Cherenkov radiation versus X-shaped localized waves

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Localized waves (LW) are nondiffracting ("soliton-like") solutions to the wave equations and are known to exist with subluminal, luminal, and superluminal peak velocities V. For mathematical and experimental reasons, those that have attracted more attention are the "X-shaped" superluminal waves. Such waves are associated with a cone, so that one may be tempted—let us confine ourselves to electromagnetism—to look [Phys. Rev. Lett. **99**, 244802 (2007)] for links between them and the Cherenkov radiation. However, the X-shaped waves belong to a very different realm: For instance, they can be shown to exist, independently of any media, even in vacuum, as localized non-diffracting pulses propagating rigidly with a peak-velocity $V>c$ [Hernández *et al.*, eds., *Localized Waves* (Wiley, 2008)]. We dissect the whole question on the basis of a rigorous formalism and clear physical considerations.

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1. INTRODUCTION

Localized waves (LW) are nondiffracting ("soliton-like") solutions to the wave equations and are known to exist with subluminal, luminal, and superluminal peak-velocities V. For mathematical and experimental reasons, those that have attracted more attention are the "X-shaped" superluminal waves. Such waves are associated with a cone, so that some authors—let us confine ourselves to electromagnetism—have been tempted to look for links between them and the Cherenkov radiation [1]. However, the X-shaped waves belong to a very different realm: For instance, they exist, independently of any media, even in vacuum as localized non-diffracting pulses propagating rigidly with a peak velocity $V>c$, as verified in a number of papers (cf., e.g., the references in the book *Localized Waves* [2]). It is our aim in this paper to dissect the question on the basis of a rigorous formalism and clear physical considerations.

In particular we show, by explicit calculations based on Maxwell equations only, that, at variance with what was assumed by some previous authors (see, e.g., [1] and references therein): (i) The "X-waves" exist in all space, and in particular inside both the front and the rear part of their double cone (which has nothing to do with Cherenkov's). (ii) The X waves are to be found not heuristically but by use of strict mathematical (or experimental) procedures, without any *ad hoc* assumptions. (iii) The ideal X-waves, as well as plane waves, are actually endowed with infinite energy, but finite-energy X-waves can be easily constructed (even without recourse to space–time truncations); and at the end of this work, by following a new technique, we construct finite-energy *exact* solutions, totally free of backward-traveling waves. (iv) The most interesting property of X-waves lies in the circumstance that they are LWs, endowed with a characteristic *self-reconstruction* property, which promises important practical applications (in part already realized, starting in 1992), quite independently of the superluminality—or not—of their peak-velocity. (v) Insistence on attempting a comparison of Cherenkov radiation with X-waves would lead one to an unconventional sphere: that of considering the rather different situation of the (X-shaped, too) field generated by a superluminal point-charge, a non-orthodox question actually exploited in previous papers [3]: We show here explicitly that in such a case the point-charge would not lose energy in the vacuum and that its field would not need to be continuously fed by incoming side-waves (as is the case, in contrast, for an ordinary X-wave).

As already said, we wish to show, by the way, that the Cherenkov–Vavilov radiation (in the following we shall write only "Cherenkov" for brevity's sake) has nothing to do with the X-shaped LWs. Happily enough, the treatment of the classic problem [4,5] of the Cherenkov radiation from a point-charge traveling in a medium with speed v such that $c_n<v<c$, where c_n and c are the speed of light in the medium and in the vacuum, respectively, has reached by now a standard form [6].

In this paper, let us address our subject in terms of rigorous mathematics and physics.

2. CHERENKOV RADIATION VERSUS X-SHAPED LOCALIZED WAVES

A. Initial Aims

One of our initial aims is showing that no "Cherenkov formulation" of the X-shaped LWs is possible. In the papers referring to the Cherenkov effect, except for some ambiguous notation [6], the analysis of the ordinary scalar-valued Cherenkov radiation is normally correct; indeed, the relevant results are well known [4,5] and generally accepted.

The ordinary Cherenkov radiation cannot be used, however, to draw conclusions about the superluminal LWs known as X-shaped waves (or simply X-waves). The latter are nondiffractive solutions to the homogeneous wave equations and propagate rigidly with superluminal *peak-velocities*. They were predicted long ago [7,8], they have been mathematically constructed [9–11], and finite-energy versions of them have been experimentally produced [12–16] even in vacuum. Let us go on to explicit calculations.

To be self-contained, let us initially address the (simpler) scalar case, in which a field ψ is governed by the inhomogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c_n^2}\frac{\partial^2}{\partial t^2}\right)\psi(\mathbf{r},t) = -\frac{4\pi}{c_n} j(\mathbf{r},t), \tag{1}$$

with $j = qv\,\delta(\rho)\delta(z-vt)/(2\pi\rho)$. Here c_n is the speed of light in the considered medium [17] and $j(\mathbf{r},t)$ is the generating source, assumed to be point-like and moving along the positive z axis with subluminal speed $c_n < v < c$, while ρ denotes the cylindrical radial coordinate.

A Green's function for Eq. (1) is given explicitly as

$$G(\mathbf{r},t,\mathbf{r}',t') = lG^{+}(\mathbf{r},t,\mathbf{r}',t') + (1-l)G^{-}(\mathbf{r},t,\mathbf{r}',t'), \tag{2}$$

where

$$G^{\pm}(\mathbf{r},t,\mathbf{r}',t') = \frac{\delta(t' - (t \mp R/c_n))}{c_n R} \tag{3}$$

and $R \equiv \sqrt{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')}$.

Quantities G^{+} and G^{-} are the retarded and the advanced Green's function, respectively.

When considering only the retarded Green's function ($l=1$), the solution to the wave equation can be expressed as

$$\psi(\mathbf{r},t) = \int dx'^3 \int dt' G^{+}(\mathbf{r},t,\mathbf{r}',t') j(\mathbf{r}',t'). \tag{4}$$

We can use the expression for G^{+} given in Eq. (3) for a direct calculation of the wave function $\psi(\mathbf{r},t)$. However, for reasons that will be made clear below, we prefer to follow the procedure adopted in previous papers, such as [1]. Specifically, we shall determine the Fourier transform of G^{+} in the variables z and t and subsequently the transform of ψ in the same variables. For $c_n < v < c$, we obtain

$$\psi(\rho,\zeta) = \frac{1}{2\pi c_n}\left[\left(\int_{-\infty}^{0} d\omega(-i\pi) q H_0^2(\rho\gamma_n^{-1}|\omega|/v) e^{i\omega\zeta/v}\right) + \left(\int_{0}^{\infty} d\omega(i\pi) q H_0^1(\rho\gamma_n^{-1}\omega/v) e^{i\omega\zeta/v}\right)\right],$$

where $\zeta \equiv z - vt$, and $H_0^{1,2}$ are the zero-order Hankel functions of the first and second kind. Using, next, the relation $H_0^2(x) = -H_0^1(-x)$ for $x \geqslant 0$, the last equation is reduced to

$$\psi(\rho,\zeta) = \frac{1}{2c_n}\int_{-\infty}^{\infty} d\omega i q H_0^1(\rho\gamma_n^{-1}\omega/v) e^{i\omega\zeta/v} \tag{5}$$

(which, incidentally, is nothing but Eq. (7) of [1]), from which one determines the well-known expression for the Cherenkov radiation

$$\psi(\rho,z,t) = \begin{cases} \dfrac{2q\beta_n}{\sqrt{\zeta^2 - \gamma_n^{-2}\rho^2}} & \text{for } \zeta < -\gamma_n^{-1}\rho \\ 0 & \text{elsewhere} \end{cases}, \tag{6}$$

where it should be noted that $\beta_n \equiv v/c_n$ and $\gamma_n \equiv 1/\sqrt{v^2/c_n^2 - 1}$. The radiation exists only inside the rear part $\zeta = -\gamma_n^{-1}\rho$ of the Cherenkov cone.

As to the vectorial case, physically more significant, let us only mention that it can be constructed by adopting the Lorentz gauge and by considering the vector potential $\mathbf{A} = A_z\hat{\mathbf{e}}_z$, with $A_z \equiv \psi$, and the current density $\mathbf{j} = j_z\hat{\mathbf{e}}_z$, with $j_z \equiv j$.

B. Cherenkov Emission and X-Shaped Waves

One might be induced to look for a connection between the Cherenkov emission and X-shaped waves. The simplest X-waves are solutions to the homogeneous scalar wave equation: They are wave functions of the type $X(\rho,z,t) = X(\rho,\zeta)$, with $\zeta \equiv z - Vt$ and $c_n < V < \infty$, and they can be obtained by suitable superpositions of axially symmetric Bessel beams, propagating in the positive z direction with the same phase-velocity (cf., e.g., [9–11]); precisely,

$$\psi_X(\rho,\zeta) = \int_0^{\infty} d\omega S(\omega) J_0\left(\rho\frac{\omega}{V}\sqrt{V^2/c_n^2 - 1}\right) e^{i\omega\zeta/V}, \tag{7}$$

where $J_0(.)$ is an ordinary zero-order Bessel function and $S(\omega)$ is the temporal frequency spectrum. Let us recall that Eq. (7) represents a particular case of the superluminal waves, which in their turn are just a particular case of the (subluminal, luminal, or superluminal) LWs. For the specific spectrum $S(\omega) = \exp[-a\omega]$, with a a positive constant, one obtains the zero-order (classic) X-wave:

$$X \equiv X(\rho,\zeta) = \frac{V}{\sqrt{(aV - i\zeta)^2 + (V^2/c_n^2 - 1)\rho^2}}. \tag{8}$$

Attempts at comparing the Cherenkov effect and the zero-order X-wave can be suggested by the apparent mathematical similarity of Eqs. (5) and (7). To be more specific: (i) we have seen that for the inhomogeneous wave equation (1), a Cherenkov solution of the type given in Eq. (6) exists only inside the cone rear part $\zeta=-\gamma^{-1}\rho$. To obtain a solution existing only inside the cone forward part $\zeta=\gamma^{-1}\rho$, one should make use of the advanced Green's function G^-: This—to go on with our example—could lead one to believe, on the basis of an incorrect analogy, that the forward part of the X-wave is non-causal. (ii) Another point that may lead to erroneous conclusions is that, if one uses $S(\omega)=i$ into the ψ_X wave synthesis in Eq. (7), that is, if one sets $a=0$ in Eq. (8) and multiplies it by i, one obtains

$$\tilde{X}(\rho,\zeta)=\frac{V}{\sqrt{\zeta^2-(V^2/c_n^2-1)\rho^2}}, \tag{9}$$

which is mathematically identical, apart from a constant, to the Cherenkov solution in Eq. (6), with a real part existing this time inside both the rear cone $\zeta=-\gamma^{-1}\rho$ and the forward cone $\zeta=\gamma^{-1}\rho$. But the statement that the advanced part of the zero-order X-wave [cf. Eq. (8)] is non-causal would again be due to an illicit extrapolation: Indeed, the X-wave, being a solution to the homogeneous wave equation, cannot possess singularities. In contrast, the solution given in Eq. (9), which was obtained from the Bessel beam superposition (7) using a *constant* spectrum $S(\omega)$, does have singularities and cannot be considered a solution to the homogeneous wave equation. Actually, the real part of the solution (9) can be an acceptable solution for the *inhomogeneous* case only. [Indeed, we shall show later on that it might represent the field of a point-charge traveling with speed $V>c_n$ when using as a Green's function the expression $G=G^+/2+G^-/2$, half-retarded and half-advanced (which means, again, that it refers to an inhomogeneous problem)].

It should be pointed out that the Bessel beam synthesis in Eq. (7) is a particular case of more general spectral representations yielding not only infinite but also finite-energy superluminal [2,9–11] LWs. Not less important, even a *finite*-energy close replica of the the classic X-wave (8) (endowed, incidentally, with a *finite* field depth) can be generated in a causal manner by means of a "dynamic" finite aperture (antennas, holographic or optical elements, etc.). For instance, it is enough to consider an array of circular elements excited according to the function $X(\rho,z=0,t)$, given by Eq. (7) or Eq. (8) on the aperture plane located at $z=0$. The emitted finite-energy X-wave can be calculated by means of the Rayleigh–Sommerfeld (II) formula [18]

$$\psi_{\mathrm{RS(II)}}(\rho,z,t)=\int_0^{2\pi}\mathrm{d}\phi'\int_0^{D/2}\mathrm{d}\rho'\rho'\frac{1}{2\pi R}\times\left\{[X]\frac{(z-z')}{R^2}+[\partial_{c_nt'}X]\frac{(z-z')}{R}\right\}. \tag{10}$$

The quantities inside the square brackets are evaluated at the retarded time $c_nt'=c_nt-R$. The distance $R=\sqrt{(z-z')^2+\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}$ is now the separation between source and observation points, and the aperture diameter is denoted by D. The depth of field of the particular solution given in Eq. (10) is known [19–21] to be $Z=D\gamma_n/2$ (at least when the aperture radius $D/2$ is much larger than the spot width $s_0=\sqrt{3}a\gamma V$).

What was stated above has been theoretically (even via numerical simulations) and experimentally verified: see, for example, [9–16,22–28,19–21,29,30,2].

We agree with previous authors such as Walker and Kuperman [1] that the superluminal spot of any X-wave is fed by the waves coming from the elements of the aperture and that these waves carry energy propagating with at most luminal speed. In such cases, the X-wave *intensity peaks* at two different locations are not causally correlated: This is indeed firmly accepted, since 1990s, by those working on LWs (cf., e.g., the references in [31,30], as well as [2,9–11,19–28]). The efforts in the area of X-waves are *not* aimed at transmitting information superluminally: On the contrary, X-waves arouse interest because of their spatio-temporal localization, unidirectionality, soliton-like nature, and self-reconstruction properties in the near-to-far zone. Such properties bear interesting consequences, from theoretical and experimental points of view, in all sectors of physics in which a role is played by a wave equation (including—*mutatis mutandis*—elementary particle physics, and even gravitation).

C. Conditions for Constructing Further X-Shaped Waves

It is also true, and again well known, that the (ideal) zero-order X-wave in Eq. (7) bears infinite energy (as well as the plane waves). The fact that such a solution needs to be fed for an infinite time has been known since the beginning: In any case, as mentioned earlier, such a problem can be overcome either by using [cf. Eq. (10)] apertures finite in space and time, i.e., by truncating the X-wave, or by constructing exact, analytical finite-energy solutions [29,22–28]. For reasons of space, we shall show *only briefly*, but in an original rigorous way, how closed-form solutions of the latter type can be actually constructed, without any recourse to the backward-traveling waves that trouble the ordinary approaches: See Subsection 2.E below.

For the moment, let us recall that X-waves endowed by themselves with finite energy even without truncation have already been constructed in the past by using, however, an (very good, by the way) *approximation*: See, for instance, the approximate solution in Eqs. (2.31) and (2.32) of [32], that is, the "SMPS pulse," which is depicted in Fig. 1. A finite-energy X-wave gets deformed while propagating: Fig. 1(b) shows the pulse in Fig. 1(a) after it has traveled 50 km.

Let us stress, as well, that the formulations leading to LWs in general, and to X-waves in particular, are not chosen *ad hoc* but are based on proper choices of the spectra (which imply a specific space–time coupling [29,22–28,33,34]) and of the Bessel functions (in order to avoid singularities both at $\rho=0$ and at $\rho=\infty$). [By contrast, the choice suggested, for instance, in Eq. (15) of [1] presents singularities]. To be clearer, let us observe that a general solution to the scalar homogeneous wave equation in free space can be written (when eliminating evanescent waves) in the form

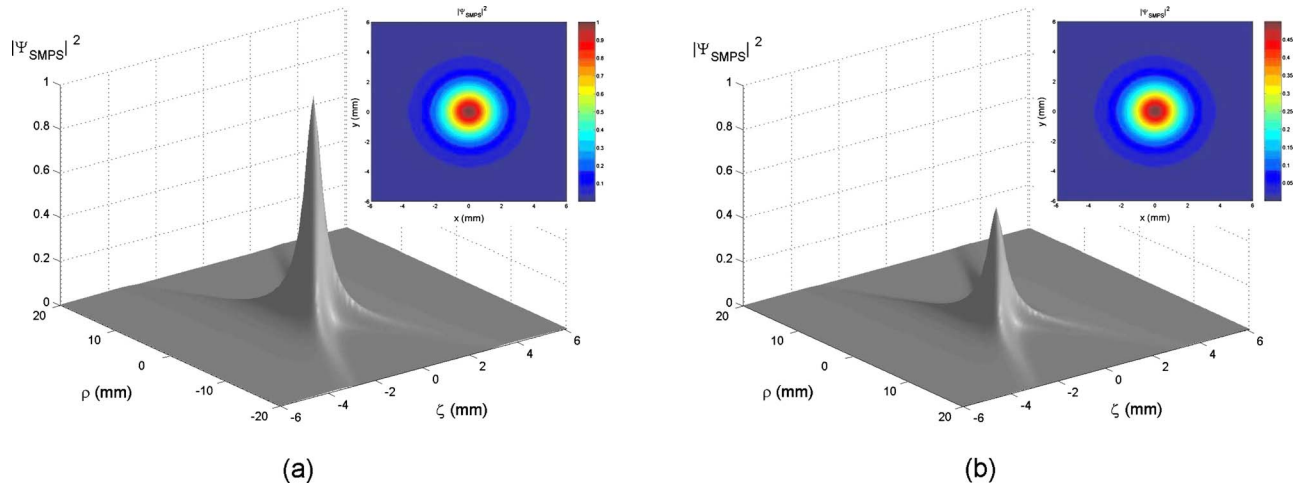

Fig. 1. (Color online) Example of an X-type LW endowed with finite energy (even without truncation), which consequently gets deformed while propagating: (b) represents the pulse depicted in (a) after it has traveled 50 km. See [32] and the text.

$$\psi(\rho,\phi,z,t)=\sum_{\nu=-\infty}^{\infty}\left[\int_0^{\infty}d\omega\int_{-\omega/c}^{\omega/c}dk_z A_\nu(k_z,\omega)\times J_\nu\left(\rho\sqrt{\frac{\omega^2}{c^2}-k_z^2}\right)e^{ik_z z}e^{-i\omega t}e^{i\nu\phi}\right] \quad (11)$$

by considering positive angular frequencies ω only. For obtaining ideal LWs propagating along the positive z direction with peak-velocity V (that can assume *a priori* any value $0\leqslant V\leqslant\infty$, as we know), the spectra A_ν must have the form [29,2,33,34]

$$A_\nu(k_z,\omega)=\sum_{\mu=-\infty}^{\infty}S_{\nu\mu}(\omega)\delta[\omega-(Vk_z+b_\mu)], \quad (12)$$

where $b_\mu=2\pi\mu V/\Delta z_0$, and it can be easily shown [29,33,34] that solution (11) possesses the important property $\psi(\rho,\phi,z,t)=\psi(\rho,\phi,z+\Delta z_0,t+\Delta z_0/V)$, where Δz_0 is a chosen space interval along z. Equations (11) and (12) already confirm that LWs are not to be found by *ad hoc* assumptions: Equation (12) does explicitly show that—as mentioned above—the ideal LWs exist only in correspondence with linear relations between ω and k_z, that is, with *specific* space–time couplings. By assuming in particular $A_0(k_z,\omega)\equiv A_\nu(k_z,\omega)=\delta_{\nu 0}S(\omega)\Theta(k_z)\delta[\omega-(Vk_z+\alpha_0)]$, where the Heaviside function Θ does eliminate, as desired, any backward components and α_0 is a constant, we can construct an infinite number of subluminal, luminal, or superluminal localized solutions, with axial symmetry and in closed form. For instance, the X-shaped waves represented in Eq. (7), correspond to the particular value $\alpha_0=0$. To get finite-energy solutions one has to abandon the strict requirement of a linear relation between ω and k_z and impose instead that the spectral functions $A(k_z,\omega)$ possess non-negligible values only *in the vicinity* of a straight line of the mentioned type [29]: This is briefly exploited in Subsection 2.E below.

D. X-Shaped Field Generated by a Really Superluminal (Point) Charge

We have already established that no analogy between the Cherenkov radiation and the X-wave solution is justified even in material media. In the case of vacuum, the aforementioned analogy should have rather led one to consider the field generated by a *really* superluminal point-charge!—an unconventional problem that was investigated in [3] and references therein. In such a situation, the point-charge superluminally traveling in the vacuum is not expected to radiate, due to physical reasons published long ago [35,8,36–38]. Such a charge does not radiate in its rest frame [8,37,38] and, consequently, neither does it radiate according to observers for whom it is superluminal [35]. We establish this result below, by explicit calculations based on Maxwell equations only.

Let us consider the wave equation (1) in vacuum ($c_n\rightarrow c$) with a superluminally moving ($v\rightarrow V>c$) point charge source. Use of the Green's function $G=(G^+ +G^-)/2$ (cf., e.g., [39–43]) yields the following integral representation for the solution [$\zeta\equiv z-Vt$]:

$$\psi(\rho,\zeta,t)=\frac{q}{c}\int_0^{\infty}d\omega N_0\left(\rho\frac{\omega}{V}\gamma^{-1}\right)\cos\left(\frac{\omega}{V}\zeta\right), \quad (13)$$

where N_0 is the zero-order Neumann function and, now, β_n and γ_n have been replaced with $\beta\equiv V/c$ and $\gamma^{-1}\equiv\sqrt{V^2/c^2-1}$. The integration can be carried out explicitly, and yields, for the field generated by a point-charge traveling superluminally in the vacuum, the expression

$$\psi(\rho,z,t)=\begin{cases}q\beta[\zeta^2-\rho^2\gamma^{-2}]^{-1/2} & \text{when } 0<\rho\gamma^{-1}<|\zeta|\\ 0 & \text{elsewhere}\end{cases}. \quad (14)$$

This solution is different from zero inside the rear *and* front parts of the unlimited double cone [3,31] generated by the rotation around the z axis of the straight lines $\rho=\pm\gamma\zeta$, in agreement with the predictions of the "extended" (or, rather, "non-restricted") theory of special relativity [8,37,38]. The expression in Eq. (14) is precisely equivalent to the solution given by Eq. (8) in our earlier paper [3] (except for a constant that was wrong therein).

Going on to the (physically more suited) vectorial formalism, adopting the Lorentz gauge, and choosing a current density $\mathbf{j}=j_z\hat{\mathbf{e}}_z$, $j_z\equiv j$, a scalar electric potential $\phi=c\psi/V$ and a vector magnetic potential $A\equiv\psi\hat{\mathbf{e}}_z$ (cf. Fig. 2

in [3], which refers to a negative point-charge), one obtains, in analogy to [3], the electric and magnetic fields

$$\mathbf{E}(\rho,\zeta) = -q\gamma^{-2}Y(\rho\hat{\mathbf{e}}_\rho + \zeta\hat{\mathbf{e}}_z), \qquad \mathbf{B} = -q\beta\gamma^{-2}Y\rho\hat{\mathbf{e}}_\theta, \tag{15}$$

in Gaussian units, where

$$Y \equiv [\zeta^2 - \rho^2\gamma^{-2}]^{-3/2}$$

inside the double cone (i.e., for $0<\rho\gamma^{-1}<|\zeta|$), while $Y=0$ outside it (that is, **E** and **B** are zero outside it). The corresponding Poynting vector is given by

$$\mathbf{S} = \frac{c}{4\pi}q^2\beta\gamma^{-4}Y^2\rho(\rho\hat{\mathbf{e}}_z - \zeta\hat{\mathbf{e}}_\rho). \tag{16}$$

The total flux, through any closed surface containing the point-charge at the considered instant of time, equals zero. Thus, a point-charge traveling at a constant superluminal speed in vacuum does not radiate energy. This fact is depicted and explained in an intuitive way in Fig. 4 of [3], which originally appeared in [7,8] and, for clarity, is reproduced in this article as Fig. 2.

We wish to emphasize that in the present case the field need not be fed, at variance with the case of the ordinary X-waves. Once more, one can see that any analogies between the Cherenkov effect and the X-waves do completely break down in the case of the vacuum.

Apparently, most authors do not address other interesting physical points. For example, usually no mention is made of the fact that a superluminal charge is expected to behave as a magnetic pole, in the sense fully clarified in [44,45,8]. One can see even from Eqs. (15) that $\mathbf{E}\to 0$, and one is left with a pure magnetic field, in the limit $V\to\infty$.

E. Finite-Energy X-Shaped Waves

At last, as anticipated in Subsection 2.C above, we are going to show how finite-energy solutions can be obtained in closed form without any recourse to the backward-traveling waves that trouble the usual approaches (even if the intervention of such components has been already minimized in [29,22–28], at the cost, however, of going on to frequency spectra with a very large bandwidth). In fact, when confining ourselves to superluminal LWs with axial symmetry, let us put in Eq. (11) $A_\nu(k_z,\omega)=\delta_{\nu 0}A(k_z,\omega)$ and adopt the "unidirectional decomposition"

$$\zeta \equiv z - Vt; \qquad \eta \equiv z - ct.$$

In terms of the new variables [46,47] and confining ourselves now to $V>c$, Eq. (11) can be rewritten as

$$\psi(\rho,\zeta,\eta) = (V-c)\int_0^\infty d\sigma \int_{-\infty}^{\sigma} d\alpha J_0(\rho\sqrt{\gamma^{-2}\sigma^2 - 2(\beta-1)\sigma\alpha}) \times \exp[-i\alpha\eta]\exp[i\sigma\zeta]A(\alpha,\sigma),$$

where $\alpha\equiv(\omega-Vk_z)/(V-c)$ and $\sigma\equiv(\omega-ck_z)/(V-c)$.

As mentioned above, ideal (infinite-energy) superluminal LWs are obtained by imposing the linear constraint $A(\alpha,\sigma)=B(\sigma)\delta(\alpha+\alpha_0)$. By contrast, finite-energy superluminal LWs are obtained by concentrating the spectrum $A(\alpha,\sigma)$ in the vicinity of the straight line $\alpha=-\alpha_0$. By choosing, for example,

$$A(\alpha,\sigma) = \frac{\Theta(-\alpha-\alpha_0)}{V-c}e^{d\alpha}e^{-a\sigma}, \tag{17}$$

we get the *finite-energy exact solutions* (free of any backward components):

$$\psi(\rho,\zeta,\eta) = \frac{X}{VZ}e^{-\alpha_0 Z}, \tag{18}$$

where X is defined in Eq. (8); quantities a, d, and α_0 are positive constants; and

$$Z \equiv (d - i\eta) - \frac{c}{V+c}(a - i\zeta - VX^{-1}).$$

In Figs. 3(a) and 3(b) we show one such finite-energy superluminal LW, corresponding to an exact, analytic solution totally free of backward-traveling components. As we know, any finite-energy X-wave gets deformed while propagating: Fig. 3(b) shows the pulse in Fig. 3(a) after it has traveled 2.78 km.

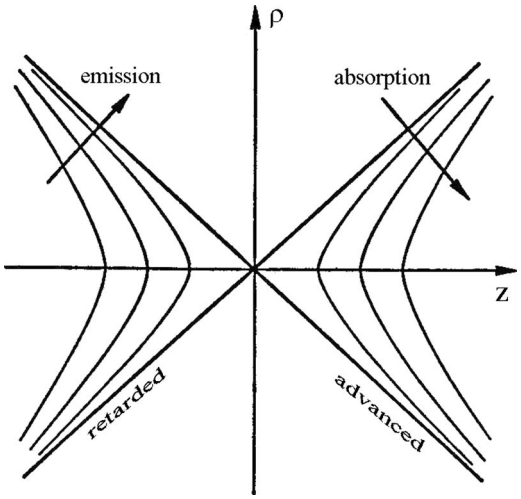

Fig. 2. This figure, which appeared in [3], but is taken from [8], intuitively shows, among other things, that a superluminal charge [35–38,7,8,30,31] traveling at constant speed in vacuum would *not* lose energy: see [3] and the text. Reprinted with kind permission of Società Italiana di Fisica.

3. SOME CONCLUSIONS

The localized waves (LWs) are non-diffracting solutions to the wave equations and are known to exist with arbitrary peak-velocities $0<V<\infty$. For mathematical and experimental reasons, those that have attracted more attention are the "X-shaped" superluminal waves. Such waves are associated with a cone, so that—let us confine ourselves to electromagnetism—it was tempting to look for links between them and the Cherenkov radiation. However, the X-shaped waves belong to a rather different realm: For instance, they exist, independently of any media, even in vacuum as localized rigidly propagating pulses. In this paper we have dissected the question, on the basis of clear physical considerations and a rigorous formalism; showing that, in spite of the seeming mathematical similarities of the solutions corresponding to the Cherenkov effect

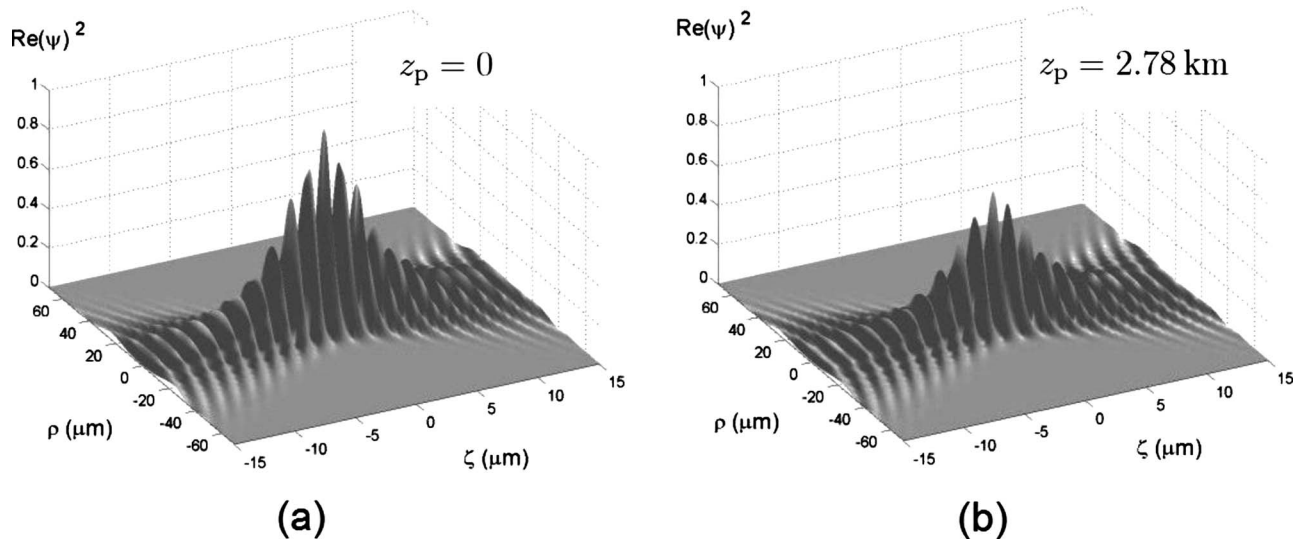

Fig. 3. Example of a finite-energy X-type LW, corresponding to an exact, analytic solution of Eq. (18), totally free of backward components. This figure represents the real part of the field, normalized at $\rho=z=0$ for $t=0$, with the choices $a=3.99\times10^{-6}$ m, $d=20$ m, $V=1.005\,c$, and $\alpha_0=1.26\times10^{7}$ m^{-1}. In this case the frequency spectrum starts at $\omega\equiv\omega_{\min}\approx3.77\times10^{15}$ Hz and afterward decays exponentially with the bandwidth $\Delta\omega\approx7.54\times10^{13}$ Hz. The value $\omega_{\min}$ can be regarded as the pulse central frequency; since $\Delta\omega/\omega_{\min}\ll1$, there exists a well-defined carrier wave, which clearly shows up in the plots. Any finite-energy LW gets deformed while propagating: (b) represents the pulse depicted in (a) after it has traveled 2.78 km.

and the zero-order X-waves, Cherenkov radiation has nothing to do with the X-shaped LWs even in material media.

Our calculations, based on Maxwell equations only, elucidate that (i) the "X-waves" exist in all space, and in particular inside both the front and the rear part of their double cone (which, as we said, has nothing to do with Cherenkov's); (ii) the X-waves are found not heuristically but by use of strict mathematical (or experimental) procedures, without *ad hoc* assumptions; (iii) the ideal X-waves, as well as plane waves, are endowed with infinite energy, but we have shown by a new technique how to construct finite-energy X-waves (even without recourse to space–time truncations), represented by *exact* solutions totally free of backward-traveling waves; (iv) the most interesting property of X-waves lies in the circumstance that they are LWs, endowed with a characteristic *self-reconstruction* property, and promise important practical applications (in part already realized), quite independently of their peak-velocity value; (v) and last, insistence on attempting a comparison of Cherenkov radiation with X-waves would lead one to a really unconventional sphere: that of considering the rather different situation of the (X-shaped, too) field generated by a superluminal point-charge, a non-orthodox question actually exploited in previous papers [3]. We show here in an explicit way, incidentally, that in such a case the point-charge would not lose energy in the vacuum, and its field would not need to be continuously fed by incoming side-waves (as is the case, in contrast, for an ordinary X-wave).

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