

Equiconvergence for perturbed Jacobi polynomial expansions

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Abstract

We show asymptotic expansions of the eigenfunctions of certain perturbations of the Jacobi operator in a bounded interval, deducing equiconvergence results between expansions with respect to the associated orthonormal basis and expansions with respect to the cosine basis. Several results for pointwise convergence then follow.

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1. Introduction

The equiconvergence principle for expansions in eigenfunctions of Sturm-Liouville operators goes back to the beginning of the 20th century, with the seminal paper of A. Haar [12, page 355]. It essentially says that the expansion of an integrable function in $[0, \pi/2]$ with respect to the eigenfunctions of the Sturm-Liouville operator in Liouville normal form

$$-u'' + q(t)u$$

with homogeneous separated boundary conditions

$$u'(0) - hu(0) = u'(\pi/2) - Hu(\pi/2) = 0$$

converges or diverges at some point if its expansion with respect to the cosine basis $\{\cos(2nx)\}_{n=0}^{+\infty}$ converges or diverges at that point. The function q is required to be of bounded variation. J. L. Walsh [24] studied the case with boundary conditions $u(0) = u(\pi/2) = 0$, realizing that the equiconvergence is uniform in the interval (although he was only able to show it for square integrable functions).

Soon after Haar's work, J. Tamarkin [23] generalized these results to expansions in eigenfunctions of certain higher order differential problems now known as Birkhoff-regular problems [2], obtaining equiconvergence for integrable functions when the coefficients of the equation are continuous. M. H. Stone [21], studied the case of integrable coefficients.

Several generalizations followed, some have been collected by A. M. Minkin in the survey [16]. Here we mention one of Minkin's own results [16, Theorem 0.1, page 3673]. He studied

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symmetric quasi differential operators of higher order, defined on any open interval I , bounded or unbounded, with locally integrable coefficients, establishing an equiconvergence result for functions in $L^2(I)$, or integrable but compactly supported away from the endpoints of I .

Here we are interested in a singular Sturm-Liouville operator again of the form

$$\ell u = -u'' + q(t)u$$

for particular choices of the function q , which will only be *locally integrable* in the interval $(0, \pi/2)$. Our goal is to obtain an equiconvergence result for all functions in $L^1((0, \pi/2))$. Unfortunately, to the best of our knowledge, none of the known results apply to our case, not even the above mentioned theorem of Minkin (as it does not apply to functions in L^1 of the *whole interval*). The reason why it is so important for us to show equiconvergence for such a large class of functions will become clear by the end of this introduction. In fact, it would be interesting to obtain results for even larger classes of functions. See also [5] on these matters in a classic context.

The present paper is divided into three parts. In the first part, §2, we show that if an orthonormal basis $\{u_n\}_{n=0}^{+\infty}$ of $L^2([0, \pi/2], dt)$, satisfies five axiomatic properties of resemblance with the basis $\{\cos(2nt)\}_{n=0}^{+\infty}$, then there is equiconvergence between the Fourier series associated respectively with the above two basis, with essentially any summability method.

More precisely, the summability method will come from a bounded double sequence $\{r_{n,N}\}$ such that $\sum_{n=0}^{+\infty} |r_{n,N} - r_{n+1,N}|$ is also bounded in N . For example, one could take the Cesàro means of order $\theta \geq 0$, given by

$$r_{n,N} = \frac{A_{N-n}^\theta}{A_N^\theta}$$

where $A_n^\theta = \binom{n+\theta}{n}$ for $n \geq 0$, and $A_n^\theta = 0$ for $n < 0$. Thus, we shall call

$$T_N f(x) := \sum_{n=0}^{+\infty} r_{n,N} \int_0^{\pi/2} f(t) u_n(t) dt u_n(x)$$

the means of the Fourier series of a function f , computed with our favourite summability method, with respect to the basis $\{u_n\}$, and

$$Q_N f(x) := \frac{2}{\pi} r_{0,N} \int_0^{\pi/2} f(t) dt + \frac{4}{\pi} \sum_{n=1}^{+\infty} r_{n,N} \int_0^{\pi/2} f(t) \cos(2nt) dt \cos(2nx)$$

the means of the Fourier series of f with respect to the basis $\{\cos(2nt)\}$ with the same summability method.

In Theorem 1 we show that if there is a dense subspace Ω of $L^1([0, \pi/2], dt)$ such that for all $t \in (0, \pi/2)$ and for all $g \in \Omega$,

$$\lim_{N \rightarrow +\infty} (T_N g(t) - Q_N g(t)) = 0,$$

then for all $t \in (0, \pi/2)$ and for all $f \in L^1([0, \pi/2], dt)$

$$\lim_{N \rightarrow +\infty} (T_N f(t) - Q_N f(t)) = 0,$$

and uniform convergence away from the endpoints in Ω implies the same type of convergence in L^1 . This result is very much in the spirit of J. E. Gilbert's work [10]. The five conditions on the

basis $\{u_n\}$ and the hypothesis on $r_{n,N}$ are basically the same here and in [10], but Gilbert proves a transplanted theorem, rather than an equiconvergence theorem. In a few words, he shows, among other results, that if the maximal operator $Q^*f(x) = \sup_N |Q_N f(x)|$ is of weak type (p, p) with respect to $L^p((0, \pi/2), dt)$ for some p , then so is $T^*f(x) = \sup_N |T_N f(x)|$.

Observe that an equiconvergence result like the one we show here completely bypasses maximal functions, and can be used to transfer basically any type of pointwise convergence result (convergence in a given point, divergence sets for continuous functions, almost everywhere convergence for L^p functions, Hausdorff dimension of divergence sets of regular functions, etc.) that one has for, say, Cesàro means of classic Fourier series of functions in any space contained in $L^1([0, \pi/2], dt)$, to the same type of means of Fourier series associated with the basis $\{u_n\}$, for functions in the same space.

Since the boundedness of the maximal operator T^* implies the almost everywhere convergence of $T_N f$, Gilbert's transplanted result can also be used to deduce almost everywhere results for $T_N f$ from well known results on the boundedness of Q^*f . Nevertheless, other types of pointwise convergence results do not seem to be deducible from Gilbert's transplanted theorem.

In the second part of the paper, §3, we consider a perturbation of the Jacobi operator

$$Ju = -u'' + \left((\alpha^2 - \frac{1}{4}) \cot^2 t + (\beta^2 - \frac{1}{4}) \tan^2 t \right) u, \quad \alpha \geq \beta > -1/2,$$

that is an operator of the form

$$\ell u = Ju - \chi(t)u,$$

where $\chi \in C^2(\mathbb{R})$ is even with respect to 0 and $\pi/2$. The unbounded operator ℓ is defined on

$$\mathcal{D} = \{f, f' \in AC_{\text{loc}}, f, \ell f \in L^2([0, \pi/2], dt)\}$$

(AC_{loc} is the class of absolutely continuous functions on all compact subintervals of $(0, \pi/2)$), and its restriction $\tilde{\ell}$ to a properly chosen subdomain of \mathcal{D} is self-adjoint and bounded below. By the spectral theorem there is a sequence of eigenfunctions of $\tilde{\ell}$, $\{u_n\}_{n=0}^{+\infty}$, that form an orthonormal basis of $L^2([0, \pi/2], dt)$. The core of this second part is condensed in Theorem 22 and consists thus in showing that these eigenfunctions have certain asymptotic expansions and therefore satisfy the five axiomatic properties of the first part, so that the equiconvergence result holds in this context. As a consequence, virtually any pointwise convergence result for, say, Cesàro means known for the cosine basis can be stated exactly in the same form for the basis $\{u_n\}$. The reader can find a short list at the end of §3.

Once again, the proofs of some of the results here follow the lines of a paper by J. E. Gilbert [11]. Gilbert also shows that the eigenfunctions of certain operators satisfy his five axioms. Notice though that he requires that his operators satisfy some sort of symmetry with respect to the center of the interval $(0, \pi/2)$. In our case, this hypothesis would require $\alpha = \beta$. Thus, our result can be seen as an extension of Gilbert's result to more general operators.

In the third and final part of the paper, §4, we consider a perturbation $A(t)$ of the function $\sin^{2\alpha+1} t \cos^{2\beta+1} t$ in the interval $(0, \pi/2)$, again with $\alpha \geq \beta > -1/2$. By this we mean that

$$A(t) = B(t) \sin^{2\alpha+1} t \cos^{2\beta+1} t,$$

where $B \in C^4(\mathbb{R})$ is strictly positive and even with respect to 0 and $\pi/2$. The Sturm-Liouville operator (now in general form)

$$Lv := \frac{1}{A(t)} (A(t)v)'$$

can be reduced, after the Liouville transformation $u(t) = A^{1/2}(t)v(t)$, to its Liouville normal form

$$\ell u = Ju - \chi(t)u,$$

where

$$\chi(t) = \left(\beta + \frac{1}{2}\right) \frac{B'(t)}{B(t)} \tan t - \left(\alpha + \frac{1}{2}\right) \frac{B'(t)}{B(t)} \cot t + \frac{1}{4} \left(\frac{B'(t)}{B(t)}\right)^2 - \frac{1}{2} \frac{B''(t)}{B(t)} + 2\alpha\beta + 2\alpha + 2\beta + \frac{3}{2}$$

is a $C^2(\mathbb{R})$ function even with respect to 0 and $\pi/2$. Thus, the results of the second part of this paper can be applied to this particular operator ℓ , to deduce that the functions

$$v_n(t) := A^{-1/2}(t)u_n(t), \quad n = 0, 1, \dots$$

form an orthonormal basis of $L^2((0, \pi/2), A(t)dt)$ consisting of eigenfunctions of L . Various means of Fourier series with respect to this type of bases have been studied widely in the past. Here we just want to mention the monograph [4] on weak type estimates and almost everywhere convergence for Cesàro means of Jacobi polynomial expansions, which in our notation corresponds to the case $B(t) \equiv 1$. The case of the unbounded interval $[0, +\infty)$ also has been studied, see [3, 6, 7].

It is perhaps worth noticing here that the class of functions $A(t)$ to which our results apply can also be described differently. Let us fix $\varepsilon > 0$ and let A_0 be any positive function in $C^\infty((0, \pi/2))$ such that $A_0(t) = t^{2\alpha+1}$ for $t \in (0, \varepsilon)$, and $A_0(t) = (\pi/2 - t)^{2\beta+1}$ for $t \in (\pi/2 - \varepsilon, \pi/2)$. Then one can easily see that

$$A(t) = B(t) \sin^{2\alpha+1} t \cos^{2\beta+1} t$$

with $B \in C^4(\mathbb{R})$, strictly positive and even with respect to 0 and $\pi/2$ if and only if

$$A(t) = \widetilde{B}(t)A_0(t)$$

with $\widetilde{B} \in C^4(\mathbb{R})$, strictly positive and even with respect to 0 and $\pi/2$. In other words, we are studying expansions in eigenfunctions of L , where A is essentially any positive smooth function that behaves as a power function with positive exponents near the endpoints of $(0, \pi/2)$.

The means operator in this context has the form

$$\begin{aligned} T_N^A f(x) &:= \sum_{n=0}^{+\infty} r_{n,N} \int_0^{\pi/2} f(t) v_n(t) A(t) dt v_n(x) \\ &= \sum_{n=0}^{+\infty} r_{n,N} \int_0^{\pi/2} f(t) A^{1/2}(t) u_n(t) dt A^{-1/2}(x) u_n(x) \\ &= A^{-1/2}(x) T_N(A^{1/2} f)(x). \end{aligned}$$

This implies that the equiconvergence results of the first two parts of the paper, between Q_N and T_N , can be transferred to $A^{-1/2}(x)Q_N(A^{1/2}f)(x)$ and $T_N^A f(x)$, as long as the function $A^{1/2}f$ belongs to $L^1((0, \pi/2), dt)$.

Concerning the almost everywhere convergence, looking at [4, Theorem 1.4] and by analogy with what happens in the case of the unbounded interval studied in [7], one would expect to obtain almost everywhere convergence for Cesàro means of order $\theta > 0$ for all functions in $L^p((0, \pi/2), A(t)dt)$ for

$$p \geq \max\left(1, \frac{4\alpha + 4}{2\alpha + 3 + 2\theta}\right).$$

Unfortunately, observe that in order to have $A^{1/2}f \in L^1((0, \pi/2), dt)$ as needed to apply equiconvergence, one needs $f \in L^p((0, \pi/2), A(t)dt)$ with

$$p > \frac{4\alpha + 4}{2\alpha + 3}.$$

This would leave the case

$$\max\left(1, \frac{4\alpha + 4}{2\alpha + 3 + 2\theta}\right) \leq p \leq \frac{4\alpha + 4}{2\alpha + 3}$$

unexplored. For this reason, here we only study the case $\theta = 0$, that is the *partial sums* of the Fourier series with respect to $\{v_n\}$, obtaining a few results on the pointwise convergence, and leave the case $\theta > 0$ for future studies. More precisely, we show that when $\theta = 0$ there is a.e. convergence in $L^p((0, \pi/2), A(t)dt)$ if and only if $p > (4\alpha + 4)/(2\alpha + 3)$ (Theorems 27 and 28) and discuss the nature of the sets of divergence for continuous and more regular functions (Theorems 29, 31 and 32)

2. An abstract equiconvergence theorem

Let $\{u_n\}_{n \geq 0}$ be an orthonormal basis of $L^2([0, \pi/2])$. We further assume that $\{u_n\}_{n \geq 0}$ satisfies the following properties, where we define $\Delta u_n(x) = u_n(x) - u_{n+1}(x)$.

There exists a constant $C > 0$ and a positive integer n_0 such that

(P1) $\sup_{0 < x < \pi/2} |u_n(x)| \leq C, n = 0, 1, \dots$

(P2) There exists a function $Y_0(x) \in L^\infty(0, \pi/4]$ and constants ν, λ such that

$$u_n(x) = \frac{2}{\sqrt{\pi}} \cos((2n + \nu)x - \lambda) + Y_0(x) \frac{2}{nx} \sin((2n + \nu)x - \lambda) + O(x^{-2}n^{-2})$$

uniformly in $[1/n, \pi/4]$ and $n \geq n_0$.

(P3) There exist functions $Z_1, \dots, Z_4 \in L^\infty(0, \pi/4]$ such that

$$\begin{aligned} \Delta u_n(x) &= x(Z_1(x) \cos(2nx) + Z_2(x) \sin(2nx)) \\ &\quad + \frac{1}{n}(Z_3(x) \cos(2nx) + Z_4(x) \sin(2nx)) + O(x^{-1}n^{-2}) \end{aligned}$$

uniformly in $[1/n, \pi/4]$ and $n \geq n_0$.

(P4) There exists $\tau > 0$ such that

$$|\Delta u_n(x)| \leq C((nx)^\tau n^{-1} + n^{-2}).$$

uniformly in $(0, 1/n]$ and $n \geq n_0$.

(P5) There is a sequence $\{U_n(x)\}_{n=0}^{+\infty}$ satisfying the above properties (P1), ..., (P4) such that

$$u_n(\pi/2 - x) = (-1)^n U_n(x) + O(n^{-2})$$

uniformly in $x \in (0, \pi/4]$. The constants ν, λ, τ and the functions Y_0, Z_1, \dots, Z_4 corresponding to the functions U_n may be different from those corresponding to the functions u_n .

Let $\{r_{n,N}\}_{n=0, N=0}^{\infty, \infty}$ be a doubly indexed sequence such that, if we define $\Delta r_{n,N} = r_{n,N} - r_{n+1,N}$, there exists a constant B (independent of n and N) such that

$$(S1) \quad |r_{n,N}| \leq B \text{ for all } n, N \geq 0$$

$$(S2) \quad \sum_{n=0}^{+\infty} |\Delta r_{n,N}| \leq B \text{ for all } N \geq 0.$$

Define the associated multiplier operator and the corresponding kernel by

$$T_N f(x) = \sum_{n=0}^{+\infty} r_{n,N} \widehat{f}(n) u_n(x), \quad T_N(x, y) = \sum_{n=0}^{+\infty} r_{n,N} u_n(x) u_n(y)$$

where

$$\widehat{f}(n) = \int_0^{\pi/2} f(x) u_n(x) dx.$$

Notice that in particular when $r_{n,N} = 1, n \leq N$ and 0 otherwise, T_N reduces to the partial sum operator. We consider the cosine basis of $L^2([0, \pi/2], dt)$,

$$\left\{ \sqrt{\frac{4 - 2\delta_{0n}}{\pi}} \cos(2nx) \right\}_{n=0}^{\infty},$$

where $\delta_{0n} = 1$ when $n = 0$ and 0 otherwise. We define the multiplier operator associated to the sequence $\{r_{n,N}\}_{n \geq 0, N \geq 0}$ with respect to the cosine basis as

$$Q_N f(x) = \frac{2}{\pi} r_{0,N} \int_0^{\pi/2} f(y) dy + \frac{4}{\pi} \sum_{n=1}^{+\infty} r_{n,N} \int_0^{\pi/2} f(y) \cos(2ny) dy \cos(2nx).$$

The corresponding kernel is given by

$$Q_N(x, y) = \frac{2}{\pi} r_{0,N} + \frac{4}{\pi} \sum_{n=1}^{+\infty} r_{n,N} \cos(2nx) \cos(2ny).$$

We prove that for a fixed $f \in L^1([0, \pi/2])$ and $x \in (0, \pi/2)$, $T_N f(x)$ converges to $f(x)$ if and only if $Q_N f(x)$ converges to $f(x)$. More precisely we prove the following theorem:

Theorem 1. *Assume that $\{u_n\}_{n=0}^{+\infty}$ satisfies (P1)-(P5), and $\{r_{n,N}\}$ satisfies (S1) and (S2). Assume also that for some dense subspace $\Omega \subseteq L^1([0, \pi/2])$, for all $x \in (0, \pi/2)$ and for all $g \in \Omega$*

$$\lim_{N \rightarrow +\infty} (T_N g(x) - Q_N g(x)) = 0,$$

then for all $x \in (0, \pi/2)$ and for all $f \in L^1([0, \pi/2])$

$$\lim_{N \rightarrow +\infty} (T_N f(x) - Q_N f(x)) = 0.$$

Furthermore, if the convergence of $\lim_{N \rightarrow +\infty} (T_N g(x) - Q_N g(x))$ is uniform on some set $\Gamma \subset (0, \pi/2)$ with positive distance from 0 and from $\pi/2$ for each $g \in \Omega$, then for all $f \in L^1([0, \pi/2])$

$$\lim_{N \rightarrow +\infty} (T_N f(x) - Q_N f(x)) = 0$$

uniformly on Γ .

2.1. Preliminary results

In this subsection we will prove some preliminary results needed to prove Theorem 1. We start with a technical lemma. It will be convenient to define some functions which would be needed in our analysis. For any given integers $M \leq N$ we set

$$\begin{aligned} C_{M,N}^0(t) &= \sum_{n=M}^N \cos(2nt), & S_{M,N}^0(t) &= \sum_{n=M}^N \sin(2nt) \\ C_{M,N}^1(t) &= \sum_{n=M}^N \frac{1}{n} \cos(2nt), & S_{M,N}^1(t) &= \sum_{n=M}^N \frac{1}{n} \sin(2nt). \end{aligned}$$

Lemma 2. *For all integers $0 \leq M \leq N$ and for all $t \in \mathbb{R}$*

$$\begin{aligned} C_{M,N}^0(t) &= \frac{\cos((N+M)t) \sin((N-M+1)t)}{\sin t}, \\ S_{M,N}^0(t) &= \frac{\sin((N+M)t) \sin((N-M+1)t)}{\sin t}. \end{aligned}$$

In particular, all the above functions are bounded by $\min(|\sin t|^{-1}, N-M+1)$.

We skip the proof of the above lemma as it simply follows by using the well known formulae for sine and cosine expansions.

Lemma 3. *There is a positive constant C' such that for all integers $1 \leq M \leq N$ and for all $t \in \mathbb{R}$*

$$\begin{aligned} |S_{M,N}^1(t)| &\leq C', \\ |C_{M,N}^1(t)| &\leq 3 + \log \left(1 + \frac{\min(|\sin t|^{-1}, N-M)}{M} \right). \end{aligned}$$

In particular, $C_{M,N}^1(t)$ is uniformly bounded for $|\sin t| \geq cM^{-1}$.

Proof. Let us begin with the inequality $|S_{1,N}^1(t)| \leq C$. The inequality for $S_{M,N}^1$ with $M > 1$ follows immediately by writing $S_{M,N}^1 = S_{1,N}^1 - S_{1,M-1}^1$. Notice that $S_{1,N}^1(t)$ is the N th partial sum of the Fourier series of the sawtooth function $\pi/2 - t$, $t \in [0, \pi]$. Assuming without loss of generality $0 < t < \pi/2$,

$$\begin{aligned} \sum_{n=1}^N \frac{1}{n} \sin(2nt) &= \int_0^t 2 \sum_{n=1}^N \cos(2nu) du = \int_0^t \left(\sum_{n=-N}^N \cos(2nu) - 1 \right) du \\ &= \int_0^t \frac{\sin((2N+1)u)}{\sin u} du - \int_0^t du. \end{aligned}$$

If $t \leq 1/(2N+1)$, the inequality

$$\left| \frac{\sin((2N+1)u)}{\sin u} \right| \leq 2N+1$$

uniformly in $(0, t)$ gives the result. If $1/(2N+1) < t < \pi/2$, then we split the above integral as

$$\int_{1/(2N+1)}^t \frac{\sin((2N+1)u)}{\sin u} du + \int_0^{1/(2N+1)} \frac{\sin((2N+1)u)}{\sin u} ds - t.$$

The last two terms are bounded. As for the first one, setting $\psi(u) = 1/\sin(u)$, a decreasing function in the interval $[1/(2N+1), t]$, integration by parts gives

$$\begin{aligned} \left| \int_{1/(2N+1)}^t \frac{\sin((2N+1)u)}{\sin u} du \right| &\leq \frac{1}{2N+1} (|\cos((2N+1)t)\psi(t)| + |\cos(1)\psi(1/(2N+1))|) \\ &\quad - \frac{1}{2N+1} \int_{1/(2N+1)}^t \psi'(s) ds \\ &\leq \frac{1}{2N+1} \left(\frac{3}{\sin(1/(2N+1))} \right) \leq C. \end{aligned}$$

Let us prove the inequality for $C_{M,N}^1$ now. It clearly suffices to assume $M < N$. It follows immediately after summation by parts,

$$|C_{M,N}^1(t)| = \left| \frac{1}{N} \sum_{n=M}^N \cos(2nt) + \sum_{j=M}^{N-1} \left(\frac{1}{j} - \frac{1}{j+1} \right) \sum_{n=M}^j \cos(2nt) \right|.$$

The sum in j involves $C_{M,j}^0(t)$. After using the estimates for $C_{M,j}^0(t)$, $M \leq j \leq N-1$ from Lemma 2 we get

$$|C_{M,N}^1(t)| \leq \frac{N-M+1}{N} + \sum_{j=M}^{N-1} \left(\frac{1}{j} - \frac{1}{j+1} \right) \min\left(\frac{1}{|\sin t|}, j-M+1 \right).$$

Clearly $(N-M+1)/N$ is bounded by 1 for all $1 \leq M < N$. Calling $[\cdot]$ the floor function, and χ_D the indicator function of the set D , the inequality above further simplifies to

$$\begin{aligned} |C_{M,N}^1(t)| &\leq 1 + \sum_{j=M}^{\min([\sin t]^{-1+M-1}, N-1)} \left(\frac{1}{j} - \frac{1}{j+1} \right) (j-M+1) \\ &\quad + \chi_{[0, N-1]}([\sin t]^{-1} + M) \sum_{[\sin t]^{-1} + M}^{N-1} \left(\frac{1}{j} - \frac{1}{j+1} \right) \frac{1}{|\sin t|} \\ &\leq 1 + \left(\sum_{j=M}^{\min([\sin t]^{-1+M-1}, N-1)} \frac{1}{j+1} \right) + \frac{1}{|\sin t|} \frac{1}{[\sin t]^{-1} + M} \\ &\leq 3 + \log \left(1 + \frac{\min([\sin t]^{-1}, N-M)}{M} \right) \end{aligned}$$

□

We will now need a few more inequalities involving sine and cosine functions. These inequalities are standard. We are mentioning them just for the sake of completeness.

Definition 4. For any given integers $M \leq N$ we set

$$\begin{aligned} C_{M,N}^{0,-}(t) &= \sum_{n=M}^N (-1)^n \cos(2nt), & S_{M,N}^{0,-}(t) &= \sum_{n=M}^N (-1)^n \sin(2nt) \\ C_{M,N}^{1,-}(t) &= \sum_{n=M}^N \frac{(-1)^n}{n} \cos(2nt), & S_{M,N}^{1,-}(t) &= \sum_{n=M}^N \frac{(-1)^n}{n} \sin(2nt). \end{aligned}$$

Lemma 5. For all integers $0 \leq M \leq N$ and for all $t \in \mathbb{R}$,

$$|C_{M,N}^{0,-}(t)| \leq \frac{1}{|\cos t|}, \quad |S_{M,N}^{0,-}(t)| \leq \frac{1}{|\cos t|}.$$

Proof. Observe that

$$C_{M,N}^{0,-}(t) = \frac{1}{2} \sum_{n=M}^N \cos\left(2n\left(t + \frac{\pi}{2}\right)\right) + \frac{1}{2} \sum_{n=M}^N \cos\left(2n\left(\frac{\pi}{2} - t\right)\right) = \frac{1}{2} C_{M,N}^0\left(t + \frac{\pi}{2}\right) + \frac{1}{2} C_{M,N}^0\left(\frac{\pi}{2} - t\right),$$

and therefore

$$|C_{M,N}^{0,-}(t)| \leq \frac{1}{2|\sin\left(t + \frac{\pi}{2}\right)|} + \frac{1}{2|\sin\left(\frac{\pi}{2} - t\right)|} \leq \frac{1}{|\cos(t)|}.$$

The estimate for $S_{M,N}^{0,-}(t)$ is similar. □

Lemma 6. There is a positive constant C such that for all integers $1 \leq M \leq N$ and for all $t \in \mathbb{R}$

$$\begin{aligned} |S_{M,N}^{1,-}(t)| &\leq C, \\ |C_{M,N}^{1,-}(t)| &\leq 3 + \log\left(1 + \frac{\min(|\cos t|^{-1}, N - M)}{M}\right). \end{aligned}$$

Proof. These inequalities follow easily from the identities

$$\begin{aligned} \sum_{n=M}^N \frac{1}{n} (-1)^n \sin(2nt) &= \frac{1}{2} S_{M,N}^1\left(t + \frac{\pi}{2}\right) + \frac{1}{2} S_{M,N}^1\left(t - \frac{\pi}{2}\right) \\ \sum_{n=M}^N \frac{1}{n} (-1)^n \cos(2nt) &= \frac{1}{2} C_{M,N}^1\left(t + \frac{\pi}{2}\right) + \frac{1}{2} C_{M,N}^1\left(t - \frac{\pi}{2}\right). \end{aligned}$$

□

2.2. Pointwise estimates on the kernels

Recall that $T_N(x, y)$ and $Q_N(x, y)$ are the kernels corresponding to the operators T_N and Q_N defined in the beginning of this section. In order to prove Theorem 1, we need some uniform estimates on the difference of the kernels $T_N(x, y)$ and $Q_N(x, y)$, for x in compact subsets of $(0, \pi/2)$ and $y \in (0, \pi/2)$. We have the following theorem:

Theorem 7. For any $0 < x < \pi/2$ there is a positive constant $K(x)$ such that for any integer N and for all $y \in (0, \pi/2)$

$$|T_N(x, y) - Q_N(x, y)| \leq K(x).$$

Furthermore, $K(x) \leq C(x^{-1} + (\pi/2 - x)^{-1})$.

Proof. We know that

$$T_N(x, y) = \sum_{n=0}^{+\infty} r_{n,N} u_n(x) u_n(y)$$

and

$$Q_N(x, y) = \frac{2}{\pi} r_{0,N} + \frac{4}{\pi} \sum_{n=0}^{+\infty} r_{n,N} \cos(2nx) \cos(2ny),$$

where $\{r_{n,N}\}$ are sequences satisfying properties S1 and S2.

Summation by parts on the difference $T_N(x, y) - Q_N(x, y)$ gives

$$\begin{aligned} T_N(x, y) - Q_N(x, y) &= \lim_{M \rightarrow +\infty} \left(r_{M,N} \sum_{n=0}^M \left(u_n(x) u_n(y) - \frac{4 - 2\delta_{0n}}{\pi} \cos(2nx) \cos(2ny) \right) \right. \\ &\quad \left. + \sum_{j=0}^{M-1} \Delta r_{j,N} \sum_{n=0}^j \left(u_n(x) u_n(y) - \frac{4 - 2\delta_{0n}}{\pi} \cos(2nx) \cos(2ny) \right) \right). \end{aligned}$$

By the assumptions on $\{r_{n,N}\}$, it is therefore enough to show that for any $0 < x < \pi/2$ there is a $K(x)$ such that for any M

$$\left| \sum_{n=0}^M \left(u_n(x) u_n(y) - \frac{4 - 2\delta_{0n}}{\pi} \cos(2nx) \cos(2ny) \right) \right| \leq K(x).$$

So, without loss of generality, we will assume $M = N$ and $r_{n,N} = 1$ for $n \leq N$ and 0 elsewhere. Thus, we can rewrite

$$Q_N(x, y) = \frac{2}{\pi} C_{0,N}^0(x - y) + \frac{2}{\pi} C_{1,N}^0(x + y).$$

Notice that for all $0 < y < \pi/2$, by Lemma 2,

$$|C_{1,N}^0(x + y)| \leq \frac{C}{|\sin(x + y)|} \leq \frac{C}{|\sin x|} + \frac{C}{|\sin(\pi/2 - x)|}.$$

So we just need to handle the term

$$T_N(x, y) - \frac{2}{\pi} C_{0,N}^0(x - y).$$

By the symmetry $x \mapsto \pi/2 - x$, $y \mapsto \pi/2 - y$, we can assume without loss of generality that $0 < x \leq \pi/4$. Assume $n_1 = n_1(x) = \max\{n_0, [2/x] + 1\}$. We can also assume without loss of generality that $N \geq n_1$, and replace $T_N(x, y)$ with the kernel

$$T_{n_1,N}(x, y) = \sum_{n=n_1}^N u_n(x) u_n(y),$$

and $Q_N(x, y)$ with $(2/\pi)C_{n_1, N}^0(x - y)$. Thus, it will be enough to study

$$T_{n_1, N}(x, y) - \frac{2}{\pi}C_{n_1, N}^0(x - y).$$

Case 1 We first consider the case $0 < x/2 \leq y \leq \pi/4$. Since we are assuming $n \geq n_1 = \max\{n_0, [2/x] + 1\}$, it follows that $y \geq x/2 \geq 1/n$ and both $u_n(x)$ and $u_n(y)$ can be expanded according to property (P2), obtaining, after some trigonometric manipulations

$$\begin{aligned} u_n(x)u_n(y) &= \left(\frac{2}{\sqrt{\pi}} \cos((2n + \nu)x - \lambda) + Y_0(x) \frac{2}{nx} \sin((2n + \nu)x - \lambda) + O(x^{-2}n^{-2}) \right) \\ &\quad \times \left(\frac{2}{\sqrt{\pi}} \cos((2n + \nu)y - \lambda) + Y_0(y) \frac{2}{ny} \sin((2n + \nu)y - \lambda) + O(y^{-2}n^{-2}) \right) \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{2}{\pi} \cos(\nu(x - y)) \cos(2n(x - y)) - \frac{2}{\pi} \sin(\nu(x - y)) \sin(2n(x - y)) \\ &\quad + \frac{2}{\pi} \cos(\nu(x + y) - 2\lambda) \cos(2n(x + y)) - \frac{2}{\pi} \sin(\nu(x + y) - 2\lambda) \sin(2n(x + y)), \end{aligned}$$

$$\begin{aligned} I_2 &= \left(\frac{2Y_0(x)}{\sqrt{\pi x}} - \frac{2Y_0(y)}{\sqrt{\pi y}} \right) \cos(\nu(x - y)) \frac{1}{n} \sin(2n(x - y)) \\ &\quad + \left(\frac{2Y_0(x)}{\sqrt{\pi x}} - \frac{2Y_0(y)}{\sqrt{\pi y}} \right) \sin(\nu(x - y)) \frac{1}{n} \cos(2n(x - y)), \end{aligned}$$

$$\begin{aligned} I_3 &= \left(\frac{2Y_0(x)}{\sqrt{\pi x}} + \frac{2Y_0(y)}{\sqrt{\pi y}} \right) \cos(\nu(x + y) - 2\lambda) \frac{1}{n} \sin(2n(x + y)) \\ &\quad + \left(\frac{2Y_0(x)}{\sqrt{\pi x}} + \frac{2Y_0(y)}{\sqrt{\pi y}} \right) \sin(\nu(x + y) - 2\lambda) \frac{1}{n} \cos(2n(x + y)) + O(x^{-2}n^{-2}). \end{aligned}$$

Summing over $n_1 \leq n \leq N$ the above expression, we obtain

$$T_{n_1, N}(x, y) - \frac{2}{\pi}C_{n_1, N}^0(x - y) = J_1 + J_2 + J_3 - \frac{2}{\pi}C_{n_1, N}^0(x - y),$$

where

$$\begin{aligned}
J_1 &= \frac{2}{\pi} \cos(\nu(x-y)) C_{n_1, N}^0(x-y) - \frac{2}{\pi} \sin(\nu(x-y)) S_{n_1, N}^0(x-y), \\
J_2 &= \frac{2}{\pi} \cos(\nu(x+y) - 2\lambda) C_{n_1, N}^0(x+y) - \frac{2}{\pi} \sin(\nu(x+y) - 2\lambda) S_{n_1, N}^0(x+y) \\
&\quad + \left(\frac{2Y_0(x)}{\sqrt{\pi}x} - \frac{2Y_0(y)}{\sqrt{\pi}y} \right) \cos(\nu(x-y)) S_{n_1, N}^1(x-y) \\
J_3 &= \left(\frac{2Y_0(x)}{\sqrt{\pi}x} - \frac{2Y_0(y)}{\sqrt{\pi}y} \right) \sin(\nu(x-y)) C_{n_1, N}^1(x-y) \\
&\quad + \left(\frac{2Y_0(x)}{\sqrt{\pi}x} + \frac{2Y_0(y)}{\sqrt{\pi}y} \right) \cos(\nu(x+y) - 2\lambda) S_{n_1, N}^1(x+y) \\
&\quad + \left(\frac{2Y_0(x)}{\sqrt{\pi}x} + \frac{2Y_0(y)}{\sqrt{\pi}y} \right) \sin(\nu(x+y) - 2\lambda) C_{n_1, N}^1(x+y) + O(x^{-1}).
\end{aligned}$$

By Lemma 2, $(\cos(\nu(x-y)) - 1) C_{n_1, N}^0(x-y)$ and $\sin(\nu(x-y)) S_{n_1, N}^0(x-y)$ are uniformly bounded in $N \geq n_1$ and $y \in [x/2, \pi/4]$. It also shows that all the other terms in the above sum with $S_{n_1, N}^0$ or with $C_{n_1, N}^0$ are uniformly bounded by $C|x|^{-1}$. The observation that for $x/2 \leq y \leq \pi/4$,

$$\left| \frac{2Y_0(x)}{\sqrt{\pi}x} \pm \frac{2Y_0(y)}{\sqrt{\pi}y} \right| \leq \frac{6}{\sqrt{\pi}x} \|Y_0\|_\infty,$$

and the estimates of Lemma 3, show that the remaining terms in the above sum are bounded by $C|x|^{-1}$ uniformly in $N \geq n_1$ and $y \in [x/2, \pi/4]$ (notice that $\pi/2 \geq x+y \geq x \geq 2n_1^{-1}$, so that Lemma 3 implies that $C_{n_1, N}^1(x+y)$ is bounded).

Case 2 We need to check what happens in the case $0 < y < x/2$. Of course, since

$$\left| \frac{2}{\pi} C_{n_1, N}^0(x-y) \right| \leq \frac{C}{|\sin(x-y)|} \leq \frac{C}{\sin|x/2|},$$

we only have to study $|T_{n_1, N}(x, y)|$. Set

$$\sum_{n=n_1}^N = \begin{cases} \sum_{n=n_1}^{\lfloor y^{-1} \rfloor + 1} + \sum_{n=\lfloor y^{-1} \rfloor + 2}^N & \text{if } \lfloor y^{-1} \rfloor + 1 < N \\ \sum_{n=n_1}^N & \text{if } N \leq \lfloor y^{-1} \rfloor + 1. \end{cases}$$

Call $M = \min(N, \lfloor y^{-1} \rfloor + 1)$. Then summation by parts gives

$$T_{n_1, M}(x, y) = \sum_{n=n_1}^M u_n(y) u_n(x) = u_M(y) \sum_{n=n_1}^M u_n(x) + \sum_{j=n_1}^{M-1} \Delta u_j(y) \sum_{n=n_1}^j u_n(x)$$

Let's begin with the term

$$u_M(y) \sum_{n=n_1}^M u_n(x).$$

We know that $u_M(y)$ is uniformly bounded in y and M by property (P1), while $|\sum_{n=n_1}^M u_n(x)| \leq C|x|^{-1}$. Indeed, by (P2),

$$\begin{aligned} \sum_{n=n_1}^M u_n(x) &= \sum_{n=n_1}^M \frac{2}{\sqrt{\pi}} \cos((2n+\nu)x-\lambda) + Y_0(x) \frac{2}{nx} \sin((2n+\nu)x-\lambda) + O(x^{-2}n^{-2}) \quad (1) \\ &= \frac{2}{\sqrt{\pi}} \cos(\nu x - \lambda) C_{n_1, M}^0(x) - \frac{2}{\sqrt{\pi}} \sin(\nu x - \lambda) S_{n_1, M}^0(x) \\ &\quad + Y_0(x) \frac{2}{x} \cos(\nu x - \lambda) S_{n_1, M}^1(x) + Y_0(x) \frac{2}{x} \sin(\nu x - \lambda) C_{n_1, M}^1(x) + O(x^{-1}), \end{aligned}$$

and the desired result follows by Lemma 2 and Lemma 3. Notice in particular that since $\pi/4 \geq x \geq 2/n_1$, it follows that $C_{n_1, M}^1(x)$ is bounded.

Since $M-1 \leq y^{-1}$, it follows that for all j between n_1 and $M-1$ we have $y \leq (M-1)^{-1} \leq j^{-1}$ and the following bound given by property (P4) holds

$$|\Delta u_j(y)| \leq C((yj)^\tau j^{-1} + j^{-2}).$$

Thus,

$$\left| \sum_{j=n_1}^{M-1} \Delta u_j(y) \sum_{n=n_1}^j u_n(x) \right| \leq C \sum_{j=n_1}^{M-1} ((yj)^\tau j^{-1} + j^{-2}) \left| \sum_{n=n_1}^j u_n(x) \right|.$$

By (1), $|\sum_{n=n_1}^j u_n(x)| = O(x^{-1})$. On the other hand,

$$\sum_{j=n_1}^{M-1} ((yj)^\tau j^{-1} + j^{-2}) \leq \left(\sum_{j=n_1}^{M-1} (yj)^\tau j^{-1} \right) + C \leq C(yM)^\tau + C \leq C.$$

Hence we get that

$$|T_{n_1, M}(x, y)| \leq Cx^{-1}.$$

It remains to deal with the case $N > [y^{-1}] + 1 = M$ and the sum

$$\sum_{n=M+1}^N u_n(y) u_n(x).$$

Replacing $u_n(x)$ with the expression given by property (P2) we have

$$\begin{aligned} \sum_{n=M+1}^N u_n(y) u_n(x) &= \sum_{n=M+1}^N u_n(y) \frac{2}{\sqrt{\pi}} \cos((2n+\nu)x-\lambda) \quad (2) \\ &\quad + \sum_{n=M+1}^N u_n(y) Y_0(x) \frac{2}{nx} \sin((2n+\nu)x-\lambda) + \sum_{n=M+1}^N u_n(y) O(x^{-2}n^{-2}). \end{aligned}$$

Clearly, by (P1)

$$\sum_{n=M+1}^N u_n(y) O(x^{-2}n^{-2}) = O(x^{-2}y) = O(x^{-1}).$$

For the remaining terms, apply summation by parts. For example,

$$\begin{aligned} \sum_{n=M+1}^N u_n(y) \frac{2}{\sqrt{\pi}} \cos((2n+\nu)x - \lambda) &= u_N(y) \sum_{n=M+1}^N \frac{2}{\sqrt{\pi}} \cos((2n+\nu)x - \lambda) \\ &+ \sum_{j=M+1}^{N-1} \Delta u_j(y) \sum_{n=M+1}^j \frac{2}{\sqrt{\pi}} \cos((2n+\nu)x - \lambda). \end{aligned}$$

By (P1), $|u_N(y)|$ is uniformly bounded in y and N , and as usual

$$\left| \sum_{n=M+1}^N \frac{2}{\sqrt{\pi}} \cos((2n+\nu)x - \lambda) \right| \leq C|x|^{-1}.$$

Thus we need to study

$$\sum_{j=M+1}^{N-1} \Delta u_j(y) \sum_{n=M+1}^j \cos((2n+\nu)x - \lambda).$$

For all indices j between $M+1$ and $N-1$ we have $j^{-1} < y < x/2 < x$ so that for $\Delta u_j(y)$ we use the expansions given by (P3),

$$\Delta u_j(y) = y(Z_1(y) \cos(2jy) + Z_2(y) \sin(2jy)) + \frac{1}{j}(Z_3(y) \cos(2jy) + Z_4(y) \sin(2jy)) + O(j^{-2}y^{-1}).$$

Thus,

$$\begin{aligned} &\sum_{j=M+1}^{N-1} \Delta u_j(y) \sum_{n=M+1}^j \cos((2n+\nu)x - \lambda) \\ &= \sum_{j=M+1}^{N-1} \left(y(Z_1(y) \cos(2jy) + Z_2(y) \sin(2jy)) + \frac{1}{j}(Z_3(y) \cos(2jy) + Z_4(y) \sin(2jy)) + O(j^{-2}y^{-1}) \right) \\ &\quad \times \left(\cos(\nu x - \lambda) C_{M+1,j}^0(x) - \sin(\nu x - \lambda) S_{M+1,j}^0(x) \right) \\ &= \sum_{j=M+1}^{N-1} \left(y(Z_1(y) \cos(2jy) + Z_2(y) \sin(2jy)) + \frac{1}{j}(Z_3(y) \cos(2jy) + Z_4(y) \sin(2jy)) + O(j^{-2}y^{-1}) \right) \\ &\quad \times \left(\cos(\nu x - \lambda) \frac{\cos((j+M+1)x) \sin((j-M)x)}{\sin x} \right. \\ &\quad \left. - \sin(\nu x - \lambda) \frac{\sin((j+M+1)x) \sin((j-M)x)}{\sin x} \right) \end{aligned}$$

Simple trigonometric identities reduce the products

$$\begin{pmatrix} \cos \\ \sin \end{pmatrix}(2jy) \begin{pmatrix} \cos \\ \sin \end{pmatrix}((j+M+1)x) \sin((j-M)x)$$

to linear combinations of terms of the form

$$\begin{pmatrix} \cos \\ \sin \end{pmatrix}(2jy), \quad \begin{pmatrix} \cos \\ \sin \end{pmatrix}((2j(x \pm y))),$$

with coefficients that do not depend on j and that are bounded in x and y . Summing over j , we obtain that

$$\sum_{j=M+1}^{N-1} \Delta u_j(y) \sum_{n=M+1}^j \cos((2n + \nu)x - \lambda)$$

equals $(\sin x)^{-1}$ times the linear combination of expressions of the form

$$yC_{M+1,N-1}^0(t), \quad yS_{M+1,N-1}^0(t), \quad C_{M+1,N-1}^1(t), \quad S_{M+1,N-1}^1(t),$$

where t can be x or y or $x \pm y$, and with coefficients that are bounded in x and y .

By Lemma 2 and Lemma 3, all these are uniformly bounded. In particular, since $M = [y^{-1}] + 1$, we have $\pi/2 \geq t \geq y \geq (M+1)^{-1}$ and, by Lemma 3, $C_{M+1,N-1}^1(t)$ is uniformly bounded.

Finally,

$$\sum_{j=M+1}^{N-1} O(j^{-2}y^{-1})O(|\sin x|^{-1}) = O(M^{-1}y^{-1})O(x^{-1}) = O(x^{-1}).$$

It remains to bound the term

$$\frac{1}{x} \sum_{n=M+1}^N \frac{u_n(y)}{n} \sin((2n + \nu)x - \lambda)$$

in formula (2). Summation by parts gives

$$\frac{1}{xN} u_N(y) \sum_{n=M+1}^N \sin((2n + \nu)x - \lambda) + \frac{1}{x} \sum_{j=M+1}^{N-1} \left(\frac{u_j(y)}{j} - \frac{u_{j+1}(y)}{j+1} \right) \sum_{n=M+1}^j \sin((2n + \nu)x - \lambda)$$

Since $x > 2y > 2/N$, we deduce that $(xN)^{-1}$ is bounded and since, as usual, $\sum_{n=M+1}^N \sin((2n + \nu)x - \lambda) = O(x^{-1})$, the first term in the above sum is settled. By (P1),

$$\frac{u_j(y)}{j} - \frac{u_{j+1}(y)}{j+1} = \frac{\Delta u_j(y)}{j} + O(j^{-2}),$$

and we are left with

$$\frac{1}{x} \sum_{j=M+1}^{N-1} \frac{\Delta u_j(y)}{j} \sum_{n=M+1}^j \sin((2n + \nu)x - \lambda) + \frac{1}{x} \sum_{j=M+1}^{N-1} O(j^{-2})O(x^{-1}).$$

The remainder term gives $O(x^{-2}M^{-1}) = O(x^{-2}y) = O(x^{-1})$. The principal part can be treated as for the previous term of (2), noticing that, by (P3),

$$\frac{\Delta u_j(y)}{j} = yZ_1(y) \frac{\cos(2jy)}{j} + yZ_2(y) \frac{\sin(2jy)}{j} + O(j^{-2}).$$

Case 3. We will deal with the case when $y \in [\pi/4, \pi/2)$. Observe that by property (P5),

setting $z = \pi/2 - y$ so that $z \in (0, \pi/4]$,

$$\begin{aligned} & T_{n_1, N}(x, y) - \frac{2}{\pi} C_{n_1, N}^0(x - y) \\ &= \sum_{n=n_1}^N \left(u_n(x) u_n(y) - \frac{4 - 2\delta_{0, n}}{\pi} \cos(2nx) \cos(2ny) \right) \\ &= \sum_{n=n_1}^N \left((-1)^n u_n(x) U_n(z) - \frac{4 - 2\delta_{0, n}}{\pi} (-1)^n \cos(2nx) \cos(2n(z)) \right) + O(1), \end{aligned}$$

and we can proceed as we did in cases 1 and 2, this time using also Lemma 5 and Lemma 6. We leave the details to the reader. \square

2.3. Proof of Theorem 1

We are now in a position to prove Theorem 1.

Proof. Fix a positive ε . Let $g \in \Omega$ be such that

$$\|f - g\|_{L^1([0, \pi/2])} \leq \frac{\varepsilon}{2K(x)}.$$

Then

$$\begin{aligned} |T_N f(x) - Q_N f(x)| &\leq |T_N(f - g)(x) - Q_N(f - g)(x)| + |T_N g(x) - Q_N g(x)| \\ &\leq \int_0^{\pi/2} K(x) |f(y) - g(y)| dy + |T_N g(x) - Q_N g(x)| \\ &\leq \frac{\varepsilon}{2} + |T_N g(x) - Q_N g(x)| < \varepsilon \end{aligned}$$

for N sufficiently big so that $|T_N g(x) - Q_N g(x)| < \varepsilon/2$. The second part of the theorem follows from the estimates on $K(x)$ in Theorem 7. \square

3. Perturbed Jacobi operator in normal form

In this section we consider the operator given by

$$\ell u := -u'' + \left(\left(\alpha^2 - \frac{1}{4} \right) \cot^2 t + \left(\beta^2 - \frac{1}{4} \right) \tan^2 t - \chi(t) \right) u$$

where χ is a twice continuously differentiable function on \mathbb{R} , even with respect to 0 and $\pi/2$, and $\alpha \geq \beta > -1/2$. We will identify the proper domain where ℓ is self-adjoint, and show that the corresponding orthonormal basis of eigenfunctions satisfies properties (P1) to (P5).

It is clear that the operator ℓ has a singularity at 0 and $\frac{\pi}{2}$. We will deal with the singularities separately to study the properties of the solution of the eigenvalue problem

$$\ell u = \mu u \tag{3}$$

We know that $\cot t$ behaves like t^{-1} near 0^+ . We try to approximate ℓ by the Bessel equation of order α near 0. After adding and subtracting $(\alpha^2 - 1/4)t^{-2}$ we get the following:

$$\ell u = -u'' + \left(\left(\alpha^2 - \frac{1}{4} \right) t^{-2} - \eta_0(t) \right) u = \mu u, \quad (4)$$

where

$$\eta_0(t) = - \left(\alpha^2 - \frac{1}{4} \right) \left(\cot^2 t - \frac{1}{t^2} \right) - \left(\beta^2 - \frac{1}{4} \right) \tan^2 t + \chi(t)$$

is even with respect to 0 and in $C^2(-\pi/2, \pi/2)$. Again $\tan t$ behaves like $(\pi/2 - t)^{-1}$ near $\pi/2^-$. Similarly, adding and subtracting $(\beta^2 - 1/4)(\pi/2 - t)^{-2}$ we get

$$\ell u = -u'' + \left(\left(\beta^2 - \frac{1}{4} \right) \left(\frac{\pi}{2} - t \right)^{-2} - \eta_1 \left(\frac{\pi}{2} - t \right) \right) u = \mu u$$

where

$$\eta_1(t) = - \left(\beta^2 - \frac{1}{4} \right) \left(\cot^2 t - \frac{1}{t^2} \right) - \left(\alpha^2 - \frac{1}{4} \right) \tan^2 t + \chi \left(\frac{\pi}{2} - t \right)$$

is also even with respect to 0 and in $C^2(-\pi/2, \pi/2)$. We need asymptotics of solutions of (3) near 0^+ and $\pi/2^-$ for obtaining the properties (P1) to (P5) of the corresponding eigenfunctions. We state the following asymptotic expansion, as found in [3] (Theorem 1.2, page 220). The result is classic, see also [9, 19].

Theorem 8. *If we set*

$$X_0(t) = \int_0^t \eta_0(s) ds,$$

then there exists a unique twice differentiable solution $V_{\mu,\alpha}$ of (3) such that

$$\left| V_{\mu,\alpha}(t) - \frac{2^\alpha \Gamma(\alpha + 1) t^{1/2}}{(\sqrt{\mu})^\alpha} \left(J_\alpha(\sqrt{\mu}t) - \frac{1}{2} X_0(t) \frac{J_{\alpha+1}(\sqrt{\mu}t)}{\sqrt{\mu}} \right) \right| \leq C \frac{t^2 \min(1, \sqrt{\mu}t)^{\alpha+5/2}}{(\sqrt{\mu})^{\alpha+5/2}} \quad (5)$$

uniformly in $t \in (0, \pi/4 + \varepsilon)$ and $\mu > 1$. Furthermore,

$$\left| V'_{\mu,\alpha}(t) - \frac{2^\alpha \Gamma(\alpha + 1)}{(\sqrt{\mu})^\alpha} (t^{1/2} J_\alpha(\sqrt{\mu}t))' \right| \leq C \frac{t \min(1, \sqrt{\mu}t)^{\alpha+1/2}}{(\sqrt{\mu})^{\alpha+1/2}}$$

uniformly in $t \in (0, \pi/4 + \varepsilon)$ and $\mu > 1$. When $\alpha = 0$, the right hand sides in both estimates above have to be multiplied by the extra factor $\log(2/\min(1, \sqrt{\mu}t))$.

Similarly, if we set

$$X_1(t) = \int_0^t \eta_1(s) ds,$$

then there exists a unique twice differentiable solution $W_{\mu,\beta}$ of (3) such that

$$\left| W_{\mu,\beta} \left(\frac{\pi}{2} - t \right) - \frac{2^\beta \Gamma(\beta + 1) t^{1/2}}{(\sqrt{\mu})^\beta} \left(J_\beta(\sqrt{\mu}t) - \frac{1}{2} X_1(t) \frac{J_{\beta+1}(\sqrt{\mu}t)}{\sqrt{\mu}} \right) \right| \leq C \frac{t^2 \min(1, \sqrt{\mu}t)^{\beta+5/2}}{(\sqrt{\mu})^{\beta+5/2}} \quad (6)$$

uniformly in $t \in (0, \pi/4 + \varepsilon)$ and $\mu > 1$. Furthermore,

$$\left| -W'_{\mu, \beta} \left(\frac{\pi}{2} - t \right) - \frac{2^\beta \Gamma(\beta + 1)}{(\sqrt{\mu})^\beta} \left(t^{1/2} J_\beta(\sqrt{\mu}t) \right)' \right| \leq C \frac{t \min(1, \sqrt{\mu}t)^{\beta+1/2}}{(\sqrt{\mu})^{\beta+1/2}}$$

uniformly in $t \in (0, \pi/4 + \varepsilon)$ and $\mu > 1$. Again, when $\beta = 0$ both right hand sides above have to be multiplied by the extra factor $\log(2/\min(1, \sqrt{\mu}t))$.

Proof. For the proof, we refer the reader to the above mentioned references. Regarding existence, we only want to mention that the proof in [3] works under the hypotheses we have here.

Concerning uniqueness, observe that the difference between two solutions satisfying, say, estimate (5) would also be a solution, too small close to 0. Indeed it is not difficult to show that there is a second solution of the equation that goes to zero as $t^{-\alpha+1/2}$ for $\alpha \neq 0$, and as $t^{1/2} \log t$ for $\alpha = 0$. Thus the difference between any two different solutions cannot be $o(t^{|\alpha+1/2|})$ as $t \rightarrow 0+$. On the other hand, the difference of two solutions satisfying estimate (5) should be $O(t^{\alpha+9/2})$ as $t \rightarrow 0+$, and this is absurd. \square

If we drop the request of uniformity for large values of μ , then we have the following.

Corollary 9. *For all $\mu \in \mathbb{R}$ there exists a unique solution V of (3) such that*

$$V(t) = t^{\alpha+1/2} - \frac{1}{4(\alpha+1)} (\mu t + X_0(t)) t^{\alpha+3/2} + O(t^{\alpha+9/2}), \quad \text{as } t \rightarrow 0,$$

$$V'(t) = (\alpha + 1/2) t^{\alpha-1/2} + O(t^{\alpha+3/2}), \quad \text{as } t \rightarrow 0.$$

When $\alpha = 0$, the remainder terms in both estimates have to be multiplied by the extra factor $\log(2/t)$. When $\mu > 1$, V coincides with $V_{\mu, \alpha}$. We will therefore call $V = V_{\mu, \alpha}$ also when $\mu \leq 1$. Similar formulas hold for $t \rightarrow \pi/2$ and $W_{\mu, \beta}$.

Proof. For a fixed $\bar{\mu} > 1$, it follows from Theorem 8 and the well known expansions of Bessel functions and their derivatives near zero that

$$V_{\bar{\mu}, \alpha}(t) = t^{\alpha+1/2} - \frac{1}{4(\alpha+1)} (\bar{\mu}t + X_0(t)) t^{\alpha+3/2} + O(t^{\alpha+9/2}), \quad \text{as } t \rightarrow 0, \quad (7)$$

$$V'_{\bar{\mu}, \alpha}(t) = (\alpha + 1/2) t^{\alpha-1/2} + O(t^{\alpha+3/2}), \quad \text{as } t \rightarrow 0. \quad (8)$$

We already showed in the proof of Theorem 8 that there is a unique solution of (3) satisfying (7). On the other hand, for a fixed $\mu \leq 1$, one can rewrite (4) as

$$-u'' + \left((\alpha^2 - 1/4) t^{-2} - \eta_0(t) - \mu + 2 \right) u = 2u.$$

Thus, applying (7) and (8) with $\bar{\mu} = 2$ and $X_0(t)$ replaced with $\int_0^t (\eta_0(s) + \mu - 2) ds$, one obtains a (unique) solution V of (3) such that

$$V(t) = t^{\alpha+1/2} - \frac{1}{4(\alpha+1)} \left(2t + \int_0^t (\eta_0(s) + \mu - 2) ds \right) t^{\alpha+3/2} + O(t^{\alpha+9/2}), \quad \text{as } t \rightarrow 0$$

$$= t^{\alpha+1/2} - \frac{1}{4(\alpha+1)} (\mu t - X_0(t)) t^{\alpha+3/2} + O(t^{\alpha+9/2}), \quad \text{as } t \rightarrow 0,$$

and

$$V'(t) = (\alpha + 1/2)t^{\alpha-1/2} + O(t^{\alpha+3/2}), \quad \text{as } t \rightarrow 0.$$

□

Recall that ℓ is a second order differential operator on $(0, \pi/2)$ with singularities at 0 and $\frac{\pi}{2}$. It is easy to check using integration by parts that ℓ is a symmetric operator when applied to smooth functions with compact support in $(0, \pi/2)$. We want to extend ℓ as a self-adjoint operator in $L^2((0, \pi/2))$ to obtain an orthonormal basis of $L^2((0, \pi/2))$ such that the basis elements are the eigenfunctions of ℓ . Once we get an orthogonal expansion we can talk about the convergence of partial sum operator with respect to that expansion.

Niessen and Zettl [18] have proved the existence of self-adjoint extensions of a general class of Sturm-Liouville operators on (a, b) with singularities at a and b . They classify all the possible self-adjoint extensions of non-oscillatory Sturm-Liouville operators. The self-adjoint extensions depend on the boundary conditions at the end points. In order to apply the result of Niessen and Zettl we have to specify the boundary conditions in such a way that when $\chi(t) \equiv 0$ the eigenfunctions are the Jacobi polynomials $\sin^{\alpha+1/2} t \cos^{\beta+1/2} t P_n^{\alpha, \beta}(\cos(2t))$. Before going further we will give some new definitions. For further details the reader is referred to [18].

Definition 10. Consider the differential equation

$$-u'' + qu = \mu u,$$

with $\mu \in \mathbb{R}$ and $q \in L_{loc}(0, \pi/2)$ (i.e. q integrable on any compact subinterval of $(0, \pi/2)$).

We say that the equation is oscillatory at an (or both) endpoint(s) if the zeros of one, and hence every, non-trivial real valued solution accumulate at the endpoint(s) (see Theorem 2.6.2 in [27]), otherwise it is called non-oscillatory.

We say that u is a principal solution at 0 or $\pi/2$ if $u(t)$ is non-zero near 0 or $\pi/2$ and for any other real valued solution y which is not a multiple of u we have $u(t) = o(y(t))$ as $t \rightarrow 0$ or $\frac{\pi}{2}$, otherwise we say it is non principal.

From the asymptotics of the solution of the differential equation (3) in Corollary 9, it is clear that ℓ is non-oscillatory at both endpoints.

By Corollary 9 it is easy to see that $V_{\mu, |\alpha|}(t)$ as defined above satisfies

$$\int_0^\varepsilon \frac{1}{|V_{\mu, |\alpha|}(t)|^2} dt \sim \int_0^\varepsilon \frac{1}{t^{2|\alpha|+1}} dt = +\infty.$$

So by Theorem 2.2 page 548 in [18], $V_{\mu, |\alpha|}(t)$ is a principal solution at 0. Similarly we can show that $W_{\mu, |\beta|}(t)$ is a principal solution at $\pi/2$.

By inspection, one can easily see that a second solution, linearly independent of $V_{\mu, |\alpha|}(t)$, is given by

$$\tilde{V}_{\mu, \alpha}(t) = V_{\mu, |\alpha|}(t) \int_t^\varepsilon \frac{1}{V_{\mu, |\alpha|}(u)^2} du.$$

Notice that, as $t \rightarrow 0+$,

$$\tilde{V}_{\mu, \alpha}(t) \sim V_{\mu, |\alpha|}(t) \int_t^\varepsilon \frac{1}{u^{2|\alpha|+1}} du \sim \begin{cases} t^{-|\alpha|+1/2} & \text{if } \alpha \neq 0 \\ -t^{1/2} \log t & \text{if } \alpha = 0. \end{cases}$$

(similar estimates hold for $\widetilde{W}_{\mu,\beta}(t)$ near $\pi/2$).

If $\alpha < 1$ then ℓ is in the limit circle case at 0, i.e. every solution of $\ell u = \mu u$ is in $L^2((0, \varepsilon))$, and if $\beta < 1$ then ℓ is in the limit circle case at $\pi/2$, i.e. every solution of $\ell u = \mu u$ is in $L^2((\pi/2 - \varepsilon, \pi/2))$. Else, by definition we say that ℓ is in the limit point case. It is known [17, §19.4 Theorem 4] that this classification is independent of μ (this was anyway transparent here, by the above considerations on the linearly independent solutions $V_{\mu,|\alpha|}$ and $\widetilde{V}_{\mu,\alpha}$).

The unbounded operator $\ell : \mathcal{D} \subset L^2 \rightarrow L^2$ is defined in

$$\mathcal{D} = \{f, f' \in \text{AC}_{\text{loc}}, f, \ell(f) \in L^2(0, \pi/2)\},$$

where AC_{loc} is the class of absolutely continuous functions on all the compact subintervals of $(0, \pi/2)$. It is known (see [17] or [18, page 549]) that \mathcal{D} is dense in $L^2(0, \pi/2)$.

Proposition 11. *The operator $\tilde{\ell}$ obtained as the restriction of ℓ to the domain*

$$\mathcal{D}(\tilde{\ell}) = \begin{cases} \{y \in \mathcal{D} : [y, V_{\mu,\alpha}](0) = [y, W_{\widetilde{\mu},\beta}](\pi/2) = 0\} & \text{if } -1/2 < \beta \leq \alpha < 1 \\ \{y \in \mathcal{D} : [y, W_{\mu,\beta}](\pi/2) = 0\} & \text{if } -1/2 < \beta < 1 \leq \alpha \\ \mathcal{D} & \text{if } 1 \leq \beta \leq \alpha \end{cases}$$

is self-adjoint and bounded from below. Here

$$[y, u](0) = \lim_{t \rightarrow 0^+} (y(t) \overline{u'(t)} - y'(t) \overline{u(t)})$$

$$[y, u](\pi/2) = \lim_{t \rightarrow \pi/2^-} (y(t) \overline{u'(t)} - y'(t) \overline{u(t)}),$$

and the domain is independent of the choice of $\mu, \widetilde{\mu}$, and the function χ . All eigenvalues are simple and can be ordered by

$$\mu_0 < \mu_1 < \dots < \mu_n < \dots$$

with $\mu_n \rightarrow +\infty$. More precisely,

$$\mu_n = 4n^2 + 4(\alpha + \beta + 1)n + O(1), \quad \text{as } n \rightarrow +\infty.$$

Proof. Assume first $0 \leq \beta \leq \alpha$ so that both $V_{\mu,\alpha}$ and $W_{\widetilde{\mu},\beta}$ are principal solutions, and consider the preminimal symmetric operator ℓ'_0 defined on

$$\mathcal{D}'_0 = \{y \in \mathcal{D} : y \text{ has compact support in } (0, \pi/2)\}$$

by $\ell'_0(y) = \ell(y)$. \mathcal{D}'_0 is dense in $L^2((0, \pi/2))$ and by Theorem 4.2 in [18], ℓ'_0 is bounded below. Theorem 4.2 and Corollary 4.1 in [18] guarantee that the Friedrichs extension of ℓ'_0 coincides with $\tilde{\ell}$. It is well known (see Definition 3.1 in [18] and the subsequent comments) that the Friedrichs extension of a densely defined symmetric bounded below operator is self-adjoint and bounded below with the same bound.

Assume now $-1/2 < \beta < 0 \leq \alpha$ so that $V_{\mu,\alpha}$ is a principal solution, but $W_{\widetilde{\mu},\beta}$ is not. This also means that ℓ is in the limit circle case at $\pi/2$. Then, by Theorem 4.4 in [18], the operator ℓ_1 defined in

$$\mathcal{D}_1 = \{y \in \mathcal{D} : [y, W_{\beta,\widetilde{\mu}}](\pi/2) = 0 \text{ and } y \text{ is identically 0 near } 0\}$$

by $\ell_1(y) = \ell(y)$, is a symmetric operator in $L^2(0, \pi/2)$ which is bounded below, and its Friedrichs extension is $\tilde{\ell}$ as defined in the statement of the proposition.

Finally, when $-1/2 < \beta \leq \alpha < 0$, then both $V_{\mu,\alpha}$ and $W_{\tilde{\mu},\beta}$ are non principal solutions, and ℓ is in the limit circle case at both endpoints. Thus, $\tilde{\ell}$ is self-adjoint and bounded below by Theorem 5.1 in [18], with the ‘‘separated boundary conditions’’ (5.21) and (5.22) with $A_1 = B_1 = 0$ and $A_2 = B_2 = 1$.

It is immediate that the subspace \mathcal{D} does not depend on μ or $\tilde{\mu}$, and that if $y \in L^2((0, \pi/2))$, since χ is bounded, then $\ell y \in L^2((0, \pi/2))$ if and only if

$$Jy := -y'' + \left(\alpha^2 - \frac{1}{4} \right) \cot^2 t + \left(\beta^2 - \frac{1}{4} \right) \tan^2(t) y \in L^2((0, \pi/2)).$$

Thus \mathcal{D} does not depend on χ either. Let us show that the boundary conditions do not depend on μ , $\tilde{\mu}$ nor χ . Let us focus on the boundary condition at 0. If $0 \leq \alpha < 1$, then $V_{\mu,\alpha}$ is a principal solution and, by Theorem 4.3 in [18] the condition $[y, V_{\mu,\alpha}](0) = 0$ is equivalent to

$$\lim_{t \rightarrow 0^+} \frac{y(t)}{\tilde{V}_{\mu,\alpha}(t)} = 0 \iff \begin{cases} \lim_{t \rightarrow 0^+} \frac{y(t)}{t^{-\alpha+1/2}} = 0, & \text{if } 0 < \alpha < 1 \\ \lim_{t \rightarrow 0^+} \frac{y(t)}{-t^{1/2} \log t} = 0, & \text{if } \alpha = 0, \end{cases}$$

and this is independent of μ and χ . When $-1/2 < \alpha < 0$, then we have to proceed differently. The asymptotic expansion of $V_{\mu,\alpha}$ and of its derivative in Corollary 9 guarantee that, calling

$$V(t) = \frac{V_{\mu_1,\alpha,\chi_1}(t)}{V_{\mu_2,\alpha,\chi_2}(t)},$$

then $V(t) = 1 + O(t^2)$ and $V'(t) = O(t)$ as $t \rightarrow 0^+$, where the dependence on $\mu_1, \mu_2, \chi_1, \chi_2$ appears only in the remainders. Assume that $[y, V_{\mu_2,\alpha,\chi_2}](0) = 0$. Then

$$\begin{aligned} y(t)V'_{\mu_1,\alpha,\chi_1}(t) - y'(t)V_{\mu_1,\alpha,\chi_1}(t) &= -V_{\mu_1,\alpha,\chi_1}^2(t) \left(\frac{y}{V_{\mu_1,\alpha,\chi_1}} \right)'(t) \\ &= -V^2(t)V_{\mu_2,\alpha,\chi_2}^2(t) \left(\frac{1}{V(t)} \frac{y}{V_{\mu_2,\alpha,\chi_2}} \right)'(t) \\ &= V'(t)V_{\mu_2,\alpha,\chi_2}(t)y(t) - V(t)V_{\mu_2,\alpha,\chi_2}^2(t) \left(\frac{y}{V_{\mu_2,\alpha,\chi_2}} \right)'(t) \end{aligned}$$

The initial assumption implies that

$$\lim_{t \rightarrow 0^+} V_{\mu_2,\alpha,\chi_2}^2(t) \left(\frac{y}{V_{\mu_2,\alpha,\chi_2}} \right)'(t) = 0.$$

Since V is bounded near 0 it only remains to show that $V'(t)V_{\mu_2,\alpha,\chi_2}(t)y(t) \rightarrow 0$ as $t \rightarrow 0^+$. Since

$$\left| \left(\frac{y}{V_{\mu_2,\alpha,\chi_2}} \right)'(t) \right| \leq Ct^{-2\alpha-1}$$

with $\alpha < 0$, it follows easily that $y/V_{\mu_2,\alpha,\chi_2}$ is bounded near 0, so that

$$|V'(t)V_{\mu_2,\alpha,\chi_2}(t)y(t)| \leq Ct^{2\alpha+2} \rightarrow 0$$

as $t \rightarrow 0^+$.

The boundary conditions guarantee that all eigenvalues are simple. Indeed, if $0 \leq \beta \leq \alpha$, then the boundary conditions can be written as (Theorem 4.3 in [18])

$$\lim_{t \rightarrow 0^+} \frac{y(t)}{\overline{V}_{\mu,\alpha}(t)} = \lim_{t \rightarrow \pi/2^-} \frac{y(t)}{\overline{W}_{\mu,\beta}(t)} = 0.$$

This shows that if y is an eigenfunction corresponding to μ , then y is a multiple of the principal solution $V_{\mu,\alpha}$. Thus, the corresponding eigenspace has dimension 1. The same argument works when $-1/2 < \beta < 0 \leq \alpha$ too. If $-1/2 < \beta \leq \alpha < 0$, the result follows from Theorem 5.3 in [18].

The orthogonality of eigenfunctions corresponding to different eigenvalues, and the separability of $L^2((0, \pi/2))$, guarantee that the eigenvalues form a (bounded below) countable subset of \mathbb{R} .

Finally, we know ([8, page 1544]) that for $n \geq 0$,

$$\mu_n = \sup_{H \in S^{(n)}} \inf_{\substack{(u,H)=0 \\ u \neq 0, u \in \mathcal{D}(\tilde{\ell})}} \frac{(u, \ell u)}{(u, u)}, \quad (9)$$

where $S^{(n)}$ denotes the family of all n -dimensional subspaces of $L^2((0, \pi/2))$. Recall also that χ is bounded on $(0, \pi/2)$,

$$m \leq -\chi(t) \leq M.$$

The Jacobi operator

$$Ju = -u'' + \left(\left(\alpha^2 - \frac{1}{4} \right) \cot^2 t + \left(\beta^2 - \frac{1}{4} \right) \tan^2 t \right) u$$

on the domain $\mathcal{D}(\tilde{\ell})$ defined as above is self-adjoint, the eigenfunctions are the Jacobi polynomials

$$\sin^{\alpha+1/2} t \cos^{\beta+1/2} t P_n^{\alpha,\beta}(\cos(2t)), \quad n \geq 0$$

with eigenvalues

$$\mu_n^J = (2n+1)^2 + 2(2n+1)(\alpha+\beta) + 2\alpha\beta + \frac{1}{2}$$

(see Theorem 4.2.2, page 61, or 4.24.2, page 67, in [22]). Also observe that these polynomials satisfy the boundary conditions by Theorem 8.21.12, page 197 in [22]), and therefore comparing

$$\ell u = Ju - \chi u$$

with

$$Ju + mu, \quad Ju + Mu$$

in (9), we obtain

$$\mu_n^J + m \leq \mu_n \leq \mu_n^J + M.$$

□

Let us denote the normalized eigenfunctions of $\tilde{\ell}$ by $\{u_n\}_{n \geq 0}$. The self-adjointness of $\tilde{\ell}$ guarantees the orthogonality of $\{u_n\}_{n \geq 0}$ and the spectral theorem for a self-adjoint operator on a Hilbert space gives us that $\{u_n\}_{n \geq 0}$ is a basis of $L^2((0, \pi/2))$. Concerning the eigenvalues, it may be convenient to rewrite the expansion in the above proposition in a slightly different form. When n is big enough, $\mu_n > 1$ and as $n \rightarrow +\infty$

$$\sigma_n := \sqrt{\mu_n} = 2n + 1 + \alpha + \beta + O\left(\frac{1}{n}\right). \quad (10)$$

3.1. Asymptotics of the eigenfunctions

For $n \geq 0$, let $\{u_n\}$ be the eigenfunctions of $\tilde{\ell}$ associated with the eigenvalue μ_n , with L^2 norm equal to 1. Once we have got an orthogonal expansion of $\tilde{\ell}$ we need some estimates for the eigenfunctions to check if they satisfy the properties (P1) to (P5). By Proposition 11, it follows that there exist constants c_n and d_n such that for all $t \in (0, \pi/2)$,

$$u_n(t) = c_n V_{\mu_n, \alpha}(t) = d_n W_{\mu_n, \beta}(t). \quad (11)$$

This is clearly true for $\alpha < 1$ due to the boundary conditions of Proposition 11, and in the other cases it follows from the uniqueness of $V_{\mu_n, \alpha}$ among the solutions of the equation $\ell u = \mu_n u$ which are in $L^2((0, \varepsilon))$, and of $W_{\mu_n, \beta}$ in $L^2(\pi/2 - \varepsilon, \pi/2)$.

Recall the asymptotic expansions, given in Theorem 8, for $V_{\mu_n, \alpha}$ and $W_{\mu_n, \beta}$ in the intervals $(0, \pi/4 + \varepsilon)$ and $(\pi/4 - \varepsilon, \pi/2)$, respectively. We will further simplify the expansions in the next lemma.

Lemma 12. *We have*

$$\begin{aligned} V_{\mu_n, \alpha}(t) &= \left(\frac{2}{\pi}\right)^{1/2} \frac{2^\alpha \Gamma(\alpha + 1)}{\sigma_n^{\alpha+1/2}} \left(\cos\left(\sigma_n t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \right. \\ &\quad \left. - (\alpha^2 - 1/4 + tX_0(t)) \frac{1}{2\sigma_n t} \sin\left(\sigma_n t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O((\sigma_n t)^{-2}) \right), \end{aligned}$$

uniformly in $[\sigma_n^{-1}, \pi/4 + \varepsilon]$ for n sufficiently big. Similarly,

$$\begin{aligned} W_{\mu_n, \beta}(t) &= \left(\frac{2}{\pi}\right)^{1/2} \frac{2^\beta \Gamma(\beta + 1)}{\sigma_n^{\beta+1/2}} \left(\cos\left(\sigma_n \left(\frac{\pi}{2} - t\right) - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) \right. \\ &\quad \left. - (\beta^2 - 1/4 + \left(\frac{\pi}{2} - t\right) X_1\left(\frac{\pi}{2} - t\right)) \frac{1}{2\sigma_n \left(\frac{\pi}{2} - t\right)} \sin\left(\sigma_n \left(\frac{\pi}{2} - t\right) - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) \right. \\ &\quad \left. + O\left(\left(\sigma_n \left(\frac{\pi}{2} - t\right)\right)^{-2}\right) \right), \end{aligned}$$

uniformly in $(\pi/4 - \varepsilon, \pi/2 - \sigma_n^{-1}]$ for n sufficiently big.

Proof. It follows directly from Theorem 8 and the asymptotic expansion of Bessel functions (see [25] page 199). \square

For proving the properties (P1) to (P5) for the eigenfunctions $\{u_n\}_{n \geq 0}$ we need asymptotic estimates of the constants c_n and d_n in (11) for n large. The following lemmas give us the desired expansion of c_n and d_n .

Lemma 13. *The following estimate holds*

$$\frac{d_n}{c_n} = (-1)^n \frac{2^{\alpha-\beta} \Gamma(\alpha + 1)}{\Gamma(\beta + 1)} \sigma_n^{\beta-\alpha} (1 + O(n^{-2})), \text{ as } n \rightarrow +\infty.$$

Proof. Let

$$t_n = \begin{cases} \frac{\pi}{4} + \frac{(\alpha - \beta)\pi}{4\sigma_n} & \text{if } n \text{ is even} \\ \frac{\pi}{4} + \frac{(\alpha - \beta + 2)\pi}{4\sigma_n} & \text{if } n \text{ is odd,} \end{cases}$$

and replace t with t_n in the asymptotic expansion of $V_{\mu_n, \alpha}(t)$ in $W_{\mu_n, \beta}(t)$ in Lemma 12. By (10), if n is even, then

$$\begin{aligned}\cos\left(t_n\sigma_n - \frac{\alpha}{2}\pi - \frac{1}{4}\pi\right) &= \cos\left(\frac{\pi}{2}n + O(n^{-1})\right) \\ \cos\left(\left(\frac{\pi}{2} - t_n\right)\sigma_n - \frac{\beta}{2}\pi - \frac{1}{4}\pi\right) &= \cos\left(\frac{\pi}{2}n + O(n^{-1})\right).\end{aligned}$$

and if n is odd then

$$\begin{aligned}\cos\left(t_n\sigma_n - \frac{\alpha}{2}\pi - \frac{1}{4}\pi\right) &= \cos\left(\frac{\pi}{2}n + O(n^{-1}) + \frac{1}{2}\pi\right) \\ \cos\left(\left(\frac{\pi}{2} - t_n\right)\sigma_n - \frac{\beta}{2}\pi - \frac{1}{4}\pi\right) &= \cos\left(\frac{\pi}{2}n + O(n^{-1}) - \frac{1}{2}\pi\right).\end{aligned}$$

In particular, $\sin(t_n\sigma_n - \alpha\pi/2 - \pi/4) = O(n^{-1})$ and $\sin((\pi/2 - t_n)\sigma_n - \beta\pi/2 - \pi/4) = O(n^{-1})$, so that

$$\begin{aligned}V_{\mu_n, \alpha}(t_n) &= \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\pi^{1/2}\sigma_n^{\alpha+1/2}} \left(\cos\left(\sigma_n t_n - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \right. \\ &\quad \left. - \left((\alpha^2 - 1/4) + t_n X_\alpha(t_n) \right) \frac{1}{2\sigma_n t_n} \sin\left(\sigma_n t_n - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(\sigma_n^{-2}) \right) \\ &= \frac{2^{\alpha+1/2}\Gamma(\alpha+1)}{\pi^{1/2}\sigma_n^{\alpha+1/2}} \left(\cos\left(\sigma_n t_n - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(\sigma_n^{-2}) \right),\end{aligned}$$

and similarly

$$W_{\mu_n, \beta}(t_n) = \frac{2^{\beta+1/2}\Gamma(\beta+1)}{\pi^{1/2}\sigma_n^{\beta+1/2}} \left(\cos\left(\sigma_n \left(\frac{\pi}{2} - t_n\right) - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) + O(\sigma_n^{-2}) \right).$$

Thus, by (11) and the above computations,

$$\begin{aligned}\frac{d_n}{c_n} &= \frac{V_{\mu_n, \alpha}(t_n)}{W_{\mu_n, \beta}(t_n)} = \frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\Gamma(\beta+1)} \frac{\cos\left(t_n\sigma_n - \frac{\alpha}{2}\pi - \frac{1}{4}\pi\right)}{\cos\left(\left(\frac{\pi}{2} - t_n\right)\sigma_n - \frac{\beta}{2}\pi - \frac{1}{4}\pi\right)} \sigma_n^{\beta-\alpha} + O(\sigma_n^{\beta-\alpha-2}) \\ &= (-1)^n \frac{2^{\alpha-\beta}\Gamma(\alpha+1)}{\Gamma(\beta+1)} \sigma_n^{\beta-\alpha} (1 + O(\sigma_n^{-2})).\end{aligned}$$

□

Lemma 14. *The following estimates hold for $n \rightarrow +\infty$*

$$\begin{aligned}c_n &= \frac{\sigma_n^{\alpha+1/2}}{2^{\alpha-1/2}\Gamma(\alpha+1)} \left(1 + O\left(\frac{1}{n^2}\right) \right) \\ d_n &= (-1)^n \frac{\sigma_n^{\beta+1/2}}{2^{\beta-1/2}\Gamma(\beta+1)} \left(1 + O\left(\frac{1}{n^2}\right) \right).\end{aligned}$$

Proof. We know that

$$1 = \int_0^{\pi/2} |u_n(t)|^2 dt = c_n^2 \left(\int_0^{\pi/4} |V_{\mu_n, \alpha}(t)|^2 dt + \int_{\pi/4}^{\pi/2} \frac{d_n^2}{c_n^2} |W_{\mu_n, \beta}(t)|^2 dt \right). \quad (12)$$

Observe that for n big, using Theorem 8 and the well known boundedness of the function $\sqrt{x}J_\alpha(x)$ for $x > 0$ and $\alpha > -1/2$,

$$\begin{aligned} \int_0^{\pi/4} |V_{\mu_n, \alpha}(t)|^2 dt &= \left(\frac{2^\alpha \Gamma(\alpha+1)}{\sigma_n^\alpha} \right)^2 \int_0^{\pi/4} \left(t^{1/2} J_\alpha(\sigma_n t) - \frac{1}{2} t^{1/2} X_0(t) \frac{J_{\alpha+1}(\sigma_n t)}{\sigma_n} + O\left(\frac{t^2}{\sigma_n^{5/2}}\right) \right)^2 dt \\ &= \left(\frac{2^\alpha \Gamma(\alpha+1)}{\sigma_n^\alpha} \right)^2 \int_0^{\pi/4} t J_\alpha^2(\sigma_n t) dt \\ &\quad - \left(\frac{2^\alpha \Gamma(\alpha+1)}{\sigma_n^\alpha} \right)^2 \frac{1}{\sigma_n} \int_{\sigma_n^{-1}}^{\pi/4} X_0(t) J_{\alpha+1}(\sigma_n t) J_\alpha(\sigma_n t) t dt + O\left(\frac{1}{\sigma_n^{2\alpha+3}}\right). \end{aligned} \quad (13)$$

Notice that

$$\int_{\sigma_n^{-1}}^{\pi/4} X_0(t) J_{\alpha+1}(\sigma_n t) J_\alpha(\sigma_n t) t dt = O(\sigma_n^{-2}).$$

Indeed, by the asymptotic expansion of Bessel functions (see [25, page 199])

$$\begin{aligned} &\int_{\sigma_n^{-1}}^{\pi/4} X_0(t) J_{\alpha+1}(\sigma_n t) J_\alpha(\sigma_n t) t dt \\ &= \int_{\sigma_n^{-1}}^{\pi/4} X_0(t) \left(\frac{2}{\pi \sigma_n t} \right) \left(\sin\left(\sigma_n t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O((\sigma_n t)^{-1}) \right) \left(\cos\left(\sigma_n t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O((\sigma_n t)^{-1}) \right) t dt \\ &= \int_{\sigma_n^{-1}}^{\pi/4} X_0(t) \left(\frac{2}{\pi \sigma_n t} \right) \left(-\frac{1}{2} \cos(2\sigma_n t - \alpha\pi) + O(|\sigma_n t|^{-1}) \right) t dt. \end{aligned}$$

Since $X_0(t)/t \in C^2(-\pi/2, \pi/2)$, integration by parts gives that the above integral is $O(\sigma_n^{-2})$. Thus, by the above, by formula (5.14.5) in [14], and by the asymptotic expansions of J_α and J'_α (see [1, page 364])

$$\begin{aligned} \int_0^{\pi/4} |V_{\mu_n, \alpha}(t)|^2 dt &= \frac{2^{2\alpha} \Gamma^2(\alpha+1) \pi^2}{\sigma_n^{2\alpha}} \frac{\pi^2}{32} \left(J_\alpha^2\left(\sigma_n \frac{\pi}{4}\right) + J_\alpha^2\left(\sigma_n \frac{\pi}{4}\right) \right) + O\left(\frac{1}{\sigma_n^{2\alpha+3}}\right) \\ &= \frac{2^{2\alpha-2} \Gamma^2(\alpha+1)}{\sigma_n^{2\alpha+1}} \left(1 + \frac{2(-1)^n}{\pi \sigma_n} \sin\left((\beta-\alpha)\frac{\pi}{2}\right) \right) + O\left(\frac{1}{\sigma_n^{2\alpha+3}}\right), \end{aligned}$$

Similarly,

$$\int_{\pi/4}^{\pi/2} |W_{\mu_n, \beta}(t)|^2 dt = \frac{2^{2\beta-2} \Gamma^2(\beta+1)}{\sigma_n^{2\beta+1}} \left(1 + \frac{2(-1)^n}{\pi \sigma_n} \sin\left((\alpha-\beta)\frac{\pi}{2}\right) \right) + O\left(\frac{1}{\sigma_n^{2\beta+3}}\right). \quad (14)$$

Plugging (13) and (14) in (12) along with Lemma 13 allows us to deduce the expansion of c_n , and again Lemma 13 gives d_n . \square

Lemma 15. *Uniformly in $t \in (0, \pi/4 + \varepsilon)$ and n sufficiently large,*

$$u_n(t) = \sqrt{2} \left((\sigma_n t)^{1/2} J_\alpha(\sigma_n t) - \frac{1}{2} X_0(t) (\sigma_n t)^{1/2} \frac{J_{\alpha+1}(\sigma_n t)}{\sigma_n} \right) + O\left(\frac{t^2 \min(1, \sigma_n t)^{\alpha+5/2}}{\sigma_n^2} \right).$$

When $\alpha = 0$ the remainder has to be multiplied by $\log(2/\min(1, \sigma_n t))$. Uniformly in $t \in (\pi/4 - \varepsilon, \pi/2)$, and n sufficiently large,

$$\begin{aligned} u_n(t) &= (-1)^n \sqrt{2} \left((\sigma_n (\pi/2 - t))^{1/2} J_\beta(\sigma_n (\pi/2 - t)) \right. \\ &\quad \left. - \frac{1}{2} X_1(\pi/2 - t) (\sigma_n (\pi/2 - t))^{1/2} \frac{J_{\beta+1}(\sigma_n (\pi/2 - t))}{\sigma_n} \right) \\ &\quad + O\left(\frac{(\pi/2 - t)^2 \min(1, \sigma_n (\pi/2 - t))^{\beta+5/2}}{\sigma_n^2} \right). \end{aligned}$$

When $\beta = 0$ the remainder has to be multiplied by $\log(2/\min(1, \sigma_n (\pi/2 - t)))$. In particular, the eigenfunctions u_n are uniformly bounded on $(0, \pi/2)$.

Proof. The expansion in the left subinterval follows from the identity $u_n(t) = c_n V_{\mu_n, \alpha}(t)$, along with the expansions of c_n in Lemma 14 and of $V_{\mu_n, \alpha}(t)$ from Theorem 8, and similarly for the right subinterval. \square

We can now write the expansion for u_n away from the endpoints in terms of sines and cosines.

Lemma 16. *We have*

$$\begin{aligned} u_n(t) &= \frac{2}{\sqrt{\pi}} \cos\left(\sigma_n t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \\ &\quad - \frac{2}{\sqrt{\pi}} \left(\alpha^2 - 1/4 + tX_0(t)\right) \frac{1}{2\sigma_n t} \sin\left(\sigma_n t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left((\sigma_n t)^{-2}\right), \end{aligned}$$

uniformly for $t \in [\sigma_n^{-1}, \pi/4 + \varepsilon)$ and n sufficiently big, and

$$\begin{aligned} u_n(t) &= (-1)^n \frac{2}{\sqrt{\pi}} \cos\left(\sigma_n \left(\frac{\pi}{2} - t\right) - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) \\ &\quad - (-1)^n \frac{2}{\sqrt{\pi}} \left(\beta^2 - 1/4 + \left(\frac{\pi}{2} - t\right) X_1\left(\frac{\pi}{2} - t\right)\right) \frac{1}{2\sigma_n \left(\frac{\pi}{2} - t\right)} \\ &\quad \times \sin\left(\sigma_n \left(\frac{\pi}{2} - t\right) - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) + O\left(\left(\sigma_n \left(\frac{\pi}{2} - t\right)\right)^{-2}\right), \end{aligned}$$

uniformly for $t \in (\pi/4 - \varepsilon, \pi/2 - \sigma_n^{-1}]$ and n sufficiently big.

Proof. One only has to use the expansions of $V_{\mu_n, \alpha}(t)$ and c_n in Lemma 12 and Lemma 14. \square

3.2. More asymptotics

In this section will prove second order asymptotics for the eigenvalues $\{\sigma_n\}$. We prove it along the same lines as in Theorem 2_J in [11].

Lemma 17. *We have*

$$X_0\left(\frac{\pi}{4}\right) + X_1\left(\frac{\pi}{4}\right) = \left(\alpha^2 + \beta^2 - \frac{1}{2}\right)\left(\frac{\pi}{2} - \frac{4}{\pi}\right) + \int_0^{\frac{\pi}{2}} \chi(t) dt.$$

Proof. This is a simple exercise. Recall (Theorem 8) that $X_i(t) = \int_0^t \eta_i(s) ds$, $i = 0, 1$. Using the expression of η_i it is enough to look at

$$\begin{aligned} & \int_0^{\frac{\pi}{4}} (\eta_0(t) + \eta_1(t)) dt \\ &= \int_0^{\frac{\pi}{4}} \left(-\left(\alpha^2 + \beta^2 - \frac{1}{2}\right) \left(\cot^2 t - \frac{1}{t^2} + \tan^2 t \right) + \chi(t) + \chi\left(\frac{\pi}{2} - t\right) \right) dt \\ &= -\left(\alpha^2 + \beta^2 - \frac{1}{2}\right) \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^{\frac{\pi}{4}} \left((\cot^2 t + 1) - \frac{1}{t^2} + (\tan^2 t + 1) - 2 \right) dt + \int_0^{\frac{\pi}{2}} \chi(t) dt \\ &= -\left(\alpha^2 + \beta^2 - \frac{1}{2}\right) \lim_{\varepsilon \rightarrow 0^+} \left[-\cot t + \frac{1}{t} + \tan t - 2t \right]_{\varepsilon}^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{2}} \chi(t) dt \\ &= \left(\alpha^2 + \beta^2 - \frac{1}{2}\right)\left(\frac{\pi}{2} - \frac{4}{\pi}\right) + \int_0^{\frac{\pi}{2}} \chi(t) dt. \end{aligned}$$

□

We prove our main estimates now.

Lemma 18. *For $n \rightarrow +\infty$*

$$\sigma_n = 2n + 1 + \alpha + \beta - \frac{\Theta}{4n} + O\left(\frac{1}{n^2}\right),$$

where

$$\Theta = \alpha^2 + \beta^2 - 1/2 + \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \chi(t) dt$$

Proof. We follow the lines of the proof of Theorem 2_l in [11]. Assume first that

$$\sin\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \text{ and } \sin\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right)$$

are both far from zero, that is $2n \pm (\alpha - \beta)$ is far from a multiple of 4. Then replace t with $\pi/4$ in both expansions of u_n in the above Lemma 16.

$$\begin{aligned} & \frac{\sqrt{\pi}}{2} u_n\left(\frac{\pi}{4}\right) \\ &= \cos\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) - \left(\alpha^2 - \frac{1}{4} + \frac{\pi}{4} X_0\left(\frac{\pi}{4}\right)\right) \frac{2}{\sigma_n \pi} \sin\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \\ & \quad + \sin\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \mathcal{R}_1(n) \\ &= (-1)^n \cos\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) - (-1)^n \left(\beta^2 - \frac{1}{4} + \frac{\pi}{4} X_1\left(\frac{\pi}{4}\right)\right) \frac{2}{\sigma_n \pi} \sin\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) \quad (15) \\ & \quad + (-1)^n \sin\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) \mathcal{R}_2(n), \end{aligned}$$

where $\mathcal{R}_i(n) = O(\sigma_n^{-2})$ as $n \rightarrow +\infty$, for $i = 1, 2$. If we set

$$\begin{aligned}x &= \sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4} \\y &= \sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4} + (1 - (-1)^n) \frac{\pi}{2} \\C &= -\left(\alpha^2 - \frac{1}{4} + \frac{\pi}{4} X_0\left(\frac{\pi}{4}\right)\right) \frac{2}{\pi} + \sigma_n \mathcal{R}_1(n) \\D &= -\left(\beta^2 - \frac{1}{4} + \frac{\pi}{4} X_1\left(\frac{\pi}{4}\right)\right) \frac{2}{\pi} + \sigma_n \mathcal{R}_2(n) \\\cos \gamma &= \frac{C}{\sqrt{C^2 + D^2}} \\\sin \gamma &= \frac{D}{\sqrt{C^2 + D^2}},\end{aligned}$$

(if $C = D = 0$ then just set $\gamma = 0$). The identity (15) can be written as

$$\cos x + \frac{C}{\sigma_n} \sin x = (-1)^n \left(\cos\left(y - (1 - (-1)^n) \frac{\pi}{2}\right) + \frac{D}{\sigma_n} \sin\left(y - (1 - (-1)^n) \frac{\pi}{2}\right) \right)$$

or equivalently

$$\cos x + \frac{C}{\sigma_n} \sin x = \cos y + \frac{D}{\sigma_n} \sin y$$

or equivalently

$$\cos x - \cos y + \frac{\sqrt{C^2 + D^2}}{\sigma_n} (\cos \gamma \sin x - \sin \gamma \sin y) = 0 \quad (16)$$

By classical trigonometric identities

$$\cos \gamma \sin x - \sin \gamma \sin y = \frac{\sqrt{2}}{2} \left(F \sin\left(\frac{x+y}{2}\right) + E \cos\left(\frac{x+y}{2}\right) \right),$$

where we have set

$$\begin{aligned}F &= \cos\left(\frac{x-y+2\gamma+\pi/2}{2}\right) + \sin\left(\frac{x-y-2\gamma+\pi/2}{2}\right) \\E &= \sin\left(\frac{x-y-2\gamma+\pi/2}{2}\right) - \cos\left(\frac{x-y+2\gamma+\pi/2}{2}\right).\end{aligned}$$

Set

$$G = -2 \sin\left(\frac{x-y}{2}\right).$$

Notice that since $2n - (\alpha - \beta)$ is far from a multiple of 4, then $(x-y)/2 = (\beta - \alpha)\pi/4 - (1 - (-1)^n)\pi/4$ cannot be a multiple of π , so that $G \neq 0$. Thus (16) becomes

$$\begin{aligned}\cos(x) - \cos(y) + \frac{\sqrt{C^2 + D^2}}{\sigma_n} \frac{\sqrt{2}}{2} \left(F \sin\left(\frac{x+y}{2}\right) + E \cos\left(\frac{x+y}{2}\right) \right) &= 0 \\\sin\left(\frac{x+y}{2}\right) + \frac{\sqrt{2}E \sqrt{C^2 + D^2}}{2G\sigma_n + \sqrt{2}F \sqrt{C^2 + D^2}} \cos\left(\frac{x+y}{2}\right) &= 0.\end{aligned}$$

Call $\tan \theta = \frac{\sqrt{2E}\sqrt{C^2+D^2}}{2G\sigma_n + \sqrt{2F}\sqrt{C^2+D^2}}$, with $\theta \in (-\pi/2, \pi/2)$. Then we can rewrite the above equation as

$$\frac{1}{\cos \theta} \left(\cos \theta \sin\left(\frac{x+y}{2}\right) + \sin \theta \cos\left(\frac{x+y}{2}\right) \right) = 0$$

$$\sin\left(\frac{x+y}{2} + \theta\right) = 0.$$

This implies that for some integer k_n

$$\frac{x+y}{2} + \theta = k_n\pi,$$

but since for $\theta \rightarrow 0$,

$$\theta = \tan \theta + O(\tan^3 \theta)$$

then

$$\frac{x+y}{2} = k_n\pi - \tan \theta + O(\tan^3 \theta)$$

$$\sigma_n \frac{\pi}{4} = \frac{(\alpha + \beta)\pi}{4} + \frac{\pi}{4} - (1 - (-1)^n) \frac{\pi}{4} + k_n\pi - \frac{\sqrt{2E}\sqrt{C^2+D^2}}{2G\sigma_n + \sqrt{2F}\sqrt{C^2+D^2}} + O(\sigma_n^{-3})$$

$$\sigma_n = \alpha + \beta + 1 - (1 - (-1)^n) + 4k_n - \frac{\sqrt{2E}\sqrt{C^2+D^2}}{\pi G n} + O(n^{-2}).$$

Now observe that, again by trigonometric identities,

$$\frac{\sqrt{2E}\sqrt{C^2+D^2}}{\pi G} = -\frac{C+D}{\pi} = \frac{1}{4} \left(\alpha^2 + \beta^2 - 1/2 + \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \chi(t) dt \right) + O(\sigma_n^{-1}).$$

This gives

$$\sigma_n = \alpha + \beta + 1 - (1 - (-1)^n) + 4k_n - \frac{\Theta}{4n} + O(n^{-2}),$$

which compared with the known asymptotic $\sigma_n = \alpha + \beta + 1 + 2n + O(n^{-1})$ gives

$$k_n = \begin{cases} n/2 & \text{if } n \text{ even} \\ (n+1)/2 & \text{if } n \text{ odd.} \end{cases}$$

Finally, if we assume that $2n \pm (\alpha - \beta)$ is close to a multiple of 4 then both

$$\sin\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \text{ and } \sin\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right)$$

are close to zero. Then replace t with $\pi/4$ in both expansions of u_n in Lemma 16, but this time in (15) we multiply $O(|\sigma_n|^{-2})$ by $\cos\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)$ and by $\cos\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right)$ rather than respectively by $\sin\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)$ and by $\sin\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right)$. That is we write the identity

$$\begin{aligned} & \cos\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) - \left(\alpha^2 - \frac{1}{4} + \frac{\pi}{4} X_0\left(\frac{\pi}{4}\right)\right) \frac{2}{\sigma_n \pi} \sin\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \\ & + \cos\left(\sigma_n \frac{\pi}{4} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) O(\sigma_n^{-2}) \\ = & (-1)^n \cos\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) - (-1)^n \left(\beta^2 - \frac{1}{4} + \frac{\pi}{4} X_1\left(\frac{\pi}{4}\right)\right) \frac{2}{\sigma_n \pi} \sin\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) \\ & + (-1)^n \cos\left(\sigma_n \frac{\pi}{4} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) O(\sigma_n^{-2}), \end{aligned}$$

and conclude the proof as before. \square

Using the second order approximations of the eigenvalues, we have the following asymptotic expansion of $\{u_n\}$.

Lemma 19. For n sufficiently big,

$$u_n(t) = \frac{2}{\sqrt{\pi}} \cos((2n + \nu)t - \lambda) - \left(\frac{\alpha^2 - 1/4 + tX_0(t) - \Theta t^2}{4\sqrt{\pi}} \right) \frac{2}{nt} \sin((2n + \nu)t - \lambda) + O((nt)^{-2})$$

uniformly for $t \in [n^{-1}, \pi/4 + \varepsilon]$, where

$$\begin{aligned} \nu &= 1 + \alpha + \beta \\ \lambda &= \frac{\alpha\pi}{2} + \frac{\pi}{4}. \end{aligned}$$

Similarly,

$$\begin{aligned} u_n(t) &= (-1)^n \frac{2}{\sqrt{\pi}} \cos\left((2n + \nu)\left(\frac{\pi}{2} - t\right) - \lambda'\right) \\ &\quad - \frac{(-1)^n}{4\sqrt{\pi}} \left(\beta^2 - 1/4 + \left(\frac{\pi}{2} - t\right) X_1\left(\frac{\pi}{2} - t\right) - \Theta \left(\frac{\pi}{2} - t\right)^2 \right) \\ &\quad \times \frac{2}{n(\pi/2 - t)} \sin\left((2n + \nu)\left(\frac{\pi}{2} - t\right) - \lambda'\right) + O\left(\left(n\left(\frac{\pi}{2} - t\right)\right)^{-2}\right), \end{aligned}$$

uniformly for $t \in (\pi/4 - \varepsilon, \pi/2 - n^{-1}]$, where

$$\lambda' = \frac{\beta\pi}{2} + \frac{\pi}{4}.$$

Proof. When $t \in [\sigma_n^{-1}, \pi/4 + \varepsilon]$ then by Lemma 16 and Lemma 18,

$$\begin{aligned} u_n(t) &= \frac{2}{\sqrt{\pi}} \cos\left(\sigma_n t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \\ &\quad - \frac{2}{\sqrt{\pi}} \left(\alpha^2 - 1/4 + tX_0(t) \right) \frac{1}{2\sigma_n t} \sin\left(\sigma_n t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O((\sigma_n t)^{-2}) \\ &= \frac{2}{\sqrt{\pi}} \cos\left(\left(2n + \nu - \frac{\Theta}{4n} + O(n^{-2})\right)t - \lambda\right) \\ &\quad - \frac{2}{\sqrt{\pi}} \frac{\alpha^2 - 1/4 + tX_0(t)}{4} \frac{1}{nt} \sin\left(\left(2n + \nu - \frac{\Theta}{4n} + O(n^{-2})\right)t - \lambda\right) + O((nt)^{-2}) \\ &= \frac{2}{\sqrt{\pi}} \cos((2n + \nu)t - \lambda) \\ &\quad - \left(\frac{\alpha^2 - 1/4 + tX_0(t) - \Theta t^2}{4\sqrt{\pi}} \right) \frac{2}{nt} \sin((2n + \nu)t - \lambda) + O((nt)^{-2}), \end{aligned}$$

where we have used that

$$\cos\left(\left(2n + \nu - \frac{\Theta}{4n} + O(n^{-2})\right)t - \lambda\right) = \cos((2n + \nu)t - \lambda) + \sin((2n + \nu)t - \lambda) \frac{\Theta t}{4n} + O((nt)^{-2}).$$

We proceed in a similar way for the interval $(\pi/4 - \varepsilon, \pi/2 - \sigma_n^{-1}]$. \square

3.3. Differences

Define $\Delta(u_n) = u_n - u_{n+1}$. Using the asymptotics of the eigenvalues in the last subsection we prove the following estimate for $\Delta(u_n)$:

Lemma 20. *There exist four bounded functions $Z_j(t)$, for $j = 1, \dots, 4$, such that*

$$\Delta u_n(t) = t(Z_1(t) \cos(2nt) + Z_2(t) \sin(2nt)) + \frac{1}{n}(Z_3(t) \cos(2nt) + Z_4(t) \sin(2nt)) + O\left(\frac{1}{n^2 t}\right)$$

uniformly for $t \in [n^{-1}, \pi/4 + \varepsilon]$ and n sufficiently big.

A similar expansion also holds in $(\pi/4 - \varepsilon, \pi/2 - n^{-1}]$.

Proof. Observe first that, by the asymptotic expansion for σ_n in Lemma 18, for any constant C , any expression of the form

$$\begin{aligned} & \sin(\sigma_{n+1}t + C), \quad \cos(\sigma_{n+1}t + C), \\ & \sin\left(\frac{\sigma_n + \sigma_{n+1}}{2}t + C\right), \quad \cos\left(\frac{\sigma_n + \sigma_{n+1}}{2}t + C\right) \end{aligned}$$

multiplied by a bounded function, and by t or by n^{-1} , fits in the desired formula. Plugging the expression of u_n from Lemma 15 in $\Delta(u_n)$ we get

$$\begin{aligned} \frac{1}{\sqrt{2}}\Delta(u_n)(t) &= (\sigma_n t)^{1/2} J_\alpha(\sigma_n t) - (\sigma_{n+1} t)^{1/2} J_\alpha(\sigma_{n+1} t) \\ &- \frac{1}{2} X_0(t) (\sigma_n t)^{1/2} \frac{J_{\alpha+1}(\sigma_n t)}{\sigma_n} + \frac{1}{2} X_0(t) (\sigma_{n+1} t)^{1/2} \frac{J_{\alpha+1}(\sigma_{n+1} t)}{\sigma_{n+1}} + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Now,

$$\begin{aligned} & (\sigma_n t)^{1/2} J_\alpha(\sigma_n t) - (\sigma_{n+1} t)^{1/2} J_\alpha(\sigma_{n+1} t) \\ &= -(\sigma_n t)^{1/2} \int_{\sigma_n t}^{\sigma_{n+1} t} J'_\alpha(s) ds + \frac{(\sigma_n - \sigma_{n+1})t}{(\sigma_n t)^{1/2} + (\sigma_{n+1} t)^{1/2}} J_\alpha(\sigma_{n+1} t) \end{aligned} \tag{17}$$

Since $\sigma_n - \sigma_{n+1} = -2 + O(n^{-2})$, after doing the first order asymptotics of J_α , when $t \in [\sigma_n^{-1}, \pi/4 + \varepsilon]$,

$$\frac{(\sigma_n - \sigma_{n+1})t}{(\sigma_n t)^{1/2} + (\sigma_{n+1} t)^{1/2}} J_\alpha(\sigma_{n+1} t) = -\sqrt{\frac{2}{\pi}} \frac{1}{2n} (1 + O(n^{-1})) \cos\left(\sigma_{n+1} t - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{\sigma_{n+1}^2 t}\right),$$

which fits in the desired formula. On the other hand, using the second order asymptotics of J'_α (see [1, page 364]) i.e.,

$$J'_\alpha(s) = -\sqrt{\frac{2}{\pi s}} \left(\sin\left(s - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) + \frac{4\alpha^2 + 3}{8s} \cos\left(s - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) \right) + O(s^{-5/2})$$

we get

$$\begin{aligned} & \int_{\sigma_n t}^{\sigma_{n+1} t} J'_\alpha(s) ds = -\int_{\sigma_n t}^{\sigma_{n+1} t} \sqrt{\frac{2}{\pi s}} \left(\sin\left(s - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) \right) ds \\ & - \int_{\sigma_n t}^{\sigma_{n+1} t} \sqrt{\frac{2}{\pi s}} \left(\frac{4\alpha^2 + 3}{8s} \cos\left(s - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) \right) ds + (\sigma_n - \sigma_{n+1}) O(\sigma_n^{-5/2} t^{-3/2}). \end{aligned}$$

Let us put $\vartheta := \frac{\pi}{2}\alpha + \frac{\pi}{4}$. Integration by parts yields

$$\begin{aligned} & -\sqrt{\frac{2}{\pi}} \int_{\sigma_n t}^{\sigma_{n+1} t} s^{-1/2} \sin(s - \vartheta) ds - \sqrt{\frac{2}{\pi}} \frac{4\alpha^2 + 3}{8} \int_{\sigma_n t}^{\sigma_{n+1} t} s^{-3/2} \cos(s - \vartheta) ds \\ &= \sqrt{\frac{2}{\pi}} \left[s^{-1/2} \cos(s - \vartheta) \right]_{\sigma_n t}^{\sigma_{n+1} t} + \sqrt{\frac{2}{\pi}} \frac{1 - 4\alpha^2}{8} \left[s^{-3/2} \sin(s - \vartheta) \right]_{\sigma_n t}^{\sigma_{n+1} t} \\ &+ \sqrt{\frac{2}{\pi}} \left(\frac{1 - 4\alpha^2}{8} \right) \frac{3}{2} \int_{\sigma_n t}^{\sigma_{n+1} t} s^{-5/2} \sin(s - \vartheta) ds \end{aligned}$$

In order to have the desired asymptotic expansion for

$$-(\sigma_n t)^{1/2} \int_{\sigma_n t}^{\sigma_{n+1} t} J'_\alpha(s) ds,$$

we first look at

$$\sqrt{\frac{2}{\pi}} \left[s^{-1/2} \cos(s - \vartheta) \right]_{\sigma_n t}^{\sigma_{n+1} t}.$$

Adding and subtracting $-\sqrt{2/\pi} (\sigma_n t)^{-1/2} \cos(\sigma_{n+1} t - \vartheta)$ in the above expression and simplifying further, we get

$$\begin{aligned} & \sqrt{\frac{2}{\pi}} (\sigma_{n+1} t)^{-1/2} \cos(\sigma_{n+1} t - \vartheta) - \sqrt{\frac{2}{\pi}} (\sigma_n t)^{-1/2} \cos(\sigma_n t - \vartheta) \\ &= \sqrt{\frac{2}{\pi}} \frac{2}{(\sigma_n t)^{1/2}} \sin\left(\frac{(\sigma_n + \sigma_{n+1})t}{2} - \vartheta\right) \sin\left(\frac{(\sigma_n - \sigma_{n+1})t}{2}\right) \\ &- \sqrt{\frac{2}{\pi}} \frac{\sigma_{n+1} - \sigma_n}{(\sigma_n t)^{1/2} \sigma_{n+1}^{1/2}} \frac{\cos(\sigma_{n+1} t - \vartheta)}{(\sigma_n^{1/2} + \sigma_{n+1}^{1/2})}. \end{aligned}$$

Similarly the second term $\sqrt{\frac{2}{\pi}} \frac{1-4\alpha^2}{8} \left[s^{-3/2} \sin(s - \vartheta) \right]_{\sigma_n t}^{\sigma_{n+1} t}$ can be written in the form

$$\sqrt{\frac{2}{\pi}} \frac{1 - 4\alpha^2}{8 (\sigma_n t)^{3/2}} 2 \cos\left(\frac{(\sigma_{n+1} + \sigma_n)t}{2} - \vartheta\right) \sin\left(\frac{(\sigma_n - \sigma_{n+1})t}{2}\right) + O\left(\frac{1}{\sigma_n^{5/2} t^{3/2}}\right)$$

For the third term it is easy to check that

$$\int_{\sigma_n t}^{\sigma_{n+1} t} s^{-5/2} \sin(s - \vartheta) ds = O\left(\sigma_n^{-5/2} t^{-3/2}\right).$$

Using once again $(\sigma_n - \sigma_{n+1}) = -2 + O(n^{-2})$, we deduce that

$$2 \sqrt{\frac{2}{\pi}} t^{-1} \sin\left(\frac{(\sigma_n - \sigma_{n+1})t}{2}\right) = -2 \sqrt{\frac{2}{\pi}} t^{-1} \sin t + O(n^{-2}).$$

If we define the bounded function $Z(t) := -2 \sqrt{\frac{2}{\pi}} t^{-1} \sin t$, then combining the above asymptotic

expansions we can finally write

$$\begin{aligned}
& -(\sigma_n t)^{1/2} \int_{\sigma_n t}^{\sigma_{n+1} t} J'_\alpha(s) ds \\
&= -\sin\left(\frac{(\sigma_n + \sigma_{n+1})t}{2} - \vartheta\right) 2\sqrt{\frac{2}{\pi}} \sin\left(\frac{(\sigma_n - \sigma_{n+1})t}{2}\right) + \sqrt{\frac{2}{\pi}} \frac{\sigma_{n+1} - \sigma_n}{\sigma_{n+1}^{1/2}} \frac{\cos(\sigma_{n+1}t - \vartheta)}{(\sigma_n^{1/2} + \sigma_{n+1}^{1/2})} \\
& - \frac{1 - 4\alpha^2}{8} \frac{1}{\sigma_n} \cos\left(\frac{(\sigma_{n+1} + \sigma_n)t}{2} - \vartheta\right) 2\sqrt{\frac{2}{\pi}} \frac{1}{t} \sin\left(\frac{(\sigma_n - \sigma_{n+1})t}{2}\right) + O\left(\frac{1}{\sigma_n^2 t}\right) \\
&= -t \sin\left(\frac{(\sigma_n + \sigma_{n+1})t}{2} - \vartheta\right) Z(t) + \sqrt{\frac{2}{\pi}} \frac{1}{2n} \cos(\sigma_{n+1}t - \vartheta) \\
& - \frac{1 - 4\alpha^2}{8} \frac{1}{2n} \cos\left(\frac{(\sigma_{n+1} + \sigma_n)t}{2} - \vartheta\right) Z(t) + O\left(\frac{1}{\sigma_n^2 t}\right).
\end{aligned}$$

This part also fits in the desired formula.

Let us handle

$$-\frac{1}{2} X_0(t) (\sigma_n t)^{1/2} \frac{J_{\alpha+1}(\sigma_n t)}{\sigma_n} + \frac{1}{2} X_0(t) (\sigma_{n+1} t)^{1/2} \frac{J_{\alpha+1}(\sigma_{n+1} t)}{\sigma_{n+1}}$$

now. We rewrite it as

$$-\frac{1}{2} X_0(t) \left((\sigma_n t)^{1/2} \frac{J_{\alpha+1}(\sigma_n t)}{\sigma_n} - (\sigma_{n+1} t)^{1/2} \frac{J_{\alpha+1}(\sigma_{n+1} t)}{\sigma_{n+1}} \right).$$

Note that the expression in the bracket above is similar to the one dealt with before in the beginning of the proof of this lemma, formula (17), except for the extra factor of order n in the denominator. After adding and subtracting $(\sigma_n t)^{1/2} \frac{J_{\alpha+1}(\sigma_{n+1} t)}{\sigma_{n+1}}$ inside the bracket above we proceed in the same way as before. Due to the extra factor of $O(n^{-1})$ we just need to use the first order approximation of the derivative of the Bessel function. We do not go into further details. \square

Lemma 21. For all $t \in (0, n^{-1}]$,

$$\Delta u_n(t) = -\frac{2\alpha + 1}{2n} u_n(t) + O(n^{-2} + t).$$

In particular,

$$\Delta u_n(t) = O\left((nt)^{\alpha + \frac{1}{2}} n^{-1} + n^{-2} + t\right),$$

so that condition (P4) holds with $\tau = \min(1, \alpha + 1/2)$. A similar expansion holds in $[\pi/2 - n^{-1}, \pi/2)$.

Proof. By Lemma 15, for $t \leq n^{-1}$ we have

$$\begin{aligned}
u_n(t) &= \sqrt{2} (\sigma_n t)^{1/2} J_\alpha(\sigma_n t) - \frac{\sqrt{2}}{2} X_0(t) (\sigma_n t)^{1/2} \frac{J_{\alpha+1}(\sigma_n t)}{\sigma_n} + O\left(\frac{t^2 (\sigma_n t)^{\alpha+1/2}}{\sigma_n^2}\right) \\
u_n(t) &= \sqrt{2} (\sigma_n t)^{1/2} J_\alpha(\sigma_n t) + O\left((t\sigma_n)^{\alpha+5/2} \sigma_n^{-2}\right).
\end{aligned}$$

Notice that, while the first identity has to be adjusted with the usual logarithmic correction when $\alpha = 0$, the second works for $\alpha = 0$ too. Thus

$$\Delta u_n(t) = \sqrt{2}\Delta(\sigma_n^{1/2})t^{1/2}J_\alpha(\sigma_n t) + \sqrt{2}(\sigma_{n+1}t)^{1/2}\Delta(J_\alpha(\sigma_n t)) + O((tn)^{\alpha+5/2}n^{-2}),$$

where

$$\Delta(\sigma_n^{1/2}) = \sigma_n^{1/2} - \sigma_{n+1}^{1/2} = -\sigma_n^{1/2}\left(\frac{1}{2n} + O(n^{-2})\right).$$

It is well known that

$$J'_\alpha(s) = \begin{cases} \frac{\alpha s^{\alpha-1}}{2^\alpha \Gamma(\alpha+1)} + O(s^{\alpha+1}) & \text{for } \alpha \neq 0 \\ O(s) & \text{for } \alpha = 0, \end{cases}$$

then, for $\alpha \neq 0$

$$\begin{aligned} \Delta(J_\alpha(\sigma_n t)) &= J_\alpha(\sigma_n t) - J_\alpha(\sigma_{n+1}t) = -\int_{\sigma_n t}^{\sigma_{n+1}t} J'_\alpha(s) ds \\ &= -\frac{1}{2^\alpha \Gamma(\alpha+1)} \int_{\sigma_n t}^{\sigma_{n+1}t} \alpha s^{\alpha-1} ds + O((\sigma_n t)^{\alpha+1}t) \\ &= \frac{-2\alpha(\sigma_n t)^\alpha}{2^\alpha \Gamma(\alpha+1)2n} + O((\sigma_n t)^\alpha n^{-2}) + O((\sigma_n t)^{\alpha+1}t) \\ &= \frac{-2\alpha}{2n} J_\alpha(\sigma_n t) + O((\sigma_n t)^\alpha n^{-2}) + O((\sigma_n t)^{\alpha+1}t). \end{aligned}$$

Finally, when $nt < 1$, after using the expansions above and rearranging terms we get

$$\begin{aligned} \Delta u_n(t) &= \sqrt{2}\Delta(\sigma_n^{1/2})t^{1/2}J_\alpha(\sigma_n t) + \sqrt{2}(\sigma_{n+1}t)^{1/2}\Delta(J_\alpha(\sigma_n t)) + O((tn)^{\alpha+5/2}n^{-2}) \\ &= \sqrt{2}\left(-\sigma_n^{1/2}\left(\frac{1}{2n} + O(n^{-2})\right)\right)t^{1/2}J_\alpha(\sigma_n t) \\ &\quad + \sqrt{2}(\sigma_{n+1}t)^{1/2}\left(\frac{-2\alpha}{2n}J_\alpha(\sigma_n t) + O((\sigma_n t)^\alpha n^{-2}) + O((\sigma_n t)^{\alpha+1}t)\right) + O((tn)^{\alpha+5/2}n^{-2}) \\ &= -\frac{2\alpha+1}{2n}\sqrt{2}(\sigma_n t)^{1/2}J_\alpha(\sigma_n t) + O((nt)^{\alpha+1/2}n^{-2} + (nt)^{\alpha+5/2}t) \\ &= -\frac{2\alpha+1}{2n}u_n(t) + O(n^{-2} + t), \end{aligned}$$

by Lemma 15. The same formula holds for $\alpha = 0$. □

3.4. Pointwise convergence

We are now ready to state the main results of this section.

Theorem 22. *The eigenfunctions of $\tilde{\ell}$, $\{u_n\}_{n=0}^{+\infty}$, satisfy the properties (P1)-(P5).*

Proof. Property (P1) is contained in Lemma 15, (P2) in Lemma 19, (P3) in Lemma 20 and (P4) in Lemma 21. Property (P5) follows from the second part of the last three mentioned lemmas. □

Theorem 23. Assume that the sequences $\{r_{n,N}\}_{n=0}^{+\infty}$ satisfy (S1) and (S2) and suppose that for any compactly supported smooth function g on $(0, \pi/2)$ and for any $t \in (0, \pi/2)$ one has

$$\lim_{N \rightarrow +\infty} (T_N g(x) - Q_N g(x)) = 0.$$

Then for any $f \in L^1((0, \pi/2), dt)$ and for any $t \in (0, \pi/2)$ one has

$$\lim_{N \rightarrow +\infty} (T_N f(t) - Q_N f(t)) = 0.$$

Furthermore, if for each compactly supported smooth function g on $(0, \pi/2)$

$$\lim_{N \rightarrow +\infty} (T_N g(x) - Q_N g(x)) = 0 \tag{18}$$

uniformly on $(0, \pi/2)$, then for each set $\Gamma \subset (0, \pi/2)$ with positive distance from 0 and from $\pi/2$ and for all $f \in L^1((0, \pi/2), dt)$

$$\lim_{N \rightarrow +\infty} (T_N f(x) - Q_N f(x)) = 0$$

uniformly on Γ .

Proof. Compactly supported smooth functions are dense in $L^1((0, \pi/2), dt)$, and since the system $\{u_n\}_{n=0}^{+\infty}$ satisfies properties (P1)-(P5), we can apply Theorem 1 and deduce both the pointwise and the uniform result. \square

There is a simple class of sequences $\{r_{n,N}\}$ for which one can guarantee condition (18).

Theorem 24. Assume the sequences $\{r_{n,N}\}_{n=0}^{+\infty}$ satisfy (S1) and (S2) and there exists a number R such that for all $n \geq 0$,

$$\lim_{N \rightarrow +\infty} r_{n,N} = R.$$

Then for any $f \in L^1((0, \pi/2), dt)$ and for any $t \in (0, \pi/2)$ one has

$$\lim_{N \rightarrow +\infty} (T_N f(t) - Q_N f(t)) = 0,$$

and the convergence is uniform on all sets $\Gamma \subset (0, \pi/2)$ with positive distance from 0 and from $\pi/2$.

Proof. Let g be a compactly supported smooth function on $(0, \pi/2)$. For any positive integer k , the function $\tilde{\ell}^k g(t)$ is bounded and we have

$$\widehat{g}(n) = \int_0^{\pi/2} g(t) u_n(t) dt = \frac{1}{\mu_n^k} \int_0^{\pi/2} g(t) \tilde{\ell}^k u_n(t) dt = \frac{1}{\mu_n^k} \int_0^{\pi/2} \tilde{\ell}^k g(t) u_n(t) dt = O(n^{-2k})$$

and by the Lebesgue dominated convergence theorem,

$$\lim_{N \rightarrow +\infty} T_N g(t) = \lim_{N \rightarrow +\infty} \sum_{n=0}^{+\infty} r_{n,N} \widehat{g}(n) u_n(t) = \sum_{n=0}^{+\infty} R \widehat{g}(n) u_n(t) = Rg(t),$$

with uniform convergence on $(0, \pi/2)$, and similarly $\lim_{N \rightarrow +\infty} Q_N g(t) = Rg(t)$ uniformly on $(0, \pi/2)$. Theorem 23 now concludes the proof. \square

Thus, since for any $\theta \geq 0$ the sequences $\{r_{n,N}\}$ defined by

$$r_{n,N} = \frac{A_{N-n}^\theta}{A_N^\theta}$$

where $A_n^\theta = \binom{n+\theta}{n}$ for $n \geq 0$, and $A_n^\theta = 0$ for $n < 0$, satisfy the hypotheses of Theorem 24, classical results for Cesàro means of Fourier series with respect to the cosine basis can be restated in exactly the same form for the Cesàro means (partial sums, when $\theta = 0$), with respect to the basis $\{u_n\}$,

$$\mathcal{T}_N^\theta f(t) = \sum_{n=0}^N \frac{A_{N-n}^\theta}{A_N^\theta} \widehat{f}(n) u_n(t).$$

Here is a non exhaustive list of results of this type.

- (M. Riesz) Let $\theta > 0$, and let $f \in L^1((0, \pi/2), dt)$. Then $\mathcal{T}_N^\theta f(t) \rightarrow f(t)$ at every point of continuity t of f . The convergence is uniform on every closed set of points of continuity with positive distance from 0 and $\pi/2$.
- (Kahane-Katznelson) For any $E \subset (0, \pi/2)$ such that $|E| = 0$ there exists a continuous function $f \in L^1((0, \pi/2), dt)$ such that $\mathcal{T}_N^0 f(t)$ diverges for all $t \in E$.
- (Kolmogorov) There is a function $f \in L^1((0, \pi/2), dt)$ such that $\mathcal{T}_N^0 f(t)$ diverges everywhere.
- (Carleson-Hunt) Let $p > 1$ and let $f \in L^p((0, \pi/2), dt)$. Then $\mathcal{T}_N^0 f(t) \rightarrow f(t)$ for almost every $t \in (0, \pi/2)$.

4. Perturbed Jacobi operator in general form

In this section we consider the operator given by

$$Lv := \frac{1}{A} (Av)'$$

Here $A(t)$ is defined on $[0, \pi/2]$ by

$$A(t) = (\sin t)^{2\alpha+1} (\cos t)^{2\beta+1} B(t),$$

where $\alpha, \beta > -1/2$. Symmetry with respect to $\pi/4$ shows that without loss of generality we can assume $\beta \leq \alpha$. The function B is a π -periodic positive function in $C^4(\mathbb{R})$, even with respect to 0 and to $\pi/2$. When $B(t) \equiv 1$, the operator L corresponds to the standard Jacobi operator.

The Liouville transformation $u(t) = A(t)^{1/2} v(t)$ applied to L gives $Lv(t) = -A(t)^{-1/2} \ell u(t)$, where

$$\ell u := -u'' + \left(\left(\alpha^2 - \frac{1}{4} \right) \cot^2 t + \left(\beta^2 - \frac{1}{4} \right) \tan^2 t - \chi(t) \right) u$$

is in normal form, and

$$\chi(t) = \left(\beta + \frac{1}{2} \right) \frac{B'(t)}{B(t)} \tan t - \left(\alpha + \frac{1}{2} \right) \frac{B'(t)}{B(t)} \cot t + \frac{1}{4} \left(\frac{B'(t)}{B(t)} \right)^2 - \frac{1}{2} \frac{B''(t)}{B(t)} + 2\alpha\beta + 2\alpha + 2\beta + \frac{3}{2}.$$

Note that χ is a twice continuously differentiable function on \mathbb{R} , even with respect to 0 and $\pi/2$, and $\alpha \geq \beta > -1/2$, so that all the results of the previous section can be applied. In particular, Proposition 11 applies, so that we can consider the orthonormal basis $\{v_n(t)\}_{n=0}^{+\infty}$ on $L^2((0, \pi/2), A(t) dt)$ defined by

$$v_n(t) := A(t)^{-1/2} u_n(t).$$

It may be worth emphasizing that $\{v_n\}_{n=0}^{+\infty}$ are the eigenfunctions of the self-adjoint restriction of the operator L to the space

$$\{z \in \mathcal{D}_A : [z, A^{-1/2} V_{\mu, \alpha}]_L(0) = [z, A^{-1/2} W_{\mu, \beta}]_L(\pi/2) = 0\}$$

if $-1/2 < \beta \leq \alpha < 1$,

$$\{z \in \mathcal{D}_A : [z, A^{-1/2} W_{\mu, \beta}]_L(\pi/2) = 0\}$$

if $-1/2 < \beta < 1 \leq \alpha$, and \mathcal{D}_A if $1 \leq \beta \leq \alpha$, where

$$\begin{aligned} \mathcal{D}_A &= \{z, Az' \in AC_{\text{loc}}, z, Lz \in L^2((0, \pi/2), A(t) dt)\} \\ [z, u]_L(0) &= \lim_{t \rightarrow 0^+} (z(t) \overline{A(t)u'(t)} - A(t)z'(t) \overline{u(t)}) \\ [z, u]_L(\pi/2) &= \lim_{t \rightarrow \pi/2^-} (z(t) \overline{A(t)u'(t)} - A(t)z'(t) \overline{u(t)}). \end{aligned}$$

This follows easily from Lemma 3.2 and Corollary 3.1 in [18] and Proposition 11.

The Fourier coefficients of a function $f \in L^2((0, \pi/2), A(t) dt)$ are denoted by

$$\mathcal{F}f(n) := \int_0^{\pi/2} f(t) v_n(t) A(t) dt.$$

It is immediate to observe that $f \in L^2((0, \pi/2), A(t) dt)$ if and only if $A^{1/2}f \in L^2((0, \pi/2), dt)$, and in this case

$$\mathcal{F}f(n) = \widehat{(A^{1/2}f)}(n).$$

We want to prove an equiconvergence result for operators of the form

$$T_N^A f(t) := \sum_{n=0}^{+\infty} r_{n,N} \mathcal{F}f(n) v_n(t),$$

where $r_{n,N}$ satisfy properties (S1) and (S2).

Theorem 25. *Assume the sequences $\{r_{n,N}\}_{n=0}^{+\infty}$ satisfy (S1) and (S2) and there exists a number R such that for all $n \geq 0$,*

$$\lim_{N \rightarrow +\infty} r_{n,N} = R.$$

Then for any $f \in L^1((0, \pi/2), A^{1/2}(t) dt)$ and for any $t \in (0, \pi/2)$ one has

$$\lim_{N \rightarrow +\infty} (T_N^A f(t) - A^{-1/2}(t) Q_N(A^{1/2}f)(t)) = 0,$$

and the convergence is uniform on all sets $\Gamma \subset (0, \pi/2)$ with positive distance from 0 and from $\pi/2$.

Proof. This follows immediately from Theorem 24 and the identity

$$T_N^A f(t) = A^{-1/2}(t) T_N (A^{1/2} f)(t).$$

□

In particular, the above theorem applies to a large class of L^p spaces. Indeed, using Holder's inequality and the assumption that $\beta \leq \alpha$, we get

Proposition 26. *Let $p > (4\alpha + 4) / (2\alpha + 3)$. Then*

$$L^p((0, \pi/2), A(t) dt) \subset L^{1+\varepsilon}((0, \pi/2), A^{1/2}(t) dt),$$

for some $\varepsilon > 0$.

4.1. Pointwise convergence of partial sums

In this section we will consider the case $r_{n,N} = 1$ for $n \leq N$ and $r_{n,N} = 0$ for $n > N$. Thus, T_N^A reduces to the partial sums operator

$$T_N^A f(t) := \sum_{n=0}^N \mathcal{F} f(n) v_n(t).$$

and similarly,

$$Q_N f(t) = \frac{2}{\pi} \int_0^{\pi/2} f(y) dy + \frac{4}{\pi} \sum_{n=1}^N \int_0^{\pi/2} f(y) \cos(2ny) dy \cos(2nt).$$

Our goal here is to apply the equiconvergence result in the previous section, namely Theorem 25, in order to transfer classical results about the pointwise convergence of the partial sums $Q_N g(t)$, to the partial sum operator $T_N^A f(t)$.

The first preliminary result is the following generalization of a well known result of C. Meany for Jacobi polynomial expansions [15].

Theorem 27. *Let $p_0 = (4\alpha + 4)/(2\alpha + 3)$. There exists a function*

$$f \in L^{p_0}((0, \pi/2), A(t) dt)$$

such that $f(t) = 0$ for all $t \in [\pi/4, \pi/2)$ and such that $T_N^A f(t)$ diverges a.e. on $(0, \pi/2)$. In particular, there is no L^{p_0} localization.

Proof. The proof goes exactly as in [15]. We sketch it here for sake of completeness. Observe first that

$$\int_0^{\pi/2} |v_n(t)|^{p'_0} A(t) dt \geq \int_{1/n}^{\pi/4} |u_n(t)|^{p'_0} A(t)^{1-p_0/2} dt \geq c \log n$$

by the expansion for u_n given by (P2).

Thus, the linear functional on $L^{p_0}([0, \pi/4], A(t) dt)$

$$f \rightarrow \int_0^{\pi/4} f(t) v_n(t) A(t) dt$$

has norm greater than $c(\log n)^{1/p'_0}$. Also, for any sequence $\varepsilon_n \rightarrow 0$, the linear functional on $L^{p_0}([0, \pi/4], A(t)dt)$

$$f \rightarrow \int_0^{\pi/4} f(t) \frac{v_n(t)}{\varepsilon_n (\log n)^{1/p'_0}} A(t) dt$$

has norm greater than $c\varepsilon_n^{-1}$. By the uniform boundedness principle, there exists a function $f \in L^{p_0}([0, \pi/4], A(t)dt)$ such that

$$\limsup_{n \rightarrow +\infty} \left| \frac{\mathcal{F}f(n)}{\varepsilon_n (\log n)^{1/p'_0}} \right| = +\infty.$$

If $T_N^A f(t) = \sum_{n=0}^N \mathcal{F}f(n)v_n(t)$ converges on a set of positive measure in $[0, \pi/2]$, then $\mathcal{F}f(n)v_n(t) \mapsto 0$ on this set, and in a subset of it with positive measure, by Egoroff's theorem, the convergence is uniform. Now observe that again by (P2), for $\varepsilon \leq t \leq \pi/2 - \varepsilon$

$$v_n(t) = A(t)^{-1/2} \frac{2}{\pi} \cos((2n + \nu)t - \lambda) + O(1/n)$$

so that by Lemma 6 in [15], $\mathcal{F}f(n) \mapsto 0$. Setting $\varepsilon_n = (\log n)^{-1/(2p'_0)}$, gives the contradiction. \square

It follows that $p > p_0$ is a necessary condition for the a.e. convergence of $T_N^A f(t)$ for all $f \in L^p((0, \pi/2), A(t)dt)$. Notice that by Proposition 26, this is precisely the range of applicability of the equiconvergence result in Theorem 25, which we may therefore use to show that in fact $p > p_0$ is also a sufficient condition.

Theorem 28. *Let $p > p_0$ and let $f \in L^p((0, \pi/2), A(t)dt)$. Then $T_N^A f(t)$ converges a.e. to $f(t)$ as $N \rightarrow +\infty$.*

Proof. Since $p > p_0$, by Proposition 26, $f \in L^{1+\varepsilon}((0, \pi/2), A^{1/2}(t)dt)$, so that by Theorem 25, $T_N^A f(t)$ is equiconvergent with $A^{-1/2}(t)Q_N(A^{1/2}f)(t)$. But again since $f \in L^{1+\varepsilon}((0, \pi/2), A^{1/2}(t)dt)$, then $A^{1/2}f \in L^{1+\varepsilon}((0, \pi/2), dt)$ so that by the Carleson-Hunt theorem $Q_N(A^{1/2}f)(t)$ converges a.e. to $A^{1/2}(t)f(t)$. Thus, $A^{-1/2}(t)Q_N(A^{1/2}f)(t)$ converges a.e. to $f(t)$, and so does $T_N^A f(t)$. \square

We can also transfer to this context the classic result of Kahane-Katznelson on the divergence of partial sums of Fourier series of continuous functions.

Theorem 29. *For any $E \subset (0, \pi/2)$ such that $|E| = 0$ there exists a continuous function $f \in L^1((0, \pi/2), A^{1/2}(t)dt)$ such that $T_N^A f(t)$ diverges for all $t \in E$.*

Proof. By the Kahane-Katznelson theorem [13, page 67], there exists a function F continuous on $[0, \pi/2]$ such that $Q_N F(t)$ diverges for all $t \in E$. Thus the function $f(t) := A^{-1/2}(t)F(t)$ is continuous in $(0, \pi/2)$ and belongs to $L^1((0, \pi/2), A^{1/2}(t)dt)$. Then, by Theorem 25, $T_N^A f(t)$ is equiconvergent with $A^{-1/2}(t)Q_N(F)(t)$, and therefore diverges in E . \square

We now want to discuss briefly the Hausdorff dimension of the sets of divergence of functions with certain L^p regularity. We will focus here on the case $p = 2$, leaving the discussion of the more delicate case $p \neq 2$ for future studies.

Following Stein's book [20], we will first define the $\Lambda_\gamma^{2,2}((0, \pi/2), A(x)dx)$ spaces, i.e. certain spaces defined in terms of the L^2 modulus of continuity. Since we need to consider translations of functions, it is better to change our point of view a bit, and think of all our generic functions

$f(x)$ as π -periodic and even functions on \mathbb{R} . This also includes $A(x) = |\sin x|^{2\alpha+1} |\cos x|^{2\beta+1} B(x)$. Thus, the integral

$$\int_0^{\pi/2} f(x+t)A(x)dx$$

will perfectly make sense, as well as the L^2 norm of a translated function

$$\|f(x+t)\|_{L^2((0,\pi/2),A(x)dx)} = \left(\int_0^{\pi/2} |f(x+t)|^2 A(x)dx \right)^{1/2}.$$

Definition 30. For any $0 < \gamma < 1$ the spaces $\Lambda_\gamma^{2,2}((0,\pi/2), A(x)dx)$ consist of all functions f in $L^2((0,\pi/2), A(x)dx)$ for which the norm

$$\|f\|_{L^2((0,\pi/2),A(x)dx)} + \int_{-\pi/2}^{\pi/2} \frac{\|f(x+t) - f(x)\|_{L^2((0,\pi/2),A(x)dx)}^2}{|t|^{1+2\gamma}} dt$$

is finite.

Theorem 31. Let $0 < \gamma < 1/2$. If $f \in \Lambda_\gamma^{2,2}((0,\pi/2), A(x)dx)$ then $T_N^A f$ diverges on a set with Hausdorff dimension less than or equal to $1 - 2\gamma$.

Proof. Let us fix a small $\varepsilon > 0$, and define χ as a smooth, even, π -periodic function which equals 1 in $[\varepsilon/2, \pi/2 - \varepsilon/2]$ and equals 0 in $[-\varepsilon/3, \varepsilon/3]$ and in $[\pi/2 - \varepsilon/3, \pi/2 + \varepsilon/3]$.

By the classic L^1 localization for Fourier series, applied to $g - g\chi$ (see e.g. [26, Vol. I, Theorem 6.2, page 52]), for all $g \in L^1((0,\pi/2), dx)$, $Q_N(g\chi)(x)$ is uniformly equiconvergent with $Q_N g(x)$ in $[\varepsilon, \pi/2 - \varepsilon]$.

Since $f \in L^2((0,\pi/2), A(x)dx)$ and $2 > p_0$, then Proposition 26 and Theorem 25 imply that for all $x \in [\varepsilon, \pi/2 - \varepsilon]$ $T_N^A f(x)$ is uniformly equiconvergent with $A^{-1/2}(x)Q_N(A^{1/2}f)(x)$, and therefore with $A^{-1/2}(x)Q_N(A^{1/2}\chi f)(x)$.

Let us now show that $A^{1/2}\chi f \in \Lambda_\gamma^{2,2}([0,\pi/2], dx)$, defined as the set of all (even, π -periodic) functions F in $L^2([0,\pi/2], dx)$ for which the norm

$$\|F\|_{L^2([0,\pi/2],dx)} + \int_{-\pi/2}^{\pi/2} \frac{\|F(x+t) - F(x)\|_{L^2([0,\pi/2],dx)}^2}{|t|^{1+2\gamma}} dt$$

is finite. Indeed, observe first that since $f \in L^2((0,\pi/2), A(x)dx)$, then obviously $A^{1/2}\chi f \in L^2([0,\pi/2], dx)$ so that all we have to show is the boundedness of the integral

$$\int_{-\varepsilon/100}^{\varepsilon/100} \frac{\|A^{1/2}\chi f(x+t) - A^{1/2}\chi f(x)\|_{L^2([0,\pi/2],dx)}^2}{|t|^{1+2\gamma}} dt.$$

Set $\tilde{\chi}$ as a smooth, even, π -periodic function which equals 1 in $[\varepsilon/10, \pi/2 - \varepsilon/10]$ and equals 0 in $[-\varepsilon/20, \varepsilon/20]$ and in $[\pi/2 - \varepsilon/20, \pi/2 + \varepsilon/20]$.

For all x in the intervals $[0, \varepsilon/10]$ or in $[\pi/2 - \varepsilon/10, \pi/2]$ and for all $|t| < \varepsilon/100$, we have

$A^{1/2}\chi f(x+t) - A^{1/2}\chi f(x) = 0$. Thus,

$$\begin{aligned}
& \int_{-\varepsilon/100}^{\varepsilon/100} \frac{\|A^{1/2}\chi f(x+t) - A^{1/2}\chi f(x)\|_{L^2([0,\pi/2],dx)}^2}{|t|^{1+2\gamma}} dt \\
&= \int_{-\varepsilon/100}^{\varepsilon/100} \frac{\|A^{1/2}\chi f(x+t) - A^{1/2}\chi f(x)\|_{L^2([0,\pi/2],\bar{\chi}(x)dx)}^2}{|t|^{1+2\gamma}} dt \\
&\leq 2 \int_{-\varepsilon/100}^{\varepsilon/100} \frac{\|(A^{1/2}(x+t)\chi(x+t) - A^{1/2}(x)\chi(x))f(x+t)\|_{L^2([0,\pi/2],\bar{\chi}(x)dx)}^2}{|t|^{1+2\gamma}} dt \\
&\quad + 2 \int_{-\varepsilon/100}^{\varepsilon/100} \frac{\|A^{1/2}(x)\chi(x)(f(x+t) - f(x))\|_{L^2([0,\pi/2],\bar{\chi}(x)dx)}^2}{|t|^{1+2\gamma}} dt \\
&\leq 2 \int_{-\varepsilon/100}^{\varepsilon/100} \frac{\|(A^{1/2}\chi)'\|_{\infty}^2 |t|^2 \|f(x)\|_{L^2([0,\pi/2],\bar{\chi}(x-t)dx)}^2}{|t|^{1+2\gamma}} dt + 2 \int_{-\varepsilon/100}^{\varepsilon/100} \frac{\|f(x+t) - f(x)\|_{L^2((0,\pi/2),A(x)dx)}^2}{|t|^{1+2\gamma}} dt
\end{aligned}$$

and this is bounded by the hypotheses on f and the estimate

$$\|f(x)\|_{L^2([0,\pi/2],\bar{\chi}(x-t)dx)}^2 \leq c \|f(x)\|_{L^2((0,\pi/2),A(x)dx)}^2$$

uniformly in $|t| \leq \varepsilon/100$.

It can be easily proven that $\Lambda_{\gamma}^{2,2}([0, \pi/2], dx)$ coincides with the potential space $\mathcal{L}_{\gamma}^2([0, \pi/2], dx)$ consisting of all even π -periodic functions F such that

$$\sum_{n=1}^{+\infty} \left| \int_0^{\pi/2} F(x) \cos(2nx) dx \right|^2 n^{2\gamma}$$

is bounded. Finally, by [26, Vol. II, Theorem 11.3, page 195], it then follows that if $\gamma < 1/2$ then $Q_N(A^{1/2}\chi f)$ diverges on a set with $1 - 2\gamma$ outer capacity equal to 0. It then follows that for all $\varepsilon > 0$, the part of the divergence set of $T_N^A f$ contained in $[\varepsilon, \pi/2 - \varepsilon]$ has $1 - 2\gamma$ outer capacity equal to 0. Finally, the outer capacity of the divergence set of $T_N^A f$ in $(0, \pi/2)$ has $1 - 2\gamma$ outer capacity equal to 0, and therefore Hausdorff dimension smaller than or equal to $1 - 2\gamma$. \square

Finally, we will show that a slightly higher regularity than required in Theorem 31 gives pointwise convergence (except perhaps in 0 and $\pi/2$).

Theorem 32. *Let $1/2 < \gamma < 1$. If $f \in \Lambda_{\gamma}^{2,2}((0, \pi/2), A(x)dx)$ then $T_N^A f(x)$ converges for all $x \in (0, \pi/2)$, and the convergence is uniform away from 0 and $\pi/2$.*

Proof. Letting $\varepsilon > 0$ and χ be as in the proof of the previous theorem, then $T_N^A f(x)$ is uniformly equiconvergent with $A^{-1/2}(x)Q_N(A^{1/2}\chi f)(x)$ in $[\varepsilon, \pi/2 - \varepsilon]$. Also, as before, $A^{1/2}\chi f \in \Lambda_{\gamma}^{2,2}([0, \pi/2], dx)$. By Bernstein's theorem [26, Vol. I, Theorem 3.1, page 240, and the remark that follows], if $\gamma > 1/2$ then $Q_N(A^{1/2}\chi f)(x)$ converges absolutely and uniformly in $(0, \pi/2)$. Thus, $T_N^A f(x)$ converges uniformly in $[\varepsilon, \pi/2 - \varepsilon]$. \square

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