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On well-posedness in vector optimization

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$\mathbb{R}$: the real numbers

$\mathbb{N}$: the natural numbers

B: closed unit ball

$B(x, r)$: closed ball centered at x with radius r

S: unit sphere

$\|x\|$: Euclidean norm

$\langle x, y \rangle$: canonical inner product

$A^c = X \setminus A$: relative complement

$\operatorname{int} A$: interior

A': derivative

$\operatorname{diam}(A)$: diameter

∂A: boundary

$\operatorname{co} A$: convex hull of set A

$\operatorname{Ls} A_n$: Kuratowski upper limits

$\operatorname{Li} A_n$: Kuratowski lower limits

$\operatorname{dom} f$: effective domain

$\operatorname{Lev}_f(y, X)$: lower level set

$\partial f(x^0)$: subgradient set

C^+: polar cone

(X, f): optimization problem

$\arg\min(X, f)$: minimal set

$\operatorname{Eff}(X, f)$: efficient solutions set

$\operatorname{Min}(X, f)$: minimal points set

$\operatorname{WEff}(X, f)$: weakly efficient solutions set

$\operatorname{WMin}(X, f)$: weakly minimal points set

$\operatorname{StMin}(X, f)$: strictly minimal points set

$d(x, C)$: distance from C

h_{a,c^0}, h_a, h: smallest monotone function

Δ_A: oriented distance function

$\xrightarrow{H}$: Hausdorff convergence

$\overset{H}{\rightharpoondown}$: upper Hausdorff convergence

$\overset{H}{\rightharpoonup}$: lower Hausdorff convergence

$\xrightarrow{K}$: Kuratowski convergence

Chapter 1

Introduction

An optimization problem represents the situation in which someone has to make an optimal choice among a set of possible alternatives, where optimality refers to different and sometimes conflicting constraints. Every day we encounter various kinds of decision problems. For example consider a young girl that wishes to buy a flat. She has many needs. She prefers a low price, as much as possible, a short distance between the flat and her office, the maximum comfort and many other factors may be listed. An optimal choice satisfying all the criteria simultaneously, does not always exist. What's to be done?
Another example is given by a group of managers preparing a programme to produce the maximum number of a certain item according to some standard of quality and the minimum cost. It is evident that several economic scenarios, and not only economic, may be represented as an optimization problem both when the decision is individual and when it is made by a group.
When the criteria of choice are translated by a vector function (objective function) and the possible alternatives are a nonempty set (feasible region), a vector optimization problem becomes a mathematical object that can be studied in a rigorous way. The two main sources of vector optimization come from economic equilibrium and welfare theories of Edgeworth and Pareto (1906) and from mathematical progress in ordered spaces by Cantor and Hausdorff ([75]). Later many authors published several works on this topic discussing general results: for instance existence of solutions (see [37] and the references therein), necessary and sufficient optimality conditions ([66], [75], [112]), convex and generalized convex optimization ([75], [50], [84], [110]), duality ([112], [84]), scalarization methods ([75], [31]), implementation of algorithms ([53], [54]) and so on.
An interesting topic, both theoretically and practically, concerns the study of stable and well-posed problems. Classically, a minimization problem is said to be well-posed if it has a unique solution which is stable. The stability condition can be made precise in different senses; for scalar functions, typically, the two fundamental approaches are associated with the names of Hadamard and Tykhonov, respectively. The first ([49]) requires existence and uniqueness of the optimal solution together

with a form of continuous dependence on the problem's data and was introduced by Hadamard for problems in mathematical physics such as boundary value problems for partial differential equations. In 1966 Tykhonov ([120]) extends to optimization problems the classical idea of Hadamard considering the convergence of every minimizing sequence to the unique minimum point. Generally speaking, some problems are more regular then others when the behavior of the objective function and of the approximate solution sequences are deeply connected. Hence, well-posedness properties play an important role in optimization theory because of their links to several theoretical issues as well as the relevance for the numerical approximation of the solution and the convergence analysis of many algorithms.

In the following years the theory of well-posedness has been widely studied (see [10], [11] and [29] for survey) and other scalar concepts have been introduced, for two main reasons. The first is the implementation of numerical methods producing sequences of points which fail to be feasible but tend asymptotically to fulfill the constraints, while the corresponding value approximate the optimal one. This is the case of Levitin-Polyak well-posedness ([69]), a strengthening of Tykhonov's property, in which the behavior of appropriate asymptotically sequences out of the feasible region is taken into account. The second is the generalization of well-posedness from scalar to vector optimization, where the uniqueness requirement is a very strong one. Thus Tykhonov well-posedness has been extended to scalar problems with several solutions ([44], [45], [10], [11]). In vector optimization, the image space is generally characterized by a partial order endowed by a closed, convex and pointed cone with nonempty interior. As consequence the concept of minimal value and thus of minimizing sequence are not uniquely determined. Hence several notions of vector well-posedness have been proposed choosing a concept of minimal value and defining an appropriate minimizing sequence, in other words, imposing some geometrical features of the solutions in the image space (see [74], [79], [102] as surveys). Actually, we expect a problem to have a solution, maybe not unique, "easy to find".

A first attempt to classify the vector well-posedness properties of Tykhonov's type may be found in [93] where two levels of analysis are identified: the pointwise notions are referred to a fixed solution point in the image set or in the feasible region, while the global notions involve the efficient frontier as a whole. Relations among pointwise notions, among global notions and between the two sets have been provided ([8], [74], [93], [102]).

Usually, every new notion of well-posedness is compared with some already existing concept and characterized metrically to obtain sufficient (or necessary and sufficient) conditions. This path suggested to identify classes of well-posed problems. In this direction some authors showed that convexity or generalized convexity for the objective function is a fundamental assumptions to name a problem well-posed according to different definitions simultaneously ([91], [92], [80], [93], [79], [102]). Another important approach to make the check faster was introduced by Miglierina et al. in [93]. The authors established a parallelism between vector and scalar well-posedness employing a nonlinear scalarization technique. Hence, a given well-posedness property of an original vector optimization problem is equivalent to some stability condition

of an associate scalar problem. In the following years this approach was developed considering both new well-posedness notions and linear scalarization methods ([101], [102]).
The identification of a class of well-posed problems allowed some researchers to approach another important issue. Roughly speaking, given a (topologized) class of minimum problems, "how many" of them are well-posed in one sense or another? This topic is known in literature as generic well-posedness in minimization problems. The words "how many" can be intended in many ways, for instance by a density result or more generally in the Baire category sense. Variational principles play a key role in this context and in particular some density results for scalar well-posedness have been proved as application of Ekeland's variational principle. Thus the goal is, given a specific problem, to find a suitable topology to apply a variational result (see [81], [61], [79], [62]). In vector optimization this topic is less developed.
A fundamental contribution in well-posedness theory was the so called well-posedness in the extended sense, introduced in the scalar case by Zolezzi ([125]) as combination of Hadamard and Tykhonov ideas. The original problem is embedded in a family of perturbed ones depending on a parameter and it is called well-posed in the extended sense when every asymptotically minimizing sequence converges to some solution. In this way the notion considers the behavior of appropriate minimizing sequences and, at the same time, realizes a continuous dependence of the solutions on the parameter. In vector optimization some generalizations have been proposed, see for example [56], [58], [21], [22].
The purpose of this work is to give an overview on the world of the vector well-posed optimization problems, in a finite dimensional setting, mainly under convexity or generalized convexity assumptions and focusing on scalarization procedures.

To this end the outline is the following. Chapters 2 and 3 are standard preliminaries. In Chapter 2 we introduce the subject of our analysis that is a vector minimization problem with abstract constraints, together with the basic material concerning vector optimization to which we refer in the sequel. Then we recall, in a separate chapter, the main approaches in scalar well-posedness and some connected results both to make possible links with scalarization results for vector problems and to underline the origin of vector well-posedness as generalization of scalar case. Thus Chapter 3 is devoted to a short, not exhaustive, survey of some scalar well-posedness notions.
Beginning with Chapter 4 the ideas of stability and well-posedness in vector optimization are presented. For the clarity of exposition we follow the classification proposed by Miglierina et al. in [93], thus we mention pointwise and global notions separately, stressing the geometrical features of the image set of each property thanks to some illustrative examples. As in the scalar case, we are able to establish the hierchical structure characterizing these concepts. For a more general overview, we investigate also the links with some global concepts considering the formulation in the nonparametric case of some extended well-posed notions.
We propose two characterizations of well-posedness properties presented in this Chapter, the first involving all global notions compared, under generalized convexity

assumption, the second involving all pointwise concepts and a variational principle. In particular, focusing on the strongest pointwise notion presented, Dentcheva-Helbig well-posedness with respect to an efficient point, we consider the idea of "how many" convex problems will have solutions and also enjoy the property of being well-posed. Here, employing a vector version of Ekeland's variational principle due to Araya in [1], we translate the idea of "many" in terms of a density result. We shall note that a density or, generally, a generic result permits to approximate a well-posed problem with a sequence of well-posed problems considering the same constraints and the same features for the objective functions.

Studying well-posedness one is naturally led to consider perturbations of functions and sets, hence Chapter 5 is dedicated to the well-posedness in the extended sense. The main contributions on this topic are due to [56], [58], [59], [57], [60], [34], both as generalization of Zolezzi and Levitin-Polyak works. In this field, first the authors provide results when only the objective functions are perturbed, then they introduce appropriate asymptotically minimizing sequences when both the objective function and the feasible region are subject to perturbation. In this last part, less developed in well-posedness literature, we propose a notion for which sufficient conditions under convexity requirements are established.

We then turn our attention on scalarization technique, linear and nonlinear. As consequence of the results in [93], a vector problem is well-posed if and only if an associate, or more then one, scalar problem satisfies a scalar stability requirement. This is the subject of Chapter 6. We mention two reasons among other for our attention on scalarization. Firstly, solving a scalar problem is less difficult than solving a vector one, secondly the generalization of the same vector approach preserves some geometrical features that these techniques emphasize in the proof of the main results. We can think this part as a window of dialogue between scalar and vector analysis of a well-posed problem.

It is not always easy to propose an application of theoretical results but it is an important effort to understand and improve an issue. Chapter 7 is dedicated to some conclusive remarks on what has been already made on this topic, in particular with reference to game theory as an application field for well-posedness properties, depicting also a potential future for the study of well-posedness and stability for a general representing model.

Chapter 2

Vector optimization

In this chapter we consider some fundamental and preliminary results in vector optimization that can be found in every detail and in more generality in the books [31] [74], [112].
As usual we begin to introduce, very briefly, preference orders and cones as the main tools to obtain a significative mathematical formulation of an abstract minimum problem.
Given a set $Y \subseteq \mathbb{R}^l$, a binary relation on Y is a subset R of $Y \times Y$ and we write $(x, y) \in R$ when the element $x \in Y$ is in relation with $y \in Y$.

Definition 2.1. *A binary relation R on Y is a* partial order *when $\forall x, y, z \in Y$:*

(i) $(x, x) \in R$ *(reflexivity);*

(ii) $[(x, y) \in R$ *and* $(y, z) \in R] \ \Rightarrow \ (x, z) \in R$ *(transitivity).*

Since Y is a subset of a linear space, we can recall the following definition.

Definition 2.2. *The partial relation R on X is said to be* linear *when $\forall x, y, x \in Y$ and $\lambda > 0$:*

(i) $(x, y) \in R \ \Rightarrow \ (x + z, y + z) \in R$*;*

(ii) $(x, y) \in R \ \Rightarrow \ (\lambda x, \lambda y) \in R$*.*

Definition 2.3. *A partial relation R on Y is called a* preorder *when $\forall x, y \in X$:*

$$[(x, y) \in R \text{ and } (y, x) \in R] \ \Rightarrow \ x = y.$$

It is known that a linear preorder R is geometrically equivalent to a convex and pointed cone. We recall that a cone is a subset $C \subseteq \mathbb{R}^l$ when $\forall x \in C$ and $\forall \lambda > 0$ one has $\lambda x \in C$. Moreover:

Definition 2.4. *A cone $C \in \mathbb{R}^l$ is called:*

(i) convex, *when $x + y \in C$, for all $x, y \in C$;*

(ii) pointed, *when* $C \cap (-C) = \{ 0 \}$.

From now on we shall consider a vector function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ and the optimization problem (X, f) given by:

$$\min_{x \in X} f(x) \tag{2.1}$$

assuming that the feasible region X is a set of $\mathbb{R}^n$ and the image space is endowed by a partial order given by a closed, convex and pointed cone C with nonempty interior. Thus we write equivalently

$$x \geq_C y \quad \text{or} \quad x - y \in C$$

$$x >_C y \quad \text{or} \quad x - y \in \operatorname{int} C.$$

2.1 Efficiency

The first interpretation of optimality was given by Vilfredo Pareto in 1896 ([103]):

> *"the members of collectivity enjoy maximum ophelity in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the optimality enjoyed by each of the individuals of that collectivity increases or decreases."*

When the image set is ordered by the paretian cone $\mathbb{R}^l_+$, the following notion identifies the Pareto efficient points, while for a generic closed, convex and pointed cone we refer to efficiency.

Definition 2.5. *A point* $\bar{x} \in X$ *is called* efficient solution *for problem* (X, f) *when*

$$(f(X) - f(\bar{x})) \cap (-C) = \{ 0 \} .$$

We denote by $\mathrm{Eff}\,(X, f)$ the set of all efficient solutions of the problem (X, f) and by $\mathrm{Min}\,(X, f)$ the set of all minimal points, that is the image of $\mathrm{Eff}\,(X, f)$ through the objective function f.
We are also interested in weakly efficiency and strictly efficiency.

Definition 2.6. *A point* $\bar{x} \in X$ *is called* weakly efficient solution *for problem* (X, f) *when*

$$(f(X) - f(\bar{x})) \cap (-\operatorname{int} C) = \emptyset.$$

We denote by $\mathrm{WEff}\,(X, f)$ the set of weakly efficient solutions of the problem (X, f) and by $\mathrm{WMin}\,(X, f)$ the set of all weakly minimal points. We stress that in the scalar case the notions of minimal point and weakly minimal point coincide, while in vector optimization the following inclusions hold:

$$\mathrm{Eff}\,(X, f) \subseteq \mathrm{WEff}\,(X, f), \qquad \mathrm{Min}\,(X, f) \subseteq \mathrm{WMin}\,(X, f)$$

This is a consequence of a partial, not complete, order in the image space that is a common characterization of vector spaces which dimension is greater or equal than two. Therefore, in vector optimization there are several degrees of efficiency and minimality (see [123]). Efficiency in strict sense, in particular, has been introduced to control the asymptotic behavior of unbounded minimizing sequences under the objective function. In 1998, Berdnarczuk et al. introduced the following concept ([12]).

Definition 2.7. *A point* $\bar{y} \in f(X)$ *is called* strictly minimal point *for problem* (X, f) *when for every* $\epsilon > 0$ *there exists* $\delta > 0$, *such that*

$$(f(X) - \bar{y}) \cap (\delta B - C) \subseteq \epsilon B.$$

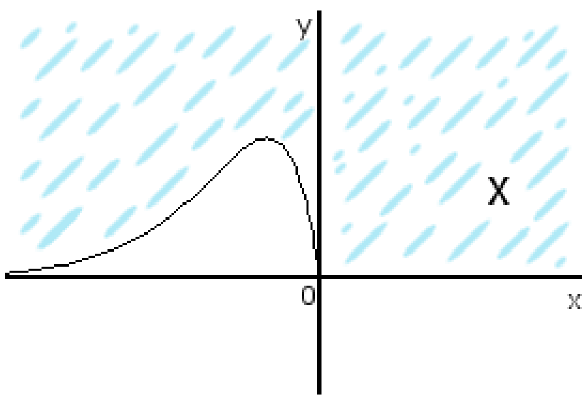
We denote by $\mathrm{StMin}\,(X, f)$ the set of strictly minimal points for problem (X, f). It is easy to see that $\mathrm{StMin}\,(X, f) \subseteq \mathrm{Min}\,(X, f)$ but the converse is not true in general. The next result emphasizes the geometrical features of strictly minimal points.

Proposition 2.8. *([9]) Let* $\bar{y} \in f(X)$. *Then* $\bar{y} \in \mathrm{StMin}\,(X, f)$ *if and only if for every sequences* $\{ z^n \}$, $\{ y^n \}$ *with* $\{ z^n \} \subseteq f(X)$, $y^n \in z^n + C$ *and* $y^n \to \bar{y}$, *it holds* $z^n \to \bar{y}$.

We conclude this section with an illustrative example of the three concepts of efficiency recalled.

Example 2.9. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, y)$ with $C = \mathbb{R}^2_+$ and

$$X = \left\{ (x, y) \in \mathbb{R}^2 : y \geq 0 \text{ if } x \geq 0 \text{ and } y \geq -xe^x \text{ if } x < 0 \right\}.$$



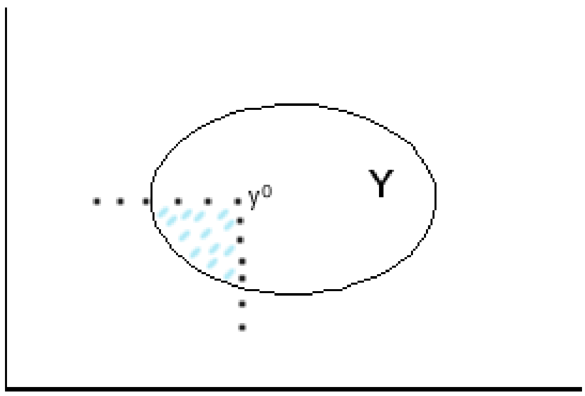
The point $(0, 0)$ is the only efficient one, but it doesn't satisfy Definition 2.7. Note that the absence of strictly minimal points is caused by the presence of an asymptote in common between image set and ordering cone.
In this example, $\mathrm{WEff}\,(X, f) = \left\{ (x, y) \in \mathbb{R}^2 : x \geq 0,\ y = 0 \right\}$.

2.2 Existence of efficient points

We devote a section to the problem of existence of optimal solutions since, as we have already pointed out in the introduction, it is deeply connected with each well-posedness approach. To say that a problem is well-posed, according to whatever definition, we need the existence of at least one optimal solution. Moreover to appreciate the geometrical nature of well-posedness notions we attempt to construct some example, satisfying the assumptions required by an existence result.
In the literature two main approaches have been developed: the first considers the geometrical properties of the efficient set, while the second is based on the asymptotic description of the objective function and the feasible region (see [37] for survey in scalar case, [31], [112]).
In the next result, introduced in 1983 by Borwein ([14]), the existence of minimum points is obtained considering sections of the image set $Y = f(X)$. For simplicity we consider $C = \mathbb{R}^l_+$ but all results presented in this section are still valid for a generic closed, convex and pointed cone C with nonempty interior.

Theorem 2.10. *Let Y be a nonempty set and suppose there is some $y^0 \in Y$ such that the section $Y^0 = \left\{ y \in Y : y \leq_{\mathbb{R}^l_+} y^0 \right\} = (y^0 - \mathbb{R}^l_+) \cap Y$ is compact. Then* $\mathrm{Min}\,(X, f) \neq \emptyset$.

The previous theorem derives the existence of minimal points from the intuitive understanding that minimal points are located in the "lower left part" of Y. The picture below shows a compact section of Y.



We recall another existence result that does not use a compact section but a condition on Y which is similar to the finite subcover property of compact sets. It was proved in 1980 by Corley ([23]).

Definition 2.11. *A set $Y \subseteq \mathbb{R}^l$ is called $\mathbb{R}^l_+$−semicompact if every open cover of Y of the form $\left\{ (y^\alpha - \mathbb{R}^l_+)^c : y^\alpha \in Y, \alpha \in A \text{ (index set)} \right\}$ has a finite subcover.*

Theorem 2.12. *Let Y be nonempty and $\mathbb{R}^l_+$−semicompact. Then* $\mathrm{Min}\,(X, f) \neq \emptyset$.

In the previous existence results $\mathrm{Min}\,(X, f)$ is considered, but using properties of f it is possible to derive results directly on $\mathrm{Eff}\,(X, f)$. Thus we take now into account existence results based on some property of the objective function. We start by discussing Weierstrass theorem.

Definition 2.13. *A function $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is called:*

(i) lower semicontinuous *(l.s.c. for short) in $x^0 \in X \cap X'$ when $\forall \epsilon > 0,\ \exists U(x^0)$ such that $f(x) > f(x^0) - \epsilon$;*

(ii) upper semicontinuous *(u.s.c. for short) in $x^0 \in X \cap X'$ when $\forall \epsilon > 0,\ \exists U(x^0)$ such that $f(x) < f(x^0) + \epsilon$;*

(iii) continuous *in $x^0 \in X \cap X'$ when it is both l.s.c. and u.s.c..*

Theorem 2.14. *(Weierstrass)*
A l.s.c. function on a compact set attains its minimum.

A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ is a vector of scalar functions $f = (f_1, \ldots, f_l)$ and its behavior in terms of continuity, derivability or differentiability, is identified by the behavior of each component f_i. This means, for example, that f is continuous if and only if each f_i is continuous.

Theorem 2.15. *Let X be a nonempty compact set, $C = \mathbb{R}^l_+$ and $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ a l.s.c. function. Then (X, f) has a Pareto optimal solution.*

To generalize the previous result considering a generic ordering cone, an extended semicontinuity concept has been introduced ([112]).

Definition 2.16. *Let C be a closed, convex and pointed cone in $\mathbb{R}^l$. A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ is said to be $C-$*semicontinuous *if the sublevel set*

$$\mathrm{Lev}_f(y, X) = \{\, x \in X : y - f(x) \in C \,\}$$

is closed for each $y \in \mathbb{R}^l$.

Theorem 2.17. *Let X be a nonempty compact set, C be a closed, convex and pointed cone in $\mathbb{R}^l$ and $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ a $C-$semicontinuous function. Then there exists an efficient solution.*

To avoid the compactness assumption, very strong both in scalar and in vector case, the authors introduced a scalar notion of coercivity ([37]).

Definition 2.18. *A function $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be* coercive *if, for every $t \in \mathbb{R}$, the set $\mathrm{Lev}_g(t)$ is bounded.*

The preceding notion is expressed in an equivalent way by

$$\lim_{\|x\| \to +\infty} g(x) = +\infty.$$

Theorem 2.19. *Let $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a l.s.c. and coercive function. Then the set of minimum points is nonempty and compact.*

For a detailed survey on scalar existence results allowing also unbounded set of minimum points see [37]. The generalization in vector optimization of coercivity was given by Ehrgott in 1997, that formally characterized Pareto optimality using level sets of the objective function.

Theorem 2.20. *Let $C = \mathbb{R}^l_+$. Then:*

(i) $\bar{x} \in X$ is Pareto optimal if and only if:

$$\bigcap_{i=1}^{l} L_{f_i}(f_i(\bar{x})) = \bigcap_{i=1}^{l} \{\, x \in X : f_i(x) = f_i(\bar{x}) \,\}\,;$$

(ii) $\bar{x} \in X$ is weakly Pareto optimal if and only if:

$$\bigcap_{i=1}^{l} \{\, x \in X : f_i(x) < f_i(\bar{x}) \,\} = \emptyset.$$

Usually, more general existence results require convexity or generalized convexity assumptions and not only (see [118], [35], [114], [36], [38], [41], [124] and the references therein). Moreover in a recent paper a unified approach characterizing efficiency without linear structure has been proposed ([40], [42]).
This short part on existence theory emphasizes the difficulties to guarantee, without strong and particular requirements, the presence of at least an efficient point and thus partially justifies the assumption $\mathrm{Eff}\,(X, f) \neq \emptyset$ in well-posedness sufficient conditions.

2.3 Convexity and generalized convexity

For several reasons, convexity and generalized convexity have a key role in multiobjective optimization. At the end of the previous section we remark the presence of generalized convexity assumptions to obtain more general existence results, and thus a potential link between convexity and continuity, but this is only an example. Another important fact is that under generalized convexity assumptions the inclusion properties (2.1) may be equalities. Deepenings on Convex Analysis can be found in [75], [108], [50] and [110].
We have already recalled the notion of convex cone, we turn now on convex sets.

Definition 2.21. *A set $X \subseteq \mathbb{R}^n$ is said to be* convex *if $\forall x, z \in X$ and $\forall t \in [0, 1]$ it holds $tx + (1 - t)z \in X$.*

Definition 2.22. *([75]) A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$, X convex, is said to be:*

(i) $C-$convex *on* X *if* $\forall x, z \in X$ *and* $t \in [0,1]$,

$$f(tx + (1-t)z) \leq_C tf(x) + (1-t)f(z)$$

(ii) $C-$quasiconvex *on* X *if* $\forall y \in \mathbb{R}^l$ *the sublevel sets* $\mathrm{Lev}_f(y)$ *are either empty or convex;*

(iii) strictly $C-$convex *on* X *if* $\forall x, z \in X,\ x \neq z$ *and* $t \in (0,1)$,

$$f(tx + (1-t)z) <_C tf(x) + (1-t)f(z)$$

(iv) strictly $C-$quasiconvex *on* X *if* $\forall y \in \mathbb{R}^l$ *and* $\forall x, z \in X,\ x \neq z,\ t \in (0,1)$,

$$f(x), f(z) \in y - C \quad \textit{implies} \quad f(tx + (1-t)y) \in y - \operatorname{int} C$$

As anticipated at the beginning of this section, we recall two important properties of $C-$convex and $C-$quasiconvex functions, widely used in the sequel.

Theorem 2.23. *([119]) Let* X *be an open set. If* $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ *is* $C-$*convex, then* f *is continuous.*

Proposition 2.24. *([75]) Let* $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ *be continuous and strictly* $C-$*quasiconvex. Then:*

(i) $\mathrm{WEff}\,(X, f) = \mathrm{Eff}\,(X, f)$*;*

(ii) for every $y \in \mathrm{Min}\,(X, f)$, $f^{-1}(y)$ *is a singleton.*

A generalization of convexity is connectedness. A convex function is defined investigating its behavior on segments belonging to its domain. Some researchers have though to ask for a regularity on paths that are not necessarily segments, for example on arcs ([75], [2], [13], [91]). Recall that a set $X \subseteq \mathbb{R}^n$ is said to be arcwise connected if for every $x_1, x_2 \in X$ there exists a curve γ of equation $r : [0,1] \to X$ such that $r(0) = x_1$ e $r(1) = x_2$.

Definition 2.25. *Let* X *be an arcwise connected set and* $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ *be a function.*

(i) The function f *is* $C-$quasiconnected *when, for every* $x_1, x_2 \in X$ *there exists a continuous path* $\gamma(t; x_1, x_2) : [0,1] \to X$ *with* $\gamma(0; x_1, x_2) = x_1$ *and* $\gamma(1; x_1, x_2) = x_2$*, such that the following implication holds:*

$$f(x_1), f(x_2) \leq_C y \;\Rightarrow\; f(\gamma(t; x_1, x_2)) \leq_C y, \quad \textit{for every} \quad t \in [0,1].$$

(ii) The function f *is* strictly $C-$quasiconnected *when, for every* $x_1, x_2 \in X,\ x_1 \neq x_2$ *there exists a continuous path* $\gamma(t; x_1, x_2) : [0,1] \to X$ *with* $\gamma(0; x_1, x_2) = x_1$ *and* $\gamma(1; x_1, x_2) = x_2$*, such that the following implication holds:*

$$f(x_1), f(x_2) \leq_C y \;\Rightarrow\; f(\gamma(t; x_1, x_2)) <_C y, \quad \textit{for every} \quad t \in (0,1).$$

When X is a convex set and $\gamma(t;x_1,x_2)=(1-t)x_1+tx_2$, we obtain respectively the definition of $C-$quasiconvex and strictly $C-$quasiconvex function (see [75]). Another important difference is that when the image set is ordered by the Paretian cone, $C-$convexity is equivalent to the convexity of each components f_i, while this very useful property is no longer true for quasiconnectedness.

Another important step in generalized convexity is the introduction of $*-$quasiconvexity by Jeyakumar et al. in [63] as a subclass of $C-$quasiconvex functions. This class of generalized convex functions is motivated by the authors with the possibility to derive a solvability theorem (also called theorem of the alternative) which applies to characterize local and global solutions of optimization problems. For its particular form, we will employ this class of functions to derive well-posedness results linked to linear scalarization.

Definition 2.26. *A function $f: X\subseteq\mathbb{R}^n\to\mathbb{R}^l$ is said to be* $*-$quasiconvex *if and only if $\forall\lambda\in C^+$ the real valued function $\langle\lambda, f(\cdot)\rangle : X\to\mathbb{R}$ is quasiconvex.*

There are many operations which preserve convexity and allow to construct new convex functions, most of them geometrically motivated. We end this section with an application of this type. We recall the notion of smallest monotone function introduced by Luc ([75]) that is nonconvex but is linked to the generalized convexity properties. This kind of functions has been employed by several authors in many fields, for instance see Rubinov [110], Luc [75], Huang [56] and Araya [1] among others.

Let $C\subset\mathbb{R}^l$ be a nonempty, closed, convex and pointed cone, let $c^0\in\operatorname{int}C$ a fixed vector and $a\in\mathbb{R}^l$. Define a smallest monotone function as follows:

$$h_{a,c^0}(y):=\min\left\{\, t\in\mathbb{R} : y\in a+tc^0-C \,\right\}.$$

We will fix a cone C and an arbitrary vector c^0 and do not explicitly mention it in the future; h_a will be written instead of h_{a,c^0}.

Function $h_a:\mathbb{R}^l\to\mathbb{R}\cup\{\,\pm\infty\,\}$ satisfies the following properties:

Lemma 2.27. *([75], [47], [48])*

(i) h_a *is continuous;*

(ii) h_a *is proper;*

(iii) h_a *is sublinear;*

(iv) h_a *is $C-$monotone (that means $y_1\leq y_2$ implies $h_a(y_1)\leq h_a(y_2)$);*

(v) $\{\,y\in Y : h_a\leq t\,\}=tc^0-C$;

(vi) $h_a(y+\lambda c^0)=h_a(y)+\lambda$ *for every $y\in Y$ and $\lambda\in\mathbb{R}$.*

Proposition 2.28. *([75]) A function $f: X\subseteq\mathbb{R}^n\to\mathbb{R}^l$ is $C-$quasiconvex if and only if $h_a\circ f$ is quasiconvex for every $a\in f(X)$ and a fixed $c^0\in\operatorname{int}C$.*

This last result, thanks to the separation properties of the smallest monotone function h_a, deals with a scalar test for $C-$quasiconvexity, in fact a vector scenary is brought back to more familiar scalar contexts. This is the spirit of scalar representations.

2.4 Scalar representations

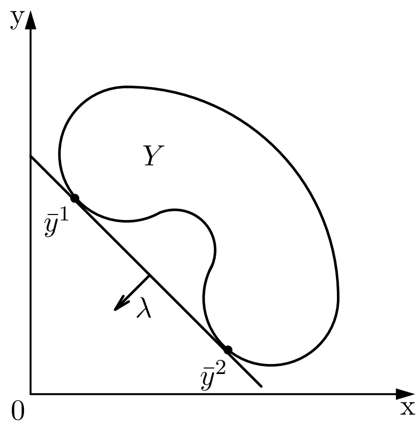
In optimization theory, a scalarization technique is a method to convert a vector problem in a scalar one preserving some geometrical features of the original problem, such as convexity, linearity, solutions, etc. The main difference between scalar and vector optimization lies in the completeness of the natural order on the real line, where the images of feasible alternatives can always compared. For this reason, to deal with a scalar problem is easier and we can say that the main goal of scalarization is to simplify the problem to handle.
In this section we mention a linear scalar representation, also called weighted sum scalarization, to characterize the solution set of a vector problem by the solutions of several scalar minimization problems. Then we remark some difficulties intrinsic in this method solved by several authors focusing on nonlinear approaches. In particular we sketch the procedure based on the oriented distance function introduced by Hiriart-Urruty in 1979 ([51], [52]) to develop stability conditions in nonsmooth analysis.
As references for this section see [75], [31].

2.4.1 Weighted sum scalarization

This method is the most popular linear technique because of its formal simplicity. The objective function of each scalar problem is constructed as convex linear combination of the components $f_i,\ i = \{ 1, \dots, l \}$ the vector function while the feasible region remains the same. Thus the problem (X, f) is investigated by solving scalarized problems of the type

$$\min_{x\in X} \sum_{i=1}^{l} \lambda_i f_i(x) = \langle \lambda, f(x) \rangle \tag{2.2}$$

where $\lambda \in C^+ := \{ y \in f(X) : \langle y, c \rangle \geq 0, \forall c \in C \}$.
Graphically, for a fixed λ the scalar problem is to minimize a linear functional, which slope is λ, on the image set $Y = f(X)$. The goal is to find the points $\bar{y}$ on the boundary of Y, intersection of the line $\langle \lambda, y \rangle = c$ and Y when c is the least value for which the intersection is nonempty. Changing $\lambda \in C^+$ a different slope of the linear functional allows to find other optimal solutions. See the picture below.



Set

$$\mathrm{Opt}\,(\lambda, Y) := \{\ \bar{y} \in Y : \langle \lambda, \bar{y} \rangle = \inf \langle \lambda, y \rangle\,, y \in Y\ \}$$

$$S(Y) := \bigcup_{\lambda \in \mathrm{int}\, C^+} \mathrm{Opt}\,(\lambda, Y)$$

$$S_0(Y) := \bigcup_{\lambda \in C^+} \mathrm{Opt}\,(\lambda, Y)$$

It is easy to see that without convexity assumption one has $S_0(Y) \subset \mathrm{Min}\,(X, f)$.

Theorem 2.29. *If the set $(Y + K)$ is convex, then $S(Y) \subseteq \mathrm{Min}\,(X, f) \subseteq S_0(Y)$.*

Convexity assumptions allow to invoke a fundamental result in convex analysis that is a separation theorem. When $(Y + K)$ is convex, the set $(Y - \bar{y})$ with $\bar{y} \in \mathrm{Min}\,(X, f)$ and the cone $-C$ can be separated by a line and thus there exists a slope λ to identify every minimum value (see [108]).
Moreover, with the same reasoning, $S_0(Y) \subseteq \mathrm{WMin}\,(X, f)$. The two following results have been proved.

Proposition 2.30. *If the set $(Y + K)$ is convex, then $S_0(Y) = \mathrm{WMin}\,(X, f)$.*

Proposition 2.31. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ a $C-$convex function.
Then $\bar{x} \in \mathrm{WEff}\,(X, f)$ if and only if $\exists \lambda \in C^+$ such that $f(\bar{x}) \in S_0(f(X))$.*

We stress that in vector optimization the image set of a convex function is not necessarily convex. For example, consider the set $X = \mathrm{co}\,\{\ (0, 0), (0, 1), (1, 0)\ \}$ that is the convex hull of three points in $\mathbb{R}^2$. Let $f(x, z) = (x^2, z^2)$ and $C = \mathbb{R}^2_+$. Obviously X is a convex set and f is a $C-$convex function but $f(X)$ is not a convex set.
The weak point of this method is that appreciable results are locked up to convexity requirements and in any case the solution set of a vector problem is the union of the solution sets of many, more than one, scalar problems. To avoid these difficulties nonlinear scalarization has been introduced. In the next section we introduce the nonlinear scalarization based on the so-called oriented distance function.

2.4.2 The oriented distance function

The notion of oriented distance function was introduced by Hiriart-Urruty in 1979 ([51],[52]) to develop stability conditions in nonsmooth analysis. Later it has been employed by several authors to compare different degrees of efficiency ([123], [93], [56], [19], [117] among others). Indeed this function allows to emphasize the geometrical features of an optimal solution considering a distance notion from a point to a fixed set.

Definition 2.32. *Let A be a subset of a normed vector space Y. The* oriented distance function *for A is $\Delta_A(y) : Y \to \mathbb{R} \cup \pm\{\infty\}$ defined as*

$$\Delta_A(y) = d_A(y) - d_{Y \setminus A}(y)$$

where $d_A(y) = \inf_{x \in A} \|y - x\|$.

The main properties of function Δ_A are gathered in the following proposition.

Proposition 2.33. *([123])*

(i) If $A \neq \emptyset$ and $A \neq Y$ then Δ_A is real valued;

(ii) Δ_A is 1-Lipschitzian;

(iii) $\Delta_A < 0, \quad \forall y \in \operatorname{int} A, \quad \Delta_A = 0, \quad \forall y \in \partial A$ and $\Delta_A > 0, \quad \forall y \in \operatorname{int} A^c$;

(iv) if A is convex, then Δ_A is convex;

(v) if A is a cone, then Δ_A is positively homogeneous;

(vi) if A is a closed convex cone, then Δ_A is nonincreasing with respect to the ordering relation induced on Y, if $y_1, y_2 \in Y$ then

$$y - z \in A \quad \Rightarrow \quad \Delta_A(y) \leq \Delta_A(z);$$

if A has nonempty interior, then

$$y - z \in \operatorname{int} A \quad \Rightarrow \quad \Delta_A(y) < \Delta_A(z).$$

To derive scalar representations in vector optimization we refer to our problem (X, f) and choose $A = -C$, thus function Δ_{-C} is deeply linked to the norm defined in $\mathbb{R}^l$ and to the cone $-C$. We recall two examples to underline the geometrical meaning of this function ([123]).

Example 2.34. Let $\|y\| = \sqrt{\sum_{i=1}^l y_i^2}$ be the Euclidean norm and $C = \mathbb{R}^l_+$. Then, for $i = 1, \ldots, l$

$$\Delta_{-C}(y) = \begin{cases} \max\ (y_i, 0) & \text{if } y \notin -C \\ \max\ \ y_i & \text{if } y \in -C \end{cases}$$

Example 2.35. Let $\|y\|_\infty = \max |y_i|$ and $C = \mathbb{R}^l_+$. Then, $\Delta_{-C}(y) = \max_i y_i$.

The first thing we ask for a scalar representation is to separate efficient and weakly efficient solutions of (X, f); in general we need a function which is sensible to different degrees of minimality. Let $p \in \mathbb{R}^l$ be a parameter, $Y = f(X)$ and define a scalar problem (Y, Δ_{-C}) as follows:

$$\min_{y \in Y} \Delta_{-C}(y - p). \tag{2.3}$$

Theorem 2.36. *([123]) Let $\bar{y} \in Y$. We have $\bar{y} \in \mathrm{Min}\,(X, f)$ if and only if $\exists p \in \mathbb{R}^l$ such that $\bar{y}$ is a strictly global minimum point for (Y, Δ_{-C}).*

Theorem 2.37. *([123]) Let $\bar{y} \in Y$. We have $\bar{y} \in \mathrm{WMin}\,(X, f)$ if and only if $\exists p \in \mathbb{R}^l$ such that $\bar{y}$ is a global minimum point for (Y, Δ_{-C}).*

We close this section with two remarks. First we note that the oriented distance function Δ_{-C} used for solving nonconvex optimization problems and the smallest monotone function h_a introduced to characterize generalized convexity are deeply connected. In fact, it was shown ([19]) that under a particular assumption on the norm of the image space, the smallest monotone function h_a is a special case of the oriented distance function, that is

$$h_a(y) = \Delta_{-C}(y).$$

The scalarization by the smallest monotone function h_a gives a characterization of every weakly efficient solution of the vector problem.

Proposition 2.38. *([16]) For every vector $e \in \operatorname{int} C$, every $x \in \mathrm{WEff}\,(X, f)$ is an optimal solution of the scalarized problem $\min_{x \in X}(h_a \circ f)(x)$.*

We have seen another application of function h_a that will be employed in the sequel to derive scalar representations preserving well-posedness properties.
The second remark justifies the next section of this chapter. The approach of both oriented distance function and h_a (and also of weighted sum scalarization), is to interpret the measure of the distance, in one sense, between a point and a set, generally between two sets. The importance of this can be found not only in vector optimization or in well-posedness theory but in general when the subject of the analysis are functions. In fact it is well known that every function, in particular under convexity assumptions, may be identified with its epigraph and the study for example of a sequence of functions may be approached considering the epiconvergence, that is the convergence of their epigraphs.

2.5 Distance between two sets

In this section we recall two important tools to measure the distance between sets from which set-convergences are derived: the Hausdorff distance and the Kuratowski-Painlevè set-convergence (for a deeper exposition on this topic see e.g. [3], [109],

[79]). Their role will be crucial in next chapters devoted to well-posedness notions and stability conditions, speaking of convergence of sets.
One of the most used ways to measure the distance between closed sets is the so-called Hausdorff distance, based on the following pseudometric h.

Definition 2.39. *Let $A, B \subseteq \mathbb{R}^l$ nonempty subsets. We define the* excess of A over B

$$e(A,B) := \sup_{a \in A} d(a,B) \in [0,\infty],$$

where $d(a,B) := \inf_{b \in B} d(a,b)$.

When $A = B = \emptyset$, we set $e(A,B) = \emptyset$. Finally, we call Hausdorff distance between A and B

$$h(A,B) := \max \{ \, e(A,B), e(B,A) \, \} \, .$$

Definition 2.40. *Let $\{ \, A_n \, \}$ be a sequence of subsets of $\mathbb{R}^l$. We say that A_n* converges in the Hausdorff sense *to $A \subseteq \mathbb{R}^l$ and write $A_n \xrightarrow{H} A$, when $h(A_n, A) \to 0$. We can distinguish between* upper and lower convergence *as follows:*

$$A_n \overset{H}{\rightharpoonup} A \iff e(A_n, A) \to 0$$

$$A_n \overset{H}{\rightharpoondown} A \iff e(A, A_n) \to 0.$$

Let us turn now on a set-convergence idea based on the limits of sets introduced by Painlevé in 1902, and popularized by Kuratowski in his book ([67]), thus often called Kuratowski lower and upper limits of sequences of sets. Let $\{ \, A_n \, \}$ be a sequence of sets of $\mathbb{R}^l$ and define the following sets:

$$\operatorname{Ls} A_n := \left\{ x \in \mathbb{R}^l \ : \ x = \lim_{k \to +\infty} x^k, \ x^k \in A_{n_k}, \ n_k \text{ a selection of the integers} \right\}$$

and

$$\operatorname{Li} A_n := \left\{ x \in \mathbb{R}^l \ : \ x = \lim_{k \to +\infty} x^k, \ x^k \in A_k, \text{ eventually} \right\}.$$

The set $\operatorname{Ls} A_n$ is called the *Limsup* of the sequence $\{ \, A_n \, \}$, while the set $\operatorname{Li} A_n$ is called the *Liminf* of the sequence $\{ \, A_n \, \}$.

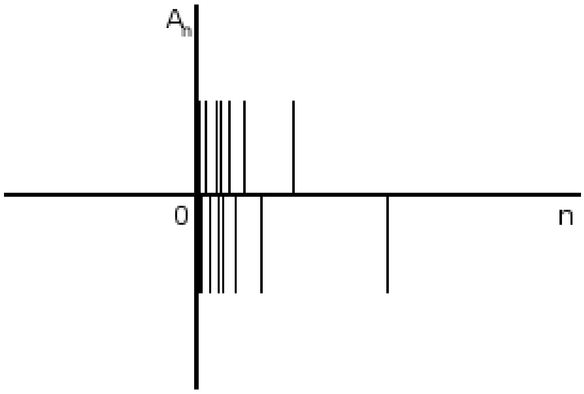
Definition 2.41. *The sequence $\{ \, A_n \, \}$ is said to* converge to A in the Kuratowski sense *if*

$$\operatorname{Ls} A_n \subseteq A \subseteq \operatorname{Li} A_n.$$

We denote this convergence by $A_n \xrightarrow{K} A$ and observe that lower and upper limits are closed, may be empty, sets.

Example 2.42. Let

$$A_n := \begin{cases} \{ \, \frac{1}{n} \, \} \times [0,1] & \text{if } n \text{ even} \\ \{ \, \frac{1}{n} \, \} \times [0,1] & \text{if } n \text{ odd} \end{cases}$$



Then $\mathrm{Li}\, A_n = \{\, 0 \,\}$, while $\mathrm{Ls}\, A_n = [-1, 1]$.

Chapter 3

Scalar well-posedness

In this chapter we focus on the presentation of well-posedness for scalar minimization problems, along with illustrative examples and remarks. We denote by (X, g), the minimization problem given by

$$\min_{x \in X} g(x)$$

where $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and X is a closed set.
The notions of well-posedness that will be crucial in the next chapters are vector generalizations of Tykhonov's idea. Thus we need to recall the notion due to Tykhonov ([120]) together with some scalar generalizations preparing this notion for a vector framework, such as generalized Tykhonov well-posedness proposed by Furi and Vignoli in [44] (see also [45]), metrically and topologically well-setness due to Bednarczuk and Penot ([10], [11]). For the sake of completeness we recall also the notion of Levitin-Polyak ([69]) well-posedness. Then in a short section we underline the geometrical and variational features of scalar well-posedness considering sufficient conditions and remarks under generalized convexity assumptions. Finally, we recall with a rephrased version of Tykhonov well-posedness in terms of the Ekeland's variational principle. Before the end, again on the variational nature of well-posed problems, we give a short review of the mixed approach by Zolezzi which combines Hadamard and Tykhonov ideas embedding (X, g) in a family of perturbed problems depending on a parameter ([125], [29]).

3.1 Tykhonov well-posedness

The basic idea behind the well-posedness of an optimization problem requires existence of a unique solution towards which every minimizing sequence converges. The mathematical formulation of this words may be realized equivalently by several tools, such as minimizing sequences, forcing maps, $\epsilon-$solutions map and so on. We decide to follow the characterization by minimizing sequences, to simplify the comparison among different concepts (for more details [29]). Recalling that a sequence $\{ x^n \} \subseteq X$ is said to be minimizing for (X, g) when $g(x^n) \to \inf g(X)$, Tykhonov introduced the following definition:

Definition 3.1. *([120]) Problem* (X, g) *is said to be* Tykhonov well-posed *(for short T-wp) or correctly posed, if and only if:*

(i) there exists a unique global minimum point $\bar{x} \in X$*;*

(ii) every minimizing sequence converges to $\bar{x}$*.*

Obviously, the existence of a unique global minimum point is not sufficient to name a minimization problem T-wp.

Example 3.2. Let $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x, & \text{if } x > 0 \\ |x-1|, & \text{if } x \leq 0 \end{cases}$$

and $X = \mathbb{R}$. Then $\arg\min(X, g) = \{ -1 \}$ but the minimizing sequence $x^n = \frac{1}{n}$ doesn't converge to -1.

An useful representation of Tykhonov well-posedness in terms of sublevel sets has been established.

Proposition 3.3. *Let* $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ *a l.s.c. function. The following are equivalent:*

(i) g *is T-wp;*

(ii) $\inf_{a > \inf g} \operatorname{diam} (\operatorname{Lev}_g(a)) = 0$.

Furi and Vignoli in [45] give a somewhat more general definition of well-posed problem which does not require the uniqueness of the minimum point; since the uniqueness requirement may be too strong and thus it is more convenient to evaluate a nonempty set of solutions. Consequently the well-posedness idea is reformulated relaxing the uniqueness of solution but saving the underlying structure of convergence.

Definition 3.4. *Problem* (X, g) *is said to be* generalized Tykhonov well-posed *(for short GT-wp) if and only if:*

(i) $\arg\min(X, g) \neq \emptyset$*;*

(ii) every minimizing sequence $\{ x^n \} \subseteq X$ *admits a subsequence* x^{n_k} *converging to some* $\bar{x} \in \arg\min(X, g)$*.*

A problem which is GT-wp is always T-wp while the converse is not true, in general.

Example 3.5. Let $g : X \subseteq \mathbb{R} \to \mathbb{R}$ given by $g(x) = ||x| - 1|$ and $X = \mathbb{R}$. One has $\arg\min(X, g) = \{ -1, 1 \}$ and $\operatorname{Min}(X, g) = \{ 0 \}$. Problem (X, g) is GT-wp but not T-wp.

The notion by Furi and Vignoli implicitly requires the compactness of the solutions set, as the following example shows.

Example 3.6. Let $g : X \subseteq \mathbb{R} \to \mathbb{R}$ given by $g(x) = \max(1 - |x|, 0)$ and $X = \mathbb{R}$. One has $\arg\min(X, g) = \{ x \in \mathbb{R} : x \leq -1 \text{ and } x \geq 1 \}$ and $\mathrm{Min}\,(X, g) = \{ 0 \}$. Problem (X, g) is not GT-wp as one can see considering the minimizing sequence $x^n = n$.

As for Tykhonov well-posedness, a useful representation of generalized Tykhonov well-posedness in terms of sublevel sets has been established by Beer et al. in [13] thanks to the property of quasi inf-compactness for a scalar function.

Definition 3.7. *Function $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be* quasi inf-compact *when for some $\alpha > \inf g(X)$ the sublevel set $\mathrm{Lev}_g(\alpha)$ is compact.*

Theorem 3.8. *([13]) Let $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ a l.s.c. function with $\mathrm{Lev}_g(\alpha)$ arcwise connected for every $\alpha \in \mathbb{R}$. The following are equivalent:*

(i) g is quasi inf-compact;

(ii) (X, g) is GT-wp;

(iii) $\arg\min(X, g) \neq \emptyset$ and compact.

The generalizations of the well-posedness concept by Bednarczuk and Penot do not impose compactness and provide a formulation based on the metric structure (metrically well-setness) and a further expression in which only topological properties are invoked. We recall that stability is usually expressed by a semicontinuity condition of a map, in fact here constructing an appropriate $\epsilon-$solutions multifunction the definitions of metrically and topologically well-setness may be rewritten in terms of upper semicontinuity (see [10], [11]). This further generalizations are due to the presence of many vector problems in which the solutions set is unbounded and thus also compactness is, in some sense, a very strong requirement.

Definition 3.9. *Problem (X, g) is said to be* metrically well-set *(for short MS) if and only if for every minimizing sequence $\{ x^n \} \subseteq X$ it holds $d(x^n, \arg\min(X, g)) \to 0$.*

Definition 3.10. *Problem (X, g) is said to be* topologically well-set *(for short TS) if and only if every minimizing sequence $\{ x^n \} \subseteq X \setminus \arg\min(X, g)$ has a cluster point $\bar{x} \in \arg\min(X, g)$.*

TS implies MS but the converse is not true, in general. The equivalence is realized when diam $(\arg\min(X, g)) = 0$ that means the problem has a unique minimum point.

Example 3.11. Let $g : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x, y) = x$ and the feasible region $X = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ y \geq 0 \right\}$.
One has $\arg\min(X, g) = \left\{ (0, y) \in \mathbb{R}^2 : y \geq 0 \right\}$ and $\mathrm{Min}\,(X, g) = \{ 0 \}$. Problem (X, g) is MS but not TS.

We end this section with a strengthening of T-wp introduced by Levitin-Polyak in 1966 ([69]) considering a notion of generalized minimizing sequences for (X, g). The authors justify their work observing that some numerical optimization methods for constrained problems produce a sequence of points which fail to be feasible, but tend asymptotically to fulfill the constraints, while the corresponding values approximate the optimal one (see [29]). Let $\{ x^n \} \subseteq \mathbb{R}^n$ a sequence such that $g(x^n) \to \inf g(X)$ and let $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$. Then $\{ x^n \}$ is called generalized minimizing sequence for (X, g) if and only if $d(x^n, X) \to 0$.

Definition 3.12. *Problem (X, g) is said to be* Levitin-Polyak well-posed *(for short LP-wp), if and only if:*

(i) $\arg\min(X, g) \neq \emptyset$*;*

(ii) every generalized minimizing sequence converges to some $\bar{x} \in \arg\min(X, g)$.

Since every minimizing sequence is also a generalized minimizing sequence, Levitin-Polyak well-posedness implies Tykhonov well-posedness while the equivalence is true assuming function g is uniformly continuous.

Example 3.13. Let $g : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = x^2 - (x^4 + x)y^2$. Problem (X, g) where $X = \{ (x, 0) : x \in \mathbb{R} \}$ is T-wp but not LP-wp. For instance the generalized minimizing sequence $\left(n, \frac{1}{n}\right)$ diverges.

3.2 Well-posedness and generalized convexity

When the objective function satisfies generalized convexity requirements, sufficient conditions for well-posedness can be derived. In this section we recall some results in this direction and we show some further generalizations.
A convex function g is T-wp assuming that there exists a unique global minimum point $\bar{x} \in \arg\min(X, g)$ ([29]). Weakening the convexity hypothesis, one can show the following proposition.

Proposition 3.14. *Let $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be quasiconnected and l.s.c. and assume $\arg\min(X, g) = \{ \bar{x} \}$. Then problem (X, g) is T-wp.*

Recalling that, by definition, T-wp implies GT-wp, MS and TS the previous proposition establishes a sufficient condition for all these properties. A similar result can be proved for LP-wp.

Proposition 3.15. *Let $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be quasiconnected and l.s.c. and assume $\arg\min(X, g) = \{ \bar{x} \}$. Moreover, assume that the continuous path $\gamma(t; \bar{x}, x^n)$ such that $g(\gamma(t; \bar{x}, x^n)) \leq \max\{ g(\bar{x}), g(x^n) \}$ satisfies the condition $d(\gamma(t; \bar{x}, x^n), X) \leq d(tx^n + (1-t)\bar{x}, X)$ for every generalized minimizing sequence $\{ x^n \}$. Then problem (X, g) is LP-wp.*

Proof: Without loss of generality, let $\bar{x} = 0$ and $g(0) = 0 = \min g(X)$.
By contradiction, (X, g) is not LP-wp. Then there exists a subsequence $\{ x^{n_k} \}$ of a generalized minimizing sequence $\{ x^n \}$ such that

$$g(x^{n_k}) \to 0, \quad d(x^{n_k}, X) \to 0, \quad \text{but } x^{n_k} \not\to \bar{x}.$$

We distinguish two cases.
Let $x^{n_k} \to c \neq \bar{x}$. Then by l.s.c.

$$0 = \liminf_{x^{n_k} \to c} g(x^{n_k}) \geq g(c),$$

that means $c \in \arg\min(X, g)$, a contradiction.
Let $\|x^{n_k}\| \to \infty$. By assumption,

$$d\left(\frac{1}{\|x^{n_k}\|} x^{n_k} + \left(1 - \frac{1}{\|x^{n_k}\|}\right)\bar{x}, X\right) \leq \frac{1}{\|x^{n_k}\|} d(x^{n_k}, X) + \left(1 - \frac{1}{\|x^{n_k}\|}\right) d(\bar{x}, X),$$

thus $\left(\frac{x^{n_k}}{\|x^{n_k}\|}, X\right) \to 0$. Since g is quasiconnected,

$$g\left(\frac{1}{\|x^{n_k}\|} x^{n_k} + \left(1 - \frac{1}{\|x^{n_k}\|}\right)\bar{x}\right) \leq g(x^{n_k}),$$

with $g\left(\frac{x^{n_k}}{\|x^{n_k}\|}\right) \to 0$. Then, $\frac{x^{n_k}}{\|x^{n_k}\|}$ is a generalized minimizing sequence such that $\frac{x^{n_k}}{\|x^{n_k}\|} \to y$ with $\|y\| = 1 \neq \|\bar{x}\| = 0$, a contradiction. $\square$

A sufficient condition for GT-wp under quasiconnectedness assumption can be derived from Theorem 3.8.

Corollary 3.16. *Let $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be quasiconnected and l.s.c. and assume $\arg\min(X, g) \neq \emptyset$ and compact. Then problem (X, g) is GT-wp.*

Proof: The proof follows by Theorem 3.8 observing that for a quasiconnected function all sublevel sets are connected. $\square$

The scenary changes when we consider well-setness, in fact the assumption of convexity for the objective function is not sufficient to name (X, g) MS.

Example 3.17. Let $g : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = \frac{x^2}{y}$ and

$$X = \left\{ (x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 1 \text{ and } y \geq x \right\}.$$

One has $\arg\min(X, g) = \left\{ (0, y) \in \mathbb{R}^2 : y \geq 1 \right\}$ and $\operatorname{Min}(X, g) = \{ 0 \}$. Function g is convex, but problem (X, g) is not MS, for instance the minimizing sequence $x^n = (n, n^3)$ doesn't satisfy the stability condition.

The previous example allows us to conclude that without further assumption on the efficient set, such as compactness, the convexity of g is not sufficient to identify a class of well-set problems. Hence the notion of well-setness allows unbounded solution sets, but a sufficient condition under convexity assumption without some compactness requirement, is not able to separate a set of MS or TS problems.

3.3 T-wp and variational analysis

As we have already pointed out in the introduction the first approach in well-posedness is associated with the name of Hadamard with reference to problems in mathematical physics such as boundary value problems for partial differential equations. These problems often have a variational origin.
In this section we intend to underline the variational nature of T-wp referring, in particular, to the Ekeland's variational principle (see [29], [79]). Ekeland's Theorem was discovered in 1974 by Ivar Ekeland ([32], [33]) and provides an approximate minimizer of a bounded from below l.s.c. function in a given neighborhood of a point. This localization property is very useful and its importance is stressed by the extensively use of this result since its discovery. Here we focus on the key role of Ekeland's variational principle to get density or genericity results.

Theorem 3.18. *(Ekeland's variational principle)*
Let $g : X \subseteq \mathbb{R}^n \to (-\infty, +\infty]$ be a l.s.c. and lower bounded function. Let $\epsilon > 0$, $r > 0$ and $\bar{x} \in X$ such that $g(\bar{x}) < \inf_{x \in X} g(x) + r\epsilon$. Then there exists $\hat{x} \in X$ enjoying the following properties:

(i) $d(\hat{x}, \bar{x}) < r$;

(ii) $g(\hat{x}) \leq g(\bar{x}) - \epsilon d(\bar{x}, \hat{x})$;

(iii) $g(\hat{x}) < g(x) + \epsilon d(\hat{x}, x)$, $\forall x \neq \hat{x}$.

Condition (iii) in Theorem 3.18 essentially states a well-posedness result, in fact it can be rewritten as

(iii) the function $g(\cdot) + \epsilon d(\hat{x}, \cdot)$ is T-wp.

Further, condition (iii) essentially states a density result for Tykhonov well-posed problems ([29], [79]). To see this Lucchetti in [79] proposed an example considering the space $\mathcal{F}$ of real valued, l.s.c. positive functions on a complete metric space. Endowing $\mathcal{F}$ with a suitable distance d compatible with uniform convergence on bounded sets he showed the following proposition.

Proposition 3.19. *In $(\mathcal{F}, d)$ the set of the functions which are T-wp is dense.*

He remarks that the same line of reasoning can be made for other classes $\mathcal{F}$ of functions, including convex functions. A reader who is interested in genericity results in the Baire category sense can see [79] and [61] and the references therein.
It is worth pointing out a little remark on terminology. Sometimes the authors speak of well-posed function instead of well-posed problem, but they refer to the minimization problem (X, g), globally considered.

3.4 Extended well-posedness

In 1996 Zolezzi ([125]) introduced a well-posedness property, under the name of extended well-posedness, considering problems with many minimizers and mixing Tykhonov and Hadamard approaches. A given original problem (X, g) is embedded in a family of perturbed ones depending on a parameter and it is extended well-posed when every asymptotically minimizing sequence converges to some solution. In this way, a strengthening of Tykhonov well-posedness is realized because more sequences are tested and, at the same time, a form of continuous dependence of the solution on the parameter, as required by Hadamard idea, is obtained. This form of well-posedness is relevant for applications, in particular to problems of the calculus of variations and optimal control and a local form of this concept is suitable for applications to mathematical programming (see [125] and the references therein).

Let (P, ρ) a metric space and p^* a fixed point of P; L is a closed ball in P of center p^* and positive radius. The functions

$$g : X \subseteq \mathbb{R}^n \to (-\infty, +\infty], \qquad I : X \times L \to (-\infty, +\infty]$$

are proper, extended real valued functions such that

$$g(x) = I(x, p^*), \quad x \in X.$$

Thus (X, g) is the original problem, while $(X, I(\cdot, p))$ model perturbations of it corresponding to the parameter p. We are interested in studying the behavior of approximate solutions of $(X, I(\cdot, p))$ if we slightly change the initial data. To check this, let us define the value function

$$V(p) := \inf \{ I(x, p) : x \in X \}, \;\; p \in L.$$

Definition 3.20. *Problem* (X, g) *is said to be* extended well-posed *(for short E-wp)* (w.r.t. I), *if and only if:*

(i) $\arg\min(X, g) \neq \emptyset$*;*

(ii) $V(p) > -\infty, \quad \forall p \in L$*;*

(iii) $\forall p^n \in P$ *with* $p^n \to p^*$ *and* $\{ x^n \} \subseteq X$ *such that* $I(x^n, p^n) - V(p^n) \to 0$ *exists a subsequence* $\{ x^{n_k} \}$ *converging to some* $\bar{x} \in \arg\min(X, g)$*.*

Zolezzi showed that E-wp (w.r.t. I) implies GT-wp while the converse is not true, in general. The equivalence is satisfied under compactness assumption for $\arg\min(X, g)$ and semicontinuity requirements.

We end this section with an example showing that E-wp (w.r.t. I) doesn't imply T-wp.

Example 3.21. Let $X = \mathbb{R}$, $L = [0, 2]$, $p^* = 1$ and $I(x, p) = (x^2 - p)^2$. Then $g(x) = (x^2 - 1)^2$, $\arg\min(X, g) = \{ -1, +1 \}$ and $\mathrm{Min}\,(X, g) = \{ 0 \}$. Problem (X, g) is E-wp (w.r.t. I) and GT-wp but it is not T-wp, the uniqueness requirement is not satisfied.

Chapter 4

Vector well-posedness

We turn now to vector results, thus we refer to the vector optimization problem (X, f) presented at the beginning of Chapter 2 assuming, here and for all next Chapters, that the feasible region X is a closed set.
We review some main notions of Tykhonov's type, classified as pointwise and global by Miglierina et al. in [93], considering also extended well-posedness introduced by Huang in the nonparametric case and we try to compare these notions constructing some examples. The hierchical structure will be established, thanks to which we shall show that the class of minimization problems in which the objective function is strictly $C-$quasiconvex is well-posed under several approaches. Thus in the main theorem of this part, assuming f strictly $C-$quasiconvex, it will be proved that problem (X, f) is well-posed according to all pointwise and global notions here presented. In the last section we propose a density result for pointwise notions generalizing the approach due to Lucchetti ([79]) in the scalar case, thanks to a vector version of Ekeland's variational principle proposed by Araya in [1]. In this way, we give a characterization of all pointwise notions here presented.

4.1 Pointwise notions

Under the label pointwise well-posedness, Miglierina et al. ([93]) classify those notions in which a minimal point or an efficient solution is fixed. In this way pointwise concepts avoid uniqueness assumption but investigate a local condition of stability, with reference to a single element. In the early eighties Bednarczuk ([7]) and Lucchetti ([78]) published the first attempts to approach well-posedness in vector optimization. In particular Lucchetti (1987) gave a first tentative definition for well-posedness in vector optimization together with initial remarks on how properties of minimum problems can be translated in a vector framework, while Bednarczuk proposed in 1994 ([8]) several definitions based on the properties of $\epsilon-$minimal solutions to vector optimization problems. Since in vector optimization there isn't a commonly accepted definition of well-posed problem, in the following years many papers have been published on this topic, and each of them can be viewed as gener-

alization of the classical approach existing in scalar optimization. In fact every new notion preserves, in some sense, a characterizing property of Tykhonov's idea.
Among the others, we focus on the concepts introduced by Bednarczuk ([8]) and by Loridan ([74]) considering a fixed minimal value, by Dentcheva and Helbig ([27]) and by Huang ([58]) considering a fixed efficient solution.

Definition 4.1. *([74]) Let $\bar{y} \in \mathrm{Min}\,(X, f)$. A sequence $\{ x^n \} \subseteq X$ is called* $\bar{y}-$minimizing *for problem (X, f), when there exists a sequence $\{ \epsilon^n \} \subseteq C, \quad \epsilon^n \to 0$, such that $f(x^n) \leq_C \bar{y} + \epsilon^n$.*

Definition 4.2. *([8]) Let $\bar{y} \in \mathrm{Min}\,(X, f)$. Problem (X, f) is said to be* B–$\bar{y}$ well-posed *(for short B–$\bar{y}$-wp) if and only if every $\bar{y}-$minimizing sequence $\{ x^n \} \subseteq X \setminus f^{-1}(\bar{y})$ admits a subsequence $\{ x^{n_k} \}$ such that $x^{n_k} \to \bar{x} \in f^{-1}(\bar{y})$.*

Definition 4.3. *([74]) Let $\bar{y} \in \mathrm{Min}\,(X, f)$. Problem (X, f) is said to be* L–$\bar{y}$ well-posed *(for short L–$\bar{y}$-wp) if and only if every $\bar{y}-$minimizing sequence admits a subsequence converging to an element of $f^{-1}(\bar{y})$.*

Let $\bar{y} \in \mathrm{Min}\,(X, f)$. If problem (X, f) is L–$\bar{y}$-wp then it is B–$\bar{y}$-wp ([74]), while the converse is not true, in general; the equivalence is satisfied when $f^{-1}(\bar{y})$ is compact.

Example 4.4. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (0, 0)$, the feasible region $X = \left\{ (x, y) \in \mathbb{R}^2 : x = y \text{ and } y \geq 0 \right\}$, with the image set ordered by $C = \mathbb{R}^2_+$. One has $\mathrm{Eff}\,(X, f) = X$ and $\mathrm{Min}\,(X, f) = \{ (0, 0) \}$. Problem (X, f) is B–$(0, 0)$-wp but not L–$(0, 0)$-wp as one can see considering the $(0, 0)-$minimizing sequence $x^n = (n, n)$.

In 1996 Dentcheva and Helbig ([27]), using new concepts of $\epsilon-$solutions, extended some variational principles to vector-valued objective functions and from this, they established a kind of well-posedness for the resulting perturbed vector problems. Thus they generalized Tykhonov's approach in order to obtain similar characterizations of their new concept in terms of sublevel sets (Proposition 3.3) and in terms of variational properties (Section 3.3).

Definition 4.5. *([27]) Let $\bar{x} \in \mathrm{Eff}\,(X, f)$. Problem (X, f) is said to be* DH–$\bar{x}$ well-posed *(for short DH–$\bar{x}$-wp) if and only if*

$$\inf_{\alpha > 0} \mathrm{diam}\,(L(\bar{x}, c, \alpha)) = 0, \quad \forall c \in C,$$

where $L(\bar{x}, c, \alpha) = \{ x \in X : f(x) \leq_C f(\bar{x}) + \alpha c \}$.

Let $\bar{x} \in \mathrm{Eff}\,(X, f)$. If problem (X, f) is DH–$\bar{x}$-wp then it is L–$\bar{y}$-wp ([93]), while the converse is not true, in general; the equivalence is satisfied when $f^{-1}(\bar{y}) = \bar{x}$.

Example 4.6. Let $f : X \subseteq \mathbb{R} \to \mathbb{R}^2$ given by

$$f(x) = \begin{cases} (0, 0) & \text{if } 0 \leq x \leq 1 \\ (x - 1, x - 1) & \text{otherwise} \end{cases}$$

and $X = \mathbb{R}_+$ with the image set ordered by $C = \mathbb{R}^2_+$. One has $\mathrm{Eff}\,(X, f) = [0, 1]$ and $\mathrm{Min}\,(X, f) = \{ (0, 0) \}$. Problem (X, f) is L–$(0, 0)$-wp but not DH–1-wp, for instance.

In 2001, Huang ([58]) proposed new pointwise notions along with sufficient conditions to guarantee these new properties of well-posedness for perturbed vector optimization problems, in connection with a vector valued variational principle. Huang stressed that the main importance in studying well-posedness of the perturbed optimization problems in connection with a vector valued variational principle is the possibility to develop approximate algorithms for vector optimization problems.

Definition 4.7. *([58]) Let $\bar{x} \in \mathrm{Eff}\,(X,f)$. Problem (X,f) is said to be* H–$\bar{x}$ well-posed *(for short H–$\bar{x}$-wp) if and only if $\forall \{ x^n \} \subseteq X$ such that $f(x^n) \to f(\bar{x})$, $x^n \to \bar{x}$.*

Let $\bar{x} \in \mathrm{Eff}\,(X,f)$. In [58] Huang showed that DH–$\bar{x}$-wp implies H–$\bar{x}$-wp while the converse is not true, in general, as one can see in the counterexample 2.1 in [58]. The same example showed that H–$\bar{x}$ implies neither L–$\bar{y}$-wp nor B–$\bar{y}$-wp. The equivalence is possible under a strengthening of the minimality degree.

Proposition 4.8. *Let $\bar{x} \in \mathrm{Eff}\,(X,f)$ such that $\bar{y} = f(\bar{x}) \in \mathrm{StMin}\,(X,f)$. Then (X,f) is H–$\bar{x}$-wp if and only if it is DH–$\bar{x}$-wp.*

Proof: We must show only one direction, that is H–$\bar{x}$-wp $\Rightarrow$ DH–$\bar{x}$-wp.
Assume by contradiction that (X,f) is not DH–$\bar{x}$-wp. Thus there exist sequences $\{ x^n \} \subseteq X$, $\{ \alpha_n \}$ with $\alpha_n \to 0^+$ and $\bar{n} \in \mathbb{N}$ such that $\forall n > \bar{n}$ and for some $c \in C$ one has

$$x^n \in L(\bar{x}, c, \alpha_n) \quad \text{but} \quad x^n \not\to \bar{x}$$

and

$$f(x^n) \leq_C f(\bar{x}) + \alpha_n c, \quad \text{but} \quad f(x^n) \not\to f(\bar{x}),$$

otherwise (X,f) is not H–$\bar{x}$-wp.
Let $z^n = f(x^n) \subseteq f(X)$. By definition of $L(\bar{x}, c, \alpha)$ follows the existence of a sequence $\{ y^n \} \subseteq f(X)$ such that $y^n \in z^n + C$ and $y^n \to \bar{y}$, contradicting the strictly minimality of $\bar{y}$ (Proposition 2.8). □

The relationships among the various pointwise concepts recalled here are summarized in the following scheme.

$$\begin{array}{rcccl}
 & & \text{H–}\bar{x}\text{-wp} & & \\
f(\bar{x}) \in \mathrm{StMin}\,(X,f) & \Downarrow & & \Uparrow & \\
 & & \text{DH–}\bar{x}\text{-wp} & & \\
 & \Downarrow & & \Uparrow & f^{-1}(\bar{y}) = \bar{x} \\
 & & \text{L–}\bar{y}\text{-wp} & & \\
 & \Downarrow & & \Uparrow & f^{-1}(\bar{y}) \text{ compact} \\
 & & \text{B–}\bar{y}\text{-wp} & &
\end{array}$$

4.2 Global notions

Global well-posedness means that the stability condition is investigated with reference to the whole solutions set. To get a tidy and complete comparison between some properties of this class, we divide them in two groups: first we consider global well-posedness and efficient solutions, then global well-posedness and weakly efficient solutions. In each section we compare the notions between them, we underline, when it is possible, the relationships with the previous concepts and finally we stress the geometrical features characterizing the image set through an example that permit us to achieve another well-posedness property which enlarge, in some way, the class of well-posed problems. Then we compare the notions belonging to the different groups in order to draw a final outline that will be our start point to study well-posedness under generalized convexity assumptions.

4.2.1 Global notions and efficient solutions

The passage from pointwise to global notions was traced by Bednarczuk ([8]), considering the concept of B–minimizing sequence and the generalization of the stability condition.

Definition 4.9. *A sequence* $\{ x^n \} \subseteq X$ *is called* B–minimizing *for problem* (X, f), *when for each* $n \in \mathbb{N}$ *there exists* $\epsilon^n \in C$ *and* $y^n \in \mathrm{Min}\,(X, f)$ *such that* $f(x^n) \leq_C y^n + \epsilon^n, \quad \epsilon^n \to 0$.

Definition 4.10. *([8]) Problem* (X, f) *is said to be* B–well-posed *(for short B-wp) if and only if:*

(i) $\mathrm{Min}\,(X, f) \neq \emptyset$;

(ii) every B–minimizing sequence $\{ x^n \} \subseteq X \setminus \mathrm{Eff}\,(X, f)$ *admits a subsequence converging to some element of* $\mathrm{Eff}\,(X, f)$.

Proposition 4.11. *([8]) Let* $\mathrm{Min}\,(X, f)$ *a compact set. If problem* (X, f) *is B–*$\bar{y}$*-wp for every* $\bar{y} \in \mathrm{Min}\,(X, f)$, *then* (X, f) *is B-wp.*

Example 4.12. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2,\ f(x_1, x_2) = (x_1, x_1)$ with $X = C = \mathbb{R}^2_+$. The only minimal value is $(0, 0)$, while $\mathrm{Eff}\,(X, f) = \{ (0, x_2) : x_2 \geq 0 \}$. Problem (X, f) is not B-wp as for example the B–minimizing sequence $x^n = \left(\frac{1}{n}, n\right)$ doesn't admit any subsequence converging to some efficient solution.

To enlarge the class of well-posed problems, Bednarczuk proposed a new definition in which the stability condition is based on the distance, in the norm sense, from the efficient solutions set instead of the convergence of appropriate minimizing subsequences.

Definition 4.13. *([8]) Problem* (X, f) *is said to be* Bw–well-posed *(for short Bw-wp) if and only if:*

(i) $\mathrm{Min}\,(X, f) \neq \emptyset$;

(ii) for every B–minimizing sequence $\{ x^n \} \subseteq X, \quad d(x^n, \mathrm{Eff}\,(X, f)) \to 0.$

B–well-posedness implies Bw–well-posedness, while the converse is in general not true. The equivalence can be stated under the compactness of the efficient set.

Proposition 4.14. *Let* $\mathrm{Eff}\,(X, f)$ *be a nonempty compact set. If* (X, f) *is Bw-wp then it is also B-wp.*

Proof: We distinguish two cases.
Assume $\mathrm{Eff}\,(X, f) = X$. Then there aren't B–minimizing sequences out of the efficient set and hence (X, f) is B-wp.
Assume $\mathrm{Eff}\,(X, f) \subset X$. By compactness assumption one has $d(x^n, \mathrm{Eff}\,(X, f)) \to 0$ if and only if $\exists x^{n_k} \to \bar{x} \in \mathrm{Eff}\,(X, f)$ for every B–minimizing sequence. □

The notion of Bw–well-posedness fails when the image set and the ordering cone have some asymptote in common as the following example shows.

Example 4.15. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, y)$, the feasible region $X = \left\{ (x, y) \in \mathbb{R}^2 : y \geq 0 \text{ or } y \geq -x \right\}$ and $C = \mathbb{R}^2_+$. Problem (X, f) is not Bw-wp as for example the B–minimizing sequence $x^n = (-n, 0)$ doesn't satisfy the stability condition.

To avoid this difficulty, Miglierina and Molho proposed to relax the requirement of convergence of the minimizing sequences also in the feasible region.

Definition 4.16. *([92]) Problem* (X, f) *is said to be* M–well-posed *(for short M-wp) when for every* $\{ x^n \} \subseteq X$ *such that* $d(f(x^n), \mathrm{Min}\,(X, f)) \to 0$, *one has* $d(x^n, \mathrm{Eff}\,(X, f)) \to 0$.

Miglierina and Molho in [92] proved that Bw–well-posedness implies M–well-posedness while the converse is not true in general as it is showed in Example 4.15. The equivalence is proved assuming a strict degree of minimality [93].

Theorem 4.17. *([93]) If* (X, f) *is M-wp and for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$(f(X) - \mathrm{Min}\,(X, f)) \cap (\delta B - C) \subseteq \epsilon B, \tag{4.1}$$

then it is also Bw-wp.

Note that Definition 4.16 is always satisfied when the efficient solution set is empty and in this particular case there is no attention to the structure of the optimization problem with reference to a weak concept of minimal points identified by the given ordering cone. In the next subsections we focus on global definitions seeking a weaker concept of efficient solution.
Till now, we can trace the following scheme:

$$
\begin{array}{ccl}
 & \text{B-wp} & \\
\Downarrow & & \Uparrow \text{Eff}\,(X,f) \text{ compact} \\
 & \text{Bw-wp} & \\
\Downarrow & & \Uparrow \text{Min}\,(X,f) = \text{StMin}\,(X,f) \\
 & \text{M-wp} &
\end{array}
$$

4.2.2 Global notions and weakly efficient solutions

We introduce three notions due to Huang ([56]) as generalization of the previous global concepts and keeping attention to the original idea of extended well-posedness published by Zolezzi ([125]). In this subsection we consider a nonparametric version, in order to compare these concepts with the others.

Definition 4.18. *A sequence* $\{ x^n \} \subseteq X$ *is called* Hs–minimizing *for problem* (X, f), *if there exists* $c \in \operatorname{int} C, \quad t_n > 0, t_n \to 0$ *such that* $f(X) - f(x^n) + t_n c \notin -C$.

Definition 4.19. *Problem* (X, f) *is said to be* Hs–well-posed *(for short Hs-wp) if and only if:*

(i) $\text{WEff}\,(X, f) \neq \emptyset$;

(ii) every Hs–minimizing sequence $\{ x^n \} \subseteq X$ *admits a subsequence converging to some element of* $\text{WEff}\,(X, f)$.

An example of minimization problem that doesn't satisfy the previous definition is the following.

Example 4.20. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, y)$, the feasible region $X = \left\{ (x, y) \in \mathbb{R}^2 : y \geq xe^{-x},\ x \geq 0 \right\}$ and $C = \mathbb{R}^2_+$. The problem (X, f) is not Hs-wp as for example the Hs–minimizing sequence $x^n = (n, ne^{-n})$ doesn't admit any subsequence converging to some weakly efficient solution.

The following notion of well-posedness is a generalization of B–well-posedness in which the stability condition is referred to the weakly efficient solutions set.

Definition 4.21. *A sequence* $\{ x^n \} \subseteq X$ *is called* H–minimizing *for problem* (X, f) *if there exist* $c \in \operatorname{int} C$, $\alpha_n > 0$, $\alpha_n \to 0$, *and* $y^n \in \text{Min}\,(X, f)$ *such that* $f(x^n) \leq_C y^n + \alpha_n c$.

Definition 4.22. *Problem* (X, f) *is said to be* H–well-posed *(for short H-wp) if and only if:*

(i) $\text{WEff}\,(X, f) \neq \emptyset$;

(ii) every H–minimizing sequence $\{ x^n \} \subseteq X$ *admits a subsequence converging to some element of* $\text{WEff}\,(X, f)$.

Huang in [56] pointed out that Hs-wp $\Rightarrow$ H-wp, but the converse is not true in general. In fact, the problem in Example 4.20 is H–well-posed but not Hs–well-posed. The geometrical feature of the image set of a problem that doesn't satisfy Definition 4.22 is the same we have already met considering Bw–well-posedness, that is the presence of some asymptote in common with the ordering cone.

Example 4.23. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x,y)$ the feasible region $X = \left\{ (x,y) \in \mathbb{R}^2 : y \geq -xe^x,\ x \leq 0 \right\}$ and $C = \mathbb{R}^2_+$. The problem (X,f) is not H-wp as for example the H–minimizing sequence $x^n = (-n, ne^n)$ doesn't admit any subsequence converging to some weakly efficient solution.

To extend the class of well-posed problems, Huang used the same trick of distance instead of convergence of every appropriate minimizing sequence in the image set.

Definition 4.24. *Problem (X,f) is said to be* Hw–well-posed *(for short Hw-wp) when for every $\{ x^n \} \subseteq X$ such that $d(f(x^n), \mathrm{WMin}\,(X,f)) \to 0$, there exists a subsequence converging to a weakly efficient solution.*

As in the parametric case, the following implications hold:

$$\text{Hs-wp} \Rightarrow \text{H-wp} \Rightarrow \text{Hw-wp}.$$

Problem (X,f) in Example 4.23 is Hw–well-posed, while the problem in Example 4.15 is neither H–well-posed nor Hw–well-posed, as for instance the H-minimizing sequence $x^n = (-n, 0)$ doesn't admit any subsequence converging to some weakly efficient solution.
As pointed out by Huang ([56]), any one of Definitions 4.19, 4.22, 4.24 implies that $\mathrm{WEff}\,(X,f)$ is compact; so to establish the equivalence of these three notions with a property in which there isn't a requirement of convergence in the domain, it is necessary to assume $\mathrm{WEff}\,(X,f)$ compact.

Theorem 4.25. *([101]) Let $\mathrm{Eff}\,(X,f) = \mathrm{WEff}\,(X,f)$ be a compact set. Then problem (X,f) is B-wp if and only if it is H-wp.*

Proof: First, we show that B-wp $\Rightarrow$ H-wp.
Suppose, to the contrary, that the problem (X,f) is not H-wp. Then there exists a sequence $\{ x^n \} \subseteq X$, $c \in \operatorname{int} C$, $\alpha_n > 0$, $\alpha_n \to 0$ and $y^n \in \mathrm{Min}\,(X,f)$ such that $f(x^n) \leq_C y^n + \alpha_n c$. We distinguish two cases.
Let $\{ x^n \} \subseteq X \setminus \mathrm{Eff}\,(X,f)$. In this case $\{ x^n \}$ is a B-minimizing sequence, so there exists a subsequence converging to some element of $\mathrm{Eff}\,(X,f)$.
Let $\{ x^n \} \subseteq \mathrm{Eff}\,(X,f)$. By compactness of $\mathrm{Eff}\,(X,f)$, there exists a subsequence converging to an efficient point.

The reverse implication is equivalent to show that for $c \in \operatorname{int} C$, $\exists \alpha_n > 0$, $\alpha_n \to 0$, such that

$$\{ x^n : f(x^n) - y^n \in \epsilon^n - C \} \subseteq \{ x^n : f(x^n) - y^n \in \alpha_n c - C \}$$

which holds if $\forall \epsilon^n \in C,\ \epsilon^n \to 0,\ \exists \alpha_n > 0, \alpha_n \to 0$ such that $\epsilon^n - C \subseteq \alpha_n c - C$. The previous inclusion can be rewritten as $\epsilon^n - \alpha_n c - C \subseteq -C$, and it is satisfied if $\Delta_{-C}(\epsilon^n - \alpha_n c - z) \leq 0,\ \forall z \in C$, as C is a closed, convex cone. Since Δ_{-C} is subadditive, we have

$$\Delta_{-C}(\epsilon^n - \alpha_n c - z) \leq \Delta_{-C}(\epsilon^n) + \Delta_{-C}(-\alpha_n c) + \Delta_{-C}(-z).$$

The righthandside in this inequality is negative requiring $\Delta_{-C}(\epsilon^n) + \Delta_{-C}(-\alpha_n c) \leq 0$ (since $\Delta_{-C}(-z) \leq 0$). Observing that $\Delta_{-C}(\epsilon^n) > 0$, while $\Delta_{-C}(-\alpha_n c) < 0$, the inequality implies $\Delta_{-C}(\epsilon^n) + \Delta_{-C}(-\alpha_n c) \leq 0$ and by homogeneity of the oriented distance function, we get

$$\Delta_{-C}(\epsilon^n) + \alpha_n \Delta_{-C}(-c) \leq 0 \text{ when } 0 < \alpha_n \leq -\frac{\Delta_{-C}(\epsilon^n)}{\Delta_{-C}(-c)}.$$

So it is always possible to choose α_n such that H-wp implies B-wp. $\square$

A comparison with the notions based on the efficient solutions, under the assumption $\text{Eff}\,(X, f) = \text{WEff}\,(X, f)$, gives the following outline

$$\begin{array}{ccc} & & \text{Hs-wp} \\ & & \Downarrow \\ \text{Bw-wp} & = & \text{H-wp} \\ \Downarrow & & \Downarrow \\ \text{M-wp} & = & \text{Hw-wp} \\ & (a) & \end{array}$$

where $(a) = \text{WEff}\,(X, f)$ compact and the link between M–well-posedness and Hw–well-posedness follows directly from definitions.
We note that Hw-wp implies H-wp under the same assumptions for which M-wp implies Bw-wp.

Remark 4.26. Huang and Yang ([60]) introduced six different types of generalized well-posedness in the extended sense inspired by the scalar notion due to Levitin-Polyak ([69]) and the scalar generalization in [59] where the constraint is specified by a function. It is worth noting that in our framework, in which the stability condition is investigated with reference to an appropriate notion of minimizing sequence when it belongs to the feasible region, the notions presented in [60] coincide with Hw, H and Hs–well-posedness.

Keeping in mind the variational nature of Tykhonov well-posedness and recalling that scalar variational inequalities provide a very general model for a wide range of problems, in particular equilibrium problems (see e.g. [65]) Crespi et al. ([20]) proposed a new notion of well-posedness. The links between variational inequalities of differential type (that means in which the operator involved is the gradient of a primitive function) and optimization problems have been studied (see [65] and more recently [17] and [18]). Moreover, by means of Ekeland's variational principle a notion of well-posed scalar variational inequality has been introduced ([29], [82]) and its relations with the well-posedness of the primitive optimization problem has been investigated. Moving from this general view, the new notion presented is an extension of Tykhonov's idea, avoiding the uniqueness requirement, for optimization problems deeply linked with a given notion of well-posed vector variational inequality (of differential type). Studying the relations with well-posedness of the primitive optimization problem, the authors show that for $C-$convex functions the two notion of well-posedness coincide. Focusing on our minimization problem, we formalize their concept in the following definition.

Definition 4.27. *([20]) A sequence* $\{ x^n \} \subseteq X$ *is called* CGR–minimizing *for problem* (X, f)*, when there exist* $c^0 \in \operatorname{int} C, \quad \epsilon_n \geq 0, \quad \epsilon_n \to 0$ *such that* $f(x) - f(x^n) + \epsilon_n c^0 \notin -\operatorname{int} C, \quad \forall x \in X$.

Definition 4.28. *([20]) Problem* (X, f) *is said to be* CGR–well-posed *(for short CGR-wp) if and only if:*

(i) $\operatorname{WEff}(X, f) \neq \emptyset$;

(ii) for every CGR–minimizing sequence $d(x^n, \operatorname{WEff}(X, f)) \to 0$ *as* $n \to +\infty$.

The problem in Example 4.23 is not CGR-wp.

Remark 4.29. The vector well-posedness in the extended sense introduced in [22] coincides with CGR-wp in the nonparametric case.

All the notions introduced till now assume the existence of a solution for the vector problem and the authors can obtain sufficient conditions under generalized convexity assumptions. In Section 2.2 we have seen as the existence may derive from a coercivity condition for the objective function. Deng in [26] studying the issue of well-posedness for vector optimization, showed that coercivity implies well-posedness without any convexity assumptions on problem data. The motivation to explore this direction is that, as showed by the same author in [25], the level-coercivity property is closely related to certain error bounds for scalar optimization problems. Thus he derived a criterion for well-posedness in terms of associated scalar problems and a consequence in terms of error bounds.

Definition 4.30. *([26]) A sequence* $\{ x^n \} \subseteq X$ *is called* D–minimizing *for problem* (X, f)*, when* $d(f(x^n), \operatorname{WMin}(X, f)) \to 0$.

Definition 4.31. *Problem* (X, f) *is said to be* D–well-posed *(for short D-wp) if and only if:*

(i) $\operatorname{WMin}(X, f)$ *is closed;*

(ii) for every D–minimizing sequence $d(x^n, \operatorname{WEff}(X, f)) \to 0$ *as* $n \to +\infty$.

We underline that Deng introduced the previous definition in the particular case of the Pareto order. In this chapter we consider the general case in which the cone satisfies the requirements specified in Chapter 2.

Theorem 4.32. *Let* $\mathrm{WMin}\,(X,f)$ *be closed. If problem* (X,f) *is CGR-wp then it is D-wp.*

Proof: To prove the statement is equivalent to show that

$$\{\, x^n \in X : d(f(x^n), \mathrm{WMin}\,(X,f)) \to 0 \,\} \subseteq \{\, x^n \in X : f(X) - f(x^n) + \epsilon_n c^0 \notin -\mathrm{int}\, C \,\}.$$

By definition $\bar{x} \in \mathrm{WEff}\,(X,f)$ when, $f(x) - f(\bar{x}) \notin -\mathrm{int}\, C,\ \forall x \in X$.
If $d(x^n, \mathrm{WEff}\,(X,f)) \to 0$, one can find a sequence $\epsilon_n \geq 0,\ \epsilon_n \to 0$ and a vector $c^0 \in \mathrm{int}\, C$ such that Definition 4.27 is satisfied. □

The assumption $\mathrm{WMin}\,(X,f)$ closed in Theorem 4.32 cannot be avoided.

Example 4.33. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2,\ f(x,y) = (x^2, e^y),\ X = \mathbb{R}^2,\ C = \mathbb{R}^2_+$. Problem (X,f) is CGR-wp but not D-wp, and $\mathrm{WMin}\,(X,f)$ is not closed.

The converse of Theorem 4.32 is not true in general, for instance problem in Example 4.23 is not CGR-wp but it is D-wp.

Proposition 4.34. *If problem* (X,f) *is D-wp and for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$(f(X) - \mathrm{WMin}\,(X,f)) \cap (\delta B - C) \subseteq \epsilon B, \tag{4.2}$$

then it is also CGR-wp.

Proof: Suppose, to the contrary, that problem (X,f) is not CGR-wp, that is $\exists \{\, x^n \,\} \subseteq X$ satisfying the following two properties:

1. $\{\, x^n \,\}$ is CGR–minimizing, thus $\exists \epsilon_n > 0,\ \epsilon_n \to 0,\ c^0 \in \mathrm{int}\, C$ such that $f(X) - f(x^n) + \epsilon_n c^0 \notin -\mathrm{int}\, C$;
2. $\exists \alpha > 0$ such that $x^n \in [\mathrm{WEff}\,(X,f) + \alpha B]^c$ for all n large enough.

Either of the two following cases occur:

(i) x^n is such that $d(f(x^n), \mathrm{WMin}\,(X,f)) \to 0$. In this case x^n is also a D-minimizing sequence and hence by the assumption problem (X,f) is D-wp, it follows it is also CGR-wp as we contradict the previous point 2.

(ii) $\exists \delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$f(x^n) \in [\mathrm{WMin}\,(X,f) + \delta B]^c, \quad \forall n > n_0. \tag{4.3}$$

Since $\{\, x^n \,\}$ is CGR–minimizing

$$f(X) - f(x^n) + \epsilon c^0 \notin -\mathrm{int}\, C$$

$$f(x^n) - f(X) - \epsilon c^0 \notin \mathrm{int}\, C$$

$$f(x^n) \in \left[f(X) + \epsilon c^0 + \mathrm{int}\, C \right]^c.$$

So, $\forall \bar{y} \in \mathrm{WMin}\,(X,f)$ one has $\bar{y} + \epsilon c^0 \in \bar{y} + \delta B,\ \forall n > n_1$ and hence

$$f(x^n) \in [\bar{y} + \delta B + \operatorname{int} C]^c$$

and also

$$f(x^n) \in f(X) \cap [\bar{y} + \delta B + \operatorname{int} C]^c\,.$$

Recalling (4.3), we have a contradiction to the assumption (4.2).

□

Now, we compare CGR–well-posedness and D–well-posedness with the previous notions, in particular with reference to the work of Huang we have the following result (see [102]).

Proposition 4.35. *Let* $\mathrm{WEff}\,(X,f)$ *be a compact set. Problem* (X,f) *is CGR-wp if and only if it is Hs-wp.*

The proof follows form the easily comparison of the two stability conditions and hence is omitted.
The compactness assumption is fundamental only to show that CGR–well-posedness implies Hs–well-posedness.

Example 4.36. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2, X = \left\{ (x,y) \in \mathbb{R}^2 : y \geq 0 \text{ or } y \geq -x \right\},\ C = \mathbb{R}^2_+$ and $f(x,y) = (x,y)$. The problem (X,f) is CGR-wp, but not Hs-wp, as, for example, the Hs–minimizing sequence $x^n = \left(n, -n + \frac{1}{n}\right)$ doesn't admit any subsequence converging to a weakly efficient solution.

The final outline, completed with all global definitions, is based on the following assumptions

(-) $\mathrm{Eff}\,(X,f) = \mathrm{WEff}\,(X,f)$

(+) $\mathrm{Eff}\,(X,f)$ compact

(*) $\mathrm{WEff}\,(X,f)$ compact

(**) $\mathrm{WMin}\,(X,f)$ closed

(***) $\mathrm{WEff}\,(X,f)$ compact and $\mathrm{WMin}\,(X,f)$ closed.

B-wp		Hs-wp	$\overset{(*)}{=}$	CGR-wp	
$(+)\Uparrow$ $\quad\Downarrow$	(-)	$\Downarrow$			
Bw-wp	$=$	H-wp		$(**)\Downarrow$	$(4.2)\Uparrow$
$(4.1)\Uparrow$ $\quad\Downarrow$		$\Downarrow$			
M-wp	$\underset{(*,-)}{=}$	Hw-wp	$\underset{(***)}{=}$	D-wp	

We have pointed out that any new notion of well-posedness of this section enlarges, in some way, the class of problems characterized by that property, but an interesting investigation about well-posedness properties consists in identifying which classes of functions satisfy, for sure, a given concept. It is known that, under appropriate generalized convexity assumptions, some well-posedness properties are satisfied (see for instance [20], [91], [92], [93]).
We end this section with a similar result involving all global notions here presented.

Theorem 4.37. *Assume* $\mathrm{WEff}\,(X, f)$ *be nonempty and bounded. If* $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ *is continuous and strictly* $C-$*quasiconvex then all global notions here presented coincide.*

Proof: The proof follows from Proposition 2.24 since under compactness of $\mathrm{WEff}\,(X, f)$ $C-$quasiconvex functions are CGR-wp. $\square$

4.3 A density result

In this section we wish to extend a density result presented in the book edited by Lucchetti ([79]) in vector optimization considering the pointwise notion introduced by Dentcheva and Helbig. To this goal we need a vector version of Ekeland's variational principle and a deepening of properties of DH–well-posed problems.
We begin to introduce the vectorial version of Ekeland's variational principle by Araya ([1]). In the sequel $c^0 \in \operatorname{int} C$ is a fixed vector such that $\left\|c^0\right\| \leq 1$.

Theorem 4.38. *([1]) Let* $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ *be a vector-valued function satisfying:*

(G) *For every* $\epsilon > 0$ *there is an initial point* $x^0 \in X$ *such that* $f(X) \cap (f(x^0) - \epsilon c^0 - \operatorname{int} C) = \emptyset$*;*

(H) $\left\{ x' \in X : f(x') + \|x' - x\|\, c^0 \leq_C f(x) \right\}$ *is closed for every* $x \in X$*.*

Then there exists $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) <_C f(x^0)$*;*

(ii) $\left\|x^0 - \bar{x}\right\| \leq 1$*;*

(iii) $f(x) + \epsilon \|x - \bar{x}\|\, c^0 \nleq_C f(\bar{x})$ *for all* $x \neq \bar{x}$*;*

(iv) (X, g) *with* $g(x) = f(x) + \epsilon \|x - \bar{x}\|\, c^0$ *is DH-*$\bar{x}$*-wp.*

Proof: We must prove only point (iv). To see this we use the smallest monotone function h_a studied by Krasnoselski and Rubinov in the sixties and seventies and later by Luc, with a equal to the zero vector together with, of course, its important properties (see Lemma 2.27). Thus, for simplicity, we write h instead of $h_0(y) := \inf \left\{ t \in \mathbb{R} : y \in tc^0 - C \right\}$.
We know that (X, g) is DH-$\bar{x}$-wp if and only if (X, h) with $h(g(x) - g(\bar{x}))$ is T-wp

([93]).
By (iii), one has $g(x) - g(\bar{x}) \not\leq_C 0,\ \forall x \neq \bar{x}$ that means $h(g(x) - g(\bar{x})) > 0,\ \forall x \neq \bar{x}$. Moreover $h(g(\bar{x}) - g(\bar{x})) = h(0) = 0$, thus $\bar{x}$ is the unique minimum point for $h(g(x) - g(\bar{x}))$.
Let $z(x) = h(g(x) - g(\bar{x})) = h(f(x) - f(\bar{x})) + \epsilon \|x - \bar{x}\|$ and let x^n be a sequence such that $\|x^n\| \to \infty$. Then

$$0 \leq \lim_{\|x^n\| \to \infty} z(x^n) = \lim_{\|x^n\| \to \infty} h(f(x^n) - f(\bar{x})) + \epsilon \|x^n - \bar{x}\| = +\infty$$

hence we can conclude that function $h(g(x) - g(\bar{x}))$ is coercive and thus T-wp. $\square$

Consider the space

$$\mathcal{F} := \Big\{ f : X \subseteq \mathbb{R}^n \to \mathbb{R}^l : f \text{ is } C - \text{convex} \Big\}.$$

We endow $\mathcal{F}$ with a distance. Fix $\theta \in X$ and set for any two functions $f, v \in \mathcal{F}$ and $n \in \mathbb{N}$,

$$\|f - v\|_n = \sup_{\|x - \theta\| \leq n} \|f(x) - v(x)\|.$$

If $\|f - v\|_n = \infty$ for some n, then set $d(f, v) = 1$, otherwise

$$d(f, v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - v\|_n}{1 + \|f - v\|_n}.$$

To introduce our density result, we need five lemmas.

Lemma 4.39. *Let $c : \mathbb{R}^n \to \mathbb{R}$ a proper convex function and $g : \mathbb{R}^n \to \mathbb{R}$ a quadratic convex function. Then $s(x) = c(x) + g(x)$ is coercive.*

Proof: Since c is proper, it is sufficient to distinguish the following two cases.
Let x^n be a sequence with $\|x^n\| \to +\infty$ and $\lim c(x^n) > -\infty$. Then, by definition of s, it holds $\lim s(x^n) = +\infty$.
Let now x^n be a sequence with $\|x^n\| \to +\infty$ and $\lim c(x^n) = -\infty$ and let $x^0 \in \mathbb{R}^n$ be a point such that $c(x^0)$ is finite. Without loss of generality, assume $g(x) = \langle x, x \rangle = \sum_{i=1}^n x_i^2$. Since $c(x)$ is convex, the set $\partial c(x^0)$ of all subgradients of c at x^0 is nonempty (see [108], Theorem 23.4) and by definition of subgradient, for every $x^* \in \partial c(x^0)$ it holds $c(x) \geq c(x^0) + \langle x^*, x - x^0 \rangle, \forall x \in \mathbb{R}^n$. Hence,

$$\begin{aligned}
\lim s(x^n) &= \lim \left[c(x^n) + \sum_{i=1}^{n} (x_i^n)^2 \right] \\
&\geq \lim \left[c(x^0) + \langle x^*, x^n - x^0 \rangle + \sum_{i=1}^{n} (x_i^n)^2 \right] \\
&= +\infty
\end{aligned}$$

showing that s is coercive. $\square$

Lemma 4.40. *Let $f \in \mathcal{F}$, $j > 0$ and $v : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ defined by $v(x) := f(x) + \frac{1}{j}\|x-\theta\|^2 c^0$. Then v is strictly $C-$convex.*

Proof: Clearly, function $x \mapsto \|x-\theta\|^2$ is strictly convex. From this, $\forall x, z \in X,\ x \neq z$ and $\tilde{x} = t(x) + (1-t)z,\ t \in (0,1)$, the following inequalities hold:

$$\begin{aligned} tv(x) + (1-t)v(z) &= t(f(x)) + (1-t)f(z) + \frac{t}{j}\|x-\theta\|^2 c^0 + (1-t)\frac{1}{j}\|z-\theta\|^2 c^0 \\ &\geq_C f(\tilde{x}) + \frac{1}{j}[t\|x-\theta\|^2 + (1-t)\|z-\theta\|^2]c^0 \\ &>_C f(\tilde{x}) + \frac{1}{j}\|\tilde{x}-\theta\|^2 c^0 \\ &= \ v(\tilde{x}) \end{aligned}$$

hence v is strictly $C-$convex. $\square$

Lemma 4.41. *Let $v : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ be a strictly $C-$quasiconvex function. If* $\mathrm{Eff}\,(X,v) \neq \emptyset$*, then (X,v) is DH-x-wp, $\forall x \in \mathrm{Eff}\,(X,v)$.*

Proof: It follows from Proposition 6.2 in [93] and Proposition 2.24. $\square$

Lemma 4.42. *Let $f \in \mathcal{F}$, $j > 0$, X unbounded and $v : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ defined by $v(x) := f(x) + \frac{1}{j}\|x-\theta\|^2 c^0$. Then $h \circ v$ is coercive and convex.*

Proof: Note that $\forall x \in X$:

$$h(v(x)) = h(f(x)) + \frac{1}{j}\|x-\theta\|^2$$

by properties of function h. To prove that $h(v(x))$ is convex, it is sufficient to prove that $h(f(x))$ is convex. For every $x, z \in X,\ t \in [0,1]$, it holds:

$$\begin{aligned} h(f(tx + (1-t)z)) &\leq h(tf(x) + (1-t)f(z)), \\ &\leq th(f(x)) + (1-t)h(f(z)), \end{aligned}$$

hence $h(v(x))$ is convex.
Coercivity follows now by Lemma 4.39. $\square$

Lemma 4.43. *Let $f \in \mathcal{F}$, $j > 0$ and $v : X \subseteq \mathbb{R}^n \to \mathbb{R}^l$ defined by $v(x) := f(x) + \frac{1}{j}\|x-\theta\|^2 c^0$. Then function v has bounded sublevel sets.*

Proof: Let $y \in \mathbb{R}^l$. Set $t = h(y)$ and $K := \{\, x \in X : h(v(x)) \leq t \,\}$. By Lemma 4.42 we know that K is a bounded set. If $\mathrm{Lev}_v(y) \subseteq K$, since y is arbitrary, v has bounded sublevel sets.
Let $x \in \mathrm{Lev}_v(y)$. Then $v(x) \leq_C y$ and by $C-$monotonicity of h it follows $h(v(x)) \leq h(y) = t$ and thus $x \in K$. $\square$

Now, we are ready to present our density result.

Theorem 4.44. *The set of functions $f \in (\mathcal{F}, d)$ such that*

(i) $\mathrm{Eff}\,(X, f) \neq \emptyset$*;*

(ii) (X, f) *is DH-x-wp,* $\forall x \in \mathrm{Eff}\,(X, f)$*;*

is dense in $(\mathcal{F}, d)$*.*

Proof: Fix $\sigma > 0$, $f \in (\mathcal{F}, d)$ and take j so large that

$$v(x) := f(x) + \frac{1}{j}\, \|x - \theta\|^2 c^0 \ \text{ satisfies } \ d(f, v) < \frac{\sigma}{2}.$$

Then $v \in (\mathcal{F}, d)$, in fact v is strictly $C-$convex by Lemma 4.40.
Since $C-$convex functions are continuous ([119]), the sets

$$\{\, x \in X : v(x) + \|x - y\|\, c^0 \leq_C v(y) \,\}$$

are closed and thus v satisfies assumption (H) in Theorem 4.38.
By Lemma 4.43, v has bounded sublevel sets and thus there exist $y \in \mathbb{R}^l$ and $M > 0$ such that $\mathrm{Lev}_v(y)$ is contained in $B(\theta, M)$ and since $B(\theta, M)$ is compact, $\mathrm{WEff}\,(\mathrm{Lev}_v(y), v) \neq \emptyset$. Moreover

$$\begin{array}{rrcl} & \mathrm{WEff}\,(\mathrm{Lev}_v(y), v) & \subseteq & \mathrm{WEff}\,(X, v) \\ \text{and} & \mathrm{Eff}\,(\mathrm{Lev}_v(y), v) & \subseteq & \mathrm{Eff}\,(X, v). \end{array}$$

Since v is strictly $C-$ convex (Lemma 4.40), by Proposition 2.24 $\mathrm{WEff}\,(X, v) = \mathrm{Eff}\,(X, v)$ and thus $\mathrm{Eff}\,(X, v) \neq \emptyset$. It follows that assumption (G) in Theorem 4.38 is trivially satisfied by function v, indeed it is enough to choose any point $x^0 \in \mathrm{Eff}\,(X, v)$.
Apply the vectorial Ekeland's variational principle with $\epsilon = \frac{\sigma}{2s}$, $s = \sum_{n=1}^{+\infty} 2^{-n}(n + M + 1)$. By Lemmas 4.42 and 4.43 there exists a bounded sublevel set of v containing θ and x^0. Thus there exists $M > 0$ such that $x^0 \in B(\theta, M)$. Find $\hat{x}$ such that $\left\|\hat{x} - x^0\right\| \leq 1$ and define

$$u(\cdot) = v(\cdot) + \epsilon\, \|\cdot - \hat{x}\|\, c^0,$$

with $u(x) \not\leq_C v(\hat{x})$, $\forall x \neq \hat{x}$. Note that $u \in (\mathcal{F}, d)$ and

$$\begin{aligned} \|x - \hat{x}\| &\leq \left\|x - x^0\right\| + \left\|x^0 - \hat{x}\right\| \\ &\leq \left\|x - x^0\right\| + 1 \\ &\leq \|x - \theta\| + \left\|\theta - x^0\right\| + 1. \end{aligned}$$

It follows:

$$\begin{aligned} \|u - v\|_n &= \sup_{\|x-\theta\|\leq n} \|u(x) - v(x)\| \\ &= \sup_{\|x-\theta\|\leq n} \left\|v(x) + \epsilon\, \|x - \hat{x}\|\, c^0 - v(x)\right\| \\ &= \sup_{\|x-\theta\|\leq n} \epsilon \left\|\|x - \hat{x}\|\, c^0\right\| \\ &\leq \epsilon(n + M + 1). \end{aligned}$$

Hence:

$$
\begin{aligned}
d(u,v) &\leq \sum_{n=1}^{+\infty} 2^{-n} \frac{\epsilon(n+M+1)}{1+\|f-v\|_n} \\
&\leq \sum_{n=1}^{+\infty} 2^{-n}(n+M+1)\epsilon \\
&= s\epsilon = s \cdot \frac{\sigma}{2s} \\
&= \frac{\sigma}{2}.
\end{aligned}
$$

Problem (X,u) is DH-$\hat{x}$-wp with $\hat{x} \in \mathrm{Eff}\,(X,u) \neq \emptyset$, by Theorem 4.38 and since u is strictly $C-$convex, by Lemma 4.41, (X,u) is DH-x-wp, $\forall x \in \mathrm{Eff}\,(X,u)$. To complete the proof invoke the triangle inequality. □

Comparing some pointwise notions we have seen that DH–well-posedness is stronger than Loridan and Bednarczuk concepts, thus Theorem 4.44 states a characterization of density also with reference to Loridan and Bednarczuk pointwise well-posedness.

Chapter 5

Extended well-posedness

Extended well-posedness was introduced in the scalar case by Zolezzi ([125]) using the embedding technique and it has been generalized in vector optimization firstly by Huang ([56]) who introduced three notions of well-posedness, employing always the embedding technique, based on several ways to understand the approximation of the objective values to the optimal value set. In particular the notion of well-posedness in the weakly extended sense is based on the understanding of approximation in terms of the norm distance, similarly to Definition 4.13 by Bednarczuk. Well-posedness in the extended sense refers to the approach of $\bar{y}-$minimizing sequence introduced by Loridan ([74]), and well-posedness in the strongly extended sense is new with respect to the previous papers.
It is worth mentioning that extended well-posedness includes usual well-posedness as a special case.

Let (P, ρ) a metric space, p^* a fixed point of P and L be a closed ball in P with center p^* and positive radius. Let $I : \mathbb{R}^n \times L \to \mathbb{R}^l$ be a vector-valued function such that

$$I(x, p^*) = f(x), \quad \forall x \in X.$$

Thus (X, f) is the original problem, while $(X, I(\cdot, p))$ model perturbations of it corresponding to the parameter p. Let

$$V(p) := \inf \{ I(x, p) : x \in X \}$$

where $y \in V(p)$ means that

(i) $y \in \mathbb{R}^l$;

(ii) $\forall x \in X,\ I(x, p) - y \notin -\operatorname{int} C$;

(iii) there exists a sequence $\{ x^n \} \subseteq$ such that $I(x^n, p) \to y$ as $n \to +\infty$.

Huang, assuming $e \in \operatorname{int} C$ arbitrarily fixed, based its definitions on the following conditions:

$$\text{Eff}\,(X, f) \neq \emptyset \tag{5.1}$$

$$V(p) \neq \emptyset, \quad \forall p \in L \tag{5.2}$$

Definition 5.1. *Problem* (X, f) *is called* well-posed in the weakly extended sense *if (5.1) and (5.2) hold and for any sequences* $p^n \to p^*$ *in* P *and* $\{ x^n \}$ *in* X *such that* $d(I(x^n, V(p^n))) \to 0$*, there exist a subsequence* $\{ x^{n_k} \}$ *of* $\{ x^n \}$ *and some point* $x^* \in \text{Eff}\,(X, f)$ *such that* $x^{n_k} \to x^*$.

Definition 5.2. *Problem* (X, f) *is called* well-posed in the extended sense *if (5.1) and (5.2) hold and for any sequences* $p^n \to p^*$ *in* P *and* $\{ x^n \}$ *in* X *such that* $\exists \{ \alpha_n \},\ \alpha_n \geq 0, \quad \alpha_n \to 0$ *and* $y^n \in V(p^n)$ *with* $I(x^n, p^n) \leq_C y^n + \alpha_n e$*, there exist a subsequence* $\{ x^{n_k} \}$ *of* $\{ x^n \}$ *and some point* $x^* \in \text{Eff}\,(X, f)$ *such that* $x^{n_k} \to x^*$.

Definition 5.3. *Problem* (X, f) *is called* well-posed in the strongly extended sense *if (5.1) and (5.2) hold and for any sequences* $p^n \to p^*$ *in* P *and* $\{ x^n \}$ *in* X *such that* $\liminf_{n \to +\infty} [\inf_{y \in V(p^n)} h(y - I(x^n, p^n))] \geq 0$*, there exist a subsequence* $\{ x^{n_k} \}$ *of* $\{ x^n \}$ *and some point* $x^* \in \text{Eff}\,(X, f)$ *such that* $x^{n_k} \to x^*$.

As pointed out by Huang, Definition 5.3 implies Definition 5.2 that implies Definition 5.1, while the converse is not true in general (see [56]).
Crespi et al. ([21], [22]) slightly generalize the previous notion of extended well-posedness in the strongly extended sense in order to consider also perturbations of the feasible region of the problem. Thus, they introduced appropriate asymptotically minimizing sequences when both the objective function and the feasible region are subject to perturbation. In particular the authors focus on convex problems, that means problems in which both the objective function and the perturbations are $C-$convex ([21]) and quasiconvex problems, assuming $C-$quasiconvexity. The main result shows that, under some assumptions, vector quasiconvex functions enjoy such well-posedness property (and a fortiori enjoy Definition 5.3).

Definition 5.4. *Let* $f_n : \mathbb{R}^n \to \mathbb{R}^l$ *be a sequence of functions, let* $f : \mathbb{R}^n \to \mathbb{R}^l$ *and let* X_n *be a sequence of subsets of* $\mathbb{R}^n$*. Problem* (X, f) *satisfies* property (P) *(with respect to the perturbations defined by the sequences* f_n *and* X_n*) when, for every sequence* $x^n \in X_n$ *such that*

$$(f_n(X_n) - f_n(x^n)) \cap (-\text{int}\, C - \epsilon_n e) = \emptyset, \tag{5.3}$$

for some sequence $\epsilon_n \to 0^+$*, there exists a subsequence* x^{n_k} *of* x^n *such that* $d(x^{n_k}, \text{WEff}\,(X, f)) \to 0$*, as* $k \to +\infty$.

It can be shown that the previous definition does not depend on the choice of the vector $e \in \text{int}\, C$. The proof of this statement can be given along the lines of Proposition 3.3 in [20].
Observe that when $\text{WEff}\,(X, f)$ is compact, the requirement $d(x^{n_k}, \text{WEff}\,(X, f)) \to 0$, amounts to the existence of a point $\bar{x} \in \text{WEff}\,(X, f)$ such that x^{n_k} converges to $\bar{x}$.

To obtain sufficient conditions for property (P), some stability results are needed. Crespi et al. gave some extentions of stability properties of vector optimization problems studied in [80] when the objective function is $C-$convex. Thus they investigated the behavior of the sets $\mathrm{WEff}\,(X_n, f_n)$, $\mathrm{Eff}\,(X_n, f_n)$, $\mathrm{WMin}\,(X_n, f_n)$, $\mathrm{Min}\,(X_n, f_n)$, when f_n and X_n "approach" to f and X respectively.

Lemma 5.5. *Let $f_n : \mathbb{R}^n \to \mathbb{R}^l$ and $f : \mathbb{R}^n \to \mathbb{R}^l$ be continuous $C-$quasiconvex functions, $y \in \mathbb{R}^l$ and $y^n \to y$. Assume*

(i) $f_n \to f$ *in the continuous convergence,*

(ii) $X_n \overset{K}{\to} X$,

(iii) $\mathrm{Lev}_f(y, X)$ *is nonempty and bounded.*

Then $\forall \epsilon > 0$ *it holds:*

$$\mathrm{Lev}_{f_n}\,(y^n, X_n) \subseteq \mathrm{Lev}_f(y, X) + \epsilon B,$$

eventually.

Proof: Assume the contrary. Then one can find a number $\bar{\epsilon} > 0$ such that $\forall n$ of some subsequence, there exists a point $x^n \in \mathrm{Lev}_{f_n}(y^n, X_n)$ with

$$x^n \notin \mathrm{Lev}_f(y, X) + \bar{\epsilon} B.$$

(i) Assume x^n is bounded. Then without loss of generality we can assume $x^n \to \bar{x}$. Since $X_n \overset{K}{\to} X$, it follows $\bar{x} \in X$ and from

$$f_n(x^n) \in y^n - C,$$

passing to the limit, and recalling $f_n \to f$ in the continuous convergence, we get $f(\bar{x}) \in y - C$, that is $\bar{x} \in \mathrm{Lev}_f(y, X)$, a contradiction.

(ii) Assume now x^n is unbounded and let $\hat{x} \in \mathrm{Lev}_f(y, X)$. Since $X_n \overset{K}{\to} X$, we can find a sequence $\hat{x}^n \in X_n$ such that $\hat{x}^n \to \hat{x}$. Since f is continuous we have $f\left(\hat{x}^n\right) \to f\left(\hat{x}\right) \in y - C$ and hence for $\alpha > 0$, we get $f_n\left(\hat{x}^n\right) \in y - C + \alpha e$, eventually, that means

$$\hat{x}^n \in \mathrm{Lev}_{f_n}\left(y + \alpha e, X_n\right). \tag{5.4}$$

Moreover, since $y^n \to y$, for $e \in \mathrm{int}\, C$ and $\alpha > 0$, we have $y^n \in y - C + \alpha e$, eventually and hence it follows $f_n(x^n) \in y - C + \alpha e$, eventually. Let $x^n(t) = tx^n + (1-t)\hat{x}^n$, $t \in [0, 1]$. From the C-quasiconvexity of f_n we obtain the existence of an integer $\bar{n} = \bar{n}(\alpha)$ such that $f_n(x^n(t)) \in y - C + \alpha e$ for every $t \in [0, 1]$ and $n > \bar{n}$.

There exists a positive number $\bar{\epsilon}$ such that for every $n > \bar{n}$, we can find a number $t_n \in [0, 1]$, which satisfies $x^n(t_n) \in \partial[\mathrm{Lev}_f(y, X) + \bar{\epsilon} B]$. Indeed, it is enough to observe that, since $\hat{x}^n \to \hat{x}$, then there exists $\bar{\epsilon}$ such that for every $\epsilon < \bar{\epsilon}$ it holds $\hat{x}^n \in \mathrm{Lev}_f(y, X) + \epsilon B$, eventually, while $x^n \notin \mathrm{Lev}_f(y, X) + \bar{\epsilon} B$. Since $\mathrm{Lev}_f(y, X) + \bar{\epsilon} B$ is compact, without loss of generality we can assume $x^n(t_n) \to \tilde{x} \in \partial[\mathrm{Lev}_f(y, X) + \bar{\epsilon} B]$ and from $f_n \to f$ in the continuous convergence, we get also $f_n(x^n(t_n)) \to f(\tilde{x}) \in y - C + \alpha e$. Since $X_n \xrightarrow{K} X$, we get $\tilde{x} \in X$ and since α is arbitrary we conclude $f(\tilde{x}) \in y - C$, or equivalently $\tilde{x} \in \mathrm{Lev}_f(y, X)$, which is a contradiction.

□

Theorem 5.6. *Let $f_n : \mathbb{R}^n \to \mathbb{R}^l$, $f : \mathbb{R}^n \to \mathbb{R}^l$ be continuous, $C-$quasiconvex functions with $f_n \to f$ in the continuous convergence and $X_n \xrightarrow{K} X$. Assume the level sets of f, $\mathrm{Lev}_f(y, X)$ are bounded when nonempty.*

(i) If $y \in \mathrm{Min}\,(X, f)$ there exists a sequence $y^n \in \mathrm{Min}\,(X_n, f_n)$, such that $y^n \to y$, that means $\mathrm{Li}\,\mathrm{Min}\,(X_n, f_n) \supseteq \mathrm{Min}\,(X, f)$.

(ii) If $y \in \mathrm{Min}\,(X, f)$ there exist $\bar{x} \in f^{-1}(y)$ and a sequence $x^n \in \mathrm{Eff}\,(X_n, f_n)$, which admits a subsequence x^{n_k} converging to $\bar{x}$.

(iii) If f is strictly $C-$quasiconvex then we have:

- *(a)* $\mathrm{Min}\,(X_n, f_n) \xrightarrow{K} \mathrm{Min}\,(X, f)$;
- *(b)* $\mathrm{Eff}\,(X_n, f_n) \xrightarrow{K} \mathrm{Eff}\,(X, f)$.

Proof:

(i) Let $y \in \mathrm{Min}\,(X, f)$ and consider the level set $\mathrm{Lev}_f(y, X) = f^{-1}(y)$. The assumptions ensure $f^{-1}(y)$ is compact. Let $\bar{x} \in f^{-1}(y)$.
From $\bar{x} \in X$, and $X_n \xrightarrow{K} X$, we get the existence of a sequence $z^n \in X_n,\ z^n \to \bar{x}$. Since $f_n \to f$ in the continuous convergence, we get $f_n(z^n) \to f(\bar{x})$ and hence, for $e \in \mathrm{int}\, C$, we can find a sequence $\alpha_n \to 0^+$, such that

$$f_n(z^n) \in y + \alpha_n e - C,$$

that means $z^n \in \mathrm{Lev}_{f_n}(w^n, X_n)$ with $w^n = y + \alpha_n e \to y$. Using Lemma 5.5, for every $\epsilon > 0$ we get

$$\mathrm{Lev}_{f_n}(w^n, X_n) \subseteq \mathrm{Lev}_f(y, X) + \epsilon B = f^{-1}(y) + \epsilon B, \tag{5.5}$$

eventually. From the assumptions we get that both $\mathrm{Lev}_{f_n}(w^n, X_n)$ and $f_n(\mathrm{Lev}_{f_n}(w^n, X_n))$ are compact. Hence $\mathrm{Min}\,(\mathrm{Lev}_{f_n}(w^n, X_n), f_n,)$ is nonempty (see [75]). From the assumptions and (5.5) we get

$$f\left(\mathrm{Lev}_{f_n}(w^n, X_n)\right) \subseteq f\left(\mathrm{Lev}_f(y, X) + \epsilon B\right),$$

eventually and hence

$$\begin{aligned}\operatorname{Min}\left(\operatorname{Lev}_{f_n}\left(w^n, X_n\right), f_n\right) &\subseteq f\left(\operatorname{Lev}_f\left(y, X\right) + \epsilon B\right)\\ &= f\left(f^{-1}(y) + \epsilon B\right),\end{aligned}$$

eventually. Since f is continuous, for every $\delta > 0$ there exists $\epsilon > 0$, such that $f\left(f^{-1}(y) + \epsilon B\right) \subseteq y + \delta B$ and hence:

$$\operatorname{Min}\left(\operatorname{Lev}_{f_n}\left(w^n, X_n\right), f_n\right) \subseteq y + \delta B,$$

eventually. Let $y^n \in \operatorname{Min}\left(\operatorname{Lev}_{f_n}\left(w^n, X_n\right), f_n\right)$. Then we can assume $y^n \to y$, and the proof is complete observing that

$$\operatorname{Min}\left(\operatorname{Lev}_{f_n}(w^n, X_n), f_n\right) \subseteq \operatorname{Min}\left(X_n, f_n\right).$$

(ii) Let $y^n \in \operatorname{Min}\left(\operatorname{Lev}_{f_n}(w^n, X_n), f_n,\right)$ be the sequence previously found at point (i) and let $x^n \in f^{-1}(y^n)$. We have

$$x^n \in \operatorname{Eff}\left(\operatorname{Lev}_{f_n}\left(w^n, X_n\right), f_n,\right) \subseteq \operatorname{Eff}\left(X_n, f_n\right).$$

Since $\operatorname{Eff}\left(\operatorname{Lev}_{f_n}\left(w^n, X_n\right), f_n\right) \subseteq \operatorname{Lev}_{f_n}\left(w^n, X_n\right) \subseteq f^{-1}(y) + \epsilon B$, eventually, ϵ is arbitrary and $f^{-1}(y)$ is compact, we obtain the existence of a subsequence of x^n converging to some point $\bar{x} \in f^{-1}(y)$.

(iii) At point (i) we have proved

$$\operatorname{Li}\operatorname{Min}\left(X_n, f_n\right) \supseteq \operatorname{Min}\left(X, f\right).$$

It remains to prove $\operatorname{Ls}\operatorname{Min}\left(X_n, f_n\right) \subseteq \operatorname{Min}\left(X, f\right)$. Let $y^n \in \operatorname{Min}\left(X_n, f_n\right)$ and assume y^n admits a convergent subsequence y^{n_k}. Since f is strictly $C-$quasiconvex we have $\operatorname{Min}\left(X, f\right) = \operatorname{WMin}\left(X, f\right)$ (see Assume by contradiction $y^{n_k} \to y \notin \operatorname{Min}(X, f)$. Hence there exists $\bar{x} \in X$ such that $f(\bar{x}) - y \in -\operatorname{int} C$. Since $\bar{x} \in X$ and $X_n \xrightarrow{K} X$, there exists a sequence $x^n \in X_n$, with $x^n \to \bar{x}$. From $f(\bar{x}) - y \in -\operatorname{int} C$, recalling $f_{n_k}(x^{n_k}) \to f(\bar{x})$, it follows easily $f_{n_k}(x^{n_k}) - y^{n_k} \in -\operatorname{int} C - \alpha e \subseteq -\operatorname{int} C$, eventually, which contradicts $y^{n_k} \in \operatorname{Min}\left(X_{n_k}, f_{n_k}\right)$ and (a) is proved.
To prove (b) it is enough to recall $f^{-1}(y)$ is a singleton and the proof easily follows from (ii).

□

Remark 5.7. One can easily check that in Theorem 5.6 the boundedness assumption on the level sets can be replaced with the weaker requirement that $f^{-1}(y)$ is bounded for every $y \in \operatorname{Min}\left(X, f\right)$. This condition is certainly satisfied when f is strictly C-quasiconvex (see Proposition 2.24).

Remark 5.8. It is known that every C-convex functions is continuous [119]. Hence, when f and f_n are C-convex functions, the continuity assumption in Theorem 5.6 is superfluous. Further, in this case, in Theorem 5.6, it is enough to require the existence of $y \in \mathbb{R}^l$ such that $\mathrm{Lev}_f(y, X)$ is nonempty and bounded. Indeed for a $C-$convex function, the boundedness of one of the nonempty level sets $\mathrm{Lev}_f(y, X)$ is equivalent to the boundedness of all the level sets (see [80]).

Now we are ready to present a sufficient condition for property (P).

Theorem 5.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ and $f_n : \mathbb{R}^n \to \mathbb{R}^l$ be continuous and C-quasiconvex, with $f_n \to f$ in the continuous convergence. Let X_n be a sequence of closed convex subsets of $\mathbb{R}^n$ such that $X_n \xrightarrow{K} X$. Assume that for every $y \in \mathbb{R}^l$, $\mathrm{Lev}_f\,(y, X)$ is bounded and let $\mathrm{WEff}\,(X, f)$ be bounded. Assume further that there exists $\bar{n} \in \mathbb{N}$ such that $\mathrm{Lev}_{f_n}\,(y, X_n)$ is bounded for every $y \in \mathbb{R}^l$ and for every $n > \bar{n}$. Then problem (X, f) satisfies property (P) with respect to the perturbations defined by the sequences f_n and X_n.*

Proof: Let

$$\mathrm{WEff}_{\epsilon_n e}(X_n, f_n) = \{\, x \in X_n : (f_n(X_n) - f_n(x)) \cap (-\mathrm{int}\, C - \epsilon_n e) = \emptyset \,\}\,.$$

Assume that (X, f) does not satisfy property (P). Then we can find sequences $\epsilon_n \to 0^+,\ x^n \in \mathrm{WEff}_{\epsilon_n e}\,(X_n, f_n)\,$, such that, for some $\delta > 0$ it holds $x^n \notin \mathrm{WEff}\,(X, f)$ $+\delta B$, eventually.
We claim that for every sufficiently large n there exists a point $z^n \in \partial[\mathrm{WEff}\,(X, f) + \delta B]$ such that $z^n \in \mathrm{WEff}_{\epsilon_n e}(X_n, f_n)$. Indeed, if such a z^n does not exist, we would have for some n

$$\mathrm{WEff}_{\epsilon_n e}\,(X_n, f_n) \subseteq \mathrm{int}\ [\mathrm{WEff}\ (X, f) + \delta B] \cup [\mathrm{WEff}\,(X, f) + \delta B]^c\,. \tag{5.6}$$

Clearly $\mathrm{WEff}_{\epsilon_n e}\,(X_n, f_n) \cap [\mathrm{WEff}\,(X, f) + \delta B]^c \neq \emptyset$. We now prove that

$$\mathrm{WEff}_{\epsilon_n e}\,(X_n, f_n) \cap \mathrm{int}\ [\mathrm{WEff}\,(X, f) + \delta B] \neq \emptyset, \tag{5.7}$$

eventually. Since

$$\mathrm{WEff}\ (X_n, f_n) \subseteq \mathrm{WEff}_{\epsilon_n e}\,(X_n, f_n)\,,$$

it is enough to prove

$$\mathrm{WEff}\ (X_n, f_n) \cap \mathrm{int}\,[\mathrm{WEff}\,(X, f) + \delta B] \neq \emptyset, \tag{5.8}$$

eventually. Let $y \in f(X)$ be fixed. The level set $\mathrm{Lev}_f(y, X)$ is nonempty since $f^{-1}(y) \subseteq \mathrm{Lev}_f(y, X)$ and from the assumptions we obtain that both $\mathrm{Lev}_f(y, X)$ and $f(\mathrm{Lev}_f(y, X))$ are compact.
It follows [75] that $\mathrm{Min}\,(\mathrm{Lev}_f(y, X), f)$ is nonempty and since $\mathrm{Min}\,(\mathrm{Lev}_f(y, X), f)$ $\subseteq \mathrm{Min}\,(X, f)$ also $\mathrm{Min}\,(X, f)$ is nonempty.
Let $y \in \mathrm{Min}\,(X, f)$. From Theorem 5.6 (ii), we get the existence of a point $\bar{x} \in$

$f^{-1}(y) \subseteq \text{Eff}\,(X,f)$ and a sequence $v^n \in \text{Eff}\,(X_n, f_n)$, which admits a subsequence converging to $\bar{x}$. Avoiding relabeling, we can assume, without loss of generality, $v^n \to \bar{x}$.
Recalling $\text{Eff}\,(X,f) \subseteq \text{WEff}\,(X,f)$, it follows easily that (5.8) holds and hence (5.7) holds.
Since there exists $\bar{n} \in \mathbb{N}$ such that $\text{Lev}_{f_n}(y, X_n)$ is bounded $\forall y \in \mathbb{R}^l$ and for all $n > \bar{n}$, the sets $\text{WEff}_{\epsilon_n e}(X_n, f_n)$ are connected, nonempty and closed for $n > \bar{n}$ (see Theorem 4 in [20]) and hence (5.6) cannot hold. It follows the existence of a sequence $z^n \in \partial\,[\text{WEff}\,(X,f) + \delta B] \cap \text{WEff}_{\epsilon_n e}(X_n, f_n)$. Since $\text{WEff}\,(X,f)$ is compact, we can assume z^n converges to a point $\bar{z}$ and since $X_n \xrightarrow{K} X$, it follows $\bar{z} \in X$. Since $z^n \in \text{WEff}_{\epsilon_n e}(X_n, f_n)$ it follows $\bar{z} \in \text{WEff}\,(X,f)$. Indeed, if $\bar{z} \notin \text{WEff}\,(X,f)$, there exists $x \in X$ such that $f(x) - f(\bar{z}) \in -\text{int}\, C$ and hence we can find a positive number $\bar{\delta}$, such that

$$f(x) - f(\bar{z}) \in -\text{int}\, C - \bar{\delta} e. \tag{5.9}$$

Since $x \in X$, there exists a sequence $w^n \to x$, $w^n \in X_n$ and from (5.9), we obtain $f_n(w^n) - f_n(z^n) \in -\text{int}\, C - \bar{\delta} e$, eventually, which contradicts to $z^n \in \text{WEff}_{\epsilon_n e}(X_n, f_n)$.
To complete the proof it is enough to observe that from $z^n \in \partial\,[\text{WEff}\,(X,f) + \delta B]$ we get the contradiction $\bar{z} \notin \text{WEff}\,(X,f)$. □

Remark 5.10. Actually, we cannot apply Lemma 5.5, to achieve boundedness of $\text{Lev}_{f_n}(y, X_n)$ from the same property for $\text{Lev}_f(y, X)$. Indeed here we require something stronger, namely that it can be fixed the same $\bar{n}$ for every y, while Lemma 5.5 implies only that such $\bar{n}$ exists for every y, possibly depending on it.

When f and f_n are C-convex functions, the assumptions of Theorem 5.9 can be simplified. Indeed, we get the following:

Corollary 5.11. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ and $f_n : \mathbb{R}^n \to \mathbb{R}^l$ be C-convex functions, with $f_n \to f$ in the continuous convergence and assume* $\text{WEff}\,(X,f)$ *is bounded. Then problem (X,f) satisfies property (P) with respect to the perturbations defined by f_n and X_n.*

Proof: It is known [119] that $C-$convex functions are continuous. If $\bar{y} = f(\bar{x})$, with $\bar{x} \in \text{WEff}\,(X,f)$, the level set $\text{Lev}_f(\bar{y}, X)$ is clearly nonempty and further we have $\text{Lev}_f(\bar{y}, X) \subseteq \text{WEff}\,(X,f)$. Indeed, assume there exists a point $x' \in \text{Lev}_f(\bar{y}, X) \setminus \text{WEff}\,(X,f)$. Hence $f(x') \in f(\bar{x}) - C$ and we can find a point $x'' \in X$ such that $f(x'') \in f(x') - \text{int}\, C$. This entails $f(x'') \in f(x') - \text{int}\, C \subseteq f(\bar{x}) - \text{int}\, C$, which contradicts to $\bar{x} \in \text{WEff}\,(X,f)$.
The inclusion $\text{Lev}_f(\bar{y}, X) \subseteq \text{WEff}\,(X,f)$ proves $\text{Lev}_f(\bar{y}, X)$ is bounded. From Lemma 5.5, we get

$$\text{Lev}_{f_n}(\bar{y}, X_n) \subseteq \text{Lev}_f(\bar{y}, X) + \epsilon B, \tag{5.10}$$

eventually. Hence there exists $\bar{n} \in \mathbb{N}$ such that $\text{Lev}_{f_n}(\bar{y}, X_n)$ is bounded for $n > \bar{n}$. Since f_n are $C-$convex, this implies that for $n > \bar{n}$ all the level sets of f_n are bounded [80]. Hence, the assumptions of Theorem 5.9 hold and the proof is complete. □

The boundedness assumption on $\mathrm{WEff}\,(X, f)$ cannot be avoided, as the following example shows.

Example 5.12. Let $f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, z) = (z^2, e^x),\ C = \mathbb{R}^2_+$ and $X = \mathbb{R}^2,\ f_n = f$, and $X_n = X$, for every n.
The objective function is $C-$convex, the set $\mathrm{WMin}\,(X, f) = \left\{\ (y_1, y_2) \in \mathbb{R}^2 : y_1 = 0\ \right\}$ while $\mathrm{WEff}\,(X, f) = \left\{\ (x, z) \in \mathbb{R}^2 : z = 0\ \right\}$.
The sequence $(x^n, z^n) = (-n, -n)$ satisfies (5.3), but doesn't admit any subsequence (x^{n_k}, z^{n_k}) such that $d(f(x^{n_k},\ z^{n_k}),\ \mathrm{WEff}\,(X, f)) \to\ 0$.

This formulation can be rewritten considering the framework of Huang. Let $\delta > 0$ and $Z : B(p^*, \delta) \rightsquigarrow \mathbb{R}^n$ be a set-valued function.
The perturbed problem corresponding to the parameter p is denoted by

$$\min_{x \in Z(p)} I(x, p).$$

In this framework we formulate a notion of extended well-posedness which is a generalization of well-posedness in the strongly extended sense formulated in [56], [57].

Definition 5.13. *Problem (X, f) is* well-posed*, with respect to the perturbations defined by the sequences $I(\cdot, p)$ and $Z(p)$, when*

(i) $\mathrm{WEff}\,(X, f) \neq \emptyset$*;*

(ii) for any sequences $p^n \to p^*$ and $x^n \in Z(p^n)$ such that $\exists \epsilon_n > 0,\ \epsilon_n \to 0^+$, with

$$(I(Z(p^n), p^n) - I(x^n, p^n) + \epsilon_n e) \cap (-\mathrm{int}\, C) = \emptyset \tag{5.11}$$

there exists a subsequence x^{n_k} of x^n such that $d(x^{n_k}, \mathrm{WEff}\,(X, f)) \to 0$, as $k \to +\infty$.

Sequences x^n satisfying (5.11) are called asymptotically minimizing sequences.

Remark 5.14. Sequences x^n and x^{n_k} in Definition 5.13 may fail to be feasible for the original problem (X, f).
We regard this feature as an extension of Levitin-Polyak approach to well-posedness.

Huang's notion is generalized in Definition 5.13 mainly by the following two facts:

(a) it allows for perturbations of the feasible region and not only of the objective function;

(b) requirement (ii) in Definition 5.13 weakens the convergence requirement of Huang's definition.

Theorem 5.15. *Let $I(\cdot, p)$ be continuous C-quasiconvex functions, and let $Z(p)$ be a closed convex subset of $\mathbb{R}^n$, for every $p \in B(p^*, \delta)$.*
Assume the following:

(i) $\forall p^n \to p^*$ *and* $x^n \to x^*$, $x^n \in Z(p^n)$, *it holds* $I(x^n, p^n) \to I(x^*, p^*) := f(x^*)$ *and* $X_n := Z(p^n) \xrightarrow{K} X$;

(ii) $\forall y \in \mathbb{R}^l$, $\mathrm{Lev}_f(y, X)$ *is bounded;*

(iii) $\forall p^n \to p^*$ *there exists* $\bar{n} \in \mathbb{N}$ *such that* $\mathrm{Lev}_{I(\cdot,p^n)}(y, Z(p^n))$ *is bounded for every* $y \in \mathbb{R}^l$ *and for every* $n > \bar{n}$;

(iv) $\mathrm{WEff}\,(X, f)$ *is nonempty and bounded.*

Then problem (X, f) *is well-posed (with respect to the perturbations defined by the sequences* $I(\cdot, p)$ *and* $Z(p)$*).*

Proof: Let $p^n \to p^*$ and set $f_n(\cdot) = I(\cdot, p^n)$ and $X_n := Z(p^n)$, $\forall n$. The proof follows easily from Theorem 5.9. $\square$

Corollary 5.16. *Assume* $I(\cdot, p)$ *are* $C-$*convex functions and let* $Z(p)$ *be a convex subset of* $\mathbb{R}^n$*, for every* $p \in B(p^*, \delta)$*. Let assumptions (i), (ii) and (iv) of Theorem 5.15 hold. Then problem* (X, f) *is well-posed (with respect to the perturbations defined by the sequences* $I(\cdot, p)$ *and* $Z(p)$*).*

Proof: It is an immediate consequence of Corollary 5.11. $\square$

It remains an open question whether, in the case of C-quasiconvex functions, the assumptions of Theorem 5.9 can be simplified. Proposition 5.20 below, shows however that, when $Z(p) = X$ for every $p \in B(p^*, \delta)$, this is the case if we strengthen the convergence requirement on the sequences $I(\cdot, p^n)$.

Lemma 5.17. *Let* $f_n : \mathbb{R}^n \to \mathbb{R}^l$ *be a sequence of functions converging to* f *in the uniform convergence. Assume that for every* $y \in \mathbb{R}^l$, $\mathrm{Lev}_f(y, X)$ *is bounded. Then there exists* $\bar{n} \in \mathbb{N}$ *such that for every* $n > \bar{n}$ *and for every* $y \in \mathbb{R}^l$, $\mathrm{Lev}_{f_n}(y, X)$ *is bounded.*

Proof: We begin observing that under the assumptions, for every $y \in \mathbb{R}^l$ we have $\Delta_{-C}(f(x) - y) \to +\infty$, as $\|x\| \to +\infty$, $x \in X$. Indeed, assume, on the contrary one can find a sequence $x^n \in X$, with $\|x^n\| \to +\infty$ and $\Delta_{-C}(f(x^n) - y) \not\to +\infty$. We distinguish two cases.

1. The set $\{\, \Delta_{-C}(f(x^n) - y), n \in \mathbb{N} \,\}$ is bounded. Then, without loss of generality, we can assume $\Delta_{-C}(f(x^n) - y) \to \beta \in \mathbb{R}$ and the following two cases are possible.

   (i) $\beta < 0$. Then it holds $f(x^n) \in y - C$, eventually, which contradicts the boundedness of the level sets.

   (ii) $\beta \geq 0$. In this case, it is easily seen that we can choose $\alpha > 0$ such that $f\,(x^n) \in y + \alpha e - C$, eventually, contradicting again the boundedness of the level sets.

2. The set $\{\, \Delta_{-C}(f(x^n)-y), n \in \mathbb{N}\,\}$ is unbounded. Since $\Delta_{-C}(f(x^n)-y) \not\to +\infty$, it is possible to find a subsequence x^{n_k} of x^n such that $\Delta_{-C}(f(x^{n_k})-y) \to -\infty$. In this case it holds again $f(x^{n_k}) \in y - C$, eventually, which contradicts the boundedness of the level sets.

Assume now, ab absurdo, that for every n there exists $y^n \in \mathbb{R}^l$, such that $\mathrm{Lev}_{f_n}(y^n, X)$ is unbounded. Hence, for a fixed $\bar{n} \in \mathbb{N}$, we can find a sequence z^k, $k \in \mathbb{N}$, with $z^k \in X$, $\forall k$, $\|z^k\| \to +\infty$, as $k \to +\infty$ and $z^k \in \mathrm{Lev}_{f_{\bar{n}}}(y^{\bar{n}}, X)$, for every k, that means

$$f_{\bar{n}}(z^k) - y^{\bar{n}} \in -C, \ \forall k.$$

We distinguish the following two cases:

(i) $f_{\bar{n}}\left(z^k\right) - y^{\bar{n}} \in -C$, for every k except a finite number. In this case we contradict the boundedness of the level set $\mathrm{Lev}_f\left(y^{\bar{n}}, X\right)$.

(ii) $f_{\bar{n}}\left(z^k\right) - y^{\bar{n}} \notin -C$ for infinitely many k. Without loss of generality we can assume $f_{\bar{n}}\left(z^k\right) - y^{\bar{n}} \notin -C$ for every k and we have

$$\begin{aligned}\left\|f(z^k) - f_{\bar{n}}(z^k)\right\| &= \left\|f(z^k) - y^{\bar{n}} - (f_{\bar{n}}(z^k) - y^{\bar{n}})\right\| \\ &\geq \Delta_{-C}(f(z^k) - y^{\bar{n}}).\end{aligned}$$

From $\Delta_{-C}(f(z^k) - y^{\bar{n}}) \to +\infty$ as $k \to +\infty$, we obtain

$$\sup_{x \in \mathbb{R}^n} \|f(x) - f_{\bar{n}}(x)\| = +\infty.$$

Since $\bar{n}$ is arbitrary, we contradict the uniform convergence of f_n to f.

□

The previous lemma does not hold (even in the quasiconvex case) if we assume $f_n \to f$ in the continuous convergence, as the following example shows.

Example 5.18. Let $f : \mathbb{R} \to \mathbb{R}$ and $f_n : \mathbb{R} \to \mathbb{R}$, be defined as:

$$f(x) = |x|;$$

$$f_n(x) = \begin{cases} |x| & x \in [-n, n] \\ |n| & \text{otherwise} \end{cases}$$

and let $X = \mathbb{R}$ and $C = \mathbb{R}_+$. We have $f_n \to f$ in the continuous convergence, but not in the uniform convergence and it can be easily seen that the level sets of f are bounded, but each function f_n admits unbounded level sets.

Proposition 5.19. *Let $f_n : \mathbb{R}^n \to \mathbb{R}^l$ and $f : \mathbb{R}^n \to \mathbb{R}^l$ be continuous C-quasiconvex functions and let* $\mathrm{WEff}(X, f)$ *be nonempty and bounded. Assume that $f_n \to f$ in the uniform convergence and that for every $y \in \mathbb{R}^l$,* $\mathrm{Lev}_f(y, X)$ *is bounded. Then problem (X, f) satisfies property (P) (with respect to the perturbations defined by the sequences f_n and X_n).*

Proof: Recalling Theorem 5.9, it is enough to prove that there exists $\bar{n} > 0$ such that for every $n > \bar{n}$ and for every $y \in \mathbb{R}^l$, $\mathrm{Lev}_{f_n}(y, X)$ is bounded. But this follows immediately from Lemma 5.17. $\square$

The proof of the next result is an immediate consequence of Proposition 2.33.

Proposition 5.20. *Let $I(\cdot, p)$ be continuous C-quasiconvex functions $\forall p \in B(p^*, \delta)$, let $Z(p) = X,\ \forall p \in B(p^*, \delta)$ and assume that $\forall p^n \to p^*,\ \sup \|I(x, p^n) - f(x)\| \to 0$, as $n \to +\infty$. If for every $y \in \mathbb{R}^l,\ \mathrm{Lev}_f(y, X)$ is bounded and $\mathrm{WEff}\,(X, f)$ is nonempty and bounded, then problem (X, f) is well-posed (with respect to the perturbations defined by the sequences $I(\cdot, p)$ and $Z(p)$).*

The research on the topic of extended well-posedness in vector and set-valued optimization is very recent and thus in progress (among others see [34], [59], [60] and the references therein).

Chapter 6

Well-posedness of scalarized problems

In this section we deal with the relationships between the well-posedness of a vector optimization problem and the well-posedness of associated scalar ones. For the clarity of exposition we divide the results of this chapter in six sections considering nonlinear and linear approach to the three main classes of pointwise, global and extended well-posedness. Sections devoted to nonlinear scalarization employing the oriented distance function; no convexity assumption is needed and a vector problem is associated to one scalar problem. In linear scalarization, the main results are showed under some generalized convexity requirements and considering a family of scalarized problems. However linear scalarization will permit us to rewrite the proof of the density theorem given in Chapter 4, in a particular case, emphasizing a strict link between scalar and vector setting in Tykhonov's approach.

6.1 Nonlinear scalarization and pointwise well-posedness

Consider the scalar problem (X, Δ_{-C}) associated to the vector problem (X, f) given by

$$\min \Delta_{-C}(f(x) - p), \quad x \in X$$

where $p \in Y = f(X)$. Using this scalar problem Miglierina et al. ([93]) derived the following results as link with the pointwise well-posedness of the vector problem (X, f).

Theorem 6.1. *Let $\bar{y} \in \operatorname{Min}(X, f)$. Problem (X, Δ_{-C}) with $p = \bar{y}$ is TS, if and only if problem (X, f) is B–$\bar{y}$-wp.*

We observe that no assumption of generalized convexity or monotonicity is required and that function Δ_{-C} doesn't imply any boundedness assumption on the feasible region X.

Corollary 6.2. *Let $\bar{y} \in \mathrm{Min}\,(X, f)$. Problem (X, Δ_{-C}) with $p = \bar{y}$ is GT-wp, if and only if problem (X, f) is L–$\bar{y}$-wp.*

Corollary 6.3. *Let $\bar{x} \in \mathrm{Eff}\,(X, f)$. Problem (X, Δ_{-C}) with $p = \bar{y}$ is T-wp, if and only if problem (X, f) is DH–$\bar{x}$-wp.*

Remark 6.4. A direct link between nonlinear scalarization and H–$\bar{x}$ well-posedness is established in [30].

Thanks to the scalarization with oriented distance function, the results of this section are equivalence, that means pointwise well-posedness of a vector problem is completely represented by a scalar model.

6.2 Linear scalarization and pointwise well-posedness

The links between well-posedness of a linearly scalarized problem and well-posedness of a vector problem, are proved under convexity or generalized convexity assumptions.
Consider the scalar problem (X, g_λ), associated to the vector problem (X, f), given by

$$\min g_\lambda(x), \quad x \in X,$$

where $g_\lambda(x) = \langle \lambda, f(x) - p \rangle$ in which $\lambda \in C^+ \cap \partial B$ and $p \in Y = f(X)$.

Theorem 6.5. *Let $\bar{y} \in \mathrm{Min}\,(X, f)$. If there exists $\bar{\lambda} \in C^+ \cap \partial B$ such that problem $(X, g_{\bar{\lambda}})$ with $p = \bar{y}$ is TS and* $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$*, then (X, f) is B–$\bar{y}$-wp.*

Proof: Recalling Theorem 6.1, if ab absurdo problem (X, f) is not B–$\bar{y}$-wp, then

$$\exists x^n \in X \setminus \arg\min(X, \Delta_{-C}) \text{ such that } \Delta_{-C}(f(x^n) - \bar{y}) \to 0,$$

$$\text{but } \nexists x^{n_k} \text{ such that } x^{n_k} \to \bar{x} \in \arg\min(X, \Delta_{-C}).$$

Since $\Delta_{-C}(f(x^n) - \bar{y}) = \max \{ \langle \lambda, f(x^n) - \bar{y} \rangle : \lambda \in C^+ \cap \partial B \}$ (see [93]), it follows

$$0 \leq \left\langle \bar{\lambda}, f(x^n) - \bar{y} \right\rangle \leq \Delta_{-C}(f(x^n) - \bar{y})$$

and recalling the assumptions, $\arg\min(X, \Delta_{-C}) = \arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$. But this means $g_{\bar{\lambda}}(x^n) \to 0$, a contradiction with topologically well-setness of (X, g_λ). □

The assumption $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$ cannot be avoided as the following example shows.

Example 6.6. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (x, 0)$ with $X = C = \mathbb{R}^2_+$. Let $\bar{\lambda} = (0, 1)$ and $g_{\bar{\lambda}}(x, y) = 0$. The set $\mathrm{Min}\,(X, f) = \{ (0, 0) \}$, all the assumptions of Theorem 6.5 are satisfied except one: $\arg\min(X, g_{\bar{\lambda}}) = X \quad \neq \quad f^{-1}(0, 0) = \{ (x, y) : x = 0,\ y \geq 0 \}$. Problem $(X, g_{\bar{\lambda}})$ is TS, but problem (X, f) is not B–$(0, 0)$-wp, for instance the B-minimizing sequence $\{ x^n \} = \left(\frac{1}{n}, n\right)$ does not converge.

Corollary 6.7. *Let $\bar{y} \in \mathrm{Min}\,(X, f)$. If there exists $\bar{\lambda} \in C^{+} \cap \partial B$ such that problem $(X, g_{\bar{\lambda}})$ with $p = \bar{y}$ is GT-wp and* $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$*, then (X, f) is L–$\bar{y}$-wp.*

Corollary 6.8. *Let $\bar{y} \in \mathrm{Min}\,(X, f)$. If there exists $\bar{\lambda} \in C^{+} \cap \partial B$ such that problem $(X, g_{\bar{\lambda}})$ with $p = \bar{y}$ is T-wp and* $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$*, then (X, f) is DH–$\bar{x}$-wp.*

Remark 6.9. The existence of a linear scalarized satisfying a well-posedness notion is a sufficient condition in order that problem (X, f) is pointwise well-posed. If the ordering cone C satisfies the geometrical requirement $C \subseteq \mathbb{R}^{l}_{+}$, the sufficient condition can be tested by a single scalar problem (X, f_i) where f_i is a component of the vector objective function f. In the most interesting case, consisting in the investigation of DH–$\bar{x}$-wp, it is possible to prove that if there exists at least one problem (X, f_i) Tykhonov well-posed, then problem (X, f) is DH–well-posed in $\bar{x}$.

Corollary 6.10. *Let $C \subseteq \mathbb{R}^{l}_{+}$. If, for some $i \in \{\, 1, \ldots, l \,\}$, (X, f_i) is T-wp in $\bar{x}$ then (X, f) is DH–$\bar{x}$-wp.*

The results of this section will permit us to present two applications. The first is the identification of a class of well-posed vector problems which satisfy a a further regularity condition, that is the existence of a vector $\bar{\lambda}$ such that the scalarized problem $(X, g_{\bar{\lambda}})$ is well-posed. We call this property $\bar{\lambda}-$well-posedness. The link between well-posedness of a linearly scalarized problem and well-posedness of the original one is weaker than the relation involving nonlinear scalarization, since the results are only in one direction, but we haven't yet imposed convexity or generalized convexity requirements.
The next result identifies the class of $*-$quasiconvex functions as satisfying a $\bar{\lambda}-$well-posedness; for these functions it is possible to replace the well-posedness analysis of the vector problem with both nonlinear and linear scalarization.

Theorem 6.11. *Let $f : X \subseteq \mathbb{R}^{n} \to \mathbb{R}^{l}$ be $*-$quasiconvex and* $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$ *be a bounded set. Then, problem (X, f) is B–$\bar{y}$-wp if and only if $\exists \bar{\lambda} \in C^{+} \cap \partial B$ such that $\langle \bar{\lambda}, y - \bar{y} \rangle \geq 0,\ \forall y \in f(X)$ and $(X, g_{\bar{\lambda}})$ is TS.*

Proof: Recalling Theorem 6.5 we only need to prove one direction. As function f is $*-$quasiconvex, the set $(f(X) + C)$ is convex ([39]) and thanks to a classical separation theorem, every $\bar{y} \in \mathrm{Min}\,(X, f)$ is unique solution of a scalarized problem. Function $g_{\bar{\lambda}}(x)$ is quasiconvex as linear combination of continuous functions and hence problem $(X, g_{\bar{\lambda}})$ is TS since $\arg\min(X, g_{\bar{\lambda}})$ is bounded. □

Corollary 6.12. *Let $f : X \subseteq \mathbb{R}^{n} \to \mathbb{R}^{l}$ be $*-$quasiconvex and* $\arg\min(X, g_{\bar{\lambda}}) = f^{-1}(\bar{y})$ *be a bounded set. Then, problem (X, f) is L–$\bar{y}$-wp if and only if $\exists \bar{\lambda} \in C^{+} \cap \partial B$ such that $\langle \bar{\lambda}, y - \bar{y} \rangle \geq 0,\ \forall y \in f(X)$ and $(X, g_{\bar{\lambda}})$ is GT.*

Corollary 6.13. *Let $f : X \subseteq \mathbb{R}^{n} \to \mathbb{R}^{l}$ be $*-$quasiconvex and* $\arg\min(X, g_{\bar{\lambda}}) = \bar{x}$*. Then, problem (X, f) is DH–$\bar{x}$-wp if and only if $\exists \bar{\lambda} \in C^{+} \cap \partial B$ such that $\langle \bar{\lambda}, y - \bar{y} \rangle \geq 0,\ \forall y \in f(X)$ and $(X, g_{\bar{\lambda}})$ is T-wp.*

We observe that Corollary 6.13 cannot be improved considering $\arg\min(X, g_{\bar{\lambda}})$ unbounded or the larger class of C-quasiconvex functions as it is possible to see in the following examples.

Example 6.14. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, the identity function and the feasible region $X = \{ (x, y) \in \mathbb{R}^2 : y \geq -x \}$ and $C = \mathbb{R}^2_+$. Function f is $*-$quasiconvex, problem (X, f) is DH–$(0, 0)$-wp, but it is not $\bar{\lambda}$-well-posed for any $\bar{\lambda} \in C^+ \cap \partial B$.

Example 6.15. Let $f : X \subseteq \mathbb{R} \to \mathbb{R}^2,\ f(x) = (x, -x^3)$ with $X = \mathbb{R}$ and $C = \mathbb{R}^2_+$. Function f is C-quasiconvex but not $*-$quasiconvex. Problem (X, f) is DH–0-wp, but it is not $\bar{\lambda}$-well-posed for any $\bar{\lambda} \in C^+ \cap \partial B$.

In Chapter 4, Theorem 4.44 is the formulation of a density result for the pointwise well-posed vector problems, according to the definition given by Dentcheva Helbig. In that result an arbitrarily closed convex and pointed cone gives the partial order in the image space. As already pointed out the proof of Theorem 4.44 is a generalization of a scalar result due to Lucchetti in [79] with reference to the notion of Tykhonov well-posedness and the Ekeland's variational principle. Now we have emphasized a deep link between Dentcheva-Helbig and Tykhonov well-posedness under certain choices of scalarization. From this we are able to show a strong connection in terms of variational principles, rewriting the proof of Theorem 4.44 in the particular case of the Paretian cone. For the reader convenience, we recall the density result.

Theorem 6.16. *Let $C = \mathbb{R}^l_+$. The set of functions $f \in (\mathcal{F}, d)$ such that*

(i) $\mathrm{Eff}\,(X, f) \neq \emptyset$*;*

(ii) (X, f) *is DH-x-wp,* $\forall x \in \mathrm{Eff}\,(X, f)$

is dense in $(\mathcal{F}, d)$.

Proof: Fix $\sigma > 0$, $f \in (\mathcal{F}, d)$ and take j so large that

$$v(x) := f(x) + \frac{1}{j} d^2(x, \theta) c^0 \ \text{ satisfies } \ d(f, v) < \frac{\sigma}{2}.$$

Then $v \in (\mathcal{F}, d)$, indeed v is strictly $C-$convex by Lemma 4.40.
Since the cone is the paretian one, $v_i(x) := f_i(x) + \gamma_i d^2(x, \theta)$ with $\gamma_i = \frac{1}{j} c^0_i$ is convex and thus continuous and by Proposition 4.39 there exists $i \in \{ 1, \ldots, l \}$ such that $\lim_{d(x,\theta) \to +\infty} v_i(x) = +\infty$. It follows that there exists $M > 0$ such that $\mathrm{Lev}_{v_i}(1) \subseteq B(\theta, M)$. Apply Ekeland's variational principle, for scalar function, to find

$$u_i(\cdot) = v_i(\cdot) + \epsilon d(\cdot, \hat{x})$$

such that (X, u_i) is Tykhonov well-posed in $\hat{x}$ and $d(u_i, v_i) \leq \frac{\sigma}{2}$. Construct $u(\cdot) = v(\cdot) + \epsilon d(\cdot, \hat{x})$. Then $u \in (\mathcal{F}, d)$, in particular u is strictly $\mathbb{R}^l_+-$convex and $d(u, v) \leq \frac{\sigma}{2}$. To complete the proof, recall Corollary 6.10 that since (X, u_i) is Tykhonov well-posed, (X, u) is DH-$\tilde{x}$-wp, $\forall \tilde{x} \in \mathrm{Eff}\,(X, u)$. $\square$

6.3 Nonlinear scalarization and global well-posedness

In Chapter 4 we distinguish global well-posedness notions involving the efficient frontier from those considering the weakly efficient solutions. Since in scalar case efficient and weakly efficient points coincide, all global notions of well-posedness generalizes the weak concept of well-setness. Hence, to get a scalarized procedure for global notions in which the efficient frontier does not include also weakly efficient solutions, we need to separate in some way the two concepts, maybe checking different properties of the scalarizing function. Only after a scalar characterization of solutions set, one can seek a scalar well-setness condition.
In [101] the authors proposed a scalar problem associated to the vector one and studied the CGR-well-posedness.
Let (X, g) the scalar problem defined as

$$\min g(x), \quad x \in X,$$

where $g(x) = -\inf_{z \in X} \Delta_{-C}(f(z) - f(x))$.
Function g is always nonnegative, in fact it is enough to observe that for $z = x$, we have $\Delta_{-C}(f(x) - f(x)) = \Delta_{-C}(0) = 0$.
We observe that function g can be rewritten as

$$g(x) = \sup_{z \in X} [-\Delta_{-C}(f(z) - f(x))]$$

and, as in [19] $\Delta_{-C}(f(z) - f(x)) = \max_{\xi \in C^{+} \cap S} \langle \xi, f(z) - f(x) \rangle$, hence

$$g(x) = \sup_{z \in X} \min_{\xi \in C^{+} \cap S} \langle \xi, f(x) - f(z) \rangle .$$

We start to show that the weak solutions of the vector problem (X, f) can be completely characterized by solutions of the scalar problem (X, g).

Theorem 6.17. *([102]) Let $\bar{x} \in X$. Then $\bar{x} \in \mathrm{WEff}\,(X, f)$ if and only if $g(\bar{x}) = 0$ (and hence $\bar{x} \in \arg\min(X, g)$).*

Proof:

$\Rightarrow$) Let $\bar{x} \in \mathrm{WEff}\,(X, f)$. By definition of weakly efficient solution, $f(z) - f(\bar{x}) \notin -\mathrm{int}\, C$, $\forall z \in X$. So

$$\Delta_{-C}(f(z) - f(\bar{x})) \geq 0, \ \forall z \in X$$

and hence $g(\bar{x}) \leq 0$, but we have just seen that $g(x)$ is always nonnegative, and then $g(\bar{x}) = 0$ which means $\bar{x} \in \mathrm{Eff}\,(X, g)$.

$\Leftarrow$) Let $g(\bar{x}) = 0$. Then, clearly, $\bar{x} \in \arg\min(X, g)$. From $g(\bar{x}) = 0$, we get $\Delta_{-C}(f(z) - f(x)) \leq 0$, $\forall z \in X$, which means $f(z) - f(x) \notin -\mathrm{int}\, C$, $\forall z \in X$ and hence $\bar{x} \in \mathrm{WEff}\,(X, f)$.

$\square$

The next result provides the equivalence between well-posedness of the vector problem (X, f) and of the related scalar problem (X, g).

Theorem 6.18. *Problem (X, f) is CGR-wp if and only if (X, g) is MS.*

Proof: From Theorem 6.17 we know that $\mathrm{WEff}\,(X, f) = \{ x \in X : g(x) = 0 \} = \mathrm{Eff}\,(X, g)$.
We shall prove that CGR-minimizing sequences for problem (X, f) are minimizing for problem (X, g) and conversely.
Indeed, let $x^n \in X$ a CGR-minimizing sequence, which means

$$f(x) - f(x^n) + \epsilon_n c^0 \notin -\mathrm{int}\, C,\ \forall x \in X,$$

that is equivalent to $\Delta_{-C}(f(x) - f(x^n) + \epsilon_n c^0) \geq 0,\ \forall x \in X$, and hence, by subadditivity of $\Delta_{-C}(\cdot)$,

$$\Delta_{-C}(f(x) - f(x^n)) \geq -\Delta_{-C}(\epsilon_n c^0) := -\gamma_n$$

where $\gamma_n \geq 0$ and $\gamma_n \to 0$. It follows

$$0 \leq g(x^n) = -\inf_{z \in X} \Delta_{-C}(f(z) - f(x^n)) \leq \gamma_n,$$

hence $g(x^n) \to 0$, that is x^n is minimizing for problem (X, g).
Conversely, assume x^n is minimizing for problem (X, g). Hence $g(x^n) \to 0$, which implies $g(x^n) \leq \beta_n$, for some sequence $\beta_n \geq 0, \beta_n \to 0$. It holds,

$$-\inf_{z \in X} \Delta_{-C}(f(z) - f(x^n)) \leq \beta_n,$$

that is $\inf_{z \in X} \Delta_{-C}(f(z) - f(x^n)) \geq -\beta_n$ and this is equivalent to

$$\Delta_{-C}(f(z) - f(x^n)) \geq -\beta_n,\ \forall z \in X.$$

Since $\Delta_{-C}(y) = \max_{\xi \in C^+ \cap S} \langle \xi, y \rangle$ (see e.g. [19]), choosing a vector $e \in \mathrm{int}\, C$, with $\langle \xi, e \rangle \geq 1,\ \forall \xi \in C^+ \cap S$ we obtain, $\forall z \in X$:

$$\begin{aligned}
\Delta_{-C}(f(z) - f(x^n) + \beta_n e) &= \max_{\xi \in C^+ \cap S} \langle \xi, f(z) - f(x^n) + \beta_n e \rangle \\
&\geq \max_{\xi \in C^+ \cap S} \langle \xi, f(z) - f(x^n) \rangle + \min_{\xi \in C^+ \cap S} \langle \xi, \beta_n e \rangle \\
&\geq \max_{\xi \in C^+ \cap S} \langle \xi, f(z) - f(x^n) \rangle + \beta_n \\
&\geq -\beta_n + \beta_n = 0.
\end{aligned}$$

Hence $\Delta_{-C}(f(z) - f(x^n) + \beta_n e) \geq 0,\ \forall z \in X$ and this is equivalent to say $f(z) - f(x^n) + \beta_n e \notin -\mathrm{int}\, C,\ \forall z \in X$.
Hence x^n is a CGR-minimizing sequence for the vector problem (X, f). Recalling $\mathrm{WEff}\,(X, f) = \{ x \in X : g(x) = 0 \} = \mathrm{Eff}\,(X, g)$, the proof of the result is easily completed. □

6.4 Linear scalarization and global well-posedness

Consider a convex vector optimization problem, that is minimize or maximize an objective function f satisfying $C-$convexity definition on a convex feasible region X. We recall that the functions $g_\lambda(x) = \langle \lambda, f(x) \rangle$ with $\lambda \in C^+ \setminus \{ 0 \}$ are convex when f is $C-$convex ([75]). Consider the family of parametric scalar problems (X, g_λ) given by

$$\min g_\lambda(x) = \langle \lambda, f(x) \rangle, \quad x \in X,$$

where $\lambda \in C^+ \cap \partial B$.

By convexity assumption follows that a point $x \in X$ is a weakly efficient solution for vector problem (X, f) if and only if it is an optimal solution for a scalar problem (X, g_λ).

Theorem 6.19. *([21]) Let $f : X \subseteq \mathbb{R}^m \to \mathbb{R}^l$ be $C-$convex on the convex set X and assume* $\mathrm{WMin}\,(X, f)$ *is closed.*
If problems (X, g_λ) are MS for every $\lambda \in C^+ \cap \partial B$, then problem (X, f) is D-wp.

Proof: We know that an asymptotically minimizing sequence for problem (X, f), is always asymptotically minimizing for problem (X, g) defined section 6.3.
Let x^n be an asymptotically minimizing sequence for problem (X, f). Then $g(x^n) \to 0$ and by the compactness of $C^+ \cap \partial B$, there exists a sequence $\lambda^n \to \lambda^* \in C^+ \cap \partial B$ such that

$$\min_{\lambda \in C^+ \cap \partial B} \langle \lambda, f(x^n) - f(x) \rangle = \langle \lambda^n, f(x^n) - f(x) \rangle,$$

and hence

$$\sup_{x \in X} \langle \lambda^n, f(x^n) - f(x) \rangle \to 0,$$

that means

$$\langle \lambda^n, f(x^n) \rangle - \inf_{x \in X} \langle \lambda^n, f(x) \rangle \to 0.$$

Since $g_\lambda(x)$ is a convex function for every $\lambda \in C^+ \cap \partial B$ ([75]) and $\lambda^n \to \lambda^*$, it follows $\langle \lambda^n, f \rangle \to \langle \lambda^*, f \rangle$. Hence (see e.g. [79]),

$$g_{\lambda^n}(x^n) = \langle \lambda^n, f(x^n) \rangle \to \inf_{x \in X} \langle \lambda^*, f(x) \rangle = \inf_{x \in X} g_{\lambda^*}(x).$$

We claim that $g_{\lambda^*}(x^n) \to \inf_{x \in X} g_{\lambda^*}(x)$.
Since $\lambda^n \to \lambda^*$, $\forall \epsilon > 0$, $\exists \bar{n}$ such that $\forall n > \bar{n}$

$$|\langle \lambda^*, f(x^n) \rangle - \langle \lambda^n, f(x^n) \rangle| < \frac{\epsilon}{2},$$

thus $|\langle \lambda^* - \lambda^n, f(x^n) \rangle| < \frac{\epsilon}{2}$. Hence, $\forall n > \bar{n}$

$$\begin{aligned}
0 &\le \langle \lambda^*, f(x^n) \rangle - \inf_{x \in X} \langle \lambda^*, f(x) \rangle \\
&= g_{\lambda^*}(x^n) - \inf_{x \in X} \langle \lambda^*, f(x) \rangle \\
&= \langle \lambda^n, f(x^n) \rangle - \inf_{x \in X} \langle \lambda^*, f(x) \rangle + \langle \lambda^* - \lambda^n, f(x^n) \rangle \\
&\le \langle \lambda^n, f(x^n) \rangle - \inf_{x \in X} \langle \lambda^*, f(x) \rangle + \frac{\epsilon}{2}.
\end{aligned}$$

Since $\langle \lambda^n, f(x^n) \rangle \to \inf_{x \in X} \langle \lambda^*, f(x) \rangle$ and ϵ is arbitrary, we prove the claim. Hence recalling the assumption of metrically well-setness on (X, g_λ) the proof is completed. $\square$

In general, the reverse of Theorem 6.19 is not true as the following example shows.

Example 6.20. Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f(x_1, x_2) = \left(\frac{x_1^2}{x_2}, x_1 \right)$, $C = \mathbb{R}^2_+$ and $X = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 1 \right\}$. The objective function is $C-$convex, $\mathrm{WMin}\,(X, f) = \{ (0, 0) \}$, $\mathrm{WEff}\,(X, f) = \{ (0, x_2) : x_2 \geq 1 \}$ the problem is D-wp since every D-minimizing sequence is identified when x_1 tends to zero, but the scalar problem (X, g_λ) with $\lambda = (1, 0)$ is not MS.

6.5 Nonlinear scalarization and extended well-posedness

We relate to (X, f) the scalar optimization problem

$$\min g(x), \quad x \in X,$$

where $g(x) = -\inf_{z \in X} \Delta_{-C}(f(z) - f(x))$ as in section 6.3 for global notions.
Further, for f_n and X_n defined like in Chapter 5, we consider the scalar perturbed problem

$$\min g_n(x), \quad x \in X_n$$

where $g_n(x) = -\inf_{z \in X_n} \Delta_{-C}(f_n(z) - f_n(x))$.
We shall investigate the behavior of solutions of problem (X, g) when it is subject to the perturbation (X_n, g_n) induced by f_n and X_n.

Definition 6.21. *A sequence $x^n \in X_n$ is called* asymptotically minimizing for problem (X, g) *when*

$$g_n(x^n) \to \inf_{x \in X} g(x).$$

Definition 6.22. *Problem (X, g) is* well-posed (with respect to the perturbations defined by the sequences f_n and X_n) *when for every asymptotically minimizing sequence x^n there exists a subsequence x^{n_k} such that $d(x^{n_k}, \arg\min(X, g)) \to 0$ as $k \to \infty$.*

Next result characterizes solutions of (X, f) in terms of solutions of the scalar problem (X, g) ([21]).

Remark 6.23. Recalling that $\bar{x} \in \mathrm{WEff}\,(X, f)$ if and only if $g(\bar{x}) = 0$, when $\mathrm{WEff}\,(f, X) \neq \emptyset$, Theorem 6.17 states $\inf_{x \in X} g(x) = 0$ and hence Definition 6.21 recalls the notion of extended well-posedness introduced in [125].

Theorem 6.24. *([21]) Problem (X, f) is well-posed (with respect to the perturbations defined by the sequences f_n and X_n) if and only if (X, g) is well-posed (with respect to the perturbations defined by the sequences g_n and X_n).*

Proof: Since $\mathrm{WEff}\,(X,f) \neq \emptyset$, it follows $\inf_{x\in X} g(x) = 0$. We show that the asymptotically minimizing sequences of the two problems coincide.
Let x^n be an asymptotically minimizing sequence for problem (X,f), that means

$$f_n(x) - f_n(x^n) + \epsilon_n e \notin -\mathrm{int}\, C,\ \forall x \in X_n.$$

This is equivalent to $\Delta_{-C}(f_n(x) - f_n(x^n) + \epsilon_n e) \geq 0,\ \forall x \in X_n$ and hence by subadditivity of $\Delta_{-C}(\cdot)$,

$$\Delta_{-C}(f_n(x) - f_n(x^n)) \geq -\Delta_{-C}(\epsilon_n e) := -\gamma_n,\ \forall x \in X_n,$$

where $\gamma_n \geq 0$ and $\gamma_n \to 0$. It follows

$$0 \leq g_n(x^n) = -\inf_{z\in X_n} \Delta_{-C}(f_n(z) - f_n(x^n)) \leq \gamma_n,$$

hence $g_n(x^n) \to 0$, that is x^n is asymptotically minimizing for problem (X,g).
Assume now x^n is an asymptotically minimizing sequence for problem (X,g), that means

$$g_n(x^n) \to 0,$$

which implies $g_n(x^n) \leq \beta_n$, for some sequence $\beta_n \geq 0, \beta_n \to 0$. It holds,

$$-\inf_{z\in X_n} \Delta_{-C}(f_n(z) - f_n(x^n)) \leq \beta_n,$$

that is $\Delta_{-C}(f_n(z) - f_n(x^n)) \geq -\beta_n,\ \forall z \in X_n$.
Since $\Delta_{-C}(y) = \max_{\lambda\in C^+\cap\partial B} \langle \lambda, y\rangle$, choosing a vector $e \in \mathrm{int}\, C$ with $\langle \lambda, e\rangle \geq 1,\ \forall \lambda \in C^+ \cap \partial B$, we obtain, $\forall z \in X_n$:

$$\begin{aligned}
\Delta_{-C}(f_n(z) - f_n(x^n) + \beta_n e) &= \max_{\lambda\in C^+\cap\partial B} \langle \lambda, f_n(z) - f_n(x^n) + \beta_n e\rangle \\
&\geq \max_{\lambda\in C^+\cap\partial B} \langle \lambda, f_n(z) - f_n(x^n)\rangle + \min_{\lambda\in C^+\cap\partial B} \langle \lambda, \beta_n e\rangle \\
&\geq \max_{\lambda\in C^+\cap\partial B} \langle \lambda, f_n(z) - f_n(x^n)\rangle + \beta_n \\
&\geq -\beta_n + \beta_n = 0.
\end{aligned}$$

Hence $\Delta_{-C}(f_n(z) - f_n(x^n) + \beta_n e) \geq 0,\ \forall z \in X_n$ and this is equivalent to say $f_n(z) - f_n(x^n) + \beta_n e \notin -\mathrm{int}\, C,\ \forall z \in X_n$.
Thus x^n is an asymptotically minimizing sequence for the vector problem (X,f). Recalling $\mathrm{WEff}\,(X,f) = \{\, x \in X : g(x) = 0 \,\} = \arg\min(X,g)$, the proof of the result is easily completed. $\square$

6.6 Linear scalarization and extended well-posedness

Let f_n and X_n be defined as in Chapter 5. Consider the scalar optimization problem (X,g)

$$\min g(\lambda, x),\quad x \in X$$

where $g(\lambda, x) = \langle \lambda, f(x) \rangle$, λ is a fixed vector belonging to the set $C^+ \cap \partial B$ and X is the feasible region of function f.
We consider the perturbed problem (X_n, g_n)

$$\min g_n(\lambda, x), \quad x \in X_n$$

where $g_n(\lambda, x) = \langle \lambda, f_n(x) \rangle$.

Definition 6.25. *A sequence $x^n \in X_n$ is called* asymptotically minimizing for problem (X, g) *when*

$$g_n(\lambda, x^n) \to \inf_{x \in X} g(\lambda, x).$$

Definition 6.26. *Problem (X, g) is* well-posed (with respect to the perturbations defined by the sequences f_n and X_n) *when*

(i) $\arg\min(X, g) \neq \emptyset$*;*

(ii) for every asymptotically minimizing sequence x^n there exists a subsequence x^{n_k} such that $d(x^{n_k}, \arg\min(X, g)) \to 0$ as $k \to \infty$.

Recalling Proposition 2.31 ([75], [112]), one has $\bar{x} \in \mathrm{WEff}\,(f, X)$ if and only if

$$\bar{x} \in \bigcup_{\lambda \in C^+ \cap \partial B} \arg\min(X, g).$$

Theorem 6.27. *Let $f : \mathbb{R}^n \to \mathbb{R}^l$ and $f_n : \mathbb{R}^n \to \mathbb{R}^l$ be $C-$convex functions with $f_n \to f$ in the continuous convergence. Let X_n be a sequence of sets such that $X_n \xrightarrow{K} X$. If, for every $\lambda \in C^+ \cap \partial B$, problem (X, g) is well-posed (with respect to the perturbations defined by the sequences g_n and X_n), then problem (X, f) is well-posed (with respect to the perturbations defined by the sequences f_n and X_n).*

Proof: Similar to the proof of Theorem 6.19. □

The following example shows that the converse of Theorem 6.27 is not true in general.

Example 6.28. Let $f : \mathbb{R} \to \mathbb{R}^2,\ f(x) = (x^2, e^x),\ C = \mathbb{R}^2_+$ and $X = \mathbb{R},\ f_n = f$ and $X_n = X, \forall n$.
Function f is $C-$convex and $\mathrm{WEff}\,(X, f) = \{\, x \in \mathbb{R} : x \leq 0 \,\}$. Problem (X, f) is well-posed (with respect to the perturbations defined by f_n and X_n), but the scalarized function $g(\lambda, x) = e^x$ obtained by $\lambda = (0, 1)$ is not well-posed, since $\arg\min(X, g) = \emptyset$.

Chapter 7

Conclusions

The motivation for studing the well-posedness notions for optimization problems is clearly inspired by practical considerations. First we stress that most numerical methods for the minimization of a real-valued functions on a feasible region provide minimizing sequences that are appreciable when the approximate solutions are not far from the minimum, thus well-posedness properties play an important role in the convergence analysis of many algorithms.
The same idea has been extended to vector optimization when an ordering cone is fixed and a notion of minimizing sequence is adopted. Under some convexity ([93],[102],[30]) or coercivity ([26]) assumptions on problem data, exploring the structure of the solution sets, a number of positive results in terms of associated scalar problems have been established.

In the introduction we have presented an optimization problem as a mathematical model to simplify and study many daily real situations in which one or more subjects need to make a decision, but usually the decision-makers interact. Game theory aims to help us understand such situations or, more in general, game theoretic reasoning can be used to understand economic, social, political and biological phenomena. For this reason game theory offers possible applications of many well-posedness properties. In the sequel we give a short survey considering some particular cases such as quadratic games, zero-sum games, noncooperative games, potential games and multicriteria games. Moreover a remind to Cournot and Stackelberg models will be stressed.
Some game theoretics idea can be traced to the 18th century, but the major development of the theory was in the last century. An introduction on game theory as theory of rational choice both historically and mathematically can be found in several handbooks such as [111], [107], [28], [4], [100], [16] among others.
By definition, an optimization problem can be viewed as special case of a game, a competitive activity in which players contend with each other according to a set of rules. Formally:

Definition 7.1. *([85]) A* game *with n players is a $2n$-tuple $G = (X_1, \dots, X_n, u_1, \dots, u_n)$ where $X_1, \dots, X_n$ are nonempty sets representing the players strategy* *spaces and* $u_i : X_1 \times \ldots \times X_n \to \mathbb{R},\ \forall i = 1, \ldots, n$ *are* n *real-valued functions representing the payoffs of the players.*

The notion of equilibrium was introduced by John Nash ([98], [99]): given a game with two players an equilibrium is a couple of strategies such that each player's strategy is an optimal reponse to other players' strategies.

Definition 7.2. *Given a game* $G = (X, Y, f, g)$ *a* Nash equilibrium (NE) *for* G *is a couple* $(\bar{x}, \bar{y}) \in X \times Y$ *such that*

$$f(\bar{x}, \bar{y}) \geq f(x, \bar{y}),\ \forall x \in X,$$

$$g(\bar{x}, \bar{y}) \geq g(\bar{x}, y),\ \forall y \in Y.$$

From Definition 7.2, it is evident the link between optimization and game theory. The Tykhonov well-posedness was generalized from minimum problems to Nash Equilibrium problems using asymptotic Nash equilibria, instead of minimizing sequences ([85], [87]).

Definition 7.3. *Given a game* $G = (X_1, \ldots, X_n, u_1, \ldots, u_n)$*, a sequence* $(x^m) = (x_1^m, \ldots, x_n^m)$ *is said to be an* asymptotic Nash equilibrium (a-NE) *if*

$$\sup_{t_i \in X_i} u_i(t_i, x_{-i}^m) - u_i(x_i^m, x_{-i}^m) \to 0, \quad \forall i = 1, \ldots, n, \quad \text{as } m \to +\infty$$

where $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_1 \times \ldots \times X_{i-1} \times x_{i+1} \times \ldots \times x_n$.

Margiocco, Patrone and Pusillo in [85], considering finite games, look for a well-posedness property for game as a combination of T-wp and generalized T-wp for minimization problem, taking also into account the additional important information provided by the value. Following this way, a natural extension of well-posedness properties for games was given:

Definition 7.4. *([87]) Given a game* $G = (X_1, \ldots, X_n, u_1, \ldots, u_n)$*, where* X_i *are topological spaces* $\forall i = 1, \ldots, n$*, we say that* G *is* Tykhonov well-posed *if there exists a unique NE* $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ *towards which every a-NE* (x^m) *converges.*

Interesting results on this topic concern the investigation of a link between already known theorems that guarantee the existence and uniqueness of NE and Tykhonov well-posedness for NE; Margiocco, Patrone and Pusillo ([87]) proved that in some particular case, (quadratic games and zero-sum games), an existence and uniqueness theorem provides also sufficient conditions for Tykhonov well-posedness property.
Several authors (see [104], [105], [122], [116], [85], [106], [86], [87] among others) studied the classical model of Cournot ([121]), reformulated as an oligopoly game, as application of theoretical results on existence and uniqueness of the equilibrium. Using the standard way to prove Tykhonov well-posedness, that is to show that the sets of $\epsilon-$equilibria are compact, Margiocco et al. ([87]) can identify the assumptions under which the Cournot oligopoly game is Tykhonov well-posed.

Given a preference system, it may be represented by bounded or unbounded utility functions and this justifies the introduction of (ϵ, k) equilibria ([83]). With reference to game theory, whenever the preferences of players have to be considered as the real data of the game, instead of their utility functions, the ordinal properties becomes relevant. In fact the problems data are the preferences of the players, not a special choice of the utility function. Recall that two games $G_1(X, Y, f_1, g_1),\ G_2 = (X, Y, f_2, g_2)$ are called ordinally equivalent games ($G_1 \sim G_2$) if there exist two functions ϕ, φ such that $\phi : I \to \mathbb{R}$, $I \supset f_1(X \times Y)$, I interval, $\varphi : J \to \mathbb{R}$, $J \supset f_2(X \times Y)$, J interval, ϕ, φ strictly increasing and continuous functions, $f_2 = \phi \circ f_1,\ g_2 = \varphi \circ g_1$. For non cooperative games, the most most plausible solution is the Nash equilibrium ([105]) for which a notion of Tykhonov well-posedness has been generalized. Focusing on the definition of (ϵ, k) equilibrium, a new well-posedness notion has been introduced ([88]) proving that it is an ordinal property if the payoff functions are bounded from below. The relations between this ordinal property and the classical Tykhonov well-posedness notion have been studied.
We remind the definition of (ϵ, k) equilibrium.

Definition 7.5. *([88]) Given* $\epsilon > 0,\ x \in X$ *is an* $\epsilon-$best reply *to* y *if*

$$f(x, y) \geq \sup_{t \in X} f(t, y) - \epsilon.$$

Given $k \in \mathbb{R}, x \in X$ *is a* $k-$guaranteeing reply *to* $y \in Y$ *if*

$$f(x, y) \geq k.$$

If $x \in X$ *is either an* $\epsilon-$*best reply or a* $k-$*guaranteeing reply (or both) to* y *then* x *is called* (ϵ, k) best reply to y.
Furthermore, we say that $(\bar{x}, \bar{y}) \in X \times Y$ *is an* (ϵ, k) *equilibrium if* $\bar{x}$ *is an* (ϵ, k) *best reply to* $\bar{y}$ *and conversely.*

This approach is a generalization of the well-posedness idea as an ordinal property, introduced with reference to optimization problem by Patrone in [104].

Another application of well-posedness for games involves the classical Stackerlberg problem introduced in 1939 by von Stackelberg ([115]) to describe many economic competitions characterized by a leader and a follower. Translated in a game with two players, the leader (player I) chooses his strategy from set X_1 and determines his strategy first, while the follower (player II), chooses from set X_2 conforming his strategy to the policies of the leader. The goal of leader and follower is to maximize their utility functions.
The problem was deeply studied by several authors (see for example [113], [68], [6], [5], [15]), in particular Morgan ([71], [95]) characterized the two level optimization problems which are well-posed and thus a notion of approximate Stackelberg solutions (very different from Nash approximate ones) and thus of Stackelberg well-posedness.

In [89], the authors considering the optimistic Stackelberg well-posedness and the pessimistic one (already known in literature also as strong and weak well-posedness respectively), established some relations between them and in particular the equivalence in the case of hierarchical potential games including a metric characterization. For the clarity of exposition we recall the preliminary notions to define a well-posedness property (see [89]).

Definition 7.6. *Let $G = (X, Y, f, g)$ be a game and consider the following problem: find $\bar{x} \in X$ such that*

$$\inf_{y \in R_{II}(\bar{x})} f(\bar{x}, y) \geq \inf_{y \in R_{II}(x)} f(x, y), \quad \forall x \in X \tag{7.1}$$

where $R_{II}(\bar{x}) = \arg\max_{y \in Y} g(\bar{x}, y)$. A pair $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x}$ satisfying (7.1) and $\bar{y} \in R_{II}(\bar{x})$ is called a pessimistic Stackelberg equilibrium *and $\bar{x}$ is called a* pessimistic Stackelberg solution.
If we write sup *instead of* inf*, we have the* optimistic Stackelberg equilibrium.

Definition 7.7. *Given $(\epsilon, \eta) \in \mathbb{R}^2$ with $\epsilon, \eta \geq 0$, $\bar{x} \in X$ is an (ϵ, η)*pessimistic Stackelberg Solution *to problem (7.1) if, $\forall x \in X$*

$$\inf_{y \in R_{II(x,\eta)}} f(x, y) - \inf_{y \in R_{II(\bar{x},\eta)}} f(\bar{x}, y) \leq \epsilon \tag{7.2}$$

where $R_{II}(x, \eta) = \{ \tilde{y} \in Y : g(x, y) - g(x, \tilde{y}) \leq \eta, \ \forall y \in Y \}$.

That is if player I is unlucky, he does not lose more than ϵ. Moreover we say that $(\bar{x}, \bar{y})$ is a pessimistic (ϵ, η) Stackelberg equilibrium if $\bar{x}$ satisfies (7.2) and $\bar{y} \in R_{II}(\bar{x}, \eta)$. Similarly for the optimistic case.

Definition 7.8. *The sequence $(x^n, y^n) \in X \times Y$ is a* pessimistic maximizing Stackelberg sequence *if there is a sequence $(\epsilon_n, \eta_n) \in \mathbb{R}_+ \times \mathbb{R}_+$ converging to $(0, 0)$ for $n \to \infty$ such that*

(i) x^n is a pessimistic (ϵ_n, η_n) Stackelberg solution;

(ii) $y_n \in R_{II}(x^n, y^n)$.

That is (x^n, y^n) is a (ϵ_n, η_n) pessimistic Stackelberg equilibrium for any n.
Recalling that in general, well-posedness means existence, uniqueness of the solution together with a notion of approximate sequences approaching the solution:

Definition 7.9. *([95]) A game G is said to be* pessimistic Stackelberg well-posed *if:*

(i) there is only one pessimistic Stackelberg equilibrium $(\bar{x}, \bar{y})$;

(ii) every (x^n, y^n) pessimistic maximizing Stackelberg sequence converges to $(\bar{x}, \bar{y})$.

Analogously for optimistic Stackelberg well-posedness.

A special class of games has been introduced by Monderer and Shapley ([94]) under the name of potential games, because their properties are dictated by a single function called the exact potential function. Formally:

Definition 7.10. *A game $G = (X, Y, f, g)$ is called an* exact potential game *if there exists a potential function $P : X \times Y \to \mathbb{R}$ such that $\forall x, x_1, x_2 \in X$ and $\forall y, y_1, y_2 \in Y$ it holds*

(i) $f(x_1, y) - f(x_2, y) = P(x_1, y) - P(x_2, y)$;

(ii) $g(x, y_1) - g(x, y_2) = P(x, y_1) - P(x, y_2)$.

The famous game called prisoner's dilemma is an exact potential game and so all symmetric game with two players.
This class of games is particular because the problem of equilibria is reduced to the study of maximum points of the potential function. Thus Margiocco et al. ([90]) related the Tykhonov well-posedness and other well-posedness properties such as the ordinal one to the same properties of the potential function as maximum problem. Moreover they studied the well-posedness of the game to establish some links with the same property of the potential function in terms of maximum problem. In this case the link between optimization and game theories is very strong.

For parametric noncooperative games and for optimization problems with constraints defined by parametric Nash equilibria, Lignola et al. ([73]) presented and studied a well-posedness concept in line with the notion introduced in [24] for optimization problems with variational inequality constraints, motivated by the numerical method due to Fukushima ([43]).
This notion originated from the definition of $\alpha-$well-posedness for variational inequalities ([72]) which had inspired the definition of $\alpha-$well-posedness for Nash equilibria which, in turns, had been inspired from the Tykhonov well-posedness for minimization problems. These links are justified by the transformation of any variational inequality into an equivalent minimization problem by using a *gap function*. We remind to the paper of Lignola et al. [73] and the references therein to more details on this topic.

Less developed is the topic of well-posedness for multicriteria games, that are games with vector payoffs. Recently, much attention has been attracted by this class of games because of their applications to real-world situations. On this argument we remind to the paper of Morgan ([96]) that firstly introduced and investigated a parametrically well-posedness notion for a multicriteria game as generalization of the well-posedness of an optimization problem with respect to a parameter considered by Zolezzi in the scalar case. The author stressed also the interests in studying the behaviour of perturbations of a multicriteria game.

In the introduction we have briefly recalled some important steps on the history of well-posedness research, from which two theoretical remarks may be emphasized: first we note that in every paper on vector well-posedness the existence of some solution is assumed, uniqueness is not considered and thus the contributions focus on the stability condition. Secondly, the optimization problem is only one of the possible models to depict easier realities. Actually the study of well-posedness for various models of equilibrium theory and variational inclusions is weakly connected. In our opinion it would be interesting to explore well-posedness through a unifying approach which allow to establish also existence of solutions. In a recent paper ([76]) a general variational relation model giving a unifying approach to many structures modeling, including vector optimization problem, has been proposed; moreover existence and stability results are provided ([76], [64]). This helps in thinking abstractly about minimization and well-posedness and in achieving a single framework for the development of properties and results.

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