On the valuation of the arbitrage opportunities[1]

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Abstract

In this paper, we present different approaches to evaluate the presence of the arbitrage opportunities in the market. In particular, we investigate empirically the well-known put-call parity no-arbitrage relation and the state price density. First, we measure the violation of the put call parity as the difference in implied volatilities between call and put options. Then, we examine the nonnegativity of the state price density. We evaluate the effectiveness of the proposed approaches by an empirical analysis on S&P 500 index options data. Moreover, we propose alternative approaches to estimate the state price density under the classical hypothesis of the Black and Scholes model. To this end, we use the classical nonparametric estimator based on *kernel* and a recent alternative the so called OLP estimator that uses a different approach to evaluate the conditional expectation consistently.

Key words

Arbitrage opportunities, put-call parity, state price density, conditional expectation estimators

JEL Classification: C14, G13

1. Introduction

The pioneering work of Black and Scholes (hereafter BS) has a central rule in modern finance and a great importance for improving research on the option pricing techniques. The main idea behind BS option pricing model is that the price of an option is defined as the least amount of initial capital that permits the construction of a trading strategy whose terminal value equals the payout of the option. In other words, if options are correctly priced in the financial market, it should not be possible for investors to set up a riskless arbitrage position and earn more than the risk free rate of return.Unfortunately, widespread empirical analyses point out that a set of assumptions under which BS model built, particularly normally distributed returns and constant volatility, result in poor pricing and hedging performance. However, using BS principle different generalizations have been proposed – see e.g. Merton [14], Heston [15] and Bates [7] for more details. Generally, most models that have been proposed so far mainly relax some assumptions of BS model and then trying to be justified via general fundamental theorem of asset pricing-FTAP, Harrison and Kreps [12].

Two fundamental entities in assets pricing theory are the put-call parity no-arbitrage relation and the so called State Price Density (hereinafter SPD). The first contribution of this

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paper is to ascertain whether arbitrage opportunities can be directly detected via the violation of the well-known put-call parity no arbitrage relation. Indeed, the first examinations of the put-call parity relationship were conducted by Stoll [17]. If the observed put or call price should deviate substantially from the parity price, an opportunity exists for investors to set up a riskless arbitrage position and earn more than the risk free rate of return. Therefore, violations of the put-call parity no-arbitrage relations contain information about the possibility of free lunch in the market. In this context, under BS model, we use the difference in implied volatility between pairs of call and put options to measure these violations. Then, one can compare this result with that obtained from the violation of the nonnegativity of the SPD. This is important, because negative values of the SPD immediately corresponds to the possibility of free-lunch in the market, e.g. Benko et al [2].

Among no-arbitrage models, the SPD is frequently called risk-neutral density, which is the density of the equivalent martingale measure with respect to the Lebesgue measure. The existence of the equivalent martingale measure follows from the absence of arbitrage opportunities, while its uniqueness demands complete markets. Breeden and Litzenberger [4] proposed an excellent framework to fully recover the SPD in an easy way. In this method, the SPD is simply equal to the second derivative of a European call option with respect to the strike price, see among others Brunner and Hafner [5] for other estimation technique. Furthermore, it is well known that option prices carry important information about market conditions and about the risk preferences of market participants. In this context, the SPD function derived from observed standard option prices have gained considerable attention in last decades. Indeed, an estimate of the SPD implicit in option prices can be useful in different contexts, see among others Ait-Sahalia and Lo [1]. The most significant application of the SPD is that it allows us computing the no-arbitrage price of complex or illiquid option simply by integration techniques.

This paper contributes to the literature in several ways. Firstly, we show that violations of the put-call parity no-arbitrage relation can be used to evaluate the presence of the arbitrage opportunities in the market. To do so, we calculate the difference in implied volatility between call and put options that have the same underlying asset, the same strike price and the same maturity. We examine the effectiveness of the proposed approach by an empirical analysis on S&P 500 index options data. Our results can be easily compared with that obtained from the nonparametric estimation of the SPD. Secondly, we propose different approaches to estimate the SPD based on futures data. Differently from previous studies we estimate SPD directly from the underlying asset under the hypothesis of the BS model. To this end we follow two distinguished approaches to recover SPD, the first one based on nonparametric estimation techniques "kernel" which are natural candidates (see among others [1], [2]), then a new method based on conditional expectation estimator proposed by [16]. First, we examine the so called real mean return function using local polynomial smoothing technique. Then, we estimate the conditional expectation under real probability density. According the hypothesis of BS model, we are able to derive a closed formula for approximating the conditional expectation under risk neutral probability. The main goal of this contribution is to examine and compare the conditional expectation method and the nonparametric technique. These methods allow us extrapolating arbitrage opportunities and relevant information from different markets (futures and options) consistently with the analysis of the underlying.

The rest of this paper is organized as follows. Section 2 reviews the main theoretical properties and describes our methodology. Section 3 presents the main empirical results on the valuation of the arbitrage opportunities, using the violations of the put-call parity relationship, and the SPDs estimation. Section 4 concludes

2. Alternative methods to evaluate the arbitrage opportunities

2.1 Black and Scholes methodology

Fisher Black and Myron Scholes [3] achieved a major breakthrough in European option pricing. In this model we assume that the price process follows a standard geometric Brownian motion defined on filtered probability space $(\Omega, \Im, \mathrm{P}, \{\Im_t\}_{t\geq 0})$, where $\{\Im_t\}_{t\geq 0}$ is the natural filtration of the process completed by the null sets. Under these assumptions we know that $\mathrm{E}(\mathrm{S}_T \mid \Im_t) = \mathrm{E}(\mathrm{S}_T \mid \mathrm{S}_t)$ as consequence of Markovian property. The model of stock price behavior used is defined as:

$$dS = \mu S dt + \sigma S dB, \tag{1}$$

where, μ is the expected rate of return, σ is the volatility of stock return and B denotes a standard Brownian motion. Under this hypothesis we know that the log price is normally distributed:

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - 0.5\sigma^2\right)\mathrm{T}, \sigma^2\,\mathrm{T}\right], \tag{2}$$

where, S_T is the stock price at future time T, S_0 is the stock price at time 0 and ϕ denotes a normal distribution. Please note that μ in equation (1) represents the expected rate of return in real world, while in BS model (risk neutral world) it becomes risk-free rate r.[3]

2.2 Put-call parity no-arbitrage relation

Under the condition of no-arbitrage, it is well known that put-call parity relation must hold for European options on non-dividend-paying stocks:

$$C - P = S_0 - Ke^{-rt}, \tag{3}$$

where, S_0 is the current stock price, C and P are the call and put prices, respectively, that have the same strike price K , the same expiration date and the same underlying asset.

To illustrate the arbitrage opportunities when equation (3) does not hold, we measure the violation of put-call parity as the difference in implied volatility between call and put options that have the same strike price, underlying asset and expiration date. In this context, it well known that the BS model satisfies put-call parity for any assumed value of the volatility parameter σ. Hence,

$$C^{BS}(\sigma) - P^{BS}(\sigma) = S_0 - Ke^{-rt} \quad \forall \sigma > 0, \tag{4}$$

where, $C^{BS}(\sigma)$ and $P^{BS}(\sigma)$ denotes BS call and put prices, respectively, as a function of the volatility parameter σ. At this point, from equation (3) and (4) we can deduce that:

$$C^{BS}(\sigma) - P^{BS}(\sigma) = \mathrm{C} - \mathrm{P} \quad \forall \sigma > 0, \tag{5}$$

By definition, the implied volatility (IV) of a call option (IV^{call}) is that value of the volatility of the underlying instrument, which matches the BS price with the price actually observed on the market. In formal way:

$$C^{BS}(IV^{call}) = C, \tag{6}$$

Now, it is straightforward form equation (5) that:

$$P^{BS}(IV^{call}) = P, \tag{7}$$

this in turn implies that:

$$IV^{call} = IV^{put}. \tag{8}$$

[3] For more details about BS assumptions we refer to Hull (2015)

In this paper, we will carry the analysis on the European options style. Since put-call parity is one of the best known no-arbitrage relations, we use the difference in implied volatility between pairs of call and put options in the spirit of equation (8) in order to detect the presence of arbitrage opportunities in the market. Intuitively, lower call implied volatilities relative to put implied volatilities means that calls are less expensive than puts, and lower put implied volatilities with respect to call implied volatilities suggest the opposite.

We compute the difference in implied volatilities between call and put options that have the same strike price, the same maturity and are written on the same underlying asset. Hence, we refer to such difference as volatility spread (VS) which may represent a valid indicator of the possibility of free lunch in the market, especially close to at-the-money options. Formally, given call and put options with the same strike price and expiration date, we compute the VS as:

$$VS = \max | IV^{call} - IV^{put} | \tag{9}$$

Of course, higher volatility spread is a significant indicator of arbitrage opportunities since put-call parity is a fundamental relation of no-arbitrage.

2.3 State price density

SPDs estimated from cross-sections of observed standard option prices have gained considerable attention during last decades. Since given an estimate of SPD, one can immediately price any path independent derivative. Clearly, the well-known arbitrage free pricing formula is of vital practical importance. In this approach, the option price is given as the expected value of its future payoff with respect to the *risk-neutral measure Q* discounted back to the present time *t*. Formally, the price $\Pi_t(\mathrm{H})$ at time *t* of a derivative with expiration date T and payoff –function $\mathrm{H}(\mathrm{S}_T)$ is given by:

$$\Pi_t(\mathrm{H}) = e^{-r(T-t)} E^Q\left[\mathrm{H} \mid \Im_t\right] = e^{-r(T-t)} \int_0^\infty \mathrm{H}(\mathrm{s}) q_{S_T}(\mathrm{s}) ds \quad \forall t \in [0,T] \tag{10}$$

where, $q_{S_T}(\mathrm{s})$ denotes the SPD. In this context, one fundamental founding in literature is the connection between SPD and implied volatility (IV), e.g. see among others Hafner and Brunner [5]. In this paper, in line with Benko et al [2], we apply local polynomial smoothing technique to estimate IVs, and then SPD. Now, we describe an alternative approach towards estimating the SPD.

2.3.1 Alternative method to estimate the SPD

For the sake of clarity, denote S^{RW} for a real world price and S^{RN} for the risk neutral price. Under the hypothesis of the BS model it is straightforward to write:

$$S_T^{RN} = S_T^{RW} e^{-(\mu - r)T}, \tag{11}$$

Since $S_t = e^{-r(\mathrm{T-t})} E(S_T^{RN} \mid \Im_t)$, we can write $S_t = e^{-r(\mathrm{T-t})} E(S_T^{RW} e^{-(\mu - r)T} \mid \Im_t)$ from which we obtain:

$$E(S_T^{RN} \mid \Im_t) = e^{-\mu T + rt} E(S^{RW} \mid \Im_t), \tag{12}$$

If we assume μ changes over time in model (1), then equation (12) becomes

$$e^{-\int_0^T (\mu(\tau) - r) d\tau} E(S_T \mid \Im_t) = E^Q(S_T \mid \Im_t), \tag{13}$$

where, $E^Q(S_T \mid \Im_t)$ denotes expectation under risk neutral world and $E(\mathrm{S}_T \mid \Im_t)$ the conditional expected price under real world. Moreover, (13) is equivalent to:

$$e^{-\int_0^T \mu(\tau)d\tau} \int_0^\infty s q_{RW}(\mathrm{s}) ds = e^{-rT} \int_0^\infty s q_{RN}(\mathrm{s})\, \mathrm{ds}, \tag{15}$$

where, $\mathrm{q}_{RW}(\mathrm{s})$ and $q_{RN}(s)$ denotes SPDs under real and risk neutral world respectively. Please note that under the BS hypothesis S_{T-t} has the same distribution as $S_T e^{-\mu t}$.

The first step in this approach is to propose a direct method of estimating the real mean return function. Therefore, we use a local estimator that automatically provides an estimate of the real mean function and its derivatives. The input data are daily prices. Denoting the intrinsic value by $\tilde{\mu}$ and the true function by $\mu(t_i)$, $i = 1,....,n,$ we assume the following regression model:

$$\tilde{\mu} \qquad\qquad + \varepsilon_i, \tag{16}$$

where, ε_i models the noise, *n* denotes the number of data considered. The local quadratic estimator $\hat{\mu}(t)$ of the regression function $\mu(t)$ in the point t is defined by the solution of the following local least squares criterion:

$$\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^{n} \left\{ \tilde{\mu} \qquad\qquad (t_i - t) - \alpha_2 (t_i - t)^2 \right\} k_h(t - t_i), \tag{17}$$

where, $k_h(t - t_i) = \frac{1}{h} k\left(\frac{t - t_i}{h} \right)$ is *kernel* function, we refer the reader to Fan and Gijbels [10] for more details. Comparing the last equation with the Taylor expansion of μ yields:

$$\alpha_0 = \hat{\mu}(t_i), \alpha_1 = \hat{\mu}^{'}(t_i),\ 2\alpha_2 = \hat{\mu}^{''}(t_i), \tag{18}$$

which make the estimation of the regression function and its two derivatives possible. The second step towards estimating state price density is to use two methodologies, namely OLP estimator and kernel estimator, to estimate the quantity $E(\mathrm{S}_T \mid \mathrm{S}_t)$.

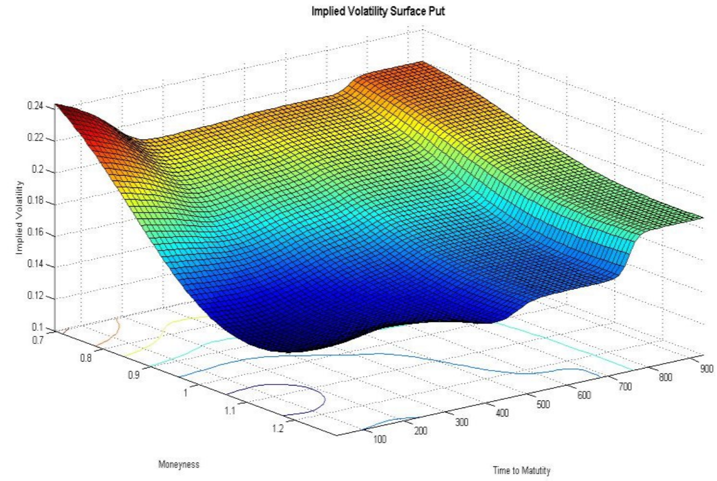
3. Emperical analysis

In this section, we report numerical experiments obtained using the methods introduced to estimate the SPD and to detect the presence of arbitrage opportunities in the market. In the first empirical application to S&P 500 index options we present the analysis concerning the estimation of IVs. For this purpose we use as dataset all options listed on May 13, 2015. The options are European style and the average daily volume during the sample day was 82.65 and 179.01 contracts for call and put respectively. Strike price is at 130 percent and barrier at 70 percent of the underlying spot price at 2098.48, while strike price intervals are 5 points. Throughout this period short-term interest rates exhibit a very low level. The options in our sample vary significantly in price and terms, for example the time-to-maturity varies from 2 days to 934 days.

The row data present some challenges that must be addressed. Clearly, in-the-money (ITM) options are rarely traded relative to at-the -money (ATM) and out-the-money (OTM) options. For example, the average daily volume for puts that are 25 points OTM is 2553 contracts, in contracts, the volume for puts that are 25 points ITM is 2. This can be justified by the strong demand of portfolio managers for protective puts.

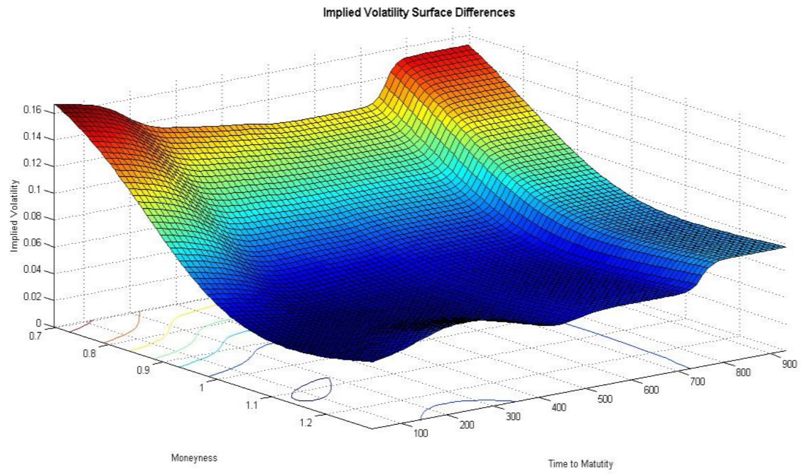
Figure 1 shows the IV surface estimated using put options for the daily data on May 13, 2015. The IV smile is very clear for small maturities and still evident as time to maturity increases.

Figure 1: implied volatility surface of S&P put options



To evaluate the presence of arbitrage opportunities, we calculate the difference in implied volatilities between call and put options that have the same strike price, the same maturity and are written on the same underlying asset. In particular, we consider the differences that are greater than 80 percent of the maximum absolute value of the differences between call and put implied volatilities. In this way, we rule out some differences due to the noisy data. Figure 2 shows the differences in implied volatilities between call and put options.

Figure 2: implied volatility surface differences

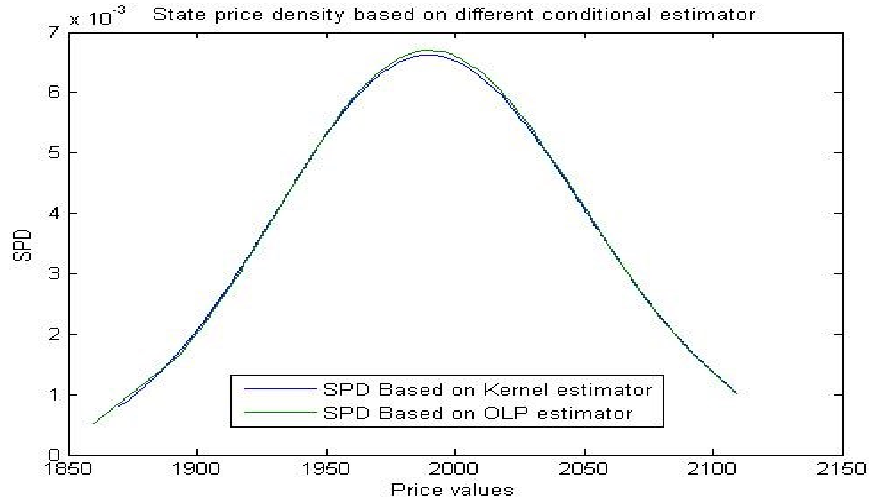


In Figure 2, it is clear that the differences are significant at lower moneyness which corresponds to OTM put options and ITM call options. However, since the market in increase and it is well known that OTM put options and ITM call options are not reliable data to evaluate arbitrage opportunities, we focus on at ATM options. From figure 2, we observe even at ATM option there are small differences, which may represent arbitrage opportunities. In particular, the differences increase as the maturities increase.

To compare the size of the arbitrage opportunities released, one can combine the IV smoothing with SPD estimation. This is important, because SPD requires some properties in order to be consistent with no-arbitrage argument. In particular, the nonnegativity property of SPD since negative values immediately corresponds to arbitrage opportunities in the market, see Benko et al [2] .

In the second application, daily price quotations from the September 2015 S&P 500 futures contrat for the period 2013 through May, 2015, have been used to estimate the SPD. In this context, we use Treasury Bond 3 months as a riskless interest rate for a period matching our selecting data. Firstly, we examine the real mean return function using local polynomial smoothing technique (17). Secondly, we evaluate the conditional expected price using both estimators, namely kernel estimator and OLP, to estimate $E(S_T \mid S_t)$ as described above. Finally, we use the relationship (13) in order to recover the SPD. The results of this analysis are reported in Figure 3.

Figure 3: State Price Densities obtained with Kernel and OLP estimators



From Figure 1 we note a slight difference in the result obtained from both estimators. This result can be explained by the nature of the two methodologies. In particular, the OLP method proposed by [16] yields a consistent estimator of the random variable $E(X \mid Y)$, while the generalized kernel method yields a consistent estimator of the distribution function of $E(X \mid Y)$.Thus, OLP method that yields consistent estimators of random variables $E(X \mid Y)$ can be used to evaluate the SPD.

4. Conclusion

In this paper, we present different methods to evaluate the presence of arbitrage opportunities. In particular, we examine the violation of the well-known put-call parity no-arbitrage relation and the nonnegativity of the SPD. Then, we propose different methods to estimate SPD. In particular, we use two distinct methodologies for estimating the conditional expectation, namely the kernel method and the OLP method recently proposed by [16]. We deviate from previous studies in that we estimate SPD directly from the underlying asset under the hypothesis of BS model. To this end, firstly we examine the real mean return function using local polynomial smoothing technique. Then, we estimate the conditional expectation under real probability density. Under the hypothesis of BS model, we are able to derive a closed formula for approximating the conditional expectation under risk neutral probability. This analysis allows us extrapolating arbitrage opportunities and relevant information from different markets (futures and options) consistently with the analysis of the underlying.

References

[1] Ait-Sahalia, Y., and Lo, A. W. (1998). Nonparametric estimation of state-price densities implicit in financial asset prices. *The Journal of Finance* 53(2), 499–547

[2] BENKO, M., FENGLER, M., HARDLE, W., KOPA, M (2007). On Extracting Information Implied in Options. *Computational statistics*, 22, 543–553

[3] BLACK, F., SCHOLES, M (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637-659

[4] Breeden, D. T., and Litzenberger, R. H. (1978). Prices of state-contingent claims implicit in option prices. *Journal of Business* 51(4), 621–51

[5] Brunner B, Hafner R (2003) Arbitrage-free estimation of the risk-neutral density from the implied volatility smile. *Journal of Computational Finance*, 7(1), 75–106

[6] Carr, P. (2001). Constraints on implied volatility. Working paper, Bank of America Securities.

[7] D. S. Bates (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *Review of Financial Studies*, 9, 69–107

[8] E. A. Nadaraya, (1964). "On estimating regression," *Theory of Probability and its Applications*, 9(1), 141-142

[9] Eli Ofek et al. (2004). Limited arbitrage and short sales restrictions: evidence from the options markets. *Journal of Financial Economics*, 74 305–342

[10] Fan J, Gijbels I (1996). Local polynomial modelling and its applications. Chapman and Hall, London

[11] G. S. Watson (1964), "Smooth regression analysis," *Sankhya, Series A*, 26(4), 359-372

[12] Harrison J, Kreps D.M (1979) Martingales and arbitrage in multiperiod securities markets. *Journal Economic Theory* 20,381–408

[13] J. Hull, (2015). Options, Futures, and Other Derivatives, 9th edition. Prentice Hall, New Jersey, USA

[14] R. C. Merton (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3, 125–144

[15] S. Heston (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6, 327–343

[16] S. Ortobelli, F. Petronio, T. Lando, (2015) "A portfolio return definition coherent with the investors preferences," to appear in IMA-*Journal of Management Mathematics*.

[17] Stoll, H, R (1969). The relationships between Put and Call options prices. *The journal of finance*, 24(5), 802- 824

[18] V.A. Epanechnikov, (1965). "Non-parametric estimation of a multivariate probability density," *Theory of Probability and its Applications,* 14(1), 153-158