Weak orderings for intersecting Lorenz curves

Tommaso Lando[1] and Lucio Bertoli-Barsotti[2(*)]

[1] University of Bergamo, via dei Caniana 2 Bergamo (Italy) and VSB Technical University of Ostrava, Sokolskà Trida 33, Ostrava (Czech Republic), tommaso.lando@unibg.it
[2] University of Bergamo, via dei Caniana 2 Bergamo (Italy), lucio.bertoli-barsotti@unibg.it
(*) Corresponding Author

Abstract

The Lorenz dominance is a primary tool for comparison of non-negative distributions in terms of inequality. However, in most of cases Lorenz curves intersect and the ordering is not fulfilled, so that some alternative (weaker) criteria need to be to introduced. In this context, the second-degree Lorenz dominance, which emphasizes the role of the left (or right) tail of the distribution, is especially suitable for ranking single-crossing Lorenz curves. We introduce a new ordering, namely *disparity dominance*, which emphasizes inequality in both of the tails, and we show that, in turn, it is especially suitable for ranking double-crossing Lorenz curves. We argue that the two approaches are basically complementary, although in both cases the Gini coefficient is crucial for the ranking. Moreover, we can use some well-known results of majorization theory to obtain classes of functionals that are consistent with the aforementioned weak preorders, and that can therefore be used as finer inequality indices.

Key words

Lorenz dominance, inequality, majorization ordering, stochastic dominance, Gini index

Acknowledgement

The research was supported by the Czech Science Foundation (GACR) under project 15-23699S and by SP2015/15, an SGS research project of VSB-TU Ostrava, and furthermore by the European Social Fund in the framework of CZ.1.07/2.3.00/20.0296 (to T.L.).

1. Introduction

The Lorenz curve (LC), which has been introduced as a representation of inequality (e.g. income inequality), is generally used to rank distributions in terms of an order of preference, namely the Lorenz ordering, or dominance (LD). In fact, the LD conforms with the idea that the higher of two non-intersecting LCs (i.e. the corresponding distribution) should be preferred, in that it shows less inequality compared with the lower one. As well known, in an economic framework, the LD is coherent with the Pigou-Dalton condition (principle of transfers): that is, the higher of two non-intersecting Lorenz curves can be obtained from the lower one by a sequence of income transfers from "richer" to "poorer" individuals (the so-called *elementary* transfers or *T-transforms*, see [25] p. 32, or *progressive* transfers, [33]). For this reason, the "coherence" with the LD represents a fundamental property of any inequality (or concentration) measure. Nevertheless, it may happen that Lorenz curves intersect or, equivalently, the LD is not verified, which implies that the Lorenz-preserving indices may disagree. In this case we can rank the distributions by relying on weaker orders of inequality. In the literature, this idea has been analysed in several works related to the concept of third-order stochastic dominance, which emphasizes the role of the left tail of the distribution [4,9,10,12,33]. Indeed, many authors agree that an elementary transfer should be more equalizing, the "lower" it occurs in the distribution. This principle has been named *aversion to downside inequality* [12]. A conceptually similar (but mathematically different) approach has been proposed by Muliere and Scarsini [26] based on the third-degree *inverse stochastic dominance* (3-ISD). Further studies in this direction has been carried out by Zoli [39,40] and, more recently, by Aaberge [1]. This idea has also been shown to be consistent with the criterion of the *Yaari social welfare functions* [34]. Using a terminology introduced by [1], we may say that the original idea of [26] is simply to move from *first-degree* to *second-degree Lorenz dominance* (2-LD) by cumulating LCs from the left (*upward*) or from the right (*downward*). This operation respectively obeys the concepts of *downside* (or *upside*) *positional inequality aversion*. In terms of income, in so doing it attaches more weighting to individuals at the bottom (or alternatively at the top) of the income scale. Indeed, while many authors in this field agree with the idea of downside inequality aversion, one may also take upside inequality into consideration [1,26], especially if one is interested in those variations occurring at the top of the income distribution (see e.g. [24]). The alternative approach proposed in this paper basically attempts to combine the main features of upside and downside inequality aversion into a single preorder.

In section 4 we review and study the second degree Lorenz dominance. In particular, slightly generalizing a result of [39], we may observe that a sufficient condition for the 2-LD (upward or downward) is that the LCs cross once. In this case, we can identify the dominant distribution by comparing the values of the Gini index. It should be stressed that, in an economic context, single-crossing LCs occur in most practical cases, as highlighted by the empirical analyses of [4] and [11], whilst multiple-crossing LCs are very rare. Nonetheless, the case of multiple-crossing LCs has been analysed in the works of Zoli [39,40], Davies and Hoy [12] and Chiu [9], e.g. an especially interesting case of double-crossing LCs was obtained under the tax reform act of 1986 in the US (see [16]). In this paper we aim at analysing all possible situations, in order to understand and clarify the mathematical properties of the orderings considered. In this framework, we show that, under an hypothetical condition of maximum uncertainty (i.e. equal values of the Gini index), if the Lorenz curves cross an even number of times, then the 2-LD (upward or downward) is unable to provide a ranking. Consequently, in this special case the preorder is also unable to rank symmetric distributions. In this regard, we show that the 2-LD is related to the concept of symmetry; and in particular, under some conditions, that it can be interpreted as an ordering of skewness. It seems that the idea of an ordering of skeweness has been introduced independently by Van Zwet [35] and Frosini [15].

Knowledge of the properties and features of the second-degree Lorenz dominance permits one to understand its possible "gaps" (by which are meant those situations where it cannot provide a ranking), and thereby introduce an alternative ordering criterion in order to fill them. In the second part of the paper we present the *second-degree disparity dominance* (2-DD), which is based on the cumulated difference between the upper and lower parts of the Lorenz curves. This ordering can serve a twofold purpose. First, the 2-DD is intended to emphasise both the tails of the distribution rather than one (left or right). Secondly, we show that it can rank distributions in those particular cases where the second-degree Lorenz dominance fails. Indeed, we find that the 2-DD is especially suitable to rank LCs that cross twice (once before 0.5 and once after), under some conditions. Also in this case the values of the Gini index are crucial for determining the dominant distribution. The 2-DD can be interpreted as complementary to the 2-LD, in that it can fill its gaps, and vice

versa. Moreover, we show that, just as the 2-LD can be related to skewness, so the 2-DD can be related to the concept of kurtosis (in particular tailweightness).
In this paper we want to stress that many of the relations, orderings and results can be equivalently expressed in terms of the majorization preorder [2]. In particular, we can propose classes of inequality measures which are coherent with the orderings analysed, based on some well-known results of majorization theory. Hence, we believe that a few basic concepts and results of majorization theory may be helpful for the reader: a brief overview is provided in the following section.

2. Preorders and majorization: a brief overview

We recall that a preorder is a binary relation $\leq$ over a set S that is reflexive and transitive. In particular, observe that a preorder $\leq$ does not generally satisfy the antisymmetry property (that is, $a \leq b$ and $b \leq a$ does not necessarily imply $a = b$) and it is generally not total (that is, each pair a, b in S is not necessarily related by $\leq$). Then, given the preorder $\leq$ (defined over S), a mapping $M: S \to \mathbb{R}$ is said to be *isotonic*, *consistent* or *order-preserving* with respect to $\leq$ whenever, for every $a, b \in S$ such that $a \leq b$, we obtain that $M(a) \leq M(b)$. In this paper, given a preorder $\leq$ and $a, b \in S$, we generally state that $\leq$ is unable to order (or rank) a, b if neither or both of the conditions $a \leq b$ and $b \leq a$ hold true.
Majorization $\prec$ is a preorder which may be used to compare vectors or functions in terms of "diversity" between their values [25]. Here we focus on continuous majorization, in particular we consider a preorder in the space of functions that are integrable with respect to the Lebesgue measure *m* on a set $(0, x_0)$.

Definition 1. Let $f, g \in \mathcal{L}^1(0, x_0)$. We say that f *is majorized by* g and write $f \prec g$ if and only if

1) $\int_0^z f_\downarrow(u)\,du \leq \int_0^z g_\downarrow(u)\,du$, $\forall\, z \in (0, x_0)$,
2) $\int_0^1 f_\downarrow(u)\,du = \int_0^1 g_\downarrow(u)\,du$,

where $f_\downarrow(u) = (m_f(x))^{-1}$ and $m_f(x) = m(\{u: f(u) > x\})$ (note that the function $f_\downarrow$ is referred to as the *decreasing re-arrangement* of f).

When condition 2) does not hold, we rely on weaker definitions of majorization.

Definition 2. Let $f, g \in \mathcal{L}^1(0, x_0)$. We say that f *is weakly majorized by* g *from below* and write $f \prec_w g$ if and only if

$$\int_0^z f_\downarrow(u)\,du \leq \int_0^z g_\downarrow(u)\,du,\ \forall\, z \in (0, x_0).$$

We say that that f *is weakly majorized by* g *from above* and write $f \prec^w g$ if and only if

$$\int_0^z f_\uparrow(u)\,du \geq \int_0^z g_\uparrow(u)\,du,\ \forall\, z \in (0, x_0),$$

where, similarly to $f_\downarrow$, $f_\uparrow$ denotes the *increasing re-arrangement* of f (for the formal definition see for instance [22]).

A key result of majorization theory [20] makes it possible to define classes of functionals that are isotonic with the strong and weak preorders [25].

Theorem 1. Let $f, g \in \mathcal{L}^1(0, x_0)$ and consider the inequality:

$$\int_0^{x_0} \phi(f(u))\,du \leq \int_0^{x_0} \phi(g(u))\,du \tag{1}$$

$f \prec g$ iff (1) holds for every convex function ϕ such that the integral exists.
$f \prec_w g$ iff (1) holds for every increasing and convex function ϕ such that the integral exists.
$f \prec^w g$ iff (1) holds for every decreasing and convex function ϕ such that the integral exists.

3. Preliminaries

Let $\mathcal{F}$ be the space of non-negative distributions F with finite expectation, $\mathcal{F} = \{F: F(z) = 0\ \forall\, z < 0\ \wedge \int_0^\infty z dF = \mu_F < \infty\}$. First, we recall that the (generalized) inverse of a distribution function $F \in \mathcal{F}$ is given by

$$F^{-1}(p) = \inf\{z: F(z) \geq p\}, p \in (0,1) \quad (2)$$

If F has finite expectation, μ_F, then the Lorenz curve $L_F: [0,1] \to [0,1]$ is defined as follows [17]:

$$L_F(p) = \frac{1}{\mu_F}\int_0^p F^{-1}(t)dt\, , p \in (0,1). \quad (3)$$

Henceforth, let X and Y be non-negative random variables, with finite expectations, with corresponding distribution functions F and G, respectively. Some basic properties of the LC are as follows: i) $L_F(p) = p$ iff $X = \mu_F$ (absence of inequality); ii) if $Y = cX$ $(c > 0)$ then $L_F(p) = L_G(p), \forall\, p$ (scale invariance; indeed, by definition, LC determines the distribution up to a scale transformation); iii) if $Y = c + X$ $(c > 0)$ then $L_F(p) < L_G(p), \forall\, p$ (in particular $p - L_G(p) = (p - L_F(p))\mu_F/(\mu_F + c)$); iv) in an empirical context, the LC is also invariant to *population replication* [30], that is, if the frequencies are multiplied by the same integer and positive number, the LC remains unchanged.

Let us also define the complementary Lorenz curve (see e.g. [13] or [25], pp. 728-730, and the references therein), $\bar{L}_F: [0,1] \to [0,1]$, given by

$$\bar{L}_F(p) = \frac{1}{\mu_F}\int_0^p F^{-1}(1-t)dt = (1 - L_F(p))_\uparrow = 1 - L_F(1-p), p \in (0,1) \quad (4)$$

(note that if $f(t)$ is a decreasing function in [0,1] then $f_\uparrow(t) = f(1-t)$). Actually, for a given percentage t, $L_F(t)$ represents the percentage of "total" possessed by the low $100t\%$ part of the distribution, while $\bar{L}_F(t)$ represents the percentage of "total" corresponding to the top $100t\%$ part of the distribution. From a geometrical point of view, $\bar{L}_F(p)$ is the 180° rotation of $L_F(p)$ with respect to the point (0.5,0.5).

In what follows we shall also need the following curves.

$$\Sigma_F(t) = \bar{L}_F(t) + L_F(t), t \in [0,1/2], \quad (5)$$

$$\Delta_F(t) = \bar{L}_F(t) - L_F(t), t \in [0,1/2]. \quad (6)$$

Note that these curves are defined in [0,1/2] because their behaviour in [1/2,1] is equivalent up to symmetric transformations.

The interpretation of Δ_F is quite simple. As $\bar{L}_F(t) \geq L_F(t)\ \forall t \in [0,1/2]$, the difference between the Lorenz curves expresses the *disparity* between the "higher" and the "lower" parts of the distribution. In terms of income distributions, Δ_F equals the difference between the proportion of the society's overall wealth that is held by the society's top (rich) $100t\%$, and the proportion of the society's overall wealth that is held by the society's low (poor) $100t\%$. On the other hand, Σ_F expresses the *conformity* between the two tails of the distribution, with respect to the $2t$ line, as we shall discuss in section 4.

Curves that express disparity between the tails, like Δ_F, should be basically equivalent for distributions that present more (less) inequality in the left (right) tail and, "symmetrically" less (more) inequality in the right (left) tail. However, note that this vague idea of symmetry can be construed in different ways: indeed, observe that Δ_F is based on a vertical distance between LCs but, as suggested by [13], we may alternatively consider similar disparity curves based on an horizontal distance or a perpendicular distance (to the line $1 - t$), that is, respectively:

$$\Delta_F^*(t) = (L_F)^{-1}(t) - (\bar{L}_F)^{-1}(t), t \in [0,1/2] \quad (7)$$

$$\Delta_F^{**}(t) = \sqrt{2}\left({K_F}^{-1}(t) - {\bar{K}_F}^{-1}(t)\right), t \in [0,1/2] \quad (8)$$

where $K_F(t) = \frac{L_F(t)+t}{2}$ and $\overline{K}_F(p) = \frac{\overline{L}_F(t)+t}{2}$ (for the proof see [13]) Clearly, we may similarly define two alternative versions for Σ_F.The three different approaches represented by (6), (7) and (8) yield different effects on the measurement of inequality: while (6) and (7) are basically equivalent from a mathematical point of view (for symmetry with respect to the line t), the distinction between (6) and (8) is more important. Indeed, it may be noted that Δ_F is equivalent for LCs obtained by rank-preserving transfers, of fixed quantity, that may occur symmetrically (i.e. same "positional" distance from the median) in the left or right tail (see the definitions of downside and upside positional transfers, [1]): this does not hold for Δ_F^{**}. On the other hand, Δ_F^{**} is equivalent for couples of LCs that are (mutually) symmetric with respect to the line $1-t$ while, in this case, Δ_F emphasizes the right tail compared to the left one.
In this paper we focus on (6) because of its mathematical simplicity (similar results may be trivially obtained for (7) by symmetry) and temporarily overlook (8). We shall show that Σ_F is related to an ordering of skewness, whilst Δ_F is related to an ordering of kurtosis (tailweightness). Moreover, we shall propose a new ordering of inequality based on Δ_F.
The Lorenz dominance (LD) $\leq_L$ is a pre-order defined over the space $\mathcal{F}$. It can be defined as follows.

Definition 3. Let $F, G \in \mathcal{F}$: we write $F \leq_L G$ if and only if $L_F(p) \geq L_G(p), \forall\, p \in (0,1)$.

It is well known that, in a discrete context, the LD is coherent with the principle of transfers, in that $F \leq_L G$ is equivalent to saying that F can be obtained from G by a sequence of (rank-preserving) progressive transfers (see e.g. [25] p. 7). Hence, any index that is isotonic with the LD is also consistent with this basic principle.
Let $l_F = F^{-1}/\mu_F$ (note that if L_F is differentiable $l_F = (L_F)'$). We can express the relation between LD and majorization as follows [6]

$$\text{if } F \leq_L G \text{ then } l_F \prec l_G. \tag{9}$$

This result makes it possible to define several classes of functionals that are isotonic with LD, based on some theorems of majorization theory: for a detailed review see [7].
In particular, if we apply Theorem 1 to l_F, we obtain that any functional of the form

$$\Upsilon(F) = \int_0^1 \varphi(l_F(t))dt, \tag{10}$$

where φ is a continuous and convex function (such that the integral exists), is isotonic with $\leq_L$. Equation (10) defines a general class of functionals that may yield suitable inequality measures. Indeed, based on the properties i)-iv) of the LC described above, it is straightforward to verify that $\Upsilon(F)$ fulfils some useful properties, such as: i) $\Upsilon(F)$ attains its minimum when $X = \mu_F$ with probability one (absence of inequality); ii) if $Y = cX$ ($c > 0$) then $\Upsilon(F) = \Upsilon(G)$ (scale invariance or *mean independence*); iii) if $Y = c + X$ ($c > 0$) then $\Upsilon(G) < \Upsilon(F)$; iv) invariance to population replication.
Several well-known indices belong to the general family defined by equation (10) (see also [29]), among which we may note the class of *additively decomposable* measures of inequality [30]:

$$I_r(F_n) = \begin{cases} \int_0^1 \ln(1/l_{F_n}(t))\, dt & r = 0 \\ \frac{1}{r(r-1)} \int_0^1 \left\{ \left(l_{F_n}(t) \right)^r - 1 \right\} dt & r \neq 0,1, \\ \int_0^1 l_{F_n}(t) \ln(l_{F_n}(t))\, dt & r = 1 \end{cases} \tag{11}$$

where F_n is the empirical distribution function. We observe that $I_r(F_n) = 0$ in case of absence of inequality, and a suitable normalization, i.e. $I_r^N(F_n) = I_r(F_n)/I_r\left(F_{n,\mu_{F_n}}^{\max}\right)$, yields $I_r^N(F_n) = 1$ iff $F_n(x) = F_{n,\mu_{F_n}}^{\max}(x)$, where $F_{n,\mu_{F_n}}^{\max}$ is the distribution in case of maximum inequality, given the couple $\left(n, \mu_{F_n}\right)$:

$$F_{n,\mu_{F_n}}^{\max}(x) = \begin{cases} 0 & x < 0 \\ \frac{n}{n-1} & 0 \leq x < 1. \\ 1 & x \geq n\mu_{F_n} \end{cases} \tag{12}$$

More interestingly, I_r also fulfils some additional attractive properties of an inequality index, in that it has been proved that it can be decomposed by population subgroups [30] and by sources (e.g. income sources) [31].

Clearly, also the coefficient of variation and the well-known Gini index [18] are consistent with the LD. We recall that the Gini index, which does not belong to the family defined by (10), is given by twice the area between the Lorenz curve and the 45° line:

$$\Gamma(F) = 1 - 2\int_0^1 L_F(t)dt. \tag{13}$$

4. Second-degree Lorenz dominance: intersections and skewness

When the Lorenz ordering is not fulfilled, i.e. when LCs intersect, we need to introduce some weaker criteria in order to obtain unambiguous rankings. Let $\mathcal{F}_\mu$ be the class of non-negative distributions with equal mean μ. Because, in $\mathcal{F}_\mu$, the LD is equivalent to the second-degree stochastic dominance (2-SD), which in turn is equivalent to the second-degree inverse stochastic dominance (2-ISD) [24], Muliere and Scarsini [26] suggest using the third-degree inverse stochastic dominance (3-ISD) to rank intersecting Lorenz curves. While the LD compares the percentages of total (wealth) corresponding to the low $100t\%$ parts of the distributions, by using the 3-ISD an integration is performed. Hence the comparison concerns the cumulated percentages of total corresponding to the low $100t\%$ parts of the distributions. In other words, by so doing we emphasize the left tail of the distribution (i.e. lower incomes). A parallel approach consists in cumulating LCs from the right: that is, attaching more weighting to top incomes. In this paper we find it more convenient to adopt the normalized (i.e. based on the LC) version of the 3-ISD: that is, the second-degree Lorenz dominance [1], defined as follows. Note that our definition slightly differs from Aaberge's definition [1] in that, coherently with the literature and our definition of LD (Def. 3), we consider "dominant" the distribution that presents greater inequality.

Definition 4. We say that G second-degree upward Lorenz dominates F, and write $F \leq_L^2 G$ iff:

$$\int_0^t L_F(p)dp \geq \int_0^t L_G(p)dp, \forall t \in [0,1] \text{ (that is } L_F \prec^w L_G).$$

We say that G second-degree downward Lorenz dominates F, and write $F \leq_{\bar{L}}^2 G$ iff any of the following equivalent conditions is true:

i) $\int_0^t \bar{L}_F(p)dp \leq \int_0^t \bar{L}_G(p)dp, \forall t \in [0,1]$

ii) $\int_t^1 1 - L_F(p)dp \leq \int_t^1 1 - L_G(p)dp, \forall t \in [0,1]$

iii) $\int_t^1 L_G(p)dp \leq \int_t^1 L_F(p)dp, \forall t \in [0,1]$ (that is $L_G \prec_w L_F$).

In this paper, we generally state that the 2-LD holds when one (both) of the orderings $\leq_L^2$ and $\leq_{\bar{L}}^2$ is (are) verified. Observe that $F \leq_L G$ implies $F \leq_L^2 G$ and $F \leq_{\bar{L}}^2 G$, but the converse is not necessarily true, i.e. $F \leq_L^2 G$ and $F \leq_{\bar{L}}^2 G$ do not imply the LD. Moreover, note that $F \leq_L^2 G$ implies that L_F starts above L_G and presents a larger (underlying) area: that is, a lower (or equal) value of the Gini index. Differently, $F \leq_{\bar{L}}^2 G$ implies that L_F starts below L_G but still has a lower (or equal) Gini. Hence it is apparent that in both cases the condition $\int_0^1 L_F(p)dp \geq \int_0^1 L_G(p)dp$ (i.e. $\Gamma(F) \leq \Gamma(G)$) is necessary to establish a dominance. In particular, Zoli [39] argues that if Lorenz curves cross only once, the value of the Gini index is crucial for determining the (upward) second-degree Lorenz dominance (this result can be trivially extended to the downward 2-LD). Actually, we recall that this is a straightforward consequence of a theorem of Hanoch and Levi [19], with regard to stochastic dominance, or an earlier result of Karlin and Novikoff [21] in a more general context. Indeed, observe that the role of the Gini index for the (first or second-degree) LD is equivalent to the role of the mean for the (first or second-degree) stochastic dominance. Hence, because of the crucial function of the Gini coefficient, it is important to investigate the special case of $\Gamma(F) = \Gamma(G)$. Although this particular situation is clearly uncommon in practical cases, it should be stressed that, in order to understand the different orderings and their relations more deeply, it is especially meaningful to analyze and compare them under conditions of maximum uncertainty, to be understood, in our case, as LCs with equal areas (that is, $\Gamma(F) = \Gamma(G)$). Indeed, preorders only represent general rules for comparison of distributions based on particular preferences (e.g. aversion to inequality). Then, once the rules have been established, it is possible to obtain functionals that are isotonic with these rules. Observe that, if a given functional is isotonic with a particular ordering under conditions of maximum uncertainty, then this functional will clearly rank distributions coherently under sharper conditions; and, in particular, we expect that it will substantially reflect our preferences, in some sense, even in those situations where we cannot establish a dominance. For this reason, in what follows we shall pay particular attention to the case of $\Gamma(F) = \Gamma(G)$. The following theorem generalizes the result of [39]. Since some of the results are stated in terms of the number of times a Lorenz curve intersects another, we provide the following formal definition (see also [9]).

Definition 5. L_F crosses L_G k times first from above if there exist a set of points namely $t_1, \dots, t_k$ (where $t_1 < t_2 \dots < t_k$) such that, if $t_0 = 0$ and $t_{k+1} = 1$, we have: $(-1)^{\mathrm{i}} L_F(\mathrm{t}) \leq (-1)^{\mathrm{i}} L_G(\mathrm{t})$ for $t_i \leq t < t_{i+1}$ and $i = 0, \dots, k$ and $(-1)^{\mathrm{i}} L_F(\mathrm{t}) < (-1)^{\mathrm{i}} L_G(\mathrm{t})$ for some $t_i \leq t_i{}' < t_{i+1}$. Similarly, L_F crosses L_G k times first from below iff $\bar{L}_F$ crosses $\bar{L}_G$ k times first from above.

Theorem 2.

1. Let L_F and L_G cross once first from above. If $\Gamma(F) \leq \Gamma(G)$ then $F \leq_L^2 G$; if $\Gamma(F) \geq \Gamma(G)$ then $G \leq_{\bar{L}}^2 F$.
2. Assume that $\Gamma(F) = \Gamma(G)$ and L_F and L_G cross an even number of times. Then the conditions $F \leq_L^2 G$, $F \leq_{\bar{L}}^2 G$, $G \leq_L^2 F$, $G \leq_{\bar{L}}^2 F$ cannot hold true.

Proof

1) In the specific context of the LC, this has been proved in [39] with regard to $\leq_L^2$. However, an alternative proof is as follows. Let us refer to the extended "distributional" version of L_F by

$$L_F^*(p) = \begin{cases} 0 & p \leq 0 \\ L_F(p) & p \in (0,1). \\ 1 & p \geq 1 \end{cases}$$

Then $F \leq_L^2 G$ is equivalent to $L_G^* \leq_{SD}^2 L_F^*$, where $\leq_{SD}^2$ indicates the 2-SD, and $\mu_{L_H^*} = \frac{\Gamma(F)+1}{2}$ (for $H = F, G$). Hence point 1), with regard to the $\leq_L^2$ dominance, is simply Theorem 3 of [19], which in turn can be seen as a special case of a result of [21]. The proof can be easily extended to $\leq_{\bar{L}}^2$ with very similar arguments.
2) Note that this result has been proved by [27] with regard to stochastic dominance.
If L_F crosses L_G first from above (or below), it is obvious that $G \leq_L^2 F$ and $G \leq_{\bar{L}}^2 F$ (or $F \leq_L^2 G$ and $F \leq_{\bar{L}}^2 G$) cannot be true.
Without loss of generality, suppose that L_F crosses L_G k times first from above (where k is even). Define $L_F(z) - L_G(z) = \delta_F(z)$.

$$\Gamma(F) - \Gamma(G) = \int_0^1 \delta_F(z)dz = \int_0^{t_k} \delta_F(z)dz + \int_{t_k}^1 \delta_F(z)dz = 0$$

thus $\int_0^{t_k} \delta_F(z)dz = -\int_{t_k}^1 \delta_F(z)dz$. By assumption $L_F(t) \geq L_G(t)$ for $t > t_k$ (and $L_F(t) > L_G(t)$ for some $t' > t_k$). Therefore $\int_0^{t_k} \delta_F(z)dz > 0$ and $\int_{t_k}^1 \delta_F(z)dz < 0$, which yields that $F \leq_L^2 G$ cannot hold (i.e. it exists at least one point t'' such that $\int_0^{t''} L_F(p)dp < \int_0^{t''} L_G(p)dp$). Similarly we can prove that $F \leq_{\bar{L}}^2 G$ cannot hold true.

For the sake of simplicity, in what follows we shall focus on $\leq_L^2$ and overlook $\leq_{\bar{L}}^2$ in most of the cases. Clearly, all the results can be easily extended to $\leq_{\bar{L}}^2$.
From an intuitive point of view, the upward (or downward) 2-LD basically expresses a preference for right (or left) skewed distributions; that is, on equal values of the Gini, distributions that present less inequality in the left tail and consequently a "heavier" (to be understood as greater inequality) right tail. With the

following results and discussion we attempt further to analyze the second-degree Lorenz dominance, especially by showing its relation with the concept of skewness. In this context, the curve Σ_F defined in section 3 turns out to be a fundamental tool for comparisons of distributions in terms of skewness.

Theorem 3. $\Sigma_F(t) = 2t, \forall t \in [0,1/2]$ iff F is symmetric.

Proof
It is well known that F is symmetric around the value $\mu_F = F^{-1}(1/2)$ iff $F(\mu_F - t) = 1 - F(\mu_F + t)$ for all t in the support of F, which can be equivalently expressed as follows, in terms of the inverse distribution F^{-1}:

$$F^{-1}(p) + F^{-1}(1-p) = 2\mu_F \text{ for all } p \in [0,1/2\,]$$

Observe that

$$\bar{L}_F(t) = \frac{1}{\mu_F}\int_0^t F^{-1}(1-p)dp,$$

then F is symmetric iff $\Sigma_F(t) = \frac{1}{\mu_F}\int_0^t \left(F^{-1}(p) + F^{-1}(1-p)\right)dp = \frac{1}{\mu_F}\int_0^t 2\mu_F = 2t.$

In view of Theorem 3, we argue that the line $2t$ in the interval $[0,1/2\,]$ can represent a watershed for the curve $\Sigma_F(t)$ and may therefore be used as a graphical tool for detecting asymmetry, that is somehow close to the idea of a skewness diagram, see Zenga [37]. The comparison between $2t$ and $\Sigma_F(t)$ makes it possible to recognize right and left asymmetric distributions, as stated in the following definition.

Definition 6. Let $F \in \mathcal{F}_0$:

1. F is symmetric iff $\Sigma_F(t) = 2t, \forall t \in [0,1/2]$
2. F is right asymmetric iff $\Sigma_F(t) \leq 2t, \forall t \in [0,1/2]$ and $\Sigma_F(t) < 2t$ for some $t \in [0,1/2]$
3. F is left asymmetric iff $\Sigma_F(t) \geq 2t, \forall t \in [0,1/2]$ and $\Sigma_F(t) > 2t$ for some $t \in [0,1/2]$

Hence, the function $\Sigma_F(t)$ expresses the form of the distribution in terms of skewness and therefore can be used to rank distributions by introducing a stochastic ordering of asymmetry (see e.g. [3,15,23,35]). In particular, according to Arnold and Groeneveld [3] we can argue that, if F, G are "centered" around the same value (that is, the median in [3], but we can take the mean as well), F is more right asymmetric than G if

$$\int_0^t \left(F^{-1}(p) + F^{-1}(1-p)\right)dp \leq \int_0^t \left(G^{-1}(p) + G^{-1}(1-p)\right)dp \text{ for all } t \in [0,1/2\,] \qquad (14)$$

If the means of F, G are equal, the above inequality is actually equivalent to

$$\Sigma_F(t) \leq \Sigma_G(t) \text{ for all } t \in [0,1/2\,] \qquad (15)$$

(see the proof of Theorem 3).
Note that, like the LC, Σ_F is scale invariant but not location invariant. Hence its behavior determines an ordering of skewness which is suitable for distributions with equal means. In particular, we can introduce the following ordering of (right) skewness in both strong and weak (i.e. integrated) versions.

Definition 7. Let $F, G \in \mathcal{F}_\mu$.

1. We say that G is more right asymmetric than F and write $F <_s G$ iff $\Sigma_F(t) \leq \Sigma_G(t), \forall t \in [0,1/2]$.
2. We say that G is weakly more right asymmetric than F and write $F <_s^2 G$ iff $\int_0^t \Sigma_F(p)dp \leq \int_0^t \Sigma_G(p)dp, \forall t \in [0,1/2]$.

If $F <_s G$ (or $F <_s^2 G$) F is equivalently less left-skewed than G. From this discussion we may also argue that the orderings $<_s, <_s^2$ can be used to rank distributions with different means (and not necessarily positive) through the use of generalized Lorenz curves [32] and a suitable standardization.
Assume that $\Gamma(F) = \Gamma(G)$: it is possible to show that if F and G are symmetric they cannot be ranked according to the second-degree Lorenz dominance. Moreover, if $\Gamma(F) = \Gamma(G)$ then the second-degree

Lorenz dominance implies the weak ordering $<_s^2$. Therefore in this case $\leq_L^2$ can also be interpreted as an ordering of skewness (stronger than $<_s^2$) within the class $\mathcal{F}_\mu$.

Theorem 4. Assume that $\Gamma(F) = \Gamma(G)$

1. Let $F, G \in \mathcal{F}_\mu$. $F \leq_L^2 G$ implies $F <_s^2 G$.
2. Assume that F, G are symmetric. Then $\leq_L^2$ (as well as $\leq_{\bar{L}}^2$) are unable to rank F, G.

Proof

As for 1), observe that $\int_0^t L_F(p)dp \geq \int_0^t L_G(p)dp$, for every t in [0,1], and $\int_0^1 L_F(p)dp = \int_0^1 L_G(p)dp$ yield $\int_t^1 L_F(p)dp \leq \int_t^1 L_G(p)dp$ which is equivalent to $\int_0^t \bar{L}_F(p)dp \geq \int_0^t \bar{L}_G(p)dp$, for every t in [0,1]. Then by summing the first and the last inequalities we obtain the thesis.

With regard to 2), it is evident that the Lorenz ordering cannot hold because $\int_0^1 L_F(p)dp = \int_0^1 L_G(p)dp$, unless $L_F = L_G$. We shall prove the thesis by contradiction. Suppose that $F \leq_L^2 G$: in this case the Lorenz curves intersect in at least one point, say $p' \neq 0.5$ (otherwise the distributions are not symmetric). Then L_F, L_G will also cross in $1 - p'$, for symmetry. Thus, we can understand that L_F, L_G can only have an even number of crossing points. Therefore the thesis follows from Theorem 2.

Theorem 2 (point 2) and Theorem 4 (point 2) are useful in order to understand the possible "blind spots" of the second-degree Lorenz dominance. Indeed, they determine some conditions under which the orderings $\leq_L^2$ and $\leq_{\bar{L}}^2$ are not able to provide a ranking. In view of these results, in the next section we introduce an alternative ordering whereby we can easily rank double-crossing Lorenz curves and also symmetric distributions.

5. A new preorder based on the difference between Lorenz curves

As an alternative approach, we may combine the basic ideas (preferences) expressed by the $\leq_L^2$ and $\leq_{\bar{L}}^2$ orderings into a single preorder, which emphasizes inequality in both the tails of the distribution. This can be done by symmetrically cumulating the Lorenz curve from both sides. Suppose $\Gamma(F) = \Gamma(G)$: in this case we might "prefer" (in terms of inequality measurement) F to G basically if L_F is above L_G in a neighborhood of 0 and 1 (i.e. F presents less inequality in the tails, while G presents less inequality in the "body").

We consider the disparity curve Δ_F, described in section 3. Clearly, $F \leq_L G$ implies that $\Delta_F(t) \leq \Delta_G(t)$ for every t in [0,1/2] ($\Delta_F(t) = 0$ iff $L_F(t) = t$). Therefore this condition may be interpreted as a strong disparity dominance. If L_F and L_G cross, it can be reasonable to wish that Δ_F is (uniformly) as small as possible: this concept can be expressed in terms of an integral inequality (i.e. weak majorization). We propose the following weak preorder of inequality based on the disparity curve.

Definition 8. We say that F second-degree disparity dominates (2-DD) G and write $F \leq_D^2 G$ iff:

$$\int_0^t \Delta_F(p)dp \leq \int_0^t \Delta_G(p)dp \ \forall t \in [0,0.5] \text{ (equivalently } \Delta_G \prec^w \Delta_F\text{).}$$

Like $F \leq_L^2 G$, also $F \leq_D^2 G$ implies the condition $\int_0^{1/2} \Delta_F(p)dp \leq \int_0^{1/2} \Delta_G(p)dp$, which is equivalent to $\Gamma(F) \leq \Gamma(G)$ ($\int_0^{1/2} \Delta_F(p)dp = \int_0^1 (p - L_F(p))dp$).

Clearly, as a complementary approach (like $\leq_L^2, \leq_{\bar{L}}^2$) we may emphasize the "body" of the distribution, rather than the tails, by integrating a suitable transformation of Δ_F (that may be $\Delta_F(0.5) - \Delta_F(0.5 - p)$ as will be shown in what follows), but for the sake of simplicity we shall not introduce another preorder in this paper.

The main result of Theorem 5 below is that we can determine a sufficient condition for the 2-DD based on the number (and the positions) of intersections and the values of the Gini index (point 1). This is somehow parallel to Theorem 2 with regard to 2-LD, and shows that the 2-DD may fill the main "gap" of the 2-LD, even when $\Gamma(F) = \Gamma(G)$. Vice versa, for the sake of completeness, we show that when Lorenz curves are single crossing after 0.5 and $\Gamma(F) = \Gamma(G)$ the 2-DD cannot hold, while the 2-LD certainly does so (upward or downward) for Theorem 2. We also observe that theorems 2 and 5 can justify the use of the Gini coefficient (which is isotonic with 2-LD and 2-DD) for ranking, respectively, single-crossing and double-crossing LCs.

Theorem 5.

1. Assume that L_F, L_G cross twice first from above in p', p'' and $\Gamma(F) \leq \Gamma(G)$. If $p' = 1 - p''$, then $F \leq_D^2 G$. If $p' > 1 - p''$ and $\nexists \bar{p} \in (1 - p'', p')$ such that $\int_0^{\bar{p}} \bar{L}_F(p) - \bar{L}_G(p)dp > 0$, then $F \leq_D^2 G$. If $p' < 1 - p''$ and $\nexists \bar{p} \in (p', 1 - p'')$ such that $\int_0^{\bar{p}} L_F(p) - L_G(p)dp < 0$, then $F \leq_D^2 G$.
2. Assume that L_F, L_G cross twice in p', p'' (so that $p' < p'' < 0.5$ or $0.5 < p' < p''$) and $\Gamma(F) = \Gamma(G)$. Then $F \leq_D^2 G$ cannot hold.
3. Assume that $\Gamma(F) = \Gamma(G)$ and L_F, L_G are single crossing in $p' > 0.5$. Hence $F \leq_D^2 G$ cannot hold.

Proof

1) Assume, for the moment, that $\Gamma(F) = \Gamma(G)$. Suppose also that $p' < 1 - p''$ (the proof can be symmetrically extended to the case of $p' > 1 - p''$, while in the case of $p' = 1 - p''$ it is trivial). We prove the thesis by contradiction. Suppose that there exists a point say p^* such that $\int_0^{p^*} \bar{L}_F(p) - L_F(p)dp > \int_0^{p^*} \bar{L}_G(p) - L_G(p)dp$, which is equivalent to $\int_{1-p^*}^1 L_G(p) - L_F(p)dp > \int_0^{p^*} L_F(p) - L_G(p)dp$. Clearly p^* must belong to the interval (p', p''). Thus the latter inequality is equivalent to

$$\int_{p''}^{1-p^*} L_G(p) - L_F(p)dp + \int_{p'}^{p^*} L_G(p) - L_F(p)dp > \int_0^{p'} L_F(p) - L_G(p)dp + \int_{p''}^1 L_F(p) - L_G(p)dp = \\ = \int_{p'}^{p''} L_G(p) - L_F(p)dp$$

which contradicts the assumption $\Gamma(F) = \Gamma(G)$. The thesis holds *a fortiori* when $\Gamma(F) \leq \Gamma(G)$.

2) We prove the thesis only for the case in $p' < p'' < 0.5$ because in the other case it can be proved similarly. Observe that

$$\int_0^{p''} L_G(p) - L_F(p)dp = \int_{p'}^{p''} L_G(p) - L_F(p)dp - \int_0^{p'} L_F(p) - L_G(p)dp = \int_{p''}^1 L_F(p) - L_G(p)dp$$

It is therefore evident that $\int_0^{p''} \bar{L}_F(p) - L_F(p)dp > \int_0^{p''} \bar{L}_G(p) - L_G(p)dp$.

3) The condition $\Delta_G \prec^w \Delta_F$ is equivalent to $\int_0^t \bar{L}_F(p) - L_F(p)dp \leq \int_0^t \bar{L}_G(p) - L_G(p)dp$, or $\int_0^t \bar{L}_F(p) - \bar{L}_G(p)dp \leq \int_0^t L_F - L_G(p)dp$. Denote with p' the crossing point of L_F, L_G. Note that $p' > 0.5$ yields $\int_0^{1-p'} \bar{L}_F(p) - \bar{L}_G(p)dp = \int_0^{p'} L_F - L_G(p)dp$. There consequently exists at least one point $p'' < 1 - p'$ such that $\int_0^{p''} \bar{L}_F(p) - \bar{L}_G(p)dp > \int_0^{p''} L_F - L_G(p)dp$.

In view of Theorem 5, we may easily argue that the 2-DD can be especially suitable for dealing with symmetric distributions, since in this case the number of crossing points must be definitely even.

Another substantial difference between the 2-LD and the 2-DD is that the former is related to skewness (as shown in section 4) while the latter is related to the general idea of "kurtosis" (see [5]). Indeed, we show that Δ_F is strictly related to the spread function originally introduced by Bickel and Lehmann [8] and defined in [5] as:

$$s_F(p) = F^{-1}\left(\frac{1}{2} + p\right) - F^{-1}\left(\frac{1}{2} - p\right) \tag{16}$$

Let $S_F(p) = \frac{1}{\mu_F}\int_0^p s_F(t)dt$. We can express S_F in terms of the disparity curve Δ_F:

$$S_F(p) = \frac{1}{\mu_F}\int_0^p F^{-1}\left(\frac{1}{2}+t\right) - F^{-1}\left(\frac{1}{2}-t\right)dt =$$

$$= \frac{1}{\mu_F}\int_{\frac{1}{2}-p}^{\frac{1}{2}} F^{-1}(1-u)du - \frac{1}{\mu_F}\int_{\frac{1}{2}-p}^{\frac{1}{2}} F^{-1}(u)du = \bar{L}_F\left(\frac{1}{2}\right) - \bar{L}_F\left(\frac{1}{2}-p\right) - L_F\left(\frac{1}{2}\right) + L_F\left(\frac{1}{2}-p\right) =$$

$$= \Delta_F\left(\tfrac{1}{2}\right) - \Delta_F\left(\tfrac{1}{2}-p\right) \tag{17}$$

(change of variables: $u = \frac{1}{2} - p$).

Note that, according to the literature, the spread function gives rise to an ordering of kurtosis (for some literature on this topic we mainly refer to [3,5,14], we also stress that the idea of measuring kurtosis based on LCs may be found in the kurtosis diagram [14, 28, 38], originally proposed by Zenga [36]). In particular, Arnold and Groeneveld [3] propose a list of possible orderings, not necessarily for symmetric distributions, among which one is based on the integrated spread curve (they also assume that the distributions are centered around the same value, but we can equivalently substitute this condition by dividing by the mean, which yields S_F). Since "kurtosis" expresses the concepts of peakedness, tailweightness or, roughly speaking, "lack of shoulders" [5], we argue that S_F is related to peakedness (because it measures concentration around the median) whilst Δ_F is related to tailweightness. Therefore S_F and Δ_F can be complementary tools for comparing distributions in terms of disparity, but also in terms of kurtosis (respectively understood as peakedness and tailweightness) within a suitable class of distributions, that is, $\mathcal{F}_\mu$. Indeed, note that Δ_F is not location invariant, but kurtosis is in fact intended to be a location and scale-free concept. Therefore, similarly to Definition 7, we can propose strong and weak orderings of kurtosis (tailweightness) in the class of distributions with equal means, based on Δ_F.

Definition 9. Let $F, G \in \mathcal{F}_\mu$.

1. We say that F is less heavy tailed than G and write $F <_k G$ iff $\Delta_F(t) \le \Delta_G(t), \forall t \in [0,1/2]$.
2. We say that F is weakly less heavy tailed than G and write $F <_k^2 G$ iff $F \le_D^2 G$.

The application of theorems 2 and 5, as well as the "complementary" relation between the 2-SD and 2-DD (on fixed values of the Gini coefficient), can be shown by a simple theoretical example. Let us consider the following vectors, with equal means and equal numbers of elements:

$$(0,10,20,30,40,50,60,70,80,90)$$
$$(5,5,20,30,40,50,60,70,75,95)$$
$$(5,5,15,35,40,50,55,75,85,85)$$

and with corresponding empirical distribution functions F, G, H, respectively, such that $\Gamma(F) = \Gamma(G) = \Gamma(H)$. As can be seen, F is a symmetric distribution, G presents less inequality in the left tail but (symmetrically, in terms of positions) more inequality in the right tail, whilst H presents less inequality in both of the tails but more inequality close to the median value. With regard to the LCs, L_F and L_G cross once, L_F and L_H cross twice and L_G and L_H cross once (after 0.5). We can verify that $G \le_L^2 F$ (Theorem 2) and $H \le_D^2 F$ (Theorem 5). Moreover, we cannot rank F, H with the 2-LD, and similarly we cannot rank F, G with the 2-DD. In other words, these couples of distributions are equivalent with respect to the orderings (note that $\Sigma_F = \Sigma_H$ while $\Delta_F = \Delta_G$). As for G, H we obtain that $G \le_L^2 H$ (Theorem 2) because G presents less inequality starting from the left tail, while conversely we can verify that $H \le_D^2 G$because H presents less inequality starting from the two tails and greater inequality in the "body".

6. Some classes of isotonic functionals

In this section we propose some classes of functionals that are isotonic with the weak preorders 2-LD and 2-DD, based on Theorem 1.
In particular, with regard to the second-degree Lorenz dominance, we propose

$$\Phi(F) = \int_0^1 \phi(L_F(t))dt, \quad (18)$$

where we assume that $\phi(0) = 0$ and $\int_0^1 |\phi(L_F(t))|dt < \infty$. Then, as straightforward consequences of Theorem 1, the following results hold.

Corollary 1.

If ϕ is decreasing and convex and $F \leq_L^2 G$, then $\Phi(F) \leq \Phi(G)$.
If ϕ is decreasing and concave and $F \leq_L^2 G$, then $\Phi(F) \leq \Phi(G)$.

For instance, we can obtain some inequality measures isotonic with $\leq_L^2$ if we set $\phi(t) = -t^a, 0 < a < 1$, or $\phi(t) = -\ln(t+1)$. Moreover, if F_n is the empirical distribution function, we can obtain a normalized version of $\Phi(F_n)$ given by

$$\Phi_N(F_n) = \frac{\Phi(F_n)}{\Phi_{\min} - \Phi_{\max}(n)} \quad (19)$$

where $\Phi_{\min} = \int_0^1 \phi(t)dt$ (min. inequality) and $\Phi_{\max}(n) = \int_{\frac{n-1}{n}}^1 \phi(-(n-1)+nt)dt$ (max. inequality). Clearly $\Phi_N(F_n) = 0$ or $\Phi_N(F_n) = 1$ respectively in the case of minimum or maximum inequality (concentration).
We can use similar arguments in order to obtain inequality measures coherent with the 2-DD. Consider:

$$\Psi(F) = \int_0^{1/2} \psi(\Delta_F(t))dt, \quad (20)$$

where we assume that $\psi(0) = 0$ and $\int_0^{1/2} |\psi(\Delta_F(t))|dt < \infty$.

Corollary 2. If ψ is increasing and concave and $F \leq_D^2 G$, then $\Psi(F) \leq \Psi(G)$.

In this case, we can still propose some inequality measures isotonic with $\leq_D^2$ by setting $\psi(t) = t^a, 0 < a < 1$, or $\psi(t) = \ln(t+1)$. The degree of concavity of ψ determines the weighting that we may attach to the tails of the distributions, i.e. the more concave is ψ, the more emphasis is given to the lowest and highest values.
The corresponding normalized index for empirical data is as follows:

$$\Psi_N(F_n) = \frac{\Psi(F_n)}{\Psi_{\max}(n)} \quad (21)$$

where $\Psi_{\max}(n) = \int_0^{1/n} \psi(nt)dt$ (note that $\Phi_{\min} = \int_0^1 \phi(0)dt = 0$). Then $\Psi_N(F_n) = 0$ or $\Psi_N(F_n) = 1$ respectively in the case of minimum or maximum inequality (concentration).
Clearly, Φ_N and Ψ_N also fulfil the elementary properties of inequality measures such as mean independence, invariance to population replication, and they decrease if the values are translated by a constant and positive quantity (these properties derive straightforward from the coherence with the LD and the basic properties of the LC described in section 3).

7. Conclusion

This paper has compared two conceptually complementary approaches to ranking intersecting Lorenz curves. The first approach is the so-called second-degree Lorenz dominance [1] or 2-LD, which is based on the third-degree inverse stochastic dominance as originally proposed by [26]. The second approach, namely the second-degree disparity dominance or 2-DD, is based on the integrated difference between the complementary LC and the LC. Essentially, the 2-LD attaches more weighting to the right or left tail of the distribution, whilst the 2-DD emphasizes both of them. We have shown that the two criteria are basically complementary: in particular, the 2-LD and the 2-DD are especially suitable for ranking, respectively, single crossing and double-crossing Lorenz curves. In both cases, the value of the Gini index is crucial. Moreover, we have examined and compared the connection between these two dominance relations and the concepts of skewness and kurtosis. We have argued that the newly introduced order, i.e. the 2-DD, may serve a twofold purpose: i) it may provide unambiguous rankings in some situations when the 2-LD fails and; ii) it may be used in order to combine the two different approaches of the 2-LD, namely upward or downward, into one single preorder. We have stressed that the 2-LD is an especially useful criterion for economic applications (income distributions) because empirical studies have shown that single-crossing Lorenz curves occur very often [4,11] compared to the multiple-crossing case. Nevertheless, this paper has mainly focused on the statistical/mathematical aspects of inequality measurement because the results may not be limited to income distributions: in fact, they may be applied to a more general context.

In the last section we proposed some classes of functionals that are isotonic with the aforementioned dominance criteria, based on some well-known results of majorization theory. Future work will involve applying the resulting inequality indices to real data, in order to show the usefulness of their properties from a practical point of view. Moreover, it would be interesting to propose and analyze similar orderings of "disparity" based on the alternative Δ-type curves defined in section 3.

References

1. Aaberge R.: Ranking intersecting Lorenz curves. Social Choice and Welfare, 33 (2), 235-259 (2009).
2. Arnold, B.: Majorization and the Lorenz order: a brief introduction, Vol. 43. Springer Science & Business Media (2012).
3. Arnold B.C., Groeneveld A.: Skewness and kurtosis orderings: an introduction. Lecture Notes-Monograph Series, 22, 17-23 (1993).
4. Atkinson B.: On the measurement of inequality. Journal of economic theory, 2 (3), 244-263 (1970).
5. Balanda K. P., MacGillivray H.: Kurtosis and spread. Canadian Journal of Statistics, 18 (1), 17-30 (1990).
6. Bertoli-Barsotti L.: Some remarks on Lorenz ordering-preserving functional. Statistical Methods and Applications, 10 (1-3), 99-112 (2001).
7. Bertoli-Barsotti L.: Maggiorazione e Schur-convessità: generalizzazioni ed applicazioni statistiche. Tech. rep., Università degli studi di Torino, Dipartimento di statistica e matematica applicata alle scienze umane "Diego de Castro" (2001).
8. Bickel P., Lehmann E.: Descriptive statistics for non-parametric model iv: spread. Contributions to Statistics, Jane Juresckova (Ed.), Reidel, Amsterdam (1979).
9. Chiu W. H.: Intersecting Lorenz curves, the degree of downside inequality aversion, and tax reforms. Social Choice and Welfare, 28 (3), 375-399 (2007).
10. Dardanoni V., Lambert P.: Welfare rankings of income distributions: a role for the variance and some insights for tax reform. Social Choice and Welfare, 5 (1), 1-17 (1988).
11. Davies J., Hoy M.: Comparing income distributions under aversion to downside inequality. London: Department of Economics, University of Western Ontario (1985).
12. Davies J., Hoy M.: Making inequality comparisons when Lorenz curves intersect. The American Economic Review, 980-986 (1995).
13. Eliazar, I: The sociogeometry of inequality: Part I. Physica A: Statistical Mechanics and its Applications, 426, 93-115 (2015).
14. Fiori A.M.: Measuring kurtosis by right and left inequality orders. Communications in Statistics-Theory and Methods, 37 (17), 2665-2680 (2008).

15. Frosini B.V.: Su alcune diseguaglianze concernenti lo scarto medio assoluto, Statistica, 33, 41-61 (1973).
16. Fullerton D., Rogers D. L.: Who bears the lifetime tax burden?. Brookings Institution (1993).
17. Gastwirth J. L.: A general defnition of the Lorenz curve. Econometrica: Journal of the Econometric Society, 1037-1039 (1971).
18. Gini C.: Variabilità e mutabilità. Reprinted in Memorie di metodologica 50 statistica (Ed. Pizetti E, Salvemini, T), Rome: Libreria Eredi Virgilio Veschi (1912).
19. Hanoch G., Levy H.: The efficiency analysis of choices involving risk. World Scientific (1969).
20. Karamata J.: Sur une inegalitè relative aux fonctions convexes. Publications de l'Institut Mathematique 1 (1) (1932) 145-147.
21. Karlin S., Novikoff A.: Generalized convex inequalities. Pacific J. 55 Math, 13 (4), 1251-1279 (1963).
22. Lando T., Bertoli-Barsotti L.: Statistical functionals consistent with a weak relative majorization ordering: applications to the mimimum divergence estimation. WSEAS Transactions on Mathematics, 13, 666-675 (2014).
23. MacGillivray H.: Skewness and asymmetry: measures and orderings. The Annals of Statistics, 994-1011 (1986).
24. Makdissi P., Yazbeck M.: On the measurement of plutonomy. Social Choice and Welfare, 44 (4), 703-717 (2014).
25. Marshall W., Olkin I., Arnold B.: Inequalities: theory of majorization and its applications, Springer Science & Business Media (2010).
26. Muliere P., Scarsini M.: A note on stochastic dominance and inequality measures. Journal of Economic Theory, 49 (2), 314-323 (1989).
27. Ortobelli S., Lando T., Petronio F., Tichy T.: Asymptotic stochastic dominance rules for sums of iid random variables. Journal of Computational and Applied Mathematics, 300, 432-448 (2016).
28. Polisicchio M., Zenga M.: Kurtosis diagram for continuous random variables. Metron, 55 (3-4), 21-41 (1997).
29. Regazzini, E.: Concentration comparisons between probability measures. Sankhyā: The Indian Journal of Statistics, Series B, 129-149 (1992).
30. Shorrocks A.F.: The class of additively decomposable inequality measures. Econometrica: Journal of the Econometric Society, 613-625 (1980).
31. Shorrocks A.F.: Decomposition by factor components. Econometrica: Journal of the Econometric Society, 50 (1), 1993-211 (1982).
32. Shorrocks, A.F.: Ranking income distributions. Economica, 3-17 (1983).
33. Shorrocks A.F., Foster J.E.: Transfer sensitive inequality measures. The Review of Economic Studies, 54 (3), 485-497 (1987).
34. Yaari M.E.: A controversial proposal concerning inequality measurement. Journal of Economic Theory 44 (2), 381-397 (1988).
35. van Zwet W.R.: Convex transformations of random variables. Mathematical Centre, Tracts No. 7. Amsterdam Mathematisch Centrum (1964).
36. Zenga M.: La curtosi, Statistica, 56, 87-102 (1996).
37. Zenga M.: Introduzione alla statistica descrittiva. Vita e Pensiero, Milano (1998).
38. Zenga M.: Kurtosis. In: Kotz S., Read C. B., Balakrishnan N., Vidakovic B., eds. Encyclopedia of Statistical Sciences. 2nd ed. New York: John Wiley and Sons (2006).
39. Zoli C.: Intersecting generalized Lorenz curves and the Gini index. Social Choice and Welfare, 16 (2), 183-196 (1999).
40. Zoli C.: Inverse stochastic dominance, inequality measurement and Gini indices. Journal of Economics, 77 (1), 119-161 (2002).