UNIVERSITA' DEGLI STUDI DI BERGAMO
DIPARTIMENTO DI INGEGNERIA GESTIONALE E DELL'INFORMAZIONE°
QUADERNI DEL DIPARTIMENTO†

Department of Management and Information Technology

Working Paper

Series "*Mathematics and Statistics*"

n. 11/MS – 2006

Generalized Bateman's expansions and finite integrals of Sonine's and Feldheim's type for hypergeometric functions

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Dedicated to the memory of Alessandro Gentilucci

Abstract The following integral formula of Bateman type

$$\int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}(\cos 2\phi)\, F_{h:\,1}^{j:\,0}\left[\begin{matrix}\underline{a}: & ; & ; \\ \underline{b}: & \alpha+1; & \beta+1;\end{matrix}\, x\sin^2\phi,\, y\cos^2\phi\right]\sin^{2\alpha+1}\phi\cos^{2\beta+1}\phi\, d\phi$$
$$=\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,(\underline{a})_n}{2\Gamma(\alpha+\beta+2n+2)\,(\underline{b})_n}(x+y)^n\ {}_jF_{h+1}\left(\underline{a}+\underline{n};\underline{b}+\underline{n},\,\alpha+\beta+2n+2;x+y\right)P_n^{\alpha,\beta}\left(\frac{y-x}{y+x}\right)$$

is proved. Here $\alpha,\beta>-1$, j and h are non negative integers, $F_{h:\,1}^{j:\,0}$ are Kampé de Fériet functions, ${}_jF_h$ the generalized hypergeometric functions, and $P_n^{\alpha,\beta}$ Jacobi polynomials. For particular values of y and n one obtains generalizations to hypergeometric functions of several formulas already known for Bessel functions, like Sonine's first and second finite integrals and Tranter's formula. Appropriate choices of the parameters j, h, $\underline{a}$ and $\underline{b}$ allow to write all these type of formulas for specific special functions, like Gegenbauer, Jacobi and Laguerre polynomials, or Jacobi functions.

Keywords Hypergeometric functions · Bateman's expansion

Mathematics Subject Classification (2000) 33C45 · 42C05

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1 Introduction

The following formula is very well-known (see, for example, [18, page 373]), and is usually called Sonine's first finite integral:

$$\int_0^{\frac{\pi}{2}} \frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi} dm_{\alpha,\beta}(\phi) = 2^\beta \Gamma(\beta+1) \frac{J_{\alpha+\beta+1}(x)}{x^{\beta+1}}. \tag{1}$$

Here J_α is the Bessel function of first kind and order α (see the next section for the precise definition of this as well as other special functions), α and β are real numbers, $\alpha, \beta > -1$, and

$$dm_{\alpha,\beta}(\phi) = \sin^{2\alpha+1}\phi \cos^{2\beta+1}\phi \, d\phi \tag{2}$$

is a finite positive measure that we will encounter often along the paper.

Sonine's first finite integral allows to express any Bessel function in terms of an integral involving a Bessel function of lower order. It can be easily proven by a simple term by term integration of the power series defining the Bessel function.

An important application of this formula is the calculation of the Fourier transform in the euclidean space $\mathbb{R}^N$ of the Bochner-Riesz multiplier

$$M(x) = \left(1 - |x|^2\right)_+^\delta.$$

Since M is a radial function, the following formula for the Fourier transform holds (see [15, page 155])

$$\widehat{M}(\xi) = \frac{2\pi}{|\xi|^{\frac{N-2}{2}}} \int_0^1 \left(1-s^2\right)^\delta J_{\frac{N-2}{2}}(2\pi|\xi|s)\, s^{\frac{N}{2}}\, ds.$$

The substitution $s = \sin\phi$ gives

$$\widehat{M}(\xi) = \frac{2\pi}{|\xi|^{\frac{N-2}{2}}} \int_0^{\frac{\pi}{2}} \frac{J_{\frac{N-2}{2}}(2\pi|\xi|\sin\phi)}{\sin^{\frac{N-2}{2}}\phi} dm_{\frac{N-2}{2},\delta}(\phi) = \frac{\Gamma(\delta+1)}{\pi^\delta |\xi|^{\frac{N}{2}+\delta}} J_{\frac{N}{2}+\delta}(2\pi|\xi|),$$

by Sonine's first finite integral.

There are two possible generalizations of formula (1). The first, known as Sonine's second finite integral (see [18, page 376]), is the formula

$$\int_0^{\frac{\pi}{2}} \frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi} \frac{J_\beta(y\cos\phi)}{\cos^\beta\phi} dm_{\alpha,\beta}(\phi) = x^\alpha y^\beta \frac{J_{\alpha+\beta+1}\left(\sqrt{x^2+y^2}\right)}{(x^2+y^2)^{\frac{\alpha+\beta+1}{2}}}. \tag{3}$$

That this is indeed a generalization of (1) can be seen by dividing both sides by y^β and letting $y \to 0$. Recall only that

$$\lim_{y\to 0} \frac{J_\beta(y\cos\phi)}{y^\beta \cos^\beta\phi} = \frac{1}{2^\beta \Gamma(\beta+1)}.$$

The second possible generalization goes in a different direction. Let $\mathscr{H}_0$ be the closed one-dimensional subspace of the Hilbert space $L^2\left(\left[0, \frac{\pi}{2}\right], dm_{\alpha,\beta}\right)$, formed by the constant functions. Then Sonine's first finite integral gives the projection onto $\mathscr{H}_0$ of the function $f_x(\phi) = J_\alpha(x\sin\phi)\sin^{-\alpha}\phi$. If we call $P_n^{\alpha,\beta}(z)$ the Jacobi polynomials, then the family $\left\{P_n^{\alpha,\beta}(\cos 2\phi)\right\}_{n=0}^\infty$ is an orthogonal basis for $L^2\left(\left[0, \frac{\pi}{2}\right], dm_{\alpha,\beta}\right)$, and therefore Sonine's first finite integral is, in other words, the 0th Fourier-Jacobi coefficient $\widehat{f_x}(0)$ of f_x. One could calculate all the other Fourier-Jacobi coefficients

$$\widehat{f_x}(n)=\int_0^{\frac{\pi}{2}}\frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi}P_n^{\alpha,\beta}(\cos 2\phi)\,dm_{\alpha,\beta}(\phi),$$

and this is what Tranter's formula [17] does:

$$\int_0^{\frac{\pi}{2}}\frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi}P_n^{\alpha,\beta}(\cos 2\phi)\,dm_{\alpha,\beta}(\phi)=\frac{2^\beta\Gamma(\beta+n+1)}{\Gamma(n+1)}\frac{J_{\alpha+\beta+2n+1}(x)}{x^{\beta+1}}. \tag{4}$$

Of course, formula (1) is obtained in the particular case $n=0$.

The regularity of f_x implies that the Fourier-Jacobi series of f_x converges pointwise to f_x. In other words, Tranter's formula can be reformulated as

$$\frac{J_\alpha(x\sin\phi)}{\sin^\alpha\phi}=\sum_{n=0}^{\infty}\frac{2^{\beta+1}(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)}\frac{J_{\alpha+\beta+2n+1}(x)}{x^{\beta+1}}P_n^{\alpha,\beta}(\cos 2\phi). \tag{5}$$

Both Sonine's second finite integral and Tranter's formula follow from a more general formula: it is known as Bateman's expansion (see [4], [5], or [18, page 370]),

$$\frac{J_\alpha(\rho\sin\phi\sin\theta)}{\rho^\alpha\sin^\alpha\phi\sin^\alpha\theta}\frac{J_\beta(\rho\cos\phi\cos\theta)}{\rho^\beta\cos^\beta\phi\cos^\beta\theta}=\sum_{n=0}^{\infty}(-1)^n\rho^{2n}\frac{J_{\alpha+\beta+2n+1}(\rho)}{\rho^{\alpha+\beta+2n+1}}p_n^{\alpha,\beta}(\cos 2\phi)\,p_n^{\alpha,\beta}(\cos 2\theta). \tag{6}$$

Here $\rho>0$, $\phi,\theta\in\left[0,\frac{\pi}{2}\right]$, and $\left\{p_n^{\alpha,\beta}(\cos 2\phi)\right\}$ are the Jacobi polynomials, properly normalized in order to form an orthonormal basis for $L^2\left(\left[0,\frac{\pi}{2}\right],dm_{\alpha,\beta}(\phi)\right)$. Observe that Tranter's formula (5) follows by taking $\theta=\pi/2$. Again, one can write the integral counterpart of (6) by evaluating in both sides the nth Fourier-Jacobi coefficient in the variable ϕ:

$$\int_0^{\frac{\pi}{2}}\frac{J_\alpha(\rho\sin\phi\sin\theta)}{\rho^\alpha\sin^\alpha\phi\sin^\alpha\theta}\frac{J_\beta(\rho\cos\phi\cos\theta)}{\rho^\beta\cos^\beta\phi\cos^\beta\theta}p_n^{\alpha,\beta}(\cos 2\phi)\,dm_{\alpha,\beta}(\phi)=(-1)^n\rho^{2n}\frac{J_{\alpha+\beta+2n+1}(\rho)}{\rho^{\alpha+\beta+2n+1}}p_n^{\alpha,\beta}(\cos 2\theta). \tag{7}$$

Sonine's second finite integral is obtained in the case $n=0$ with the change of variables

$$\begin{cases} y=\rho\cos\theta\\ x=\rho\sin\theta.\end{cases}$$

Bateman's expansion has a geometric interpretation, at least when α and β are half positive integers. Let $f\in\mathscr{C}_c^\infty\left(\mathbb{R}^{N+M}\right)$ be radial in the first N variables and in the last M variables. In other words, assume that there is a function F defined on $[0,+\infty)\times[0,+\infty)$ such that for all $(x,y)\in\mathbb{R}^{N+M}$

$$f(x,y)=F(|x|,|y|).$$

For this particular class of functions, the spherical harmonic decomposition has the following simple form

$$f(x,y)=\sum_{n=0}^{\infty}f_n(\rho)P_n^{\alpha,\beta}(\cos 2\phi)\rho^{2n},$$

where

$$\begin{cases} |x|=\rho\cos\phi\\ |y|=\rho\sin\phi,\end{cases}$$

with $(\theta, \rho) \in [0, \pi/2] \times [0, \infty)$,

$$\alpha = \frac{M-2}{2}, \quad \beta = \frac{N-2}{2},$$

and $P_n^{\alpha,\beta}$ are the above mentioned Jacobi polynomials. In particular, observe that the functions

$$P_n^{\alpha,\beta}(\cos 2\phi)\,\rho^{2n}$$

are solid spherical harmonics of degree $2n$. Let us compute the Fourier transform $\widehat{f}(\xi, \eta)$. For our purposes, it will be sufficient to assume that

$$f(x, y) = F(|x|, |y|) = f_n(\rho)\,P_n^{\alpha,\beta}(\cos 2\phi)\,\rho^{2n},$$

for a non-negative integer n. Then, using the radiality of f in x and in y,

$$\begin{aligned}
&\widehat{f}(\xi, \eta) \\
&= \int_{\mathbb{R}^M} \int_{\mathbb{R}^N} f(x, y)\, e^{-2\pi i x\cdot\xi}\, dx\, e^{-2\pi i y\cdot\eta}\, dy \\
&= 4\pi^2 \int_0^\infty \int_0^\infty F(s, t)\, \frac{J_\beta(2\pi s|\xi|)}{s^\beta |\xi|^\beta} \frac{J_\alpha(2\pi t|\eta|)}{t^\alpha |\eta|^\alpha} s^{N-1} t^{M-1}\, ds\, dt \\
&= 2\pi \int_0^\infty f_n(r) \left[2\pi \int_0^{\pi/2} P_n^{\alpha,\beta}(\cos 2\phi) \frac{J_\beta(2\pi r\cos\phi\,|\xi|)}{r^\beta \cos^\beta\phi\,|\xi|^\beta} \frac{J_\alpha(2\pi r\sin\phi\,|\eta|)}{r^\alpha \sin^\beta\phi\,|\eta|^\alpha}\, dm_{\alpha,\beta}(\phi) \right] r^{2n+N+M-1}\, dr.
\end{aligned}$$

On the other hand, since $P_n^{\alpha,\beta}(\cos 2\phi)\,\rho^{2n}$ is a solid spherical harmonic of degree $2n$ in $\mathbb{R}^{N+M}$, then (see [15, page 158])

$$\widehat{f}(\xi, \eta) = 2\pi \int_0^\infty f_n(r) \left[(-1)^n \frac{J_{\alpha+\beta+2n+1}(2\pi r\rho)}{(r\rho)^{\alpha+\beta+2n+1}} r^{2n} \rho^{2n} P_n^{\alpha,\beta}(\cos 2\theta) \right] r^{2n+N+M-1}\, dr,$$

where

$$\begin{cases} |\xi| = \rho\cos\theta \\ |\eta| = \rho\sin\theta. \end{cases}$$

By the arbitrarity of f_n, the two expressions in square brackets must coincide, the equality thus obtained being Bateman's formula (7).

In the literature there are several formulas analog to Sonine's first finite integral, where the Bessel functions are replaced by other special functions. For example, for the Gegenbauer polynomials $P_k^{\alpha+\frac{1}{2}}$ the following formula by Feldheim (see [9, page 278] or [16, page 95]) holds

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \left(1 - \sin^2\psi\cos^2\phi\right)^{\frac{k}{2}} P_k^{\alpha+\frac{1}{2}} \left(\frac{\cos\psi}{\sqrt{1-\sin^2\psi\cos^2\phi}} \right) dm_{\alpha,\beta}(\phi) \\
&\qquad = \frac{\Gamma\left(\alpha+\frac{k}{2}+1\right)\Gamma\left(\alpha+\frac{k}{2}+\frac{1}{2}\right)\Gamma(\beta+1)\,\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{2\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\alpha+\beta+\frac{k}{2}+2\right)\Gamma\left(\alpha+\beta+\frac{k}{2}+\frac{3}{2}\right)} P_k^{\alpha+\beta+\frac{3}{2}}(\cos\psi). \quad (8)
\end{aligned}$$

The above formula is a particular case of the more general formula for Jacobi polynomials

$$\int_0^{\frac{\pi}{2}} \left[(1+x)+(1-x)\sin^2\phi\right]^k P_k^{\alpha,\gamma}\left(\frac{(1+x)-(1-x)\sin^2\phi}{(1+x)+(1-x)\sin^2\phi}\right) dm_{\alpha,\beta}(\phi)$$
$$= \frac{2^{k-1}\Gamma(\beta+1)\Gamma(\alpha+k+1)}{\Gamma(\alpha+\beta+k+2)} P_k^{\alpha+\beta+1,\gamma}(x), \quad (9)$$

proved by Askey and Fitch (see [3, formula (3.7)]). The following formula, due to Koshlyakov, for the Laguerre polynomials L_k^α can be found in [11, page 94] or in [12, page 462, formula 2]

$$\int_0^{\frac{\pi}{2}} L_k^\alpha\left(x\sin^2\phi\right) dm_{\alpha,\beta}(\phi) = \frac{\Gamma(\alpha+k+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+k+2)} L_k^{\alpha+\beta+1}(x). \quad (10)$$

A natural question one could raise in this context is: can these formulas be generalized in a similar way as, in the case of Bessel functions, Bateman's expansion generalizes Sonine's first finite integral? In other words, is there a "Bateman expansion" for Gegenbauer, Laguerre, or Jacobi polynomials, or other special functions? We could for example look for a formula of this type

$$F(\alpha,\rho\sin\theta\sin\phi)F(\beta,\rho\cos\theta\cos\phi) = \sum_{n=0}^{\infty} c_n F(\alpha+\beta+2n+1,\rho)\rho^{2n} P_n^{\alpha,\beta}(\cos 2\phi) P_n^{\alpha,\beta}(\cos 2\theta). \quad (11)$$

A first, but unfortunately negative answer comes from a theorem of Al-Salam and Carlitz [1].

Theorem 1 *The functional equation*

$$F(\alpha,\rho\sin\theta\sin\phi)F(\beta,\rho\cos\theta\cos\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n n!\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+2n+1)}$$
$$\times F(\alpha+\beta+2n+1,\rho)\rho^{2n} A_n(\cos 2\phi) A_n(\cos 2\theta),$$

where A_n is a polynomial of degree n, the parameters $\alpha,\beta>-1$, and $F(\alpha,\cdot)$ is analytic, is satisfied if and only if

$$F(\alpha,z) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha}$$

(or a dilation of the same function).

Thus, if a generalized Bateman expansion holds, its structure must be subtler than what we expected in our first guess (11). In this paper we show that such a generalization indeed exists, but before we state and prove it, we need to introduce some definitions and notation. The next section is precisely devoted to that goal.

2 Definitions

We begin with Gauss' generalized hypergeometric function. Let j and h be non-negative integers satisfying the condition $j \le h+1$, then the functions ${}_jF_h$ are given by (see e.g. [7], [8], [11], or [14])

$${}_jF_h(a_1,\ldots,a_j;b_1,\ldots,b_h;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k\ldots(a_j)_k}{(b_1)_k\ldots(b_h)_k k!} z^k, \quad (12)$$

where $(a)_k = a(a+1)\dots(a+k-1) = \Gamma(a+k)/\Gamma(a)$ for $k \geq 1$, $(a)_0 = 1$, is the Pochhammer symbol. The numbers a_i and b_j are arbitrary parameters, except that $b_j \neq 0, -1, -2, \dots$ Using the ratio test, we see that the series (12) converges for all $z \in \mathbb{C}$ when $j < h+1$, and for $|z| < 1$ when $j = h+1$. If one of the a_i's is a negative integer or zero, the series terminates. In this case only, the b_j's can be negative integers, as long as the series terminates before the denominator vanishes.

Let $\alpha, \beta > -1$. The Jacobi polynomials are given by (see [16, page 62])

$$P_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)\Gamma(\alpha+1)} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right). \tag{13}$$

The following equality holds (see [16, page 68])

$$P_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\right). \tag{14}$$

Observe that (see [16, page 59])

$$P_n^{\alpha,\beta}(-1) = (-1)^n \frac{\Gamma(\beta+n+1)}{\Gamma(n+1)\Gamma(\beta+1)}.$$

The Jacobi polynomials form an orthogonal basis of the Hilbert space $L^2\left([-1, 1], (1-x)^\alpha (1+x)^\beta\, dx\right)$. If we define

$$p_n^{\alpha,\beta}(x) = \left(\frac{2(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)\Gamma(n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right)^{\frac{1}{2}} P_n^{\alpha,\beta}(x) \tag{15}$$

one can see in [16, page 68] that $\left\{p_n^{\alpha,\beta}(\cos 2\phi)\right\}_{n=0}^{\infty}$ form an orthonormal basis for $L^2\left(\left[0, \frac{\pi}{2}\right], dm_{\alpha,\beta}(\phi)\right)$, where $dm_{\alpha,\beta}$ is the measure we defined in the introduction (2)

$$dm_{\alpha,\beta}(\phi) = \sin^{2\alpha+1}\phi \cos^{2\beta+1}\phi\, d\phi.$$

We will also use a different normalization of the Jacobi polynomials. Define

$$R_n^{\alpha,\beta}(x) = {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} P_n^{\alpha,\beta}(x) = \frac{P_n^{\alpha,\beta}(x)}{P_n^{\alpha,\beta}(1)}. \tag{16}$$

This gives $R_n^{\alpha,\beta}(1) = 1$ for all n.

If in the definition of Jacobi polynomials we assume that $\mu = n$ be not necessarily an integer, we obtain the so-called Jacobi functions (see [10])

$$\begin{aligned} P_\mu^{\alpha,\gamma}(x) &= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\mu+1)\Gamma(\alpha+1)} {}_2F_1\left(-\mu, \mu+\alpha+\gamma+1; \alpha+1; \frac{1-x}{2}\right) \\ &= \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\mu+1)\Gamma(\alpha+1)} \left(\frac{1+x}{2}\right)^\mu {}_2F_1\left(-\mu, -\mu-\gamma; \alpha+1; \frac{x-1}{x+1}\right). \end{aligned}$$

The Bessel function of the first kind is defined by (see [11, page 102] or [18, page 40])

$$J_\alpha(x) = \frac{x^\alpha}{2^\alpha \Gamma(\alpha+1)} {}_0F_1\left(\alpha+1, -\frac{x^2}{4}\right), \tag{17}$$

the Laguerre polynomials by (see [16, page 103])

$$L_n^\alpha(x) = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)\Gamma(\alpha+1)} {}_1F_1(-n; \alpha+1; x), \tag{18}$$

and the Gegenbauer (or ultraspherical) polynomials by (see [16, page 80])

$$P_k^{\alpha+\frac{1}{2}}(x) = \frac{\Gamma(\alpha+1)\Gamma(2\alpha+k+1)}{\Gamma(\alpha+k+1)\Gamma(2\alpha+1)} P_k^{\alpha,\alpha}(x) \tag{19}$$

$$= \frac{\Gamma(2\alpha+k+1)}{\Gamma(k+1)\Gamma(2\alpha+1)}\, {}_2F_1\left(-k, k+2\alpha+1; \alpha+1; \frac{1-x}{2}\right) \tag{20}$$

$$= \frac{\Gamma(2\alpha+k+1)}{\Gamma(k+1)\Gamma(2\alpha+1)} \left(\frac{1+x}{2}\right)^k {}_2F_1\left(-k, -k-\alpha; \alpha+1; \frac{x-1}{x+1}\right). \tag{21}$$

The following formula gives a different expression of the Gegenbauer polynomials in terms of the Jacobi polynomials (see [16, page 59])

$$P_k^{\alpha+\frac{1}{2}}(x) = \begin{cases} \frac{\sqrt{\pi}\Gamma\left(\alpha+m+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)} P_m^{\alpha,-\frac{1}{2}}\left(2x^2-1\right) & \text{if } k=2m \\ \\ \frac{\sqrt{\pi}\Gamma\left(\alpha+m+\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(m+\frac{3}{2}\right)} x P_m^{\alpha,\frac{1}{2}}\left(2x^2-1\right) & \text{if } k=2m+1. \end{cases} \tag{22}$$

Let j, h, p, q be non-negative integers and $\underline{a} = (a_1, \ldots, a_j)$, $\underline{b} = (b_1, \ldots, b_h)$, $\underline{c} = (c_1, \ldots, c_p)$, $\underline{c'} = \left(c'_1, \ldots, c'_p\right)$, $\underline{d} = (d_1, \ldots, d_q)$, $\underline{d'} = \left(d'_1, \ldots, d'_q\right)$. The Kampé de Fériet function is defined as (see [8], or [14])

$$F_{h:\,q}^{j:\,p}\left[\begin{matrix} \underline{a} : \underline{c};\ \underline{c'}; \\ \underline{b} : \underline{d};\ \underline{d'}; \end{matrix}\ x, y\right] = \sum_{r,s=0}^{\infty} \frac{\Pi_{i=1}^{j}(a_i)_{r+s}\,\Pi_{i=1}^{p}(c_i)_r\,(c'_i)_s\, x^r y^s}{\Pi_{i=1}^{h}(b_i)_{r+s}\,\Pi_{i=1}^{q}(d_i)_r\,(d'_i)_s\, r!s!}.$$

We are interested in a particular class of Kampé de Fériet functions, that is when $p=0$ and $q=1$, thus

$$F_{h:\,1}^{j:\,0}\left[\begin{matrix} \underline{a} : \quad ; \quad ; \\ \underline{b} : \alpha+1;\ \beta+1; \end{matrix}\ x, y\right] = \sum_{r,s=0}^{\infty} \frac{(a_1)_{r+s}\cdots(a_j)_{r+s}\, x^r y^s}{(b_1)_{r+s}\cdots(b_h)_{r+s}\,(\alpha+1)_r\,(\beta+1)_s\, r!s!}.$$

Unless one of the a_i is a negative integer, by Horn's theory (see [7, page 227], [8, page 37], or [14, page 56]), the series defining $F_{h:\,1}^{j:\,0}$ converges only for $x=y=0$ when $j>h+2$, converges absolutely for $\sqrt{|x|}+\sqrt{|y|}<1$ and diverges for $\sqrt{|x|}+\sqrt{|y|}>1$ when $j=h+2$, and converges absolutely for all x and y when $j<h+2$. The result on convergence that we are actually going to use can be readily proved.

Proposition 1 *Let $j \le h+2$, and $\alpha, \beta > -1$. The partial sums*

$$\sum_{N=0}^{M}\left[\sum_{r=0}^{N} \frac{(a_1)_N\cdots(a_j)_N\, x^r y^{N-r}}{(b_1)_N\cdots(b_h)_N\,(\alpha+1)_r\,(\beta+1)_{N-r}\, r!\,(N-r)!}\right],$$

converge to $F_{h:\,1}^{j:\,0}\left[\begin{matrix} \underline{a} : \quad ; \quad ; \\ \underline{b} : \alpha+1;\ \beta+1; \end{matrix}\ x, y\right]$ *as $M\to\infty$ uniformly on any compact set contained in the region of convergence, that is $\mathbb{C}^2$ when $j<h+2$, and $\left\{\sqrt{|x|}+\sqrt{|y|}<1\right\}$ when $j=h+2$.*

Proof Let us first prove that

$$\frac{1}{(\alpha+1)_r\,(\beta+1)_{N-r}\, r!\,(N-r)!} \le C\frac{4^N N^2}{(2r)!\,(2N-2r)!}.$$

Indeed, using the duplication formula for the gamma function,

$$
\begin{aligned}
&\frac{(2r)!\,(2N-2r)!}{(\alpha+1)_r\,(\beta+1)_{N-r}\,r!\,(N-r)!4^N N^2} \\
&= \frac{(2r)\,\Gamma(2r)\,(2N-2r)\,\Gamma(2N-2r)\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+r+1)\,\Gamma(\beta+N-r+1)\,r\Gamma(r)\,(N-r)\,\Gamma(N-r)\,4^N N^2} \\
&= \frac{4\Gamma(\alpha+1)\,\Gamma(\beta+1)\,2^{2r-1}\Gamma(r)\,\Gamma(r+1/2)\,2^{2N-2r-1}\Gamma(N-r)\,\Gamma(N-r+1/2)}{\pi\Gamma(\alpha+r+1)\,\Gamma(\beta+N-r+1)\,\Gamma(r)\,\Gamma(N-r)\,4^N N^2} \\
&= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,(r-1/2)\,\Gamma(r-1/2)\,(N-r-1/2)\,\Gamma(N-r-1/2)}{\pi\Gamma(\alpha+r+1)\,\Gamma(\beta+N-r+1)\,N^2} \\
&\le \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\pi}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left|\sum_{r=0}^{N} \frac{(a_1)_N \ldots (a_j)_N\, x^r y^{N-r}}{(b_1)_N \ldots (b_h)_N\,(\alpha+1)_r\,(\beta+1)_{N-r}\,r!\,(N-r)!}\right| \\
&\le \sum_{r=0}^{N} \frac{\left|(a_1)_N \ldots (a_j)_N\right|\,|x|^r\,|y|^{N-r}}{\left|(b_1)_N \ldots (b_h)_N\right|\,(\alpha+1)_r\,(\beta+1)_{N-r}\,r!\,(N-r)!} \\
&\le C\frac{\left|(a_1)_N \ldots (a_j)_N\right| 4^N N^2}{\left|(b_1)_N \ldots (b_h)_N\right|(2N)!} \sum_{r=0}^{N} \frac{(2N)!}{(2r)!\,(2N-2r)!}\,|x|^r\,|y|^{N-r} \\
&\le C\frac{\left|(a_1)_N \ldots (a_j)_N\right| 4^N N^2 \left(\sqrt{|x|}+\sqrt{|y|}\right)^{2N}}{\left|(b_1)_N \ldots (b_h)_N\right|(2N)!}.
\end{aligned}
$$

If we call the last expression A_N, the ratio test gives

$$
\frac{A_{N+1}}{A_N} \sim N^{j-h-2}\left(\sqrt{|x|}+\sqrt{|y|}\right)^2.
$$

The thesis follows by Weierstrass M-test. □

Particular cases of the functions $F^{j:\,0}_{h:\,1}$ are

$$
F^{0:\,0}_{0:\,1}\left[\begin{matrix} : & ; & ; \\ : \alpha+1; & \beta+1; & \end{matrix}\; x, y\right] = {}_0F_1(\alpha+1; x)\,{}_0F_1(\beta+1; y), \tag{23}
$$

$$
F^{1:\,0}_{0:\,1}\left[\begin{matrix} a: & ; & ; \\ : \alpha+1; & \beta+1; & \end{matrix}\; x, y\right] = \Psi_2(a; \alpha+1, \beta+1; x, y), \tag{24}
$$

where Ψ_2 is the Humbert function (see [8, page 28]), and

$$
F^{2:\,0}_{0:\,1}\left[\begin{matrix} a_1, a_2: & ; & ; \\ : \alpha+1; & \beta+1; & \end{matrix}\; x, y\right] = F_4(a_1, a_2; \alpha+1, \beta+1; x, y),
$$

where F_4 denotes the Appell function (see [2], [7, page 224], [8, page 23], or [14, page 22]) . Also observe that

$$
F^{j:\,0}_{h:\,1}\left[\begin{matrix} \underline{a}: & ; & ; \\ \underline{b}: \alpha+1; & \beta+1; & \end{matrix}\; x, 0\right] = {}_jF_{h+1}(\underline{a}; \underline{b}, \alpha+1; x), \tag{25}
$$

$$
F^{j:\,0}_{h:\,1}\left[\begin{matrix} \underline{a}: & ; & ; \\ \underline{b}: \alpha+1; & \beta+1; & \end{matrix}\; 0, y\right] = {}_jF_{h+1}(\underline{a}; \underline{b}, \beta+1; y).
$$

In the sequel we shall write $\underline{a}+\underline{n}$ instead of $(a_1+n,\ldots,a_j+n)$, and $(\underline{a})_n$ instead of $(a_1)_n\cdots(a_j)_n$, with similar notations for $(b_1,\ldots,b_h)$.

A final remark before we proceed with the theorem. By replacing the terms $(\underline{a})_{r+s}$ and $(\underline{b})_{r+s}$ in the definition of $F^{j:\,0}_{h:\,1}$ with the terms $(\underline{a})_r(\underline{a})_s$ and $(\underline{b})_r(\underline{b})_s$ respectively, one obtains the product

$$ {}_jF_{h+1}\left(\underline{a};\underline{b},\alpha+1;x\right)\,{}_jF_{h+1}\left(\underline{a};\underline{b},\beta+1;y\right). \tag{26}$$

We can therefore look at the function $F^{j:\,0}_{h:\,1}$ as a modification of the product of two hypergeometric functions (26). Formulas (25) show that this modification leaves invariant the behaviour along the coordinate axes, while formula (23) shows that in the simplest case ($j=h=0$) there is no modification whatsoever.

3 The main theorem and first consequences

We are now ready to state and prove our generalization of Bateman's expansion. Recall that our first model formula (11) was

$$F(\alpha,\rho\sin\theta\sin\phi)\,F(\beta,\rho\cos\theta\cos\phi)=\sum_{n=0}^{\infty}c_nF(\alpha+\beta+2n+1,\rho)\,\rho^{2n}P_n^{\alpha,\beta}(\cos2\phi)\,P_n^{\alpha,\beta}(\cos2\theta).$$

All the special functions that admit an integral formula of Sonine's type, like Bessel functions, or Gegenbauer or Laguerre polynomials, and that therefore ask for some kind of expansion of Bateman's type, can be written in terms of a hypergeometric function ${}_jF_{h+1}$. For this reason, on the right hand side of our formula it would be interesting to have $F={}_jF_{h+1}$. On the other hand, we already know from Theorem 1 that on the left hand side we cannot hope to have, in the general case, the product ${}_jF_{h+1}(\ldots)\,{}_jF_{h+1}(\ldots)$. We will now see that the right substitute for this product will be precisely the function $F^{j:\,0}_{h:\,1}$ introduced in the previous section.

Theorem 2 *The following generalization of Bateman's formula and its equivalent integral form hold*

$$\begin{aligned}&F^{j:\,0}_{h:\,1}\left[\begin{matrix}\underline{a}: & ; & ; \\ \underline{b}: & \alpha+1; & \beta+1;\end{matrix}\ \rho\sin^2\theta\sin^2\phi,\rho\cos^2\theta\cos^2\phi\right]\\ &=\sum_{n=0}^{\infty}\frac{(\underline{a})_n\Gamma(\alpha+1)\Gamma(\beta+1)}{(\underline{b})_n2\Gamma(\alpha+\beta+2n+2)}\rho^n\,{}_jF_{h+1}\left(\underline{a}+\underline{n};\underline{b}+\underline{n},\alpha+\beta+2n+2;\rho\right)p_n^{\alpha,\beta}(\cos2\theta)\,p_n^{\alpha,\beta}(\cos2\phi),\end{aligned} \tag{27}$$

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}P_n^{\alpha,\beta}(\cos2\phi)\,F^{j:\,0}_{h:\,1}\left[\begin{matrix}\underline{a}: & ; & ; \\ \underline{b}: & \alpha+1; & \beta+1;\end{matrix}\ x\sin^2\phi,y\cos^2\phi\right]dm_{\alpha,\beta}(\phi)\\ &=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\underline{a})_n}{2\Gamma(\alpha+\beta+2n+2)(\underline{b})_n}(x+y)^n\,{}_jF_{h+1}\left(\underline{a}+\underline{n};\underline{b}+\underline{n},\alpha+\beta+2n+2;x+y\right)P_n^{\alpha,\beta}\left(\frac{y-x}{y+x}\right).\end{aligned} \tag{28}$$

Here $F^{j:\,0}_{h:\,1}$ are the Kampé de Fériet functions, ${}_jF_h$ the generalized hypergeometric functions, $P_n^{\alpha,\beta}$ the Jacobi polynomials, and $p_n^{\alpha,\beta}$ the normalized Jacobi polynomials (15). Also $\alpha,\beta>-1$ and $dm_{\alpha,\beta}(\phi)=\sin^{2\alpha+1}\phi\cos^{2\beta+1}\phi\,d\phi$. Finally, the above equalities hold for all the values of the parameters and the variables for which the series defining $F^{j:\,0}_{h:\,1}$ and ${}_jF_{h+1}$ converge.

Proof Let us first show that the two formulas are equivalent. Calling $x=\rho\sin^2\theta$, and $y=\rho\cos^2\theta$, formula (28) becomes

$$\int_0^{\frac{\pi}{2}} p_n^{\alpha,\beta}(\cos 2\phi)\, F_{h:\,1}^{j:\,0}\left[\begin{matrix}\underline{a}: & ; & ; \\ \underline{b}: & \alpha+1; & \beta+1;\end{matrix}\ \rho\sin^2\theta\sin^2\phi,\ \rho\cos^2\theta\cos^2\phi\right] dm_{\alpha,\beta}(\phi)$$
$$=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\underline{a})_n}{2\Gamma(\alpha+\beta+2n+2)(\underline{b})_n}\rho^n\, {}_jF_{h+1}(\underline{a}+\underline{n};\ \underline{b}+\underline{n},\ \alpha+\beta+2n+2;\ \rho)\, p_n^{\alpha,\beta}(\cos 2\theta).$$

This shows immediately that (28) follows from (27). Conversely, observe that the Fourier-Jacobi expansion of

$$F_{h:\,1}^{j:\,0}\left[\begin{matrix}\underline{a}: & ; & ; \\ \underline{b}: & \alpha+1; & \beta+1;\end{matrix}\ \rho\sin^2\theta\sin^2\phi,\ \rho\cos^2\theta\cos^2\phi\right]$$

in θ and ϕ is

$$\sum_{k,n=0}^{\infty} G_{k,n}(\rho)\, p_k^{\alpha,\beta}(\cos 2\theta)\, p_n^{\alpha,\beta}(\cos 2\phi),$$

where

$$G_{k,n}(\rho)=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} F_{h:\,1}^{j:\,0}\left[\begin{matrix}\underline{a}: & ; & ; \\ \underline{b}: & \alpha+1; & \beta+1;\end{matrix}\ \rho\sin^2\theta\sin^2\phi,\ \rho\cos^2\theta\cos^2\phi\right]$$
$$\times\, p_k^{\alpha,\beta}(\cos 2\theta)\, p_n^{\alpha,\beta}(\cos 2\phi)\, dm_{\alpha,\beta}(\theta)\, dm_{\alpha,\beta}(\phi)$$
$$=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\underline{a})_n}{2\Gamma(\alpha+\beta+2n+2)(\underline{b})_n}\rho^n\, {}_jF_{h+1}(\underline{a}+\underline{n};\ \underline{b}+\underline{n},\ \alpha+\beta+2n+2;\ \rho)$$
$$\times\int_0^{\frac{\pi}{2}} p_n^{\alpha,\beta}(\cos 2\theta)\, p_k^{\alpha,\beta}(\cos 2\theta)\, dm_{\alpha,\beta}(\theta)$$
$$=\delta_{k,n}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)(\underline{a})_n}{2\Gamma(\alpha+\beta+2n+2)(\underline{b})_n}\rho^n\, {}_jF_{h+1}(\underline{a}+\underline{n};\ \underline{b}+\underline{n},\ \alpha+\beta+2n+2;\ \rho).$$

Let us now prove (28). A term by term integration, along with formula (14) to express the Jacobi polynomials in terms of Gauss' hypergeometric series, yields

$$\int_0^{\frac{\pi}{2}} R_n^{\alpha,\beta}(\cos 2\phi)\, F_{h:\,1}^{j:\,0}\left[\begin{matrix}\underline{a}: & ; & ; \\ \underline{b}: & \alpha+1; & \beta+1;\end{matrix}\ x\sin^2\phi,\ y\cos^2\phi\right] dm_{\alpha,\beta}(\phi)$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\underline{a})_{r+s}\, x^r y^s}{(\underline{b})_{r+s}(\alpha+1)_r(\beta+1)_s\, r!s!}\int_0^{\frac{\pi}{2}} R_n^{\alpha,\beta}(\cos 2\phi)\sin^{2\alpha+2r+1}\phi\cos^{2\beta+2s+1}\phi\, d\phi$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\underline{a})_{r+s}\, x^r y^s}{(\underline{b})_{r+s}(\alpha+1)_r(\beta+1)_s\, r!s!2^{\alpha+\beta+s+r+2}}\int_{-1}^{1} R_n^{\alpha,\beta}(t)(1-t)^r(1+t)^s(1-t)^\alpha(1+t)^\beta\, dt$$
$$=\sum_{s=n}^{\infty}\sum_{k=0}^{s}\frac{(\underline{a})_s\, x^{s-k}y^k}{(\underline{b})_s(\alpha+1)_{s-k}(\beta+1)_k(s-k)!k!2^{\alpha+\beta+s+2}}\int_{-1}^{1} R_n^{\alpha,\beta}(t)(1-t)^{s-k}(1+t)^k(1-t)^\alpha(1+t)^\beta\, dt$$
$$=\sum_{s=n}^{\infty}\sum_{k=0}^{s}\frac{(\underline{a})_s\,\Gamma(\alpha+1)\Gamma(\beta+k+n+1)\, x^{s-k}y^k}{2(\underline{b})_s(\beta+1)_k(s-k)!k!\Gamma(\alpha+\beta+n+s+2)}$$
$$\times\, {}_3F_2(-n,\ -\beta-n,\ \alpha+s-k+1;\ \alpha+1,\ -\beta-k-n;\ 1).$$

On the other hand

$$\begin{aligned}
&\frac{(\underline{a})_n}{(\underline{b})_n}(x+y)^n\,{}_jF_{h+1}\left(\underline{a}+\underline{n};\underline{b}+\underline{n},\,\alpha+\beta+2n+2;x+y\right)R_n^{\alpha,\beta}\left(\frac{y-x}{y+x}\right)\\
&=\frac{(\underline{a})_n}{(\underline{b})_n}(x+y)^n\sum_{s=0}^{\infty}\frac{(\underline{a}+\underline{n})_s(x+y)^s}{(\underline{b}+\underline{n})_s(\alpha+\beta+2n+2)_s\,s!}\sum_{l=0}^{n}\frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l\,l!}\left(\frac{x}{x+y}\right)^l\\
&=\sum_{s=0}^{\infty}\frac{(\underline{a})_{s+n}}{(\underline{b})_{s+n}(\alpha+\beta+2n+2)_s\,s!}\sum_{l=0}^{n}\frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l\,l!}x^l(x+y)^{s+n-l}\\
&=\sum_{s=0}^{\infty}\frac{(\underline{a})_{s+n}}{(\underline{b})_{s+n}(\alpha+\beta+2n+2)_s\,s!}\sum_{l=0}^{n}\frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l\,l!}\sum_{k=0}^{s+n-l}\binom{s+n-l}{k}x^{s+n-k}y^k\\
&=\sum_{s=n}^{\infty}\frac{(\underline{a})_{s}}{(\underline{b})_{s}(\alpha+\beta+2n+2)_{s-n}(s-n)!}\sum_{l=0}^{n}\frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l\,l!}\sum_{k=0}^{s-l}\binom{s-l}{k}x^{s-k}y^k\\
&=\sum_{s=n}^{\infty}\sum_{k=0}^{s}\frac{(\underline{a})_{s}x^{s-k}y^k}{(\underline{b})_{s}(\alpha+\beta+2n+2)_{s-n}(s-n)!}\sum_{l=0}^{\min(s-k,n)}\frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l\,l!}\binom{s-l}{k}\\
&=\sum_{s=n}^{\infty}\sum_{k=0}^{s}\frac{(\underline{a})_{s}x^{s-k}y^k}{(\underline{b})_{s}(\alpha+\beta+2n+2)_{s-n}(s-n)!}\sum_{l=0}^{\infty}\frac{(-n)_l(n+\alpha+\beta+1)_l}{(\alpha+1)_l\,l!}\binom{s-l}{k}\\
&=\sum_{s=n}^{\infty}\sum_{k=0}^{s}\frac{s!\,(\underline{a})_{s}x^{s-k}y^k}{(\underline{b})_{s}(\alpha+\beta+2n+2)_{s-n}(s-k)!k!(s-n)!}\,{}_3F_2\left(-n,\,\alpha+\beta+n+1,\,k-s;\,\alpha+1,\,-s;\,1\right).
\end{aligned}$$

Now, by Thomae's identity (see [19]),

$$\begin{aligned}
&{}_3F_2\left(-n,\,-\beta-n,\,\alpha+s-k+1;\,\alpha+1,\,-\beta-k-n;\,1\right)\\
&=\frac{(-1)^n(-s)_n}{(\beta+k+1)_n}\,{}_3F_2\left(-n,\,\alpha+\beta+n+1,\,k-s;\,\alpha+1,\,-s;\,1\right),
\end{aligned}$$

and the theorem follows. □

Certain particular cases of Theorem 2 deserve to be properly emphasized, because they generalize Sonine's integrals of first and second type and Tranter's formula to this general context of hypergeometric functions. Some of these formulas are already known.

Corollary 1 *For $\theta=\pi/2$, that is $y=0$, Bateman's expansion becomes a generalization of Tranter's formula*

$$\begin{aligned}
{}_jF_{h+1}\left(\underline{a};\underline{b},\,\alpha+1;x\sin^2\phi\right)&=\sum_{n=0}^{\infty}\frac{(\underline{a})_n\Gamma(\alpha+\beta+n+1)}{(\underline{b})_n(\alpha+1)_n\Gamma(\alpha+\beta+2n+1)}(-x)^n\\
&\quad\times{}_jF_{h+1}\left(\underline{a}+\underline{n};\underline{b}+\underline{n},\,\alpha+\beta+2n+2;x\right)P_n^{\alpha,\beta}(\cos2\phi),\qquad(29)
\end{aligned}$$

or, equivalently

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}P_n^{\alpha,\beta}(\cos2\phi)\,{}_jF_{h+1}\left(\underline{a};\underline{b},\,\alpha+1;x\sin^2\phi\right)dm_{\alpha,\beta}(\phi)\\
&=\frac{\Gamma(\alpha+1)\Gamma(\beta+n+1)(\underline{a})_n}{2\Gamma(\alpha+\beta+2n+2)\Gamma(n+1)(\underline{b})_n}(-x)^n\,{}_jF_{h+1}\left(\underline{a}+\underline{n};\underline{b}+\underline{n},\,\alpha+\beta+2n+2;x\right).\qquad(30)
\end{aligned}$$

For $n=0$, formula (28) becomes a generalized version of Sonine's second finite integral

$$\int_0^{\frac{\pi}{2}} F_{h:\,1}^{j:\,0}\left[\begin{matrix}\underline{a}: & ; & ; \\ \underline{b}: \alpha+1; & \beta+1; & \end{matrix}\, x\sin^2\phi,\, y\cos^2\phi\right] dm_{\alpha,\beta}(\phi)$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+2)}\, {}_jF_{h+1}\left(\underline{a};\underline{b},\alpha+\beta+2;x+y\right), \quad (31)$$

which becomes a generalized version of Sonine's first finite integral when $y=0$

$$\int_0^{\frac{\pi}{2}} {}_jF_{h+1}\left(\underline{a};\underline{b},\alpha+1;x\sin^2\phi\right) dm_{\alpha,\beta}(\phi) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+2)}\, {}_jF_{h+1}\left(\underline{a};\underline{b},\alpha+\beta+2;x\right). \quad (32)$$

The next diagram describes the reciprocal implications between the above formulas. In it, we use the following notations:

BE. The generalized Bateman Expansion formula (27).

BI. The generalized Bateman Integral formula (28).

TE. The generalized Tranter Expansion formula (29).

TI. The generalized Tranter Integral formula (30).

S2. The generalized Sonine's second finite integral (31).

S1. The generalized Sonine's first finite integral (32).

$$\begin{tikzcd}
BE \arrow[rr, Leftrightarrow] \arrow[d, Rightarrow, "\theta=\frac{\pi}{2}"'] & & BI \arrow[dl, Rightarrow, "y=0"'] \arrow[dr, Rightarrow, "n=0"] & \\
TE \arrow[r, Leftrightarrow] & TI \arrow[dr, Rightarrow, "n=0"'] & & S2 \arrow[dl, Rightarrow, "y=0"] \\
& & S1 &
\end{tikzcd}$$

Implications between formulas.

Remark 1 Formula (32) can be proved directly by an easy term by term integration, and is originally due to Bateman (see [6, page 184]), at least for the function ${}_2F_1$. It also appears as an exercise in [11, page 277].

Remark 2 Formula (31) follows as a particular case of a formula of G. P. Srivastava and Saran (see [13, formula (1a)], by taking $h=2$, $n=1$, $\mu=0$, $\rho=1$, $s=2\delta$ and $\sigma=\delta'$. Although the authors study several particular cases of their formula, they seem to miss this particular one, as well as all the formulas of S2-type that we show in the next section. We also want to emphasize that their formula (1a) generalizes (31) in a sense that is different from what our formula (27) does.

4 Particular cases

The formulas we proved in the last section are very general. Nevertheless, it may be useful to write all these formulas in terms of the precise special function we are interested in. For instance, we have already said that, when one takes $j=h=0$ and $\rho=-r^2/4$, formula (27) becomes

$$
\begin{aligned}
&{}_0F_1\left(;\alpha+1;-\frac{r^2\sin^2\theta\sin^2\phi}{4}\right){}_0F_1\left(;\beta+1;-\frac{r^2\cos^2\theta\cos^2\phi}{4}\right)\\
&\quad=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+2n+2)4^n}(-1)^n r^{2n}\,{}_0F_1\left(;\alpha+\beta+2n+2;-\frac{r^2}{4}\right)p_n^{\alpha,\beta}(\cos 2\theta)\,p_n^{\alpha,\beta}(\cos 2\phi)
\end{aligned}
$$

which, by the definition of Bessel function in terms of hypergeometric functions (17), gives precisely the original Bateman's expansion (6)

$$
\frac{J_\alpha(r\sin\phi\sin\theta)}{r^\alpha\sin^\alpha\phi\sin^\alpha\theta}\frac{J_\beta(r\cos\phi\cos\theta)}{r^\beta\cos^\beta\phi\cos^\beta\theta}=\sum_{n=0}^{\infty}(-1)^n r^{2n}\frac{J_{\alpha+\beta+2n+1}(r)}{r^{\alpha+\beta+2n+1}}p_n^{\alpha,\beta}(\cos 2\theta)\,p_n^{\alpha,\beta}(\cos 2\phi).
$$

In the upcoming subsections we will reproduce this type of arguments for Laguerre polynomials, Jacobi polynomials/functions, and Gegenbauer polynomials.

4.1 Laguerre polynomials

Due to formula (18), we know that a Laguerre polynomial can be expressed in terms of a hypergeometric function ${}_jF_{h+1}$ with $j=1$ and $h=0$. Therefore Bateman's generalized expansion, in this case, involves the Humbert function

$$
F_{0:\,1}^{1:\,0}\left[\begin{matrix} a: & ; & ; \\ & :\alpha+1; & \beta+1; \end{matrix}\; x,y\right]=\Psi_2(a;\alpha+1,\beta+1;x,y).
$$

When, as here, a is a negative integer or zero, $a=-k$, with $k\geq 0$, the function $\Psi_2(a;\alpha+1,\beta+1;x,y)$ is a polynomial of degree k in the variables x and y and can therefore be expressed in a more friendly fashion, as a linear combination of $L_n^\alpha(x)L_m^\beta(y)$ with $n+m\leq k$. The next propositions deal with this task.

Proposition 2 *For any $\theta\in[0,\pi/2]$, the polynomials $\Psi_2\left(-k;\alpha+1,\beta+1;x\sin^2\theta,y\cos^2\theta\right)$ solve the differential equation*

$$
x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}+(\alpha+1-x)\frac{\partial u}{\partial x}+(\beta+1-y)\frac{\partial u}{\partial y}+ku=0 \tag{33}
$$

The proof is a trivial term by term differentiation of the finite sum defining Ψ_2.

Proposition 3 *The following equality holds*

$$
\Psi_2\left(-k;\alpha+1,\beta+1;x\sin^2\theta,y\cos^2\theta\right)=\sum_{m=0}^{k}\frac{\Gamma(k+1)}{(\alpha+1)_m(\beta+1)_{k-m}}\sin^{2m}\theta\cos^{2k-2m}\theta\,L_m^\alpha(x)L_{k-m}^\beta(y).
$$

Proof Since $\Psi_2\left(-k;\alpha+1,\beta+1;x\sin^2\theta,y\cos^2\theta\right)$ is a polynomial of degree k in the x variable, we have

$$
\Psi_2\left(-k;\alpha+1,\beta+1;x\sin^2\theta,y\cos^2\theta\right)=\sum_{m=0}^{k}A_{k,m}(\theta,y)L_m^\alpha(x),
$$

and this must be a solution of (33) for all θ. Thus

$$
\begin{aligned}
0=\sum_{m=0}^{k}A_{k,m}(\theta,y)x\frac{\partial^2 L_m^\alpha}{\partial x^2}(x)+y\frac{\partial^2 A_{k,m}}{\partial y^2}(\theta,y)L_m^\alpha(x)+(\alpha+1-x)A_{k,m}(\theta,y)\frac{\partial L_m^\alpha}{\partial x}(x)\\
+(\beta+1-y)\frac{\partial A_{k,m}}{\partial y}(\theta,y)L_m^\alpha(x)+kA_{k,m}(\theta,y)L_m^\alpha(x).
\end{aligned}
$$

It is a well-known fact that $x\frac{\partial^2 L_m^\alpha}{\partial x^2}(x)+(\alpha+1-x)\frac{\partial L_m^\alpha}{\partial x}(x)=-mL_m^\alpha(x)$ (see [16, page 100]). Thus

$$0=\sum_{m=0}^{k}-mA_{k,m}(\theta,y)L_m^\alpha(x)+y\frac{\partial^2 A_{k,m}}{\partial y^2}(\theta,y)L_m^\alpha(x)$$
$$+(\beta+1-y)\frac{\partial A_{k,m}}{\partial y}(\theta,y)L_m^\alpha(x)+kA_{k,m}(\theta,y)L_m^\alpha(x),$$

and therefore, for all $m=1,\ldots,k$, it must be

$$y\frac{\partial^2 A_{k,m}}{\partial y^2}(\theta,y)+(\beta+1-y)\frac{\partial A_{k,m}}{\partial y}+(k-m)A_{k,m}(\theta,y)=0.$$

It is also a well known fact (see again [16, page 101]) that the only polynomial solution to the above equation is

$$A_{k,m}(\theta,y)=C_{k,m}(\theta)L_{k-m}^\beta(y).$$

Thus

$$\Psi_2\left(-k;\alpha+1,\beta+1;x\sin^2\theta,y\cos^2\theta\right)=\sum_{m=0}^{k}C_{k,m}(\theta)L_{k-m}^\beta(y)L_m^\alpha(x).$$

In order to determine $C_{k,\,m}(\theta)$, observe that the coefficient of the term $x^m y^{k-m}$ is

$$\frac{(-k)_k\sin^{2m}\theta\cos^{2k-2m}\theta}{(\alpha+1)_m(\beta+1)_{k-m}m!\,(k-m)!}$$

in the left hand side, and

$$C_{k,m}(\theta)\frac{(-1)^k}{m!\,(k-m)!}$$

in the right hand side, and these two must coincide. □

We are now ready to write Bateman's expansion for Laguerre polynomials, in two different versions: one involving the Humbert function ${}_1S_0=\Psi_2$, and the other involving just Laguerre polynomials.

Theorem 3 (Bateman's expansion for Laguerre polynomials) *The following BE-type expansions hold*

$$\Psi_2\left(-k;\alpha+1,\beta+1;\rho\sin^2\phi\sin^2\theta,\rho\cos^2\phi\cos^2\theta\right)$$
$$=\sum_{n=0}^{k}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(k+1)}{2\Gamma(\alpha+\beta+n+k+2)}(-\rho)^n L_{k-n}^{\alpha+\beta+2n+1}(\rho)\,p_n^{\alpha,\beta}(\cos 2\theta)\,p_n^{\alpha,\beta}(\cos 2\phi),\quad(34)$$

$$\sum_{m=0}^{k}\frac{1}{(\alpha+1)_m(\beta+1)_{k-m}}\sin^{2m}\theta L_m^\alpha\left(\rho\sin^2\phi\right)\cos^{2k-2m}\theta L_{k-m}^\beta\left(\rho\cos^2\phi\right)$$
$$=\sum_{n=0}^{k}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+n+k+2)}(-\rho)^n L_{k-n}^{\alpha+\beta+2n+1}(\rho)\,p_n^{\alpha,\beta}(\cos 2\theta)\,p_n^{\alpha,\beta}(\cos 2\phi),\quad(35)$$

as well as their integral counterparts of BI-type

$$\int_0^{\frac{\pi}{2}}P_n^{\alpha,\beta}(\cos 2\phi)\Psi_2\left(-k;\alpha+1,\beta+1;\rho\sin^2\phi\sin^2\theta,\rho\cos^2\phi\cos^2\theta\right)dm_{\alpha,\beta}(\phi)$$
$$=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(k+1)}{2\Gamma(\alpha+\beta+n+k+2)}(-\rho)^n L_{k-n}^{\alpha+\beta+2n+1}(\rho)\,P_n^{\alpha,\beta}(\cos 2\theta),\quad(36)$$

$$\sum_{m=0}^{k} \frac{\sin^{2m}\theta \cos^{2k-2m}\theta}{(\alpha+1)_m (\beta+1)_{k-m}} \int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}(\cos 2\phi)\, L_m^{\alpha}\left(\rho \sin^2\phi\right) L_{k-m}^{\beta}\left(\rho \cos^2\phi\right) dm_{\alpha,\beta}(\phi)$$
$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+n+k+2)} (-\rho)^n L_{k-n}^{\alpha+\beta+2n+1}(\rho)\, P_n^{\alpha,\beta}(\cos 2\theta), \quad (37)$$

$$\sum_{m=0}^{k} \frac{L_m^{\alpha}\left(\rho \sin^2\phi\right) L_{k-m}^{\beta}\left(\rho \cos^2\phi\right)}{(\alpha+1)_m (\beta+1)_{k-m}} \int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}(\cos 2\theta) \sin^{2m}\theta \cos^{2k-2m}\theta\, dm_{\alpha,\beta}(\theta)$$
$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+n+k+2)} (-\rho)^n L_{k-n}^{\alpha+\beta+2n+1}(\rho)\, P_n^{\alpha,\beta}(\cos 2\phi), \quad (38)$$

where $n = 0, \ldots, k$.

Proof The first expansion is just an application of Theorem 2 with $j = 1$, $h = 0$, and $a_1 = -k$, while the second follows from it and from Proposition 3. The integral formulas follow from the two expansions, after an integration against $p_n^{\alpha,\beta}(\cos 2\phi)\, dm_{\alpha,\beta}(\phi)$ or $p_n^{\alpha,\beta}(\cos 2\theta)\, dm_{\alpha,\beta}(\theta)$. □

Corollary 2 (Particular cases of Bateman's expansion for Laguerre polynomials) *For* $\theta = \pi/2$ *one obtains a TE-type expansion*

$$L_k^{\alpha}\left(\rho \sin^2\phi\right) = \sum_{n=0}^{k} \frac{(\alpha+\beta+2n+1)\Gamma(\alpha+k+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+n+k+2)\Gamma(\alpha+n+1)} \rho^n L_{k-n}^{\alpha+\beta+2n+1}(\rho)\, P_n^{\alpha,\beta}(\cos 2\phi), \quad (39)$$

or its equivalent TI-type integral form

$$\int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}(\cos 2\phi)\, L_k^{\alpha}\left(\rho \sin^2\phi\right) dm_{\alpha,\beta}(\phi) = \frac{\Gamma(\alpha+k+1)\Gamma(\beta+n+1)}{2\Gamma(\alpha+\beta+n+k+2)\Gamma(n+1)} \rho^n L_{k-n}^{\alpha+\beta+2n+1}(\rho) \quad (40)$$

where $0 \le n \le k$. *When* $n = 0$, *formulas (36) and (37) give analogs of Sonine's second finite integral (S2-type formulas)*

$$\int_0^{\frac{\pi}{2}} \Psi_2\left(-k; \alpha+1, \beta+1; \rho \sin^2\phi \sin^2\theta, \rho \cos^2\phi \cos^2\theta\right) dm_{\alpha,\beta}(\phi)$$
$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(k+1)}{2\Gamma(\alpha+\beta+k+2)} L_k^{\alpha+\beta+1}(\rho), \quad (41)$$

$$\sum_{m=0}^{k} \frac{2\Gamma(\alpha+\beta+k+2)}{\Gamma(\alpha+m+1)\Gamma(\beta+k-m+1)} \sin^{2m}\theta \cos^{2k-2m}\theta \int_0^{\frac{\pi}{2}} L_m^{\alpha}\left(\rho \sin^2\phi\right) L_{k-m}^{\beta}\left(\rho \cos^2\phi\right) dm_{\alpha,\beta}(\phi)$$
$$= L_k^{\alpha+\beta+1}(\rho), \quad (42)$$

while formula (38) gives (see [11, page 96])

$$L_k^{\alpha+\beta+1}(x+y) = \sum_{m=0}^{k} L_m^{\alpha}(x)\, L_{k-m}^{\beta}(y).$$

These two last formulas become for $\theta = \pi/2$ *and* $y = 0$ *two S1-type formulas, that is the already mentioned Koshlyakov formula (10)*

$$\int_0^{\frac{\pi}{2}} L_k^{\alpha}\left(x \sin^2\phi\right) dm_{\alpha,\beta}(\phi) = \frac{\Gamma(\alpha+k+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+k+2)} L_k^{\alpha+\beta+1}(x)$$

and (see [11, page 96])

$$L_k^{\alpha+\beta+1}(x) = \sum_{m=0}^{k} \frac{\Gamma(\beta+k-m+1)}{\Gamma(\beta+1)\Gamma(k-m+1)} L_m^{\alpha}(x).$$

4.2 Jacobi functions and polynomials

Apply Theorem 2 in the case $j=2$, $h=0$, $a_1=-\mu$, $a_2=-\mu-\gamma$ and obtain (BI)

$$\int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}(\cos 2\phi)(\rho+1)^{\mu} F_4\left(-\mu,-\mu-\gamma;\alpha+1,\beta+1;\frac{\rho-1}{\rho+1}\sin^2\theta\sin^2\phi,\frac{\rho-1}{\rho+1}\cos^2\theta\cos^2\phi\right) \times dm_{\alpha,\beta}(\phi)$$
$$=\frac{2^{\mu-n-1}\Gamma(\alpha+1)\Gamma(\beta+1)(-\mu)_n(-\gamma-\mu)_n\Gamma(\mu-n+1)}{\Gamma(\alpha+\beta+n+\mu+2)}(\rho-1)^n P_{\mu-n}^{\alpha+\beta+2n+1,\gamma}(\rho)P_n^{\alpha,\beta}(\cos 2\theta) \quad (43)$$

where $0\le n\le\mu$ if μ is a non-negative integer. For $n=0$ we have an S2-type formula

$$\int_0^{\frac{\pi}{2}}(\rho+1)^{\mu} F_4\left(-\mu,-\mu-\gamma;\alpha+1,\beta+1;\frac{\rho-1}{\rho+1}\sin^2\theta\sin^2\phi,\frac{\rho-1}{\rho+1}\cos^2\theta\cos^2\phi\right)dm_{\alpha,\beta}(\phi)$$
$$=\frac{2^{\mu-1}\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\mu+1)}{\Gamma(\alpha+\beta+\mu+2)}P_{\mu}^{\alpha+\beta+1,\gamma}(\rho); \quad (44)$$

for $\theta=\pi/2$ in (43) we obtain a TI-type formula

$$\int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}(\cos 2\phi)\left[(1+\rho)+(1-\rho)\sin^2\phi\right]^{\mu} P_{\mu}^{\alpha,\gamma}\left(\frac{(1+\rho)-(1-\rho)\sin^2\phi}{(1+\rho)+(1-\rho)\sin^2\phi}\right)dm_{\alpha,\beta}(\phi)$$
$$=\frac{2^{\mu-n-1}\Gamma(\mu+\gamma+1)\Gamma(\beta+n+1)\Gamma(\alpha+\mu+1)}{\Gamma(\mu+\gamma-n+1)\Gamma(\alpha+\beta+n+\mu+2)\Gamma(n+1)}(1-\rho)^n P_{\mu-n}^{\alpha+\beta+2n+1,\gamma}(\rho), \quad (45)$$

where $0\le n\le\mu$ if μ is a non-negative integer, or equivalently, calling $\rho=\frac{1+r}{1-r}$,

$$\int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}(\cos 2\phi)\left[1-r\sin^2\phi\right]^{\mu} P_{\mu}^{\alpha,\gamma}\left(\frac{1+r\sin^2\phi}{1-r\sin^2\phi}\right)dm_{\alpha,\beta}(\phi)$$
$$=\frac{(-1)^n\Gamma(\mu+\gamma+1)\Gamma(\beta+n+1)\Gamma(\alpha+\mu+1)}{2\Gamma(\mu+\gamma-n+1)\Gamma(\alpha+\beta+n+\mu+2)\Gamma(n+1)}r^n(1-r)^{\mu-n}P_{\mu-n}^{\alpha+\beta+2n+1,\gamma}\left(\frac{1+r}{1-r}\right); \quad (46)$$

taking $n=0$ we get an S1-type formula

$$\int_0^{\frac{\pi}{2}}\left[(1+\rho)+(1-\rho)\sin^2\phi\right]^{\mu} P_{\mu}^{\alpha,\gamma}\left(\frac{(1+\rho)-(1-\rho)\sin^2\phi}{(1+\rho)+(1-\rho)\sin^2\phi}\right)dm_{\alpha,\beta}(\phi)$$
$$=\frac{2^{\mu-1}\Gamma(\beta+1)\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+\beta+\mu+2)}P_{\mu}^{\alpha+\beta+1,\gamma}(\rho), \quad (47)$$

(this is Askey and Fitch's formula (9), when μ is a non-negative integer), or equivalently

$$\int_0^{\frac{\pi}{2}}\left[1-r\sin^2\phi\right]^{\mu} P_{\mu}^{\alpha,\gamma}\left(\frac{1+r\sin^2\phi}{1-r\sin^2\phi}\right)dm_{\alpha,\beta}(\phi)$$
$$=\frac{\Gamma(\beta+1)\Gamma(\alpha+\mu+1)}{2\Gamma(\alpha+\beta+\mu+2)}(1-r)^{\mu}P_{\mu}^{\alpha+\beta+1,\gamma}\left(\frac{1+r}{1-r}\right). \quad (48)$$

The expansions are (BE)

$$(\rho+1)^{\mu} F_4\left(-\mu,-\mu-\gamma;\alpha+1,\beta+1;\frac{\rho-1}{\rho+1}\sin^2\theta\sin^2\phi,\frac{\rho-1}{\rho+1}\cos^2\theta\cos^2\phi\right)$$
$$=\sum_{n=0}^{\infty}\frac{2^{\mu-n-1}\Gamma(\alpha+1)\Gamma(\beta+1)(-\mu)_n(-\gamma-\mu)_n\Gamma(\mu-n+1)}{\Gamma(\alpha+\beta+n+\mu+2)}(\rho-1)^n P_{\mu-n}^{\alpha+\beta+2n+1,\gamma}(\rho)$$
$$\times p_n^{\alpha,\beta}(\cos 2\theta)\,p_n^{\alpha,\beta}(\cos 2\phi), \quad (49)$$

and (TE)

$$\begin{aligned}&\left[(1+\rho)+(1-\rho)\sin^2\phi\right]^{\mu} P_{\mu}^{\alpha,\gamma}\left(\frac{(1+\rho)-(1-\rho)\sin^2\phi}{(1+\rho)+(1-\rho)\sin^2\phi}\right)\\ &\quad=\sum_{n=0}^{\infty}\frac{2^{\mu-n}\left(\alpha+\beta+2n+1\right)\Gamma\left(\alpha+\beta+n+1\right)\left(-\gamma-\mu\right)_n\Gamma\left(\alpha+\mu+1\right)}{\Gamma\left(\alpha+\beta+n+\mu+2\right)\Gamma\left(\alpha+n+1\right)}\left(\rho-1\right)^n P_{\mu-n}^{\alpha+\beta+2n+1,\gamma}(\rho)\\ &\qquad\times P_n^{\alpha,\beta}\left(\cos 2\phi\right),\end{aligned}\tag{50}$$

or equivalently

$$\begin{aligned}&\left[1-r\sin^2\phi\right]^{\mu} P_{\mu}^{\alpha,\gamma}\left(\frac{1+r\sin^2\phi}{1-r\sin^2\phi}\right)\\ &\quad=\sum_{n=0}^{\infty}\frac{\left(\alpha+\beta+2n+1\right)\left(-\gamma-\mu\right)_n\Gamma\left(\alpha+\beta+n+1\right)\Gamma\left(\alpha+\mu+1\right)}{\Gamma\left(\alpha+\beta+n+\mu+2\right)\Gamma\left(\alpha+n+1\right)}\\ &\qquad\times r^n\left(1-r\right)^{\mu-n} P_{\mu-n}^{\alpha+\beta+2n+1,\gamma}\left(\frac{1+r}{1-r}\right)P_n^{\alpha,\beta}\left(\cos 2\phi\right),\end{aligned}\tag{51}$$

where the last three series become a finite sum (with $n=0,\ldots,\mu$) when μ is a non-negative integer.

4.3 Gegenbauer polynomials

The formulas from this subsection follow from those in the previous subsection, by means of the identities

$$P_k^{\alpha+\frac{1}{2}}(r)=\begin{cases}\frac{\sqrt{\pi}\Gamma\left(\alpha+m+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}P_m^{\alpha,-\frac{1}{2}}\left(2r^2-1\right) & \text{if } k=2m\\[2ex] \frac{\sqrt{\pi}\Gamma\left(\alpha+m+\frac{3}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(m+\frac{3}{2}\right)}rP_m^{\alpha,\frac{1}{2}}\left(2r^2-1\right) & \text{if } k=2m+1.\end{cases}$$

Thus, taking $\gamma=-\frac{1}{2}$ and $k=2\mu$, or $\gamma=\frac{1}{2}$ and $k=2\mu+1$ in formulas (43), (44), (45), (47), (49) and (50) according to whether k is even or odd respectively, and letting $\rho=2r^2-1$ one obtains (BI)

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}\left(\cos 2\phi\right) r^k F_4\left(-\frac{k}{2},-\frac{k-1}{2};\alpha+1,\beta+1;\frac{r^2-1}{r^2}\sin^2\theta\sin^2\phi,\frac{r^2-1}{r^2}\cos^2\theta\cos^2\phi\right)dm_{\alpha,\beta}(\phi)\\ &\quad=\frac{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)\left(-\frac{k}{2}\right)_n\left(-\frac{k-1}{2}\right)_n\Gamma\left(\frac{k}{2}-n+1\right)\Gamma\left(\frac{k}{2}-n+\frac{1}{2}\right)\Gamma\left(\alpha+\beta+2n+\frac{3}{2}\right)}{2\sqrt{\pi}\Gamma\left(\alpha+\beta+n+\frac{k}{2}+2\right)\Gamma\left(\alpha+\beta+n+\frac{k}{2}+\frac{3}{2}\right)}\\ &\qquad\times\left(r^2-1\right)^n P_{k-2n}^{\alpha+\beta+2n+\frac{3}{2}}(r)\,P_n^{\alpha,\beta}\left(\cos 2\theta\right)\end{aligned}\tag{52}$$

if $0\le 2n\le k$; for $n=0$ we have (S2)

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} r^k F_4\left(-\frac{k}{2},-\frac{k-1}{2};\alpha+1,\beta+1;\frac{r^2-1}{r^2}\sin^2\theta\sin^2\phi,\frac{r^2-1}{r^2}\cos^2\theta\cos^2\phi\right)dm_{\alpha,\beta}(\phi)\\ &\quad=\frac{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{2\sqrt{\pi}\Gamma\left(\alpha+\beta+\frac{k}{2}+2\right)\Gamma\left(\alpha+\beta+\frac{k}{2}+\frac{3}{2}\right)}P_k^{\alpha+\beta+\frac{3}{2}}(r);\end{aligned}\tag{53}$$

calling $r=\cos\psi$, and letting $\theta=\pi/2$ in (52), we obtain (TI)

$$\int_0^{\frac{\pi}{2}} P_n^{\alpha,\beta}(\cos 2\phi)\left(1-\sin^2\psi\cos^2\phi\right)^{\frac{k}{2}} P_k^{\alpha+\frac{1}{2}}\left(\frac{\cos\psi}{\sqrt{1-\sin^2\psi\cos^2\phi}}\right) dm_{\alpha,\beta}(\phi)$$
$$=\frac{\Gamma\left(\alpha+\beta+2n+\frac{3}{2}\right)\Gamma(\beta+n+1)\Gamma\left(\alpha+\frac{k}{2}+1\right)\Gamma\left(\alpha+\frac{k}{2}+\frac{1}{2}\right)}{2\Gamma\left(\alpha+\beta+n+\frac{k}{2}+2\right)\Gamma\left(\alpha+\beta+n+\frac{k}{2}+\frac{3}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(n+1)}\sin^{2n}\psi P_{k-2n}^{\alpha+\beta+2n+\frac{3}{2}}(\cos\psi) \quad (54)$$

if $0\le 2n\le k$; for $n=0$ we have (S1)

$$\int_0^{\frac{\pi}{2}} \left(1-\sin^2\psi\cos^2\phi\right)^{\frac{k}{2}} P_k^{\alpha+\frac{1}{2}}\left(\frac{\cos\psi}{\sqrt{1-\sin^2\psi\cos^2\phi}}\right) dm_{\alpha,\beta}(\phi)$$
$$=\frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)\Gamma(\beta+1)\Gamma\left(\alpha+\frac{k}{2}+1\right)\Gamma\left(\alpha+\frac{k}{2}+\frac{1}{2}\right)}{2\Gamma\left(\alpha+\beta+\frac{k}{2}+2\right)\Gamma\left(\alpha+\beta+\frac{k}{2}+\frac{3}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}P_k^{\alpha+\beta+\frac{3}{2}}(\cos\psi) \quad (55)$$

(this was Feldheim's formula (8)); as for the expansion formulas, if we denote by $\left[\frac{k}{2}\right]$ the greatest integer smaller than or equal to $\frac{k}{2}$, we obtain (BE)

$$r^k F_4\left(-\frac{k}{2},-\frac{k-1}{2};\alpha+1,\beta+1;\frac{r^2-1}{r^2}\sin^2\theta\sin^2\phi,\frac{r^2-1}{r^2}\cos^2\theta\cos^2\phi\right)$$
$$=\sum_{n=0}^{\left[\frac{k}{2}\right]}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\left(-\frac{k}{2}\right)_n\left(-\frac{k-1}{2}\right)_n\Gamma\left(\frac{k}{2}-n+1\right)\Gamma\left(\frac{k}{2}-n+\frac{1}{2}\right)\Gamma\left(\alpha+\beta+2n+\frac{3}{2}\right)}{2\sqrt{\pi}\Gamma\left(\alpha+\beta+n+\frac{k}{2}+2\right)\Gamma\left(\alpha+\beta+n+\frac{k}{2}+\frac{3}{2}\right)}$$
$$\times\left(r^2-1\right)^n P_{k-2n}^{\alpha+\beta+2n+\frac{3}{2}}(r)\, p_n^{\alpha,\beta}(\cos 2\theta)\, p_n^{\alpha,\beta}(\cos 2\phi); \quad (56)$$

and for $\theta=\pi/2$ and $r=\cos\psi$,we have (TE)

$$\left(1-\sin^2\psi\cos^2\phi\right)^{\frac{k}{2}} P_k^{\alpha+\frac{1}{2}}\left(\frac{\cos\psi}{\sqrt{1-\sin^2\psi\cos^2\phi}}\right)$$
$$=\sum_{n=0}^{\left[\frac{k}{2}\right]}\frac{(\alpha+\beta+2n+1)\Gamma\left(\alpha+\beta+2n+\frac{3}{2}\right)\Gamma(\alpha+\beta+n+1)\Gamma\left(\alpha+\frac{k}{2}+1\right)\Gamma\left(\alpha+\frac{k}{2}+\frac{1}{2}\right)}{\Gamma\left(\alpha+\beta+n+\frac{k}{2}+2\right)\Gamma\left(\alpha+\beta+n+\frac{k}{2}+\frac{3}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha+n+1)}$$
$$\times\sin^{2n}\psi P_{k-2n}^{\alpha+\beta+2n+\frac{3}{2}}(\cos\psi)\, P_n^{\alpha,\beta}(\cos 2\phi). \quad (57)$$

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