



Discrepancy and approximation of absolutely continuous measures with atomic measures

Luca Brandolini¹ · Leonardo Colzani² · Giancarlo Travaglini² 

Received: 9 November 2023 / Accepted: 3 February 2025
© The Author(s) 2025

Abstract

We prove several results concerning the discrepancy, tested on balls in the d -dimensional torus \mathbb{T}^d , between absolutely continuous measures and finite atomic measures.

Keywords Geometric discrepancy · Roth's theorem · Fourier transforms · Cassels-Montgomery lemma · Morrey spaces · Jittered sampling · Gates of Hell Problem

Mathematics Subject Classification Primary 11K38 · 42B10

1 Introduction

Is there any measure which is more difficult to approximate by finite atomic measures with equal weights than the Lebesgue measure?

This intriguing question has been raised by Aistleitner, Bilyk, and Nikolov in [2] (where it has been termed *Gates of Hell* problem, by referring to Rodin's sculpture in Stanford), and of course it can be specified in more than one way. See also [10, 18, 19, 29, 48]. In this paper we focus on the related problem of estimating the irregularities of distribution in the case of absolutely continuous measures on \mathbb{T}^d , and we prove sharp order of decay for the discrepancies, tested on balls. Each one of these bounds depends explicitly on the number of points, on the square norm of the finite sequence

✉ Giancarlo Travaglini
giancarlo.travaglini@unimib.it

Luca Brandolini
luca.brandolini@unibg.it

Leonardo Colzani
leonardo.colzani@unimib.it

¹ Dipartimento di Ingegneria Gestionale, dell'Informazione e della Produzione, Università degli Studi di Bergamo, Viale Marconi 5, 24044 Dalmine, BG, Italy

² Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy

of weights associated to the points, and on the L^p or Morrey norm of the function appearing in the absolutely continuous measure. Anyhow we acknowledge that the above cited question of C. Aistleitner, D. Bilyk, and A. Nikolov is, in general, more delicate, since it asks whether the optimal discrepancy is actually minimized by the Lebesgue measure. In particular, Theorem 4 by W. Chen and Theorem 6 below provide the correct order of decay for the discrepancy, but leave open the possibility that some absolutely continuous measure may yield a discrepancy strictly smaller than that of the Lebesgue measure. In short we only consider the order of decay of the discrepancy for large N and our results are sharp up to a constant.

The need for an understandable introduction leads us to take a step back.

In his celebrated 1954 *Mathematika* paper [39] K. Roth proved the following result, which is the starting point for the field termed *Geometric Discrepancy*.

Here and thereafter we identify $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ with the unit cube $[-1/2, 1/2)^d$.

Theorem 1 (Roth) *Let $d \geq 2$. There exists a constant $c = c(d) > 0$ such that, for every choice of N points z_1, \dots, z_N , not necessarily distinct, in the torus \mathbb{T}^d , we have the following lower bound for the L^2 norm of the discrepancy.*

$$\int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N \chi_{I_t}(z_j) - |I_t| \right|^2 dt_1 \cdots dt_d \geq c \frac{\log^{d-1}(N)}{N^2}, \tag{1}$$

where $t = (t_1, \dots, t_d) \in [0, 1)^d$, $I_t = [0, t_1] \times \cdots \times [0, t_d]$, $|G|$ denotes the volume of a measurable set G , and

$$\chi_G(x) = \begin{cases} 1 & \text{if } x \in G, \\ 0 & \text{if } x \notin G, \end{cases} \tag{2}$$

denotes its characteristic function.

H. Davenport [25] and K. Roth [40] have shown that the lower bound in (1) cannot be improved (see also [20]).

The monographs [5, 35] are basic references for Roth’s theorem and its developments. See also [17, 22, 28, 36, 46].

Roth’s theorem can be rewritten in the following equivalent form (see e.g. [27, 36, 41, 43]), where the parallelepipeds anchored at the origin are replaced by translated and dilated copies of a given cube Q .

Theorem 2 *Let $d \geq 2$ and let $Q = [0, 1)^d$. Then, for every choice of N points z_1, \dots, z_N , not necessarily distinct, in the torus \mathbb{T}^d , we have*

$$\int_0^1 \int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N \chi_{\lambda Q+x}(z_j) - \lambda^d \right|^2 dx d\lambda \geq c \frac{\log^{d-1}(N)}{N^2}.$$

A basic development of geometric discrepancy came from a seminal paper by W. Schmidt [42], who observed that replacing parallelepipeds or cubes with balls leads

to a much higher lower bound. This has suggested an investigation of the relation between the quality of the sampling points z_1, z_2, \dots, z_N and the geometry of the sets (cubes, balls, convex bodies, fractals, ...) we use to test them. See e.g. [3, 7, 10, 15, 16, 24, 30, 35].

H. Montgomery [36] and J. Beck [4] have sharpened Schmidt’s result and have independently proved the first part of the following theorem (see also [9]). See e.g. [12, 14, 33] for proofs of the second part.

Theorem 3 (*Schmidt - Montgomery - Beck*) *Let $d \geq 2$ and let $B \subset \mathbb{T}^d$ be a ball of diameter < 1 . Then there exists a constant $c_1 > 0$ such that, for every choice of N points z_1, \dots, z_N , not necessarily distinct, in \mathbb{T}^d , we have*

$$\int_0^1 \int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N \chi_{\lambda B+x}(z_j) - \lambda^d |B| \right|^2 dx d\lambda \geq c_1 N^{-1-1/d} ,$$

where $|B|$ is the volume of the ball B . Moreover, for every $0 < \lambda \leq 1$ there exists a constant c_2 such that, for every positive integer N , there exist points $\tilde{z}_1, \dots, \tilde{z}_N$ in \mathbb{T}^d such that

$$\int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N \chi_{\lambda B+x}(\tilde{z}_j) - |B| \right|^2 dx \leq c_2 N^{-1-1/d} .$$

Perhaps it is interesting to recall that the first part of Theorem 3 does not hold true if we forget the integration in the dilation variable λ , that is we fix λ . Indeed, in this case the results may depend on the dimension d in an unexpected way. See e.g. [8, 12, 37]. See also [47] for a curious phenomenon concerning rotated polygons.

As in [21], we reconsider the discrepancy in the following more general setting.

We want to estimate the quality of the distribution of N points z_1, \dots, z_N , not necessarily distinct, with respect to a Borel probability measure μ on \mathbb{T}^d . We choose a reasonably large family \mathcal{R} of measurable sets and, for $R \in \mathcal{R}$, we study the discrepancy

$$D_N(R) = \frac{1}{N} \sum_{j=1}^N \chi_R(z_j) - \mu(R) .$$

In [10] we have investigated the case of possibly singular measures μ and a family \mathcal{R} of possibly fractal sets. Here we move in the opposite direction and consider absolutely continuous measures $f(x)dx$. Hence we meet again the question raised at the beginning of this paper, which we modify as follows.

Is there any absolutely continuous measure $f(x) dx$ which is easier to approximate by finite atomic measures than the Lebesgue measure dx ?

The above question is perhaps motivated by the following theorem of W. Chen [18], which shows that the decay of the discrepancy of an integrable function with respect to rectangle is, up to a constant, not better than the decay of Lebesgue measure.

Theorem 4 (Chen) *Let $f(x)$ be a non-zero Lebesgue integrable real-valued function on \mathbb{T}^d and let $h(x)$ be any real-valued function on \mathbb{T}^d . Let us consider N points z_1, \dots, z_N , not necessarily distinct in $[0, 1]^d$. For every $t = (t_1, \dots, t_d) \in [0, 1]^d$ we consider the parallelepiped (anchored at the origin)*

$$\mathcal{R}_t = \{(u_1, u_2, \dots, u_d) : 0 \leq u_j \leq t_j, \quad j = 1, 2, \dots, N\} .$$

Then there exists a constant $c > 0$ depending only on the dimension d and on the function f such that

$$\int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N \chi_{\mathcal{R}_t}(z_j) h(z_j) - \int_{\mathbb{T}^d} \chi_{\mathcal{R}_t}(x) f(x) dx \right|^2 dt \geq c \frac{\log^{d-1}(N)}{N^2} . \quad (3)$$

Observe that by letting $h(u) = f(u) = \text{constant} \neq 0$ we are back to Roth’s theorem.

Actually Chen’s result is more general, since it deals with the L^p norm ($p > 1$) of the discrepancy.

In this paper we are going to prove some analogs of Chen’s result in the spirit of the Schmidt-Montgomery-Beck theorem (see Theorem 6 and Theorem 10), that is, testing the discrepancy on balls or on convex bodies having boundaries with positive curvature. We point out that the constant c in (3) depends on f and, in our setting, one of our goals is to make such a dependence explicit. We also prove some sharp results on the discrepancies associated to absolutely continuous measures (see Theorem 7 and Theorem 12). See Section 3 below.

Our arguments are essentially Fourier analytic and (a variation of) a classical lemma of Cassels and Montgomery (see [36, Chapter 6], see also [10, 11]) is a basic tool.

2 Notation

- c, c_1, c_2, \dots positive constants which may change from step to step.
- $E \approx F$ when both $E \leq c_1 F$ and $F \leq c_2 E$ are true.
- $|G|$ volume of a measurable set G .
- $\|\alpha\|$ normalized ℓ^2 norm of the weights, see (4).
- B_r $= \{x \in \mathbb{T}^d : |x| < r\}$, ball of radius r .
- $B(z, r)$ $= \{x \in \mathbb{T}^d : |x - z| < r\}$, ball of radius r centered at z .
- $\|f\|_p$ L^p space norm, see (7).
- $\|f\|_{1,\lambda}$ Morrey norm, see (13).
- $\text{diam}(G)$ diameter of a set G .
- $\mathcal{D}_N(x)$ discrepancy (translations), see (6).
- $\mathcal{D}_N(x, r)$ discrepancy (translations and dilations), see (5).
- $\mathcal{D}_N(x, r, \sigma)$ discrepancy (translations, dilations, and rotations) see (17).
- $\widehat{\mathcal{D}}_N(m, r)$ Fourier coefficient of $\mathcal{D}_N(x, r)$, see (23).
- E_j see (18).
- $\|f\|_p$ L^p space norm, see (7).

$\ f\ _{1,\lambda}$	Morrey norm, see (13).
f_z	$= (f(z_1), \dots, f(z_N))$, see (40).
$J(N, f, r)$	see (19).
$K_M(x)$	see (30).
\mathcal{M}_{E_j}	see (41).
$\chi_G(x)$	characteristic function of a set G , see (2).

3 Main results

We first define the discrepancy that we are going to consider in this paper.

Definition 5 Let $f \in L^1(\mathbb{T}^d)$ and let z_1, \dots, z_N be N points in \mathbb{T}^d , not necessarily distinct. Let $\alpha_1, \dots, \alpha_N$ be non-negative real numbers (we call them weights), let $\alpha = (\alpha_1, \dots, \alpha_N)$, and let

$$\|\alpha\| = \left\{ \frac{1}{N} \sum_{j=1}^N \alpha_j^2 \right\}^{1/2}. \tag{4}$$

For $x \in \mathbb{T}^d$, $0 < r < 1/2$, and $B_r = \{x \in \mathbb{T}^d : |x| < r\}$, we define the discrepancy

$$\mathcal{D}_N(x, r) = \frac{1}{N} \sum_{j=1}^N \alpha_j \chi_{-x+B_r}(z_j) - \int_{-x+B_r} f(y) dy. \tag{5}$$

When B is a ball with fixed radius, we write $\mathcal{D}_N(x)$ in place of $\mathcal{D}_N(x, r)$, that is

$$\mathcal{D}_N(x) = \frac{1}{N} \sum_{j=1}^N \alpha_j \chi_B(z_j + x) - \int_B f(y + x) dy. \tag{6}$$

The following theorem is our first result. Here

$$\|f\|_p = \left\{ \int_{\mathbb{T}^d} |f(t)|^p dt \right\}^{1/p}. \tag{7}$$

Theorem 6 Let $1 < p \leq 2$ and let $1/p + 1/q = 1$. Let $0 < a < b < 1/2$. Then there exists a constant $c > 0$ such that for every complex-valued function $f \in L^p(\mathbb{T}^d)$, for every choice of N points z_1, z_2, \dots, z_N in \mathbb{T}^d , not necessarily distinct, and non-negative weights $\{\alpha_j\}_{j=1}^N$, not all zero, we have

$$\int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr \geq c N^{-1-q/(2d)} \|\alpha\|^{2+q/d} \|f\|_p^{-q/d}. \tag{8}$$

In particular, when $p = q = 2$ we have

$$\int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr \geq c N^{-1-1/d} \|\alpha\|^{2+2/d} \|f\|_2^{-2/d}. \tag{9}$$

In particular observe that for equal weights $\alpha_j = 1$ and for the Lebesgue density $f(x) = 1$ for every x , if $p = q = 2$ the above Theorem 6 reduces to the Theorem 3 of Schmidt, Montgomery and Beck,

$$\int_a^b \int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N \chi_{B_r+x}(z_j) - r^d |B| \right|^2 dx dr \geq c_1 N^{-1-1/d}.$$

Also observe that ratios of different L^p norms can be viewed as a quantification of concentration or localization of the density function. Thus if $\|f\|_1 = 1$ is fixed, i.e. the measure is a probability measure, then $\|f\|_2$ can be viewed as such a quantification, and the right hand side of (9) is minimized when $f(x) = 1$, i.e. the measure is Lebesgue. See also the following Remark 8. In this sense the above result can be considered a contribution to the question raised at the very beginning of this paper.

The next result shows that the RHS in (8) cannot be improved if $p = 2$. We do not know if (8) is sharp in the range $1 < p < 2$. Note that if $p \rightarrow 1^+$ and $\|\alpha\| = \|f\|_p = 1$, then the RHS in (8) vanishes. Actually, the second part of the next theorem shows that for $p = 1$ the discrepancy can be as small as we wish.

Theorem 7 1) Let $p > 1$. Let H be a positive integer; let $N = H^d$ and let

$$\{z_1, \dots, z_N\} = \frac{1}{H} \mathbb{Z}^d \cap \left[-\frac{1}{2}, \frac{1}{2}\right)^d. \tag{10}$$

Assume $\alpha_1 = \dots = \alpha_N = 1$ (hence $\|\alpha\| = 1$). Then there exist functions $f \in L^p(\mathbb{T}^d)$ of arbitrarily large L^p norms, such that, for any given ball B in \mathbb{T}^d ,

$$\int_{\mathbb{T}^d} |\mathcal{D}_N(x)|^2 dx \leq c N^{-1-1/d} \|f\|_p^{-q/d}. \tag{11}$$

2) Let $p = 1$. Then for every $\varepsilon > 0$, for every sequence z_1, \dots, z_N of points in \mathbb{T}^d , not necessarily distinct, and for every sequence $\alpha_1, \alpha_2, \dots, \alpha_N$ of weights with

$$\frac{1}{N} \sum_{j=1}^N \alpha_j = 1,$$

there exists a function $f \in L^1(\mathbb{T}^d)$ with $\|f\|_1 \geq 1$ such that

$$\int_{\mathbb{T}^d} |\mathcal{D}_N(x)|^2 dx < \varepsilon. \tag{12}$$

Observe that at least for $p = q = 2$ the upper bound in (11) matches the lower bound in (9). Also observe that the assumption $\alpha_j \geq 0$ for every j in Theorem 6 cannot be dropped. See Remark 17 in the next section.

Remark 8 The exponents in the term $\|\alpha\|^{2+q/d} \|f\|_p^{-q/d}$ in (8) may have the following explanation. First we observe that the discrepancy associated to the Dirac delta δ_0 can be zero. Indeed, if $z_1 = \dots = z_N = 0$ and $N^{-1} \sum_{j=1}^N \alpha_j = 1$, then, for any measurable set $R \subset \mathbb{T}^d$, we have

$$D_N(R) = \chi_R(0) - \delta(R) = 0.$$

The above trivial observation suggests that the “closer” is a function f to a Dirac delta, the smaller its discrepancy can be. Since an L^p function close to the Dirac delta has large L^p norm (if $p > 1$), then it is natural to expect that a small discrepancy should be associated to a negative power of the L^p norm of the function f . The exponent $2 + q/d$ of $\|\alpha\|$ comes from the following simple homogeneity argument. Assume we multiply both the vector α and the function f by the same constant t . Then

$$\int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr$$

gets multiplied by t^2 . Therefore the sum of the exponents of $\|\alpha\|$ and $\|f\|_p$ must be 2. In particular, if $\|f\|_p$ is raised to the power $-q/d$, then $\|\alpha\|$ needs the power $2 + q/d$.

Besides L^p spaces, we can consider other function spaces. We are going to consider Morrey spaces (see e.g. [1]), which seem to be quite tailored for the problems considered in this paper.

Definition 9 Let $B(z, r)$ be the ball centered at $z \in \mathbb{T}^d$, with radius r , let $0 \leq \lambda \leq d$, and let $f \in L^1(\mathbb{T}^d)$. The Morrey spaces $\mathcal{M}_{1,\lambda}(\mathbb{T}^d)$ are defined by the norms

$$\|f\|_{1,\lambda} = \sup_{z \in \mathbb{T}^d, 0 < r < 1/2} \frac{1}{r^\lambda} \int_{B(z,r)} |f(x)| dx. \tag{13}$$

Observe that $\mathcal{M}_{1,0}(\mathbb{T}^d) = L^1(\mathbb{T}^d)$ and, by the Lebesgue differentiation theorem, $\mathcal{M}_{1,d}(\mathbb{T}^d) = L^\infty(\mathbb{T}^d)$. Also observe that the Hölder inequality yields the imbedding $L^p(\mathbb{T}^d) \subseteq \mathcal{M}_{1,d/q}(\mathbb{T}^d)$, where $1 < p < +\infty$, and p, q are conjugate indices. Indeed,

$$\begin{aligned} \|f\|_{1,d/q} &= \sup_{z \in \mathbb{T}^d, 0 < r < 1/2} \frac{1}{r^{d/q}} \int_{B(z,r)} |f(x)| dx \\ &\leq \sup_{z \in \mathbb{T}^d, 0 < r < 1/2} \frac{1}{r^{d/q}} |B(z, r)|^{1/q} \|f\|_p = c \|f\|_p. \end{aligned} \tag{14}$$

The following result concerns the discrepancies associated to functions in Morrey spaces.

Theorem 10 *Let $0 < a < b < 1/2$ and $0 < \lambda \leq d$. Then there exists a constant $c > 0$ such that for every choice of N points z_1, z_2, \dots, z_N , not necessarily distinct, in \mathbb{T}^d , for all choices of non-negative weights $\alpha_1, \dots, \alpha_N$ and for every non-negative function $f \in \mathcal{M}_{1,\lambda}(\mathbb{T}^d)$, we have*

$$\int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr \geq c N^{-1-1/\lambda} \|\alpha\|^{2+1/\lambda} \|f\|_{1,\lambda}^{-1/\lambda} . \tag{15}$$

Remark 11 The above theorem and the imbedding of $L^p(\mathbb{T}^d)$ in the Morrey space $\mathcal{M}_{1,d/q}(\mathbb{T}^d)$ (see (14)) yield

$$\begin{aligned} \int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr &\geq c N^{-1-q/d} \|\alpha\|^{2+q/d} \|f\|_{1,d/q}^{-q/d} \\ &\geq c N^{-1-q/d} \|\alpha\|^{2+q/d} \|f\|_p^{-q/d} . \end{aligned} \tag{16}$$

The above argument is quite simple, and it is not surprising that the inequality (16), with the term $N^{-1-q/d}$, is weaker than the inequality (8), with the term $N^{-1-q/(2d)}$.

The next result shows that the lower bound (15) is sharp.

Theorem 12 *Let $0 < \lambda < d$, let H be a positive integer, and let $N = H^d$. Let*

$$\{z_1, \dots, z_N\} = \frac{1}{H} \mathbb{Z}^d \cap \left[-\frac{1}{2}, \frac{1}{2}\right)^d ,$$

and let $\alpha_1 = \dots = \alpha_N = 1$ (hence $\|\alpha\| = 1$). Then there exist functions $f \in \mathcal{M}_{1,\lambda}(\mathbb{T}^d)$ of arbitrarily large norms, such that, for every ball B ,

$$\int_{\mathbb{T}^d} |\mathcal{D}_N(x)|^2 dx \leq c N^{-1-1/\lambda} \|f\|_{1,\lambda}^{-1/\lambda} .$$

Remark 13 The above results can be extended with no difficulties to the case where the balls B_r are replaced with the dilates of a convex body $C \subset \mathbb{T}^d$, and smooth boundary with positive curvature. The key ingredients are classical estimates on the decay of the Fourier transform of a convex body, see e.g. [44] and the references therein. If we drop the smoothness and curvature assumptions, we can obtain similar results by averaging over rotations of the body (which then needs to have diameter < 1). More precisely, for $\sigma \in SO(d)$ we define

$$\mathcal{D}_N(x, r, \sigma) = \frac{1}{N} \sum_{j=1}^N \alpha_j \chi_{r\sigma(C)}(z_j + x) - \int_{r\sigma(C)} f(y + x) dy . \tag{17}$$

Then (8) can be replaced with

$$\int_{SO(d)} \int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r, \sigma)|^2 dx dr d\sigma \geq c N^{-1-q/(2d)} \|\alpha\|^{2+q/d} \|f\|_p^{-q/d} ,$$

and (15) can be replaced with

$$\int_{SO(d)} \int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r, \sigma)|^2 dx dr d\sigma \geq c N^{-1-1/\lambda} \|\alpha\|^{2+1/\lambda} \|f\|_{1,\lambda}^{-1/\lambda} .$$

Moreover, there are analogues of Theorem 7 and Theorem 12. We omit the details. The key observation is that the classical estimates for the pointwise decay of the Fourier transforms of a convex body can be replaced by L^2 spherical averages of the decay of these Fourier transforms. See [9, 10, 13].

For the classical case, that is $f(x) dx$ replaced with the Lebesgue measure dx , sharp estimates for the discrepancy can be obtained by using the so-called *jittered sampling*, which in this paper consists in partitioning \mathbb{T}^d into N disjoint measurable sets E_1, \dots, E_N , having volume N^{-1} and diameter $\approx N^{-1/d}$ (as an example, let $N = H^d$, where H is a positive integer, and split $\mathbb{T}^d = [-1/2, 1/2)^d$ as the union of N equal half-open disjoint cubes with side-length $N^{-1/d}$ and volume N^{-1}). We recall that the existence of this decomposition is known in the very general setting of metric measure spaces (see [31]). Next we take a random point z_j inside each E_j , and we let $\alpha_j = f(z_j)$. See e.g. [6, 12, 23, 26, 34, 38, 46].

We have the following result.

Theorem 14 *Let N be a positive integer, and consider a decomposition of \mathbb{T}^d into a disjoint union of measurable sets E_j satisfying*

$$\mathbb{T}^d = \bigcup_{j=1}^N E_j , \text{ with } |E_j| = \frac{1}{N} \text{ and } \text{diam}(E_j) \approx N^{-1/d} . \tag{18}$$

Let $0 < a < b < 1/2$, let f be a non-negative function in $L^2(\mathbb{T}^d)$, and let

$$\begin{aligned} & J(N, f, r) \\ &= N \int_{E_1} \dots N \int_{E_N} \int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^N f(z_j) \chi_{-x+B_r}(z_j) - \int_{-x+B_r} f(y) dy \right|^2 dx dz_1 \dots dz_N \end{aligned} \tag{19}$$

denote the discrepancy of the so-called jittered sampling. Then there exists $c_1 > 0$, independent of N and f , such that

$$\int_a^b J(N, f, r) dr \geq c_1 N^{-1-1/d} \|f\|_2^2 . \tag{20}$$

Moreover there exist $c_2 > 0$, independent of N, f and r , such that

$$J(N, f, r) \leq c_2 N^{-1} \|f\|_2^2 . \tag{21}$$

These inequalities are best possible. That is, there exist positive constants c_3 and c_4 such that for every N there exist functions f_1 and f_2 in $L^2(\mathbb{T}^d)$ such that, for every r ,

$$J(N, f_1, r) \leq c_3 N^{-1-1/d} \|f_1\|_2^2, \quad J(N, f_2, r) \geq c_4 N^{-1} \|f_2\|_2^2. \tag{22}$$

Finally, the upper bound $J(N, f, r) \leq c_2 N^{-1} \|f\|_2^2$ in (21), which holds true for every function in $L^2(\mathbb{T}^d)$, can be improved under suitable regularity assumptions on f . Indeed we have the following result.

Theorem 15 *We keep the notation of the previous theorem. Let f be a non-negative and Hölder continuous function of order $0 < \beta \leq 1$ on \mathbb{T}^d , that is $|f(x) - f(y)| \leq |x - y|^\beta$ for all $x, y \in \mathbb{T}^d$. Then, for every r ,*

$$J(N, f, r) \leq \begin{cases} cN^{-1-2\beta/d} & \text{if } \beta < 1/2, \\ cN^{-1-1/d} & \text{if } \beta \geq 1/2. \end{cases}$$

Observe that in the limiting case $\beta = 0$ we obtain the bound cN^{-1} which coincides with the bound in (21), while when $\beta \geq 1/2$ we obtain the estimate $cN^{-1-1/d}$ which is independent of β but it cannot be improved because of the estimate (20).

4 Proofs

The starting point in our proofs is the Fourier expansion of the discrepancy function $x \mapsto \mathcal{D}_N(x, r)$. For every $m \in \mathbb{Z}^d$ the Fourier coefficients of this function are given by

$$\widehat{\mathcal{D}}_N(m, r) = \int_{\mathbb{T}^d} \mathcal{D}_N(x, r) e^{-2\pi imx} dx. \tag{23}$$

Lemma 16 *For every $m \in \mathbb{Z}^d$ we have*

$$\widehat{\mathcal{D}}_N(m, r) = \left(\frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi imz_j} - \widehat{f}(-m) \right) \widehat{\chi}_{B_r}(m). \tag{24}$$

Therefore, by Parseval’s formula,

$$\int_{\mathbb{T}^d} |\widehat{\mathcal{D}}_N(x, r)|^2 dx = \sum_{m \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi imz_j} - \widehat{f}(-m) \right|^2 |\widehat{\chi}_{B_r}(m)|^2. \tag{25}$$

Moreover, for every $0 < a < b < 1/2$,

$$\int_a^b \int_{\mathbb{T}^d} |\widehat{\mathcal{D}}_N(x, r)|^2 dx dr$$

$$\approx \sum_{m \in \mathbb{Z}^d} (1 + |m|)^{-d-1} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2. \tag{26}$$

Proof By Definition 5, the discrepancy $\mathcal{D}_N(x, r)$ is the convolution of χ_{B_r} with the signed measure

$$d\mu(y) = \frac{1}{N} \sum_{j=1}^N \alpha_j \delta_{z_j}(-y) - f(-y) dy.$$

Hence

$$\widehat{\mathcal{D}}_N(m, r) = \widehat{\chi_{B_r}}(m) \widehat{d\mu}(m).$$

Since

$$\begin{aligned} \widehat{d\mu}(m) &= \int_{\mathbb{T}^d} e^{-2\pi i m y} \frac{1}{N} \sum_{j=1}^N \alpha_j \delta_{z_j}(-y) - f(-y) dy \\ &= \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m). \end{aligned}$$

we have

$$\widehat{\mathcal{D}}_N(m, r) = \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} \widehat{\chi_{B_r}}(m) - \widehat{f}(-m) \widehat{\chi_{B_r}}(m).$$

Then Parseval’s identity yields (25). Since

$$\widehat{\chi_{B_r}}(m) = r^d \frac{J_{d/2}(2\pi |rm|)}{|rm|^{d/2}}, \tag{27}$$

the asymptotic expansions of the Bessel functions (see e.g. [45]) readily give

$$\int_a^b |\widehat{\chi_{B_r}}(m)|^2 dr \approx |m|^{-d-1} \tag{28}$$

for every $0 \neq m \in \mathbb{Z}^d$. Then we obtain (26). □

We also need the following simple inequality. For all complex numbers a, b , we have

$$|a - b|^2 \geq \frac{1}{2} |a|^2 - |b|^2. \tag{29}$$

The proof of (29) is straightforward,

$$|a|^2 \leq (|a - b| + |b|)^2 = |a - b|^2 + |b|^2 + 2|a - b||b| \leq 2|a - b|^2 + 2|b|^2.$$

Proof of Theorem 6 Let $k \in L^1(\mathbb{R}^d)$ be a radial non-negative function such that its Fourier transform

$$\widehat{k}(\xi) = \int_{\mathbb{R}^d} k(x) e^{-2\pi i x \xi} dx$$

is smooth, with $0 \leq \widehat{k}(\xi) \leq 1$, with $\widehat{k}(0) = 1$, and $\widehat{k}(\xi) = 0$ if $|\xi| \geq 1$. Let M be a positive integer to be chosen later, and define the periodic function

$$K_M(x) = \sum_{m \in \mathbb{Z}^d} M^d k(M(x+m)) = \sum_{m \in \mathbb{Z}^d} \widehat{k}(M^{-1}m) e^{2\pi i m x}. \tag{30}$$

Observe that

$$K_M(0) \geq k(0) M^d. \tag{31}$$

Moreover, by the assumptions on $\widehat{k}(\xi)$ we have $k(0) > 0$ and

$$0 \leq \widehat{K}_M(m) = \widehat{k}(M^{-1}m) \leq 1.$$

Since we are going to prove an estimate from below, then we may assume f real-valued. Then (26), (29) and (28) yield

$$\begin{aligned} & \int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr \\ & \approx \sum_{m \in \mathbb{Z}^d} (1 + |m|)^{-d-1} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\ & \geq c \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) (1 + |m|)^{-d-1} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\ & \geq c (1 + M)^{-d-1} \sum_{|m| \leq M} \widehat{K}_M(m) \left(\frac{1}{2} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} \right|^2 - |\widehat{f}(m)|^2 \right) \\ & \geq c M^{-d-1} \left(\frac{1}{2} \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} \right|^2 - \sum_{|m| \leq M} |\widehat{f}(m)|^2 \right) \\ & = c M^{-d-1} \left(\frac{1}{2} I - II \right). \tag{32} \end{aligned}$$

The non-negativity of the α_j and (31) yield

$$I = \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} \right|^2$$

$$\begin{aligned}
 &\geq \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) \frac{1}{N^2} \sum_{k=1}^N \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} \alpha_k e^{-2\pi i m z_k} \\
 &= \frac{1}{N^2} \sum_{j,k=1}^N \alpha_j \alpha_k K_M(z_j - z_k) \geq \frac{1}{N^2} \sum_{k=1}^N \alpha_k^2 K_M(0) \\
 &\geq k(0) M^d N^{-1} \|\alpha\|^2 .
 \end{aligned} \tag{33}$$

(The above nice argument is essentially due to H. Montgomery, see [36, Theorem 5.12]). In order to deal with II we set $r = q/2$ and $1/r + 1/r' = 1$ (hence $r' = q/(q - 2)$ and $r' = \infty$ when $q = 2$). Then the inequalities of Hölder and Hausdorff–Young yield

$$\begin{aligned}
 II &= \sum_{|m| \leq M} |\widehat{f}(m)|^2 \leq \left\{ \sum_{|m| \leq M} 1 \right\}^{1/r'} \left\{ \sum_{|m| \leq M} |\widehat{f}(m)|^{2r} \right\}^{1/r} \\
 &\leq cM^{d/r'} \left\{ \sum_{|m| \leq M} |\widehat{f}(m)|^q \right\}^{2/q} \leq cM^{d(1-2/q)} \|f\|_p^2 .
 \end{aligned}$$

Then

$$\begin{aligned}
 \frac{1}{2}I - II &\geq \frac{1}{2}k(0) M^d N^{-1} \|\alpha\|^2 - cM^{d(1-2/q)} \|f\|_p^2 \\
 &= M^d \left(\frac{1}{2}k(0) N^{-1} \|\alpha\|^2 - cM^{-2d/q} \|f\|_p^2 \right) .
 \end{aligned}$$

Now we set

$$M = \left(\frac{4cN \|f\|_p^2}{k(0) \|\alpha\|^2} \right)^{q/(2d)} ,$$

so that

$$cNM^{-2d/q} \|f\|_p^2 = \frac{1}{4}k(0) \|\alpha\|^2 .$$

Then we obtain

$$\begin{aligned}
 &\int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr \geq c_1 M^{-d-1} M^d N^{-1} \|\alpha\|^2 \\
 &= c_1 N^{-1} \left(\frac{4cN \|f\|_p^2}{k(0) \|\alpha\|^2} \right)^{-q/(2d)} \|\alpha\|^2 \\
 &= cN^{-1-q/(2d)} \|\alpha\|^{2+q/d} \|f\|_p^{-q/d} .
 \end{aligned}$$

□

Proof of Theorem 7 By Lemma 16, we have

$$\int_{\mathbb{T}^d} |\widehat{\mathcal{D}}_N(x)|^2 dx = \sum_{m \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 |\widehat{\chi}_B(m)|^2 .$$

We now choose

$$\{z_1, \dots, z_N\} = \frac{1}{H} \mathbb{Z}^d \cap \left[-\frac{1}{2}, \frac{1}{2} \right)^d$$

and $\alpha_j = 1$ for every j (hence $\|\alpha\| = 1$). We observe that

$$\sum_{j=1}^N e^{2\pi i m \cdot z_j} = \begin{cases} N & \text{if } m = Hk \text{ with } k \in \mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

Let F be any function in $L^p(\mathbb{T}^d)$ and let $f(x) = F(Hx)$. It is easy to check that $\|f\|_p = \|F\|_p$ and

$$f(x) = F(Hx) = \sum_{k \in \mathbb{Z}^d} \widehat{F}(k) e^{2\pi i Hkx} .$$

Hence

$$\widehat{f}(m) = \begin{cases} \widehat{F}(H^{-1}m) & \text{if } m = Hk, \text{ with } k \in \mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} & \sum_{m \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^N e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 |\widehat{\chi}_B(m)|^2 \\ &= \sum_{k \in \mathbb{Z}^d} |1 - \widehat{F}(-k)|^2 |\widehat{\chi}_B(Hk)|^2 . \end{aligned}$$

The above equality holds for every $F(x)$. Now we choose a smooth function $\phi(x)$ on \mathbb{R}^d satisfying

$$\widehat{\phi}(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1, \\ 0 & \text{if } |\xi| > 2. \end{cases}$$

For every $M > 0$ we set

$$F(x) = \sum_{k \in \mathbb{Z}^d} M^d \phi(M(x+k)) .$$

Then the Poisson summation formula gives

$$F(x) = \sum_{k \in \mathbb{Z}^d} M^d \phi(M(x+k)) = \sum_{m \in \mathbb{Z}^d} \widehat{\phi}(M^{-1}m) e^{2\pi i m x}. \tag{34}$$

Observe that under our assumptions $\phi(x)$ has fast decay at infinity, that is $|\phi(x)| \leq c(1+|x|)^{-L}$ for every $L > 0$. It follows that for every $x \in (-\frac{1}{2}, \frac{1}{2}]^d$ we have

$$|F(x)| \leq M^d |\phi(Mx)| + cM^{d-L} \sum_{0 \neq k \in \mathbb{Z}^d} |k|^{-L} \leq M^d |\phi(Mx)| + cM^{d-L},$$

$$|F(x)| \geq M^d |\phi(Mx)| - cM^{d-L} \sum_{0 \neq k \in \mathbb{Z}^d} |k|^{-L} \geq M^d |\phi(Mx)| - cM^{d-L}.$$

Hence

$$\begin{aligned} \|f\|_p &= \|F\|_p \approx M^d \left(\int_{(-\frac{1}{2}, \frac{1}{2}]^d} |\phi(Mx)|^p dx \right)^{1/p} \\ &\approx M^{d(1-1/p)} \left(\int_{\mathbb{R}^d} |\phi(y)|^p dy \right)^{1/p} = cM^{d/q}. \end{aligned}$$

Since $\widehat{F}(m) = \widehat{\phi}(M^{-1}m) = 1$ for $|m| \leq M$, then (28) yields

$$\sum_{m \in \mathbb{Z}^d} |1 - \widehat{F}(-m)|^2 |\widehat{\chi}_{B_r}(Hm)|^2 \leq c \sum_{|m| \geq M} |\widehat{\chi}_{B_r}(Hm)|^2 \leq cH^{-d-1}M^{-1}.$$

Since $H^d = N$ and since $M^{-1} \leq c\|f\|_p^{-q/d}$, one finally obtains

$$\sum_{m \in \mathbb{Z}^d} |1 - \widehat{F}(m)|^2 |\widehat{\chi}_{B_r}(Hm)|^2 \leq cN^{-1-1/d} \|f\|_p^{-q/d}.$$

Now we prove the second part of this theorem. Let $\delta > 0$ to be chosen later, and, as in the first part of the proof, construct a function $F(x)$ such that $\widehat{F}(m) = 1$ if $|m| \leq \delta$. Observe that $\|F\|_1$ can be bounded independently of δ . Finally define

$$f(x) = \frac{1}{N} \sum_{j=1}^N \alpha_j F(x - z_j).$$

Observe that

$$\widehat{f}(m) = \left(\frac{1}{N} \sum_{j=1}^N \alpha_j e^{-2\pi i m z_j} \right) \widehat{F}(m).$$

In particular

$$\|f\|_1 \geq |\widehat{f}(0)| = \left| \left(\frac{1}{N} \sum_{j=1}^N \alpha_j \right) \widehat{F}(0) \right| = 1.$$

Then (27) yields the upper bound

$$\begin{aligned} \int_{\mathbb{T}^d} |\mathcal{D}_N(x)|^2 dx &\leq c \sum_{m \in \mathbb{Z}^d} |m|^{-(d+1)} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\ &= c \sum_{m \in \mathbb{Z}^d} |m|^{-(d+1)} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} \right|^2 |1 - \widehat{F}(-m)|^2 \\ &\leq c \sum_{m \in \mathbb{Z}^d} |m|^{-(d+1)} |1 - \widehat{F}(-m)|^2. \end{aligned}$$

Hence, since $\widehat{F}(m) = 1$ when $|m| \leq \delta$, letting δ large enough yields

$$\begin{aligned} \int_{\mathbb{T}^d} |\mathcal{D}_N(x)|^2 dx &\leq c \sum_{|m| > \delta} |m|^{-(d+1)} |1 - \widehat{F}(-m)|^2 \\ &\leq c \sum_{|m| > \delta} |m|^{-(d+1)} \leq c \delta^{-1} < \varepsilon. \end{aligned}$$

□

Remark 17 Let us briefly discuss the assumption $\alpha_j \geq 0$. For instance, if the α_j are not all zero and we assume $\sum_{j=1}^N \alpha_j = 0$, together with $z_j = 0$ for every j , then

$$\begin{aligned} &\int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr \\ &\approx \sum_{m \in \mathbb{Z}^d} (1 + |m|)^{-d-1} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\ &= \sum_{m \in \mathbb{Z}^d} (1 + |m|)^{-d-1} |\widehat{f}(-m)|^2. \end{aligned}$$

Now let $f(x) = e^{2\pi i k x}$. Then $\|f\|_p = 1$ and

$$\int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr \approx (1 + |k|)^{-d-1}.$$

This quantity can be made arbitrarily small as $|k|$ gets large.

Proof of Theorem 10 Let $M > 0$ be a constant to be chosen later. By (26) we have

$$\begin{aligned} & \int_a^b \int_{\mathbb{T}^d} |\widehat{D}_N(x, r)|^2 dx dr \\ & \approx \sum_{m \in \mathbb{Z}^d} (1 + |m|)^{-d-1} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\ & \geq (1 + M)^{-d-1} \sum_{|m| \leq M} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 . \end{aligned}$$

Let $K_M(x)$ be as in (30). Observe that, for every $h > 0$, there exists a positive constant c such that

$$0 \leq K_M(x) \leq c M^d (1 + M|x|)^{-h} . \tag{35}$$

If $h > \lambda$, then (35) yields

$$\begin{aligned} 0 \leq K_M * f(x) &= \int_{\mathbb{T}^d} K_M(y) f(x - y) dy \\ &= \int_{\{|y| \leq M^{-1}\}} K_M(y) f(x - y) dy \\ &\quad + \sum_{k=0}^{+\infty} \int_{\{2^k M^{-1} < |y| \leq 2^{k+1} M^{-1}\} \cap \mathbb{T}^d} K_M(y) f(x - y) dy \\ &\leq c M^d \int_{\{|y| \leq M^{-1}\}} f(x - y) dy \\ &\quad + c M^d \sum_{k=0}^{+\infty} 2^{-hk} \int_{\{|y| \leq 2^{k+1} M^{-1}\} \cap \mathbb{T}^d} f(x - y) dy \\ &\leq c M^{d-\lambda} \frac{1}{(M^{-1})^\lambda} \int_{\{|y| \leq M^{-1}\}} |f(x - y)| dy \\ &\quad + c M^d \sum_{k=0}^{+\infty} 2^{-hk} (2^{k+1} M^{-1})^\lambda \\ &\quad \times \frac{1}{(2^{k+1} M^{-1})^\lambda} \int_{\{|y| \leq 2^{k+1} M^{-1}\} \cap \mathbb{T}^d} |f(x - y)| dy \\ &\leq c M^{d-\lambda} \|f\|_{1,\lambda} + c 2^\lambda M^{d-\lambda} \sum_{k=0}^{+\infty} 2^{-(h-\lambda)k} \|f\|_{1,\lambda} \\ &\leq c M^{d-\lambda} \|f\|_{1,\lambda} . \end{aligned} \tag{36}$$

Since $0 \leq \widehat{K}_M(m) \leq 1$, we have

$$\begin{aligned}
 & \sum_{|m| \leq M} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\
 & \geq \sum_{|m| \leq M} \widehat{K}_M(m) \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\
 & = \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) \left(\frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right) \\
 & \quad \times \left(\frac{1}{N} \sum_{k=1}^N \alpha_k e^{-2\pi i m z_k} - \overline{\widehat{f}(-m)} \right) \\
 & = \frac{1}{N^2} \sum_{j,k=1}^N \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) \alpha_k \alpha_j e^{2\pi i m(z_j - z_k)} \\
 & \quad - \frac{1}{N} \sum_{j=1}^N \alpha_j \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) e^{2\pi i m z_j} \overline{\widehat{f}(-m)} \\
 & \quad - \frac{1}{N} \sum_{k=1}^N \alpha_k \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) \widehat{f}(-m) e^{-2\pi i m z_k} \\
 & \quad + \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) |\widehat{f}(m)|^2 \\
 & = \frac{1}{N^2} \sum_{j,k=1}^N \alpha_k \alpha_j K_M(z_j - z_k) - \frac{2}{N} \operatorname{Re} \left(\sum_{j=1}^N \alpha_j K_M * f(z_j) \right) \\
 & \quad + \sum_{m \in \mathbb{Z}^d} \widehat{K}_M(m) |\widehat{f}(m)|^2 . \tag{37}
 \end{aligned}$$

We are left with the three terms in (37). For the first one, as in (33), the non-negativity assumption for the α_j yields

$$\frac{1}{N^2} \sum_{j,k=1}^N \alpha_k \alpha_j K_M(z_j - z_k) \geq \frac{1}{N^2} \sum_{k=1}^N \alpha_k^2 K_M(0) \geq c_1 \frac{1}{N} M^d \|\alpha\|^2 .$$

As for the second term, (36) and the Cauchy-Schwarz inequality yield

$$\frac{2}{N} \operatorname{Re} \left(\sum_{j=1}^N \alpha_j K_M * f(z_j) \right) \leq \frac{2}{N} \left| \sum_{j=1}^N \alpha_j K_M * f(z_j) \right|$$

$$\leq 2 \|\alpha\| \sup_{z \in \mathbb{T}^d} |K_M * f(z)| \leq c_2 \|\alpha\| M^{d-\lambda} \|f\|_{1,\lambda} .$$

We observe that the third term is non-negative. Then

$$\begin{aligned} & \sum_{|m| \leq M} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\ & \geq c_1 \frac{1}{N} M^d \|\alpha\|^2 - c_2 \|\alpha\| M^{d-\lambda} \|f\|_{1,\lambda} . \end{aligned}$$

Setting

$$M = \left(\frac{2c_2 N \|f\|_{1,\lambda}}{c_1 \|\alpha\|} \right)^{1/\lambda}$$

we have

$$c_1 \frac{1}{N} M^d \|\alpha\|^2 = 2c_2 \|\alpha\| M^{d-\lambda} \|f\|_{1,\lambda} .$$

Then

$$\begin{aligned} & \int_a^b \int_{\mathbb{T}^d} |\mathcal{D}_N(x, r)|^2 dx dr \\ & \geq c M^{-d-1} \sum_{|m| \leq M} \left| \frac{1}{N} \sum_{j=1}^N \alpha_j e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 \\ & \geq c M^{-1} N^{-1} \|\alpha\|^2 = c N^{-1-1/\lambda} \|f\|_{1,\lambda}^{-1/\lambda} \|\alpha\|^{2+1/\lambda} . \end{aligned} \tag{38}$$

Then (15) is proved. □

The proof of Theorem 12 is not very different from the proof of Theorem 7.

Proof of Theorem 12 As in the proof of Theorem 7, let $N = H^d$, where H is a positive integer. Let

$$\{z_1, \dots, z_N\} = \frac{1}{H} \mathbb{Z}^d \cap \left[-\frac{1}{2}, \frac{1}{2} \right)^d ,$$

let $\alpha_j = 1$ for every j , let $M > 0$ and let $f(x) = F(Hx)$ with $F \in L^p(\mathbb{T}^d)$ such that $\widehat{F}(m) = 1$ if $|m| \leq M$. As in (34) let

$$F(x) = \sum_{k \in \mathbb{Z}^d} M^d \phi(M(x+k)) = \sum_{m \in \mathbb{Z}^d} \widehat{\phi}(m/M) e^{2\pi i m x} .$$

By the fast decay of $\phi(x)$, i.e. $|\phi(x)| \leq c(1+|x|)^{-L}$, for every $L > 0$, we have

$$\|F\|_{1,\lambda} = \sup_{z \in [-1/2, 1/2)^d, 0 < r < 1/2} \frac{1}{r^\lambda} \int_{B(z,r)} \left| M^d \sum_{k \in \mathbb{Z}^d} \phi(M(x+k)) \right| dx$$

$$\begin{aligned} &\leq \sup_{z \in \mathbb{R}^d, r > 0} \frac{1}{r^\lambda} \int_{B(z,r)} |M^d \phi(Mx)| dx + cM^{d-L} \sum_{0 \neq k \in \mathbb{Z}^d} |k|^{-L} \\ &\leq M^\lambda \sup_{z \in \mathbb{R}^d, r > 0} \frac{1}{(Mr)^\lambda} \int_{B(z,Mr)} |\phi(y)| dy + cM^{d-L} \leq cM^\lambda. \end{aligned}$$

We claim that

$$\|f\|_{1,\lambda} \leq c \|F\|_{1,\lambda} \leq cM^\lambda.$$

Indeed,

$$\begin{aligned} \|f\|_{1,\lambda} &= \sup_{z \in \mathbb{T}^d, 0 < r < 1/2} \frac{1}{r^\lambda} \int_{B(z,r)} |F(Hx)| dx \\ &= \sup_{z \in [-1/2, 1/2]^d, 0 < r < 1/2} \frac{1}{r^\lambda} \int_{B(Hz, Hr)} |F(y)| H^{-d} dy \\ &= \sup_{z \in [-1/2, 1/2]^d, 0 < r < 1/2} \frac{1}{r^\lambda} \int_{B(z, Hr)} |F(y)| H^{-d} dy \\ &= H^{\lambda-d} \sup_{z \in [-1/2, 1/2]^d, 0 < r < 1/2} \frac{1}{(Hr)^\lambda} \int_{B(z, Hr)} |F(y)| dy. \end{aligned}$$

Let $Hr < 1/2$, then

$$\begin{aligned} &\frac{1}{(Hr)^\lambda} \int_{B(z, Hr)} |F(y)| dy \\ &\leq \sup_{z \in [-1/2, 1/2]^d, 0 < s < 1/2} \frac{1}{s^\lambda} \int_{B(z, s)} |F(y)| dy \leq \|F\|_{1,\lambda}. \end{aligned}$$

Let $Hr \geq 1/2$. Since $r < 1/2$ and $\lambda \leq d$, then

$$\begin{aligned} \frac{1}{(Hr)^\lambda} \int_{B(z, Hr)} |F(y)| dy &\leq \frac{c}{(Hr)^\lambda} (Hr)^d \int_{[-1/2, 1/2]^d} |F(y)| dy \\ &\leq cr^{d-\lambda} H^{d-\lambda} \|F\|_{1,\lambda} \leq cH^{d-\lambda} \|F\|_{1,\lambda}. \end{aligned}$$

Hence,

$$\|f\|_{1,\lambda} \leq c \max \left(H^{\lambda-d} \|F\|_{1,\lambda}, \|F\|_{1,\lambda} \right) \leq c \|F\|_{1,\lambda} \leq cM^\lambda.$$

Since $\widehat{F}(m) = 1$ for $|m| \leq M$, then

$$\begin{aligned} \sum_{m \in \mathbb{Z}^d} |1 - \widehat{F}(-m)|^2 |\widehat{\chi}_{B_r}(Hm)|^2 &\leq c \sum_{|m| > M} |\widehat{\chi}_{B_r}(Hm)|^2 \\ &\leq cH^{-d-1} M^{-1}. \end{aligned}$$

Since $H^d = N$ and since $M^{-1} \leq c \|f\|_{1,\lambda}^{-1/\lambda}$ one finally obtains

$$\sum_{m \in \mathbb{Z}^d} |1 - \widehat{F}(-m)|^2 |\widehat{\chi}_{B_r}(Hm)|^2 \leq c N^{-1-1/d} \|f\|_{1,\lambda}^{-1/\lambda}.$$

□

Proof of Theorem 14 We recall (see (24)) that

$$\widehat{D}_N(m) = \widehat{\chi}_{B_r}(m) \left[\frac{1}{N} \sum_{j=1}^N f(z_j) e^{2\pi i m \cdot z_j} - \widehat{f}(-m) \right].$$

Since f is real-valued we have $\widehat{f}(-m) = \overline{\widehat{f}(m)}$. Then, see (19), Parseval’s identity yields

$$\begin{aligned} & J(N, f, r) \\ &= N \int_{E_1} \cdots N \int_{E_N} \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \\ &\quad \times \left| \frac{1}{N} \sum_{j=1}^N f(z_j) e^{2\pi i m z_j} - \widehat{f}(-m) \right|^2 dz_1 \cdots dz_N \\ &= \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 N \int_{E_1} \cdots N \int_{E_N} \\ &\quad \times \left(\frac{1}{N^2} \sum_{j=1}^N f^2(z_j) + \frac{1}{N^2} \sum_{j \neq k} f(z_j) f(z_k) e^{-2\pi i m(z_j - z_k)} \right. \\ &\quad \left. - \frac{2}{N} \operatorname{Re} \left(\overline{\widehat{f}(-m)} \sum_{j=1}^N f(z_j) e^{2\pi i m z_j} \right) + |\widehat{f}(m)|^2 \right) dz_1 \cdots dz_N \\ &= \frac{1}{N} \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \sum_{j=1}^N \int_{E_j} f^2(z_j) dz_j \\ &\quad + \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \sum_{j \neq k} \int_{E_j} f(z_j) e^{-2\pi i m z_j} dz_j \\ &\quad \times \int_{E_k} f(z_k) e^{2\pi i m z_k} dz_k \\ &\quad - 2 \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \operatorname{Re} \left(\widehat{f}(m) \sum_{j=1}^N \int_{E_j} f(z_j) e^{2\pi i m z_j} dz_j \right) \\ &\quad + \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 |\widehat{f}(m)|^2 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{N} \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \|f\|_2^2 + \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \sum_{j,k=1}^N \left(\int_{E_j} f(z_j) e^{-2\pi i m z_j} dz_j \right. \\
 &\quad \left. \times \int_{E_k} f(z_k) e^{2\pi i m z_k} dz_k \right) - \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \sum_{j=1}^N \left| \int_{E_j} f(z_j) e^{-2\pi i m z_j} dz_j \right|^2 \\
 &\quad - 2 \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 |\widehat{f}(m)|^2 + \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 |\widehat{f}(m)|^2 \\
 &= \frac{1}{N} \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \|f\|_2^2 + \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 |\widehat{f}(m)|^2 \\
 &\quad - \sum_{j=1}^N \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 \left| \int_{E_j} f(z) e^{-2\pi i m z} dz \right|^2 \\
 &\quad - \sum_{m \in \mathbb{Z}^d} |\widehat{\chi}_{B_r}(m)|^2 |\widehat{f}(m)|^2 \\
 &= \frac{1}{N} |B_r| \|f\|_2^2 - \sum_{j=1}^N \|\chi_{B_r} * (f \chi_{E_j})\|_2^2 .
 \end{aligned}$$

In short, we have the nice identity

$$J(N, f, r) = N^{-1} |B_r| \|f\|_2^2 - \sum_{j=1}^N \|\chi_{B_r} * (f \chi_{E_j})\|_2^2 , \tag{39}$$

which holds true for every $f \in L^2(\mathbb{T}^d)$. Then we readily obtain

$$J(N, f, r) \leq c N^{-1} \|f\|_2^2 .$$

In order to prove the inequality (20), that is

$$\int_a^b J(N, f, r) dr \geq c_1 N^{-1-1/d} \|f\|_2^2 ,$$

we appeal to the lower estimate in (9), which holds true for every choice of the points z_1, \dots, z_N , and then we integrate z_1, \dots, z_N . Let

$$f_z = (f(z_1), \dots, f(z_N)) . \tag{40}$$

Then the Hölder inequality yields

$$\int_a^b J(N, f, r) dr$$

$$\begin{aligned}
 &= N \int_{E_1} \cdots N \int_{E_N} \int_{\mathbb{T}^d} \int_a^b \\
 &\quad \times \left| \frac{1}{N} \sum_{j=1}^N f(z_j) \chi_{-x+B_r}(z_j) - \int_{-x+B_r} f(y) dy \right|^2 dr dx dz_1 \cdots dz_N \\
 &\geq cN \int_{E_1} \cdots N \int_{E_N} N^{-1-1/d} \|f_z\|^{2+2/d} \|f\|_2^{-2/d} dz_1 \cdots dz_N \\
 &= cN \int_{E_1} \cdots N \int_{E_N} N^{-1-1/d} \left(\frac{1}{N} \sum_{j=1}^N f^2(z_j) \right)^{1+1/d} \\
 &\quad \times \|f\|_2^{-2/d} dz_1 \cdots dz_N \\
 &\geq cN^{-1-1/d} \|f\|_2^{-2/d} \\
 &\quad \times \left(N \int_{E_1} \cdots N \int_{E_N} \frac{1}{N} \sum_{j=1}^N f^2(z_j) dz_1 \cdots dz_N \right)^{1+1/d} \\
 &= cN^{-1-1/d} \|f\|_2^{-2/d} \left(\frac{1}{N} \sum_{j=1}^N N \int_{E_j} f^2(z_j) dz_j \right)^{1+1/d} \\
 &= cN^{-1-1/d} \|f\|_2^{-2/d} \|f\|_2^{2+2/d} = cN^{-1-1/d} \|f\|_2^2 .
 \end{aligned}$$

So far we have proved (20) and (21). We still have to prove (22). We recall that

$$J(N, f, r) = N^{-1} |B_r| \|f\|_2^2 - \sum_{j=1}^N \|\chi_{B_r} * (f \chi_{E_j})\|_2^2 .$$

We choose f supported in E_1 and such that $\|f\|_2^2 \geq 2N \|f\|_1^2$. Then

$$\begin{aligned}
 J(N, f, r) &= N^{-1} |B_r| \|f\|_2^2 - \|\chi_{B_r} * f\|_2^2 \\
 &\geq N^{-1} |B_r| \|f\|_2^2 - \|\chi_{B_r}\|_2^2 \|f\|_1^2 = N^{-1} |B_r| (\|f\|_2^2 - N \|f\|_1^2) \\
 &\geq \frac{1}{2} N^{-1} |B_r| \|f\|_2^2 .
 \end{aligned}$$

Finally, the existence of a function f satisfying

$$J(N, f, r) \leq c_3 N^{1-1/d} \|f\|_2^2$$

is a particular case of Theorem 15. □

Proof of Theorem 15 In this proof the integration in r plays no role. Hence we may forget r and replace B_r with B . The RHS in (39) equals

$$\sum_{j=1}^N \left(N^{-1} |B| \|f \chi_{E_j}\|_2^2 - \|\chi_B * (f \chi_{E_j})\|_2^2 \right)$$

We are going to show that all the terms in the above sum give the same contribution. For every function $g \in L^1(\mathbb{T}^d)$ and every $j = 1, \dots, N$ let

$$\mathcal{M}_{E_j}(g) = \frac{1}{|E_j|} \int_{E_j} g. \tag{41}$$

Then, for every $j = 1, \dots, N$

$$\begin{aligned} & N^{-1} |B| \|f \chi_{E_j}\|_2^2 - \|\chi_B * (f \chi_{E_j})\|_2^2 \\ &= N^{-2} |B| \left(\mathcal{M}_{E_j}(f^2) - \mathcal{M}_{E_j}^2(f) \right) \\ &\quad + N^{-2} \left(|B| \mathcal{M}_{E_j}^2(f) - \left\| \chi_B * \frac{f \chi_{E_j}}{|E_j|} \right\|_2^2 \right) \\ &= N^{-2} |B| \left(\mathcal{M}_{E_j} \left((f - \mathcal{M}_{E_j}(f))^2 \right) \right) \\ &\quad + N^{-2} \left(|B| \mathcal{M}_{E_j}^2(f) - \left\| \chi_B * \frac{f \chi_{E_j}}{|E_j|} \right\|_2^2 \right). \end{aligned}$$

We recall that $\text{diam}(E_j) \approx N^{-1/d}$. Since f is Hölder continuous with exponent β , then, for any $x \in E_j$, we have

$$|f(x) - \mathcal{M}_{E_j}(f)| \leq c \text{diam}(E_j)^\beta \leq c N^{-\beta/d}.$$

Hence

$$N^{-2} |B| \mathcal{M}_{E_j} \left((f - \mathcal{M}_{E_j}(f))^2 \right) \leq c N^{-2-2\beta/d}. \tag{42}$$

Let us consider the second term, that is

$$N^{-2} \left(|B| \mathcal{M}_{E_j}^2(f) - \left\| \chi_B * \frac{f \chi_{E_j}}{|E_j|} \right\|_2^2 \right).$$

Let $t_j \in E_j$ and $B_j = t_j + B$. First observe that if $x \in B_j$ and $\text{dist}(x, \partial B_j) \geq c N^{-1/d}$ then

$$\chi_B * \frac{f \chi_{E_j}}{|E_j|}(x) = \frac{1}{|E_j|} \int_{E_j} f(y) dy = \mathcal{M}_{E_j}(f).$$

Moreover if $x \notin B_j$ and $\text{dist}(x, \partial B_j) \geq cN^{-1/d}$, then

$$\chi_B * \frac{f \chi_{E_j}}{|E_j|}(x) = 0.$$

In short, the assumption $\text{dist}(x, \partial B_j) \geq cN^{-1/d}$ yields

$$\chi_B * \frac{f \chi_{E_j}}{|E_j|}(x) = \mathcal{M}_{E_j}(f) \chi_{B_j}(x). \tag{43}$$

Now let

$$\mathcal{C}_j = \left\{ x \in \mathbb{T}^d : \text{dist}(x, \partial B_j) \leq cN^{-1/d} \right\}.$$

Observe that $|\mathcal{C}_j| \leq cN^{-1/d}$. Then (43) and the Cauchy-Schwartz inequality $\mathcal{M}_{E_j}^2(f) \leq \mathcal{M}_{E_j}(f^2)$ yield

$$\begin{aligned} & |B| \mathcal{M}_{E_j}^2(f) - \left\| \chi_B * \frac{f \chi_{E_j}}{|E_j|} \right\|_2^2 \\ &= |B| \mathcal{M}_{E_j}^2(f) - \left(\mathcal{M}_{E_j}^2(f) |B_j \setminus \mathcal{C}_j| + \int_{\mathcal{C}_j} \left(\chi_B * \frac{f \chi_{E_j}}{|E_j|}(x) \right)^2 dx \right) \\ &= \mathcal{M}_{E_j}^2(f) |B_j \cap \mathcal{C}_j| - \int_{\mathcal{C}_j} \left(\chi_B * \frac{f \chi_{E_j}}{|E_j|}(x) \right)^2 dx \\ &\leq \mathcal{M}_{E_j}^2(f) |B_j \cap \mathcal{C}_j| \leq \mathcal{M}_{E_j}^2(f) |\mathcal{C}_j| \leq cN^{-1/d} \mathcal{M}_{E_j}(f^2). \end{aligned} \tag{44}$$

Then, using (42), (44), and summing over all E_j we obtain

$$\begin{aligned} & N \int_{E_1} \dots N \int_{E_N} \int_{\mathbb{T}^d} |\mathcal{D}_N(x)|^2 dx dz_1 \dots dz_N \\ &\leq c \sum_{j=1}^N \left(N^{-2-2\beta/d} + N^{-2-1/d} \mathcal{M}_{E_j}(f^2) \right) \\ &\leq c \left(N^{-1-2\beta/d} + N^{-1-1/d} \|f\|_2^2 \right). \end{aligned}$$

This ends the proof of Theorem 15. □

Funding Open access funding provided by Università degli Studi di Milano - Bicocca within the CRUI-CARE Agreement.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Adams, D.: Morrey spaces. Birkhäuser (2015)
2. Aistleitner, C., Bilyk, D., Nikolov, A.: Tusnády's problem, the transference principle, and non-uniform QMC sampling. in: Monte Carlo and quasi-Monte Carlo methods, 169–180, Springer (2018)
3. Albrecher, H., Matoušek, J., Tichy, R.: Discrepancy of point sequences on fractal sets. *Publ. Math. Debrecen* **56**, 233–249 (2000)
4. Beck, J.: Irregularities of point distribution I. *Acta Math.* **159**, 1–49 (1988)
5. Beck, J., Chen, W.: Irregularities of distribution. Cambridge University Press (1987)
6. Brandolini, L., Chen, W., Colzani, L., Gigante, G., Travaglini, G.: Discrepancy and numerical integration on metric measure spaces. *J. Geom. Anal.* **29**, 328–369 (2019)
7. Brandolini, L., Colzani, L., Gariboldi, B., Gigante, G., Travaglini, G.: Discrepancy for convex bodies with isolated flat points. *Rev. Mat. Iberoam.* **36**, 1597–1626 (2020)
8. Brandolini, L., Colzani, L., Gigante, G., Travaglini, G.: L^p and weak- L^p estimates for the number of integer points in translated domains. *Math. Proc. Cambridge Philos. Soc.* **159**, 471–480 (2015)
9. Brandolini, L., Colzani, L., Travaglini, G.: Average decay of Fourier transforms and integer points in polyhedra. *Ark. Mat.* **35**, 253–275 (1997)
10. Brandolini, L., Colzani, L., Travaglini, G.: Irregularities of distribution for bounded sets and half-spaces. *Mathematika* **69**, 68–89 (2023)
11. Brandolini, L., Gariboldi, B., Gigante, G.: On a sharp lemma of Cassels and Montgomery on manifolds. *Math. Ann.* **379**, 1807–1834 (2021)
12. Brandolini, L., Gigante, G., Travaglini, G.: Irregularities of distribution and average decay of Fourier transforms. In: A panorama of discrepancy theory, 159–220, Lecture Notes in Math., 2107, Springer (2014)
13. Brandolini, L., Hofmann, S., Iosevich, A.: Sharp rate of average decay of the Fourier transform of a bounded set. *Geom. Funct. Anal.* **13**, 671–680 (2003)
14. Brandolini, L., Iosevich, A., Travaglini, G.: Planar convex bodies, Fourier transform, lattice points, and irregularities of distribution. *Trans. Amer. Math. Soc.* **355**, 3513–3535 (2003)
15. Brandolini, L., Travaglini, G.: Fourier analytic techniques for lattice point discrepancy. *Radon Ser. Comput. Appl. Math.* **26**, De Gruyter, Berlin, 173–216 (2020)
16. Brandolini, L., Travaglini, G.: Irregularities of distribution and geometry of planar convex sets. *Adv. Math.* **396**: 108162 (2022)
17. Chazelle, B.: The Discrepancy Method. Cambridge University Press, Randomness and complexity (2000)
18. Chen, W.: On irregularities of distribution and approximate evaluation of certain functions. *Quart. J. Math. Oxford* **36**, 173–182 (1985)
19. Chen, W.: On irregularities of distribution and approximate evaluation of certain functions. II. in: Analytic number theory and Diophantine problems, 75–86, Birkhäuser (1987)
20. Chen, W.: Davenport's theorem in geometric discrepancy theory. *Int. J. Number Theory* **11**, 1437–1449 (2015)
21. Chen, W.: Results and problems old and new in discrepancy theory, in: Discrepancy Theory, 21–42, De Gruyter (2020)
22. Chen, W.: Lectures on Discrepancy Theory. <https://www.williamchen-mathematics.info/ln.html>
23. Chen, W., Travaglini, G.: Deterministic and probabilistic discrepancies. *Ark. Mat.* **47**, 273–293 (2009)
24. Cristea, L., Tichy, R.: Discrepancies of point sequences on the Sierpinski carpet. *Math. Slovaca* **53**, 351–367 (2003)

25. Davenport, H.: Note on irregularities of distribution. *Mathematika* **3**, 131–135 (1956)
26. Doerr, B.: A sharp discrepancy bound for Jittered sampling. *Math. Comp.* **91**:1871–1892 (2022)
27. Drmota, M.: Irregularities of distribution with respect to polytopes. *Mathematika* **43**, 108–119 (1996)
28. Drmota, M., Tichy, R.F.: *Sequences, discrepancies and applications*. Springer (1997)
29. Fairchild, S., Goering, M., Weiss, C.: Families of well approximable measures. *Unif. Distrib. Theory* **16**, 53–70 (2021)
30. Gariboldi, B.: Discrepancy of a convex set with zero curvature at one point. *Mathematika* **66**, 458–474 (2020)
31. Gigante, G., Leopardi, P.: Diameter bounded equal measure partitions of Ahlfors regular metric measure spaces. *Discrete Comput. Geom.* **57**, 419–430 (2017)
32. Katznelson, Y.: *An Introduction to Harmonic Analysis*. Cambridge University Press, Cambridge Mathematical Library (2004)
33. Kendall, D.G.: On the number of lattice points in a random oval. *Quart. J. Math. Oxford Series* **19**, 1–26 (1948)
34. Kiderlen, M., Pausinger, F.: On a partition with a lower expected L^2 -discrepancy than classical jittered sampling. *J. Complexity* **70**: 101616 (2022)
35. Matousek, J.: *Geometric Discrepancy*. Springer, An illustrated guide (2010)
36. Montgomery, H.: *Ten Lectures on the interface between Analytic Number Theory and Harmonic Analysis*. American Mathematical Society (1994)
37. Parnowski, L., Sobolev, A.: On the Bethe-Sommerfeld conjecture for the polyharmonic operator. *Duke Math. J.* **107**, 209–238 (2001)
38. Pausinger, F., Steinerberger, S.: On the discrepancy of jittered sampling. *J. Complexity* **33**, 199–216 (2016)
39. Roth, K.: On irregularities of distribution. *Mathematika* **1**, 73–79 (1954)
40. Roth, K.: On irregularities of distribution IV. *Acta Arith.* **37**, 67–75 (1980)
41. Ruzsa, I.: The discrepancy of rectangles and squares. *Österreichisch-Ungarisch-Slowakisches Kolloquium über Zahlentheorie* (Maria Trost, 1992), 135–140, *Grazer Math. Ber.*, **318**, Karl-Franzens-Univ. Graz, Graz (1993)
42. Schmidt, W.: Irregularities of distribution IV. *Invent. Math.* **7**, 55–82 (1969)
43. Schmidt, W.: Irregularities of distribution X. in: *Number Theory and algebra*, 311–329. Academic Press (1977)
44. Stein, E.: *Harmonic Analysis - Real variable methods, orthogonality, and oscillatory integrals*. Princeton University Press (1993)
45. Stein, E., Weiss, G.: *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press (1971)
46. Travaglini, G.: *Number Theory*. Cambridge University Press, *Fourier analysis and Geometric Discrepancy* (2014)
47. Travaglini, G., Tupputi, M.: A characterization theorem for the L^2 -discrepancy of integer points in dilated polygons. *J. Fourier Anal. Appl.* **22**, 675–693 (2016)
48. Weiss, C.: Approximation of discrete measures by finite point sets. *Unif. Distrib. Theory* **16**, 31–38 (2023)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.