. and N. Topaloglou (2017). Testing for prospect and markowitz stochastic dominance efficiency. *Journal of Econometrics 198*(2), 253–270.

Banz, R. W. (1981). The relationship between return and market value of common stocks. *Journal of Financial Economics 9*(1), 3–18.

Barberis, N. and R. Thaler (2003). A survey of behavioral finance. *Handbook of the Economics of Finance 1*(B), 1053–1128.

Barrett, G. F. and S. G. Donald (2003). Consistent tests for stochastic dominance. *Econometrica 71*(1), 71–104.

Baucells, M. and F. H. Heukamp (2006). Stochastic dominance and cumulative prospect theory. *Management Science 52*(9), 1409–1423.

Bawa, V. S. (1975). Optimal rules for ordering uncertain prospects. *Journal of Financial Economics 2*(1), 95–121.

Bawa, V. S. (1976). Admissible portfolios for all individuals. *The Journal of Finance 31*(4), 1169–1183.

Bawa, V. S., J. N. J. Bodurtha, M. R. Rao, and H. L. Suri (1985). On determination of stochastic dominance optimal sets. *Journal of Finance 40*(2), 417–431.

Beach, C. M. and R. Davidson (1983). Distribution-free statistical inference with lorenz curves and income shares. *The Review of Economic Studies 50*(4), 723–735.

Biglova, A., S. Ortobelli, S. T. Rachev, and S. Stoyanov (2004). Different approaches to risk estimation in portfolio theory. *The Journal of Portfolio Management 31*(1), 103–112.

Black, F. (1972). Capital market equilibrium with restricted borrowing. *The Journal of Business 45*(3), 444–455.

Borch, K. (1969). A note on uncertainty and indifference curves. *The Review of Economic Studies 36*(1), 1–4.

Broadie, M. (1993). Computing efficient frontiers using estimated parameters. *Annals of Operations Research 45*(1), 21–58.

Bruni, R., F. Cesarone, A. Scozzari, and F. Tardella (2017). On exact and approximate stochastic dominance strategies for portfolio selection. *European Journal of Operational Research 259*(1), 322–329.

Ceriani, L. and P. Verme (2012). The origins of the gini index: extracts from variabilità e mutabilità (1912) by corrado gini. *The Journal of Economic Inequality* (10), 3.

Chamberlain, G. (1983). A characterization of the distributions that imply mean—variance utility functions. *Journal of Economic Theory 29*(1), 185–201.

Chopra, V. K. and W. T. Ziemba (1993). The effect of errors in means, variances, and covariances on optimal portfolio choice. *Journal of Portfolio Management 19*(2), 6–12.

Choueifaty, Y. and Y. Coignard (2008). Toward maximum diversification. *The Journal of Portfolio Management 35*(1), 40–51.

Clarke, R., H. De Silva, and S. Thorley (2011). Minimum-variance portfolio composition. *The Journal of Portfolio Management 37*(2), 31–45.

Clarke, R., H. De Silva, and S. Thorley (2013). Risk parity, maximum diversification, and minimum variance: An analytic perspective. *The Journal of Portfolio Management 39*(3), 39–53.

Crowder, M. (1986). On consistency and inconsistency of estimating equations. *Econometric Theory 2*(3), 305–330.

Davidson, R. and J. Duclos (2000). Statistical inference for stochastic dominance and for the measurement of poverty and inequality. *Econometrica 68*(6), 1435–1464.

de Athayde, G. M. and R. G. J. Flôres (2004). Finding a maximum skewness portfolio: A general solution to three-moments portfolio choice. *Journal of Economic Dynamics and Control 28*(7), 1335–1352.

De Giorgi, E. (2005). Reward-risk portfolio selection and stochastic dominance. *Journal of Banking & Finance 29*(4), 895–926.

De Giorgi, E. and T. Post (2008). Second-order stochastic dominance, reward-risk portfolio selection, and the capm. *Journal of Financial and Quantitative Analysis 43*(2), 525–546.

DeMiguel, V., L. Garlappi, and R. Uppal (2009). Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *Review of Financial Studies 22*(5), 1915–1953.

Deng, X.-T., Z.-F. Li, and S.-Y. Wang (2005). A minimax portfolio selection strategy with equilibrium. *European Journal of Operational Research 166*(1), 278–292.

Dentcheva, D. and A. Ruszczyński (2006). Portfolio optimization with stochastic dominance constraints. *Journal of Banking & Finance 30*(2), 433–451.

Dhaene, J., D. Linders, W. Schoutens, and D. Vyncke (2012). The herd behavior index: A new measure for the implied degree of co-movement in stock markets. *Insurance: Mathematics and Economics 50*(3), 357–370.

Dopfel, F. E. (2003). Asset allocation in a lower stock-bond correlation environment. *The Journal of Portfolio Management 30*(1), 25–38.

Dybvig, P. H. (1988). Distributional analysis of portfolio choice. *Journal of Business 61*(3), 369–393.

Dybvig, P. H. and S. A. Ross (1982). Portfolio efficient sets. *Econometrica 50*(6), 1525–1546.

Eeckhoudt, L., H. Schlesinger, and I. Tsetlin (2009). Apportioning of risks via stochastic dominance. *Journal of Economic Theory 144*(3), 994–1003.

Egozcue, M., L. F. García, W.-K. Wong, and R. Zitikis (2011). Do investors like to diversify? a study of markowitz preferences. *European Journal of Operational Research 215*(1), 188–193.

Egozcue, M. and W.-K. Wong (2010). Gains from diversification on convex combinations: A majorization and stochastic dominance approach. *European Journal of Operational Research 200*(3), 893–900.

Fábián, C. I., G. Mitra, D. Roman, and V. Zverovich (2011). An enhanced model for portfolio choice with ssd criteria: a constructive approach. *Quantitative Finance 11*(10), 1525–1534.

Fagiuoli, E., F. Pellerey, and M. Shaked (1999). A characterization of the dilation order and its applications. *Statistical papers 40*(4), 393.

Fama, E. and K. French (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics 33*(1), 3–56.

Fastrich, B., S. Paterlini, and P. Winker (2014). Cardinality versus q-norm constraints for index tracking. *Quantitative Finance 14*(11), 2019–2032.

Feldstein, M. S. (1969). Mean-variance analysis in the theory of liquidity preference and portfolio selection. *The Review of Economic Studies 36*(1), 5–12.

Fishburn, P. C. (1964). *Decision and Value Theory.* Wiley.

Fishburn, P. C. (1976). Continua of stochastic dominance relations for bounded probability distributions. *Journal of Mathematical Economics 3*(3), 295–311.

Fishburn, P. C. (1980). Continua of stochastic dominance relations for unbounded probability distributions. *Journal of Mathematical Economics 7*(3), 271–285.

Flores, Y. S., R. J. Bianchi, M. E. Drew, and S. Trück (2017). The diversification delta: A different perspective. *The Journal of Portfolio Management 43*(4), 112–124.

Gaivoronski, A. A. and G. Pflug (2005). Value-at-risk in portfolio optimization: properties and computational approach. *Journal of Risk 7*(2), 1–31.

Gibbons, M., S. Ross, and J. Shanken (1989). A test of the efficiency of a given portfolio. *Econometrica 57*(5), 1121–1152.

Gini, C. (1921). Measurement of inequality of incomes. *The Journal of Economic Inequality 31*(124–126), 3.

Godambe, V. and M. Thompson (1989). An extension of quasi-likelihood estimation. *Journal of Statistical Planning and Inference 22*(2), 137–152.

Goetzmann, W. N. and A. Kumar (2008). Equity portfolio diversification. *Review of Finance 12*(3), 433–463.

Hadar, J. and W. R. Russell (1969). Rules for ordering uncertain prospects. *American Economic Review 59*(1), 140–146.

Hakansson, N. H. (1971). Capital growth and the mean-variance approach to portfolio selection. *Journal of Financial and Quantitative Analysis 6*(1), 517–557.

Hanoch, G. and H. Levy (1969). The efficiency analysis of choices involving risk. *Review of Economic Studies 36*, 335–346.

Harris, R. G. (1972). A general equilibrium analysis of the capital asset pricing model. *Journal of Financial and Quantitative Analysis 15*(1), 99–122.

Hodder, J. E., J. C. Jackwerth, and O. Kolokolova (2014). Improved portfolio choice using second-order stochastic dominance. *Review of Finance 19*(4), 1623–1647.

Ingersoll, J. E. (1987). *Theory of financial decision making*. Rowman & Littlefield.

Jagannathan, R. and T. Ma (2003). Risk reduction in large portfolios: Why imposing the wrong constraints helps. *The Journal of Finance 58*(4), 1651–1684.

Kahneman, D. and A. Tversky (1979). Prospect theory: An analysis of decision under risk. *Econometrica 47*(1), 263–291.

Kempf, A., O. Korn, and S. Saßning (2015). Portfolio optimization using forward-looking information. *Review of Finance 19*(1), 467–490.

Konno, H. and H. Yamazaki (1991). Mean-absolute deviation portfolio optimization model and its applications to tokyo stock market. *Management Science 37*(5), 519–531.

Kopa, M. and P. Chovanec (2008). A second-order stochastic dominance portfolio efficiency measure. *Kybernetika 44*(2), 243–258.

Kopa, M. and T. Post (2015). A general test for ssd portfolio efficiency. *OR Spectrum 37*(3), 703–734.

Kuosmanen, T. (2004). Efficient diversification according to stochastic dominance criteria. *Management Science 50*(10), 1390–1406.

Landsberger, M. and I. Meilijson (1993). Mean-preserving portfolio dominance. *The Review of Economic Studies 60*(2), 479–485.

Ledoit, O. and M. Wolf (2004). Honey, i shrunk the sample covariance matrix. *The Journal of Portfolio Management 30*(4), 110–119.

Levy, H. (1992). Stochastic dominance and expected utility: survey and analysis. *Management Science 38*(4), 555–593.

Levy, H. and M. Sarnat (1970). International diversification of investment portfolios. *The American Economic Review 60*(4), 668–675.

Levy, M. and H. Levy (2002). Prospect theory: much ado about nothing? *Management Science 48*(10), 1334–1349.

Linders, D., J. Dhaene, and W. Schoutens (2015). Option prices and model-free measurement of implied herd behavior in stock markets. *International Journal of Financial Engineering 2*(2), 1–35.

Lintner, J. (1964). Optimal dividends and corporate growth under uncertainty. *The Quarterly Journal of Economics 78*(1), 49–95.

Linton, O., E. Maasoumi, and Y.-J. Whang (2005). Consistent testing for stochastic dominance under general sampling schemes. *The Review of Economic Studies 72*(3), 735–765.

Linton, O., T. Post, and Y. Whang (2014). Testing for the stochastic dominance efficiency of a given portfolio. *Econometrics Journal 17*(2), 59–74.

Lizyayev, A. and A. Ruszczyński (2012). Tractable almost stochastic dominance. *European Journal of Operational Research 218*(2), 448 – 455.

Loistl, O. (2015). Does mean-variance portfoliomanagement deserve expected utility's approximative affirmation? *European Journal of Operational Research 247*(2), 676–680.

Luc, T. D. (2016). *Multiobjective Linear Programming*. Springer.

Ma, C. and W.-K. Wong (2010). Stochastic dominance and risk measure: A decision-theoretic foundation for var and c-var. *European Journal of Operational Research 207*(2), 927–935.

Mangasarian, O. L. (1979). Uniqueness of solution in linear programming. *Linear Algebra and its Applications 25*, 151–162.

Mansini, R., W. Ogryczak, and M. G. Speranza (2007). Conditional value at risk and related linear programming models for portfolio optimization. *Annals of Operations Research 152*(1), 227–256.

Mao, J. C. T. (1970). Models of capital budgeting, ev vs es. *Journal of Financial and Quantitative Analysis 4*(5), 657–675.

Markowitz, H. (1952a). Portfolio selection. *The Journal of Finance 7*(1), 77–91.

Markowitz, H. (1952b). The utility of wealth. *Journal of Political Economy 60*(2), 151–158.

Markowitz, H. (2014). Mean–variance approximations to expected utility. *European Journal of Operational Research 234*(2), 346–355.

Marshall, A. W., I. Olkin, and B. C. Arnold (1943). *Inequalities: theory of majorization and its applications*. Springer.

Mencia, J. and E. Sentana (2009). Multivariate location-scale mixtures of normals and mean-variance-skewness portfolio allocation. *Journal of Econometrics 153*(2), 105–121.

Meucci, A. (2009). Managing diversification. *Risk 22*(5), 74–79.

Miettinen, K. (2012). *Nonlinear Multiobjective Optimization.* Springer.

Mosler, K. and M. Scarsini (1991). Some theory of stochastic dominance. In *Stochastic Orders and Decision Under Risk*, Volume 19, pp. 261–284. Institute of Mathematical Statistics.

Mossin, J. (1966). Equilibrium in a capital asset market. *Econometrica 34*(4), 768–783.

Müller, A. and D. Stoyan (2002). *Comparison Methods for Stochastic Models and Risks.* Wiley.

Ogryczak, W. and A. Ruszczyński (1999). From stochastic dominance to mean-risk models: Semideviations as risk measures. *European Journal of Operational Research 116*(1), 33–50.

Ogryczak, W. and A. Ruszczyński (2002). Dual stochastic dominance and quantile risk measures. *International Transactions in Operational Research 9*(5), 661–680.

Ortobelli, S. (2001). The classification of parametric choices under uncertainty: analysis of the portfolio choice problem. *Theory and Decision 51*(2-4), 297–328.

Ortobelli, S., S. T. Rachev, H. Shalit, and F. J. Fabozzi (2009). Orderings and probability functionals consistent with preferences. *Applied Mathematical Finance 16*(1), 81–102.

Ortobelli, S., H. Shalit, and F. J. Fabozzi (2013). Portfolio selection problems consistent with given preference order. *International Journal of Theoretical and Applied Finance 16*(5), 121–159.

Ortobelli, S., F. J. Wong, Wing-Keung Fabozzi, and M. Egozcue (2018). Diversification versus optimality: is there really a diversification puzzle? *Applied Economics 50*(43), 4671–4693.

Owen, J. and R. Rabinovitch (1983). On the class of elliptical distributions and their applications to the theory of portfolio choice. *The Journal of Finance 38*(3), 745–752.

Pedersen, C. S. and S. E. Satchell (2002). On the foundation of performance measures under asymmetric returns. *Quantitative Finance 2*(3), 217–223.

Pflug, G. C. (2000). Some remarks on the value-at-risk and the conditional value-at-risk. In S. Uryasev (Ed.), *Probabilistic Constrained Optimization. Nonconvex Optimization and Its Applications*, Volume 49, pp. 272–281. Springer.

Porter, B. R., J. R. Wart, and D. L. Ferguson (1973). Efficient algorithms for conducting stochastic dominance tests on large number of portfolios. *Journal of Financial and Quantitative Analysis 8*(1), 71–81.

Post, T. (2003). Empirical tests for stochastic dominance efficiency. *The Journal of Finance 58*(5), 1905–1932.

Post, T. and M. Kopa (2013). General linear formulations of stochastic dominance criteria. *European Journal of Operational Research 230*(2), 321–332.

Post, T. and M. Kopa (2016). Portfolio choice based on third-degree stochastic dominance. *Management Science 63*(10), 3147–3529.

Post, T. and V. Potì (2017). Portfolio analysis using stochastic dominance, relative entropy and empirical likelihood. *Management Science 63*(1), 153–165.

Post, T. and P. Versijp (2007). Multivariate tests for stochastic dominance efficiency of a given portfolio. *Journal of Financial and Quantitative Analysis 42*(2), 489–516.

Preis, T., D. Y. Kenett, H. E. Stanley, D. Helbing, and E. Ben-Jacob (2012). Quantifying the behavior of stock correlations under market stress. *Scientific Reports 2.*

Quirk, J. P. and R. Saposnik (1962). Admissibility and measurable utility functions. *Review of Economics Studies 29*(2), 140–146.

Rachev, S. T. (1991). *Probability Metrics and the Stability of Stochastic Models.* Wiley.

Rachev, S. T., S. Ortobelli, S. Stoyanov, F. J. Fabozzi, and A. Biglova (2008). Desirable properties of an ideal risk measure in portfolio theory. *International Journal of Theoretical and Applied Finance 11*(1), 19–54.

Rachev, S. T., S. V. Stoyanov, and F. J. Fabozzi (2011). *A Probability Metrics Approach to Financial Risk Measures.* Wiley.

Rockafellar, T. R. and S. Uryasev (2000). Optimization of conditional value-at-risk. *Journal of Risk 2*(3), 21–41.

Rockafellar, T. R. and S. Uryasev (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance 26*(7), 1443–1471.

Roll, R. (1976). A critique of the asset pricing theory's tests part i: On past and potential testability of the theory. *Journal of Financial Economics 4*(2), 129–176.

Roll, R. (1978). Ambiguity when performance is measured by the securities market line. *The Journal of Finance 33*(4), 1051–1069.

Roman, D., K. Darby-Dowman, and G. Mitra (2006). Portfolio construction based on stochastic dominance and target return distributions. *Mathematical Programming 108*(2), 541–569.

Rothschild, M. and J. E. Stiglitz (1970). Increasing risk: I. a definition. *Journal of Economic Theory 2*(3), 225–243.

Rothschild, M. and J. E. Stiglitz (1971). Increasing risk ii: Its economic consequences. *Journal of Economic Theory 3*(1), 66–84.

Rudin, A. M. and J. S. Morgan (2006). A portfolio diversification index. *The Journal of Portfolio Management 32*(1), 81–89.

Ruttiens, A. (2013). Portfolio risk measures: The time's arrow matters. *Computational Economics 41*(3).

Samuelson, P. A. (1967). General proof that diversification pays. *Journal of Financial and Quantitative Analysis 2*(1), 1–13.

Scaillet, O. and N. Topaloglou (2010). Testing for stochastic dominance efficiency. *Journal of Business & Economic Statistics 28*(1), 169–180.

Schulmerich, M. and S. Trautmann (2003). Local expected shortfall-hedging in discrete time. *Review of Finance 7*(1), 75–102.

Shalit, H. and S. Yitzhaki (1994). Marginal conditional stochastic dominance. *Management Science 40*(5), 670–684.

Shalit, H. and S. Yitzhaki (2005). The mean-gini efficient portfolio frontier. *Journal of Financial Research 28*(1), 59–75.

Shanken, J. (1985). Multivariate tests of the zero-beta capm. *Journal of Financial Economics 14*(3), 327–348.

Shanken, J. (1986). Testing portfolio efficiency when the zero-beta rate is unknown: A note. *Journal of Finance 41*(1), 269–276.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance 19*(3), 425–442.

Silvapulle, P. and C. W. Granger (2001). Large returns, conditional correlation and portfolio diversification: a value-at-risk approach. *Quantitative Finance 1*(5), 542–551.

Simaan, Y. (1993). Portfolio selection and asset pricing—three-parameter framework. *Management Science 39*(5), 568–577.

Simaan, Y. (2014). The opportunity cost of mean–variance choice under estimation risk. *European Journal of Operational Research 234*(2), 382–391.

Skintzi, V. D. and A.-P. N. Refenes (2005). Implied correlation index: A new measure of diversification. *Journal of Futures Markets: Futures, Options, and Other Derivative Products 25*(2), 171–197.

Skorokhod, A. (1977). On a representation of random variables. *Theory of Probability & Its Applications 21*(3), 628–632.

Statman, M. (2004). The diversification puzzle. *Financial Analysts Journal 60*(4), 44–53.

Stoyanov, S. V., S. T. Rachev, and F. J. Fabozzi (2007). Optimal financial portfolios. *Applied Mathematical Finance 14*(5), 401–436.

Tasche, D. (2006). Measuring sectoral diversification in an asymptotic multi-factor framework. *Journal of Credit Risk 2*(3), 33–55.

Tversky, A. and D. Kahneman (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty 5*(4), 297–323.

Vermorken, M. A., F. R. Medda, and T. Schröder (2012). The diversification delta: a higher-moment measure for portfolio diversification. *The Journal of Portfolio Management 39*(1), 67–74.

Von Neumann, J. and O. Morgenstern (2007). *Theory of games and economic behavior (commemorative edition).* Princeton University Press.

Wong, W.-K. (2007). Stochastic dominance and mean–variance measures of profit and loss for business planning and investment. *European Journal of Operational Research 182*(2), 829–843.

Young, M. R. (1998). A minimax portfolio selection rule with linear programming solution. *Management Science 44*(5), 673–683.

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