University of Bergamo

Department of Management, Economics and Quantitative Methods

Doctoral Thesis

Pension Fund Management in a Stochastic Optimization Framework

Author:

Sebastiano Vitali

Supervisor:

Prof. Vittorio Moriggia

External Supervisor:

Prof. Jitka Dupačová

Doctor of Philosophy in

Economics, Applied Mathematics and Operational Research

Academic Year 2014-2015

to my uncles and aunts,

praesidii iuventutis meae

ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to my supervisor Prof. Vittorio Moriggia, a mentor and a friend, for his trust and continuous encouragement. A special thank to my external supervisor Prof. Jitka Dupačová for her advices and immense knowledge.

I am grateful to Prof. Miloš Kopa for guiding my research and for pushing me to my best.

A special thank to the director of the Ph.D. School, Prof. Marida Bertocchi, for promoting and supporting me.

I would like to thank Dr. Sandra L. Schwartz and Prof. William T. Ziemba for their useful advice concerning the whole thesis.

I would like to thank Prof. Adriana Gnudi for her caring and patience, Prof. Giorgio Consigli and Prof. Sergio Ortobelli for helping me with their useful suggestions.

I want also to thank my Ph.D. colleague Marco Cassader, my friends at Charles University in Prague, in particular Václav Kozmík and Karel Lavička, and the RiskLab team for fruitful discussions.

Great thanks to Andrea, Battista and Stefano for being there in good and rough times.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

PREFACE

The pension system has become more and more complex and structured all over Europe in the last decades. Because of the financial and social crisis, several countries implemented strong reforms in the state welfare in order to reduce the pension costs on the state budget balance. Furthermore, they allowed and encouraged the establishment of private pension facilities. We have to consider the three pillars. The first one concerns the state pension system. The second concerns the relationship between the employer and the employee. The third pillar consists in individual investment plans typically issued by insurance companies.

In general, a private pension fund is an investment fund which receives periodical contributions from a private investor and then provides an annuity during the retirement. The main function of a pension plan is to avoid the risk that the subscriber might survive his/her savings. Then, a reasonable aim for this kind of investment would be to guarantee an integration of the public retirement pension so that the total income before and after retirement does not differ substantially. To satisfy this target is crucial to understand the difference between Defined Benefit Fund (DBF) and Defined Contribution Fund (DCF). The DBFs are characterized by a mutual structure meaning that the benefit during the retirement is fixed and independent from the total amount of the contribution accumulated in the fund. This implies that if the final wealth is not enough to support the benefit, the fund draws money from the current contributions of younger investors. The DCFs establishes the level of the contribution. Then the benefit during retirement depends on the total accumulated wealth and on some actuarial considerations. The DCF thus becomes similar to private savings in the sense that the subscriber bears the risk. A key issue is that there is a contribution by the employer and an aspect of forced saving.

Italy decided to move the first pillar from the DBF to the DCF system. The reason is based on the evidence that the DCF structure is more sustainable considering the increasing life expectancy perspective even if it usually provides a lower financial benefit. The second and third pillars instead are historically based predominantly on DCFs.

In this thesis, we analyze a pension fund from the point of view of the three main actors involved in it: the pension plan provider, who decides the tactical allocation of the pension funds, the fund manager who takes care of the strategical investment problem, and the individual investor who faces the problem to allocate his/her savings in a retirement perspective. All these problems involve the analysis of a long-term choice and require to take into consideration some elements of uncertainty. Therefore, it is natural to face them using the instruments provided by Stochastic Programming.

In Chapter 1 a theoretical overview of the pension problem is proposed. In particular, Section 1.1 analyzes the main reasons which strengthen and highlight the urgency and the complexity of this issue, while in Sections 1.2 and 1.3 we report the state-of-the-art literature quoting the milestone works which made the history of the ALM and of the pension investment field.

Chapter 2 is dedicated to the stochastic optimization methods. Section 2.1 focuses on the two-stage and successively on the multistage stochastic problems giving a few hints about their formulation and classification. In Section 2.2 the main scenario generation techniques are listed. Section 2.3 gives an introduction to the multiple criteria objective formulation, exploring the ways to make it tractable both in an algebraic and in a computational point of view. In Section 2.4 the chance constraint setting is analyzed with particular reference to the definition of the Value at Risk and the Average Value at Risk constraints as well as the stochastic dominance constraints.

Chapter 3 briefly focuses on the pension manager problem which implies the adoption of an Asset and Liability Management model for which we propose a general description. Moreover, we propose a portfolio replication pricing approach for the liability side in case the annuities are inflation linked.

Chapter 4 proposes a stochastic formulation of the problem of the pension plan provider who has to decide the allocation guidelines for each pension funds proposed by the pension plan. In Section 4.1 a statistical approach is introduced in order to analyze the population for whom the pension funds have been issued. In Section 4.2 we propose a linear stochastic model in order to define the optimal pension funds which should compose the pension plan.

Chapter 5 deals the individual investment problem in a retirement perspective. In Section 5.1 we propose an extension of the model introduced in Section 4.2. We add other constraints to define specific regulatory features and to consider the individual investor's behavior as well as a stochastic dominance framework which characterizes the model in three different formulations. Here a single case study is analyzed and the results according to the three proposed formulations are shown in Sections 5.3.1, 5.3.2 and 5.3.4.

Chapter 6 summarizes the main conclusions.

1 THE PENSION PROBLEM

Pension problems concern a wide part of society. Many of us rely on the public retirement pension system, others have a specific pension protection due to their profession. Nevertheless, because of many welfare state cutbacks, a growing number of people are aware that pension savings must be increased penalizing current consumption and having a great concern about owns future. Furthermore, the retirement savings have to be administered adopting typical investment strategies and the implemented one must consider several features which are usually ignored in an individual portfolio optimization.

1.1 INTRODUCTION

World population is getting older. As we observe in Table 1, the average population age has constantly been increasing through the decades. This evidence is more remarkable in the developed countries where the average age has increased by nine years from 1950 to 2010 and in some countries even by eleven (Italy) or eighteen (Japan) years.

The retirement issue has not always been a problem. Indeed, until fifty years ago, the working age almost coincided with the lifetime expectancy: it is not like this any longer. The lifetime expectancy has prolonged and the wealth conditions have bet-

Region/Country	1950	1970	1990	2000	2010
World	26,88	26,00	27,51	29,09	30,98
Developed countries	30,95	32,72	35,86	37,82	39,81
Less developed countries	24,95	23,47	25,21	26,98	29,04
Japan	26,18	30,97	37,07	40,86	44,34
Germany	34,58	35,70	38,83	40,41	43,04
Italy	31,30	34,21	38,30	41,05	42,93
France	34,81	34,43	36,49	38,26	39,84
UK	34,98	35,48	37,47	38,46	39,57
USA	31,60	31,92	34,76	35,79	37,34
China	26,79	24,16	27,81	30,95	34,69

Table 1: Average age

Source: United Nations - Department of Economic and Social Affairs

tered, but not with the same ratio. This means that the welfare system can increase the retirement age, but not in the same measure as the life expectancy grows. However, it is acceptable to retire at 55 considering to live up to 65, is becomes hard imagining to to retire at 85 expecting to live up to 95. As shown in Table 2, the expected life is increasing too and this means that in the future we must expect a longer period of our retirement lives. The increasing life expectancy is not a problem in its own. Indeed, in a mutualistic system in which the pension savings of the workers succeed in paying the annuity of the pensioners, a suitable ratio workers-pensioners ensures a stable sustainability. Derangements arise if the number of pensioners is not properly balanced by the number of working people. Some types of work are physically heavier than others and some people react differently to work efforts, therefore, let assume that the average threshold age to retire is 65

Region	1950	1970	1990	2000	2009
World	45,51	57,93	64,43	66,31	69,14
Developed countries	63,17	70,68	74,03	75,10	77,17
Less developed countries	40,21	54,62	62,27	64,47	67,40
Japan	60,18	72,23	79,02	81,17	82,93
Italy	65,27	71,48	76,86	79,45	81,75
France	66,05	71,81	76,65	78,90	81,17
Germany	66,90	70,88	75,44	77,90	80,05
UK	68,56	71,90	75,59	77,69	79,91
USA	68,00	70,68	75,24	76,77	78,35
China	43.31	63,00	69,45	72,12	74,72

Table 2: Life expectancy

Source: United Nations - Department of Economic and Social Affairs

years. Then, a worrying issue is the growing proportion of the population older than 65, as shown in Table 3. And in particular, the reducing ration between the working population (20-65 years) and the non-working population (older than 65 years), as shown in Table 4. These tables show that in some countries (Japan, Italy and Germany) more than one person out of five is already retired and that for each pensioner there are only three workers. In a mutualistic framework, typical of a Defined Benefit system, this value is unsustainable because each month the pension savings of three workers are barely enough to sustain financially the life costs of a pensioner. In a Defined Contribution structure the issue is not mutualistically relevant but it still is individually. Indeed, if the pension saving is personal, what everyone saves in forty working years must be sufficient to pay others thirty years of life expenses.

Region	1950	1970	1990	2000	2010
World	5.08	5.36	6.21	6.90	7.67
Developed countries	7.71	9.88	12.54	14.32	16.07
Less developed countries	3.84	3.66	4.48	5.10	5.83
Japan	4,95	7,03	11,95	17,18	22,96
Germany	9,57	13,56	14,99	16,33	20,81
Italy	8,09	11,07	14,93	18,26	20,29
France	11,39	12,87	14,13	16,02	16,80
UK	10,83	13,03	15,70	15,80	16,59
USA	8,26	9,76	12,47	12,36	13,06
China	4,48	3,95	5,78	6,87	8,35

Table 3: Population aged over 65, %

Source: United Nations - Department of Economic and Social Affairs

Region	1950	1970	1990	2000	2010
World	10.05	8.79	8.23	7.78	7.42
Developed countries	7.33	5.62	4.76	4.22	3.82
Less developed countries	12.65	12.01	10.91	10.19	9.59
Japan	9,98	8,53	5,15	3,62	2,57
Germany	6,26	4,15	4,22	3,81	2,91
Italy	6,99	5,16	4,07	3,40	3,00
France	5,13	4,21	4,13	3,64	3,50
UK	5,58	4,30	3,72	3,73	3,59
USA	6,97	5,40	4,71	4,78	4,59
China	11,55	11,32	9,43	8,61	7,85

Table 4: Ratio of population aged 20-65 to population aged over 65

Source: United Nations - Department of Economic and Social Affairs

A major purpose for a pension system should be to offer to each pensioner a decent and perpetual annuity. The relation between last salary and retirement annuity amount changes from country to country and should consider the whole welfare system, especially health care costs. In any case, the population dynamics generate implicitly a risk in everyone's retirement future. Clearly, this risk can be measured, modeled and managed. Similarly, also the financial risk can be measured, modeled and managed. It would be shortsighted to think that this problem concerns only a pension fund provider or a pension fund manager. We believe that everyone must play his/her part, included the worker who, too often, does not care properly about his/her retirement issue.[1]

1.2 PENSION MANAGER ALM PROBLEM

Several models have been implemented in order to solve financial real life problem. Probably, the first one is BONDS model proposed by Bradley and Crane (1972)[2]. With this model the authors help the bond portfolio manager to take decision in an uncertain environment, but they do not explore the liabilities side in depth. Thus, we consider it mainly as an asset model. At the end of the eighties and the beginning of the nineties, the MIDAS model has been developed and several other ALM models have been proposed, cf. Kusy and Ziemba (1986). In Dempster and Ireland (1988) we find the description of the first real asset and liability model focused on the immunization of the liabil-

1 Cf. Schwartz and Ziemba (2007)
2 See also Bradley and Crane (1980).

ity side. The model is formulated in order to be applied to a real electricity company and to handle its debt management properly (see also Dempster and Ireland (1989) and Dempster and Ireland (1991)). One of the most cited is the Russell-Yasuda Kasai Model described firstly in Cariño et al. (1994) and then analyzed in Cariño and Ziemba (1998a) and Cariño et al. (1998b). The model is based on a multistage stochastic programming. The solution of the model is an optimal strategy which defines risks in a practical operative way. It also considers the complex Japanese insurance laws in order to produce high returns adequate to pay annual interest on savings-type insurance policies without sacrificing the aim of maximizing the long-term wealth. The main features of the model are the following:

- to represent the book and market value goals of the company's business coherently,
- to incorporate the regulation,
- to reflect multiple and conflicting targets: maximizing the profit but still offering a high quality service to the customers,
- to reflect the multiperiod nature of the goals and constraints,
- to reflect the uncertainty of the financial markets,
- to solve the problem in less than three hours,
- to have a solution believable and understandable by managers.

It is clear that these goals are shared by almost all the pension fund models developed for insurance companies and banks.

For this reason the Russell-Yasuda Kasai Model represents a milestone in the pension ALM field.

In the same years interesting and complete analysis of the multiperiod stochastic problem applied to ALM problems considering the fixed income investment has been done by Mulvey and Zenios.[3] Later on, Mulvey's research on the Tower Perrin scenario generation system evolved into the well known Towers Perrin-Tillinghast ALM model (see Mulvey et al. (2000)).

A fundamental model proposed by Consigli and Dempster is the CALM model.[4] It is specifically designed to handle the pension fund management problem covering a long-term period and considering as liabilities five different pension contracts. A similar approach proposed again by Dempster can be found in Dempster et al. (2003). Another author who proposed innovative and path-breaking solutions is Ziemba who formulates an optimal planning under uncertainty in Kallberg et al. (1982). Then, his research[5] produced, among others[6], the milestone InnoALM model for the multistage managing of the pension fund of an electricity company (see Geyer and Ziemba (2008), Ziemba (2007)). A specific focus for the defined benefit pension fund can be found in Dert (1998). Pflug and Świetanowski (1999) pay a particular attention to both the asset and the liability side modeling. Therefore, the suitability of multistage approach to deal with ALM problems has been proved correct on multiple occasions during the last twenty years. Neverthe-

3 See e.g. Mulvey (1994b), Mulvey (1989), Mulvey (1994a), Mulvey and Zenios (1994), Nielsen and Zenios (1996), Zenios (1993), Zenios (1995) and Zenios et al. (1998).

4 See e.g. Consigli and Dempster (1998b) and Consigli and Dempster (1998a).

5 For an overview refer to Ziemba (2003).

6 Ziemba is one of the author of the Russell-Yasuda Kasai Model, see Cariño et al. (1994), Cariño and Ziemba (1998a) and Cariño et al. (1998b).

less, we still need Mulvey et al. (2006) who suggest again a multiperiod model to increase the understanding of risks and rewards in a multistage framework for pension plans and other long-term investors (see also Mulvey et al. (2007) and Mulvey et al. (2008)). Recent and innovative formulations of the ALM problem for pension fund are proposed by Consigli in Consigli and di Tria (2012) and by Consigli and Moriggia in Consigli et al. (2011) and Consigli and Moriggia (2014).

Each country required an adjustment of a general model in order to consider the specificity of the country's regulations. Some models have been built starting from the country pension system. Høyland and Wallace (2001a) analyze the Norwegian regulations; Dupačová and Polívka (2009) focus on the Czech Republic scheme; Hilli et al. (2007) study the case of a Finnish pension company; Kouwenberg (2001), Bovenberg and Knaap (2005), Streutker et al. (2007), Haneveld et al. (2009) and Haneveld et al. (2010) explore the Dutch system; Dondi et al. (2007) analyze the Swiss setting; in Fabozzi et al. (2005) there is a comparison among 28 defined benefit pension funds of the Netherlands, Switzerland, the United Kingdom, and the United States.

ALM for pension fund must consider the evolution of the asset universe according to regulatory updates (e.g. McKendall et al. (1994), Davis (2000), Gollier (2008) and the books of Guiso et al. (2002), Hardy (2003) and Broeders et al. (2009)). However, the fixed income securities are still the widest used asset class. A particular focus on bond sensitivity can be found in Bertocchi et al. (2000a), Bertocchi et al. (2000b) and Abaffy et al.

(2007), while an example of bond portfolio strategy is in Ortobelli Lozza et al. (2013).[7]

We cannot forget, among others, the comprehensive collections of Ziemba and Mulvey (1998) and Zenios and Ziemba (2006, 2007).

1.3 INDIVIDUAL PENSION PROBLEM

The individual asset allocation problem has first been investigated by Merton (1969, 1971) who introduced the concept of consumption and optimal investment through a dynamic programming approach in order to maximize the utility for a private investor in a fixed time horizon. Richard (1975) introduces the concept of lifetime uncertainty, labor and insured wealth as further elements to take into account[8]. Berger and Mulvey (1998) propose a tool named Home Account Advisor, a multistage model which optimizes the investor financial objective considering simultaneously investments, savings and borrowing. The main innovative features of the model which have become a standard for future works are:

- the stochastic system generates stochastic scenarios for a long-term run,

7 For a unifying work on bond market refer to Bertocchi et al. (2013)

8 See also Geyer et al. (2009) who replicate Richard's model by using a multistage stochastic programming approach and argue that dynamic programming and stochastic linear programming can be combined and integrated into one SLP formulation.

- multiobjective preference function for inter temporal trade-off among savings, consumption and wealth accumulation,
- well defined set of rules for investing, borrowing and income management,
- optimization of the decision rules to recommend an individual investment strategy.

Several contributions deal with the individual asset allocation problem not necessarily in a pension perspective. The main feature of this class of models is to consider jointly all the variables which characterize the investor's investment, i.e. the salary process, the consumption, the borrowings, etc. See for example Consiglio et al. (2004, 2007) Consigli (2007), Moriggia et al. (2013) and Medova et al. (2008). Moreover, institutional investors usually have only one objective (low risk, high returns, passive or active strategy, etc.) while individual investors have more goals, for example to obtain a suitable retirement annuity minimizing the risk. For this reason, often these models face multiple targets problems. A common approach is to divide the portfolio in sub-portfolios in order to dedicate each sub-portfolio to each target (e.g. Brunel (2003) and Chhabra (2005)). More recent works propose a unique model without sub-portfolios. For example Cai and Ge (2012) consider a loss-aversion objective which minimizes a loss risk measure and maximizes the final wealth, having as constraint a minimum level of wealth in order to maintain the life style. In this way they also consider dynamically the investment behavior.

Indeed, for individual investor's strategies, it is imperative to know the way in which the investor takes a decision (see

Kahneman and Tversky (1979)). Many works deal with behavioral finance and propose models which translate behaviors into constraints. For example Blake et al. (2013) assume that under loss-aversion the investor's strategy is target-driven, i.e. the risky allocation increases if the accumulating fund is below the target and decreases if it is above. Moreover, if the fund is sufficiently above the target, the optimal investment strategy switches to portfolio insurance. The description of the investors' investment attitude is not trivial. The authors show that investors tend to become risk averse when winning and so they sell winning investments too quickly; on the other hand they take extra risks when they are making losses. For this reason the investment strategy is target-driven while the risk attitude is path-driven.

Lim and Shin (2011) investigate the borrowing constraints supposing that the private investor has limited opportunities to borrow in the market and so he/she cannot totally insure the risk of income fluctuation. Moreover, they suppose that the labor income is continuous, the disutility comes from labor and that the immortal agent can decide when to retire.

Munk and Sørensen (2004) propose a model for optimal consumption in an uncertain interest rates environment. The authors characterize the solution to the consumption and investment problem of a power utility investor in a continuous-time dynamically complete market with stochastic changes in the opportunity set. Under stochastic interest rates the investor optimally hedges against changes in the term structure of interest rates by investing in a coupon bond, or portfolio of bonds, with a payment schedule that equals the forward-expected (i.e. certainty equivalent) consumption pattern. They found that the hedged portfolio is more sensitive to the form of the term structure than to the dynamics of interest rates. Cairns et al. (2006) investigate the impact of uncertain interest rates in a private investor asset allocation problem.

A specific focus on household investment strategies is proposed in Bruhn and Steffensen (2011) where the authors optimize the consumption of the household underlining several differences among its members, and lead to closed form solutions for the optimal control of their investments, for the consumption and for the purchase of a life insurance for each member. An overview about householders can be found in Guiso et al. (2002). Another analysis about the optimal investment-consumption strategy with mortality risk and environment uncertainty in a multiperiod framework is discussed in Li et al. (2008).

In the last decade, many articles have dealt with the private investor problem keeping in mind a pension perspective.

For individual pension strategies not only do several papers focus on the accumulation phase but also on the decumulation phase (e.g. Milevsky and Young (2007), Horneff et al. (2008), Gerrard et al. (2004, 2006, 2012) and Consigli et al. (2012). In particular, in Gerrard et al. (2012) the authors formulate and solve a problem of finding the optimal time of annuitization for a retiree who has the possibility to choose his/her own investment and consumption strategy. They formulate the model as a combined stochastic control and optimal stopping problem. As a criterion for the optimization they select a loss function that penalizes both the deviance of the running consumption rate from a desired consumption rate and the deviance of the final wealth at the time of the annuitization from a desired target.

The financial risk is then split into two parts: the investment risk, during the accumulation phase, and the annuity risk, during the retirement, where as annuity risk they consider the risk that high annuity prices (driven by low bond yields) at retirement can lead to a pension income lower than expected. As in the ALM for pension fund, in the individual pension planning is fundamental to take care of the evolution of the available products which can be used to implement an optimal strategy. For example, Consigli et al. (2012) show an extension of a typical ALM individual investor model to include a pension strategy. They investigate a selection of different retirement products with different risk exposures and financial payoffs. In particular they peruse three asset classes: pension funds, unit-linked products and variable life annuities (life insurance lifecase). One of the purpose of the analysis is to explain that a pension fund is not enough to provide a suitable pension strategy but it is necessary to include also other asset classes. They focus on the sensitivity of the optimal choice according to male and female pension coefficients, to an increasing planning horizon, to an objective function based on pension wealth target and to an increasing inflation in individual consumption.

Several further features of the individual pension problem have been discussed in the last decade. An individual optimal asset allocation under a regime switching model is explored in Cheung and Yang (2007) and Siu (2011). Chen et al. (2006) and Pliska and Ye (2007) study the adoption of the life insurance to hedge human capital against death. Yaari (1965) has been the first author to include an insurance decision in a personal finance optimization problem. A survey of the individual pension problem can be found in Bertocchi et al. (2010).

2 STOCHASTIC OPTIMIZATION

Real life problems are complex. The number of variables, the structure of the decisional process and the hugeness of the uncertainty regarding the future state of a system require a suitable approach. Quantitative economical and financial issues, energy production, logistic scheduling are just some examples of problems which ask mathematical programming for answers. When the problem consists in the research of an optimal choice in an uncertain environment the most used approach is the Stochastic Programming. The different classes of stochastic problems and their properties are formalized and classified in Birge and Louveaux (1997, 2011). In this chapter we are going to describe two general settings of stochastic problems according to the formulation proposed in Shapiro et al. (2009): the two-stage and the multistage formulation and some of their properties. Moreover, some techniques about the stochasticity representation are investigated. After that, we are going to define the multiobjective problems as suggested in Dupačová et al. (2002) and to introduce the stochastic dominance framework as investigated in Dupačová and Kopa (2014). In the chance constraint section, we are going to face the risk issue. Indeed, in financial mathematics, one of the main challenges is modeling and measuring risk. The definition of risk itself has been revised several times.[1] Intuitively, uncertainty and risk are associatively linked. A financial meaning of risk could correspond

1 Cf. Hanoch and Levy (1969).

to the idea of loss and in general to the idea of shortfall as well. Our belief is that there is something certain about uncertainty and, as Pflug and Römisch say, that is: real applications need to manage risk, risk can only be measured if it is modeled and can only be managed if it is measured.[2]

2.1 STOCHASTIC PROBLEM FORMULATION

Stochastic optimization problems occur in almost all science areas: finance, engineering, medicine, physics, aeronautics, logistics[3]. Whenever someone needs to solve an optimal problem according to some stochastic conditions, the most used tool is stochastic optimization. It represents a unifying framework in which mathematics, statistics and optimization theory work together. The first models considering a decision and a future reaction to a stochastic observation appear in Beale (1955) and in Dantzig (1955). Subsequent analysis, according to the mathematical properties of this class of models, was made by Wets (1974). On the other hand, the class of models which introduces probability in the constrains and not in the future setting was also developed quite early, mainly by Charnes and Cooper (1959). Convexity conditions for chance constrained problems were derived by Prékopa (1971). Since these path-breaking works, researches started to analyze mathematical properties and advanced solution techniques more deeply.[4] A well known book which developed and spread out this topic is Birge and

2 Cf. Pflug and Römisch (2007)

3 One of the earliest application includes an airline fleet-assignment model by Ferguson and Dantzig (1956).

4 For an overview of the literature until the end of nineties see Birge (1997).

Louveaux (1997)[5]. Referring to the financial world, one of the first work which applies the stochastic programming is Shapiro (1988) who suggests an evolution of the classical immunization techniques in a dynamic perspective, see also Hiller and Shapiro (1989) and Hiller and Eckstein (1993). A recent unifying work is Powell (2014). For an overview of the stochastic modeling for financial application refer to Dupačová et al. (2002) and to Ziemba and Vickson (1975, 2006).

2.1.1 Two-stage Stochastic Problem

The simplest case of *stochastic program* refers to a problem in which we take a decision today and adjust the choice in the future after the observation of an uncertain variable. This kind of formulation is called *two-stage stochastic problem*. The first stage decision x must be made before the realization $\tilde{\xi}$ of a stochastic element ξ is known. Typically, ξ affects some constraints of the model, then we can distinguish between deterministic and stochastic constraints. When the objective function and all the constraints are linear, the model assumes the following form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{T}(\xi)\mathbf{x} = \mathbf{h}(\xi), \\ & \mathbf{x} \geqslant 0 \end{aligned}$$

where $\mathbf{x} \in \mathfrak{R}^n$, $\mathbf{T}(\xi) \in \mathfrak{R}^{r \times n}$, $\mathbf{h}(\xi) \in \mathfrak{R}^r$ and $\xi \in \Xi \subseteq \mathfrak{R}^n$ with Ξ the support of ξ and P its probability distribution. To deal with this kind of problem we need a method to measure the error on

5 For a more recent overview of the topic see Kall and Mayer (2005), Shapiro and Philpott (2007) and Birge and Louveaux (2011).

the second stage once we know the realization $\tilde{\xi}$. We define the recourse variable $\mathbf{y}(\tilde{\xi}) \in \mathfrak{R}^m$ and the recurse matrix $\mathbf{W} \in \mathfrak{R}^{r \times m}$ to quantify the compensation:

$$\mathbf{W}\mathbf{y}(\tilde{\xi}) = \mathbf{h}(\tilde{\xi}) - \mathbf{T}(\tilde{\xi})\mathbf{x}$$

Generally, also $\mathbf{W}$ could depend on ξ but we assume that it is deterministic (*fixed recourse problem*). Then, we introduce the vector $\mathbf{q}(\tilde{\xi})$ to compute the cost of the compensation and we formulate the second stage optimization problem in order to minimize this cost:

$$\begin{aligned} \min_{\mathbf{y}} \quad & \mathbf{q}(\tilde{\xi})^\top \mathbf{y}(\tilde{\xi}) \\ \text{s.t.} \quad & \mathbf{W}\mathbf{y}(\tilde{\xi}) = \mathbf{h}(\tilde{\xi}) - \mathbf{T}(\tilde{\xi})\mathbf{x}, \\ & \mathbf{y}(\tilde{\xi}) \geqslant 0 \end{aligned}$$

The function $Q(\mathbf{x}, \tilde{\xi})$ represents the optimal value of the second stage problem:

$$Q(\mathbf{x}, \tilde{\xi}) = \min_{\mathbf{y}} \left\{ \mathbf{q}(\tilde{\xi})^\top \mathbf{y} : \mathbf{W}\mathbf{y} = \mathbf{h}(\tilde{\xi}) - \mathbf{T}(\tilde{\xi})\mathbf{x},\ \mathbf{y} \geqslant 0 \right\}$$

Finally, we can include this cost function in the first stage objective function as a penalty element. Of course, me must take the expectation of $Q(\mathbf{x}, \tilde{\xi})$ with respect to ξ:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} + \mathbb{E}_{\xi}\left[Q(\mathbf{x}, \xi)\right] \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & Q(\mathbf{x}, \tilde{\xi}) = \min_{\mathbf{y}} \left\{ \mathbf{q}(\tilde{\xi})^\top \mathbf{y} : \mathbf{W}\mathbf{y} = \mathbf{h}(\tilde{\xi}) - \mathbf{T}(\tilde{\xi})\mathbf{x},\ \mathbf{y} \geqslant 0 \right\}, \\ & \mathbf{x} \geqslant 0 \end{aligned}$$

This formulation is called Implicit Representation of the stochastic linear program[6]. Here on, we deal with fixed recourse problems and the stochasticity arises only from parameters $\mathbf{q}, \mathbf{T}$ and

6 Cf. Birge and Louveaux (1997, 2011).

$\mathbf{h}$, even if not explicitly written. The expectation is taken with respect to the probability distribution of the random variable ξ. Let us consider that this distribution has finite support. Thus, ξ can assume a finite number K of realizations called scenarios, each with a probability p_k. Then the expectation assumes the following form

$$\mathbb{E}_\xi\left[Q(\mathbf{x},\xi)\right] = \sum_{k=1}^{K} p_k Q(\mathbf{x},\xi_k)$$

For a given $\mathbf{x}$, the expectation $\mathbb{E}_\xi\left[Q(\mathbf{x},\xi)\right]$ is the optimal value of the second stage problem

$$\begin{aligned} \min_{\mathbf{y}_1,\dots,\mathbf{y}_k} \quad & \sum_{k=1}^{K} p_k \mathbf{q}_k^\top \mathbf{y}_k \\ \text{s.t.} \quad & \mathbf{W}_k\mathbf{y}_k = \mathbf{h}_k - \mathbf{T}_k\mathbf{x}, \\ & \mathbf{y}_k \geqslant 0 \end{aligned}$$

The whole two stage problem becomes equivalent to the following linear programming problem

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}_1,\dots,\mathbf{y}_k} \quad & \mathbf{c}^\top\mathbf{x} + \sum_{k=1}^{K} p_k \mathbf{q}_k^\top \mathbf{y}_k \\ \text{s.t.} \quad & \mathbf{W}_k\mathbf{y}_k = \mathbf{h}_k - \mathbf{T}_k\mathbf{x}, \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant 0, \mathbf{y}_k \geqslant 0 \end{aligned}$$

According to the exposed formulation, the properties of the expected recourse cost problem with a discrete distribution of the stochastic variable derive directly from the properties of parametric linear programming problems.

2.1.2 Nonanticipativity for two-stage problems

Dealing with two-stage stochastic problems we usually adopt the scenario formulation considering a finite number K of realizations of the stochastic variable ξ_k, each with probability p_k. In order to implement the problem in a relatively easy formulation, we habitually structure the code relaxing the here-and-now decision by replacing the vector x with K vectors x_k, one for each scenario. Therefore, the problem assumes the form

$$\begin{aligned} \min_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_K \\ \mathbf{y}_1,\dots,\mathbf{y}_K}} \quad & \mathbf{c}^\top \mathbf{x_k} + \sum_{k=1}^{K} p_k \mathbf{q}_k^\top \mathbf{y}_k \\ \text{s.t.} \quad & \mathbf{W}_k \mathbf{y}_k = \mathbf{h}_k - \mathbf{T}_k \mathbf{x}_k, \\ & \mathbf{A}\mathbf{x}_k = \mathbf{b} \\ & \mathbf{x}_k \geqslant 0, \mathbf{y}_k \geqslant 0 \end{aligned}$$

Moreover, we could split the problem and solve it for each $\mathbf{x}_k$ as follows

$$\begin{aligned} \min_{\substack{\mathbf{x}_k \\ \mathbf{y}_1,\dots,\mathbf{y}_k}} \quad & \mathbf{c}^\top \mathbf{x_k} + \sum_{k=1}^{K} p_k \mathbf{q}_k^\top \mathbf{y}_k \\ \text{s.t.} \quad & \mathbf{W}_k \mathbf{y}_k = \mathbf{h}_k - \mathbf{T}_k \mathbf{x}_k, \\ & \mathbf{A}\mathbf{x}_k = \mathbf{b} \\ & \mathbf{x}_k \geqslant 0, \mathbf{y}_k \geqslant 0 \end{aligned}$$

This formulation is not suitable for solving the whole two-stage problem because the decision variable $\mathbf{x}_k$ depends on the realization of the stochastic variable, i.e. on the values of $\mathbf{q}_k$, $\mathbf{W}_k$, $\mathbf{h}_k$, $\mathbf{T}_k$. Therefore, we have to fix this distortion introducing additional constraints: the *nonanticipativity constraints*. They can assume several forms. The most used form is the following

$$(\mathbf{x}_1, \dots, \mathbf{x}_K) \in \mathcal{L}$$

where $\mathcal{L} = \{\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_K) : \mathbf{x}_1 = \cdots = \mathbf{x}_K\}$.

A second way considers the probability distribution associated to each scenario by imposing each decision equal to the expected decision:

$$\mathbf{x}_k = \sum_{i=1}^{K} p_i \mathbf{x}_i$$

A further formulation of the nonanticipativity condition consists in a very sparse system which is very convenient for numerical methods and it is composed of the following set of equations

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{x}_2,\\ \mathbf{x}_2 &= \mathbf{x}_3,\\ &\vdots\\ \mathbf{x}_{K-1} &= \mathbf{x}_K.\end{aligned}$$

2.1.3 Multistage Stochastic Problem

The extension of two-stage stochastic problems to multistage stochastic problems arises quite naturally. Real life applications often consider to take a decision, and to revise it, stage by stage, according to the observation of a stochastic process developing in the future. Hence, we obtain a decision process adapted to the realization of the stochastic phenomenon. We denote ξ_t the observation of the stochastic variable at time t and $\xi_{[t]}$ its history up to time t. Therefore, the nonanticipativity of the choice process means that the decision $\mathbf{x}_t$ may depend on $\xi_{[t]}$, but not on future observations. In a generic form we can represent a T-stage stochastic problem with its nested formulation[7]:

$$\begin{aligned}\min_{\mathbf{x}_1 \in \mathcal{X}_1} f_1(\mathbf{x}_1) \;&+\; \mathbb{E}_{\xi_2}\left[\inf_{\mathbf{x}_2 \in \mathcal{X}_2(\mathbf{x}_1,\xi_2)} f_2(\mathbf{x}_2, \xi_2) \right.\\ &+\; \left.\mathbb{E}_{\xi_t}\left[\ldots + \mathbb{E}_{\xi_T}\left[\inf_{\mathbf{x}_T \in \mathcal{X}_T(\mathbf{x}_{T-1},\xi_T)} f_T(\mathbf{x}_T, \xi_T) \right]\right]\right]\end{aligned}$$

where $\xi_2, \ldots, \xi_T$ is the random process, $\mathbf{x}_t \in \mathfrak{R}^{n_t}, t = 1, \ldots, T$ are the decision variables, $f_t : \mathfrak{R}^{n_t} \times \mathfrak{R}^{d_t} \to \mathfrak{R}$ are continuous functions and $\mathcal{X}_t : \mathfrak{R}^{n_{t-1}} \times \mathfrak{R}^{d_t} \to \mathfrak{R}, t = 2, \ldots, T$ are measurable

7 Cf. Shapiro et al. (2009).

closed values multifunctions. If a problem assumes the following form, we define it a multistage linear stochastic problem

$$\min_{\substack{\mathbf{A}_1\mathbf{x}_1=\mathbf{b}_1\\ \mathbf{x}_1\geqslant 0}} \mathbf{c}_1^\top \mathbf{x}_1 \;+\; \mathbb{E}\left[\min_{\substack{\mathbf{B}_2\mathbf{x}_1+\mathbf{A}_2\mathbf{x}_2=\mathbf{b}_2\\ \mathbf{x}_2\geqslant 0}} \mathbf{c}_2^\top \mathbf{x}_2 \right.$$
$$\left. +\; \mathbb{E}\left[\ldots + \mathbb{E}\left[\min_{\substack{\mathbf{B}_T\mathbf{x}_{T-1}+\mathbf{A}_T\mathbf{x}_T=\mathbf{b}_T\\ \mathbf{x}_T\geqslant 0}} \mathbf{c}_T^\top \mathbf{x}_T\right]\right]\right]$$

where $\mathbf{c}_t$, $\mathbf{A}_t$, $\mathbf{B}_t$ and $\mathbf{b}_t$ for $t = 2, ..., T$ are stochastic depending on the realization of the random process ξ_t. The usual way to approach this problem is with a backward perspective. Let us start defining the optimal problem on the last stage which is a simple linear programming problem:

$$\begin{aligned} \min_{\mathbf{x}_T} \quad & \mathbf{c}(\xi_T)^\top \mathbf{x}_T \\ \text{s.t.} \quad & \mathbf{B}(\xi_T)\mathbf{x}_{T-1} + \mathbf{A}(\xi_T)\mathbf{x}_T = \mathbf{b}(\xi_T), \\ & \mathbf{x_T} \geqslant 0 \end{aligned}$$

This problem is not deterministic because the earlier decision vector $\mathbf{x}_{T-1}$ depends on the stochastic variable ξ_T. The optimal value is denoted by $Q_T(\mathbf{x}_{T-1}, \xi_T)$. Recursively, we can compute the optimal value of $Q_{T-1}(\mathbf{x}_{T-2}, \xi_{T-1})$ by solving

$$\begin{aligned} \min_{\mathbf{x}_{T-1}} \quad & \mathbf{c}(\xi_{T-1})^\top \mathbf{x}_{T-1} + \mathbb{E}_{\xi_T}\left[Q_T(\mathbf{x}_{T-1}, \xi_T)|\xi_{[T-1]}\right] \\ \text{s.t.} \quad & \mathbf{B}(\xi_{T-1})\mathbf{x}_{T-2} + \mathbf{A}(\xi_{T-1})\mathbf{x}_{T-1} = \mathbf{b}(\xi_{T-1}), \\ & \mathbf{x}_{T-1} \geqslant 0 \end{aligned}$$

Finally, proceeding backward, we have the implicit representation of the multistage linear stochastic problem

$$\begin{aligned} \min_{\mathbf{x}_1} \quad & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbb{E}_{\xi_2}\left[Q_2(\mathbf{x}_1, \xi_2)\right] \\ \text{s.t.} \quad & \mathbf{A}_1\mathbf{x}_1 = \mathbf{b}_1, \\ & \mathbf{x}_1 \geqslant 0 \end{aligned}$$

So far, the expectation is taken with respect to the distribution of the stochastic variable without making any assumption about the distribution but the existence of its expectation. If we consider that the distribution of the stochastic variable has finite support, then ξ_t can assume a finite number K of values ξ_t^k, each with probability p_t^k. Then the multistage linear stochastic problem becomes

$$\min_{\substack{\mathbf{A}_1\mathbf{x}_1=\mathbf{b}_1\\ \mathbf{x}_1\geqslant 0}} \mathbf{c}_1^\top \mathbf{x}_1 \quad + \quad \sum_{k=1}^{K} p_2^k \left[\min_{\substack{\mathbf{B}_2^k\mathbf{x}_1+\mathbf{A}_2^k\mathbf{x}_2=\mathbf{b}_2^k\\ \mathbf{x}_2\geqslant 0}} \mathbf{c}_2^{k\top}\mathbf{x}_2 + \right.$$
$$\left. + \quad \sum_{k=1}^{K} p_3^k \left[\dots + \sum_{k=1}^{K} p_T^k \left[\min_{\substack{\mathbf{B}_T^k\mathbf{x}_{T-1}+\mathbf{A}_T^k\mathbf{x}_T=\mathbf{b}_T^k\\ \mathbf{x}_T\geqslant 0}} \mathbf{c}_T^{k\top}\mathbf{x}_T \right]\right]\right]$$

2.1.4 Nonanticipativity for multistage problems

As done with two-stage stochastic problems, also for multistage stochastic problems we can relax the choice structure assuming that for each scenario we can take a particular decision. Then, the decision process depends on the scenario, i.e. $\mathbf{x}^k = (\mathbf{x}_1^k, \mathbf{x}_2^k, ..., \mathbf{x}_T^k)$. Of course the solution of the relaxed problem is not correct because we allow the decision to depend on the future values of the stochastic variable:

$$\min_{\substack{\mathbf{A}_1\mathbf{x}_1^k=\mathbf{b}_1\\ \mathbf{x}_1^k\geqslant 0}} \mathbf{c}_1^\top \mathbf{x}_1^k \quad + \quad \sum_{k=1}^{K} p_2^k \left[\min_{\substack{\mathbf{B}_2^k\mathbf{x}_1+\mathbf{A}_2^k\mathbf{x}_2^k=\mathbf{b}_2^k\\ \mathbf{x}_2^k\geqslant 0}} \mathbf{c}_2^{k\top}\mathbf{x}_2^k + \right.$$
$$\left. + \quad \sum_{k=1}^{K} p_3^k \left[\dots + \sum_{k=1}^{K} p_T^k \left[\min_{\substack{\mathbf{B}_T^k\mathbf{x}_{T-1}^k+\mathbf{A}_T^k\mathbf{x}_T^k=\mathbf{b}_T^k\\ \mathbf{x}_T^k\geqslant 0}} \mathbf{c}_T^{k\top}\mathbf{x}_T^k \right]\right]\right]$$

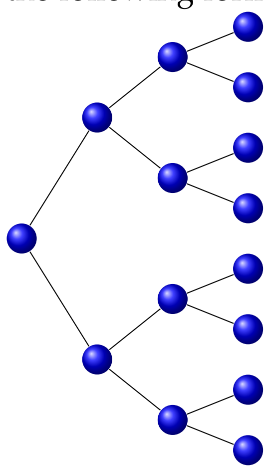
To fix the effect of the relaxation we include another constraint in order to obtain the same value for the decision variables referred to the same stage if they originate from the same information history. For the first stage decision we have

$$x_1^k = x_1^l, \quad \forall k, l \in 1, ..., K$$

while for $1 < t \leqslant T$

$$x_t^k = x_t^l, \quad \forall k, l \text{ s.t. } \xi_{[t]}^k = \xi_{[t]}^l$$

The sparse matrix representation depends on the branching structure of the problem. Let us consider a simple four-stage problem with binomial structure of the stochastic process. The scenario tree assumes the following form



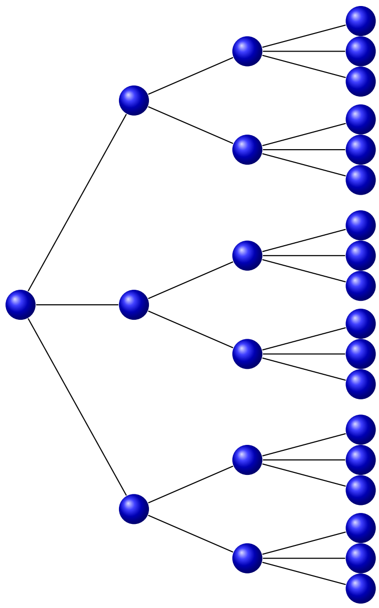
and it is clearly composed of eight scenarios (cf. next section). Thus, its nonanticipativity matrix includes the following equations

first stage			
$\mathbf{x}_1^1 = \mathbf{x}_1^2,\ \mathbf{x}_1^2 = \mathbf{x}_1^3,\ \mathbf{x}_1^3 = \mathbf{x}_1^4,\ \mathbf{x}_1^4 = \mathbf{x}_1^5,\ \mathbf{x}_1^5 = \mathbf{x}_1^6,\ \mathbf{x}_1^6 = \mathbf{x}_1^7,\ \mathbf{x}_1^7 = \mathbf{x}_1^8,$			
second stage			
$\mathbf{x}_2^1 = \mathbf{x}_2^2,\ \mathbf{x}_2^2 = \mathbf{x}_2^3,\ \mathbf{x}_2^3 = \mathbf{x}_2^4,$		$\mathbf{x}_2^5 = \mathbf{x}_2^6,\ \mathbf{x}_2^6 = \mathbf{x}_2^7,\ \mathbf{x}_2^7 = \mathbf{x}_2^8,$	
third stage			
$\mathbf{x}_3^1 = \mathbf{x}_3^2,$	$\mathbf{x}_3^3 = \mathbf{x}_3^4,$	$\mathbf{x}_3^5 = \mathbf{x}_3^6,$	$\mathbf{x}_3^7 = \mathbf{x}_3^8,$

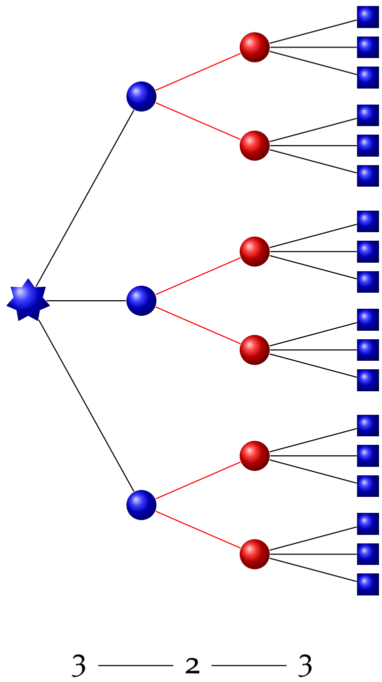
while on the fourth stage we do not need to impose any constraints because any decision variable refers to a particular node and no choice variable shares the same information up to its node with another one.

2.1.5 Tree Structure Notation

The classical structure used to represent the evolution of the stochastic processes which characterize the problem is the tree structure. In the following example we assume that the time direction is from left to right.

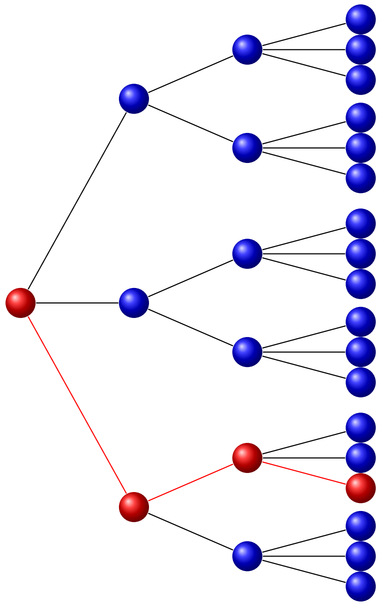


Each point represents a particular time and place in the stochastic space and it is called *node*. The time instant of the node is called *stage*. In general, in each node we observe the realization of the stochastic variable. Then, we must define if we just account the observation or if we take a decision according to it. The way in which the tree spreads over the stages is basically defined by the number of *children* generated by each node. The structure can be irregular and difficult to describe, or it could be regular if the nodes of each stage have the same number of children. In this case the tree structure is called *branching* and it represents precisely the number of children that each node has in each stage. In the following tree the branching is 3-2-3 and the nodes at the third stage are highlighted in red.



The points on the last stage of the tree are called *leaves* (square points), while the point on the first stage which contains the deterministic information available up to now is called *root* of the tree (star point). The line connecting a couple of consecutive nodes is the *period* in which the realization of the uncertainty takes place. We remark in red the branching of the second period.

The *scenario* is the path that connects the root with a particular leaf and every scenario contains only one node for each stage.



2.2 SCENARIO GENERATION

Real world complexity manifests continuously, while the decisions we have to make are frequently discrete or even boolean. Discrete decisions, e.g. to switch on/off a plant, are often relaxed to be linear because of computational limits to solve large scale integer problems. On the other hand, the continuity issue is related to the impossibility to represent numerically the continuous behavior of a stochastic variable and then, always, a discrete representation via scenarios is needed. Therefore, a suitable scenario generation is a key element for a reasonable definition of a stochastic problem. Hypothesis on the distribution, correlation of the stochastic processes, cardinality of stages, structure of the branching, etc., make the generation theoretically and computationally very challenging, see Dupačová et al. (2009). Hence, theorems and numerical algorithms are required to handle this issue.

Scenario generation techniques can be used in all those fields which need to manage uncertainty, such as: investment strategy evaluation[8], portfolio immunization[9], energy production[10], logistics problems[11], option and insurance products pricing, etc. In general, a scenario generation model deals with three data sources that are: historical data, simulation and experts opinion. For financial applications is relatively easy to collect historical data, while for energy and logistic application a simulation method based on a theoretical model proposed by the literature or by experts is often the best choice. On the other hand, the experts opinion can be directly translated in a set of scenario describing the uncertain future.

Independently from the chosen scenario generation method, we expect that the algorithm influences the solution as less as possible. Indeed, the optimal choice must be related only to the stochastic structure of the problem and it should be independent from the way we use to represent the stochastic framework, otherwise, we would obtain a algorithm-dependent decision and not a stochastic-dependent decision. Moreover, the solution must converge to the true optimum value as the number of scenario increases and, in the meanwhile, the stochastic representation must be as good as possible for a given number of scenarios.

8 Cf. Ziemba and Ziemba (2007).
9 Cf. Hiller and Schaack (1990) and Dembo (1991).
10 Cf. Greenhall (2013) and Dembo (1991).
11 Cf. Powell and Topaloglu (2003).

Several methods have been proposed in the literature to handle the scenario generation. We propose an overview of the four most used methods.

Sample Average Approximation

The Sample Average Approximation (SAA) method was proposed in Kleywegt et al. (2002) to solve stochastic problems of the form

$$\min_{x \in \mathcal{X}} g(x) = \mathbb{E}_P[G(x, \xi)] \tag{1}$$

where ξ is the random variable with probability distribution P, $\mathcal{X}$ is a finite set and G is a real valued function. The authors suggest to implement their method when the expected valued function is difficult to compute or to write in closed form, when $G(x, \xi)$ is easily computable and when the size of the feasible set grows exponentially with the number of variables. The basic idea is to generate a relatively small random sample ξ and solve the problem. Then the procedure is repeated N times and we take the expected value of the resulting values. We obtain the sample average approximation by

$$\hat{g}_N(x) = \frac{1}{N} \sum_{j=1}^{N} G(x, \xi^j)$$

and then solve the problem

$$\min_{x \in \mathcal{X}} \hat{g}_N(x) \tag{2}$$

Observe that $\mathbb{E}[\hat{g}_N(x)] = g(x)$ and that the solutions of (1) and (2) converge for $N \to \infty$.[12]

12 For more detail about properties and convergence refer to Kleywegt et al. (2002) and Ahmed and Shapiro (2002); for applications with chance constraints refer to Pagnoncelli et al. (2009) and Luedtke and Ahmed (2008).

Optimal discretization

The optimal discretization method was proposed in Pflug (2001). It is mainly based on the concept of distance between probability distributions defined as

$$d_H(P, \tilde{P}) = \sup \left\{ \left| \int h(w) dP(w) - \int h(w) d\tilde{P}(w) \right| : h \in H \right\}$$

where H is a class of functions. Pflug proposes several functions to compute the distance measure, the two main functions are the Kantorovich distance:

$$d_1(P, \tilde{P}) = \sup \left\{ \int h dP - \int h d\tilde{P} : |h(u) - h(v)| \leqslant \|u - v\| \right\}$$

and the Wasserstein distance:

$$d(P, \tilde{P}) = \sup \left\{ \int f(w) dP(w) - \int f(w) d\tilde{P}(w) : L_1(f) \leqslant 1 \right\}$$

where

$$L_1(f) = \inf\{L : |f(u) - f(v)| \leqslant L|u - v|, \forall u, v\}$$

Then, for a given distribution P the optimal discretization method consists in finding the mass point of the distribution $\tilde{P}$ such that a specific distance is minimized. Using this method, the whole trajectories are generated at once and the scenario set is optimal in a specific sense. This algorithm can be extended for a multistage tree generation using a nested distance. Several examples can be found for financial applications[13] and for energy problems[14]. This algorithm is nowadays improved thanks to the research of Pflug and Pichler.[15]

13 Cf. Hochreiter and Pflug (2007).
14 Cf. Hochreiter et al. (2006).
15 Cf. Pflug and Pichler (2011) and the overview of Pflug and Pichler (2014).

Moment matching

The moment matching method has its roots in several studies.[16] The method has been finally formalized for a scenario generation suitable for stochastic problems in Høyland and Wallace (2001b) and Høyland et al. (2003). This approach is the most reliable one in case we have a limited knowledge of the real distribution of the stochastic variables and the available information is restricted to the distribution moments. On the other hand, in case we know the entire distribution, using this method we would lose information. Then, the approach consists in build a discrete distribution satisfying those properties. Moreover, the moment matching method allows the construction of multistage trees with multivariate distributions.

Path-based methods

These methods assume that the stochastic processes are described by a set of stochastic differential equations and thus it is possible to generate paths starting from a given root. Clearly, the result of this approach is not a tree but a set of paths, often called *fan*. Therefore, it is necessary somehow to cluster or to bucket the paths in order to generate values for each node of the tree.[17]

The last proposed method needs a clustering in order to produce a tree. We want to remark that the clustering/bucketing approach must not be confused with the *reduction* issue. The reduction takes place if the produced tree is very bushy and

16 Cf. Vale and Maurelli (1983), Smith (1993), Lurie and Goldberg (1998) and Lyhagen (2001).

17 Cf. Dupačová et al. (2000).

computationally tough to handle. Then, we should reduce the cardinality of the scenarios (or, better, the number of nodes in each stage) but meanwhile preserve the information within the tree. Scenario reduction methods are proposed in Dupačová et al. (2003) and Heitsch and Römisch (2003). The approach could be forward or backward looking and can be applied to single scenarios or to scenarios subsets. The triple crucial aim of these methods is to define which nodes are close to each other in some sense, to reduce them in a single node, to redistribute the probability in a reasonable way among the remaining nodes of the stage.

Specific scenario generation models for ALM problems have been proposed during the last twenty years. A dedicated approach for ALM for pension fund can be found in Boender (1997) where a model for the Netherlands is developed.[18] An intensive study has been proposed by Mulvey in order to generate a suitable scenario setting for the Towers Perrin, see Mulvey (1996) and Mulvey and Thorlacius (1998). More recent studies for ALM scenario generation are to be found in Kouwenberg (2001).

Finally we remark the importance of testing the whatever chosen scenario generation method in order to analyze the suitability of the scenarios to the problem we are dealing with. A thorough analysis of such topic has been done by Dupačová in several works. In particular, the relation between the data and the structure of the scenarios is found in Dupačová (1996), Bertocchi et al. (2000b) and Dupačová and Bertocchi (2001) and the optimal solution sensitivity to the scenario structure is studied in Dupačová et al. (1998) and Dupačová (1999). A specific

18 Cf. the application in Boender et al. (1997).

discussion would be required in order to introduce the *contamination technique* which is widely used to stress the solution. The method proposes to introduce some noise in the stochastic variables distribution via another distribution which contaminates the original one. The issue is challenging and promising and it is currently studied by Dupačová and Kopa.[19]

2.3 MULTIPLE CRITERIA

An economic decision must be taken considering several criteria, such as the maximization of the net gain, the minimization of the costs, the achievement of some economic targets. Often, the whole problem is constituted by conflicting goals. Different targets may regard different time horizons. Typically, financial choices involve return and risk and it is well known that in an efficient market riskier positions correspond to lower returns and vice versa. Problems which consider multiple aims are called multi-objective programming. In this section we refer to Dupačová et al. (2002) to introduce suitable methods to deal with such class of problems.

In general, we consider as a multi-objective problem the minimization of a multifunction $\mathbf{f}(x)$ consisting of two or more functions $f_k(x) : \mathfrak{R}^n \to \mathfrak{R}$ on an acceptance set $\mathcal{X}$, i.e.

$$\min \mathbf{f}(x) \text{ on } \mathcal{X} \tag{3}$$

An *ideal solution* of such problem is defined by

$$\tilde{x} \in \bigcap_{k=1}^{K} \operatorname*{argmin}_{x \in \mathcal{X}} f_k(x)$$

19 Cf. Dupačová and Kopa (2012, 2014).

Clearly, it is very rare to find an ideal solution for the whole problem, then we define a feasible solution $\hat{x}$ as *efficient solution* if there is no other element $x \in X$ for which $\mathbf{f}(x) \leqslant \mathbf{f}(\hat{x})$ and $\mathbf{f}(x) \neq \mathbf{f}(\hat{x})$. Multi-objective programming aims at locating efficient solutions through various formulations.

Method 1

Let us consider a continuous function $h : \mathfrak{R}^K \to \mathfrak{R}$ non decreasing in its arguments. Then at least one solution belonging to

$$X_h = \underset{x \in X}{\operatorname{argmin}}\, h(f_1(x), ..., f_K(x)) \tag{4}$$

is an efficient solution for the multi-objective problem. In particular if h is increasing then all optimal solutions of (4) are efficient solutions for (3). A widely used choice of h is a linear and positive combination of the functions $f_k(x)$. Then, (4) becomes

$$X_h = \underset{x \in X}{\operatorname{argmin}} \sum_{k=1}^{K} t_k f_k(x) \tag{5}$$

It is possible to prove that if $t_k > 0$ and (5) consists in a unique solution, then this solution is an efficient solution for (3). If $f_k(x)$ are convex and X is compact and convex, for any efficient solution $\hat{x}$ of (3), there exists a set of $t_k > 0$ such that $\hat{x}$ is an optimal solution for (5). Moreover, if $f_k(x)$ are linear and X is a polyhedral set, then the previous statement becomes an *if and only if*. The adoption of a linear combination implies that the magnitude of the final value of each f_k is somehow predictable and then it is possible to use t_k to weigh f_k and in the meanwhile rescale them making the quantities comparable. In case of a very strong mismatch between the shape feature of the functions, h should be a composition of rescaling functions (e.g. logarithm, exponential, power) and a linear combination might not be suitable anymore.

Method 2 - The ϵ-Constrained Approach

Select one objective function, for example f_1 and a threshold vector $\epsilon \in \mathfrak{R}^{K-1}$ and solve the classical optimization problem

$$\min_{x \in \mathcal{X}} f_1(x) \text{ subject to } f_k(x) \leqslant \epsilon_k, k = 2, \dots, K \tag{6}$$

If there exists a unique solution $\bar{x}$ of (6), then $\bar{x}$ is also an efficient solution for (3). On the other hand, for every efficient solution $\tilde{x}$ of (3), there exists a vector $\epsilon \in \mathfrak{R}^{K-1}$ such that $\tilde{x}$ is an optimal solution for (3). This method is reasonable if it is possible to associate to each f_k a specific ϵ_k in an adequate way. Moreover, it is possible to calibrate the values of ϵ_k and test the problem sensitivity in order to build an handleable but still reliable model.

Method 3

It is possible to mix Methods 1 and 2 choosing a set of functions to be included in the objective and the remaining set to be considered as constrained functions. This choice depends on the features of the problem. Often, for some functions a satisfactory level or an acceptance threshold is known and then the choice is easier. The problem assumes the following form

$$\min_{x \in \mathcal{X}} \sum_{k=1}^{K_0} t_k f_k(x) \text{ subject to } f_k(x) \leqslant \epsilon_k, \quad k = K_0 + 1, \dots, K \tag{7}$$

Method 4 - The Goal Programming Approach

In this method, the main idea is to get previously a vector of optimal values minimizing each single function, i.e. $f_k^* =$

$\min_{x \in \mathcal{X}} f_k(x), k = 1, ..., K$ and then find the solution which is as close as possible to this vector solving the following problem

$$\min_{x \in \mathcal{X}} ||\mathbf{T}(\mathbf{f}^* - \mathbf{f}(x))||_p \tag{8}$$

assuming a specific weight matrix $\mathbf{T} = \text{diag}\{t_1, ..., t_K\}, t_k > 0$. It is possible to prove that if $\bar{x}$ is an optimal solution of (8) then it is an efficient solution for (3). Moreover, for $p = \infty$, at least one solution of the minimax problem

$$\min_{x \in \mathcal{X}} \max_k t_k |(f_k^* - f_k(x))| \tag{9}$$

is an efficient solution for (3).

A classical multiple criteria example is the Markowitz investment model. This model finds out a portfolio for a rational investor who prefers lower risks to higher ones and larger returns to smaller ones. Let us assume to have $x_i, i = 1, ..., n$ asset fractions, each of them with return r_i and characterized by a covariance matrix $\mathbf{V} = [\text{cov}(r_i, r_j), i, j = 1, ..., n]$, then the portfolio return is given by

$$\mathbf{r}^\top \mathbf{x}$$

while the portfolio variance, assumed to be the risk measure, is computed as

$$\mathbf{x}^\top \mathbf{V} \mathbf{x}$$

Then, each of the mean-variance efficient portfolio can be obtained by solving one of the following problems (10), (11), (12) and (13).

The objective function can express jointly the maximization or the portfolio return and the minimization of the portfolio risk thanks to a linear combination weighed by the value of a coefficient λ which measures the investor's risk aversion:

$$\max_{x \in \mathcal{X}} \quad \lambda \mathbf{r}^\top \mathbf{x} - \frac{1}{2} \mathbf{x}^\top \mathbf{V} \mathbf{x} \tag{10}$$

where $\mathcal{X}$ is the set of all feasible portfolios for which short selling is not allowed, i.e. $\mathcal{X} = \{\mathbf{x} | x_i \geqslant 0, \sum_{i=1}^{n} x_i = 1\}$.

Alternatively, we can move the return function as constraint if we know a minimal acceptable return level $\bar{r}$:

$$\min_{\mathbf{x} \in \mathcal{X}} \mathbf{x}^\top \mathbf{V} \mathbf{x} \text{ subject to } \mathbf{r}^\top \mathbf{x} \geqslant \bar{r} \tag{11}$$

Similarly, the problem can be formulated by maximizing only the portfolio return and by imposing a risk upper bound:

$$\max_{\mathbf{x} \in \mathcal{X}} \mathbf{r}^\top \mathbf{x} \text{ subject to } \mathbf{x}^\top \mathbf{V} \mathbf{x} \leqslant \bar{\sigma}^2 \tag{12}$$

Lastly, we can introduce another well known aggregating function in portfolio decision problems, i.e. $h(\mathbf{r}^\top \mathbf{x}, \mathbf{x}^\top \mathbf{V} \mathbf{x}) = \frac{\mathbf{r}^\top \mathbf{x}}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}}$. Hence, we obtain the Sharpe ratio maximization problem

$$\max_{\mathbf{x} \in \mathcal{X}} \frac{\mathbf{r}^\top \mathbf{x}}{\sqrt{\mathbf{x}^\top \mathbf{V} \mathbf{x}}} \tag{13}$$

Further evolutions of the portfolio management suggest the introduction of more complex and suitable performance measures, but still the multiple criteria problem is an important issue to deal with. Mainly, because of the aim to stay in a convex environment from the point of view both of the adopted objective function and of the feasible region described by the constraints. Several paths have been explored in order to define such risk measures and the most modern applications apply the quantiles analysis of the return distribution. In some cases this approach leads to the evaluation of the portfolio risk considering the loss which can occur at a fixed confidence level. Therefore, it is necessary to introduce the concept and the properties of a specific set of constraints which require to be satisfied only at a desired probability threshold, the so called *chance constraints*.

2.4 CHANCE CONSTRAINTS

In a stochastic constraints setting, especially when the stochastic support is very large, the request that all the constraints must be satisfied for every possible realization of the stochastic variable ξ might be too strict. A simple form for a stochastic problem which requires that each constraint is fulfilled for each realization of ξ is the following

$$\begin{aligned} \min \quad & c(\mathbf{x}) \\ \text{s.t.} \quad & g_j(\mathbf{x}, \xi) \leqslant 0, j \in \mathcal{J}, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

where $\mathcal{X} \subset \mathfrak{R}^n$ is a nonempty set, $c : \mathfrak{R}^n \to \mathfrak{R}$, $g_j : \mathfrak{R}^n \times \mathfrak{R}^s \to \mathfrak{R}$ and $\xi \in \Xi \subseteq \mathfrak{R}^s$ is an s-dimensional random vector which induces a probability measure P_ξ on the support set Ξ. In order to avoid some very hard constraints it is possible to introduce the expected value. The problem assumes the following form

$$\begin{aligned} \min \quad & c(\mathbf{x}) \\ \text{s.t.} \quad & \mathbb{E}[g_j(\mathbf{x}, \xi)] \leqslant 0, j \in \mathcal{J}, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

On the other hand, if the reliability of the solution is a fundamental issue, the expected value might not be enough to express the real attitude adverse to undesirable events not considering the shape of the whole distribution of ξ. Therefore, it makes sense to accept that the constraints hold for a subset of the random support with a fixed confidence level, i.e. with at least a desired probability p:

$$\begin{aligned} \min \quad & c(\mathbf{x}) \\ \text{s.t.} \quad & P\{g_j(\mathbf{x}, \xi) \leqslant 0, j \in \mathcal{J}\} \geqslant p, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

Hence, the problem requires that the event $A(x) = \{g_j(\mathbf{x}, \xi) \leqslant 0, j \in \mathcal{J}\}$ has a probability p, i.e. $P\{A(x)\} = p$, with respect to the probability distribution P_ξ. The constraint

$$P\{g_j(\mathbf{x}, \xi) \leqslant 0, j \in \mathcal{J}\} \geqslant p$$

is called a *joint probabilistic constraint*, while constraints such as

$$P\{g_j(\mathbf{x}, \xi) \leqslant 0\} \geqslant p_j, j \in \mathcal{J}, p_j \in [0, 1]$$

are called *individual probabilistic constraints*. This topic has been widely investigated from a theoretical point of view and adopted for applications, in particular in energy and finance fields.[20] A unifying overview can be found in Prékopa (2003). Some recent results are Henrion and Strugarek (2008), van Ackooij and Henrion (2014), Henrion et al. (2013), Henrion (2007), van Ackooij et al. (2011, 2014) and Fabian and Veszprémi (2007).

Consider again the investment problem. In (12) we describe a general portfolio optimization considering the return and the risk jointly. We assume that the risk is appropriately represented by the variance of the distribution. This hypothesis can be true if we consider the historical information as the best future forecast. A stochastic framework introduces the idea that

20 One of the first application to pension plan management is Brockett et al. (1993), a specific focus to pension fund is proposed in Dert (1995) and an application to a Dutch pension fund is in Haneveld et al. (2010).

the returns future distributions are random and the history can help in the characterization of such distributions but not entirely. Assuming to have uncertain yields, i.e. r_i depends on the realization of ξ, we could still adopt the variance as risk measure, but it would be more consistent to consider as risk the event of losing money and not just to stay far from the mean. Therefore, we can reformulate the problem as a stochastic optimization with a probabilistic constraint

$$\begin{aligned} \max \quad & \mathbb{E}\left[\sum_{i=1}^{n} r_i(\xi)x_i\right] \\ \text{s.t.} \quad & P\left\{\sum_{i=1}^{n} r_i(\xi)x_i \geqslant \bar{r}\right\} \geqslant p, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

The level of $\bar{r}$ is typically a negative value, i.e. the maximum potential loss we do not want to overstep, and p is the confidence probability usually greater than 95%. This means, for example with $\bar{r} = -0.1$ and $p = 0.97$, that we allow the final wealth to lose more than 10% only in 3% of the cases, or that we expect to have a return grater than -10% with a probability of 97%. Such kind of probabilistic constraint, the widest used risk measure in the financial sphere, corresponds to the α-quantile and it is called *value at risk*, $V@R_\alpha$.

Let us now consider a random variable X, representing for example the investment final return given by $r_i(\xi)x_i$, and a random variable Y which is the random yield of a benchmark. Considering their cumulative distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, we can show that infinitely many individual probabilistic constraints appear naturally interpretable as stochastic order dominance constraints.[21] Stochastic dominance is a stochastic

21 Cf. the overview of Levy (2006).

ordering which provides a partial order. Instead of focusing only on a set of moments of the distributions, the stochastic dominance criteria involves the whole distributions shape. Furthermore, they reflect the investor's preferences features.
We define that X dominates Y *in the first order* if

$$F_X(\eta) \leqslant F_Y(\eta), \forall \eta \in \mathfrak{R}$$

which is a probabilistic constraint. The first order stochastic dominance assumes non-satiable investor's preferences, as mentioned in Kopa and Post (2009). We can define the cumulative of the cumulative distribution function as

$$F_X^{(2)}(\eta) = \int_{-\infty}^{\eta} F_X(\alpha) d\alpha = \mathbb{E}\left[(\eta - X)_+\right]$$

Thus, we define that X dominates Y *in the second order* if

$$F_X^{(2)}(\eta) \leqslant F_Y^{(2)}(\eta), \forall \eta \in \mathfrak{R}$$

which is

$$\mathbb{E}\left[(\eta - X)_+\right] \leqslant \mathbb{E}\left[(\eta - Y)_+\right], \forall \eta \in \mathfrak{R}$$

The second order stochastic dominance assumes non-satiation and risk aversion for the investor's preferences considering the entire distribution, see Kopa and Post (2015). The application in a financial problem is evident when we want to link the random realization of the optimal portfolio with the value of a benchmark and we assume that the risk arises when the position goes below the benchmark value.[22] Therefore, if we introduce the first order stochastic dominance, the optimization problem becomes

$$\begin{aligned} \max \quad & \mathbb{E}\left[\sum_{i=1}^{n} r_i(\xi) x_i\right] \\ \text{s.t.} \quad & F_{r_i(\xi)x_i}(\eta) \leqslant F_Y(\eta), \forall \eta \in \mathfrak{R} \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

22 Cf. Diamond and Stiglitz (1974).

where the set $\mathcal{X}$ corresponds again to the set of all feasible portfolios when short sales are not allowed, i.e.

$$\mathcal{X} = \left\{ \mathbf{x} | x_i \geqslant 0, \sum_{i=1}^{n} x_i = 1 \right\}$$

Similarly, the introduction of the second order stochastic dominance leads to the following formulation

$$\begin{aligned} \max \quad & \mathbb{E}\left[\sum_{i=1}^{n} r_i(\xi) x_i \right] \\ \text{s.t.} \quad & F^{(2)}_{r_i(\xi)x_i}(\eta) \leqslant F^{(2)}_Y(\eta), \forall \eta \in \mathfrak{R} \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

If the random space of the realizations of ξ is finite and if the realizations are equiprobable, then both the first and the second order stochastic dominance constraints can be formulated with a vector representation as proposed in Kuosmanen (2004). Post and Kopa (2013) develop a linear formulation of general N-th order stochastic dominance criteria for discrete probability distributions. A second order stochastic dominance constraint on the portfolio return represents a constraint on the shortfall function and can be viewed as a continuum of the *Average Value at Risk* constraint (AV@R). In the particular case of a constant Y we would obtain the classical AV@R constraint.

Clearly, the AV@R$_\alpha$ can be derived by the V@R$_\alpha$. For a given cumulative probability distribution function G, the value at risk is defined as

$$V@R_\alpha(X) = G_X^{-1}(\alpha), 0 < \alpha < 1$$

In the literature, according to the considered field, it is possible to have $V@R_\alpha = -G_\alpha^{-1}$ and the confidence level α could be interpreted as $1-\alpha$. Therefore, a correct specification of the value at risk we are referring to is needed. A simple modification of the risk measure which includes the expected value of the distribution is the value at risk deviation defined as

$$V@RD_\alpha(X) = \mathbb{E}[X] - V@R_\alpha(X), 0 < \alpha < 1.$$

The $V@R_\alpha$ fulfills several important properties: translation invariance, isotonicity with respect to the first order stochastic dominance, positive homogeneity, comonotone additivity. While some other crucial properties, as the convexity, do not hold. Nevertheless, it is a simple and meaningful measure used to identify the maximum loss or the minimum gain that can occur with a given probability α. In case of neither smooth nor plain distribution the information about the loss tail must be much more complete and the $V@R_\alpha$ becomes less reliable. The average value at risk, $AV@R_\alpha$, fulfills this request giving the average value of the tail beyond the $V@R_\alpha$ threshold. In the literature it is also called conditional value at risk, $CV@R_\alpha$, underlining the conditioned distribution of the tail. If the support of X is continuous, the $AV@R_\alpha$ is defined as

$$AV@R_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha G_X^{-1}(u)du, 0 < \alpha < 1$$

while, if the support of X is finite and then it assumes values $X_s,\ s = 1, ..., S$, the $AV@R_\alpha$ corresponds to the optimal objective value of the following linear problem as formulated in Rockafellar and Uryasev (2000, 2002)

$$AV@R_\alpha(X) = \max\left\{a - \frac{1}{\alpha}\sum_{s=1}^{S}(z_s \cdot p_s) : -a + X_s + z_s \geqslant 0, z_s \geqslant 0\right\}$$

Similarly to the value at risk deviation, the average value at risk deviation is given by

$$AV@RD_\alpha(X) = \mathbb{E}[X] - AV@R_\alpha(X), 0 < \alpha < 1.$$

The $AV@R_\alpha$ has several properties: translation invariance, positive homogeneity, concavity, isotonicity with respect to the second order stochastic dominance, strictness, Lipschitz continuity, comonotone additivity, convex composition. For an overview of the topic and a comparison of the statistical and mathematical properties between the $V@R_\alpha$ and the $AV@R_\alpha$ refer to the unifying study made in Pflug (2000), Pflug and Römisch (2007) and Gaivoronski and Pflug (2005). This property set makes the $AV@R_\alpha$ very suitable to deal with stochastic financial problem represented with a discrete tree. In fact, we typically want to reach a wealth target and then care about a risk measure able to capture the whole distribution of the tree leaves wealth. Recent studies involve complex scenario generation structures which produce a non smooth distribution of the final wealth. Then, the $AV@R_\alpha$ represents a good answer to such issue. Several applications which consider the $AV@R_\alpha$ as risk measure have been proposed. Among them, we suggest: an optimal multicurrency asset allocation done in Topaloglou et al. (2002), a pension fund ALM management optimization proposed in Bogentoft et al. (2001), an individual pension plan asset optimization developed in Kilianová and Pflug (2009).

3 PENSION MANAGER ALM PROBLEM

The main goal of a pension fund manager is sustainability. Very rarely a pension fund represents a gain source for the pension plan sponsor. Clearly, the investors have to pay fees to the fund. These costs constitute the fund income, but the pension fund should not be managed in a profit oriented way. Indeed, the ALM ensures that the fund assets are and will be able to pay back the annuities to the pensioners. The correct evaluation of the liability actual value is fundamental in order to guarantee a proper funding ratio of the pension fund. The ALM problem consists in an investment portfolio problem which takes all internal and regulatory constraints into account and relies on a market-based valuation approach in order to guarantee the long term sustainability. In this chapter we outline the main fundamental features of a general ALM problem applied to a defined benefit pension fund in a stochastic optimization framework. Then, we propose a liability side pricing based on a portfolio replication approach.

3.1 MODEL SURVEY

In this framework, a multistage stochastic model is a suitable approach considering as stochastic both the assets and the liabilities processes. Moreover, in defined benefit pension funds, risk management involves the measurement and assessment of pension fund risks and the design, revision and monitoring of the fund's parameters (contributions, benefits, and investments). The main risks that defined benefit pension funds are exposed to are: investment, inflation, and longevity risk. We analyze a multicriteria objective model as suggested in Consigli et al. (2011). The long term horizon is a crucial feature of a pension fund ALM model. We assume a final target after twenty years, as well as three intermediate targets after one year, three years and five years respectively. Nevertheless, we consider the existing liability information which has a perspective of eighty years because of the future annuity that will be paid to each active member of the fund who has just started to work, assuming forty years of contribution and forty retirement years. The pension fund population could be divided into several classes. We adopt three different classes according to the status of the member:

- paying member, who is still paying the contribution to the fund;
- pending member, who terminated the contributions and is waiting for receiving the annuity;
- pensioner, who is receiving the annuity.

Paying and pending members could be subdivided according to the year in which they will become pensioners in order to consider different features in terms of annuity and indexation. Therefore, the liability structure depends on the overall pension fund membership structure, on the considered age distribution of paying members, on the categories of paying members, and on the structure of pending and pensioner members. A fair actual value of the liability is needed to evaluate the funding gap and the funding ratio of the fund which typically represent a target of the management. Moreover, we evaluate the perspective net payments the fund will pay to the pensioners. Depending on the scheme features, such cash flows may be reevaluated with specific indexes (e.g. inflation) for each members' category.

The asset universe should consider a much more detailed fragmentation than the pension fund provider universe which considers only macro asset classes. In particular, we can identify as asset classes:

- cash;
- treasury and corporate bonds (for different maturity baskets);
- treasury inflation protected securities (for different maturity baskets);
- private equity;
- public equity;
- defensive and cyclical infrastructure;
- real estate;
- renewable.

Each class could be divided according to the rating level or to the country segmentation. The more the asset universe is populated, the more flexible will be the optimization, but, on the other hand, the more difficult will be the tractability of the stochastic problem and of the fund management itself. The asset model includes the specification of the asset return model for long-term scenario forecasting, the estimation model and all the functionalities needed to generate, from relevant data histories, scenarios for the coefficients to be included in the decision problem. The tree representation of the asset classes returns requires specific forecast models where the usual stochastic processes are not suitable. Moreover, each asset class has some characteristic features, as dividend or coupon payments, decay factors, inflation indexation, etc., which have to be considered directly in the constraints formulation. The asset universe can include also derivative contracts, e.g. interest rate swaps and inflation swaps.

In general, the model structure follows the major ALM formulations proposed in the literature in terms of cash balance constraints, duration mismatch evaluation, investment risk capital allocation, turnover and liquidity constraints. The main innovation of the model is to consider the expiring asset and the coupon payments which can occur also in an intra-period instant. Especially in a long term scheduling, to consider each time period as decisional can make the problem impossible to handle in a computational sense, even if linear. Therefore, we assume to have non-decisional intra-periods in which we evaluate the coupon payments and the liability net payments and we compute the resulting net cash flow that will be included in the cash balance of the next decisional stage. Adopting this setting it is possible to stay closer to the real features of the problem and still have an handleable formulation. Moreover, in the intra-period stages, it is possible to evaluate the target variables even if the decision variables are not defined, for instance if they depend on the risk allocation of the portfolio or on the mismatch between cash account and net payments.

The target variables regards:

- the mismatch between liquidity and net payments (T^1);
- a risk exposure measure (T^2) which includes both the duration mismatch between asset and liability and a measure of the market risk associated to the portfolio composition;
- a return measure (T^3);
- the funding ratio of the pension fund given by the assets fair value over the actual value of the future liabilities (T^4).

The objective function is multicriteria and it is formulated as a linear combination between expected values of the target variables and expected shortfall of each variables with the respective target. Moreover, to each variable we assign a specific weight, again as a linear combination. Therefore, the objective assumes the following form

$$\min \left[\alpha \cdot \sum_{k=1}^{4} \lambda_k \mathbb{E}\left[T^k_{t_k}\right] + (1-\alpha) \cdot \sum_{k=1}^{4} \lambda_k \mathbb{E}\left[\tilde{T}^k_{t_k} - T^k_{t_k} | T^k_{t_k} \leqslant \tilde{T}^k_{t_k}\right]\right]$$

with $\sum_{k=1}^{4} \lambda_k = 1$. It is possible to have jointly minimizing and maximizing variables, and the model user is free to set arbitrarily the stage where each target variable plays a role.

3.2 LIABILITY PRICING MODEL

In this section, we provide a simple description of the employed liability pricing model for the Defined Benefit Obligation (DBO) estimation. The liability model can be developed under alternative assumptions on the adopted indexation scheme.

We consider a case of inflation adjustment. The pension fund always knows the future net pension payments over a long-term horizon, depending on the retirement of current employees as defined in the pension agreement. Since these payments are inflation linked and the known amount is nominal we need to construct a replication portfolio. A sequence of inflation adjusted pension payments can be replicated with an appropriate portfolio of discount bonds and options. Within a twenty year horizon problem we need to compute the evolution of the Pension Funds liability - specifically its Defined Benefit Obligation (DBO) - as discounted values of inflation adjusted pension payments. At year zero, the DBO will thus reflect the discounted value of pension payments from year one to the far future. We may assume that, due to discounting and progressive reduction of passive members, a truncation year will be considered. Assuming to trace pension payments up to fifty years and a unique indexation scheme (e.g. inflation based), the replicating portfolio should include discount bonds maturing at $t = 1, 2, 3, ..., 50$ years and carrying a nominal value equal to the payment of pensions in those years. Further, we assume that the pensioners benefit from an indexation scheme which includes a cap and a floor set of rates. In the case of inflation adjustment, in each year, pensions are revalued according to the occurred inflation but with a maximum possible revaluation - the cap rate - and also with a protection of minimum revaluation - the floor rate. From the perspective of portfolio replication not only should we then consider a set of discount bonds maturing in the years $t = 1, 2, ..., 50$ but also, for each year, we should consider a set of options, caps and floors, also maturing in those years. At each time point the replication portfolio will evolve so to span the entire set of residual maturities and discarding the liabilities already matured.

Let us consider an underlying interest rate process consistent with the log-normal assumptions at the grounds of Black's pricing model for interest-rate sensitive instruments. In which case considering a nominal liability of $l(t)$ at time t, the fair value of a caplet contract is

$$\begin{aligned}\mathrm{Caplet}(t,T) = \; l(T) \; \cdot \; & \Big[e^{-r_R(t,T)\cdot(T-t)}N(d_1^C) \\ & -(1+k^C)^{T-t}e^{-r_N(t,T)\cdot(T-t)}N(d_2^C)\Big]\end{aligned}$$

$$d_1^C = \frac{\ln\left(\frac{1}{(1+k^C)^{T-t}}\right) + \left(r_N(t,T) - r_R(t,T) + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2^C = d_1^C - \sigma\sqrt{T-t}$$

where $r_R(t,T)$ is the real spot interest rate at t for payments at T, $r_N(t,T)$ is the corresponding spot curve, σ is the volatility of the inflation process (here assumed to be constant), while k^C is the cap level expressed on annual basis and compounded over $T-t$.

Similarly, for the floorlet contract we have

$$\begin{aligned}\mathrm{Floorlet}(t,T) = \; l(T) \; \cdot \; & \Big[-e^{-r_R(t,T)\cdot(T-t)}N(-d_1^F) \\ & +(1+k^F)^{T-t}e^{-r_N(t,T)\cdot(T-t)}N(d_2^F)\Big]\end{aligned}$$

$$d_1^F = \frac{\ln\left(\frac{1}{(1+k^F)^{T-t}}\right) + \left(r_N(t,T) - r_R(t,T) + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2^F = d_1^F - \sigma\sqrt{T-t}$$

where k^F is the floor level expressed in annual basis and compounded over $T-t$.

Finally, the computation of the zero coupon bond (ZCB) at time t for the payments at T follows the equation

$$ZCB(t,T) = liab(T) \cdot (1 + r_R(t,T))^{-(T-t)}$$

Then the replicating portfolio at time t is given by the following formula

$$\sum_{T=t+1}^{M} ZCB(t,T) - Caplet(t,T) + Floorlet(t,T)$$

where we assume to have information about the pension net payments until the year M.

A correct pricing of the liability side is a fundamental matter for a pension fund in order not to run into an underfunded balance. That is true for any pension fund, in particular for defined benefit ones.

4 PENSION PLAN SPONSOR PROBLEM

In general, a private pension fund is an investment fund which receives periodical contributions from the private investor and provides an annuity during the retirement. The main function of a pension plan is to help the subscribers survive their savings. Then, a reasonable aim for this kind of investment would be to guarantee an integration of the public retirement pension such that the total income before and after the retirement does not vary relevantly. Typically, the pension plan is composed of pension funds which differ sufficiently from each other in order to let the investors choose the optimal pension perspective investment among a well diversified asset universe. Often, such pension funds are issued following some standard investment allocations: a guaranteed capital, a low risk profile, an high yield investment, etc. The competition among the private pension plan sponsors is becoming stronger and stronger. They all would like to offer suitable and reliable pension funds for their contributors. This means that the offer could be somehow standard for huge providers, but for small and medium ones the pension funds composition should be defined according to the needs of future subscribers. The case of pension plans issued only for some workers categories or for the employees of a single company is a typical situation in which a policy decision following a rigorous statistical analysis of the members is required. For a given pension plan members population, our goal is to identify the best pension funds that should be issued, i.e. their optimal asset allocations. The project consists of two steps. The former is a precise statistical analysis of a database containing the subscribers of the fund. The aim of this preliminary study is to cluster the population and identify a set of representative members. Thus, we assume that the optimal investment portfolio for each of these representatives will define each pension fund of the pension plan. Therefore, the second step consists in the formulation and implementation of a multistage stochastic program (MSP) in order to define the optimal asset allocation for each representative member.

The statistical analysis of the first step is briefly investigated in Section 4.1. In Section 4.2 the second step explores the modelization of the MSP.

4.1 FIRST STEP – POPULATION ANALYSIS

The population analysis consists in a statistical description of a dataset. The dataset we are considering has 5577 individuals. We assume they belong to an homogeneous population and represent the active population of the pension plan, i.e. those who are currently contributing to the pension fund. The focus of the study is twofold: give to the pension plan sponsor a complete and rigorous view of the actual fund participants and investigate the most characterizing features in order to have a reliable starting point for the sequent clusterization. The analized members' features are:

- age and remaining working life
- gender

- accumulated wealth
- average annual contribution
- percentage of the salary chosen for contribution
- diversification attitude
- withdraw and switch behavior

The age analysis uses as input data the year when each member joined the fund. For the considered dataset the result shows a uniform distribution in the last decades. The male and female cardinality is almost equal. The accumulated wealth analysis highlights a huge variety starting from the younger employees with almost null wealth to the top managers positions which create a heavy right tail. The mean value is 70,000 euros, the standard deviation is 46,000 euros. To analyze better the accumulation process, we introduce a contribution ratio given by the accumulated wealth per year passed in the fund. The contribution ratio distribution is highly concentrated between 3,000 euros and 6,000 euros per year.

A particular focus is dedicated to the diversification choice. Up to now, the fund is composed of seven pension funds and we analyze the number of positions opened for each pension funds and for each member. A first analysis investigates the favorite funds, a second one shows the individual inclination to invest simultaneously in more than one fund, i.e. to adopt for the pension perspective savings the same diversification strategies that are usually performed for investment portfolios.

The withdraw decision is studied both in terms of frequency and in terms of quantity. In our dataset, according to the pension plan regulation, the minimum number of years between the entering date and the time of the first withdraw is eight. Considering the calendar year in which each member required the withdraw, it is clear that he/she uses this option either because the regulatory eight years have expired or due to the 2008/2014 financial crisis. The average amount withdrawn depends on the age of the members: as far as the older ones are concerned they withdraw 45% of the amount, while the younger ones withdraw 62% of the amount. However, only the 22% of the pension fund population requires a withdraw.

The switch option among the pension funds is not widely used. Only the 4% of the pension plan population moved the accumulated wealth at least once. In those cases, the switch occurs typically from risky funds to low risk ones.

From a cross analysis, we observe a strong correlation between the switch and the diversification attitude. The participants who require a switch are experiencing a diversified portfolio. Generally, we can distinguish between members with static and concentrated portfolios and members implementing dynamic and diversified strategies.

The strategies choice study brings out also the risk attitude of the pension plan members. The lowest risk pension funds represent the main investment. Only a few contributors switch to riskier positions due mainly to the following two reasons: the perspective of a long investment window, if they are young members, or a natural attitude for risk which leads them to invest the savings in order to seek for some extra gain during the market high volatility periods.

Thanks to the previous results, we start the clustering analysis having as main characterizing features three elements: the accumulated wealth, the portfolio risk level, the remaining working years. The aim of the clusterization is to extract a set of representative members among the whole population. For each of them, we propose an optimal portfolio allocation considering the investors' features and the stochastic environment. The pension plan sponsor wants to offer the best pension funds to the active members. Then, the optimal obtained portfolios (one for each representative) will be suggested to the pension plan sponsor to become the pension funds that should be offered. Clearly, the cardinality of the representative members set and the number of pension funds will coincide. Therefore, the cluster cardinality must be decided by the provider, taking into account the suitability for the members and the manageability for the pension fund manager.

In order to create the cluster sets, we adopt the k-means Lloyd's algorithm using the cityblock distance which measures the distance between two elements x_i, x_j having p attributes as $d(x_i, x_j) = \sum_{k=1}^{p} |x_i^k - x_j^k|$, i.e., each centroids is a component-wise median of the points in that cluster, see Lloyd (1982) and Kaufman and Rousseeuw (2009). As already mentioned, the number of clusters is a sponsor's decision. The choice should consider the representativeness of the clusters. Therefore, we propose the cluster analysis assuming sequentially three, four and five clusters, i.e., three, four and five centroids (the representative members) and three, four and five optimal portfolios (the pension funds) respectively.

4.1.1 Clusterization Results

The results of the cluster analysis are reported in Table 5 in which for each representative we propose the wealth, the risk/reward profile and the expected remaining working years. In order to identify the best cluster cardinality choice we de-

	wealth in €	risk profile	years to retirement
Three clusters			
Repres. 1	105,000	very low	13
Repres. 2	43,000	very low	28
Repres. 3	41,000	high	32
Four clusters			
Repres. 1	104,000	very low	13
Repres. 2	43,000	very low	27
Repres. 3	66,000	medium	28
Repres. 4	38,000	high	33
Five clusters			
Repres. 1	132,000	very low	9
Repres. 2	78,500	very low	17
Repres. 3	66,800	medium	28
Repres. 4	35,000	very low	31
Repres. 5	38,000	high	33

Table 5: Centroids features for each cluster cardinality choice

velop a silhouette analysis. The silhouette value $s(x_i)$ describes how each point i is similar to the points in its own cluster and it is defined as

$$s(x_i) = \frac{m(x_i) - a(x_i)}{\max[m(x_i), a(x_i)]}$$

where $a(x_i)$ is the average distance from the $i-$th point to the points in the same cluster and $m(x_i)$ is the minimum average distance from the $i-$th point to the points in different clusters, see Kaufman and Rousseeuw (2009). As distance measure we adopt the cityblock distance. The silhouette analysis of the cluster cases produces: Figure 1 for three clusters, Figure 2 for four clusters, Figure 3 for five clusters. In Rousseeuw (1987) the sil-

Figure 1: Silhouette of the three clusters case

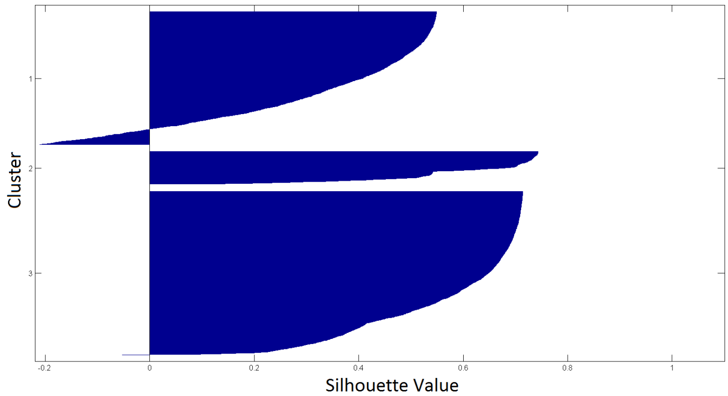
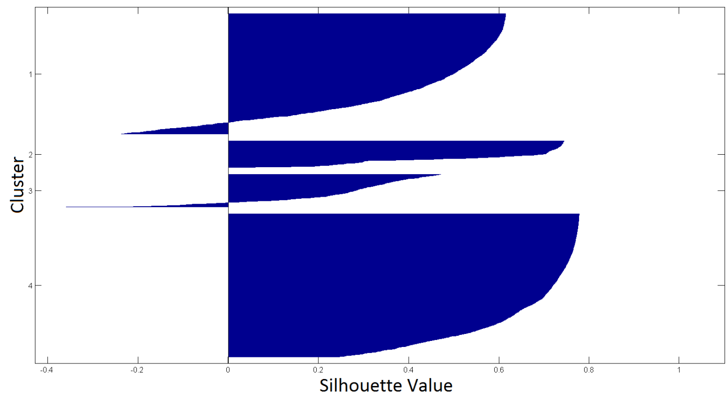
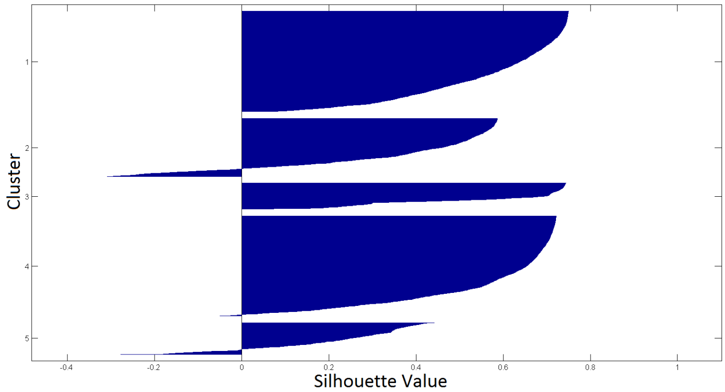


Figure 2: Silhouette of the four clusters case



houette analysis is used to compare different choices of cluster cardinality. A negative silhouette value for some elements of the population suggests a non efficient choice of the cluster cardinality. Moreover, de Amorim and Hennig (2015) suggests to use the silhouette index defined as $1/N \sum_{i=1}^{N} s(x_i)$ to quantify the validity of the whole clustering. In our case, we compute

Figure 3: Silhouette of the five clusters case



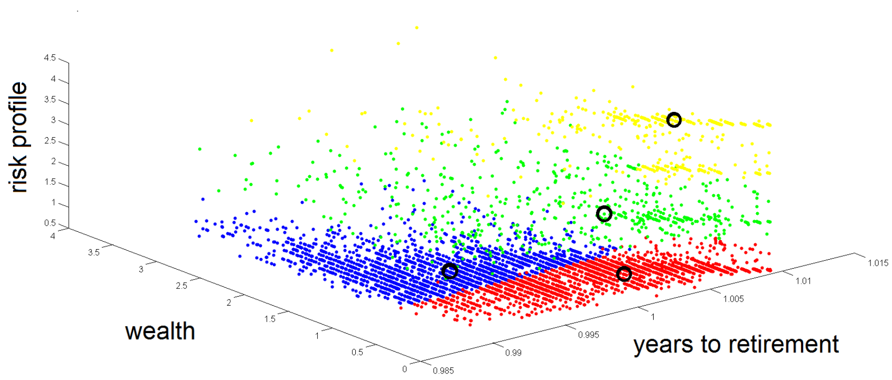
the values reported in Table 6. The results highlight the quality

cluster cardinality	silhouette index
3	0.4818
4	0.5145
5	0.4946

Table 6: Silhouette index value for each cluster cardinality choice

of the four-cluster choice. Therefore, we go on studying only the four-cluster case and its representatives reported in Table 5. Figure 4 graphically represents the clusters (differentiated by the color) and the centroids (the black dots) of the four-cluster case. The x-axis represents the wealth, the y-axis the remaining working years and the z-axis the risk profile. The values are normalized.

Figure 4: Cluster analysis of the four clusters case



4.2 SECOND STEP – PENSION PLANS OPTIMAL ALLOCATION

Given the representative employees defined in the previous section, each pension funds allocation will coincide with each representative member optimal portfolio allocation. Therefore, we need a model to describe the pension problem for a private investor. The aim of the procedure is to define the optimal asset allocation for an employee in a retirement perspective. We have to deal with two main features: a long-term horizon with a fixed and given sequence of portfolio rebalancing stages and an uncertainty environment regarding the assets returns and the salary evolution. These elements lead naturally to a multistage stochastic approach, see Dupačová et al. (2002). The considered framework is a defined contribution pension fund. The distinction between defined contribution and defined benefit pension funds, especially in terms of securities included, is described and analyzed in Consiglio et al. (2015).

4.2.1 Multistage Stochastic Model

We suppose that the decision times correspond to all the stages but the last one in which we just compute the accumulated final wealth. The stochasticity arises from two sources: the assets returns and the salary process. The investment universe is composed of n assets which are the benchmarks that the fund manager is able to replicate. Thus, $\rho_{i,t,s}$ are the asset returns where the index $i = 1, ..., n$ represents the assets, $t = t_0, ..., T$ represents the stages and $s = 1, ..., S$ represents the scenarios; $\rho^{sal}_{t,s}$ is the salary growth rate. Both processes are modeled as Geometric Brownian motions. In particular, the salary stochasticity is crucial in the definition of a consistent model for the private investor optimal allocation. Several pension fund ALM models use the salary modeling also to consider the stochastic contributions which are the inflows generated by the members of the fund. A special focus on the impact of salary uncertainty in a private asset allocation model is proposed in Cairns et al. (2006). Considering the approach suggested in Cairns et al. (2006) we assume that the salary is correlated with the riskiest assets. Then, having the mean μ_i and the standard deviation σ_i for each process and a correlation matrix $corr_{i,j}$, we assume the following structure for the assets price evolution

$$dP^i_t = \mu_i P^i_t dt + \sigma_i P^i_t dW^i_t, \quad \forall i, \ \forall t$$

$$\mathbb{E}(dW^i_t, dW^j_t) = corr_{i,j} dt \quad \forall i, \ \forall j, \ \forall t$$

where W_t is the Wiener process.

The stochasticity is represented by a discrete scenario tree composed of S paths and characterized by a regular branching. We define the non negative decision variables: $c_{i,t,s}$, $r^+_{i,t,s}$ and $r^-_{i,t,s}$. Respectively, $c_{i,t,s}$ expresses the level of contribution we want to invest in the asset i, on the stage t, in the scenario s, while the rebalancing variables $r^+_{i,t,s}$ and $r^-_{i,t,s}$ allow the redistribution of the accumulated wealth among assets, indeed, they quantify respectively how much we buy and how much we sell of each asset at the beginning of each stage, i.e. before adding the contribution.

Finally, we can list the set of constraints in order to express the regulatory bounds and the cash balance conditions.

Salary process

Fixing the initial level $sal_{t_0,s}$ equal to the actual salary of the employee, we can easily describe the salary process

$$sal_{t,s} = sal_{t-1,s} \cdot (1 + \rho^{sal}_{t,s}), \quad \forall t > t_0, \forall s. \tag{14}$$

Maximum contribution level

In each stage the employee does not want to invest more than a certain maximum percentage of his/her salary. Therefore, we introduce the parameter *propensity to save* (ps) and the parameter *employer contribution* (e) which represents a supplementary contribution added from the employer as a percentage of the employee's contribution. Moreover, the time structure of the problem defines the stages every Δt years, but in real life the contribution is added in the pension fund yearly (sometimes also monthly). Therefore, considering the growth rate of the salary constant over each period, we compute the actual value of a growing annuity paying one euro for Δt years multiplying by Δt. In particular, we assume that the contribution would be paid at the beginning of each year. Thus, the constraint describing the maximum contribution level assumes the following form

$$\sum_{i=1}^{n} c_{i,t,s} \leqslant sal_{t,s} \cdot ps \cdot (1+e) \cdot \Delta t, \quad \forall t, \forall s. \tag{15}$$

Portfolio Balance

We define the set of constraints that describes the portfolio allocation, the rebalancing decisions and the wealth account. For this purpose, we introduce the holding variable $h_{i,t,s}$ which represents the amount we hold in each asset, and the total wealth variable $w_{t,s}$. Moreover, we define the *initial portfolio* vector $h_{i,0}$ in case the investor already has a position in the pension fund and the *initial cash* parameter w_0 if the investor wants to add a further amount of money, e.g. a shift from another pension fund and/or an initial extra contribution:

$$h_{i,t_0,s} = h_{i,0} + r^+_{i,t_0,s} - r^-_{i,t_0,s} + c_{i,t_0,s}, \quad \forall i, \forall s. \tag{16}$$

$$\sum_{i=1}^{n} r^+_{i,t_0,s} = \sum_{i=1}^{n} r^-_{i,t_0,s} + w_0, \quad \forall s. \tag{17}$$

$$r^-_{i,t_0,s} \leqslant h_{i,0}, \quad \forall i, \forall s. \tag{18}$$

$$\sum_{i=1}^{n} r^-_{i,t_0,s} \leqslant \theta \sum_{i=1}^{n} h_{i,0}, \quad \forall s. \tag{19}$$

$$h_{i,t,s} = h_{i,t-1,s} \cdot (1+\rho_{i,t,s}) + r^+_{i,t,s} - r^-_{i,t,s} + c_{i,t,s}, \quad \forall i, t_0 < t < T, \forall s. \tag{20}$$

$$\sum_{i=1}^{n} r^+_{i,t,s} = \sum_{i=1}^{n} r^-_{i,t,s}, \quad t_0 < t < T, \forall s. \tag{21}$$

$$r^-_{i,t,s} \leqslant h_{i,t-1,s} \cdot (1+\rho_{i,t,s}), \quad \forall i, t_0 < t < T, \forall s. \tag{22}$$

$$\sum_{i=1}^{n} r^-_{i,t,s} \leqslant \theta \cdot w_{t,s}, \quad t_0 < t < T, \forall s. \tag{23}$$

$$w_{t,s} = \sum_{i=1}^{n} (h_{i,t-1,s} \cdot (1+\rho_{i,t,s})), \quad t > t_0, \forall s. \tag{24}$$

Equation (16) defines the holding in the first stage for each asset as the sum of the initial portfolio allocation, $h_{i,0}$, and of the buying/selling of the initial portfolio and of the initial wealth, $r^+_{i,t_0,s}$ and $r^-_{i,t_0,s}$, and of the first period contribution, $c_{i,t_0,s}$. The initial portfolio reallocation is defined using equations (18) and (19), while equation (17) defines the buying as reallocation of the initial portfolio plus the allocation of the initial wealth. As far as the other stages are concerned, equation (20) defines the holding as capitalization of the previous holding for each asset plus the reallocation of the accumulated wealth and plus the contribution. The portfolio reallocation follows equations (21), (22) and (23). Equations (19) and (23) define the turnover constraints through the parameter θ which states that it is not possible to sell more than a fixed percentage θ of the portfolio. Finally, equation (24) computes the accumulated wealth in each stage for each scenario. According to this wealth variable we build the target constraints and the objective function. Moreover, we want to include a risk exposure constraint. In order to keep the problem linear, we assume that each asset has an associated risk coefficient, rc_i, and we set a risk level R that the portfolio cannot exceed in average:

$$\sum_{i=1}^{n} h_{i,t,s} \cdot rc_i \leqslant R \cdot \sum_{i=1}^{n} h_{i,t,s}, \quad \forall t, \forall s. \tag{25}$$

Since we use a stochastic tree structure, we include the set of all the nonanticipativity constraints on the decision variables. As suggested in Kilianová and Pflug (2009), we define a multicriteria objective function including two wealth targets and the Average Value at Risk Deviation (AV@RD) as risk measure, where $AV@RD(x) = \mathbb{E}(x) - AV@R(x)$. In particular, we include the multicriteria objective as suggested in Dupačová et al. (2002)

and considering the investor risk adverse, the optimal portfolio allocation depends on the minimization of the AV@RD of the final wealth. A wealth target can be chosen in an intermediate stage, $\Pi_{t_{int}}$, and in the final stage, Π_T, according to investor's policy. Adopting the ϵ-Constrained Approach the objective assumes the following form

$$\begin{aligned}
\min \quad & \sum_{s=1}^{S} (w_{T,s} \cdot p_s) - a + \frac{1}{\alpha} \sum_{s=1}^{S} (z_s \cdot p_s) && (26) \\
\text{s.t.} \quad & -a + w_{T,s} + z_s \geqslant 0, \quad z_s \geqslant 0 && (27) \\
& \sum_{s=1}^{S} w_{t_{int},s} \cdot p_s \geqslant \Pi_{t_{int}} && (28) \\
& \sum_{s=1}^{S} w_{T,s} \cdot p_s \geqslant \Pi_T && (29) \\
& (14) - (25) && (30)
\end{aligned}$$

In (26) we minimize the AV@RD on the last stage, i.e. on the final wealth, for a given confidence level α. According to Rockafellar and Uryasev (2000, 2002), the discrete definition of the AV@RD needs the inequality (27) in order to define jointly the variables a and z_s. The wealth target constraints (28) and (29) force the average of the accumulated wealth on the intermediate stage and on the final stage to be greater than or equal to a fixed amount $\Pi_{t_{int}}$ and Π_T, respectively. For this purpose, the definition of a benchmark wealth is needed. We assume that the returns $\rho^b_{t,s}$ of the benchmark wealth are strictly linked with the evolution of the returns of the assets. Indeed, we define the benchmark returns equal to the average[1] of the assets returns which singly satisfy the risk exposure constraint, i.e. $I = \{i | rc_i \leqslant R\}$, then $\rho^b_{t,s} = 1/|I| \sum_{i \in I} \rho_{i,t,s}$, $\forall t$, $\forall s$. Then, assuming that the contribution touches the bound in (15), i.e.

1 Cf. DeMiguel et al. (2009).

$C^b_{t,s} = sal_{t,s} \cdot ps \cdot (1+e) \cdot \Delta t$, $\forall t$, $\forall s$, and starting from the initial wealth, i.e. $w^b_{t_0,s} = \sum_{i=1}^{n} h_{i,0} + w_0$, $\forall s$, the evolution of the benchmark wealth is

$$h^b_{t_0,s} = w^b_{t_0,s} + C^b_{t_0,s} \quad \forall s \tag{31}$$

$$h^b_{t,s} = h^b_{t-1,s} \cdot (1+\rho^b_{t,s}) + C^b_{t_0,s} \quad t > t_0, \forall s \tag{32}$$

$$w^b_{t,s} = h^b_{t-1,s} \cdot (1+\rho^b_{t,s}) \quad t > t_0, \forall s. \tag{33}$$

Then, the values of the two targets become as follows

$$\Pi_{t_{int}} = \mathbb{E}[w^b_{t_{int},s}] \tag{34}$$

$$\Pi_T = \mathbb{E}[w^b_{T,s}] \tag{35}$$

The whole described formulation is a linear programming problem.

4.2.2 Problem Settings

The proposed model is applied to the four representative members defined with the cluster analysis in order to identify the four optimal pension funds that should be issued by the pension plan sponsor. Let us assume that the pension fund manager is able to replicate artificially six different securities which are the investment universe we deal with. The pension funds are a combination of these assets: a guaranteed capital security, two low risk, a medium risk and two high risk assets. Their risky level is described by the associated risk coefficient

$$rc_i = [0 \;\; 1 \;\; 2 \;\; 3 \;\; 7 \;\; 8]$$

We assume that the returns processes for the assets and the salary (*) follow a multivariate normal distribution characterized by the following statistics

$$\mu = \begin{bmatrix} 0\% \\ 1.5\% \\ 2.0\% \\ 4.5\% \\ 5.0\% \\ 5.5\% \\ 1.0\%^* \end{bmatrix} \qquad \sigma = \begin{bmatrix} 0\% \\ 1.5\% \\ 2.0\% \\ 9.5\% \\ 10.0\% \\ 10.5\% \\ 1.0\%^* \end{bmatrix}$$

$$\mathrm{corr} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0^* \\ 0 & 1 & 0.9 & -0.1 & -0.1 & -0.1 & 0^* \\ 0 & 0.9 & 1 & 0 & 0 & 0 & 0^* \\ 0 & -0.1 & 0 & 1 & 0.9 & 0.8 & 0.9^* \\ 0 & -0.1 & 0 & 0.9 & 1 & 0.9 & 0.9^* \\ 0 & -0.1 & 0 & 0.8 & 0.9 & 1 & 0.9^* \\ 0^* & 0^* & 0^* & 0.9^* & 0.9^* & 0.9^* & 1^* \end{bmatrix}$$

In (15) the propensity to save parameter ps is 7%, while the employer contribution e is 50%. Moreover, we let the solver free to find the best here-and-now solution by setting null the initial portfolio, i.e. $h_{i,0} = 0 \;\; \forall i$, and accumulating the whole wealth as extra initial contribution w_0. The initial salary is settled for each representative to 15,000 euros, i.e. $sal_{t_0,s} = 15000$. This choice is driven by the evidence of an highly dishomogeneous salary level of the clusters elements, thus, we adopt as fixed initial salary the average net salary of the whole population.

In the multicriteria objective function (26) the Average Value at Risk Deviation (AV@RD) is computed considering a confidence level $\alpha = 5\%$. The intermediate wealth target constraint (28) is always defined at the second stage. The stochastic tree grows on six stages and the tree branching is 8-5-5-5-5, then 5,000 scenarios. We propose different time lengths between stages according to the representative member we are going to consider.

4.2.3 Results

The first representative participant is characterized by an initial wealth of 104,000 euros, $w_0 = 104000$, by a very low risk profile, $R = 1$, and by thirteen remaining working years, then we define that the time length between the six stages is 2, 2, 2, 2 and 5 years respectively. The optimal dynamic allocation is shown in Figure 5, where the white asset is the guaranteed capital security, and then the asset risk level is identified by the color: from dark green the less risky, to dark red the riskiest.

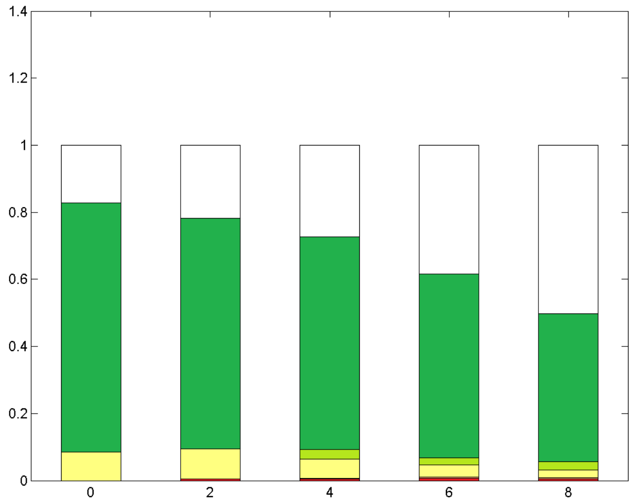


Figure 5: Allocation percentage - First representative member

The dynamic evolution of the asset allocation highlights a slow shift to a less risky portfolio in the final stage. The main used asset is the low risk one with a partial constant diversification adopting some more risky assets. The allocation in the guaranteed capital asset is remarkable in the here-and-now solution and increases through the stages reaching more than the 50% of the portfolio. The evolution of the wealth through the stages is shown in Figure 6.

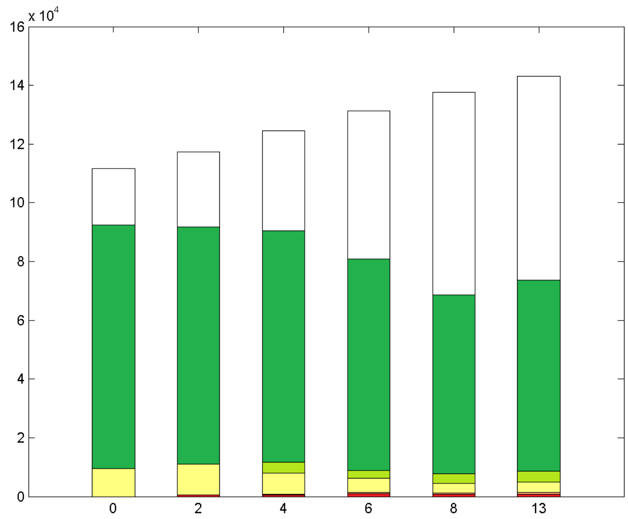
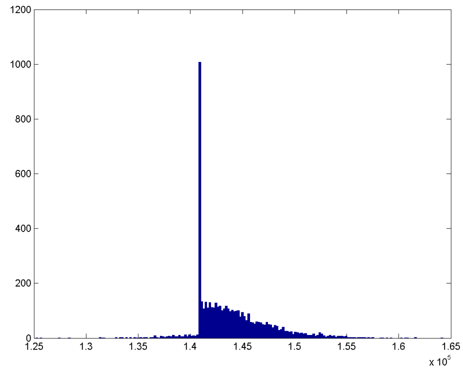


Figure 6: Wealth evolution - First representative member

The wealth slightly increases principally because of the investor's contribution and residually because of the financial returns. The allocation is mainly determined by the relatively short horizon and the very low risk/reward profile of the first representative. The final wealth distribution and its statistics are shown in Figure 7.



mean	143,930
median	143,150
st. dev.	3,578
$V@R_{0.05}$	140,850
$AV@R_{0.05}$	140,310
kurtosis	5.17
skewness	0.94

Figure 7: Final wealth distribution and related statistics, First representative member

The distribution is highly concentrated around the final wealth target. Indeed, we observe that mean, median, V@R and AV@R are very close each other. This feature express both a very safe profile in terms of risk and a remarkable results in terms of AV@RD reduction.

The second representative member is characterized by an initial wealth of 43,000 euros, $w_0 = 43000$, by a very low risk profile, $R = 1$, and by twenty seven remaining working years, then we define that the time length between the six stages is 5, 5, 5, 5 and 7 years respectively. The optimal dynamic allocation is shown in Figure 8. The optimal allocation is very similar to the first representative one. The whole dynamic brings the investor's choice from an initial prudential allocation to a very low risk portfolio. With respect to the previous representative the here-and-now solution is slightly more risky with a lower allocation in the guaranteed capital asset. The evolution of the wealth is shown in Figure 9.

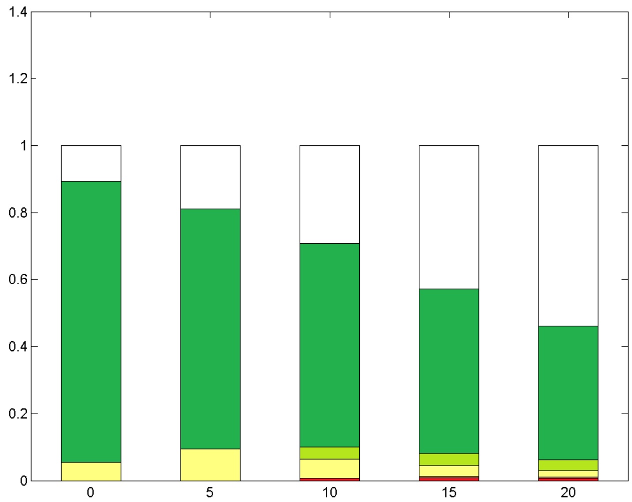


Figure 8: Allocation percentage - Second representative member

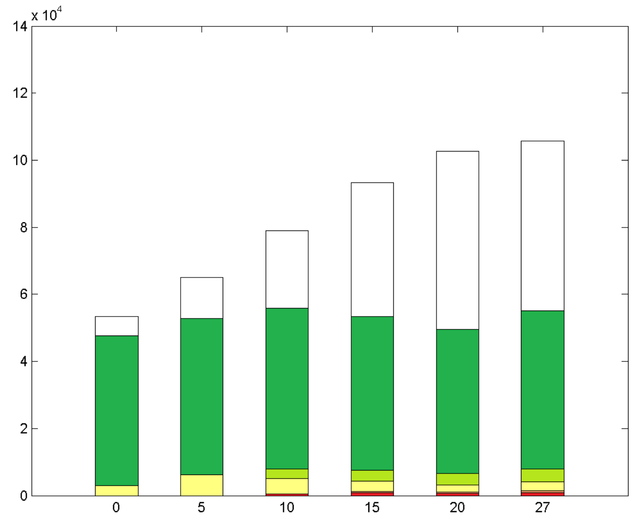
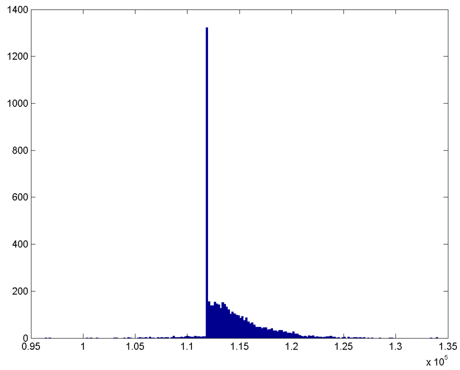


Figure 9: Wealth evolution - Second representative member

The wealth dynamic seems to be more rewarding than the first representative. The reason of this feature is mainly the longer horizon which allows an higher contribution level and increases the financial effect despite of the low risk profile. The final wealth distribution and its statistics are in Figure 10.



mean	114,180
median	113,370
st. dev.	2,933
$V@R_{0.05}$	111,930
$AV@R_{0.05}$	111,640
kurtosis	6.97
skewness	1.21

Figure 10: Final wealth distribution and related statistics, Second representative member

As the previous representative, the distribution is concentrated on the final target and the three location values are very close each other. The standard deviation is even less than the first representative and also the AV@RD is lower as well. The risk/reward investor's profile is fully respected in terms both of allocation and of final wealth distribution.

The third representative member is characterized by an initial wealth of 66,000 euros, $w_0 = 66000$, by a medium risk profile, $R = 3$, and by twenty eight remaining working years, then we define that the time length between the six stages is 5, 5, 5, 6 and 7 years respectively. The optimal dynamic allocation is shown in Figure 11.

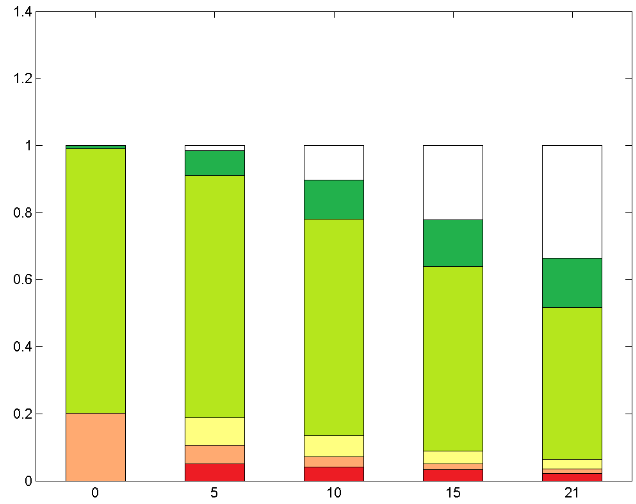


Figure 11: Allocation percentage - Third representative member

The allocation is significantly more risky than the two previous representative. The guaranteed capital asset enters the portfolio only from the second stage. The here-and-now solution involves only risky assets. The dynamic moves again the portfolio on a more safe allocation getting close to the final stage. The riskiest asset is included within the allocation from the second stage and its allocation remains almost constant till the end. The most used asset is the medium risk/reward one which fully reflects the investor's profile. The evolution of the wealth through the stages is shown in Figure 12.

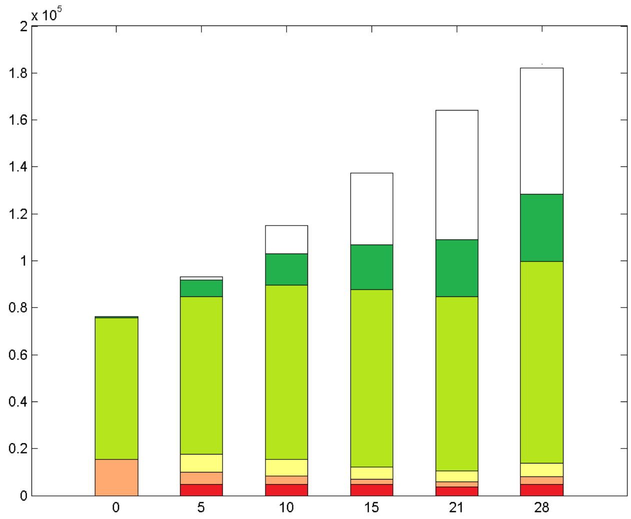
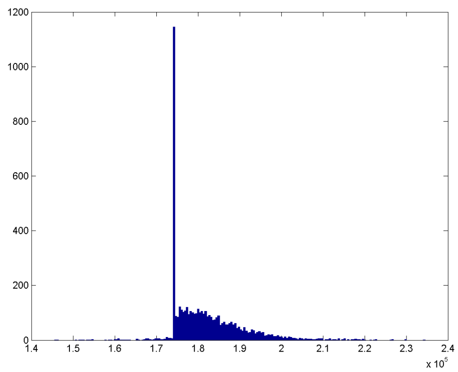


Figure 12: Wealth evolution - Third representative member

The wealth evolution benefits from both a long-term horizon and a quite risky allocation. The allocation in all the assets is almost constant in terms of wealth. It seems that the new contribution and the financial gains are used to increase the proportion of guaranteed capital asset stage by stage. The final wealth distribution and its statistics are shown in Figure 13.



mean	181,940
median	179,920
st. dev.	8,833
$V@R_{0.05}$	174,210
$AV@R_{0.05}$	173,560
kurtosis	6.01
skewness	1.30

Figure 13: Final wealth distribution and the related statistics, Third representative member

The final wealth statistics highlight again a very concentrated distribution. With respect to the previous ones, the standard deviation is higher but it is compensated by an higher returns. The AV@RD is still low guaranteeing a good quality of the solution. The fourth representative member is characterized by an initial wealth of 38,000 euros, $w_0 = 38000$, by a high risk profile, $R = 10$, and by thirty three remaining working years, then we define that the time length between the six stages is 6, 6, 7, 7 and 7 years respectively. The optimal dynamic allocation is in Figure 14.

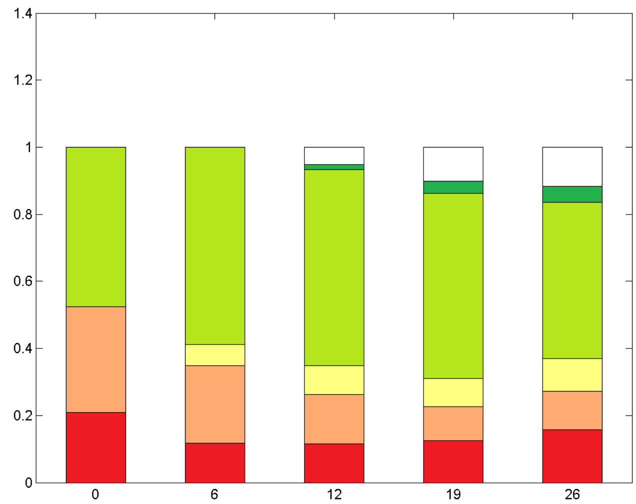


Figure 14: Allocation percentage - Fourth representative member

The fourth representative is the youngest and the most risk lover among the four representatives. The allocation reflects the risk attitude including a huge portion of the riskiest assets in the here-and-now solution and moving only a residual part of the portfolio to the guaranteed capital asset in the last stages. The evolution of the wealth through the stages is shown in Figure 15.

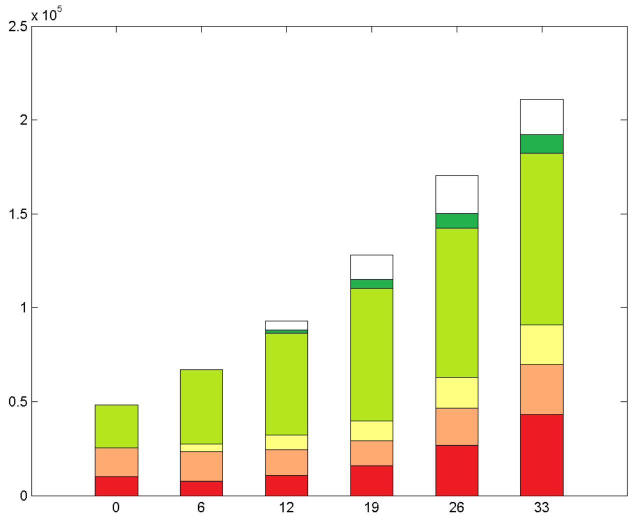
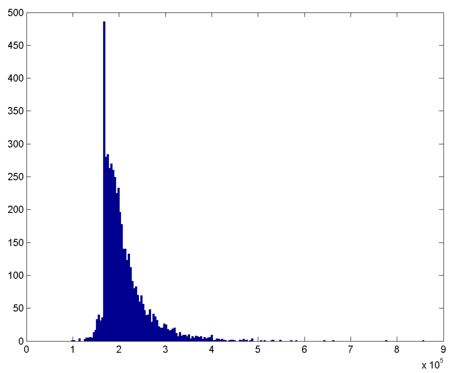


Figure 15: Wealth evolution - Fourth representative member

The wealth gain is huge thanks to the financial gains which largely affect the solution. The long-term horizon helps and enlarge this feature. The final wealth distribution and its statistics are shown in Figure 16.



mean	211,110
median	196,260
st. dev.	52,947
$V@R_{0.05}$	166,670
$AV@R_{0.05}$	160,086
kurtosis	17.56
skewness	2.77

Figure 16: Final wealth distribution and related statistics, Fourth representative member

The final wealth statistics fully reflects the investor's profile. The distribution is less concentrated on the final target than the previous representatives ones. The standard deviation is quite high but the AV@R value is still satisfying and the AV@RD is not too high as well. Clearly, the investor risk/reward profile determines the allocation and the final statistics are its natural consequence.

To summarize, the four optimizations produce four dynamic investment strategies, one for each of the representative members. According to the aim of the proposed analysis, the optimal pension funds portfolios are the here-and-now solutions of each strategy and are called pension funds A, B, C and D. The pension funds compositions are reported in Table 7. The percentage allocation is almost similar in fund A and fund B which invest the 91% and the 95% in the two lower risk assets and only a residual percentage in the medium risk asset. Fund C moves to a more balanced allocation by investing the 20% in a high risk security and nothing in the guaranteed capital asset. The most aggressive fund is D which allocates more than 50% in the two riskiest assets. Analyzing the wealth evolution, it is clear that young representative members can afford riskier positions and achieve higher returns than older investors. Consequently, the final wealth distribution reflects the portfolio risk attitude. For the first and the second fund the distance between the mean and the AV@R is smaller than the other two. The kurtosis values highlight fatter tails for the third and the fourth investor. The features of the dynamic allocations and the statistics of the final wealth distributions represent the risk/reward level of each pension funds.

pension fund	A	B	C	D
guaranteed capital	17%	11%	0%	0%
low risk 1	74%	84%	1%	0%
low risk 2	0%	0%	79%	48%
medium risk	9%	5%	0%	0%
high risk 1	0%	0%	20%	31%
high risk 2	0%	0%	0%	21%

Table 7: Pension funds allocations

The constitution of a suitable set of pension funds is a crucial goal for a pension plan sponsor. In particular, in case the pension plan is offered to a homogeneous group of people, the sponsor should analyze the population and offer a product accordingly to its features and needs. In the proposed case study, the optimal portfolios which are the pension funds that the pension plan sponsor should issue, have been created after the clusterization of the population in four classes. The number of pension funds follows the number of clusters. Clearly, the pension plan sponsor has to decide if the proposed investment strategies are sufficiently different from each other, in order to justify the implementation of all of them. The correct balance between the pension fund effort and the members satisfaction is hard to reach. In our case, pension funds A and B have a similar composition, therefore the sponsor could decide to join them in a single fund or to maintain all of them to better fit the population features. Indeed, the next step would be to implement an individual portfolio optimization in order to define for each real member, the optimal allocation using as asset universe the issued pension funds.

5 INDIVIDUAL PENSION PROBLEM

The purpose of this chapter is to define the optimal asset allocation for an employee in a retirement perspective having as asset universe the pension funds identified in the previous chapter. This aim is achieved by a dynamic stochastic programming approach. We describe a formulation that partially follows the pension plan sponsor multistage stochastic problem. Therefore, we investigate only the new parts of the model proposing an extension both in the constraints setting and in the targets construction.

5.1 MULTISTAGE STOCHASTIC MODEL

The main innovative elements are the investor's behavior formulation and the stochastic dominance constraints which join and substitute the deterministic ones. As before, we suppose that the decision times correspond to all the stages but the last one in which we just compute the accumulated final wealth. The stochasticity arises from three sources: the pension funds returns, the salary process and the investor's behavior. In general, the investment universe is composed of n different pension funds which are the portfolios we found in the previous chapter.

In particular, $\rho_{i,t,s}$ are the pension funds returns where the index $i = 1, ..., n$ represents the fund, $t = t_0, ..., T$ represents the stages and $s = 1, ..., S$ represents the scenarios, and $\rho^{sal}_{t,s}$ is the salary growth rate. The pension funds returns and the salary stochastic processes are modeled as correlated Geometric Brownian Motions having mean μ_i and standard deviation σ_i for each process and correlation matrix $corr_{i,j}$. The investor's behavior is the choice of the investor to withdraw an amount of money from the pension fund. We assume that this decision can be made only in one predetermined stage with a certain probability. The stochasticity is introduced using again a discrete scenario tree.
We define the non negative decision variables: $l_{i,t,s}$, $c_{i,t,s}$, $r^+_{i,t,s}$ and $r^-_{i,t,s}$. Respectively, $l_{i,t,s}$ is a binary variable and represents the allocation choice, i.e. it is equal to 1 only if we choose to invest in the pension fund i, on the stage t, in the scenario s; then $c_{i,t,s}$ expresses the level of contribution we want to invest in this fund; the rebalancing variables $r^+_{i,t,s}$ and $r^-_{i,t,s}$ allow the redistribution of the accumulated wealth among the chosen pension funds quantifying respectively how much we buy and how much we sell of each of them at the beginning of each stage, i.e. before adding the contribution.
Finally, we can list the set of constraints in order to express the regulatory bounds and the cash balance conditions. The constraints according to the salary process (14) and the contribution amount (15) are now referred to the pension funds instead of the assets, except this only difference they are equal. We must include some specific constraints in addition to the previous ones, considering the reduced investment universe

and some regulatory bounds.

Diversification bound

At most we can invest in L pension funds up to the n funds available:

$$\sum_{i=1}^{n} l_{i,t,s} \leqslant L, \quad \forall t, \forall s. \tag{36}$$

Total minimum contribution

The individual who decides to enter in the pension plan must save at least a minimum amount (m_1):

$$\sum_{i=1}^{n} l_{i,t,s} \cdot c_{i,t,s} \geqslant m_1, \quad \forall t, \forall s. \tag{37}$$

Single pension fund minimum contribution

The pension plan requires a minimum amount (m_2) invested in each chosen pension fund:

$$l_{i,t,s} \cdot c_{i,t,s} \geqslant m_2 \cdot l_{i,t,s}, \quad \forall i, \forall t, \forall s. \tag{38}$$

Allowed contribution

Clearly, $c_{i,t,s}$ is positive only if we choose the corresponding pension fund, in this case it must satisfy (15), (37) and (38) as well:

$$c_{i,t,s} \leqslant l_{i,t,s} \cdot M, \quad \forall i, \forall t, \forall s. \tag{39}$$

Portfolio Balance

The following constraints describe the portfolio allocation, the rebalancing decisions and the wealth account. They are almost equal to the corresponding constraints in the pension plan sponsor model. We only need to reformulate them according to the new investment universe and the behavior variable. Therefore, we introduce the holding variable $h_{i,t,s}$ which represents the amount we hold in each pension fund, and the total wealth variable $w_{t,s}$. Moreover, we define the *initial portfolio* vector $h_{i,0}$ in case the investor has already a position in the pension fund, and the *initial cash* parameter w_0 if the investor wants to add an amount of money, e.g. a shift from another pension fund and/or an initial extra contribution. The coefficient $d_{t,s}$ represents the stochastic invested wealth after withdraw, in percentage. Thus, $d_{t,s} = 1 - b_d$, where b_d is the withdraw percentage, for the nodes in a predetermined stage and $d_{t,s} = 1$ otherwise:

$$h_{i,t_0,s} = r^+_{i,t_0,s} - r^-_{i,t_0,s} + c_{i,t_0,s} + h_{i,0}, \quad \forall i, \forall s. \tag{40}$$

$$\sum_{i=1}^{n} r^+_{i,t_0,s} = \sum_{i=1}^{n} r^-_{i,t_0,s} + w_0, \quad \forall s. \tag{41}$$

$$r^-_{i,t_0,s} \leqslant h_{i,0}, \quad \forall i, \forall s. \tag{42}$$

$$\sum_{i=1}^{n} r^-_{i,t_0,s} \leqslant \theta \sum_{i=1}^{n} h_{i,0}, \quad \forall s. \tag{43}$$

$$h_{i,t,s} = h_{i,t-1,s} \cdot (1 + \rho_{i,t,s}) \cdot d_{t,s} + r^+_{i,t,s} - r^-_{i,t,s} + c_{i,t,s}, \quad \forall i, t_0 < t < T, \forall s. \tag{44}$$

$$w_{t,s} = \sum_{i=1}^{n} (h_{i,t-1,s} \cdot (1 + \rho_{i,t,s})) \cdot d_{t,s}, \quad t > t_0, \forall s. \tag{45}$$

$$\sum_{i=1}^{n} r^+_{i,t,s} = \sum_{i=1}^{n} r^-_{i,t,s}, \quad t_0 < t < T, \forall s. \tag{46}$$

$$r^-_{i,t,s} \leqslant h_{i,t-1,s} \cdot (1 + \rho_{i,t,s}), \quad \forall i, t_0 < t < T, \forall s. \tag{47}$$

$$\sum_{i=1}^{n} r^-_{i,t,s} \leqslant \theta \cdot w_{t,s}, \quad t_0 < t < T, \forall s. \tag{48}$$

$$r^+_{i,t,s} \leqslant l_{i,t,s} \cdot M, \quad \forall i, t < T, \forall s. \tag{49}$$

Equations (40)-(43) and (46)-(48) reproduce exactly equations (16)-(19) and (21)-(23), respectively. Equation (44) defines the holding as capitalization of the previous holding for each pension fund (decreased by the withdraw $d_{t,s}$) plus the reallocation of the accumulated wealth and plus the contribution. With

(49) we ensure that the buying follows the strategy choice. Finally, equation (45) computes the accumulated wealth in each stage for each scenario considering again the withdraw variable $d_{t,s}$. According to this wealth variable we build the target constraints and the objective function.
Using a stochastic tree structure, we include the set of all the nonanticipativity constraints on the decision variables.
We define the multicriteria objective function including two wealth targets and the Average Value at Risk Deviation (AV@RD) as risk measure. We refer to this model as the Deterministic Wealth Target (DWT). We adopt the ϵ-Constrained Approach fixing two wealth targets:

$$\begin{aligned}
\min \quad & \sum_{s=1}^{S} (w_{T,s} \cdot p_s) - a + \frac{1}{\alpha} \sum_{s=1}^{S} (z_s \cdot p_s) && (50)\\
\text{s.t.} \quad & -a + w_{T,s} + z_s \geqslant 0, \quad z_s \geqslant 0 && (51)\\
& \sum_{s=1}^{S} w_{t_{int},s} \cdot p_s \geqslant \Pi_{t_{int}} && (52)\\
& \sum_{s=1}^{S} w_{T,s} \cdot p_s \geqslant \Pi_T && (53)\\
& (14)-(15), \quad (36)-(49) && (54)
\end{aligned}$$

In (50) we minimize the AV@RD on the last stage, i.e. on the final wealth, for a given confidence level α. According to Rockafellar and Uryasev (2000, 2002), the discrete definition of the AV@RD needs the inequality (51) in order to define jointly the variables a and z_s. The first wealth target (52) forces the average of the accumulated wealth on an intermediate stage (t_{int}) to be greater than or equal to a fixed amount $\Pi_{t_{int}}$. Similarly, the final wealth target (53) is fixed at the level Π_T. As in the previous model, these two targets assume at first a deterministic value. Further, we propose two more formulations using the Stochastic Dominance (SD) concept. In particular, (52) and (53) are replaced with the First order Stochastic Dominance (FSD) and the Second order Stochastic Dominance (SSD) constraints in two distinct models. For this purpose, the definition of a benchmark wealth is needed and we adopt the same formulation used for the definition of the benchmark in Section 4.2.1 including the withdraw variable $d_{t,s}$. Then we replace (32) with

$$h^b_{t,s} = h^b_{t-1,s} \cdot (1 + \rho^b_{t,s}) \cdot d_{t,s} + C^b_{t_0,s} \quad t > t_0, \forall s \tag{55}$$

We define the two targets of the DWT simply as the expected value of the benchmark wealth. Then, the values of the two targets become as follows

$$\Pi_{t_{int}} = \mathbb{E}[w^b_{t_{int},s}] \tag{56}$$

$$\Pi_T = \mathbb{E}[w^b_{T,s}] \tag{57}$$

In order to implement the Stochastic Dominance constraints, since we deal with a finite number of scenarios and they are equiprobable, it is useful to define the FSD conditions on an intermediate stage and on the final stage using the vector formulation as proposed in Kuosmanen (2004):

$$\mathbf{w}_{t_{int}} \geqslant \mathbf{P}_{t_{int}} \cdot \mathbf{w}^b_{t_{int}} \tag{58}$$

$$\mathbf{w}_T \geqslant \mathbf{P}_T \cdot \mathbf{w}^b_T \tag{59}$$

$\mathbf{P}_{t_{int}}$ and $\mathbf{P}_T$ represent two square permutation matrices where all elements are binary variables satisfying the following conditions

$$\sum_i P_{i,j} = 1 \tag{60}$$

$$\sum_j P_{i,j} = 1 \tag{61}$$

The SSD formulation is basically the same:

$$\mathbf{w}_{t_{int}} \geqslant \mathbf{W}_{t_{int}} \cdot \mathbf{w}^b_{t_{int}} \tag{62}$$

$$\mathbf{w}_T \geqslant \mathbf{W}_T \cdot \mathbf{w}_T^b \tag{63}$$

And the elements of $\mathbf{W}_{t_{int}}$ and of $\mathbf{W}_T$ again must satisfy the following conditions

$$\sum_i W_{i,j} = 1 \tag{64}$$

$$\sum_j W_{i,j} = 1 \tag{65}$$

The only difference between matrices **W** and matrices **P** is that in **W** the elements do not have to be binary, but they just have to belong to the interval $[0, 1]$, i.e. each row and each column represent a convex combination.
The formulation with SD produces a Mixed Integer Program. It is still computationally manageable but we remark a time consuming optimization for the FSD case because of the high number of binary variables in the permutation matrices.

5.2 PROBLEM SETTINGS

The aim of this section is to propose an individual portfolio tool able to suggest the optimal investment in a retirement perspective for any of the pension fund participants, i.e. for any of the 5577 members we considered for the pension fund provider problem and for whom we defined the pension funds of the pension plan. Therefore, we choose randomly one of the active member and we define the setting of the model according to his/her features. The investment universe is composed of the four pension funds defined previously, in addition we include a pure guaranteed capital fund as imposed by the Italian regularization.

According to the pension funds composition reported in Table 7, we assume that the returns processes for the pension funds and the salary (*) follow a multivariate normal distribution characterized by the following statistics

$$\mu = \begin{bmatrix} 0\% \\ 1.51\% \\ 1.48\% \\ 2.64\% \\ 3.75\% \\ 1.0\%^{*} \end{bmatrix} \qquad \sigma = \begin{bmatrix} 0\% \\ 1.39\% \\ 1.29\% \\ 2.74\% \\ 5.41\% \\ 1.0\%^{*} \end{bmatrix}$$

$$corr = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0^{*} \\ 0 & 1 & 0.975 & 0.817 & 0.679 & 0.678^{*} \\ 0 & 0.975 & 1 & 0.746 & 0.545 & 0.529^{*} \\ 0 & 0.817 & 0.746 & 1 & 0.937 & 0.820^{*} \\ 0 & 0.679 & 0.545 & 0.937 & 1 & 0.908^{*} \\ 0^{*} & 0.678^{*} & 0.529^{*} & 0.820^{*} & 0.908^{*} & 1^{*} \end{bmatrix}$$

Moreover, their risky level is described by the associated risk coefficient, computed accordingly to the pension funds composition

$$rc_i = [0 \;\; 1 \;\; 1 \;\; 3 \;\; 5]$$

The chosen investor is a 25 year woman with a remaining working life of 40 years. She wants to achieve two wealth goals: a final pension benefit in 40 years and an intermediate objective after 8 years. For this reason and because of the Italian frequency rebalance rules, we propose a time structure composed of five periods, each of eight years, i.e. $\Delta t = 8$ and then $t = 0, 8, 16, 24, 32, 40$. She has a medium/high risk profile, $R = 5$. In (36) we assume that she cannot invest in more than four pension funds up to the five available, i.e. $L = 4$. The initial net salary is 15,000 euros, i.e. $sal_{t_0,s} = 15000$. In (15) the propensity to save parameter ps is 7%, while the employer contribution e is 50%. The single pension fund minimum contribution m_1 in (38) is 200 euros, while the total minimum contribution m_2 in (37) is 300 euros. We suppose that the investor just started to work, then her initial wealth is null, i.e. $w_0 = 0$, and the initial portfolio is null as well, i.e. $h_{i,0} = 0, \ \forall i$. The turnover coefficient θ is equal to 20%. Using this setting we define the portfolio balance equations (16)-(49).

In the multicriteria objective function (26) the Average Value at Risk Deviation (AV@RD) is computed considering a confidence level $\alpha = 5\%$. Moreover, the investor wants to achieve the intermediate target after eight years, i.e. $t_{int} = 8$. As particular case of the SSD formulation, we analyze the sensitivity of the optimal allocation with respect to the investment behavior, in particular we assume that the probability to observe a withdraw is 50%, the value of the withdraw is $d_{t,s} = 0.70$ and that this process is independent from the other stochastic processes.

In order to compare the three formulations (DWT, FSD, SSD), we propose several experiments. Thus, the model changes accordingly to the formulation:

- DWT: (14)-(15) and (36)-(57);
- SSD: (14)-(15), (36)-(51), (31)-(33) and (62)-(65);
- FSD: (14)-(15), (36)-(51), (31)-(33) and (58)-(61).

Moreover, for the DWT case we adopt a regular branching equal to 5-5-3-3-3 for the five periods, i.e. 675 scenarios, for the SSD and the FSD we have to reduce it to 5-3-3-3-3, i.e. 405 scenarios, for computational complexity issues.

The choice of the branching is confirmed by the following stability analysis. For each scenarios cardinality we run the model 100 times and we represent the optimal values in a box-plot form. The results for the DWT case is shown in Figure 17.

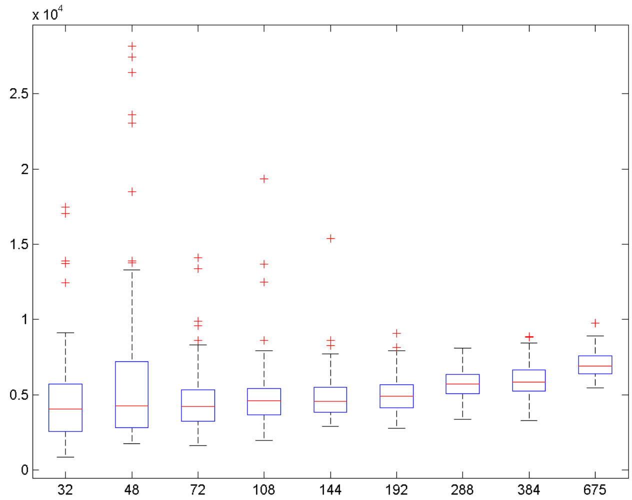


Figure 17: Stability Analysis - DWT

The goodness of the 675 scenarios choice is evident in terms of reduction of the optimal value volatility with respect to lower cardinalities. The results for the SSD case is shown in Figure 18.

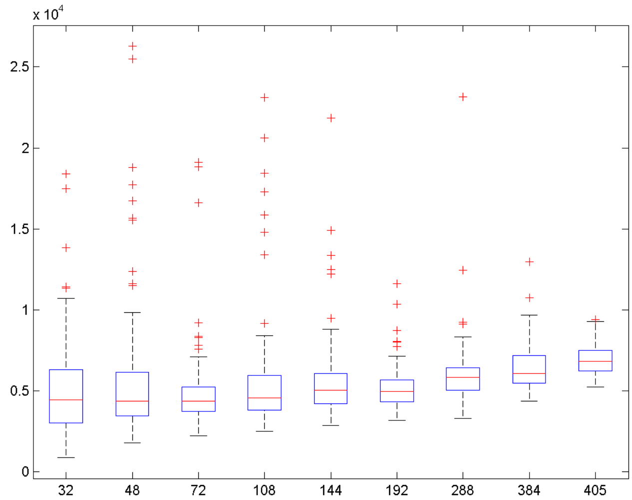


Figure 18: Stability Analysis - SSD

The 405 scenarios choice represents a good balance between volatility of the optimal value and manageability of the problem from a computational point of view.

For both cases we stop the analysis when the difference between the minimum and the maximum value is low enough to guarantee to the investor a comparable level of satisfaction.
We do not run the stability analysis for the FSD case because it is very time consuming taking more than one hour for each run in the 405 scenarios case. Thus, we adopt the maximum number of scenarios with which the model is still handleable.

5.3 RESULTS

5.3.1 Deterministic Wealth Target

Figure 19 shows the solution of the DWT case in terms of percentage optimal allocation adopting the wealth targets $\Pi_{t_{int}}$ and Π_T described in (56) and (57). The represented solution for each stage is the average solution of all the nodes of that stage.

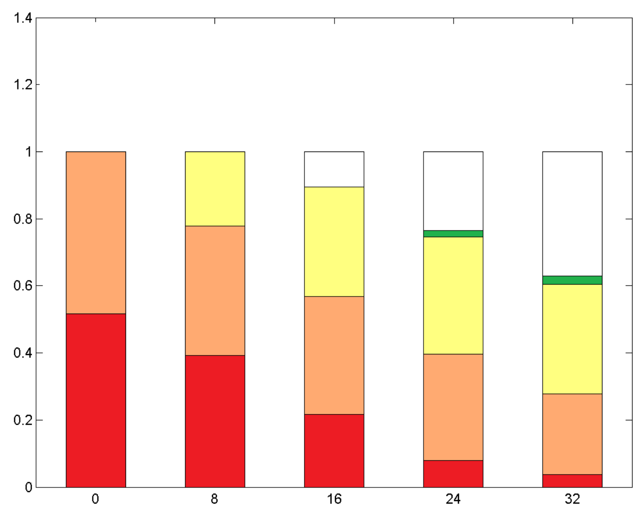


Figure 19: Dynamic Allocation - DWT

The allocation reflects the high risk/reward profile of the investor. The here-and-now solution involves only the two most risky pension funds. The guaranteed capital fund is included in the portfolio only since the third stage. The strategy moves to a safer allocation through the stages getting close to the final horizon. In the last decisional stage almost the 40% is invested in the guaranteed capital fund and less than the 30% in the two riskiest funds. In Figure 20 we present the average optimal allocation showing the increasing wealth.

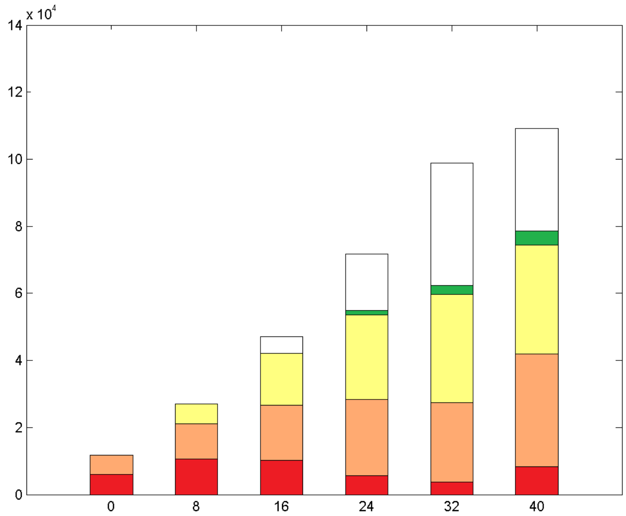
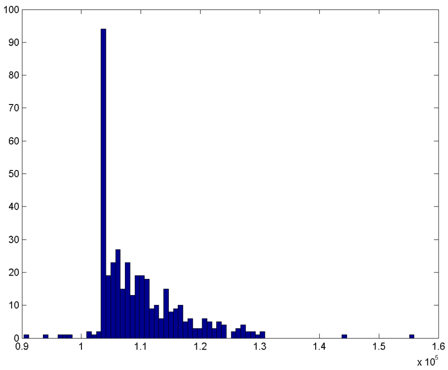


Figure 20: Dynamic Wealth - DWT

The wealth process benefits from both an aggressive allocation and a long-term horizon. The contribution and the financial gains define a remarkable increase through the stages.The portfolio portion invested in the riskiest funds increase till the third stage, then it stays constant and the gains and contributions are invested in the safer funds in order to reduce the risk in the last stages. The final wealth distribution and its statistics are shown in Figure 21.



mean	109,940
median	108,100
st. dev.	7,192
$V@R_{0.05}$	103,790
$AV@R_{0.05}$	103,230
kurtosis	7.68
skewness	1.53

Figure 21: Final wealth distribution, DWT

Comparing the whole time horizon, the wealth return is huge but in the last stage the riskiness, i.e. the kurtosis and the standard deviation of the final wealth distribution, is not too high. Also the difference between the mean and the AV@R is small. In general, we observe a well balanced diversification in the five pension funds and through the stages. The risk/reward profile is quite aggressive, but it takes advantage of the double diversification (by investing in a combination of pension funds which are themselves diversified portfolios of assets) to achieve the risk targets.

5.3.2 Second order Stochastic Dominance

In Figure 22 we present the average optimal solution of the SSD case as percentage allocation. As in the previous case, the represented solution for each stage is the average solution of all the nodes of that stage. The here-and-now solution is more aggressive than the DWT case. More than the 90% is invested in the riskiest pension fund. The main difference with the DWT case is a fast shift on a more safe and a more diversified portfolio which guarantees the second order stochastic dominance and a great risk reduction in the last stages. The turnover coefficient is fully exploited. Since the second stage the guaranteed capital fund is included in the portfolio and in the last decisional stage it represents more than the 40% of the portfolio. In Figure 23 we present the average optimal allocation showing the increasing wealth.

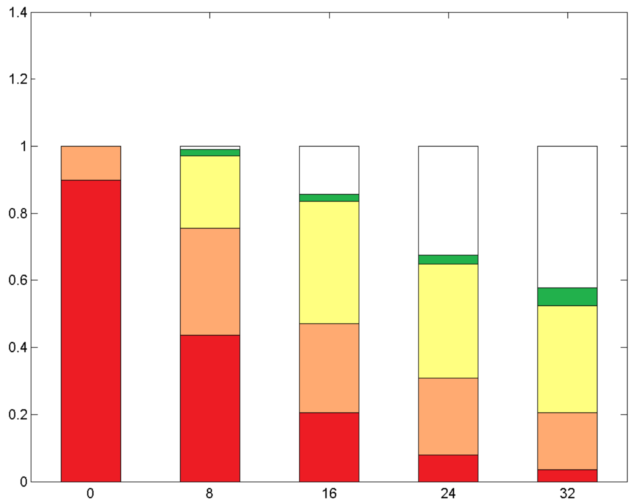


Figure 22: Dynamic Allocation - SSD

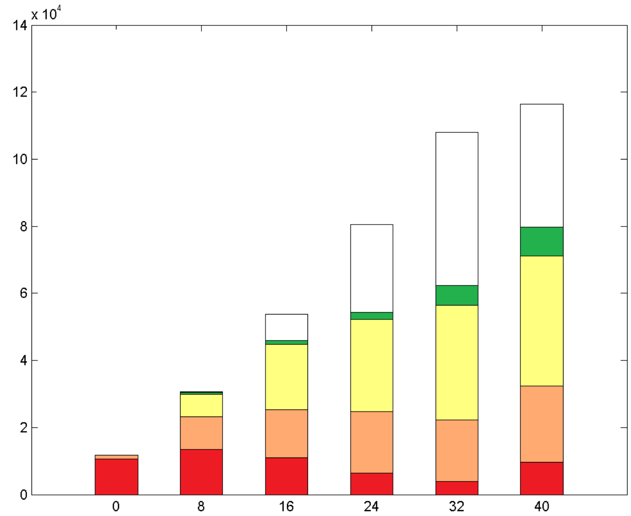
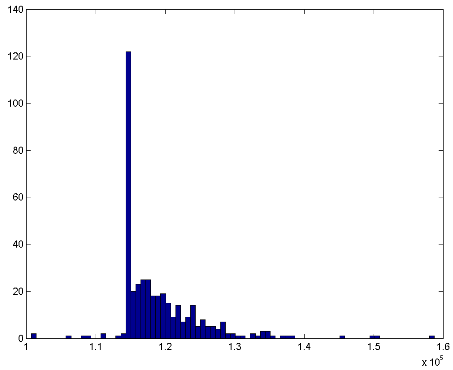


Figure 23: Dynamic Wealth - SSD

As in the DWT case, the wealth process grows significantly through the stages. The portfolio portion allocated in the riskiest fund decrease not only in percentage terms but also in monetary terms along the decisional stages. The final wealth distribution and its statistics are shown in Figure 24.



mean	118,970
median	117,220
st. dev.	6,165
$V@R_{0.05}$	114,410
$AV@R_{0.05}$	113,890
kurtosis	10.85
skewness	2.03

Figure 24: Final wealth distribution, SSD

The portfolio is more balanced in terms of composition and this leads to higher values of V@R and AV@R with respect to the DWT case. We observe a distribution shifted to the right having all the location statistics higher than the DWT case. Nevertheless, the final wealth distribution has a lower standard deviation remarking the high quality of the second order stochastic dominance riskier features in terms of kurtosis and standard deviation. The risk/reward profile is still coherent with the investor's characteristics.

In Figure 25 we propose a comparison between the distribution of the benchmark wealth and the final wealth achieved by the investor. In Figure 26 we show their cumulative distribution. The stochastic dominance of the second order is clearly satisfied as shown in Figure 26, while the first order stochastic dominance in Figure 25 is neither requested and then nor fulfilled.

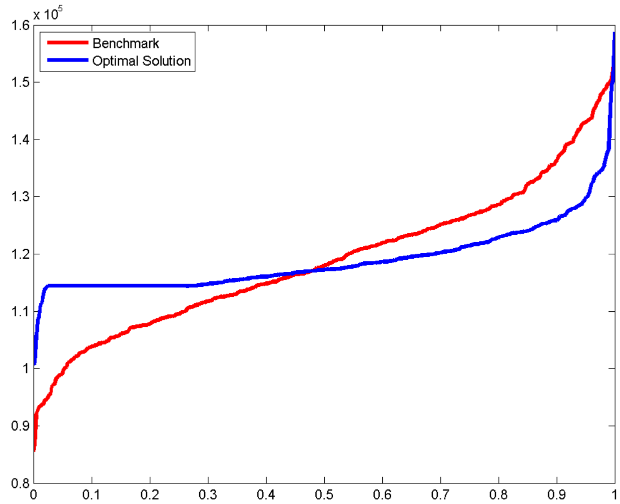


Figure 25: First Order Stochastic Dominance - SSD

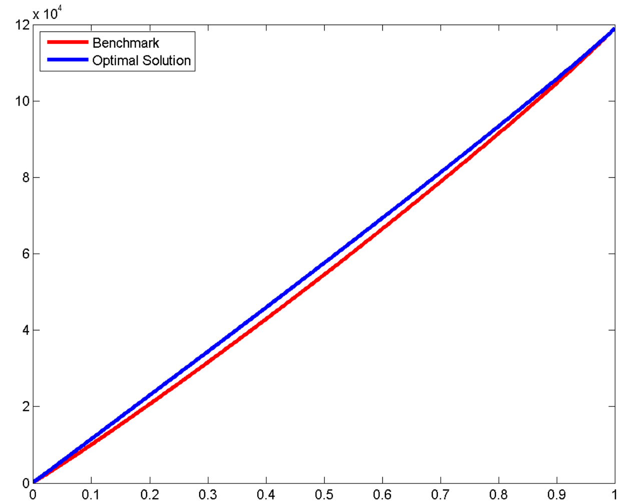


Figure 26: Second Order Stochastic Dominance - SSD

The optimal solution has a better left tail than the benchmark one but it suffers lower returns in the right part of the distribution. This behavior is due to the fact that the optimal allocation moves to the safer funds getting closer to the final horizon, while the benchmark portfolio represents an equidistributed choice which turns out to be much riskier.

5.3.3 Second order Stochastic Dominance – Withdraw case

In the Italian private pension system it is often possible to withdraw an amount of money from the existing pension account after a given minimum contribution period. A low percentage (around 30%) can be withdrawn for any reason at any time, while a larger amount (till 75%) is allowed for specific issues, e.g. health or housing, and only after eight years. For this reason, we propose a further formulation of the SSD framework in which the variable $d_{t,s}$, representing the withdraw choice, assumes an active role. In particular, we fix $d_{2,s} = 70\%$ to include the possibility that the investor decides to withdraw the 70% of the wealth accumulated on the second stage; the probability that this event occurs is fixed at 50% and it is independent from other stochastic processes. The second stage has been chosen because it occurs eight years after the first stage and, accordingly to the pension plan regulation, a huge withdraw is possible only after at least eight years. Clearly, we assume that the same event occurs also for the benchmark wealth in order to have a fair comparison, otherwise it would be impossible to reach the targets. We test also the cases in which the investor withdraws accordingly to the salary level, considering the case in which she withdraws both in the lower salary scenarios and in the higher ones. The results are almost the same, therefore, we show only the outputs of the independence case. In Figure 27 we present the optimal solution of the SSD case including the withdraw event as percentage allocation. As in the previous case, the represented solution for each stage is the average solution of all the nodes of that stage. The here-and-now allocation is completely concentrated in the riskiest pension fund. The withdraw in the second stage makes the portfolio seeking for a huge return in the first stages in order to compensate the withdraw and somehow to be prepared to that event. After that, in order to satisfy the second order stochastic dominance constraint and to reduce the objective risk measure, the portfolios quickly moves to the guaranteed capital pension fund till the last decisional stage in which more than the 60% of the portfolio is invested in it.

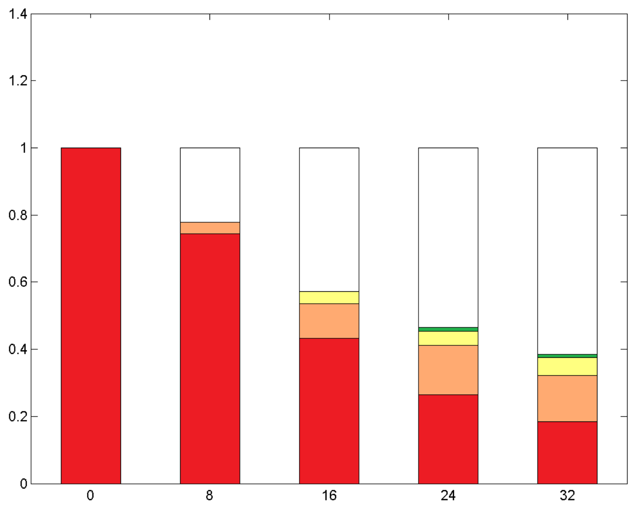


Figure 27: Dynamic Allocation - SSD withdraw

In Figure 28 we present the average optimal allocation showing the increasing wealth.

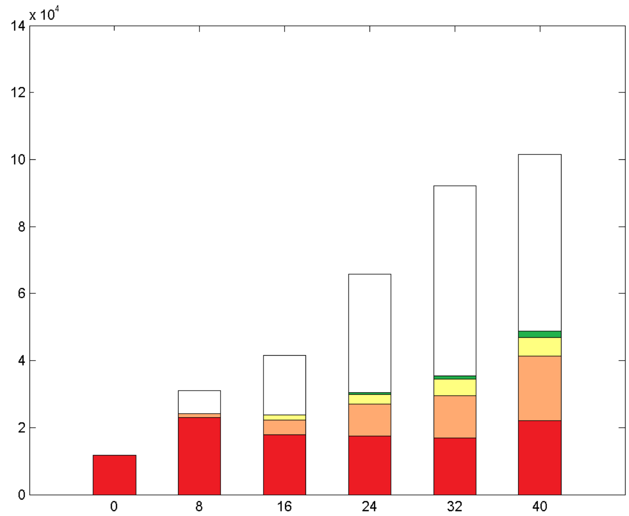
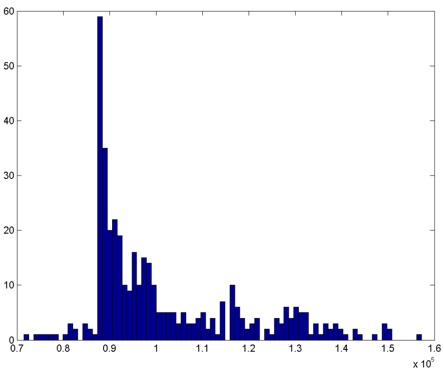


Figure 28: Dynamic Wealth - SSD withdraw

Clearly, the wealth evolution suffers the withdraw event which occurs after the computation of the wealth on the second stage. The dynamic is driven by the following steps: in the second stage the investor makes the contribution an the reallocation choice, he accounts the accumulated wealth, he decides if he might withdraw or not and, eventually, the withdraw is done at the end of the second stage inducing a visible effect on the third one. Indeed, the wealth of the third stage appears a step below its natural trend. The withdraw consists in the 70% of the wealth accumulated in the second stage and the event occurs with a probability of 50%. The remaining portfolio is still invested mainly in the most risky pension fund and in the third stage the investor makes the related contribution. Therefore, in the third stage we observe a reduction of the wealth but the trend is still positive. The final wealth distribution and its statistics are shown in Figure 29.



mean	101,880
median	95,433
st. dev.	16,837
$V@R_{0.05}$	87,928
$AV@R_{0.05}$	85,722
kurtosis	3.25
skewness	1.09

Figure 29: Final wealth distribution, SSD withdraw

Comparing the dynamic solution with the no withdraw case, we observe a more polarized allocation: more weight on the riskiest pension fund and the inclusion of the guaranteed capital fund since the second stage. The final wealth statistics highlights an aggressive portfolio which has an higher standard deviation and also an higher AV@RD both the absolute terms and in relative ones. In Figure 30 we propose a comparison between the distribution of the benchmark wealth and the final wealth accumulated by the investor. In Figure 31 we show their cumulative distribution.

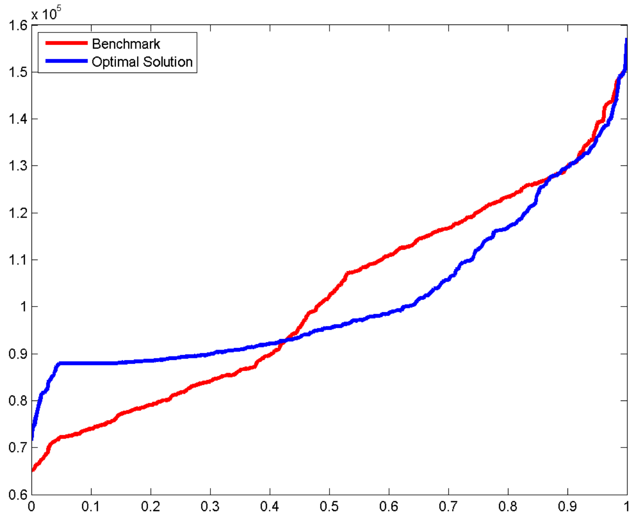


Figure 30: First Order Stochastic Dominance - SSD withdraw

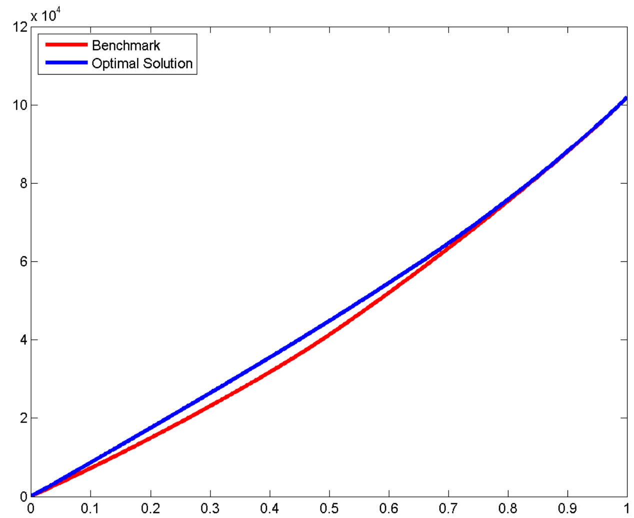


Figure 31: Second Order Stochastic Dominance - SSD withdraw

The second order stochastic dominance is fulfilled even if the two cumulated distribution are very close each other in the right part of Figure 31. The aggressive allocation allows the portfolio to compensate and reach the same top returns as shown in Figure 30 avoiding the worst losses on the left tail suffered by the benchmark portfolio. Therefore, even in the withdraw case, the second order stochastic dominance and the AV@RD reduction produce an optimal portfolio less riskier than the benchmark one but still able to reach the same returns along the best scenarios.

5.3.4 First order Stochastic Dominance

In Figure 32 we present the optimal solution of the FSD case as percentage allocation. As in the previous case, the represented solution for each stage is the average solution of all the nodes of that stage.

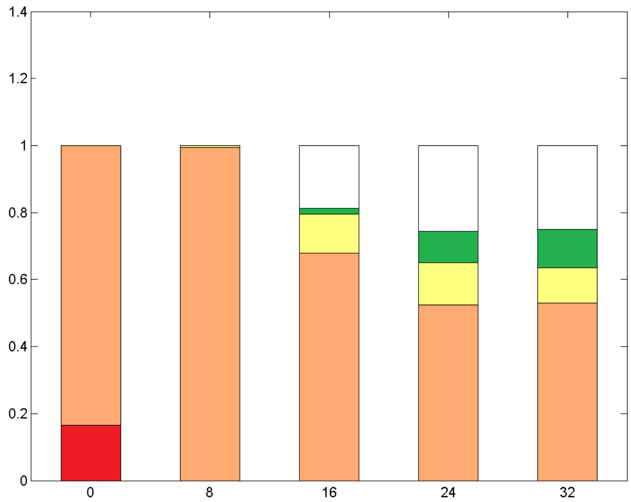


Figure 32: Dynamic Allocation - FSD

The dynamic allocation has as favorite pension fund one of the two most risky. The riskiest fund is used only in the here-and-now solution and the guaranteed capital pension fund enters the portfolio only since the third stage and slightly increases its portion which never goes above 25%. The other two low risk pension funds are used mainly as diversification assets getting closer to the final horizon. In Figure 33 we present the average optimal allocation showing the increasing wealth.

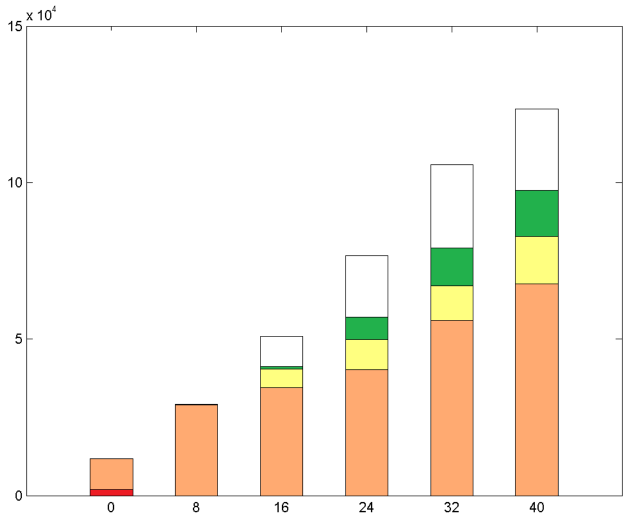
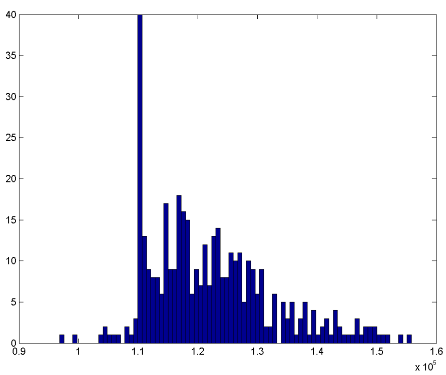


Figure 33: Dynamic Wealth - FSD

The wealth process highlights how the pension fund proportion is constantly fed through the stages while in the previous cases the wealth invested in the most risky funds remained fixed. The final wealth distribution and its statistics are shown in Figure 34.



mean	122,260
median	120,760
st. dev.	10,494
$V@R_{0.05}$	110,570
$AV@R_{0.05}$	109,270
kurtosis	3.29
skewness	0.78

Figure 34: Final wealth distribution, FSD

The final wealth statistics remark an aggressive portfolio which reach an higher final average wealth than the other cases. The V@R and the AV@R are higher than the DWT case but lower than the SSD ones. In this sense, the portfolio is riskier than the SSD one and the standard deviation confirms this feature. The reason of the higher average returns and meanwhile of the riskier allocation lies in the first order stochastic dominance constraint.

In Figure 35 we propose a comparison between the distribution of the benchmark wealth and the final wealth accumulated by the investor. In Figure 35 we show their cumulative distribution.

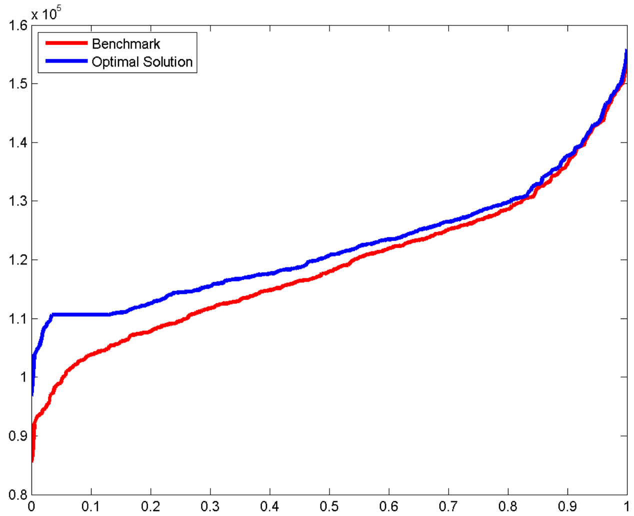


Figure 35: First Order Stochastic Dominance - FSD

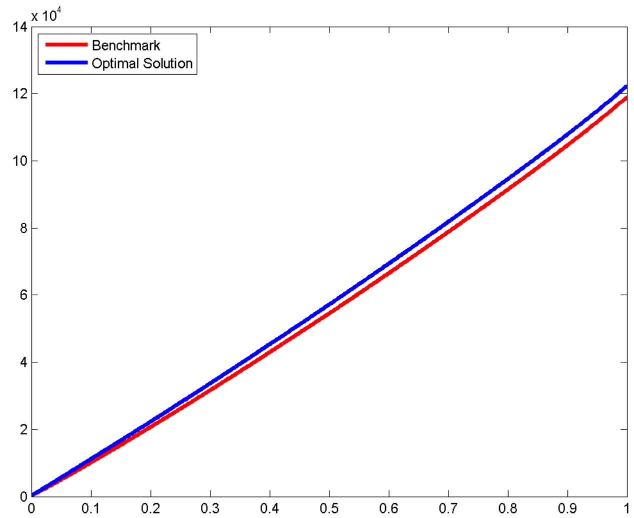


Figure 36: Second Order Stochastic Dominance - FSD

The first order stochastic dominance constraint is stricter than the second order one and then the optimal allocation has to move to a more aggressive portfolio in order to be sure to dominate the benchmark portfolio for all possible realizations of the stochastic variables, i.e. along all scenarios. The drawback is a more uncertain final wealth distribution.

5.3.5 Summary results

We faced the problem of a private investor who has to choose how to allocate his savings in a pension perspective. The stochastic dominance constraints produce a remarkable impact in the investment allocation. The evidences shown in Table 8 highlight that both the second order stochastic dominance and the first order stochastic dominance induce the here-and-now solution to be more and more aggressive in order to satisfy the target constraints in the second stage and then beat the benchmark in the stochastic dominance sense. Getting close to the final horizon, and in particular in the last decisional stage, the stochastic dominance constraints bring to a huge risk reduction which is reflected in a greater diversification and in a wealth shift to a more conservative allocation. Through the stages, the FSD is more aggressive than the SSD to beat the benchmark for all possible scenarios. Indeed, even in the last decisional stage, more than 50% is allocated in one of the most risky pension fund.

	Here-and-now				Last decisional stage			
	DWT	SSD	SSDw	FSD	DWT	SSD	SSDw	FSD
Fund 1	0	0	0	0	38	41	61	25
Fund 2	0	0	0	0	2	5	1	12
Fund 3	0	0	0	0	32	33	6	10
Fund 4	49	9	0	83	25	18	13	53
Fund 5	51	91	100	17	3	3	19	0

Table 8: First stage and last stage solution for each model (in %)

	DWT	SSD	SSDw	FSD
mean (e+04)	11.0	11.9	10.2	12.2
median (e+04)	10.8	11.7	9.5	12.1
st. dev. (e+04)	0.7	0.6	1.7	1.0
$V@R_{0.05}$ (e+04)	10.4	11.4	8.8	11.1
$AV@R_{0.05}$ (e+04)	10.3	11.4	8.6	10.9
kurtosis	7.68	10.85	3.25	3.29
skewness	1.53	2.03	1.09	0.78

Table 9: Summary statistics of the final wealth for each model

The withdraw event largely affect the portfolio choice in the first stage in which the allocation is completely concentrated in the most risky pension fund. The same case shown the most polarized final stage allocation with the 61% invested in the guaranteed capital pension fund and almost the 20% in the riskiest one.

The final wealth distribution statistics are reported in Table 9. The allocation choice is coherently reflected in the statistics. The more aggressive is the allocation, the higher are the mean and the median, the higher is the standard deviation. The only exception is the SSD case which is less riskier than the DWT and the FSD. The risk reduction is highlighted not only by the lower standard deviation, but also by the V@R and AV@R values which are remarkable higher for the SSD allocation.

To conclude, the stochastic dominance has a double effect according to the dominance order we want to impose. The FSD produces a more aggressive portfolio which is able to beat the benchmark for all the scenarios, the drawback is a slightly more risky performance. The SSD cannot surpass the benchmark for every scenarios but produces a well diversified portfolio which pushes the wealth higher then the DWT with a huge risk reduction. The V@R and the AV@R point out the high quality of the stochastic dominance framework and in particular of the SSD case. The SD formulation is better than the DWT one from almost all points of view. The choice to introduce the SSD or the FSD depends mainly on what the investor is looking for in terms of risk/reward targets.

6 CONCLUSION

In this thesis we analyzed the pension problem from the point of view of the three main actors involved: the fund manager who takes care of the strategical investment problem dealing with an ALM problem; the pension plan sponsor, who decides the tactical allocation of the pension funds issued; the individual investor who faces the problem to allocate his/her savings in a retirement perspective, .
All these problems consider a long-term choice and require to face some elements of uncertainty. Therefore, we dealt with them applying the state-of-the-art techniques provided by the Stochastic Programming.
We proposed an overview of the literature quoting the milestone works which made the history of the ALM and of the pension investment field, and we outlined the mathematical setting of the two-stage and multistage stochastic problems exploring some risk measures representable as chance constraints and the multicriteria setting.
The ALM problem for a pension fund has been widely investigated in the literature. After a brief description of an ALM model, our contribution has been to suggest a methodology to price the liability side of a pension fund in case the net payments need to be adjusted according to the inflation and in case we want to estimate the actual value of all the future payments.
The main innovative parts are the formulations proposed for the pension plan sponsor problem and for the individual investor's problem in a pension perspective. Our belief is that the importance and the impact of the plan sponsor problem is nowadays underestimated. Thus, we proposed a two-step approach in order to define the optimal pension plan offer for a homogeneous population of members.

The output is the allocation of the pension funds that the sponsor should issue. The concrete implementation of such pension funds has to be decided considering the suitability for the members and the manageability for the pension fund manager.
The individual investor's problem was already well known in the literature, our innovative proposal has been to include the stochastic dominance formulation within the model. To adopt stochastic dominance constraints is more demanding from a model, formula and computation points of view. However, the solutions highlighted a tangible difference both in the allocation and in the quality of the risk/return profile of the portfolio. The difference between the DWT, the SSD and the FSD solutions is clear and well define, and the advantages of the SD formulation have been proved in the related chapter. The SD framework induces a better balanced portfolio than the DWT case. The risk diminution is more evident in particular observing the V@R and the AV@R values which represent the left tail features and then the SD reduces the huge losses problem. Therefore, we strongly recommend a stochastic dominance formulation for an optimal individual allocation in a pension perspective.

BIBLIOGRAPHY

Jozsef Abaffy, Marida Bertocchi, Jitka Dupačová, Vittorio Moriggia, and Giorgio Consigli. Pricing nondiversifiable credit risk in the corporate Eurobond market. *Journal of Banking & Finance*, 31(8): 2233–2263, 2007.

Shabbir Ahmed and Alexander Shapiro. The sample average approximation method for stochastic programs with integer recourse. *Submitted for publication*, 2002.

Evelyn M. L. Beale. On minimizing a convex function subject to linear inequalities. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 173–184, 1955.

Adam J. Berger and John M. Mulvey. The Home Account Advisor: asset and liability management for individual investors, 1998.

Marida Bertocchi, Jitka Dupačová, and Vittorio Moriggia. Sensitivity analysis of a bond portfolio model for the Italian market. *Control and Cybernetics*, 29:595–615, 2000a.

Marida Bertocchi, Vittorio Moriggia, and Jitka Dupačová. Sensitivity of bond portfolio's behavior with respect to random movements in yield curve: A simulation study. *Annals of Operations Research*, 99 (1-4):267–286, 2000b.

Marida Bertocchi, Sandra L. Schwartz, and William T. Ziemba. *Optimizing the aging, retirement, and pensions dilemma*, volume 477. John Wiley & Sons, 2010.

Marida Bertocchi, Giorgio Consigli, Rita D'Ecclesia, Rosella Giacometti, Vittorio Moriggia, and Sergio Ortobelli. Euro bonds: Markets, infrastructure and trends. *World Scientific Books*, 2013.

John R. Birge. State-of-the-art-survey-stochastic programming: Computation and applications. *INFORMS journal on computing*, 9(2): 111–133, 1997.

John R. Birge and Francois Louveaux. *Introduction to stochastic programming*. Springer, 1997.

John R. Birge and Francois Louveaux. *Introduction to stochastic programming*. Springer Science & Business Media, 2011.

David Blake, Douglas Wright, and Yumeng Zhang. Target-driven investing: Optimal investment strategies in defined contribution pension plans under loss aversion. *Journal of Economic Dynamics and Control*, 37(1):195–209, 2013.

Guus C. E. Boender. A hybrid simulation/optimisation scenario model for asset/liability management. *European Journal of Operational Research*, 99(1):126–135, 1997.

Guus C. E. Boender, Paul C. van Aalst, and Fred Heemskerk. *Modelling & management of assets & liabilities of pension plans in the Netherlands*. Erasmus University Rotterdam, 1997.

Erik Bogentoft, Edwin H. Romeijn, and Stanislav Uryasev. Asset/liability management for pension funds using CVaR constraints. *The Journal of Risk Finance*, 3(1):57–71, 2001.

Lans Bovenberg and Thijs Knaap. Ageing, funded pensions and the Dutch economy. 2005.

Stephen P. Bradley and Dwight B. Crane. A dynamic model for bond portfolio management. *Management Science*, 19(2):139–151, 1972.

Stephen P. Bradley and Dwight B. Crane. Managing a bank bond portfolio over time. *Stochastic Programming"(MAH Dempster, ed.), Academic Press, London*, pages 449–471, 1980.

Patrick Brockett, Abraham Chranes, and Li Sun. A chance-constrained programming approach to pension plan management. *Contents of Actuarial Research Clearing House*, 3:324–357, 1993.

Dirk Broeders, An Chen, and Birgit Koos. *An institutional evaluation of pension funds and life insurance companies*. De Nederlandsche Bank, 2009.

Kenneth Bruhn and Mogens Steffensen. Household consumption, investment and life insurance. *Insurance: Mathematics and Economics*, 48(3):315–325, 2011.

Jean L. P. Brunel. Revisiting the asset allocation challenge through a behavioral finance lens. *The Journal of Wealth Management*, 6(2): 10–20, 2003.

Jun Cai and Chenliang Ge. Multi-objective private wealth allocation without subportfolios. *Economic Modelling*, 29(3):900–907, 2012.

Andrew J. G. Cairns, David Blake, and Kevin Dowd. Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans. *Journal of Economic Dynamics and Control*, 30(5): 843–877, 2006.

David R. Cariño and William T. Ziemba. Formulation of the Russell-Yasuda Kasai financial planning model. *Operations Research*, 46(4): 433–449, 1998a.

David R. Cariño, Terry Kent, David H. Myers, Celine Stacy, Mike Sylvanus, Andrew L. Turner, Kouji Watanabe, and William T. Ziemba. The Russell-Yasuda Kasai model: An asset/liability model for a Japanese insurance company using multistage stochastic programming. *Interfaces*, 24(1):29–49, 1994.

David R. Cariño, David H. Myers, and William T. Ziemba. Concepts, technical issues, and uses of the Russell-Yasuda Kasai financial planning model. *Operations research*, 46(4):450–462, 1998b.

Abraham Charnes and William W. Cooper. Chance-constrained programming. *Management science*, 6(1):73–79, 1959.

Peng Chen, Roger G. Ibbotson, Moshe A. Milevsky, and Kevin X. Zhu. Human capital, asset allocation, and life insurance. *Financial Analysts Journal*, pages 97–109, 2006.

Ka Chun Cheung and Hailiang Yang. Optimal investment-consumption strategy in a discrete-time model with regime switching. *Discrete and continuous dynamical systems Series B*, 8(2):315, 2007.

Ashvin B. Chhabra. Beyond Markowitz: a comprehensive wealth allocation framework for individual investors. *The Journal of Wealth Management*, 7(4):8–34, 2005.

Giorgio Consigli. Individual asset liability management for individual investors. In Stavros A. Zenios and William T. Ziemba, editors, *Handbook of Asset and Liability Management: Applications and case studies*, pages 752–827. North-Holland Finance Handbook Series, Elsevier, 2007.

Giorgio Consigli and Michael A. H. Dempster. The CALM stochastic programming model for dynamic asset-liability management. *Worldwide asset and liability modeling*, 10:464, 1998a.

Giorgio Consigli and Michael A. H. Dempster. Dynamic stochastic programming for asset-liability management. *Annals of Operations Research*, 81:131–162, 1998b.

Giorgio Consigli and Massimo di Tria. Optimal long-term property and casualty ALM with risk capital control. *Asset-Liability Management for Financial Institutions*, 1:137–150, 2012.

Giorgio Consigli and Vittorio Moriggia. Applying stochastic programming to insurance portfolios stress-testing. *Quantitative Finance Letters*, 2(1):7–13, 2014.

Giorgio Consigli, Massimo di Tria, Michele Gaffo, Gaetano Iaquinta, Vittorio Moriggia, and Angelo Uristani. Dynamic portfolio management for property and casualty insurance. In *Stochastic Optimization Methods in Finance and Energy*, pages 99–124. Springer, 2011.

Giorgio Consigli, Gaetano Iaquinta, Vittorio Moriggia, Massimo di Tria, and Davide Musitelli. Retirement planning in individual asset-liability management. *IMA Journal of Management Mathematics*, 23(4):365–396, 2012.

Andrea Consiglio, Flavio Cocco, and Stavros A. Zenios. www. Personal_Asset_Allocation. *Interfaces*, 34(4):287–302, 2004.

Andrea Consiglio, Flavio Cocco, and Stavros A. Zenios. Scenario optimization asset and liability modelling for individual investors. *Annals of Operations Research*, 152(1):167–191, 2007.

Andrea Consiglio, Michele Tumminello, and Stavros A. Zenios. Designing guarantee options in defined contributions pension plans. *The Wharton Financial Institutions Center*, 2015.

George B. Dantzig. Linear programming under uncertainty. *Management science*, 1(3-4):197–206, 1955.

Philip E. Davis. Portfolio regulation of life insurance companies and pension funds. *Economics*, 2000.

Renato Cordeiro de Amorim and Christian Hennig. Recovering the number of clusters in data sets with noise features using feature rescaling factors. *Information Sciences*, 324:126–145, 2015.

Ron S. Dembo. Scenario optimization. *Annals of Operations Research*, 30(1):63–80, 1991.

Victor DeMiguel, Lorenzo Garlappi, and Raman Uppal. Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *Review of Financial Studies*, 22(5):1915–1953, 2009.

Michael A. H. Dempster, Matteo Germano, Elena A. Medova, and Michael Villaverde. Global asset liability management. *British Actuarial Journal*, 9(01):137–195, 2003.

Micheal A. H. Dempster and Alice M. Ireland. MIDAS: An expert debt management advisory system. In *Data, expert knowledge and decisions*, pages 116–127. Springer, 1988.

Micheal A. H. Dempster and Alice M. Ireland. Object-oriented model integration in MIDAS [Manager's Intelligent Debt Advisory System]. In *System Sciences, 1989. Vol. III: Decision Support and Knowledge Based Systems Track, Proceedings of the Twenty-Second Annual Hawaii International Conference on*, volume 3, pages 612–620. IEEE, 1989.

Micheal A. H. Dempster and Alice M. Ireland. Object-oriented model integration in a financial decision support system. *Decision Support Systems*, 7(4):329–340, 1991.

Cees Dert. *Asset liability management for pension funds: a multistage chance constrained programming approach*. PhD thesis, Erasmus School of Economics (ESE), 1995.

Cees L. Dert. A dynamic model for asset liability management for defined benefit pension funds. *Worldwide asset and liability modeling*, 10:501–536, 1998.

Peter A. Diamond and Joseph E. Stiglitz. Increases in risk and in risk aversion. *Journal of Economic Theory*, 8(3):337–360, 1974.

Gabriel Dondi, Florian Herzog, Lorenz M. Schumann, and Hans P. Geering. Dynamic asset and liability management for Swiss pension funds. In Stavros A. Zenios and William T. Ziemba, editors, *Handbook of Asset and Liability Management: Applications and case studies*, pages 963–1028. North-Holland Finance Handbook Series, Elsevier, 2007.

Jitka Dupačová. Uncertainty about input data in portfolio management. In *Modelling Techniques for Financial Markets and Bank Management*, pages 17–33. Springer, 1996.

Jitka Dupačová. Portfolio optimization via stochastic programming: Methods of output analysis. *Mathematical methods of operations research*, 50(2):245–270, 1999.

Jitka Dupačová and Marida Bertocchi. From data to model and back to data: A bond portfolio management problem. *European Journal of Operational Research*, 134(2):261–278, 2001.

Jitka Dupačová and Miloš Kopa. Robustness in stochastic programs with risk constraints. *Annals of Operations Research*, 200(1):55–74, 2012.

Jitka Dupačová and Miloš Kopa. Robustness of optimal portfolios under risk and stochastic dominance constraints. *European Journal of Operational Research*, 234(2):434–441, 2014.

Jitka Dupačová and Jan Polívka. Asset-liability management for Czech pension funds using stochastic programming. *Annals of Operations Research*, 165(1):5–28, 2009.

Jitka Dupačová, Marida Bertocchi, and Vittorio Moriggia. Postoptimality for scenario based financial planning models with an application to bond portfolio management. *World wide asset and liability modeling*, pages 263–285, 1998.

Jitka Dupačová, Giorgio Consigli, and Stein W. Wallace. Scenarios for multistage stochastic programs. *Annals of operations research*, 100 (1-4):25–53, 2000.

Jitka Dupačová, Jan Hurt, and Josef Štěpán. *Stochastic Modeling in Economics and Finance*. Applied Optimization. Springer, 2002. ISBN 9781402008405.

Jitka Dupačová, Nicole Gröwe-Kuska, and Werner Römisch. Scenario reduction in stochastic programming. *Mathematical programming*, 95 (3):493–511, 2003.

Jitka Dupačová, Marida Bertocchi, and Vittorio Moriggia. Testing the structure of multistage stochastic programs. *Computational Management Science*, 6(2):161–185, 2009.

Csaba I. Fabian and Anna Veszprémi. Algorithms for handling CVaR-constraints in dynamic stochastic programming models with applications to finance. 2007.

Frank J. Fabozzi, Sergio M. Focardi, and Caroline L. Jonas. Market experience with modeling for defined-benefit pension funds: evidence from four countries. *Journal of Pension Economics and Finance*, 4(03):313–327, 2005.

Allen R. Ferguson and George B. Dantzig. The allocation of aircraft to routes-an example of linear programming under uncertain demand. *Management science*, 3(1):45–73, 1956.

Alexei A. Gaivoronski and Georg Ch. Pflug. Value-at-risk in portfolio optimization: properties and computational approach. *Journal of Risk*, 7(2):1–31, 2005.

Russell Gerrard, Steven Haberman, and Elena Vigna. Optimal investment choices post-retirement in a defined contribution pension scheme. *Insurance: Mathematics and Economics*, 35(2):321–342, 2004.

Russell Gerrard, Steven Haberman, and Elena Vigna. The management of decumulation risks in a defined contribution pension plan. *North American Actuarial Journal*, 10(1):84–110, 2006.

Russell Gerrard, Bjarne Højgaard, and Elena Vigna. Choosing the optimal annuitization time post-retirement. *Quantitative Finance*, 12(7):1143–1159, 2012.

Alois Geyer and William T. Ziemba. The Innovest Austrian pension fund financial planning model InnoALM. *Operations Research*, 56 (4):797–810, 2008.

Alois Geyer, Michael Hanke, and Alex Weissensteiner. Life-cycle asset allocation and consumption using stochastic linear programming. *Journal of Computational Finance*, 12(4):29–50, 2009.

Christian Gollier. Intergenerational risk-sharing and risk-taking of a pension fund. *Journal of Public Economics*, 92(5):1463–1485, 2008.

Adam Greenhall. *Wind scenarios for stochastic energy scheduling*. PhD thesis, 2013.

Luigi Guiso, Michael Haliassos, and Tullio Jappelli. *Household portfolios*. MIT press, 2002.

Willem K. K. Haneveld, Matthijs H. Streutker, and Maarten H. van der Vlerk. Indexation of Dutch pension rights in multistage recourse ALM models. *IMA Journal of Management Mathematics*, page dpp010, 2009.

Willem K. K. Haneveld, Matthijs H. Streutker, and Maarten H. van der Vlerk. An ALM model for pension funds using integrated chance constraints. *Annals of Operations Research*, 177(1):47–62, 2010.

Giora Hanoch and Haim Levy. The efficiency analysis of choices involving risk. *The Review of Economic Studies*, pages 335–346, 1969.

Mary Hardy. *Investment guarantees: modeling and risk management for equity-linked life insurance*, volume 215. John Wiley & Sons, 2003.

Holger Heitsch and Werner Römisch. Scenario reduction algorithms in stochastic programming. *Computational optimization and applications*, 24(2-3):187–206, 2003.

René Henrion. Structural properties of linear probabilistic constraints. *Optimization*, 56(4):425–440, 2007.

René Henrion and Cyrille Strugarek. Convexity of chance constraints with independent random variables. *Computational Optimization and Applications*, 41(2):263–276, 2008.

René Henrion, Andris Möller, Stefan Vigerske, and Thomas Arnold. A mixed-integer stochastic nonlinear optimization problem with joint probabilistic constraints. 2013.

Randall S. Hiller and Jonathan Eckstein. Stochastic dedication: Designing fixed income portfolios using massively parallel benders decomposition. *Management Science*, 39(11):1422–1438, 1993.

Randall S. Hiller and Christian Schaack. A classification of structured bond portfolio modeling techniques. *The Journal of Portfolio Management*, 17(1):37–48, 1990.

Randall S. Hiller and Jeremy F. Shapiro. *Stochastic programming models for asset/liability management problems*. International Financial Services Research Center, Sloan School of Management, Massachusetts Institute of Technology, 1989.

Petri Hilli, Matti Koivu, Teemu Pennanen, and Antero Ranne. A stochastic programming model for asset liability management of a Finnish pension company. *Annals of Operations Research*, 152(1): 115–139, 2007.

Ronald Hochreiter and Georg Ch. Pflug. Financial scenario generation for stochastic multi-stage decision processes as facility location problems. *Annals of Operations Research*, 152(1):257–272, 2007.

Ronald Hochreiter, Georg Ch. Pflug, and David Wozabal. Multi-stage stochastic electricity portfolio optimization in liberalized energy markets. In *System modeling and optimization*, pages 219–226. Springer, 2006.

Wolfram J. Horneff, Raimond H. Maurer, and Michael Z. Stamos. Optimal gradual annuitization: Quantifying the costs of switching to annuities. *Journal of Risk and Insurance*, 75(4):1019–1038, 2008.

Kjetil Høyland and Stein W. Wallace. Analyzing legal regulations in the Norwegian life insurance business using a multistage asset–liability management model. *European Journal of Operational Research*, 134(2):293–308, 2001a.

Kjetil Høyland and Stein W. Wallace. Generating scenario trees for multistage decision problems. *Management Science*, 47(2):295–307, 2001b.

Kjetil Høyland, Michal Kaut, and Stein W. Wallace. A heuristic for moment-matching scenario generation. *Computational optimization and applications*, 24(2-3):169–185, 2003.

Daniel Kahneman and Amos Tversky. Prospect theory: An analysis of decision under risk. *Econometrica: Journal of the Econometric Society*, pages 263–291, 1979.

Peter Kall and Janos Mayer. Stochastic linear programming: Models, theory, and computation, volume 80 of international series in operations research & management science. *Springer*, 2005.

Jarl G. Kallberg, R. W. White, and William T. Ziemba. Short term financial planning under uncertainty. *Management Science*, 28(6): 670–682, 1982.

Leonard Kaufman and Peter J. Rousseeuw. *Finding groups in data: an introduction to cluster analysis*, volume 344. John Wiley & Sons, 2009.

Soňa Kilianová and Georg Ch. Pflug. Optimal pension fund management under multi-period risk minimization. *Annals of Operations Research*, 166(1):261–270, 2009.

Anton J. Kleywegt, Alexander Shapiro, and Tito Homem-de Mello. The sample average approximation method for stochastic discrete optimization. *SIAM Journal on Optimization*, 12(2):479–502, 2002.

Miloš Kopa and Thierry Post. A portfolio optimality test based on the first-order stochastic dominance criterion. *Journal of Financial and Quantitative Analysis*, 44(05):1103–1124, 2009.

Miloš Kopa and Thierry Post. A general test for SSD portfolio efficiency. *OR Spectrum*, 37(3):703–734, 2015.

Roy Kouwenberg. Scenario generation and stochastic programming models for asset liability management. *European Journal of Operational Research*, 134(2):279–292, 2001.

Timo Kuosmanen. Efficient diversification according to stochastic dominance criteria. *Management Science*, 50(10):1390–1406, 2004.

Martin I. Kusy and William T. Ziemba. A bank asset and liability management model. *Operations Research*, 34(3):356–376, 1986.

Haim Levy. *Stochastic dominance: Investment decision making under uncertainty*, volume 12. Springer Science & Business Media, 2006.

Zhongfei Li, Ken S. Tan, and Hailiang Yang. Multiperiod optimal investment-consumption strategies with mortality risk and environment uncertainty. *North American Actuarial Journal*, 12(1):47–64, 2008.

Byung Hwa Lim and Yong Hyun Shin. Optimal investment, consumption and retirement decision with disutility and borrowing constraints. *Quantitative Finance*, 11(10):1581–1592, 2011.

Stuart Lloyd. Least squares quantization in PCM. *Information Theory, IEEE Transactions on*, 28(2):129–137, 1982.

James Luedtke and Shabbir Ahmed. A sample approximation approach for optimization with probabilistic constraints. *SIAM Journal on Optimization*, 19(2):674–699, 2008.

Philip M. Lurie and Matthew S. Goldberg. An approximate method for sampling correlated random variables from partially-specified distributions. *Management Science*, 44(2):203–218, 1998.

Johan Lyhagen. A method to generate multivariate data with moments arbitrary close to the desired moments. Technical report, SSE/EFI Working Paper Series in Economics and Finance, 2001.

Raymond McKendall, Stavros A. Zenios, and Martin Holmer. Stochastic programming models for portfolio optimization with mortgage backed securities: Comprehensive research guide. In *Operations Research Models in Quantitative Finance*, pages 134–171. Springer, 1994.

Elena A. Medova, James K. Murphy, Andrew P. Owen, and Kasim Rehman. Individual asset liability management. *Quantitative Finance*, 8(6):547–560, 2008.

Robert C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, pages 247–257, 1969.

Robert C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory*, 3(4):373–413, 1971.

Moshe A. Milevsky and Virginia R. Young. Annuitization and asset allocation. *Journal of Economic Dynamics and Control*, 31(9):3138–3177, 2007.

Vittorio Moriggia, Giorgio Consigli, and Gaetano Iaquinta. Optimal stochastic programming-based personal financial planning with intermediate and long term goals. *Stochastic Programming: Applications in Finance, Energy, Planning and Logistics*, 4:43–68, 2013.

John M. Mulvey. A surplus optimization perspective. *Investment Management Review*, 3(8):31–39, 1989.

John M. Mulvey. An asset-liability investment system. *Interfaces*, 24 (3):22–33, 1994a.

John M. Mulvey. Financial planning via multi-stage stochastic programs. *Mathematical Programming: State of the Art*, 1994b.

John M. Mulvey. Generating scenarios for the Towers Perrin investment system. *Interfaces*, 26(2):1–15, 1996.

John M. Mulvey and Eric A. Thorlacius. The Towers Perrin global capital market scenario generation system. *Worldwide asset and liability modeling*, 10:286, 1998.

John M. Mulvey and Stavros A. Zenios. Capturing the correlations of fixed-income instruments. *Management Science*, 40(10):1329–1342, 1994.

John M. Mulvey, Gordon Gould, and Clive Morgan. An asset and liability management system for Towers Perrin-Tillinghast. *Interfaces*, 30(1):96–114, 2000.

John M. Mulvey, Koray D. Simsek, and Zhuojuan Zhang. Improving investment performance for pension plans. *Journal of Asset Management*, 7(2):93–108, 2006.

John M. Mulvey, Cenk Ural, and Zhuojuan Zhang. Improving performance for long-term investors: wide diversification, leverage, and overlay strategies. *Quantitative Finance*, 7(2):175–187, 2007.

John M. Mulvey, Koray D. Simsek, Zhuojuan Zhang, Frank J. Fabozzi, and William R. Pauling. Assisting defined-benefit pension plans. *Operations research*, 56(5):1066–1078, 2008.

Claus Munk and Carsten Sørensen. Optimal consumption and investment strategies with stochastic interest rates. *Journal of Banking & Finance*, 28(8):1987–2013, 2004.

Soren S. Nielsen and Stavros A. Zenios. A stochastic programming model for funding single premium deferred annuities. *Mathematical Programming*, 75(2):177–200, 1996.

Sergio Ortobelli Lozza, Sebastiano Vitali, and Marco Cassader. Reward and risk in the fixed income markets. In *Financial regulation and supervision in the after-crisis period. Proceedings of 14th international conference on finance and banking*, pages 329–340, 2013.

Bernardo K. Pagnoncelli, Shabbir Ahmed, and Alexander Shapiro. Sample average approximation method for chance constrained programming: theory and applications. *Journal of optimization theory and applications*, 142(2):399–416, 2009.

Georg Ch. Pflug. Some remarks on the value-at-risk and the conditional value-at-risk. In *Probabilistic constrained optimization*, pages 272–281. Springer, 2000.

Georg Ch. Pflug. Scenario tree generation for multiperiod financial optimization by optimal discretization. *Mathematical programming*, 89(2):251–271, 2001.

Georg Ch. Pflug and Alois Pichler. Approximations for probability distributions and stochastic optimization problems. In Marida Bertocchi, Giorgio Consigli, and Michael A. H. Dempster, editors, *Stochastic Optimization Methods in Finance and Energy*, volume 163 of *International Series in Operations Research & Management Science*, pages 343–387. Springer New York, 2011.

Georg Ch. Pflug and Alois Pichler. *Multistage Stochastic Optimization*. Springer, 2014.

Georg Ch. Pflug and Werner Römisch. *Modeling, measuring and managing risk*, volume 20. World Scientific, 2007.

Georg Ch. Pflug and Artur Świetanowski. Dynamic asset allocation under uncertainty for pension fund management. *Control and Cybernetics*, 28:755–777, 1999.

Stanley R. Pliska and Jinchun Ye. Optimal life insurance purchase and consumption/investment under uncertain lifetime. *Journal of Banking & Finance*, 31(5):1307–1319, 2007.

Thierry Post and Miloš Kopa. General linear formulations of stochastic dominance criteria. *European Journal of Operational Research*, 230 (2):321–332, 2013.

Warren B. Powell. Clearing the jungle of stochastic optimization. *Informs TutORials*, 2014.

Warren B. Powell and Huseyin Topaloglu. Stochastic programming in transportation and logistics. *Handbooks in operations research and management science*, 10:555–635, 2003.

András Prékopa. Logarithmic concave measures with application to stochastic programming. *Acta Scientiarum Mathematicarum*, 32(197): 301–316, 1971.

András Prékopa. Probabilistic programming. *Handbooks in operations research and management science*, 10:267–351, 2003.

Scott F. Richard. Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. *Journal of Financial Economics*, 2(2):187–203, 1975.

Tyrrell R. Rockafellar and Stanislav Uryasev. Optimization of conditional value-at-risk. *Journal of risk*, 2:21–42, 2000.

Tyrrell R. Rockafellar and Stanislav Uryasev. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26 (7):1443–1471, 2002.

Peter J. Rousseeuw. Silhouettes: a graphical aid to the interpretation and validation of cluster analysis. *Journal of computational and applied mathematics*, 20:53–65, 1987.

Sandra L. Schwartz and William T. Ziemba. ALM issues in social security. In Stavros A. Zenios and William T. Ziemba, editors, *Handbook of Asset and Liability Management: Applications and case studies*, pages 1069–1117. North-Holland Finance Handbook Series, Elsevier, 2007.

Alexander Shapiro and Andy Philpott. A tutorial on stochastic programming. *Manuscript. Available at www2.isye.gatech.edu/~ashapiro/publications.html*, 2007.

Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. Lectures on stochastic programing. *Modeling and Theory. SIAM and Mathematical Programming Society*, 2009.

Jeremy F. Shapiro. Stochastic programming models for dedicated portfolio selection. In *Mathematical models for decision support*, pages 587–611. Springer, 1988.

Tak K. Siu. Long-term strategic asset allocation with inflation risk and regime switching. *Quantitative Finance*, 11(10):1565–1580, 2011.

James E. Smith. Moment methods for decision analysis. *Management science*, 39(3):340–358, 1993.

Matthijs Streutker, Maarten H. van der Vlerk, and Willem K. K. Haneveld. Implementation of new regulatory rules in a multistage ALM model for Dutch pension funds. Technical report, University of Groningen, Research Institute SOM (Systems, Organisations and Management), 2007.

Nikolas Topaloglou, Hercules Vladimirou, and Stavros A. Zenios. CVaR models with selective hedging for international asset allocation. *Journal of Banking & Finance*, 26(7):1535–1561, 2002.

David C. Vale and Vincent A. Maurelli. Simulating multivariate nonnormal distributions. *Psychometrika*, 48(3):465–471, 1983.

Wim van Ackooij and René Henrion. Gradient formulae for nonlinear probabilistic constraints with gaussian and gaussian-like distributions. *SIAM Journal on Optimization*, 24(4):1864–1889, 2014.

Wim van Ackooij, Riadh Zorgati, René Henrion, and Andris Möller. Chance constrained programming and its applications to energy management. *Stochastic Optimization-Seeing the Optimal for the Uncertain*, pages 291–320, 2011.

Wim van Ackooij, René Henrion, Andris Möller, and Riadh Zorgati. Joint chance constrained programming for hydro reservoir management. *Optimization and Engineering*, 15(2):509–531, 2014.

Roger J.-B. Wets. Stochastic programs with fixed recourse: the equivalent deterministic program. *SIAM review*, 16(3):309–339, 1974.

Menahem E. Yaari. Uncertain lifetime, life insurance, and the theory of the consumer. *The Review of Economic Studies*, pages 137–150, 1965.

Stavors A. Zenios. A model for portfolio management with mortgage-backed securities. *Annals of Operations Research*, 43(6):337–356, 1993.

Stavros A. Zenios. Asset/liability management under uncertainty for fixed-income securities. *Annals of Operations Research*, 59(1):77–97, 1995.

Stavros A. Zenios and William T. Ziemba. *Handbook of Asset and Liability Management: Theory and Methodology*, volume 1. North-Holland Finance Handbook Series, Elsevier, 2006.

Stavros A. Zenios and William T. Ziemba. *Handbook of Asset and Liability Management: Applications and case studies*, volume 2. North-Holland Finance Handbook Series, Elsevier, 2007.

Stavros A. Zenios, Martin R. Holmer, Raymond McKendall, and Christiana Vassiadou-Zeniou. Dynamic models for fixed-income portfolio management under uncertainty. *Journal of Economic Dynamics and Control*, 22(10):1517–1541, 1998.

Rachel E. S. Ziemba and William T. Ziemba. *Scenarios for risk management and global investment strategies*. John Wiley & Sons, 2007.

William T. Ziemba. *The stochastic programming approach to asset, liability, and wealth management*. Charlottesville, VA:AIMR, USA, 2003.

William T. Ziemba. The Russell-Yasuda, InnoALM and related models for pensions, insurance companies and high net worth individuals. In Stavros A. Zenios and William T. Ziemba, editors, *Handbook of Asset and Liability Management: Applications and case studies*, pages 861–962. North-Holland Finance Handbook Series, Elsevier, 2007.

William T. Ziemba and John M. Mulvey, editors. *Worldwide asset and liability modeling*, volume 10. Cambridge University Press, UK, 1998.

William T. Ziemba and Raymond G. Vickson, editors. *Stochastic optimization models in finance*. Academic Press, New York, 1975.

William T. Ziemba and Raymond G. Vickson, editors. *Stochastic optimization models in finance*. World Scientific Handbook in Financial Economics Series, 2006.