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EDITORIAL

This special issue of Operations Research and Decisions is dedicated to Gianfranco Gambarelli, on his retirement at … the age prescribed by the Italian rules.

He is a mathematician, but not only. He has provided contributions in Game Theory, but not only. He has authored hundreds of papers and books on Game Theory, but not only. He is a person whose interests span several areas, topics and disciplines. He is the author of a number of poems, a lot of which are in "Bergamasco", the local dialect of Bergamo, his city. He is a singer and everybody that has attended a meeting with Gianfranco may well have witnessed the quality of his performances. He likes good friends, good jokes, good food. In summary, he likes good life! But there is something that he likes more, his family: his wife Gabriella, daughter Gretel and son Daniele.

Last November, we had the idea of this special issue in his honor and decided to invite his coauthors to contribute a paper. We collected the following contributions.

Cesarino Bertini, Jacek Mercik and Izabella Stach survey and criticize some models of indirect control of corporations, mainly referring to the evaluation of the power of shareholders and companies with cross-shareholdings. A comparison of various approaches is carried out using two examples, one with and one without a cyclic structure.

Francesc Carreras, Josep Freixas and Antonio Magaña study the voting systems adopted in the European Union Council of Ministers, where major discussions took place during each enlargement of the Union, paying particular attention to the dimension, the egalitarianism and the decisiveness of each system.

Marco Dall'Aglio, Vito Fragnelli and Stefano Moretti consider the criticality of players in a simple game, formalizing their power to blackmail by threatening to leave a winning coalition which they are not critical for. They also analyse the credibility of such threatening behaviour, by evaluating the possibility of forming a different majority.

Michel Grabisch and Agnieszka Rusinowska propose a model of opinion formation based on aggregation functions. They generalize the notion of influential player to influential coalitions and represent it using a (hyper)graph. They propose simple algorithms for determining influential coalitions in various cases.

Manfred Holler and Hannu Nurmi deal with the notion of monotonicity in the theory of voting and the measurement of a priori voting power, focusing on the relationship between opinion aggregation and voting power, with the aim of improving the understanding of the elements that determine the outcome of voting.

Hans Peters, Judith Timmer and Rene van den Brink analyse the distribution of power on a particular communication structure, the invariant digraph, by characterizing some power indices that under suitable hypotheses on the effects of arc addition are related to three well-known indices: the Copeland score, the β-measure and the apex power index.

Joaquín Sanchez-Soriano, Natividad Llorca and Encarnación Algaba apply a bankruptcy approach to the apportionment problem. They show that the constrained equal losses rule coincides with the greatest remainder method. They also propose new properties for governability and test them on apportionment methods in the case of the 2015 Spanish elections.

We thank all the authors for their important contributions and for their availability as reviewers that guaranteed a valuable issue and an efficient publishing process. Particular thanks go to Fioravante Patrone, a friend more than a colleague, who retired several years ago, but is still active in research. He agreed to review a paper, even if he had refused this role at all other times.

An important role was played by David Ramsey whose language expertise was extremely helpful.

Last, but not least, special thanks go to Jacek Mercik, Editor in Chief of Operations Research and Decisions, for his constant presence "in the background" and for his useful suggestions.

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Guest Editors

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INDIRECT CONTROL AND POWER

To determine who has the power within a stock corporate company can be a quite complex problem, especially when control is achieved through alliances between shareholders. This problem arises especially in cases of indirect control of corporations, that is, in situations involving shareholders and companies with cross-shareholdings. The first to solve the problem of measuring power in the case of indirect share control were Gianfranco Gambarelli and Guillermo Owen in [10]. In the following years, numerous other models were introduced. In this paper, we critically examine the models of: Gambarelli and Owen, Denti and Prati, Crama and Leruth, Karos and Peters, as well as Mercik and Lobos, taking into account two well-known, illustrative examples, one with an acyclic corporate structure and the other with a cyclic structure.

Keywords: *game theory, indirect control, corporations, power indices*

1. Introduction

To determine who has the power within a stock corporate company can be a quite complex problem, especially when control is achieved through alliances between shareholders. This problem arises especially in the cases of indirect control of corporations, that is, in situations involving shareholders and companies with cross-shareholdings. In these circumstances, there is the need to know what coalitions of firms can control a given company.

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It is a problem as complex as it is crucial. It is indeed easy to find situations in which a holding company controls other companies, resulting in a pyramidal construction that allows the holding company to gradually multiply the capital controlled, starting from a relatively low initial investment. The effects of this phenomenon are also accentuated by the dispersion of actions made by an ocean of small investors. Therefore, in practice, to acquire control of a company, it is often sufficient to possess a much lower percentage of the capital than the theoretical proportion (50% of the shares plus one share).

The work of Gambarelli and Owen [10], published in 1994, is one of the pioneering papers in this field. In the following years, numerous other models were introduced. For us it is difficult, if not impossible, to cite all the articles on indirect control of corporations in one paper, because there is a vast literature on this topic. Anyway, Crama and Leruth [5] and Karos and Peters [13] made a vast literature review of the most relevant and illustrative references in this field. For this reason, we refer interested readers to those sources and here we limit ourselves to a brief review of the relevant literature related to measuring power in corporate structures.

Gambarelli and Owen [10] and Denti and Prati [8, 9] focused on determining the winning coalitions in a control structure. Kołodziej and Stach [14] proposed a computer program based on the approach of Denti and Prati to enable simulations. On the other hand, the works of Hu and Shapley [11], Crama and Leruth [4, 5] and Crama, Leruth and Wang [6], Karos and Peters [13], as well as Mercik and Lobos [15], are dedicated to modelling indirect control relationships in corporate structures and using power indices to evaluate the power of players.

This paper is organized as follows. In Section 2, some preliminary definitions and notation are given. Section 3 provides two illustrative examples. In Section 4, brief overviews of some approaches to measuring indirect control are described. Section 5 gives some conclusions and open problems.

2. Preliminaries

A game is given by a set of rules describing a strategic situation. In cooperative games, players can collaborate to obtain common benefits. Let $N = \{1, 2, ..., n\}$ be the set of all players, indexed by the first n natural numbers. A cooperative n-person game in characteristic function form is an ordered pair (N, v), where $v: 2^N \to R$ is a real-valued function on the family 2^N of all subsets of N such that $v(\emptyset) = 0$. The real-valued function v is called the characteristic function of the game. Any subset S of N is called a coalition and $v(S)$ is the worth of the coalition S in the game. By $|S|$ we denote the cardinality of the set S. In this paper, we denote a cooperative game (N, v) simply by its characteristic function v.

A cooperative game v is monotonic if $v(S) \le v(T)$ when $S \subset T$. A simple game is a monotonic game v (in N, omitted hereafter), which assumes values in the set {0, 1}: i.e. $v(S) = 0$ or $v(S) = 1$ for all the coalitions $S \subseteq N$. In the first case, a coalition is said to be losing, in the second – winning. Let $W(v)$ denote the set of all winning coalitions in game v. A player i is critical in a winning coalition S if $v(S\backslash\{i\}) = 0$. A simple game is said to be *proper*, if and only if the following is satisfied: for all $T \subset N$, if $v(T) = 1$, then $v(N \setminus T) = 0$. A coalition S is called a minimal winning coalition if $v(S) = 1$, but $v(T) = 0$ for all $T \subset S$, $T \ne S$. $W^m(v)$ denotes the set of all minimal winning coalitions in v. Such a game can be defined either by the family of winning coalitions W or equivalently by the set of minimal winning coalitions W^m. Every player i who belongs to a minimal winning coalition $S \subseteq N$ is critical in S. We will use the following notation: $C_i = \{S \subseteq N : i \text{ is critical in } S\}$. A coalition S is called vulnerable if S contains at least one critical player, i.e. $\exists i \in S$ such that $v(S \setminus \{i\}) = 0$.

Let ($w_1, ..., w_n$) be a vector with non-negative components such that $\sum_{i \in N} w_i = 1$. For any coalition S, $w(S) = \sum_{i \in S} w_i$ is the weight of the coalition. Let $q > 0$ be the majority quota that establishes winning coalitions (usually $q > w(N)/2$). We call the simple game: $v(S) = 1$ if $w(S) > q$ and otherwise $v(S) = 0$ a weighted majority game and denote it by $[q; w_1, ..., w_n]$. Weighted majority games are suitable for describing many voting situations: the weights can be shares owned in a company, seats of political parties, etc.

A power index is a function that maps an n-person simple game v, to an n-dimensional real vector and is a measure of the influence of the players in such games. The literature has proposed many power indices based on diverse axiomatic assumptions and/or models of bargaining. Below, we recall the definitions of only those power indices that are used in the models considered; namely, the definitions of the Shapley –Shubik, Banzhaf–Penrose, and Johnston indices.

The Shapley–Shubik index was introduced by Shapley and Shubik in [19]. For the sake of simplicity, we set $|S| = s$. The Shapley–Shubik index σ, for any v and $i \in N$ is expressed as

$$\sigma_i(v) = \sum_{S \in C_i} \frac{(s-1)!(n-s)!}{n!}$$

For further explanations see, e.g., [20].

The measure called the absolute Banzhaf index β by Banzhaf [1] is sometimes called the Penrose–Banzhaf index, since it actually goes back to Penrose [17, 18]. The absolute Banzhaf index, for any v and $i \in N$, is defined as:

$$\beta_i(v) = \frac{|C_i|}{2^{n-1}}$$

For further explanations, see e.g., [3].

The Johnston index was introduced by Johnston in [12]. Denote by *VC* the set of all vulnerable coalitions. For any $S \in VC$, by $r(S)$ we denote the reciprocal of the number of critical players in *S* and we define $r_i(S)$ in the following way: if *i* is critical in *S* then $r_i(S) = r(S)$, otherwise $r_i(S) = 0$. The raw (absolute) Johnston index is defined as: $\bar{\gamma}_i(S) = \sum_{S \in VC} r_i(S)$ and the Johnston index is obtained after normalization, i.e.

$$\gamma_i(S) = \frac{\sum_{S \in VC} r_i(S)}{\sum_{i=1}^{n} \sum_{S \in VC} r_i(S)}$$

Note that the Johnston index treats all coalitions with critical voters equally, and within each coalition the power is divided equally among critical voters.

We can now analyse the problem of measuring the power of indirect control in corporations by means of these power indices. Naturally, the problem of determining the percentage of shares in a certain company indirectly owned by investors affects the measurement of power. For example, suppose that companies A, B, C, and D have a share capital composed of 100 shares each. Now assume that A holds 40 shares of company B and the remaining shares are equally divided between two other investors. Suppose also that B owns 51 shares of C, which has in turn 25 shares of D, and that the remaining shares of D are divided equally among three other investors. In the above situation, we could say that A has (1/3)×(1)×(1/4) = 1/12 of the power in company D. In general, it would seem logical to assign to each shareholder a measure of "power via indirect control" given by the product of the appropriate power indices. Unfortunately, this method can lead to situations where the total quota of shares in the controlled company exceeds 100%. It is therefore necessary to consider other approaches that will be covered in the next sections.

But firstly it is necessary to analyse typical situations that may arise in the case of indirect control. The simplest case is characterized by the absence of cycles (loops). In this situation, each investor holds shares in a number of companies; investee companies may themselves hold voting rights in other companies, but there can be no cross-shareholdings. The presence of a loop occurs when there are two or more companies with cross-shareholdings.

Although in each country the legislator regulates the indirect control of corporations more or less decisively, this does not mean that the phenomenon of indirect control does not exist. The models recalled here do not consider legislative constraints. Hereafter, by an investor we mean a firm which is not controlled either directly or indirectly by any other firm in a corporate network and by a company we mean a stock corporation i.e. a corporation which has shareholders.

3. Illustrative examples

In our treatment, we will refer to two illustrative examples of corporate control structures. In the first example, there are no cycles, whereas in the second one, looped relationships among shareholders exist. In both examples, we use networks (weighted directed graphs) to show corporate control structures: firms (corporation and investors) are represented by vertices, while share ownership is represented by weighted edges. The number corresponding to an edge connecting, e.g., firm *i* to firm *j* represents, in percentage terms, how many voting rights firm *i* holds in firm *j*.

As in this paper we adopt a game-theoretical approach, where corporate structures are often described by simple voting games (in particular weighted majority games), in defining the examples we call the firms players and we talk about the majority quota *q*. Our illustrative examples refer to real corporate groups but reflect their past shareholding structures. Although over time the real corporate structures presented in both examples have changed, we have chosen not to update the situation to maintain the effectiveness of the examples.

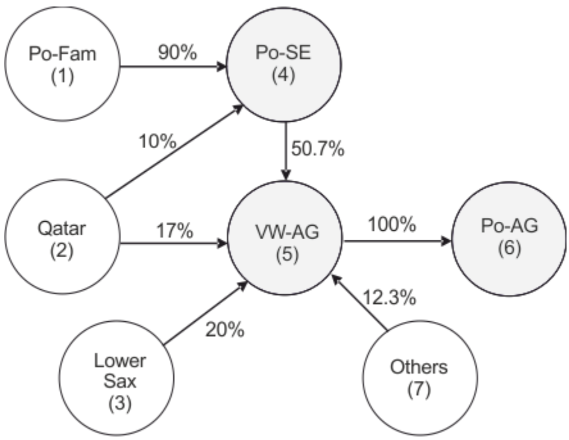

Fig. 1. The shareholding structure of the Porsche–Volkswagen case

Example 1. Let us consider the corporate structure presented in Fig. 1 with majority quota *q* = 80%. The players are: Po-Fam (1), Qatar (2), Lower Sax (3), Po-SE (4), VW--AG (5), Po-AG (6), and Others (7). In this case, we have seven players, where four of which (1, 2, 3, 7) are investors and three, (4, 5, 6), are companies. This example refers to the Porsche–Volkswagen case, which was considered by Karos and Peters [13]. Here, we modify a little the corporate structure, in order to be able to compare all the models considered. This means that we aggregate the 12.3% of voting rights of all the undefined shareholders to player 7 (Others), in accordance with the Volkswagen shareholder structure [21]. For a full description of this real case, see [13] and [21].

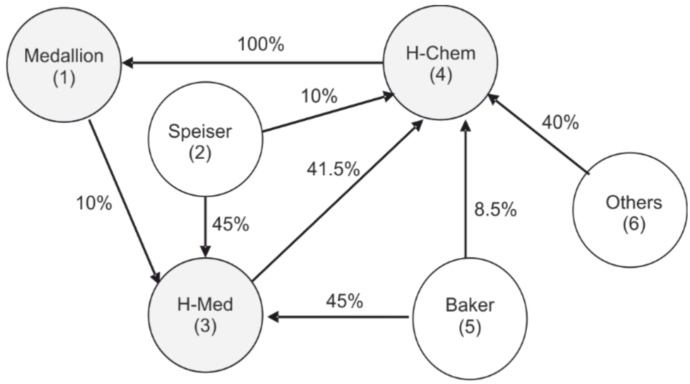

Fig. 2. The shareholding structure of the Speiser-Baker case;
H-Med – HealthMed, H-Chem – HealthChem

Example 2. This example deals with the Speiser and Baker case [15]. In this case, we also slightly simplify the corporate structure (Fig. 2). In this network, there are six players, where four of which are investors. Namely, there are the following players: Medallion (1), Speiser (2), HealthMed (3), HealthChem (4), Baker (5), Others (6). For this case, we consider a simple majority quota $q = 50\%$.

4. Approaches for measuring indirect control

In the literature on this topic, some general game-theoretic models to describe control relationships in corporate structures have been developed. In the following subsections, we limit ourselves to only consider those proposed by the following: Gambarelli and Owen, Denti and Prati, Crama and Leruth, Karos and Peters, as well as Mercik and Lobos. For each of these approaches, we provide only a brief overview, necessary to perform the calculation of the power indices considered by these models. Then, we try to compare these models taking into account the two illustrative examples. In describing these models, we try to keep the original notation, but sometimes, for clarity, we make some minor changes.

4.1. The Gambarelli and Owen model

Gambarelli and Owen [10] were first to define a power measure in the case of indirect control of corporations. They developed a mathematical model to determine the control (direct or indirect) that coalitions of investors have in the firms within a closed shareholding system. In particular, these two authors devised a very refined process capable of transforming a set of various linked majority games into a single game. This method is based on the concept of a multilinear extension, introduced by Owen [16]. One of the advantages of this methodology is that it can be used with any power index. In fact, this method constructs a resultant game, in which the index considered most suitable to describe the situation under consideration can be applied. The Gambarelli and Owen model enables us to solve all concatenated games without cycles and a class of cyclic games. The model recognizes other games as being unstable [2, 10]. In cases of instability, the Gambarelli and Owen model can only be interpreted via the intervention of exogenous factors to provide a statistical value; for example: the mean power index of the players involved in the cycle.

Let us introduce some necessary concepts. Let N be the set of all companies and M be the set of all investors in a shareholding system. Direct control in such a stock system can be described by a formal game system (f.g.s.), i.e. a n-tuple $[W_1, ..., W_n]$, of simple monotonic games over the set $N \cup M$. Let W_j denote the weighted majority game played in company j and in accordance with the notation in Section 2, it can be described by the set of all winning coalitions of shareholders in firm j, i.e. coalitions which hold the required majority of voting rights in j. In order to consider indirect control, Gambarelli and Owen proposed a reduction operation. A reduction for firms N and investors M is an n-tuple $(V_1, ..., V_n)$ of voting games over a set of investors M. In cases without loops, a recursive procedure to find the so-called effective reduction $(V_1, ..., V_n)$ starting from the original f.g.s. is provided [10]. In the cyclic case, such a procedure would repeat itself forever. Therefore, Gambarelli and Owen introduced a more general concept, namely a so-called consistent reduction which, however, is not necessarily unique. In a shareholding structure without loops, a consistent reduction coincides with the effective reduction. For this reason, hereafter we will only use the name "consistent reduction". Gambarelli and Owen provided a technique to obtain a consistent reduction. This technique relies on the idea of multilinear extensions. For details see [10].

Now, let us apply the Gambarelli and Owen approach to Example 1. In this example, {4, 5, 6} is the set of firms, and {1, 2, 3, 7} – the set of individual investors. Considering direct control, we have the formal game system: W_4 = {{1} and supersets}, W_5 = {{2, 3, 4}, {3, 4, 7} and supersets}, W_6 = {{5} and supersets}. Note that player 1 has control over player 4. Thus, concerning indirect control, we can see that investors 1, 2 and 3 jointly have indirect control over all the firms, 4, 5, and 6. Also, coalition {1, 3, 7} has control over all the companies. Considering multilinear extensions, we obtain the same result and the so-called, in the Gambarelli–Owen terminology, consistent reduction. Namely, the multilinear extensions for firms 4, 5, and 6 are:

$$MLE_4 = x_1(1-x_2) + x_1x_2 = x_1$$

$$MLE_5 = x_2x_3x_4(1-x_7) + x_3x_4x_7(1-x_2) + x_2x_3x_4x_7 = x_4(x_2x_3 + x_3x_7 - x_2x_3x_7)$$

$$MLE_6 = x_5$$

Solving this system of 3 equations with 7 variables by substituting $MLE_j = x_j$ (j = 4, 5, 6), we obtain: $x_4 = x_1$, $x_5 = x_6 = x_3x_4(x_2 + x_7 - x_2x_7)$ and the reduced extension: $RE_4 = x_1$, $RE_5 = RE_6 = = x_1x_3(x_2 + x_7 - x_2x_7)$. Since $x_1x_3(x_2 + x_7 - x_2x_7) = 1$ when $x_1 = x_2 = x_3 = 1$ or $x_1 = x_3 = x_7 = 1$ or $x_1 = x_2 = x_3 = x_7 = 1$, then we obtain the following consistent reduction: V_4 = {{1}}, $V_5 = V_6$ = {{1, 2, 3}, {1, 3, 7}, {{1, 2, 3, 7}}.

Now, for any firm j (j = 4, 5, 6), we can calculate the power of the investors using the three power indices considered in Section 2. Therefore, regarding company 4, we see that player 1 has total control over this player and the other investors have power measures equal to zero. The power of investors is presented in Table 1.

Table 1. Power of investors in companies 4, 5, and 6 in Example 1

Power index	Company 4	Companies 5 and 6			
	Player 1	Player 1	Player 2	Player 3	Player 7
Shapley–Shubik index	1	0.4166	0.0833	0.4166	0.0833
Absolute Banzhaf index		0.3750	0.1250	0.3750	0.1250
Johnston index		0.3889	0.1111	0.3889	0.1111

Player 3 (Lower Saxony) with only 20% of the voting rights in company 5 has the same power as player 1 (Porsche Families) in both companies 5 and 6. This result does not depend on the power index applied. It is sufficient that the index satisfies the symmetry condition (Table 1).

Let us see how the Gambarelli and Owen approach works in a stock system with loops. Namely, let us consider Example 2 (the Baker-Speiser case). Firstly, we calculate the multilinear extensions for players 1, 3, and 4:

$$MLE_1 = x_4, \quad MLE_3 = x_1x_2(1-x_5) + x_1x_5(1-x_2) + x_2x_5(1-x_1) + x_1x_2x_5$$

$$MLE_4 = x_2x_3(1-x_5)(1-x_6) + x_2x_{35}(1-x_6) + x_2x_3x_6(1-x_5) + x_2x_5x_6(1-x_3) + x_3x_6(1-x_2)(1-x_5) + x_3x_5x_6(1-x_2) + x_2x_3x_5x_6$$

After some algebraic steps, we obtain the following functions:

$$MLE_1 = x_4$$

$$MLE_3 = x_1x_2 + x_1x_5 + x_2x_5 - 2x_1x_2x_5$$

$$MLE_4 = x_2x_3 + x_3x_6 + x_2x_5x_6 - x_2x_3x_6 - x_2x_3x_5x_6$$

Now we solve this system of 3 equations with 6 variables by substituting $MLE_j = x_j$ (j = 1, 3, 4). This structure has loops. To solve the problem of cycles, Gambarelli and Owen proposed calculating a reduced multilinear extension (*RMLE*) for each company, thus we set $x_j = RMLE_j$ and solve the resulting system of equations for these variables. Hence, we obtain the following reduced extensions:

$$RE_1 = RE_4 = \frac{x_2x_5(x_2 + 2x_6 - x_2x_6 - x_2x_5x_6)}{X_1 + X_2}$$

$$RE_3 = \frac{x_2x_5(1 + x_2x_6 + x_5x_6 - 2x_2x_5x_6)}{X_1 + X_2}$$

where

$$X_1 = 1 - x_2^2 - x_2x_5 + 2x_2^2x_5 - x_2x_6 - x_5x_6 + 3x_2x_5x_6$$

$$X_2 = x_2^2x_6 - 2x_2^2x_5x_6 + x_2^2x_5x_6 + x_2x_5^2x_6 - 2x_2^2x_5^2x_6$$

As $x_j^2 = x_j$ for $x_j \in \{0, 1\}$, these functions can be reduced by lowering the exponents of the variables x_j (here, j = 2, 5, 6) to the first degree, i.e.:

$$RE_1 = RE_4 = \frac{x_2x_5(x_2 + 2x_6 - x_2x_6 - x_2x_5x_6)}{1 - x_2 + x_2x_5 - x_5x_6 + x_2x_5x_6}$$

$$RE_3 = \frac{x_2x_5(1 + x_2x_6 + x_5x_6 - 2x_2x_5x_6)}{1 - x_2 + x_2x_5 - x_5x_6 + x_2x_5x_6}$$

It is now necessary to evaluate the function RE_j. Because we are trying the find the set of winning coalitions, it is interesting to know when this function takes the value 1.

Considering the values of the numerators and denominators of RE_j, we have the following cases (Table 2).

Table 2. Value of RE_j (j = 1, 3, 4) for $x_i \in \{0, 1\}$, i = 2, 5, 6,
i.e. at the vertices of the unit hypercube

Case	x_2	x_5	x_6	Numerator of RE_j	Denominator of RE_j	RE_j
1	0	0	0	0	1	0
2	0	0	1	0	1	0
3	0	1	0	0	1	0
4	0	1	1	0	0	0/0
5	1	0	0	0	0	0/0
6	1	1	0	1	1	1
7	1	0	1	0	0	0/0
8	1	1	1	1	1	1

Source: Authors' calculations.

1. If the numerator of RE_j equals 1, then the denominator also equals 1, and in consequence the quotient equals 1 and the set of all winning coalitions required to control the three companies is composed of {2, 5} and {2, 5, 6}. Thus, investors 2 and 5 together have full control over all three companies.

2. If the numerator of RE_j equals 0 then

2.1. The denominator equals 0 for coalitions {2}, {2, 6}, and {5, 6} and the quotient becomes the indeterminate form 0/0, which means that hidden solutions are possible. Investors 5 and 6 alone do not have control. Investor 2 may or may not have control. The situation is unclear. On the one hand, if player 2 could convince the management of any of the companies 1, 3 or 4 to cooperate with him, this would give him full control over all the companies. On the other hand, if investor 5 could get firm 3's management to oppose investor 2, then 5 could keep player 2 indefinitely from getting full control over all the companies. From the theoretical point of view, a coalition between investors 5 and 6 would get company 3's management to cooperate with them. This would give coalition {5, 6} full control over all the companies. Another possibility is that the coalition of investors {2, 6} could act jointly and convince company 3's management to cooperate with them and thus they would obtain full control over all the companies.

2.2. The denominator equals 1 for other coalitions, so the ratio is 0. These coalitions are therefore losing.

Thus we see that in the presence of loops the calculation becomes more complicated and consistent reduction may not be unique. We find that the three simple games in the consistent reduction must all be equivalent and contain the following winning coalitions: {2, 5}, {2, 5, 6}. In addition to this, they might contain one, two, all, or none of the three following coalitions: {2}, {2, 6}, {5, 6}. Hence, there are eight possible consistent reductions. But some of these consistent reductions are improper, e.g. those with the two coalitions {2}, {5, 6} as in any consistent reduction with these coalitions, the coalition {2, 5, 6} is also present. However, a much more serious problem is the occurrence of conflicting consistent reductions, as in this example. More precisely, a consistent reduction containing {2} but not {5, 6}, and another one with {5, 6} but not {2} are conflicting. If investor 2 can put his creatures in control of the three companies, he will be able to keep control indefinitely. If {5, 6} can, acting jointly, put their creatures in control, they can effectively shut investor 2 out. Hence, the result seems to hinge on which investor(s) manage to move first.

4.2. The Denti and Prati model

To the authors' knowledge, the first publication about the Denti and Prati model dates back to 1996 [7]. This model was developed and improved in [8, 9]. Denti and Prati, compared to the Gambarelli and Owen approach, extend the set of winning coalitions to all alliances able to achieve control of the "target" firm. Namely, such a model is not limited to coalitions of investors alone, but also considers coalitions formed by companies, and by companies and investors together. In [8], they proposed an algorithm to check whether a preset coalition of firms, in a corporate shareholding structure either with or without loops, is winning or not. Then, they extended their approach and assumed that shareholders can abstain or oppose others [9]. Therefore, not all the winning coalitions have the same relevance, i.e. controlling power. Hence, Denti and Prati [9] proposed three algorithms to determine the winning coalitions of various levels of relevance. More precisely, they proposed algorithms to calculate so-called: potentially winning coalitions, potentially stably winning coalitions and stably winning coalitions. For details, see [8, 9]. These algorithms have exponential computational complexity. Thus computational problems can arise if the number of firms is large.

Based on Denti and Prati's algorithm [8], a computer program was implemented, which enables to perform simulations [14]. More precisely, it is possible to find all the minimal winning coalitions which control a preset coalition of firms, or check whether a certain coalition is able to control a preset coalition of firms. So regarding Example 1, we find that there are two minimal winning coalitions ({1, 2, 3} and {1, 3, 7}) that control companies 4, 5, and 6. On the other hand, player 6 can be controlled directly or indirectly by five minimal winning coalitions: {5}, {1, 2, 3}, {1, 3, 7}, {2, 3, 4}, {3, 4, 7}. Of course, by considering only coalitions of investors, the results obtained by the Denti and Prati algorithm coincide with the results obtained by the Gambarelli and Owen approach.

Considering Example 2, player 2 and 5 jointly control companies 1, 3, and 4. This is a unique such minimal winning coalition. In addition, there are five minimal winning coalitions that control company 3: {1, 2}, {1, 5}, {2, 4}, {2, 5}, {4, 5}, and also five that have control over player 4: {1, 2}, {2, 3}, {2, 5}, {3, 6}, {1, 5, 6}. Thanks to the computer program, we can provide, of course, more simulations.

Denti and Prati, as mentioned before, focus only on the determination of winning coalitions in a corporate network, without considering power indices. But once the winning coalitions have been obtained, it is possible to measure the control power by power indices.

4.3. The Crama and Leruth approach

Crama and Leruth [4, 5] focus on the use of power indices to model control relationships in corporate structures. More precisely, they focus on an algorithmic approach to estimating the Banzhaf index (Section 2) in corporate networks, both with and without the presence of loops. It should be added that they also took into consideration in their algorithmic approach some difficult issues that exist in complex corporate structures, like modelling the set of small shareholders, etc. In large corporate networks, where there are a lot of firms involved, they proposed a Monte-Carlo approach to compute control power. They model a corporate structure by a network, i.e. direct graph. A precise graph-theoretic model is given in [4] and an intuitive description is provided in [5]. Therefore, here we only provide the necessary details.

Let V be the set of all firms involved in a corporate structure. Each $j \in V$ is associated with a 0–1 variable x_j. If $x_j = 1$, then firm j votes "yes" and $x_j = 0$ means that firm i votes "no". Let N be a set of n investors. Thus, in order to measure the control power of investor j in target firm t by power indices, they define indirect games v_j. However, it should be noted that in order to define an indirect game, they used an equivalent definition of a simple game. Namely, for the set of players N, they modelled a simple game v as a Boolean function $f_v : \{0, 1\}^n \to \{0, 1\}$, where the value of the function reflects the outcome of the vote for each vector of individual votes. More precisely, for all $X = (x_1, x_2, ..., x_n) \in \{0, 1\}^n$, if $v(\{i:\ x_i = 1\}) = 1$, then $f_v(X) = 1$, otherwise $f_v(X) = 0$. The direct game g_j for firm $j \in V \setminus N$ is the weighted majority game with the player-set composed of all direct shareholders of firm j. Now, for any acyclic network and any firm $j \in V$, the indirect game v_j is defined as the composition of the direct weighted majority games (g) associated with the direct shareholders of firm j. In particular, for all $X = (x_1, x_2, ..., x_n) \in \{0, 1\}^n$ and $j \in V$, the corresponding indirect game is defined recursively as follows: if $j \in N$, then $v_j(X) = x_j$, otherwise $v_j(X) = g_j(v_{i_1}(X),\ v_{i_2}(X), ..., v_{i_k}(X))$, where $i_1, i_2, ..., i_k$ denote the direct shareholders of firm $j \in V \setminus N$. For details see [4].

The Z index, proposed by Crama and Leruth [4] to measure the amount of a priori voting power held by a firm $j \in N$ in target company t, is given as follows:

$$Z_t(j) = \frac{1}{2^{n-1}} \left(\sum_{X \in \{0,1\}^n : x_j = 1} v_t(X) - \sum_{X \in \{0,1\}^n : x_j = 0} v_t(X) \right) \tag{1}$$

$Z_t(j)$ is simply the Banzhaf index of firm j in the indirect game v_t associated with firm t.

Let us apply the Crama and Leruth approach to Example 1 (the Porsche –Volkswagen case). In this example, there are 4 investors or sources in the terminology of the Crama and Leruth approach: players 1, 2, 3 and 7. Thus $N = \{1, 2, 3, 7\}$ and $n = 4$. The majority quota q equals 80%. There are $2^4 = 16$ different patterns (x_1, x_2, x_3, x_7) of possible votes of investors (Table 3). Calculating index Z, in this example, is rather simple, as in the games corresponding to companies 4 and 6, there are dictator players. For firm 4, player 1 is a dictator. Thus, the result of the vote in game v_4 depends only on player 1, so $v_4 = x_1$. For firm 5, we have three winning and vulnerable coalitions: {2, 3, 4}, {3, 4, 7}, and {2, 3, 4, 7}. Considering only direct control in company 5, we see that players 3 and 4 are indispensable to passing any decision. In game v_6, we also have a dictator, player 5, so $v_6 = v_5$. Thus, players 2 and 7 have the same control power, as these players are symmetric in game v_5 and so also in v_6.

Table 3. Calculation of v_i (i = 4, 5, 6) based on the voting pattern

Voting pattern				Game		
x_1	x_2	x_3	x_7	v_4	v_5	v_6
0	0	0	0	0	0	0
1	0	0	0	1	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
1	1	0	0	1	0	0
1	0	1	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	0	0	0
0	1	0	1	0	0	0
0	0	1	1	0	0	0
1	1	1	0	1	1	1
1	1	0	1	1	0	0
1	0	1	1	1	1	1
0	1	1	1	0	0	0
1	1	1	1	1	1	1

Source: Authors' calculations.

Hence, based on the results in Table 3, we obtain: $Z_4(1)=1$, $Z_4(j)=0$, for j = 2, 3, 7, and $Z_i(1)=Z_i(3)=3/8$, $Z_i(2)=Z_i(7)=1/8$, for i = 5, 6. The same power indices were obtained for these investors based on calculating β according to the Gambarelli and Owen approach (Table 1).

For a shareholding structure with loops, the index (1) is not well defined and, for this reason, Crama and Leruth [4] proposed a heuristic approach to calculating the influence of a firm-investor in a company. More precisely, in the presence of cycles, they proposed an iterative procedure called MIX, in order to attempt to find a stable voting pattern and estimate the value of the game when the outcome of the game g_i $(j \in V \setminus N)$ is not perfectly defined. A voting pattern $X=(x_1, x_2, ..., x_n) \in \{0,1\}^n$ is stable if $x_j = g_j(X)$ for each $j \in V$. The concept of a stable pattern is closely related to the concept of a "consistent reduction" as introduced by Gambarelli and Owen [10].

Let us see how this method works in Example 2 (the Speiser and Baker case), where a loop of shareholding companies exists. Namely, let us calculate $Z_i(j)$, the influence of investor j (j = 2, 5, 6) on firm i (i = 1, 3, 4) in game $g_i(x_2, x_5, x_6)$, using the Banzhaf index. In this example, there are eight possible voting patterns for the combination of players 2, 5, and 6, see Table 6. We have to find $g_i(x_2, x_5, x_6)$ for each i = 1, 3, 4. For some voting patterns (x_2, x_5, x_6), the values of $g_i(x_2, x_5, x_6)$ for each i = 1, 3, 4 are perfectly defined, as in cases 1, 3, 4, 5, and 8 in Table 6. Let us consider case 5 when (x_2, x_5, x_6) = (1, 1, 0), then necessarily g_3 = 1 (2 and 5 form a winning coalition in g_3, since the combined voting rights of 2 and 5 in company 3 are greater than 50%). Also, g_4 = 1 and g_1 = 1. Hence, in this case, $g_i(x_2, x_5, x_6) = g_i(1, 1, 0)$ is perfectly determined for each i = 1, 3, 4 and this reasoning is valid independently of the initial votes of firms 1, 3, 4 and thus does not require the consideration of stable patterns. But in the other cases (2, 6, or 7 in Table 6), we cannot immediately deduce the vote of firms (1, 3, 4), and hence we must resort to the MIX procedure. The problem is that g_i is not uniquely defined when there is a cycle. So the underlying idea is to replace $g_i(x_2, x_5, x_6)$ by its expected value, assuming that all the firms whose votes are not entirely determined by (x_2, x_5, x_6) initially vote randomly and that they keep updating their votes until a stable pattern emerges. Since convergence to a stable pattern is not guaranteed, we estimate (by simulation) the expected value of $g_i(x_2, x_5, x_6)$ over all the values that it can take. More precisely, if, for example, (x_2, x_5, x_6) = (0, 1, 1), and the initial pattern of firms 1, 3, 4 is (x_1, x_3, x_4) = (1, 1, 0), then, using the MIX procedure, firms 1, 3, 4 will change their votes to (0, 1, 1), then to (1, 0, 1), then back to the initial state (1, 1, 0). As a stable pattern does not exist in this case, the value of $g_3(0, 1, 1)$ is taken to be 2/3, meaning that x_3 = 1 in 2/3 of the states of the cycle. However, we still have to take into account all the other possible initial votes for (x_1, x_3, x_4), and average over all these initial votes, which in this case gives 4/8 (Tables 4 and 5).

In Table 4, we present the results of the MIX procedure for cases when the votes of firms 2, 5, 6 are $(x_2, x_5, x_6) = (1, 0, 0), (1, 0, 1)$, or $(0, 1, 1)$. The outcomes $g_i(x_2, x_5, x_6)$ obtained for any initial pattern (x_1, x_3, x_4) and $i = 1, 3, 4$ are shown in Table 5, and subsequently the average of $g_i(x_2, x_5, x_6)$ is calculated over all initial patterns for each i, $i = 1, 3, 4$. This average is defined to be the outcome of $g_\mathrm{i}(x_2, x_5, x_6)$, $i = 1, 3, 4$, for a fixed voting pattern (x_2, x_5, x_6) (Table 6).

Table 4. The MIX procedure for $g_i(x_2, x_5, x_6)$, $i = 1, 2, 3$, $(x_2, x_5, x_6) = (1, 0, 0), (1, 0, 1), (0, 1, 1)$, and initial voting patterns: (0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 1, 1) for firms 1, 3, 4

Step k	Voting pattern			Voting pattern			Voting pattern			Voting pattern		
	x_1^k	x_3^k	x_4^k	x_1^k	x_3^k	x_4^k	x_1^k	x_3^k	x_4^k	x_1^k	x_3^k	x_4^k
0	0	0	0	1	0	0	1	1	0	1	1	1
1	0	0	0	0	1	0	0	1	1	1	1	1
2				0	0	1	1	0	1			
3				1	0	0	1	1	0			
$g_i(x_2, x_5, x_6)$	0	0	0	1/3	1/3	1/3	2/3	2/3	2/3	1	1	1
Stable pattern (ST)	ST exists			ST does not exist. The same result occurs for initial patterns:						ST exists		
				(0, 1, 0) and (0, 0, 1)			(1, 0, 1) and (0, 1, 1)					

Source: Authors' calculations.

Table 5. Calculation of the expected values of $g_i(x_2, x_5, x_6)$ for $i = 1, 3, 4$ and $(x_2, x_5, x_6) = (1, 0, 0), (1, 0, 1), (0, 1, 1)$

Initial pattern			Game			Summary of MIX procedure
x_1	x_3	x_4	$g_1(x_2, x_5, x_6)$	$g_3(x_2, x_5, x_6)$	$g_4(x_2, x_5, x_6)$	
0	0	0	0	0	0	Stable pattern exists.
1	0	0	1/3	1/3	1/3	Stable pattern does not exist.
0	1	0	1/3	1/3	1/3	
0	0	1	1/3	1/3	1/3	
1	1	0	2/3	2/3	2/3	
1	0	1	2/3	2/3	2/3	
0	1	1	2/3	2/3	2/3	
1	1	1	1	1	1	Stable pattern exists.
Total			4	4	4	
Average			4/8	4/8	4/8	

Source: Authors' calculations.

The values of the game $g_i(x_2, x_5, x_6)$, $i = 1, 3, 4$ for all possible voting patterns of players 2, 5, and 6 are summarized in Table 6. We can calculate the Banzhaf indices

($Z_i(j)$) of players j = 2, 5, 6 in game i = 1, 3, 4. Thus: $Z_i(2) = 5/8$, $Z_i(5) = 3/8$, $Z_i(6) = 1/8$, i = 1, 3, 4.

Table 6. Values of g_i (i = 1, 3, 4) for all possible voting patterns of players 2, 5, and 6

Case	Player vote			Game $g_i(x_2, x_5, x_6)$			Summary
	x_2	x_5	x_6	g_1	g_3	g_4	
1	0	0	0	0	0	0	$g_i(0, 0, 0) = 0$ is perfectly determined for all i.
2	1	0	0	1/2	1/2	1/2	We cannot immediately deduce how firms (1, 3, 4) vote, and must resort to the MIX procedure.
3	0	1	0	0	0	0	$g_i(0, 1, 0) = 0$ is perfectly determined for all i.
4	0	0	1	0	0	0	$g_i(0, 0, 1) = 0$ is perfectly determined for all i.
5	1	1	0	1	1	1	$g_i(1, 1, 0) = 1$ is perfectly determined for all i.
6	1	0	1	1/2	1/2	1/2	We cannot immediately deduce how firms (1, 3, 4) vote, and must resort to the MIX procedure.
7	0	1	1	1/2	1/2	1/2	We cannot immediately deduce how firms (1, 3, 4) vote, and must resort to the MIX procedure.
8	1	1	1	1	1	1	$g_i(1, 1, 1) = 1$ is perfectly determined for all i.

Source: Authors' calculations.

Firms 2 and 3 both have 45% of the stocks in company 3 but the influence of investor 2 (Speiser) in company 3, as measured by the Z index, is greater than the power of investor 5 (Baker). Player 6 has many more stock rights in company 4 than players 2 and 5 but the power of player 6 in this company is much lower than the power of the others.

In Example 2, player 6 (others) is an undefined set of small shareholders. As we mentioned at the beginning of this Section, Crama and Leruth [4] proposed an algorithmic method to take such a set into account. Thus the power of such a set could be calculated in a similar way (cf. also [5]).

4.4. The Karos and Peters approach

Karos and Peters [13] model relations of indirect control in a shareholding structure in two equivalent ways: by the so called invariant mutual control structure (a map which assigns the set of controlled players to each coalition), and by a simple game structure where each simple game indicates who controls the corresponding player. Hence, they propose a large class of indices, based on the concept of dividends that satisfy four axioms and can measure the power of players in a shareholding network. By adding one more axiom, called the controlled player condition, they obtain a uniquely defined power index Φ. Everything is rigorously defined in [13]. Here, we only provide the details necessary to estimate the power of players, using the index Φ, in the two examples.

A mutual control structure represents direct control in a shareholding structure. Formally, a mutual control structure C is a function assigning to each nonempty coalition $T \subseteq N$ $(T \neq \varnothing)$ another coalition $C(T) = S$ such that each player of S is controlled by the coalition T and the following monotonicity condition holds: if T controls S, then any coalition containing T also controls S. In order to capture the idea of indirect control in a shareholding structure, a mutual control structure should be invariant, i.e. satisfy the condition of indirect control, which states that for all coalitions R, S, T, if T controls S, and S and T jointly control R, then T indirectly controls R. An invariant mutual control structure is denoted by C^*.

Equivalently, a mutual control structure can be characterized by a simple game structure, i.e. a vector of simple games. More precisely, given a mutual control structure C for any player i, the simple game v_i^C is defined as follows: $v_i^C(S) = 1$ if $i \in C(S)$ and $v_i^C(S) = 0$ otherwise, i.e. the winning coalitions of v_i^C are exactly those that control player i.

Let $N = \{1, 2, ..., n\}$ be a set of players, and C^* be the set of all invariant mutual control structures based on N. The power index $\Phi : C \to R^n$, which satisfies the following five axioms: null player, constant sum, anonymity, transfer, and controlled player, is given by the following formula:

$$\Phi_i(C) = \sum_{k \in N} \sigma_i(v_k^C) - v_i^C(N) \qquad \text{for each } i \in N$$

where σ is nothing else than the Shapley–Shubik index. Note that the range of the index is $\Phi \geq -1$. The minimum value (–1) is obtained by the least powerful players, i.e. players who do not control any firm but are controlled by at least one coalition. Moreover, for all investors the value of Φ is non-negative. Finally, the sum of this index over all the players is equal to 0.

Let us calculate the power of the players in Example 1 (i.e. the Porsche–Volkswagen case) applying the index Φ. In this case, the mutual control structure C is defined as follows: for any coalition $S \subseteq N = \{1, ..., 7\}$, we have:

$$4 \in C(S) \Leftrightarrow 1 \in S, \quad 5 \in C(S) \Leftrightarrow \{2, 3, 4\} \subseteq S \text{ or } \{3, 4, 7\} \subseteq S, \text{ and } 6 \in C(S) \Leftrightarrow 5 \in S$$

Now, let us consider indirect control. Applying the updating procedure to C [13], we obtain the invariant mutual control structure C^* as follows: $4 \in C^*(S) \Leftrightarrow 1 \in S$,

$$5 \in C^*(S) \Leftrightarrow \{2, 3, 4\} \subseteq S \text{ or } \{1, 2, 3\} \subseteq S \text{ or } \{3, 4, 7\} \subseteq S \text{ or } \{1, 3, 7\} \subseteq S$$

$$6 \in C^*(S) \Leftrightarrow 5 \in S \text{ or } \{2, 3, 4\} \subseteq S \text{ or } \{1, 2, 3\} \subseteq S \text{ or } \{3, 4, 7\} \subseteq S \text{ or } \{1, 3, 7\} \subseteq S$$

Since players 1, 2, 3, and 7 are not controlled by any coalition, for i = 1, 2, 3, 7 and any coalition $S \subseteq N$ 7/60 we have $v_i^{C^*}(S) = 0$. For players i = 4, 5, 6, the simple games $v_i^{C^*}$ are defined by the sets of minimal winning coalitions W_i^m, where W_4^m = {{1}}, W_5^m = {{1, 2, 3}, {1, 3, 7}, {2, 3, 4}, {3, 4, 7}}, W_6^m = {{5}, {1, 2, 3}, {1, 3, 7}, {2, 3, 4}, {3, 4, 7}}. In Table 7, we present the values of the Shapley–Shubik power index (cf. Section 2) calculated for each player in the simple game $v_i^{C^*}$, i = 1, 2, ..., 7.

Table 7. The σ index calculated for each player and a simple game defined for Example 1

Simple game $v_i^{C^*}$	Player						
	1	2	3	4	5	6	7
i = 4	1	0	0	0	0	0	0
i = 5	7/60	7/60	32/60		0	0	7/60
i = 6	3/60	3/60	10/60	3/60	38/60	0	
i = 1, 2, 3, 7	0	0	0	0	0	0	0

Source: Authors' calculations.

Now, we calculate the index Φ taking into account the results from Table 7 and thus obtaining $\Phi_1(C^*) =$ (60 + 7 + 3)/60 = 70/60, $\Phi_2(C^*) = \Phi_7(C^*) = 1060$, $\Phi_3(C^*)$ = 42/60, $\Phi_4(C^*) = 10/60 - 1 = -50/60$, $\Phi_5(C^*) = 38/60 - 1 = -22/60$, and $\Phi_6(C^*) = -1$.

Note that Karos and Peters excluded player 7 (others) in the Porsche–Volkswagen case. Others mean investors who hold less than 3% of the shares and are therefore not mentioned in any reports. Thus they assigned to others a power equal to zero. Here, it is different. Keeping player 7 results in a slight increase in the power of player 1 and, what is interesting, a significant difference in the powers of players 2 and 3. In [13], players 2 and 3 had equal power and here $\Phi_2(C^*) < \Phi_3(C^*)$ and the difference is significant.

Consider Example 2 (the Speiser–Baker case). Taking into consideration the Karos and Peters model, we can describe direct control by a mutual control structure C. For any coalition $S \subseteq N = \{1, ..., 6\}$, we have:

$$1 \in C(S) \Leftrightarrow 4 \in S, \quad 3 \in C(S) \Leftrightarrow \{1, 2\} \subseteq S \text{ or } \{1, 5\} \subseteq S \text{ or } \{2, 5\} \subseteq S$$

$$4 \in C(S) \Leftrightarrow \{2, 3\} \subseteq S \text{ or } \{3, 6\} \subseteq S \text{ or } \{2, 5, 6\} \subseteq S$$

Now, taking into account indirect relationships, we obtain the invariant mutual structure C^*:

$$1 \in C^*(S) \Leftrightarrow 4 \in S \text{ or } \{2, 3\} \subseteq S \text{ or } \{3, 6\} \subseteq S$$
$$\text{or } \{2, 5, 6\} \subseteq S \text{ or } \{1, 2\} \subseteq S \text{ or } \{1, 5, 6\} \subseteq S$$

$$3 \in C^*(S) \Leftrightarrow \{1, 2\} \subseteq S \text{ or } \{1, 5\} \subseteq S \text{ or } \{2, 5\} \subseteq S \text{ or } \{2, 4\} \subseteq S$$
$$\text{or } \{4, 5\} \subseteq S \text{ or } \{2, 3\} \subseteq S \text{ or } \{3, 5\} \subseteq S,$$

$$4 \in C^*(S) \Leftrightarrow \{2, 3\} \subseteq S \text{ or } \{3, 6\} \subseteq S \text{ or } \{2, 5, 6\} \subseteq S$$
$$\text{or } \{1, 2\} \subseteq S \text{ or } \{1, 5, 6\} \subseteq S \text{ or } \{2, 4\} \subseteq S \text{ or } \{4, 5, 6\} \subseteq S$$

Given C^*, for any $i \in N$ we define the following simple game: $v_i^{C^*}(S) = 1$ if $i \in C^*(S)$ and $v_i^{C^*}(S) = 0$ otherwise. Since players 2, 5, and 6 are not controlled by any coalition, for i = 2, 5, 6 and any coalition $S \subseteq N$ we have $v_i^{C^*}(S) = 0$. For players i = 1, 3, and 4, the simple games $v_i^{C^*}$ are defined by the following sets of minimal winning coalitions W_i^m : W_1^m = {{4}, {1, 2}, {2, 3}, {3, 6}, {2, 5, 6}, {1, 5, 6}}, W_3^m = {{1, 2}, {1, 5}, {2, 5}, {2, 4}, {4, 5}, {2, 3}, {3, 5}}, W_4^m = {{1, 2}, {2, 3}, {2, 4}, {3, 6}, {2, 5, 6}, {1, 5, 6}, {4, 5, 6}}. In Table 8, we present the values of the Shapley–Shubik power index calculated for each player in the simple games $v_i^{C^*}$, i = 1, ..., 6.

Table 8. The σ index calculated for each player and simple game defined for Example 2

Simple game $v_i^{C^*}$	Player					
	1	2	3	4	5	6
i = 1	5/60	10/60	8/60	28/60	2/60	7/60
i = 3	6/60	21/60	6/60	6/60	21/60	0
i = 4	5/60	22/60	10/60	5/60	14/60	4/60
i = 2, 5, 6	0	0	0	0	0	0

Source: Authors' calculations.

Now, we apply the index Φ to Example 2 with invariant mutual structure C^*. Taking into account the results from Table 8, we obtain $\Phi_1(C^*) = -44/60$, $\Phi_2(C^*) = 53/60$, $\Phi_3(C^*) = -36/60$, $\Phi_4(C^*) = -21/60$, $\Phi_5(C^*) = 37/60$, $\Phi_6(C^*) = 11/60$.

Players 2 and 5 both have 45% of the stocks of firm 3. Player 5 has only a 1.5% lower share of stocks in company 4 than player 2, but in the whole shareholder structure, the difference in power, calculated according to the Karos and Peters index, seems to be greater.

4.5. The Mercik and Lobos approach

Mercik and Lobos in [15] proposed a measure of reciprocal ownership, called the index of implicit power, as a modification of the Johnston power index [12]. Also, they focused on application of this index to measure indirect power in cyclic shareholder structures. The implicit power index takes into account not only the power of the individual entities constituting the companies (investors), but also the impact of the companies themselves on implicit relationships.

Mercik and Lobos suggested a three-step algorithm to calculate the implicit power index. They assumed that there should be at least two companies, i.e. stock corporations, in the corporate structure. Here, we only give a brief sketch of this algorithm, for a full description see [15]. Namely, in step 1, the absolute value of the Johnston index is calculated for each company, taking into account only direct ownership. In step 2, for each shareholder–company, each value of the power index calculated in step 1 must be divided equally among all its shareholders. They call this first degree regression. In step 3, for each company, the absolute value of the implicit power index is calculated by summing up the appropriate values in the whole corporate network. For each investor, the absolute value of the implicit power index is calculated by summing up the appropriate values across the entire system of companies. Then, these absolute values are appropriately standardized to obtain the implicit power index of each shareholder.

Table 9. Fractional critical defections and the value of the raw Johnston index

Vulnerable coalition	Investors in company						
	4		5				6
	1	2	2	3	4	7	5
{1}	1	0					
{1, 2}	1	0					
{5}							
{2, 3, 4}			1/3	1/3	1/3	0	1
{3, 4, 7}			0	1/3	1/3	1/3	
{2, 3, 4, 7}			0	1/2	1/2	0	
Raw Johnston index	2	0	1/3	7/6	7/6	1/3	

Source: Authors' calculations.

Let us calculate the implicit power index for Example 1 (the Porsche–Volkswagen case) following the three-step algorithm mentioned above. Table 9 illustrates the necessary calculations to realize step 1. More precisely, for any shareholder of a company, the raw Johnston index is calculated as the sum of the fractional critical defections over all the vulnerable coalitions in which the given shareholder is critical.

The last row of Table 9 gives the distributions of absolute power in companies 4, 5, and 6, taking into account only direct control by shareholders. But indirect control is also considered in the calculation of the implicit power index. Thus, in step 2, the power of investors is augmented by a fraction according to indirect control. For example, company 4 is a direct shareholder of company 5 with absolute power 7/6. Since investors 1 and 2 directly control company 4 and, consequently, indirectly control company 5, according to this approach, 7/6 units of power are shared equally between investors 1 and 2. Hence, in consequence, the indirect absolute power of investor 1 in company 5 is equal to 7/12, and the power of investor 2 in the same company is 11/12 . The results of all the necessary calculations to complete steps 2 and 3 are provided in Table 10. The Mercik and Lobos approach allows us to measure not only the influence of investors on companies but also the absolute and standardized power of all the firms involved in corporate network.

Table 10. Absolute and standardized values of the implicit power index in Example 1

Company	Investors (members of companies)				Implicit index of a company	
	1	2	3	7	Absolute	Standardized
4	2	0	0		2	$\frac{24}{69} \approx 0.348$
5	$\frac{7}{12}$	$\frac{11}{12}$	$\frac{7}{6}$	$\frac{1}{3}$	3	$\frac{36}{69} \approx 0.522$
6	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4} = 0.75$	$\frac{9}{69} \approx 0.130$
Absolute implicit index of an investor	$\frac{31}{12} \approx 2.583$	$\frac{14}{12} \approx 1.667$	$\frac{17}{12} \approx 1.417$	$\frac{71}{12} \approx 0.583$	$\frac{69}{12}$	
Standardized implicit index of an investor	$\frac{31}{69} \approx 0.449$	$\frac{14}{69} \approx 0.203$	$\frac{17}{69} \approx 0.246$	$\frac{7}{69} \approx 0.101$		1

Source: Authors' calculations.

Mercik and Lobos assessed the implicit power of investors and companies in the Speiser and Baker case (Example 2). Here, we only recall, in Table 11, the values of this index. For detailed calculations see [15].

Table 11. Values of the implicit power index in Example 2

Power index	Player (company)			Player (investor)		
	1	3	4	2	5	6
Standardized implicit index	0.088	0.324	0.588	0.451	0.333	0.216
Absolute implicit index	0.75	2.75	5	3.833	2.833	1.833

Source: [15].

5. Conclusions

In the context of game theory, various studies on indirect control have been made in corporate shareholding systems. In this paper, we have critically examined the models of Gambarelli and Owen [10], Denti and Prati [8, 9], Crama and Leruth [4, 5], Karos and Peters [13], and Mercik and Lobos [15], taking into account two examples of shareholding structure, one with cycles and one without. The reason that we chose these and not other models is that we wanted to compare the Gambarelli and Owen approach (one of the oldest pioneering methods) with those most recently presented in the literature. It should be noted that Crama and Leruth [4–6] presented the broadest approach to measuring indirect power/control in corporate networks. In particular, they took into account not only the presence of cyclic shareholding relationships but also the collection of small, unidentified shareholders called the float. Then, they also considered an aspect of computing indirect power in real-world financial networks. Also, their algorithmic approach allows us to efficiently deal with the complexity of computing power indices in corporate networks, regardless of their size.

Reviewing the literature on the topic, it can be said that most methods use two popular indices (Shapley–Shubik and Banzhaf) or their modifications to measure indirect control. An exception is the Mercik and Lobos [15] approach, which uses the Johnston index. A different approach was proposed by Karos and Peters [13], who do not start with a particular proposition of an index, but from axioms that determine a large class of indices. The Denti and Prati approach only considers the determination of winning coalitions.

There are some similarities between the considered approaches. The concept of a stable pattern in the method of Crama and Leruth is closely related to the concept of a consistent reduction in the method of Gambarelli and Owen. Also, the procedure of making a mutual control structure invariant, as defined by Karos and Peters, shows some resemblance to a reduction operation. As we saw in Example 2, a consistent reduction based on the Gambarelli–Owen approach is not necessarily unique (in contrast to the minimal invariant extensions of Karos and Peters), but a consistent reduction (a vector of voting games over a set of investors) based on the Gambarelli–Owen approach, as well as the simple games in a simple game structure according to the Karos and Peters approach, can be improper. According to the Crama–Leruth and Mercik–Lobos approaches, simple games are considered proper. By the way, simple games in a simple game structure based on the Karos–Peters model are called command games [11].

There can be many differences between these approaches, and they might even result from using different power indices; but above all, however, from the proposed game-theoretical structures describing indirect control. Tables 12 and 13 summarize the measures of power in Examples 1 and 2, respectively.

Table 12. Power of players in Example 1

Power index	Player (company)			Player (investor)			
	4	5	6	1	2	3	7
Shapley–Shubik	–	–	–	0.417	0.083	0.417	0.083
Absolute Banzhaf and Crama–Leruth				0.375	0.125	0.375	0.125
Johnston				0.389	0.111	0.389	0.111
Karos–Peters (Φ)	–0.833	–0.367	–1.000	1.167	0.167	0.700	0.167
Standardized implicit	0.348	0.522	0.130	0.449	0.203	0.246	0.101

Table 13. Power of players in Example 2

Power index	Player (company)			Player (investor)		
	1	3	4	2	5	6
Karos–Peters (Φ)	–0.733	–0.600	–0.350	0.883	0.617	0.183
Standardized implicit	0.088	0.324	0.588	0.451	0.333	0.216
Crama–Leruth (Z)	–	–	–	0.625	0.375	0.125

However, it is difficult to compare the results obtained in these two examples, and there may well be many reasons (even if the indices proposed have different ranges, for example). The Karos–Peters and Mercik–Lobos approaches take into account all of the firms involved in a corporate system in the calculation of a power index, and other methods (Gambarelli–Owen and Crama–Leruth) only consider investors. Subsequently, the Gambarelli–Owen and Crama–Leruth methods calculate the power of investors in a target company, while the Karos–Peters and Mercik–Lobos approaches consider the entire system. We hope that considering the calculation of indices of indirect control using different models in one paper is of value in itself. Of course, in our examples, we noticed that all of these models rank the investors (in terms of control power over companies) or companies (in terms of power in the whole system) in the same way. We suspect that these indices always rank the players in the same way but this is an open problem. Moreover, based on the methods of Gambarelli–Owen, Denti–Prati, and Crama–Leruth, players 1 and 3 are symmetric, as players 2 and 7 are. However, taking into account the whole system and the approach of Karos and Peters, as well as the Mercik–Lobos approach, we see that player 1 is more powerful than player 3, which intuitively seems to be correct. Regarding players 2 and 7, the index Φ confirms the previous statement regarding the symmetry of these players, while the implicit index gives more power to player 2 (which also seems to be the most-expected result).

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DIMENSION, EGALITARIANISM AND DECISIVENESS OF EUROPEAN VOTING SYSTEMS

An analysis of three major aspects has been carried out that may apply to any of the successive voting systems used for the European Union Council of Ministers, from the first one established in the Treaty of Rome in 1958 to the current one established in Lisbon. We mainly consider the voting systems designed for the enlarged European Union adopted in the Athens summit, held in April 2003 but this analysis can be applied to any other system. First, it is shown that the dimension of these voting systems does not, in general, reduce. Next, the egalitarian effects of superposing two or three weighted majority games (often by introducing additional consensus) are considered. Finally, the decisiveness of these voting systems is evaluated and compared.

Keywords: *voting systems, simple games, weighted majority games, Shapley–Shubik power index, dimension, egalitarianism, decisiveness*

1. Introduction

The successive enlargements undergone by the European Union raise many interesting questions concerning not only politics, but also the mechanisms used to make decisions. Cooperative game theory, and more particularly simple games, provide suitable tools to analyze some of them. Among the decision–making organisms of the Union, the Council of Ministers appears, each time, as one of the main battlefields in the design of the enlarged structure.

In this paper, we are interested in the study of three major aspects of the sophisticated voting rules that concern the Council of Ministers. These rules are defined

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by combining in each case two or three elementary mechanisms (weighted majority games), giving rise to much more complicated and restrictive ones. We will adopt here a normative viewpoint, so that no strategic behavior of the involved countries will be assumed.

First, we shall deal with the dimension of the simple games defining these voting rules and the possibility of simplifying them. Second, we will focus on a delicate point, particularly since the agents are countries and not individual people: the egalitarianism level of these rules, rather than the specific fraction of power they allocate in the Council to each of the countries that will form the future Union. Third, we will evaluate the (structural) decisiveness that the rules show as decision-making procedures, given that their structure suggests a strong component of inertia. In all cases, the effect of the imposed restrictions will be our main interest.

These three aspects, which are the object of analysis in this work, complement many others treated in several references, among them: Bertini et al. [1, 2], Chakravarty et al. [7], Freixas and Pons [14], Freixas et al. [18], Freixas and Gambarelli [16], Gambarelli [20, 21] or Owen [28].

The organization of this article is as follows. In Section 2, technical preliminaries concerning the three points of our research are given. Section 3 provides a summary of the voting rules adopted in Athens and the corresponding simple games. Section 4 is devoted to the study of the dimension. Section 5 refers to egalitarianism. Section 6 deals with decisiveness. Finally, the conclusions are given in Section 7.

2. Preliminaries

In this section, in order to make a self-contained article, we recall some basic definitions and properties of simple games, dimension, egalitarianism and decisiveness.

2.1. Simple games

Definition 2.1. A (monotonic) simple game is a pair (N, v) where $N = \{1, 2, ..., n\}$ is a finite set of players, every $S \subseteq N$ is a coalition, 2^N is the set of all coalitions, and $v : 2^N \to \{0, 1\}$ is a characteristic function which satisfies $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \le v(T)$ if $S \subset T \subseteq N$. A coalition S is winning if $v(S) = 1$ and losing otherwise. If W denotes the set of winning coalitions in v, then $\emptyset \notin W$, $N \in W$, and $T \in W$ whenever $S \subset T \subseteq N$ and $S \in W$. A coalition S is blocking if $N - S \notin W$. Wherever N is clearly fixed, we will simplify the notation and speak of the game v. For additional material on simple games, the reader is referred to Shapley [31], Carreras and Freixas [5], Taylor and Zwicker [36] and Carreras [3].

Definition 2.2. A simple game (N, v) is a weighted majority game (WMG, for short) if there are nonnegative weights $w_1, ..., w_n$ attached to the players and a positive quota $q \le w_N$ such that

$$v(S) = \begin{cases} 1 & \text{if} \quad w_S \ge q \\ 0 & \text{if} \quad w_S < q \end{cases}$$

where $w_S = \sum_{i \in S} w_i$ for every $S \subseteq N$. We then write $(N, v) = [q; w_1, ..., w_n]$. It is well known that only for $n \le 3$ every simple game is a WMG. In the sequel, we will always assume that $w_1 \ge \ldots \ge w_n$ and, in the case of having different weights $w_1 > ... > w_r$ repeated $k_1, ..., k_r$ times, respectively (so that $k_1 + ... + k_r = n$), we will often write $(N, v) = [q; w_1(k_1), ..., w_r(k_r)]$ for short. In particular, a *k*-out-of-*n* game is a special case of WMG: in fact, the expression k-out-of-n refers to the description of the game in which each one of the n players is given a weight of 1 and the quota is set at *k*, i.e. $(N, v) = [k; 1(n)]$.

Taylor and Zwicker [34] established that, among simple games, the WMGs are precisely those where winningness is "robust" with respect to general trades (appropriate changes of coalition members, see below).

Definition 2.3. A simple game (N, v) is *k*-trade robust for some positive integer *k* if there is no exchange of members among any collection of $j \le k$ winning coalitions $R_1, ..., R_j$ that leads to losing coalitions $T_1, ..., T_j$ in such a way that

$$|\{p : i \in R_p\}| = |\{p : i \in T_p\}| \quad \text{for each} \quad i \in N$$

A simple game is trade robust if it is *k*-trade robust for all k.

For instance, if $N = \{1, 2, 3, 4\}$ and v is the simple game in which the winning coalitions are: $R_1 = \{1, 2\}$ and $R_2 = \{3, 4\}$ plus those extended by monotonicity, it follows that both $T_1 = \{1, 3\}$ and $T_2 = \{2, 4\}$ are losing and can be obtained from R_1 and R_2 by swapping players 2 and 3 between them. Thus this game is not 2-trade robust and therefore it is not weighted.

Theorem 2.4 [34]. A simple game is a WMG if and only if it is trade robust.

2.2. Dimension

The following notion was introduced for graphs in the late 1970s. Its extension to hypergraphs (equivalent to simple but not necessarily monotonic games) is due to Jereslow [23]. Nevertheless, the definition of dimension for simple games is reminiscent of the definition of the dimension of a partially ordered set as the minimum number of linear orderings whose intersection is the given partial ordering [11].

Definition 2.5. The dimension of a simple game (N, v) is the least k for which there exist k WMGs $(N, v_1), ..., (N, v_k)$ such that $v = v_1 \cap ... \cap v_k$.

Theorem 2.6. Every simple game has a dimension which is bounded by the number of maximal losing coalitions of the game (see, for example, [33] or [35]) . □

The dimension of a simple game can be seen as a measure of its complexity. In the books by Taylor [33] and Taylor and Zwicker [35], the authors deal with dimension theory for simple games. In Freixas and Puente [15], the dimensions of several types of composite games are computed. Most real voting systems are described by simple games of dimension one or two: the United Nations Security Council is of dimension 1, and interesting examples of dimension 2 are the United States federal system (see, for instance, Taylor [33]) and the Victoria Proposal, the procedure to amend the Canadian Constitution (see [24, 33]).

2.3. Linear games and egalitarianism

Definition 2.7. The individual desirability relation D, introduced by Isbell [22] and generalized later on by Maschler and Peleg [26], is the partial preorder on the player set N defined, for each game v on N, by iDj in v if and only if $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for every $S \subseteq N - \{i, j\}$.

Definition 2.8. Games for which D is complete (or total, i.e. satisfying that for every $i, j \in N$ either iDj or jDi or both) have been given various names in the literature (ordered, complete). We will refer to them here as linear games. It is clear that every WMG is linear, but for any $n \geq 6$ there are linear games that are not WMGs. In the sequel, when considering linear games, we will always assume that $1D2$, $2D3$, ..., $(n-1)Dn$. For a characterization of any linear simple game in terms of numerical invariants, the reader is referred to [5].

Definition 2.9. A simple game v on N is a linear game with consensus if

$$v = u \cap [q; 1(n)]$$

where u is a simple game on N such that v becomes linear. This notion, introduced in [6], is slightly more general than the one considered by Peleg [29].

Notice that u is not required to be linear. If iDj in u, then iDj in v, but the converse is not true. From the fact that every WMG is linear, it follows that if u is a WMG then v is a linear game with consensus. Furthermore, if u is the intersection of two WMGs u^1 and u^2 and the weights, respectively, defining them satisfy $w_i^k \ge w_{i+1}^k$ for $i = 1, 2, ..., n-1$ and $k = 1, 2$ (as mentioned in Definition 2.2), then u is linear and v becomes a linear game with consensus. This is especially important for the voting systems we will study below.

Definition 2.10. The well-known Shapley–Shubik index of power, introduced in [32] (see also [30]), is the allocation rule that assigns to every simple game (N, v) the n-vector $\Phi[v] = (\Phi_1[v], ..., \Phi_n[v])$ defined by

$$\Phi_i[v] = \sum_{S \subseteq N: S \ni i} \gamma_n(s)[v(S) - v(S - \{i\})] \quad \text{for each} \quad i \in N$$

where $s = |S|$ and $\gamma_n(s) = \dfrac{(s-1)!(n-s)!}{n!}$. It is worth mentioning the axiomatic characterization of this allocation rule by means of the efficiency, symmetry, null player and transfer properties stated by Dubey [10]. In addition, Shapley and Shubik interpreted $\Phi_i[v]$ as the probability of player i being pivotal when all permutations of the players are equally likely (player $i = \pi_k$ is pivotal in permutation $\pi = (\pi_1, ..., \pi_n)$ for game v if $\{\pi_1, ..., \pi_k\}$ is winning in v but $\{\pi_1, ..., \pi_{k-1}\}$ is not).

Definition 2.11. An n-vector $x = (x_1, ..., x_n)$ Lorenz-dominates $y = (y_1, ..., y_n)$ if $\sum_{i=j}^{n} x_i \ge \sum_{i=j}^{n} y_i$ for $j = 1, ..., n$. In symbols, xLy.

Let u be a linear game on N, $v^1 = u \cap [q_1; 1(n)]$ and $v^2 = u \cap [q_2; 1(n)]$, with $1 \le q_1 < q_2 \le n$. Peleg [29] proved that $\Phi[v^2]\ L\ \Phi[v^1]$ (see also [39]). From the efficiency and Lorenz-domination, it follows that

$\Phi_1[v^1] \ge \Phi_1[v^2]$ and

$\Phi_n[v^2] \ge \Phi_n[v^1]$

(For any other player, i.e. $i \ne 1, n$, there are no generally valid inequalities like these; for details, see Proposition 3.1 in [6].) This can be interpreted as reflecting that, from the viewpoint of the Shapley–Shubik index, the game v^2 is more “egalitarian” than v^1, in the sense that

$$\Phi_1[v^1] - \Phi_n[v^1] \geq \Phi_1[v^2] - \Phi_n[v^2]$$

To cope with this idea, we introduce some notions.

Definition 2.12. Let (N, v) be a linear simple game.

- The range of (N, v) is the range of the set of numbers $\Phi_1[v], ..., \Phi_n[v]$, i.e.

$$\text{rang}[v] = \Phi_1[v] - \Phi_n[v]$$

- The egalitarianism of (N, v) is

$$\text{egal}[v] = 1/\text{rang}[v]$$

Notice that $1 \leq \text{egal}[v] \leq \infty$ for all v. In fact, $\text{egal}[v] = 1$ iff v is a dictatorship and $\text{egal}[v] = \infty$ iff v is a k-out-of-n game. We will be interested in studying the increase in egalitarianism when passing from a linear game with consensus v^1 to another linear game with a higher level of consensus v^2. The over-egalitarianism percentage, defined by

$$\text{oep}\ [v^1, v^2] = \frac{\text{egal}\ [v^2] - \text{egal}\ [v^1]}{\text{egal}\ [v^1]} \times 100$$

reflects this increase. The definition makes sense unless v^1 is a k-out-of-n game. In this case, v^2 would also be a k'-out-of-n game, and we could take $\text{oep}\,[v^1, v^2] = 0$ as a convention.

At this point, we recall from [6] an important result to be used below.

Theorem 2.13 [6]. Let $v^1 = u \cap [q_1; 1(n)]$ and $v^2 = u \cap [q_2; 1(n)]$ be linear games with consensus with $1 \leq q_1 < q_2 \leq n$. Then:

- $0 \leq \Phi_1[v^1] - \Phi_1[v^2] \leq \dfrac{q_2 - q_1}{n}$
- $0 \leq \Phi_n[v^2] - \Phi_n[v^1] \leq \dfrac{1}{n}$ □

2.4. Structural decisiveness

Let us finally refer to the notion of decisiveness, introduced in [4] (see also [9]). As real life experience shows, two main tendencies arise in the design of voting systems.

The first one tries to strengthen the agility of the mechanism in order to take decisions, and usually applies to national and regional parliaments, town councils, and many other committee systems. The second tendency is rather interested in protecting the rights of certain minorities, even at the cost of introducing a remarkable inertia into the mechanism, and is found particularly often in supranational organizations. It seems therefore interesting to measure, and of course to compare, the agility/inertia of such decision--making procedures, and the decisiveness index is intended to this end.

Definition 2.14 [4]. The (structural) decisiveness index is the map that assigns to every simple game (N, v) the number

$$\delta(N, v) = 2^{-n} \, |W|$$

The number $\delta(N, v)$, or simply $\delta[v]$, will be called the decisiveness degree of game (N, v).

If f is the multilinear extension of game v [27], then $\delta[v] = f(1/2, ..., 1/2)$. Thus, $\delta[v]$ merely gives the probability of a proposal being socially accepted by N under the acceptance rules stated by v when each agent votes independently of each other for the motion with probability 1/2. Nevertheless, it is precisely this formal approach, that does not take into account any strategic behavior by the players, which is the tool best suited to analyze voting systems from just a structural viewpoint.

The decisiveness index is a normalized measure, as $0 < \delta[v] < 1$ for any simple game v. More precisely, for a given N the minimum decisiveness degree is attained for the unanimity game u_N (where N is the only winning coalition) and is $\delta[u_N] = 2^{-n}$, whereas the maximum degree is attained for the individualistic game u_N^* (the dual game of u_N, where any $S \neq \varnothing$ is winning) and is $\delta[u_N^*] = 1 - 2^{-n}$. Notice that all so-called decisive games (that is, those where $S \in W$ iff $N - S \notin W$) show a decisiveness degree of 1/2. In general, all proper (i.e. superadditive simple) games have a degree lower than, or equal to, 1/2. The lower $\delta[v]$, the more difficult it is to take decisions in v. For the main properties of the decisiveness index, in particular referring to standard ways of combining simple games, several axiomatic characterizations, and an alternative computation procedure, we refer the interested reader to [4].

3. Provisions of the Accession Treaty on voting in the Council

As was pointed out in the first part of this article, the normative methodology proposed may be used for any binary voting system or simple game resulting from the intersection of at least two WMGs. Dimension and consensus then become interesting issues to be analyzed, while decisiveness applies to all simple games, no matter whether they decompose or not as the intersection of two or more WMGs.

As Taylor [33] noted in his book entitled *Mathematics and Politics*: *the interest of dimension lies in the fact that all known voting simple games in practice have small dimension: either one or two*. This observation makes dimension a very interesting notion, since a dimensionally efficient representation is a compact, intuitive and simple way to represent almost any real voting simple game.

Two voting systems of the European Union Council, that entered into effect on February 1st 2003 under the Nice rules, became the first known real-world examples of dimension 3 [13]. Other real-world examples with dimension 3 appeared later on. Indeed, Cheung and Ng [8] proved that the voting system in Hong Kong, which is not a complete simple game, also has dimension 3. Kurz and Napel [25] have proven that the Lisbon voting system of the Council of the European Union, which became effective in November 2014, cannot be represented as the intersection of six or fewer weighted games, i.e., its dimension is at least 7 and determination of the exact dimension has been posed as a challenge to the community. This sets a new record for real-world voting bodies.

The Athens treaty was signed on April 16th 2003 in Athens, Greece and came into force on May 1st 2004, the day of the enlargement of the European Union. It modified a significant number of points that originally dealt with the Treaty of Nice. This treaty, chronologically situated between the treaties of Nice and Lisbon, will be taken as the basis for the theoretical discussions that follow in this article.

The Athens Treaty amended the system of qualified majority voting to apply from 2004. We consider rules regarding two different scenarios for enlargement: the transitional period and the period from November 1st 2004. For each assumed scenario, a WMG is at the core of the system, but some additional conditions must also be met in terms of the number of countries supporting a proposal and, in some cases, their population.

Using the appropriate terminology from game theory, the players are: Germany, United Kingdom, France, Italy, Spain, Poland, The Netherlands, Greece, Czech Republic, Belgium, Hungary, Portugal, Sweden, Austria, Slovak Republic, Denmark, Finland, Ireland, Lithuania, Latvia, Slovenia, Estonia, Cyprus, Luxembourg, Malta, which we will represent by the set $\{1, 2, ..., 25\}$, where 1 stands for Germany, 2 stands for the United Kingdom, and so on.

3.1. Transitional period: May 1st 2004–October 31st 2004

We quote the relevant text from [37], article 26:

For their adoption, acts of the Council shall require at least:

• 88 votes in favour (of a total of 124 votes) where this Treaty requires to be adopted on a proposal from the Commission,

• 88 votes in favour (of a total of 124 votes), cast by at least two-thirds of the members, in other cases.

In the event that fewer than ten new Member States accede to the Union, the threshold for the qualified majority for the period until October 31st 2004 shall be fixed by Council decision so as to correspond as closely as possible to 71.26% of the total number of votes.

The first game

$$u_1 = [88; 10(4), 8(2), 5(6), 4(2), 3(8), 2(3)] \tag{1}$$

corresponds to a given vote distribution among countries and a majority of 70.97%, i.e. the threshold for the qualified majority is as close as possible to 71.26%. Let $v_1 = [13; 1(25)]$ and $v_2 = [17; 1(25)]$ be the games that correspond to a simple majority and a two-thirds majority of the members, respectively. Notice that $u_1 \cap v_1 = u_1$. The second game is

$$u_2 = u_1 \cap v_2 \tag{2}$$

3.2. From November 1st 2004

We quote the relevant text from [37], article 12:

Acts of the Council shall require for their adoption at least 232 votes in favour cast by a majority of the members where this Treaty requires them to be adopted on a proposal from the Commission.

In other cases, for their adoption acts of the Council shall require at least 232 votes in favour, cast by at least two-thirds of the members.

When a decision is to be adopted by the Council by a qualified majority, a member of the Council may request verification that the Member States constituting the qualified majority represent at least 62% of the total population of the Union. If that condition is shown not to have been met, the decision in question shall not be adopted.

In the event of fewer than ten new Member States acceding to the European Union, the threshold for the qualified majority shall be fixed by Council decision by applying a strictly linear, arithmetical interpolation, rounded up or down to the nearest vote, between 71% for a Council with 300 votes and the level of 72.27% for an European Union of 25 Member States.

The first game is

$$v_3 = [232; 29(4), 27(2), 13(1), 12(5), 10(2), 7(5), 4(5), 3(1)]$$

Let

$$v_4 = [620; 182, 132, 131, 128, 87, 86, 35, 23(3), 22(2), 20, 18, 12(2), 11, 8(2), 5, 4, 3, 2, 1(2)],$$

where weights are proportional to populations and a majority of 62% is demanded.

The voting systems to be used will correspond to

$$u_3 = v_3 \cap v_4 \cap v_1 \tag{3}$$

and

$$u_4 = v_3 \cap v_4 \cap v_2 \tag{4}$$

Systems (3) and (4) should therefore be thought of as requiring triple majorities: (a) weights must meet or exceed the threshold (a super-majority about 72.27% of the sum of voting weights); (b) a super-majority of 62% of the total EU population with the weights and quota being based on the appropriate rate per thousand; and (c) either a simple majority of the number of countries or a super–majority of 2/3 of this number.

In this model, we assume that the planned referenda will result in all the 10 candidate countries joiningthe European Union.

The two Athens rules from November 1st 2004 that we deal with here, u_3 and u_4, require the agreement of three sorts of majorities. Among other results, we shall prove that the noted complexity of both systems is irreducible, thus proving the existence of real voting systems of dimension three. As for the rule u_2, used in the transitional period, we will also prove that this system is irreducible.

4. On the dimension of the Council

In this section, we prove that the dimension of u_2 is two and the dimension of u_3 and u_4 is three and, therefore, none of these games can be described using fewer WMGs. Similar calculations have already been done in Freixas [13] for the initially foreseen enlargement to 27 members, agreed in December 2000 at the summit of Nice.

For variants, e.g., the notion of codimension and theoretical background on the notion of dimension, we refer the reader to Freixas and Marciniak [17].

4.1. The dimension for the transitional period

Theorem 4.1. The dimension of game u_2 is 2.

Proof. Obviously, the dimension of $u_2 = u_1 \cap v_2$ is at most 2. If $i \leq j$ in *N*, let us denote $[i,\ j] = \{k \in N\ :\ i \leq k \leq j\}$. Consider the following coalitions: $A = [5,\ 25]$, $B = [1, 16]$, $A' = A - \{24,\ 25\} \cup \{4\}$, and $B' = B - \{4\} \cup \{24,\ 25\}$. The weights of these coalitions in games u_1 and v_2 are as follows:

	A	B	A'	B'
u_1	84	100	90	94
v_2	21	16	20	17

Assume now that u_2 has dimension 1, i.e. that u_2 is a WMG. Coalition *A* is losing in u_1 and *B* is losing in v_2 and, hence, both coalitions are losing in u_2. But, after the described trades, *A* and *B* convert into the winning coalitions A' and B'. Consequently, the game u_2 cannot be a WMG according to Theorem 2.4.□

4.2. The dimension from November 1st 2004

Theorem 4.2. The dimension of game u_3 is 3.

Proof. Since $u_3 = v_3 \cap v_4 \cap v_1$, its dimension is at most 3. Consider the following coalitions:

- $A = [1,\ 3] \cup [7,\ 23]$, $B = [1, 12]$ and $C = \{1\} \cup [5,\ 25]$.
- $A' = A - \{13,\ 20,\ 21\} \cup \{4\}$, $B' = B - \{4\} \cup \{13,\ 20,\ 21\}$, and $C' = C - \{5,\ 25\} \cup \{3\}$.
- $A'' = A - \{3\} \cup \{5,\ 25\}$, $B'' = B - \{2\} \cup \{13, 14\}$, and $C'' = C - \{13, 14\} \cup \{2\}$.

The weights of these coalitions in games v_1, v_3 and v_4 are stated as follows:

	A	B	C	A'	B'	C'	A''	B''	C''
v_1	20	12	22	18	14	21	21	13	21
v_3	231	243	234	242	232	233	232	234	243
v_4	696	894	608	795	795	651	653	800	702

Assume first that u_3 has dimension 1. Coalition A is losing in v_3 and so is B in v_1. Consequently, A and B are losing in u_3. However, after trades, A and B convert into A' and B' that are both winning in u_3.

Assume now that u_3 has dimension 2, i.e. it can be represented as the intersection of two WMGs. It follows that at least one of the following statements should be true:

1. A and B are losing in the same WMG.
2. A and C are losing in the same WMG.
3. B and C are losing in the same WMG.

Statement 1 cannot be true because, as we have seen above, A' and B' are both winning in u_3, which is not possible in a WMG.

Statement 2 is impossible, because A'' and C' are both winning in v_3, v_1 and v_4 and, thus, coalitions A, C, A'' and C' show that trade robustness does not hold.

Finally, statement 3 is impossible for the same reason by considering coalitions B, C, B'' and C''. □

Theorem 4.3. The dimension of game u_4 is 3.

Proof. This proof follows the same approach as that of the proof of Theorem 4.2. For the sake of completeness, we indicate the coalitions we use to make trades and their corresponding weights in games v_2, v_3 and v_4:

- $A = [1, 2] \cup [4, 5] \cup [7, 12] \cup [16, 23]$, $B = [1, 16]$ and $C = [2, 3] \cup [6, 25]$.
- $A' = A - \{22, 23\} \cup \{3\}$, $B' = B - \{3\} \cup \{22, 23\}$ and $C' = C - \{6, 25\} \cup \{4\}$.
- $A'' = A - \{4\} \cup \{6, 25\}$, $B'' = B - \{1\} \cup \{24, 25\}$ and $C'' = C - \{24, 25\} \cup \{1\}$.

	A	B	C	A'	B'	C'	A''	B''	C''
v_2	18	16	22	17	17	21	19	17	21
v_3	231	277	236	252	256	235	232	255	258
v_4	730	956	602	856	830	643	689	776	782 □

4.3. Two surprising facts about the dimension of the Council

Let us analyze the initial enlargement of the European Union planned at the summit of Nice. There, 27 countries were supposed to form the future EU: the countries considered in Athens with the addition of Romania and Bulgaria. These two countries were assigned 14 and 10 votes in the Council, respectively. The 25 other countries were assigned the same number of votes as in game v_3.

We quote the relevant text from [38], p. 164.

Acts of the Council shall require for their adoption at least 258 votes in favour, cast by a majority of members, where this Treaty requires them to be adopted on a proposal from the Commission. [...] *When a decision is to be adopted by the Council by a qualified majority, a member of the Council may request verification that the Members States constituting the qualified majority represent at least 62% of the total population of the Union. If that condition is shown not to have been met, the decision in question shall not be adopted.*

If we consider $N = \{1, 2, ..., 27\}$ and games

$$\begin{aligned} v^1 &= [14; 1(27)], \\ v^3 &= [258; 29(4), 27(2), 14(1), 13(1), 12(5), 10(3), 7(5), 4(5), 3(1)], \\ v^4 &= [620; 170, 123(2), 120, 82, 80, 47, 33, 22, 21(4), 18, 17(2), \\ &\quad 11(3), 8(2), 5, 4, 3, 2, 1(2)], \end{aligned}$$

then the game that represents the full rules is $u = v^3 \cap v^4 \cap v^1$ (notice that, e.g. the notation for the voting rule based on a weight per thousand inhabitants changes from v_4 to v^4 when including both new countries). For simplicity, we assume that no relevant population changes have occurred since 2001. Notice that the threshold of 258 implies a super–majority of 74.78%, which is quite high and, therefore, it is rather difficult to reach agreements. Game u reduces to a single WMG [12], so that its dimension is 1.

Assume for a while that some states (Romania and Bulgaria) delay their incorporation into the EU, but the ratio between the threshold and the sum of weights used in each of the three games is not modified: 50.01% for game v^1, 74.78% for v^3 and 62% for v^4. This means that the *spirit* of the rule is maintained. How does the reduction of the number of countries affect the dimension? Will the dimension of this game with two fewer players be necessarily 1 or, instead, can it be greater than 1?

Notice that, in general, if a simple game has dimension k, i.e. it can be expressed as an intersection of k WMGs, and some players are removed, but the threshold is left invariant in each WMG, then the dimension of the reduced game is at most k. However, we will see that the dimension of the reduced game may be greater than k if the thresholds are modified in order to preserve the corresponding proportion of the sum of weights required in each of the WMGs. Game u will help us to show this fact.

Let us consider the following game with the 25 Member States

$$u' = (v^3)' \cap v_4 \cap v_1$$

where $(v^3)' = [240; 29(4), 27(2), 13(1), 12(5), 10(2), 7(5), 4(5), 3(1)]$ and v_1 and v_4 are the games that we described in Section 3.

Game u' represents the reduction of u in the way we mentioned above. In fact, the threshold for the 25-player game $(v^3)'$, 240, is a proportion 100×240/321 = 74.77% of the number of votes, and this ratio almost coincides with that of the 27-player game v^3. Games v_1 and v_4 also have ratios that almost coincide with those in games v^1 and v^4. The first, rather surprising, fact is shown in the next result.

Theorem 4.4. The dimension of game u' is 2.

Proof. This comes from the following properties:

1. Each winning coalition in $(v^3)'$ is also winning in v_4, i.e. $(v^3)' \cap v_4 = (v^3)'$ and hence $u' = (v^3)' \cap v_1$.

2. $(v^3)' \cap v_1$ does not reduce to a single WMG.

To see property 1, it suffices to check that if S is a coalition with a weight lower than 82 in $(v^3)'$, then its weight in v_4 is lower than 380, and this allows us to considerably reduce the number of coalitions to examine to 106 relevant ones (of course we omit this tedious but easy part). Thus, $(v^3)' \cap v_4 = (v^3)'$.

Property 2 follows from the fact that coalitions $A = [1, 3] \cup [7, 25]$ and $B = [1, 12]$ are both losing in u' but, after trades, convert into the winning coalitions $A' = A - \{13, 14, 24\} \cup \{6\}$ and $B' = B - \{6\} \cup \{13, 14, 24\}$. This proves that $u' = (v^3)' \cap v_1$ does not reduce to a WMG. □

Let us explain now a second surprising phenomenon that might happen. If we consider that significant changes in the populations are possible, imagine, for instance, that the population of each state tends to be proportional to the weight assigned to this state in the original weighted game. Thus it will be possible that the weighted–votes game and the population game are the same WMG. In consequence, if the populations change, it is theoretically possible to reach a game with the same member states, but with a smaller dimension than the original one.

In conclusion, the behavior of the dimension is sensitive to both the addition/removal of members and small changes in the population percentages. Therefore, eventual reductions of the dimension hardly justify simplification of the voting mechanisms intended for the Council.

5. On the egalitarianism of the Council

In this section, we are interested in the effect of requiring consensus in the voting systems planned for the enlargement of the EU to 25 members. We will quantify the egalitarianism of each voting system and how much it changes when the level of consensus required by the voting rule increases. As in Section 3, we study both of the foreseen scenarios separately. Three articles dealing with the issue of egalitarianism are by Peleg [29], Carreras and Freixas [6] and Freixas and Marciniak [19].

5.1. On the effect of the consensus required in the transitional period

Recall that game u_1 represents the basic WMG to be applied for a proposal coming from the European Commission (if a straight majority is also required, u_1 does not change: i.e. $u_1 = u_1 \cap v_1$), and u_2 is the same game, but with a threshold of 2/3 $(u_2 = u_1 \cap v_2)$: this applies for motions not coming from the European Commission.

Table 1. Distribution of power
and egalitarianism for the transitional period

Weight	$\Phi_i[u_1]$	$\Phi_i[u_2]$
10	0.0830	0.0759
8	0.0651	0.0602
5	0.0397	0.0391
4	0.0325	0.0337
3	0.0234	0.0265
2	0.0157	0.0207
Egalitarianism	14.8588	18.1159

Table 1 shows the distribution of power, according to the Shapley–Shubik index of power, and the egalitarianism index for games u_1 and u_2. The players are represented by their weights and the power indices are rounded to four decimal places.

As intuition would predict, the higher level of consensus required in game u_2 makes it more egalitarian than u_1. The over-egalitarianism percentage quantifies how much more:

$$\text{oep}[u_1, u_2] = 21.92\%$$

Notice that the fall in the power index of Germany, United Kingdom, France and Italy (the main players) is 0.0071, while that of Malta's (the weakest player) increases by 0.0050. According to Theorem 2.13, these two quantities could reach 0.1600 and 0.0400, respectively.

5.2. On the effect of the consensus required from November 1st 2004

Table 2 shows the distribution of power among the players for the most interesting games involved in the rules of the voting systems applied since November 1st 2004. Again, the players are represented by their weights, but now the Shapley–Shubik index of power is rounded to six decimal places because, if only four decimals were taken into account, certain differences in power that really exist would appear to be zero for some non-equivalent players.

Table 2. Distribution of power and egalitarianism from November 1st 2004

Population	Weight	$\Phi_i[v_4]$	$\Phi_i[v_3]$	$\Phi_i[v_4 \cap v_3]$	$\Phi_i[u_3]$	$\Phi_i[u_4]$
182	29	0.197955	0.092926	0.094941	0.094930	0.085770
132	29	0.134660	0.092926	0.093708	0.093698	0.084537
131	29	0.133462	0.092926	0.093704	0.093693	0.084532
128	29	0.129934	0.092926	0.093694	0.093683	0.084523
87	27	0.085358	0.086136	0.086715	0.086705	0.078377
86	27	0.084452	0.086136	0.086715	0.086705	0.078377
35	13	0.033029	0.039829	0.039515	0.039511	0.038448
23	12	0.021359	0.036479	0.036142	0.036138	0.035901
23	12	0.021359	0.036479	0.036142	0.036138	0.035901
23	12	0.021359	0.036479	0.036142	0.036138	0.035901
22	12	0.020384	0.036479	0.036142	0.036138	0.035901
22	12	0.020384	0.036479	0.036142	0.036138	0.035901
20	10	0.018452	0.030241	0.029926	0.029927	0.031159
18	10	0.016505	0.030241	0.029903	0.029904	0.031135
12	7	0.011085	0.020984	0.020655	0.020662	0.024126
12	7	0.011085	0.020984	0.020655	0.020662	0.024126
11	7	0.010137	0.020984	0.020651	0.020657	0.024121
8	7	0.007297	0.020984	0.020646	0.020652	0.024116
8	7	0.007297	0.020984	0.020646	0.020652	0.024116
5	4	0.004468	0.011892	0.011695	0.011704	0.017524
4	4	0.003615	0.011892	0.011695	0.011704	0.017524
3	4	0.002723	0.011892	0.011695	0.011704	0.017524
2	4	0.001819	0.011892	0.011695	0.011704	0.017524
1	4	0.000911	0.011892	0.011695	0.011704	0.017524
1	3	0.000911	0.008938	0.008741	0.008750	0.015410
	Egalitarianism	5.0750	11.9065	11.6009	11.6036	14.2126

Let us first refer to games v_4 (population game) and v_3 (weight game). The egalitarianism of game v_4, given by $\text{egal}[v_4] = 5.0750$, reflects the great difference in power between the main and the weakest players. Game v_3 is more egalitarian, since

$\text{egal}[v_3] = 11.9065$. Note that there is an increase of 134.61% in egalitarianism when passing from v_4 to v_3.

If we cross v_3 and v_4, we can see that v_4 is highly affected by v_3, as $\text{egal}[v_4 \cap v_3] = 11.6009$. Nevertheless, the intersection of v_4 and v_3 is somewhat superficial, since the egalitarianism of v_3 remains almost unvaried when intersected with v_4.

We can also see that the requirement of a majority by means of v_1 is almost negligible, because $\text{egal}[u_3] = 11.6036$. In fact, the weight game v_3 is more egalitarian than the one obtained when there are also population and majority requirements.

The demand of a 2/3 consensus changes the situation a little bit. In this case, we get a higher level of egalitarianism, $\text{egal}[u_4] = 14.2126$, which represents an increase of 22.48% with respect to u_3. As shown in Table 2, there are visible modifications in the players' power with respect to the other games.

6. On the decisiveness of the Council

We finally analyze the (structural) decisiveness of the different voting systems involved in the Council's decision-making procedures. All of these games are proper (i.e., do not contain disjoint winning coalitions) and most of them are weak (i.e., admit blocking coalitions). As a matter of comparison, we note that the previous 15-member voting systems of the Council show decisiveness degrees of 0.0778 and 0.0704 depending on whether the proposal at stake comes from the European Commission or not (for details, see [4]).

Table 3. Decisiveness degree of several games

Game	Description (when necessary)	Structural decisiveness
u_N	unanimity game	3×10^{-8} (minimum)
v_1	transitional 1/2-majority	0.5000 (maximum)
v_2	transitional 2/3-majority	0.0539
$u_1 = u_1 \cap v_1$	transitional qualified majority	0.0349
$u_2 = u_1 \cap v_2$	transitional qualified majority + 2/3-majority	0.0259
v_3	qualified majority of weights	0.0359
v_4	qualified majority of population	0.2397
$v_3 \cap v_1$		0.0359
$v_3 \cap v_2$		0.0222
$v_4 \cap v_1$		0.1988
$v_4 \cap v_2$		0.0404
$v_3 \cap v_4$		0.0359
$u_3 = v_3 \cap v_4 \cap v_1$		0.0359

Game	Description (when necessary)	Structural decisiveness
$u_4 = v_3 \cap v_4 \cap v_2$		0.0222
u_1'	straight majority of transit. weights	0.4863
v_3'	straight majority of weights	0.5000
v_4'	straight majority of population	0.5000

Table 4. Loss in decisiveness [%]

	v_1	u_1'	u_1	v_3'	v_4'	v_3	v_4	$v_3 \cap v_4$	u_3
v_2	89								
$u_1 = u_1 \cap v_1$		93							
$u_2 = u_1 \cap v_2$			26						
v_3				93					
v_4					52				
$v_3 \cap v_1$						0			
$v_3 \cap v_2$						38			
$v_4 \cap v_1$							17		
$v_4 \cap v_2$							83		
$v_3 \cap v_4$						0	85		
$u_3 = v_3 \cap v_4 \cap v_1$						0	85	0	
$u_4 = v_3 \cap v_4 \cap v_2$						38	91	38	38

Table 3 displays the decisiveness degrees of several games. Among them, we have included the games v_1, v_2, u_1, u_2, v_3, v_4, u_3, u_4 and some combinations of these, as well as the unanimity game u_N for $n = |N| = 25$, which gives the minimum degree of decisiveness, and the games u_1', v_3' and v_4' that correspond to u_1, v_3 and v_4 by replacing the appropriate qualified majority with a straight one.

Table 4 presents the percentage loss in decisiveness obtained when passing from a given game (to be found in the upper row) to a less decisive one (to be found in the left column). Together with the results from Table 3, these results will be the basis for our subsequent comments.

6.1. Decisiveness for the transitional period

Although the decisiveness degrees of games u_1 and u_2 are far from the minimum (attained by u_N), they are less than 1/2 of those corresponding to the previous Council. This decrease seems hard to justify, due to the provisional nature of the transitional period.

Incidentally, notice that using the straight majority game u_1' instead of u_1 would take the decisiveness degree to 0.4863, while the maximum for proper games is 0.5 (as, for example, in game v_1): this difference is due to the fact that u_1' admits blocking coalitions, since the total number of votes is even. The percentage loss in decisiveness when passing from u_1' to u_1 is 93%.

When comparing the two real procedures given by u_1 and u_2, we should first notice that passing from v_1 to v_2 implies a loss of 89% in decisiveness. However, passing from $u_1 = u_1 \cap v_1$ to $u_2 = u_1 \cap v_2$ only gives a loss of 26%, so that the negative effect on decisiveness derived from the additional requirement of a 2/3-consensus could be considered, after all, quite reasonable.

6.2. Decisiveness after November 1st 2004

We first note that the very low degree of decisiveness derived from imposing a qualified majority on the weights (game v_3) represents a loss of 93% with regard to the straight majority game v_3'. Instead, the qualified majority based on population (game v_4) gives rise to a decisiveness degree of 0.2397 and hence to a clearly smaller loss of 52% with regard to the straight majority game v_4'.

Among the intermediate intersections, $v_3 \cap v_1$, $v_3 \cap v_2$, $v_4 \cap v_1$, $v_4 \cap v_2$ and $v_3 \cap v_4$, only $v_4 \cap v_1$ presents a decisiveness degree which is clearly greater than the others (0.1988), with a loss of 17% with respect to v_4. Especially striking are the losses of decisiveness for the games $v_4 \cap v_2$ and $v_3 \cap v_4$ compared to v_4 (83% and 85%, respectively).

The actual procedures u_3 and u_4 are also interesting to analyze. First, their decisiveness degrees are again very small and hardly 1/2 of the corresponding previous procedures. Thus, the enlarged Union does not seem designed to be especially effective in decision-making processes. The equalities

$$\delta[u_3] = \delta[v_3] = \delta[v_3 \cap v_4] = \delta[v_3 \cap v_1] = 0.0359$$

are also worth mentioning, and mean that intersections often cause no loss of decisiveness. Finally, u_3 (respectively, u_4) implies a loss of 85% (respectively, 91%) with respect to v_4, whereas the loss of u_4 with respect to v_3, $v_3 \wedge v_4$ and u_3 is 38%.

7. Conclusions

Several features of the voting rules for the Council of Ministers of the European Union adopted at the Athens summit have been analyzed here. We have studied, from a strictly normative viewpoint, dimension, egalitarianism and decisiveness. Two periods have been considered in each case: the transitional one (until October 31st 2004) and the definitive one (from November 1st 2004).

As to dimension, one of the transitional voting rules has been found to be of dimension 2, as its definition suggests, but the other reduces to a one-dimensional game (i.e., a WMG). On the other hand, both of the rules adopted in the definitive period were shown to be of dimension 3 and, therefore, provide examples of real voting systems of this dimension (not easy to find). A rather surprising fact is also stated, namely that changes of dimension are possible in two cases: the first by either adding or removing countries but maintaining the proportion between the quota and the total weight, and the second resulting from changes in populations. We conclude that dimension is a very sensitive notion, and hence eventual changes in its value do not justify simplification of the voting mechanisms.

With regard to egalitarianism, for the transitional period we find that increasing the required level of consensus implies, of course, increasing the egalitarianism of the rule, but this difference is not especially relevant. Only the power of the four main countries and the single least powerful one changes appreciably with the required change in the level of consensus. Things seem therefore balanced enough for this period. Instead, in the definitive period, v_3 is much more egalitarian than v_4. Intersecting these two games, there is a great increase in egalitarianism with respect to v_4 and a small decrease with respect to v_3. A further intersection with v_1 does not affect egalitarianism, whereas intersecting with v_2 clearly increases this characteristic. In general, there are only noticeable differences in individual power indices between game v_4 and the other games: v_3, $v_4 \cap v_3$, u_3 and u_4.

Finally, concerning decisiveness, in the transitional period we find very low degrees, even less than 1/2 of the decisiveness degrees of the previous rules used for the 15-member Union (already very low). The increase in consensus implies a loss in decisiveness of 26%. The decisiveness of the corresponding rules adopted since November 1st 2004 are even lower. Only v_4 shows a relatively high degree of 0.2397, but there are drastic losses for u_3 and u_4 with regard to that population game. When intersected with other voting rules, game v_1 does not affect the degree of $v_3 \cap v_4$ but there is a decrease if the majority rule used in game v_2 is adopted. The loss from u_3 to u_4 is 38%. In our opinion, if the voting procedures intended for the European Union Council of Ministers have to be really useful for taking decisions, then the best games for the definitive period would be $v_4 \cap v_1$ and $v_4 \cap v_2$. The Athens rules should thus be subjected to new, sound analysis and maybe modified by the European Union. Notice that our conclusion is similar to the proposal contained in the first draft of the European Constitution.

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ORDERS OF CRITICALITY IN VOTING GAMES

The authors focus on the problem of investigating the blackmail power of players in simple games, which is the possibility of players of threatening coalitions to cause them loss using arguments that are (apparently) unjustified. To this purpose, the classical notion of the criticality of players has been extended, in order to characterize situations where players may gain more power over the members of a coalition thanks to collusion with other players.

Keywords: *voting game, blackmailing power, semivalue*

1. Introduction

We consider a parliament that has produced a majority coalition. If this majority corresponds to a minimal winning coalition, then all the coalition parties are critical, i.e. each of them is able to destroy the majority by leaving. However, we may face a different situation in which not all the parties are critical, i.e. the majority corresponds to a quasi-minimal winning coalition. A similar situation was typical in the eighties when the Italian governments included five parties, namely Christian Democracy (*Democrazia Cristiana* – DC), Italian Socialist Party (*Partito Socialista Italiano* – PSI), Italian Social-Democratic Party (*Partito Socialista Democratico Italiano* – PSDI), Italian Republican Party (*Partito Repubblicano Italiano* – PRI), Italian Liberal Party

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(*Partito Liberale Italiano* – PLI); for some years, only DC was critical (in the last years of the eighties PSI also became critical) but all the parties received ministries and/or departments, and in 1981 the premiership of the government was given to Giovanni Spadolini, the leader of PRI.

At a first glance, this situation may seem unusual, because all the parties received a quota of the power, even if the non-critical parties should have received nothing. On the other hand, it is possible to notice that the total number of seats of the coalition parties meant that a minimal winning coalition would still exist even after a non-critical party left. The presence of non-critical parties can be explained by their role in making other parties marginal. For instance, in the case of a minimal winning coalition of four parties, each of them is critical and should receive a quarter of the power; but if the majority is formed by a quasi-minimal winning coalition including five parties, only one of which is critical, each non-critical party may reclaim some power from the unique critical party for its role in keeping other parties non-critical. In fact, its exit from the governing coalition would enlarge the group of critical parties, thus reducing the power of the original unique critical party. This situation was considered from a different viewpoint in [3]; they accounted for the possibility of parties forming a different majority coalition that excludes another party, which, in its turn, may propose another majority coalition that does not include the party that started the process. In this way, the hypotheses on which the bargaining set [1] relies are satisfied, so the elements in the bargaining set are suitable for measuring the power of the parties.

In the present Italian political situation, we may consider the group *Alleanza Liberal-popolare-Autonomie* (ALA) that is represented only in the Senate, where the majority supporting the Italian government is very unstable, as opposed to the Lower Chamber, *Camera dei Deputati*, where the majority is stable. The ALA group had in mind to support the approval of some reforms proposed by the government, thus acting as a critical party. However, the presence of this group had the consequence that other members of the majority, from the Democratic Party, decided to support the reforms in order to avoid the ALA group becoming critical. We can say that critical parties have a first order of criticality, while non-critical ones have a higher order of criticality.

The aim of this paper is to provide a formal definition of second order critical players, which may be extended to higher orders, and analyze some properties, resulting in a proposal for the allocation of power.

The paper is organized as follows. We start by recalling some notation and general definitions in Section 2. Section 3 deals with higher orders of criticality, where player i becomes critical for coalition M only if other players (not critical for the same coalition) leave M before i. In Section 4, we strengthen the notion of the criticality of player i to coalition M adding the further constraint that the threat of i to leave M is made “credible” only if i has an opportunity to form a winning coalition with players outside of M. Section 5 concludes.

2. Preliminaries

A cooperative game with transferable utility (TU-game) is a pair (N, v), where $N = \{1, 2, ..., n\}$ denotes the finite set of players and $v: 2^N \to \mathbb{R}$ is the characteristic function, with $v(\varnothing) = 0$. $v(S)$ is the worth of coalition $S \subseteq N$, i.e. what players in S may obtain by standing alone.

A TU-game (N, v) is simple when $v: 2^N \to \{0, 1\}$, with $S \subseteq T \Rightarrow v(S) \le v(T)$ [4] and $v(N) = 1$. If $v(S) = 0$ then S is a losing coalition, while if $v(S) = 1$ then S is a winning coalition. Given a winning coalition S, if $S \setminus \{i\}$ is losing then $i \in N$ is a critical player for S. When a coalition S contains at least one critical player, S is a quasi-minimal winning coalition; when all the players of S are critical, it is a minimal winning coalition. A simple game may also be defined by giving the set of winning coalitions or the set of minimal winning coalitions.

A particular class of simple games is represented by weighted majority games. A vector of weights $(w_1, w_2, ..., w_n)$ is associated to the players that leads to the following definition of the characteristic function of the corresponding weighted majority game (N, w):

$$w(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \ge q \\ 0 & \text{otherwise} \end{cases}, \; S \subseteq N$$

where q is the majority quota. A weighted majority situation is often denoted as $[q; w_1, w_2, ..., w_n]$. Usually, we require that the game is proper or N-proper, i.e. if S is winning then $N \setminus S$ is losing; for this aim, it is sufficient to choose $q > \frac{1}{2} \sum_{i \in N} w_i$. Note that a simple game may not correspond to any weighted majority situation.

Given a TU-game (N, v), an allocation is an n-dimensional vector $(x_i)_{i \in N} \in \mathbb{R}^N$ assigning to player $i \in N$ the amount x_i; an allocation $(x_i)_{i \in N}$ is efficient if $x(N) = \sum_{i \in N} x_i = v(N)$. A solution is a function ψ that assigns an allocation $\psi(v)$ to each TU-game (N, v) belonging to a given class of games G with player set N.

For simple games, and in particular for weighted majority games, a solution is often called a power index, as each component x_i may be interpreted as the percentage of

[4] This property is called monotonicity.

power assigned to player $i \in N$. In the literature, several power indices have been introduced; among others, we recall the following definitions.

The Shapley-Shubik index [7], φ, is the natural version for simple games of the Shapley value [6]. It is defined as the average of the marginal contributions of player i w.r.t. all the possible orderings of players and it can be written as:

$$\varphi_i(v) = \sum_{S \subseteq N, S \ni i} \frac{(s-1)!(n-s)!}{n!} \; m_i(S), \; i \in N$$

where n and s denote the cardinalities of the set of players N and of the coalition S, respectively, and $m_i(S) = v(S) - v(S \setminus \{i\})$ denotes the marginal contribution of player $i \in N$ to coalition $S \subseteq N$, $S \ni i$.

The normalized Banzhaf index [2], β, is similar to the Shapley–Shubik index but it considers the marginal contributions of a player to all possible coalitions, independently of the order of the players; first, we define:

$$\beta_i^*(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N, S \ni i} m_i(S), \; i \in N$$

then, by normalization we get:

$$\beta_i(v) = \frac{\beta_i^*(v)}{\sum_{j \in N} \beta_j^*(v)}, \; i \in N$$

Let $\mathbf{p} = (p_1, ..., p_n)$ be a vector of n non-negative numbers such that

$$\sum_{k=1}^{n} p_k \binom{n}{k} = 1$$

with the interpretation that p_s is the probability that a coalition of size s forms. We denote by $\pi^{\mathbf{p}}$ the semivalue [4] engendered by the vector $\mathbf{p}$. Hence,

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subseteq N, S \ni i} p_s m_i(S) \tag{1}$$

Notice that both the Shapley–Shubik index φ and the Banzhaf index β^* can be defined as particular semivalues $\pi^{\mathbf{p}^{\varphi}}$ and $\pi^{\mathbf{p}^{\beta}}$, respectively, where the probability vector $\mathbf{p}^{\varphi}$ is such that $p_s^{\varphi} = (s-1)!(n-s)!/n!$ and $\mathbf{p}^{\beta}$ is such that $p_s^{\beta} = 1/2^{n-1}$. Notice that for a (monotonic) simple game *v*, relation (1) can be simply written as

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \in W_i(v)} p_s m_i(S) \quad (2)$$

where $W_i(v)$ is the set of winning coalitions containing player $i \in N$. Moreover, the value $m_i(S) = 1$ only if i is critical for S in *v*, for each $S \in W_i(v)$. Consequently, the value $\pi_i^{\mathbf{p}}(v)$ can be interpreted as the probability of player i playing a critical role in *v*. Every semivalue $\pi^{\mathbf{p}}$ satisfies the symmetry property (i.e., $\pi_i^{\mathbf{p}}(v) = \pi_j^{\mathbf{p}}(v)$ for each game v and every pair of symmetric players $i, j \in N$, i.e. such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$)) and the null player property (i.e., $\pi_i^{\mathbf{p}}(v) = 0$ for each game v and every null player $i \in N$, i.e. such that $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$).

In the following, we mainly refer to the Banzhaf index because it is more probability-oriented, as it is not based on the order in which players enter a coalition which is reasonable for majority coalitions.

3. Second and higher orders of criticality

In this section, we introduce the formal definition of order of criticality for a player. Given a winning coalition $M \subseteq N$, a critical player $i \in M$ may be called first order critical for coalition *M*. Now we deal with the other players in *M*.

Definition 1. Let $M \subseteq N$, with $|M| \geq 3$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. We say that player i is second order critical (SOC) for coalition *M*, via player $j \in M \setminus \{i\}$ iff $v(M \setminus \{i, j\}) = 0$ with $v(M \setminus \{j\}) = 1$.

The interpretation is the following: player i is not critical for *M*, but there exists in *M* another player *j*, different from *i*, s.t. *M* becomes a losing coalition when both the players leave. From this definition, we immediately get the following proposition.

Proposition 1. If player $i \in M$ is second order critical for coalition *M*, via player $j \in M$, then player j is second order critical for coalition *M*, via player *i*.

Remark 1. Note that when there are critical players of second order, there are at least two but they can be greater in number, as shown in the following example.

Example 1. Consider the weighted majority situation [51; 40, 8, 5, 5, 5]; considering the grand coalition of all players, the first party is the unique critical one, while the other four parties are second order critical, even if the last three parties are critical only via the second party. Definition 1 can be extended to higher orders as follows.

Definition 2. Let $k \geq 2$ be an integer, let $M \subseteq N$, with $|M| \geq k+2$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. We say that player i is order $k+1$ critical for coalition *M*, via coalition $K \subseteq M \setminus \{i\}$, with $|K| = k$ iff

$$v(M \setminus K) - v(M \setminus (K \cup \{i\})) = 1 \tag{3}$$

and *K* is the set of minimal cardinality satisfying (3), i.e.

$$v(M \setminus (T \cup \{i\})) = 1 \tag{4}$$

for any $T \subset K$ with $|T| < k$.

The interpretation is similar to the previous one: Player i is not critical for *M* but there exists in M a coalition *K*, not including *i*, s.t. M becomes losing when all the players in $K \cup \{i\}$ leave.

It should also be noticed that the notion of minimal cardinality is crucial to unambiguously assign the order of criticality of a player in a coalition.

Example 2. Consider the weighted majority situation [31, 21, 5, 3, 3, 3, 3, 2]; player 7 becomes critical whenever either a coalition involving player 2 and any one of the players 3–6 are involved (they are the coalition *K* in the definition), or three of the players 3–6 are involved. According to Definition 2, in the grand coalition of all players, player 7 is third order critical.

Notice also that the above definition encompasses the definitions for lower orders. In particular, we obtain first order criticality when (3) is satisfied by $K = \varnothing$ of 0-cardinality, leading to $v(M) - v(M \setminus \{i\}) = 1$. For second order criticality, consider $K = \{j\}$, then, by (3), $v(M \setminus \{j\}) = 1$ and $v(M \setminus \{i, j\}) = 0$. Moreover, since $\{j\}$ is the set of minimal cardinality which makes i critical, then $v(M \setminus \{i\}) = 1$.

We can derive a more general result than Proposition 1.

Proposition 2. If player $i \in M$ is order $k+1$ critical for coalition *M*, via coalition $K \subset M$, then each player $j \in K$ is order $k+1$ critical for coalition *M*, via coalition $K \cup \{i\} \setminus \{j\}$.

Proof. Define $K' = K \setminus \{j\}$. Since i is critical for *M* via coalition $K = K' \cup \{j\}$, then

$$v(M \setminus (K' \cup \{j\})) - v(M \setminus (K' \cup \{i, j\})) = 1 \tag{5}$$

We want to show that j is critical for M via $K' \cup \{i\}$.

Now $v(M \setminus (K' \cup \{i, j\})) = 0$ by (5), and, since $|K'| < k$, $v(M \setminus (K' \cup \{i\})) = 1$ by (4). Therefore,

$$v(M \setminus (K' \cup \{i\})) - v(M \setminus (K' \cup \{i, j\})) = 1$$

We need to verify the minimality of $K' \cup \{i\}$. Consider $T \subset K' \cup \{i\}$. There are two cases:

A. $i \in T$, then

$$v(M \setminus (T \cup \{j\})) = v(M \setminus [T \cup \{j\} \setminus \{i\}] \cup \{i\}) = 1$$

since $|T \cup \{j\} \setminus \{i\}| < k$.

B. $i \notin T$, then $T \cup \{j\} \subset K$ and $v(M \setminus (T \cup \{j\})) \geq v(M \setminus K) = 1$ and Eq. (4) is always satisfied.

Remark 2. A null player is never critical.

Remark 3. Note that when there are critical players of order $k+1$, there are at least $k+1$, but they can be greater in number as in the following example.

Example 3. Consider the weighted majority situation [51; 44, 3, 3, 3, 3, 3, 3]; in this case, the first party is the unique critical one for the grand coalition, while the other six parties are fourth order critical; note that there are no critical parties of order 2 or 3.

In view of Propositions 1 and 2 and Remark 2, we obtain the following corollary.

Corollary 1. Let $M \subseteq N$ be a winning coalition, then the players in M may be partitioned according to their order of criticality, plus those who are never critical.

Proposition 3. Let $i \in M$ be a player critical of order $k+1$ for coalition M, via coalition $K \subset M$; if a player $j \in K$ leaves the coalition, then i is a critical player of order k for coalition $M \setminus \{j\}$, via coalition $K \setminus \{j\}$.

Proof. It is sufficient to note that $M \setminus (K \cup \{i\}) = (M \setminus \{j\}) \setminus ((K \setminus \{j\}) \cup \{i\})$, so both of them are losing and $|K \setminus \{j\}| = |K| - 1$.

After defining the various orders of criticality, we want to provide an index to measure how much a player may profit from being critical. The first step is to measure the power of a player w.r.t. a given coalition, taking into account his order; then we may aggregate the power of a player w.r.t. all the coalitions he may belong to.

We want now to compute the probability of a player $i \in N$ being SOC in v for some coalitions via another player $j \in N$. First, consider a coalition $S \in 2^{N \setminus \{i, j\}}$ with $v(S \cup \{i, j\}) = 1$ and define $C_{ij}(S)$ as follows:

$$C_{ij}(S) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\} - v(S)$$

By the monotonicity of v, we have four possible cases, as shown in Table 1.

Table 1. Possible cases for player i (j) to be SOC for coalition $S \cup \{i, j\}$ via player j (i)

No.	$v(S \cup \{i\})$	$v(S \cup \{j\})$	$v(S)$	$C_{ij}(S)$
1	0	1	0	0
2	1	0	0	0
3	1	1	1	0
4	1	1	0	1

The only case in which i is SOC for $S \cup \{i, j\}$ via j is the last one (4) and $C_{ij}(S) = 1.$ Note also that, in general, $C_{ij}(S) = C_{ji}(S).$

Let $\mathbf{p} = (p_0, ..., p_{n-1})$ be a probability vector as defined in Section 2. If we want to compute the probability that i is SOC for some coalitions via j, we should compute the following expression:

$$\Gamma_{ij}^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i, j\}}} p_{s+1} C_{ij}(S)$$

By Proposition 1, it immediately follows that $\Gamma_{ij}^{\mathbf{p}}(v) = \Gamma_{ji}^{\mathbf{p}}(v)$ for each $i, j \in N.$ Following the same approach used to define semivalues (see relation (1)), we can also compute the total probability that player i is SOC for some coalition via some other player using the following summation:

$$C_i^{\mathbf{p}}(v) = \sum_{j \in N \setminus \{i\}} \Gamma_{ij}^{\mathbf{p}}(v)$$

Now, consider the game (N, v^{ij}) such that for each $S \in 2^{N \setminus \{i, j\}}$:

$$v^{ij}(S \cup \{i, j\}) = v^{ij}(S \cup \{i\}) = v^{ij}(S \cup \{j\}) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\}$$

and

$$v^{ij}(S) = v(S)$$

The game (N, v^{ij}) represents a coalitional situation where the role of i (j) is negatively influenced by j (i), that is the value of each coalition M containing either i or j is lowered to the worst value from the pair $v(M \cup \{j\} \setminus \{i\})$ and $v(M \cup \{i\} \setminus \{j\})$. It is easy to check that $\Gamma_{ij}^{\mathbf{p}}(v) = \pi_i^{\mathbf{p}}(v^{ij})$.

Example 4. Consider a simple game (N, v) with $N = \{1, 2, 3, 4\}$ whose minimal winning coalitions are $\{1, 2, 3\}$ and $\{1, 2, 4\}$. Note that 3 (4) is SOC for $\{1, 2, 3, 4\}$ via 4 (3), and no other player is SOC via another player for any coalition. Taking the vector $\mathbf{p}^{\beta}$ as the probability vector yielding the Banzhaf index $\pi^{\mathbf{p}^{\beta}}$ (see Section 2), we obtain $\Gamma_{34}^{\mathbf{p}^{\beta}}(v) = \Gamma_{43}^{\mathbf{p}^{\beta}}(v) = 1/8$ and $\Gamma_{ij}^{\mathbf{p}^{\beta}}(v) = 0$ for all the other i and j in N with $\{i, j\} \neq \{3, 4\}$. We have $v^{34}(S) = v^{43}(S) = v(S)$ for each $S \in 2^N$. So, $\Gamma_{34}^{p^{\beta}}(v) = \Gamma_{43}^{p^{\beta}}(v)$ $= \pi_3^{p^{\beta}}(v^{34}) = \pi_4^{p^{\beta}}(v^{43}) = 1/8$. In addition, we also have $v^{ij}(S) = 0$ for each $S \in 2^N$ and for all i and j with $\{i, j\} \neq \{3, 4\}$; hence, $\Gamma_{ij}^{\mathbf{p}^{\beta}}(v) = \pi_i^{\mathbf{p}^{\beta}}(v^{ij}) = 0$ for all i and j with $\{i, j\} \neq \{3, 4\}$.

In this example, the total probabilities of being SOC are $C_1^{\mathbf{p}^{\beta}}(v) = C_2^{\mathbf{p}^{\beta}}(v) = 0$ and $C_3^{\mathbf{p}^{\beta}}(v) = C_4^{\mathbf{p}^{\beta}}(v) = 1/8$.

Now, consider a game (N, v) and take a coalition M such that $i \in M$ and $K \subseteq M \setminus \{i\}$ with $|K| = k$, $k \geq 2$. Similarly to the above, we can compute the value

$$C_{iK}(M) = \min\{v(M \setminus T) : T \subset K \cup \{i\}\} - v(M \setminus (K \cup \{i\}))$$

that is equal to 1 iff i is order $k+1$ critical for coalition M via coalition $K \subset M$. Consequently, if we want to compute the probability that i is order $k+1$ critical for some coalitions via coalition K, we should compute the following expression:

$$\Gamma_{iK}^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i\}} \text{ with } K \subseteq S} p_s C_{iK}(S)$$

We conclude this section with an example of a possible application of the notions of criticality of first and second order to the analysis of the power of political parties in a realistic scenario.

Example 5. Consider the political situation described in Section 1 concerning the Italian Senate during the eighties. More precisely, the distribution of seats among the political parties of the largest alliance in the Italian Senate during the IX Legislature (1979–1983) was the one shown in Table 2.

Table 2. The distribution of seats in the largest alliance in the Italian Senate during the IX Legislature (1979–1983)

Party	Seats
Democrazia Cristiana (DC)	145
Partito Socialista Italiano (PSI)	32
Partito Socialdemocratico Italiano (PSDI)	9
Partito Repubblicano Italiano (PRI)	6
Partito Liberale Italiano (PLI)	2

At that time, the quota needed to have a majority within the Senate was 162. This leads to the weighted majority situation $[162; 145, 32, 9, 6, 2]$ on the player set $\{\text{DC, PSI, PSDI, PRI, PLI}\}$. Notice that a coalition is winning if it contains one of the following minimal winning coalitions $\{\{\text{DC, PSI}\}, \{\text{DC, PSDI, PRI, PLI}\}\}$. The symmetric relation of SOC exists between several pairs of players for the coalition $\{\text{DC, PSI, PSDI, PRI, PLI}\}$, specifically: PSI vs. PSDI, PSI vs. PRI and PSI vs. PLI.

Using the probability vector $\mathbf{p}^{\beta}$, we can compute the probability of being first order critical (i.e., the Banzhaf index) and of being SOC (using the index $C^{\mathbf{p}^{\beta}}$), as shown in Table 3.

Table 3. The Banzhaf index and the total probability of being SOC in the Italian Senate

Party	DC	PSI	PSDI	PRI	PLI
$\pi^{\mathbf{p}^{\beta}}$	$\frac{9}{16}$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
$C^{\mathbf{p}^{\beta}}$	0	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

For the sake of completeness, we recall that in 1983 the PSI threatened to leave the five-party alliance unless Bettino Craxi, the PSI party's leader, was made Prime Minister. The DC party accepted this compromise, in order to avoid a new election.

Maybe, the DC party had evaluated the threat of the PSI as credible in view of the high $C^{\mathbf{P}^{\beta}}$ index for the PSI party.

4. Credible criticality

We now consider an alternative notion of first order criticality where the fact that player i can threaten coalition M in a credible way is made possible by the fact that there exists another opportunity for i to be winning without the help of the players in M. We want to remark that this hypothesis is different from that in [3], where i could ask for the help of some players in M, but not all.

Definition 3. Let $M \subseteq N$ with $v(M)=1$. A player $i \in M$ is said to be first order credibly critical (or simply credibly critical) for coalition M iff it satisfies the following two conditions: (i) i is critical for M (i.e. $v(M \setminus \{i\}) = 0$) and (ii) there exists another coalition $S \subseteq N \setminus M$ that becomes winning with the addition of i, i.e. $v(S \cup \{i\}) = 1$.

Example 6. Consider a simple game (N, v) with $N = \{1, 2, 3\}$ whose minimal winning coalitions are $\{1, 2\}$ and $\{2, 3\}$. Player 2 is credibly critical for coalition $\{1, 2\}$ (in fact $v\,(1, 2) = 1,\ v\,(1) = 0$ and $v\,(2, 3) = 1$) but players 1 and 3 are never credibly critical.

Suppose $M \cup \{i\}$ is a winning coalition, where $M \subseteq N \setminus \{i\}$. When player i is in negotiations with the coalition M, the property of the credible criticality of player $i \in N$ can affect the ability of player i to gain power over $M \cup \{i\}$ by defeating the resistance of the other members of M to assigning the marginal contribution $v(M \cup \{i\}) - v(M)$ to i.

More in general, we can think of a situation where the marginal contribution $v(M \cup \{i\}) - v(M)$ is assigned to player i only if i could potentially take part in another coalition $S \subseteq N \setminus M$ at least as powerful as $M \cup \{i\}$, such that $M \cap S = \varnothing$. For a simple game, this leads us to the following definition of the *credible* marginal contribution of player i to coalition $M \subseteq N \setminus \{i\}$:

$$\hat{m}_i^v(M) = \begin{cases} v(M \cup \{i\}) - v(M) & \text{iff } \ v(N \setminus M) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Remark 4. A credible marginal contribution exists and can be computed for each player i and each coalition $M \subseteq N \setminus \{i\}$ on each possible game (N, v). However, in the remainder of the paper we will focus on the computation of the quantity $\hat{m}_i^v(M)$ for simple games.

Remark 5. A more concise way to define the quantity $\hat{m}_i^v(S)$ for each $i \in N$ and $M \in 2^{N\setminus\{i\}}$ is as follows*:*

$$\hat{m}_i^v(M) = \min\{v(M \cup \{i\}) - v(M), v(N \setminus M)\} \tag{7}$$

Now, consider again a probability vector $\mathbf{p} = (p_0, ..., p_{n-1})$ as in the previous section. As a measure of the power that players may credibly claim in a simple game, we define the following *credible semivalue* engendered by the vector $\mathbf{p}$. Hence,

$$\hat{\pi}_i^{\mathbf{p}}(v) = \sum_{S \in 2^{N\setminus\{i\}}} p_s \hat{m}_i^v(S) = \sum_{S \in W_i(v)} p_s \hat{m}_i^v(S) \tag{8}$$

For simple games, $\hat{\pi}_i^{\mathbf{p}}(v)$ can be interpreted as the probability of player i being credibly critical (under the probability vector $\mathbf{p}$). The next examples show that the vector of indices given by a semivalue $\pi^{\mathbf{p}}$ engendered by a probability vector $\mathbf{p}$ can be drastically different from the one given by the credible semivalue $\hat{\pi}^{\mathbf{p}}$ engendered by the same probability vector.

Example 7. Consider the simple game (N, v) with $N = \{1, 2, 3\}$ whose unique minimal winning coalition is $\{1, 2\}$. By the null player property, for any semivalue $\pi^{\mathbf{p}}$, $\pi_3^{\mathbf{p}}(v) = 0$ and by symmetry $\pi_1^{\mathbf{p}}(v) = \pi_2^{\mathbf{p}}(v)$. Notice also that no player $i \in \{1, 2, 3\}$ is credibly critical. So each credible semivalue yields $\hat{\pi}_i^{\mathbf{p}}(v) = 0$ for each $i \in \{1, 2, 3\}$.

The next example shows that not only can the values $\pi^{\mathbf{p}}$ and $\hat{\pi}^{\mathbf{p}}$ be very different, but even the ranking of players according to $\pi^{\mathbf{p}}$ and $\hat{\pi}^{\mathbf{p}}$ need not be preserved.

Example 8. Consider the simple game (N, v) with $N = \{1, 2, 3, 4, 5\}$ whose minimal winning coalitions are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$ and $\{3, 4, 5\}$. Consider the semivalue $\pi^{\mathbf{p}^\beta}$ corresponding to the Banzhaf value. Thus, the Banzhaf value gives $\pi_1^{\mathbf{p}^\beta} = \pi_2^{\mathbf{p}^\beta} = 6/16$ and $\pi_3^{\mathbf{p}^\beta} = \pi_4^{\mathbf{p}^\beta} = \pi_5^{\mathbf{p}^\beta} = 4/16$. Now consider the notion of credible criticality. Note that 1 and 2 are never credibly critical, whereas 3, 4 and 5 are credibly critical whenever they are critical. Consequently, $\hat{\pi}_1^{\mathbf{p}^\beta} = \hat{\pi}_2^{\mathbf{p}^\beta} = 0$ and $\hat{\pi}_3^{\mathbf{p}^\beta} = \hat{\pi}_4^{\mathbf{p}^\beta} = \hat{\pi}_5^{\mathbf{p}^\beta} = 4/16$.

Proposition 4. A credible semivalue $\hat{\pi}^{\mathbf{p}}$ satisfies the null player property and the symmetry property.

Proof. Both properties follow from the definition of credible marginal contribution as for the case of classical semivalues.

Proposition 5. Consider a weighted majority game (N, v) with quota and weights $[q; w_1, ..., w_n]$. Let $\hat{\pi}^{\mathbf{p}}$ be defined according to relation (8) on any probability vector **p**. It follows that

$$w_i \geq w_j \Rightarrow \hat{\pi}_i^{\mathbf{p}} \geq \hat{\pi}_j^{\mathbf{p}}$$

for each $i, j \in N$.

Proof Let $i, j \in N$ with $w_i \geq w_j$ and $S \in 2^{N \setminus \{i, j\}}$. It follows from the definition of a weighted majority game that $v(S \cup \{i\}) \geq v(S \cup \{j\})$. So, by Eq. (7), it immediately follows that

$$\begin{aligned} \hat{m}_i^v(S) &= \min\{v(S \cup \{i\}) - v(S), v(N \setminus S)\} \\ &\geq \min\{v(S \cup \{j\}) - v(S), v(N \setminus S)\} = \hat{m}_j^v(S) \end{aligned} \tag{9}$$

Now take $S \in 2^{N \setminus \{i, j\}}$ and consider the credible marginal contribution $\hat{m}_i^v(S \cup \{j\})$ and $\hat{m}_j^v(S \cup \{i\})$. We obtain $v(S \cup \{i\}) \geq v(S \cup \{j\})$. Again by Eq. (7) it follows that

$$\begin{aligned} \hat{m}_i^v(S \cup \{j\}) &= \min\{v(S \cup \{i, j\}) - v(S \cup \{j\}), v(N \setminus (S \cup \{j\}))\} \\ &\geq \min\{v(S \cup \{i, j\}) - v(S \cup \{i\}), v(N \setminus (S \cup \{i\}))\} = \hat{m}_j^v(S \cup \{i\}) \end{aligned} \tag{10}$$

where the inequality follows from the fact that by the definition of a weighted majority game,

$$v(S \cup \{i, j\}) - v(S \cup \{j\}) \geq v(S \cup \{i, j\}) - v(S \cup \{i\})$$

and $v(N \setminus (S \cup \{j\})) \geq v(N \setminus (S \cup \{i\}))$, since $N \setminus (S \cup \{j\})$ is obtained by substituting *j* by *i* in $N \setminus (S \cup \{i\})$.

The proof follows by relations (9) and (10) and the fact that by relation (8)

$$\hat{\pi}_i^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i,j\}}} (\hat{m}_i^v(S) + \hat{m}_i^v(S \cup \{j\}))$$

for each $i \in N$.

The converse of Proposition 5 is not true, as shown by Example 7, where the game (N, v) can be generated by the weighted majority situation [3; 2, 2, 0] and $\hat{\pi}^{\mathbf{p}}(v) = (0, 0, 0)$ for any credible semivalue $\hat{\pi}^{\mathbf{p}}$.

5. Concluding remarks

We have introduced the concept of order of criticality for a party in a winning coalition, based on the minimal number of parties who must leave the coalition to make it losing such that the party of interest is the last to leave. Then we defined a measure of criticality that describes the relevance of a party in a majority coalition; this measure is given by the probability that a party is of a given order of criticality. Finally, we studied the credibility of a threat made by a party considering the possibility of forming an alternative majority by joining some of the parties in the opposition. Possible developments of this concept of credibility may account for the effectiveness of possible alternative majorities. For instance, it is possible to take into account the ideological contiguity of parties on a left-right axis [5], or majorities that had been formed in the past.

Another possible direction of investigation might be an analysis of credible criticality of higher orders; in this case, when player $i \in N$ is second (or higher) order critical for a winning coalition $M \ni i$, we cannot use the same approach as for first order criticality because this implies that the game is not proper, as the two disjoint coalitions $M \setminus \{i\}$ and $(N \setminus M) \cup \{i\}$ are both winning. Consequently, such a definition should allow some players in $M \setminus \{i\}$ to join an alternative majority.

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DETERMINING MODELS OF INFLUENCE

We consider a model of opinion formation based on aggregation functions. Each player modifies his opinion by arbitrarily aggregating the current opinion of all players. A player is influential on another player if the opinion of the first one matters to the latter. Generalization of an influential player to a coalition whose opinion matters to a player is called an influential coalition. Influential players (coalitions) can be graphically represented by the graph (hypergraph) of influence, and convergence analysis is based on properties of the hypergraphs of influence. In the paper, we focus on the practical issues of applicability of the model w.r.t. a standard framework for opinion formation driven by Markov chain theory. For a qualitative analysis of convergence, knowing the aggregation functions of the players is not required, one only needs to know the set of influential coalitions for each player. We propose simple algorithms that permit us to fully determine the influential coalitions. We distinguish three cases: a symmetric decomposable model, an anonymous model, and a general model.

Keywords: social network, opinion formation, aggregation function, influential coalition, algorithm

1. Introduction. Dynamic models of opinion formation

Models of opinion formation are widely studied in psychology, sociology, economics, mathematics, computer sciences, among others; for overviews, see, e.g., [32, 1]. A seminal model of opinion formation and imitation was introduced in [16]. In that model, individuals in a society start with initial opinions on a subject. The interaction patterns are described by a stochastic matrix, whose entry in row j and column k represents the weight that player j places on the current belief of player k in forming j's belief for the next period. These beliefs are updated over time.

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While Degroot assumes that players update their opinion by taking weighted averages of the opinions of all players [16], Grabisch and Rusinowska investigated a model of opinion formation in which players update their beliefs according to arbitrary aggregation functions [27]. Foerster et al. study a model of opinion formation in which ordered weighted averages are used in the process of updating information [21]. In this paper, we consider the model of influence based on aggregation functions [27] and discuss practical issues of applying this model w.r.t. the standard framework for opinion formation driven by Markov chain theory. For a full qualitative analysis of the convergence of opinions, i.e., determining all the terminal classes (without their probabilities), it is sufficient to identify influential coalitions, which can be easily obtained by interviewing the agents. The aim of this paper is to show that a full qualitative analysis of convergence is feasible in practical situations. We introduce simple algorithms that permit us to fully determine the influential coalitions in three cases: the symmetric decomposable model (influential coalitions reduce to individuals), the anonymous model (only the number of agents matters, not their identity), and the general model. We show how clues on convergence can be obtained in a simple way, even without determining the reduced transition matrix.

There exists a vast literature that presents other variations and extensions of the DeGroot model. We briefly recall some of them. In particular, Jackson [32] and Golub and Jackson [26] investigate a model, in which players update their beliefs by repeatedly taking weighted averages of their neighbors' opinions. According to these authors, one of the issues regarding the DeGroot framework concerns necessary and sufficient conditions for convergence of the social influence matrix and reaching a consensus (see additionally [9]). Jackson also examines the speed of convergence of beliefs [32], and Golub and Jackson analyze, in the context of the DeGroot model, whether consensus beliefs are correct, i.e., whether beliefs converge to the right probability, expectation, etc [26]. The authors consider a sequence of societies, where each society is strongly connected and convergent in opinions, and described by its updating matrix. In each social network of the sequence, the belief of each player converges to the consensus limit belief. There is a true state of nature, and the sequence of networks is wise if the consensus limit belief converges in probability to the true state as the number of societies grows.

Several other generalizations of the DeGroot model can be found in the literature, e.g., models in which the updating of beliefs can vary in time and circumstances (see e.g., [17, 35, 37, 23, 24]). In the model described by Demarzo et al., players in a network try to estimate some unknown parameter [17]. This model allows updating to vary over time, i.e., a player may place more or less weight on his own belief over time. The authors study the case of multidimensional opinions, in which each player has a vector of beliefs. They show that, in fact, individuals' opinions can often be well approximated by a one-dimensional line, where a player's position on the line determines his position on all issues. Friedkin and Johnsen study a similar framework, in which social attitudes depend on the attitudes of neighbors and evolve over time [23, 24]. In their model, players start with initial attitudes and then mix in some of their neighbors' recent attitudes with their starting attitudes.

Also, other works in sociology related to influence are worth mentioning, e.g., the eigenvector-like notions of centrality and prestige [33, 10, 11], and models of social influence and persuasion by French [22] and Harary [29] (see also [39]). A sociological model of interactions on networks is also presented by Conlisk [13] (see also [14, 15, 36]), who introduces interactive Markov chains, in which every entry in a state vector at each time represents the fraction of the population with some attribute. The matrix depends on the current state vector, i.e., the current social structure is taken into account to model how sociological dynamics evolve. Threshold models of collective behavior are discussed by Granovetter [28]. In these models, agents have two alternatives and the costs and benefits of each depend on how many other agents choose which alternative. The author focuses on the effect of individual thresholds (i.e., the proportion or number of others that make their decision before a given agent) on collective behavior, discusses an equilibrium in a process occurring over time and the stability of equilibrium outcomes. Another model of influence is studied by Asavathiratham [2] and Asavathiratham et al. [3]. This model consists of a network of nodes, each with a status evolving over time. The evolution of status acts according to an internal Markov chain, but the transition probabilities depend not only on the current status of the node, but also on the statuses of the neighboring nodes.

More research on interaction is presented by Hu and Shapley [31, 30]. The authors apply the command structure of [38] to model players' interactions using simple games. For each player, boss sets and approval sets are introduced, and based on these sets, a simple game called the command game for a player is built. Hu and Shapley introduce an authority distribution over an organization and the (stochastic) power transition matrix, in which the entry in row j and column k is interpreted as agent j's power transfered to k [30]. The authority equilibrium equation is defined. In [30], multi-step commands are considered, where commands can be implemented through command channels.

There is also a vast literature on learning in the context of social networks; see e.g. [6, 18–20, 4, 5, 25, 12, 7]. In general, in models of social learning, agents observe choices over time and update their beliefs accordingly, which is different from models where choices depend on the influence of others.

The paper is organized as follows. Section 2 presents fundamental material on models of influence based on aggregation functions, as well as establishing notation and terminology, and recalls an essential result, which is the basis for determining the qualitative part of the model of influence. Section 3 addresses the problem of determining a model of influence in practice and focuses on determining its qualitative part, which is sufficient for a qualitative analysis of convergence. Section 4 gives some concluding remarks.

2. A model of influence based on aggregation functions

In this section, we recapitulate a model of influence based on aggregation functions [27]. Consider a set $N := \{1, ..., n\}$ of players that have to make a yes-no decision on a certain issue. Each player has an initial opinion, which may change due to mutual interaction (influence) between players. By $b_{S,T}$ we denote the probability that the set S of yes-voters becomes T after one step of influence. We assume that the process of influence may iterate, and therefore obtain a stochastic process of influence, depicting the evolution of the coalition of yes-players in time. We assume that the process is Markovian ($b_{S,T}$ depends on S and T, but not on the whole history) and stationary ($b_{S,T}$ is constant over time). The states of this finite Markovian process are all subsets $S \subseteq N$ representing the set of yes-players, and we also have the transition matrix $\mathbf{B} := [b_{S,T}]_{S,T \subseteq N}$, which is a $2^n \times 2^n$ row-stochastic matrix.

For a qualitative description of the convergence of the process, it is sufficient to know the reduced matrix $\tilde{\mathbf{B}}$ given by

$$\tilde{b}_{S,T} = \begin{cases} 1, & \text{if } b_{S,T} > 0 \\ 0, & \text{otherwise} \end{cases}$$

and equivalently represented by the transition graph $\Gamma = (2^N, E)$, where E is the set of arcs, its vertices are all possible coalitions, and the arc (S, T) from state S to state T exists if and only if $\tilde{b}_{S,T} = 1$.

Definition 1. An n-place aggregation function is a mapping $A: [0, 1]^n \to [0, 1]$ satisfying

(i) $A(0, ..., 0) = 0$, $A(1, ..., 1) = 1$ (boundary conditions),

(ii) If $\mathbf{x} \le \mathbf{x}'$ then $A(\mathbf{x}) \le A(\mathbf{x}')$ (nondecreasingness).

To each player $i \in N$ we associate an aggregation function A_i, which specifies the way player i modifies his opinion based on the opinions of the other players. Let $\mathbf{A} := (A_1, ..., A_n)$ denote the vector of aggregation functions. We compute $\mathbf{A}(1_S) = (A_1(1_S), ..., A_n(1_S))$, where 1_S is the characteristic vector of S, and $A_i(1_S)$ indicates the probability of player i saying yes at the next step when the set of agents presently saying yes is S. We assume that these probabilities are independent over the set of agents. Hence the probability of transition from the yes-coalition S to the yes-coalition T is given by

$$b_{S,T} = \prod_{i\in T} A_i(1_S) \prod_{i\notin T}(1 - A_i(1_S)). \tag{1}$$

A detailed study of convergence under this model is provided in [27]. It is shown, in particular, that three types of terminal class[2] can exist: singletons, cycles, and regular terminal classes. The first case occurs when a class is reduced to a single coalition (called the terminal state). The second one is the case where no convergence occurs because the process endlessly cycles over a sequence of coalitions, and the last case occurs when the class is a Boolean lattice of the form $\{S \in 2^N \mid K \subseteq S \subseteq L\}$ for some sets $K, L.$ In any case, N and $\varnothing$ are terminal states (called trivial terminal states).

We emphasize two particular aggregation functions. The first one is the well-known weighted arithmetic mean (WAM), defined by

$$\mathrm{WAM}_w(x_1, ..., x_n) = \sum_{i=1}^{n} w_i x_i$$

where $w = (w_1, ..., w_n)$ is a weight vector, i.e., $w \in [0, 1]^n$ with the property $\sum_{i=1}^{n} w_i = 1.$ Weighted arithmetic means are used in most models of opinion formation, e.g., the DeGroot model. Another noteable aggregation function is the ordered weighted arithmetic mean (OWA) [40], defined by

$$\mathrm{OWA}_w(x_1, ..., x_n) = \sum_{i=1}^{n} w_i x_{(i)}$$

where w is a weight vector, and the inputs have been arranged in decreasing order: $x_{(1)} \geq ... \geq x_{(n)}$. Note that, unlike in the case of WAM, these weights do not act on inputs, but on the rank of the inputs, so that the minimum and the maximum are particular cases, by taking $w = (0, ..., 0, 1)$ and $w = (1, 0, ..., 0),$ respectively. Applied to our context of influence where the input vectors are binary, if each agent aggregates his opinions according to an OWA, we obtain a model of anonymous influence, because each agent

[2] A class is a maximal collection of coalitions such that for any two distinct coalitions S, T in the class, there exists a sequence of transitions inside the class leading from S to T. A class is terminal if no transition to a coalition outside that class is possible.

updates his opinion according to the number of agents saying yes, not to which agents say yes. Models of anonymous influence have been studied in detail in [21].

Definition 2. Let A_i be the aggregation function of agent *i*. A nonempty coalition $S \subseteq N$ is yes-influential for *i* if

(i) $A_i(1_S) > 0$,

(ii) for all $S' \subset S$, $A_i(1_{S'}) = 0$.

Similarly, a coalition *S* is no-influential for i if

(i) $A_i(1_{N \setminus S}) < 1$,

(ii) for all $S' \subset S$, $A_i(1_{N \setminus S'}) = 1$.

We denote by $\mathcal{C}_i^{\text{yes}}$ and $\mathcal{C}_i^{\text{no}}$ the collections of yes- and no-influential coalitions for *i*. Coalition S is yes-influential for player i if, when the players in S say yes and every other player says no, i has a positive probability of saying yes (and similarly for no-influential coalitions). Moreover, S has no superfluous player. If an influential coalition is formed by only one player, then we call it an influential player. Note that these collections are never empty, since if no proper subcoalition of N were yes- or no-influential, then N would be both yes- and no-influential by Definition 2. More importantly, each such collection is an antichain in 2^N, that is, for any two distinct members of the collection S, S', $S \not\subset S'$ and $S' \not\subset S$.

Influential players can easily be represented in a directed graph. Define $G_{\mathbf{A}}^{\text{yes}}$, the graph of yes-influence, as follows: the set of nodes is the set of agents *N*, and there is an arc (j, i) from j to i if j is yes-influential on *i*. The graph of no-influence $G_{\mathbf{A}}^{\text{no}}$ is defined similarly. The representation of influential coalitions requires the more complex notion of a hypergraph.

Definition 3. We define the following concepts:

(i) A hypergraph [8] H is a pair $(N, \mathcal{E})$ with N being the set of nodes and $\mathcal{E}$ the set of hyperedges, where a hyperedge $S \in \mathcal{E}$ is a nonempty subset of *N*. If $|S| = 2$ for all $S \in \mathcal{E}$, then we have a classical graph.

(ii) A directed hypergraph on N is a pair $N, \mathcal{E}$ with $\mathcal{D}$ being the set of directed hyperedges, where a directed hyperedge is an ordered pair (S', S'') (called an hyperarc from S' to S''), with S', S'' both being nonempty.

(iii) A directed hyperpath from i to j is a sequence $i_0(S_1', S_1'')i_1(S_2', S_2'')i_2 \ldots i_{q-1}(S_q', S_q'')i_q$, where $i_0 := i, i_1, \ldots, i_{q-1}, j =: i_q$ are nodes, $(S_1', S_1''), \ldots, (S_q', S_q'')$ are hyperarcs such that $S_k' \ni i_{k-1}$ and $S_k'' \ni i_k$ for all $k = 1, \ldots, q$.

We define the hypergraphs $H_{\mathbf{A}}^{\text{yes}}$, $H_{\mathbf{A}}^{\text{no}}$ of yes-influence and no-influence as follows: for $H_{\mathbf{A}}^{\text{yes}}$, the set of nodes is N, and there is a hyperarc $(C, \{i\})$ for each $C \in \mathcal{C}_i^{\text{yes}}$ (similarly for $H_{\mathbf{A}}^{\text{no}}$).

Grabisch and Rusinowska [27] prove that the hypergraphs $H_{\mathbf{A}}^{\text{yes}}$, $H_{\mathbf{A}}^{\text{no}}$ (equivalently, the collections $\mathcal{C}_i^{\text{yes}}$ and $\mathcal{C}_i^{\text{no}}$ for all $i \in N$) are equivalent to the reduced matrix $\tilde{\mathbf{B}}$, and therefore contain the entire qualitative description of convergence.

Theorem 1. Consider an influence process $\mathbf{B}$ based on the aggregation functions $\mathbf{A}$. Then $\tilde{\mathbf{B}}$ can be reconstructed from the hypergraphs $H_{\mathbf{A}}^{\text{yes}}$ and $H_{\mathbf{A}}^{\text{no}}$ as follows: for any $S, T \in 2^N$, $\tilde{b}_{S,T} = 1$ if and only if

1. For each $i \in T$, there exists a nonempty $S_i' \subseteq S$ such that S_i' is yes-influential on *i*, i.e., $S_i' \in \mathcal{C}_i^{\text{yes}}$,

2. For each $i \notin T$, there exists a nonempty S_i'' such that $S_i'' \cap S = \varnothing$ and S_i'' is no-influential on *i*, i.e., $S_i'' \in \mathcal{C}_i^{\text{no}}$.

In particular, $\tilde{b}_{\varnothing,T} = 0$ for all $T \neq \varnothing$, $b_{\varnothing,\varnothing} = 1$, and $\tilde{b}_{N,T} = 0$ for all $T \neq N$, $b_{N,N} = 1$. Recall that (1) is valid only if the probabilities of saying yes are independent over the set of agents. Therefore, non-independence in this sense makes the determination of the transition matrix difficult. However, $\tilde{\mathbf{B}}$ is insensitive to possible correlation between agents, because $\tilde{b}_{S,T} = 1$ if and only if $A_i(1_S) > 0$ for every $i \in T$ and $A_i(1_S) < 1$ for every $i \notin T$, regardless of the correlation between agents.

3. Determination of the model

An important issue concerns the determination of a model of influence of the above type in a practical situation. This implies that we are making essentially two assumptions:

1. Each agent aggregates the opinion of all the other agents to form his opinion in the next step.
2. The aggregation function is monotonically increasing.

The latter assumption implies that anti-conformist behaviors (i.e., the more individuals say yes, the more I am inclined to say no) cannot be modeled in this framework.

3.1. General considerations

Complete determination of the model amounts to identifying either the transition matrix **B** or all the aggregation functions $A_1, ..., A_n$ (assuming the absence of correlation). Considering the size of the matrix $\mathbf{B}$ ($2^n \times 2^n$), statistical determination of $\mathbf{B}$ seems to be nearly impossible, unless a huge number of observations are made. As for the determination of the aggregation functions, the situation is even worse, since questioning agents about their aggregation functions (type, parameters) appears to be quite unrealistic. We know from Section 2 that knowledge of the reduced matrix $\tilde{\mathbf{B}}$ is enough to obtain a qualitative description of the convergence of the model, which is insensitive to possible correlations between agents. Moreover, knowledge of $\tilde{\mathbf{B}}$ (size 2^{2n}) is equivalent by Theorem 1 to knowledge of the collections of all yes- and no-influential coalitions of the size at most $2n\binom{n}{\lfloor \frac{n}{2} \rfloor}$, which is, in turn, equivalent to knowledge of the hypergraphs of yes- and no-influence. In some favorable cases (e.g., the WAM model), the hypergraphs reduce to ordinary graphs. This immediately indicates two ways of identifying the (qualitative part of the) model: either by observation of the transitions, i.e., the evolution of the coalition of the yes agents, or by interviewing the agents. In the first case, observing a transition from S to T yields $\tilde{b}_{S,T} = 1.$ In the second case, interviews permit us to determine influential coalitions or graphs of influence.

In the remaining part of this section, we mainly focus on the second approach. Concerning the first one, we only mention an important fact. The underlying assumptions of the model mean that the reduced matrix $\tilde{\mathbf{B}}$ is not arbitrary and has specific properties. Recall that $\tilde{b}_{S,T} = 1$ if and only if for all $i \in T$, $A_i(1_S) > 0$ and for all $i \notin T$, $A_i(1_S) < 1.$ This implies the following fact:

Fact 1. For a given $S \subseteq N$, $S \neq \varnothing, N,$ the candidates transitions are all sets of the form $T = K \cup L,$ where

$$K = \{i \in N \mid A_i(1_S) = 1\}$$

$$L \subseteq \{i \in N \mid 0 < A_i(1_S) < 1\}$$

In other words, $\bigcap T$, the intersection of all possible transitions from S yields the set $K = \{i \in N \mid A_i(1_S) = 1\},$ while $N \setminus \bigcup T$ yields $K' = \{i \in N \mid A_i(1_S) = 0\}.$ When S

increases, K increases, while K' decreases. This fact permits us to detect, when $\tilde{\mathbf{B}}$ is constructed from observations, possible deviations from the model (e.g., presence of anti-conformists).

3.2. Determination of influential coalitions

We may distinguish three cases, according to the type of underlying model:

1. WAM model (symmetric decomposable model): all aggregation functions are weighted arithmetic means.
2. OWA model (anonymous model): all aggregation functions are ordered weighted averages.
3. General model (no special assumptions).

The symmetric decomposable model. The case of the WAM model is particularly simple and has been studied in depth in [27]. It has been proved to be equivalent to a symmetric decomposable model. An aggregation model is decomposable if for every agent $i \in N$, every yes- and no-influential coalition for agent i is a singleton. Now, an aggregation model is symmetric if a yes-influential coalition for i is also no-influential for i and vice versa, for every $i \in N$. Note that the first property implies that the hypergraphs of yes- and no-influence reduce to ordinary graphs, while the second property implies that the two graphs are identical, and therefore the whole (qualitative) model is represented by a single graph representing influence. This makes interviewing agents particularly simple: it suffices to ask to every agent whom he asks for advice. Then, *i* asks *j* for advice is translated into the graph representing influence by an arc from *j* to *i*.

We applied this technique to a real case [27], namely the manager network of Krackhardt [34]. The agents are the 21 managers of a small manufacturing firm in the USA, and the network is obtained as follows: each agent *k* is asked if he/she thinks that agent *i* asks agent *j* for advice. An arc from *j* to *i* is placed in the graph representing influence if a majority of agents think that *i* asks *j* for advice. From the graph, and due to the properties of symmetric decomposable models, many conclusions can be easily drawn on the convergence of the model. In particular, it is possible to detect the presence of regular terminal classes (Theorem 8 in [27]). There is also a simple criterion to determine if there is no regular terminal class: it suffices that for each agent i, there is an arc in the influence graph from cl(i) to every agent outside cl(i), where cl(i), the closure of *i*, is the set of agents who can reach i by a path in the influence graph.

The anonymous model. According to the OWA model, agents do not change their opinion due to particular individuals but due to the number of individuals saying yes.

Therefore, in general, these are not decomposable models, and one needs to determine influential coalitions as in the general case. However, because according to these models the players are anonymous, a collection $\mathcal{C}_i^{\text{yes}}$ or $\mathcal{C}_i^{\text{no}}$ is composed of all sets of a given size s, $1 \le s \le n$, and this is characteristic of an anonymous model. Therefore, under the assumption of anonymity, it suffices to ask to every agent i the following questions:

Q1. Suppose that your opinion on some question is yes. What is the minimal number of agents saying no that may make you change your opinion?

Q2. Suppose that your opinion on some question is no. What is the minimal number of agents saying yes that may make you change your opinion?

Assuming that the answers are respectively s and s', it follows that

$$\mathcal{C}_i^{\text{no}} = \{S \in 2^N \,\|\, S \,|= s\}, \qquad \mathcal{C}_i^{\text{yes}} = \{S \in 2^N \,\|\, S \,|= s'\}$$

Now, it is easy to see that given s, s' for agent i, one can get the form of the weight vector w in the aggregation function OWA_w of agent i (Proposition 2 in [21]):

$$w = (\underbrace{00\cdots 0}_{s'-1 \text{ zeros}}, \bullet\bullet\cdots\bullet, \underbrace{00\cdots 0}_{s-1 \text{ zeros}})$$

where $\bullet$ indicates any nonzero weight. In particular, all agents are yes-influential (no--influential) on i if and only if $w_1 > 0$ ($w_n > 0$).

As for convergence under this model, it is shown in [21] (Proposition 3) that no cycle can occur but the two other types of terminal classes may occur. Terminal states are easily detected as follows: S of size s is a terminal state if and only if for every $i \in S$, the size of a no-influential coalition is at least $n - s + 1$, and for every $i \notin S$, the size of a yes-influential coalition is at least $s + 1$. The absence of regular terminal classes can also be characterized only through influential coalitions but this condition is more complex (see Corollary 3 in the aforementioned paper).

The general model. We now address the general case, where no special assumption is made on the model, except the following: we assume that each agent is yes- and no--influential on himself, which means that $A_i(1_i) > 0$, $A_i(1_{N\setminus i}) < 1$ (in other words, the agent trusts his opinion to a nonnull extent). This induces some simplification in the algorithm, but it would not be difficult to generalize it, in order to overcome this limitation.

Interview for Agent i

0. Set $\mathcal{C}_i^{\text{yes}} = \{\{i\}\}$, $\mathcal{C}_i^{\text{no}} = \{\{i\}\}$, $\mathcal{N}_i^{\text{yes}} = \mathcal{N}_i^{\text{no}} = 2^{N\setminus i}$

% $\mathcal{N}_i^{\text{yes}}, \mathcal{N}_i^{\text{no}}$ are the sets of candidate coalitions.

1. For each agent $j \in N$, $j \neq i$, ask[3]:

Q. Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent j says no?

If the answer is positive:

- add $\{j\}$ to $\mathcal{C}_i^{\text{no}}$, and discard $\{j\}$ and all sets containing j from $\mathcal{N}_i^{\text{no}}$.
- If $\mathcal{N}_i^{\text{no}} = \varnothing$, STOP (GO TO STEP 3).

Otherwise, discard $\{j\}$ from $\mathcal{N}_i^{\text{no}}$.

2. For $\ell = 2$ to $n-1$, do:

2.1. Define $\mathcal{S} = \{S \in \mathcal{N}_i^{\text{no}} : |S| = \ell\}$.

2.2. Ask Q: Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if one of the coalitions in $\mathcal{S}$ says no? In the case of affirmative answer, for which ones?

For every set S answered, do:

- add S to $\mathcal{C}_i^{\text{no}}$ and discard all supersets of S from $\mathcal{N}_i^{\text{no}}$
- If $\mathcal{N}_i^{\text{no}} = \varnothing$ or if $|\mathcal{C}_i^{\text{no}}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, STOP (GO TO STEP 3).

2.3. Set $\mathcal{N}_i^{\text{no}} \leftarrow \mathcal{N}_i^{\text{no}} \setminus \mathcal{S}$.

3. Exactly as in Steps 1 and 2 for $\mathcal{C}_i^{\text{yes}}$, Question 1 becomes: Suppose that your opinion on some question is no. Would you be inclined to change your opinion if Agent j says yes, etc.?

We give some examples.

Example 1. (braces are omitted when denoting coalitions) Consider $N = \{1, 2, 3, 4, 5\}$. We detail the process of interviewing Agent 1.

1. We have $\mathcal{C}_1^{\text{no}} = \{1\}$. We take agent 2.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent 2 says no?

Answer: Yes. Hence, $\mathcal{C}_1^{\text{no}} = \{1, 2\}$, and $\mathcal{N}_1^{\text{no}} = \{3, 4, 5, 34, 35, 45, 345\}$.

2. Agent 3.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent 3 says no?

[3]As in Step 2, it is possible to gather all these questions into a single one: Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if one of the agents in $N \setminus i$ says no? In the case of an affirmative answer, for which ones?

Answer: No. Thus $N_1^{no} = \{4, 5, 34, 35, 45, 345\}$.

3. Agent 4.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent 4 says no?

Answer: No. Thus $N_1^{no} = \{5, 34, 35, 45, 345\}$.

4. Agent 5.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if Agent 5 says no?

Answer: No. Thus $N_1^{no} = \{34, 35, 45, 345\}$.

5. Coalitions of size 2.

Suppose that your opinion on some question is yes. Would you be inclined to change your opinion if one of the coalitions in $\{34, 35, 45\}$ says no? In the case of an affirmative answer, for which ones?

Answer: Yes, 34. Thus $C_i^{no} = \{1, 2, 34\}$, and $N_1^{no} = \{35, 45\}$. It follows that $N_1^{no} = \varnothing$, since all coalitions of size 2 are discarded. STOP.

Finally, $C_1^{no} = \{1, 2, 34\}$. We do the same for C_1^{yes}.

1. For all individual agents.

Suppose that your opinion on some question is no. Would you be inclined to change your opinion if one of the agents 2, 3, 4, 5 says yes? In the case of an affirmative answer, for which ones?

Answer: No. Thus $N_1^{yes} = \{23, 24, 25, 34, 35, 45, 234, 235, 245, 345, 2345\}$.

2. Coalitions of size 2.

Suppose that your opinion on some question is no. Would you be inclined to change your opinion if one of the coalitions in $\{23, 24, 25, 34, 35, 45\}$ says yes? In the case of an affirmative answer, for which ones?

Answer: No. Thus $N_1^{yes} = \{234, 235, 245, 345, 2345\}$.

3. Coalitions of size 3.

Suppose that your opinion on some question is no. Would you be inclined to change your opinion if one of the coalitions in $\{234, 235, 245, 345\}$ says yes? In the case of an affirmative answer, for which ones?

Answer: Yes: 234, 235, 245. Thus $N_1^{yes} = \varnothing$, STOP.

Finally, $C_1^{yes} = \{1, 234, 235, 245, 345\}$.

We now give another example to illustrate how the reduced transition matrix $\tilde{\mathbf{B}}$ can be obtained from the influential coalitions using Theorem 1. To this end, we suppose that the above algorithm has been applied to each agent, in order to obtain all influential coalitions. The condition that every agent is self-influential permits us to simplify the application of the theorem to determine each term $\tilde{b}_{S,T}$. Indeed, the following facts are easy to show.

Fact 2. Suppose that $\{i\} \in \mathcal{C}_i^{\text{yes}}$ and $\{i\} \in \mathcal{C}_i^{\text{no}}$ for every $i \in N$. It follows that:

1. $\tilde{b}_{S,S} = 1$ for every $S \in 2^N$.

2. If $T \subseteq S$, condition (1) in Theorem 1 is always sufficient to check that $\tilde{b}_{S,T} = 1$, moreover, if $\tilde{b}_{S,T} = 0$, then $\tilde{b}_{S,T'} = 0$ for every $T' \subset T$. Similarly, if $\tilde{b}_{S,T} = 1$, then $\tilde{b}_{S,T'} = 1$ for every $T \subset T' \subseteq S$.

3. If $T \supseteq S$, condition (2) in Theorem 1 is always sufficient to check that $\tilde{b}_{S,T} = 1$. Moreover, if $\tilde{b}_{S,T} = 0$, then $\tilde{b}_{S,T'} = 0$ for every $T' \supset T$; similarly, if $\tilde{b}_{S,T} = 1$, then $\tilde{b}_{S,T'} = 1$ for every $S \subseteq T' \subset T$.

Example 2. Consider a society $N = \{1, 2, 3, 4\}$ of 4 agents. Suppose that the following collections have been obtained (braces are omitted when denoting coalitions):

$$\mathcal{C}_1^{\text{no}} = \{1, 2, 34\}, \qquad \mathcal{C}_1^{\text{yes}} = \{1, 234\}$$

$$\mathcal{C}_2^{\text{no}} = \{2, 34\}, \qquad \mathcal{C}_2^{\text{yes}} = \{2, 134\}$$

$$\mathcal{C}_3^{\text{no}} = \{2, 3\}, \qquad \mathcal{C}_3^{\text{yes}} = \{3, 12\}$$

$$\mathcal{C}_4^{\text{no}} = \{12, 4\}, \qquad \mathcal{C}_4^{\text{yes}} = \{4\}$$

Observe that agent 4 is stubborn when he supports yes (no influence is possible when agent 4 thinks yes).

Let us apply Theorem 1. Using Fact 2, one easily finds that $\tilde{b}_{S,T} = 1$ only for the following S, T (braces omitted):

$$S = 1: \qquad T = \varnothing, 1$$

$$S = 2: \qquad T = \varnothing, 2$$

$$S = 3: \qquad T = \varnothing, 3$$

$$S = 4: \qquad T = \varnothing, 4$$

$$S = 12: \qquad T = \emptyset, 1, 2, 3, 12, 13, 23, 123$$

$$S = 13: \qquad T = \emptyset, 1, 3, 13$$

$$S = 14: \qquad T = 4, 14$$

$$S = 23: \qquad T = 23$$

$$S = 24: \qquad T = 24$$

$$S = 34: \qquad T = \emptyset, 3, 4, 34$$

$$S = 123: \qquad T = 123$$

$$S = 124: \qquad T = 124, 1234$$

$$S = 134: \qquad T = 4, 14, 24, 34, 124, 134, 234, 1234$$

$$S = 234: \qquad T = 234, 1234.$$

We detail the case $S = 12$ for illustrative purposes. We can see from condition (2) of Theorem 1 that $T = \emptyset$ is possible (i.e., $\tilde{b}_{S,T} = 1$). Indeed, for $i = 1, 2, 3, 4,$ there exists a set in $\mathcal{C}_i^{\text{no}}$ which is disjoint from 12. Thus, by Fact 2.2, it follows that $T = 1, 2, 12$ are also possible. Now, for $T = 13, 23$ both conditions of Theorem 1 must be checked, while for $T = 123,$ only condition (1) has to be checked. Lastly, observe that all the remaining sets contain 4. Thus condition (1) of Theorem 1 is never satisfied, since there is no $S' \in \mathcal{C}_4^{\text{yes}}$ which is included in 12.

One can check that Fact 1 is satisfied. Observe that this approach is very useful to identify quickly all the possible *T*s: it suffices to find the minimal one (*K*) and the maximal one ($K \cup L$). The corresponding transition graph Γ is given in Fig. 1. It is seen that, apart from the trivial terminal classes, 23, 24 and 123 are terminal states. There is no regular nor cyclic class.

We now show that it is possible to get conclusions on convergence without computing $\tilde{\mathbf{B}}$, by solely examining the hypergraphs, thanks to results presented in [27]. To this end, we need the notion of an ingoing hyperarc. We say that a coalition S has an ingoing hyperarc (T', T'') in hypergraph H if $T' \subseteq N \setminus S$ and $T'' \subseteq S$ (and vice versa for an outgoing hyperarc).

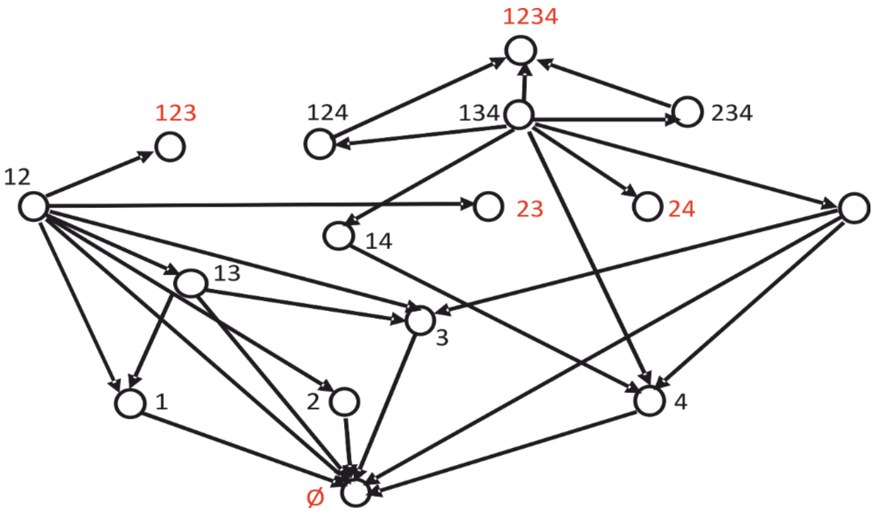

Fig. 1. Transition graph (loops are omitted)

Now, Theorem 3 in the aforementioned paper establishes that a nonempty $S \neq N$ is a terminal state if and only if S has no ingoing arc in the hypergraph $(\hat{H}_{\mathbf{A}}^{\text{yes}})^* \cup \hat{H}_{\mathbf{A}}^{\text{no}}$, where $()^*$ indicates that the hyperarcs have been inverted, and $\hat{H}$ indicates that only normal hyperarcs are considered[4]. This result can be translated in terms of influential collections as follows:

Fact 3. A nonempty $S \neq N$ is a terminal state if and only if

3.1. For every $i \notin S$, there is no $T \in C_i^{\text{yes}}$ such that $T \subseteq S$.

3.2. For every $i \in S$, there is no $T \in C_i^{\text{no}}$ such that $T \cap S = \varnothing$.

Applying this fact to Example 1, we indeed find that the only terminal states are 23, 24 and 123. For example, 23 is a terminal state, because none of 1, 234 are subsets of 23 (condition (1) for $i = 1$), 4 is not a subset of 12 (condition (1) for $i = 4$), none of 2, 34 are disjoint from 23 (condition (2) for $i = 2$), and none of 2, 3 are disjoint from 23 (condition (2) for $i = 3$).

The advantage of Fact 3 is that it is not necessary to find all S, T such that $\tilde{b}_{S,T} = 1$ (i.e., it is not necessary to know the transition graph) to check whether a given coalition is a terminal state (or to find all of them).

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[4] A hyperarc (T', T'') is normal if $T' \cap T'' = \varnothing$. Note that due to our assumption that every player is self-influential, all hyperarcs are normal.

4. Concluding remarks

We have shown how, in a practical situation, one can determine a model of influence based on aggregation functions. Exact determination of such a model, yielding the type and parameters of the aggregation function of each agent, appears to be out of reach without using complex procedures. What we show is that, on the contrary, it is easy to obtain the qualitative part of the model, which permits a full qualitative analysis of the convergence of opinions, that is, to determine all terminal classes. This is sufficient to predict whether a consensus will occur or, on the contrary, society will become polarized, or a cycle will appear, etc. Simple criteria are available to detect terminal states or the presence of regular terminal classes, without even determining the reduced transition matrix. We believe that this study will make the use of models of influence based on aggregation functions more familiar and easier to use.

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ON TYPES OF RESPONSIVENESS IN THE THEORY OF VOTING

In mathematics, monotonicity is used to denote the nature of the connection between variables. Hence for example, a variable is said to be a monotonically increasing function of another variable if an increase in the value of the latter is always associated with an increase in the other variable. In the theory of voting and the measurement of a priori voting power one encounters, not one, but several concepts that are closely related to the mathematical notion of monotonicity. We deal with such notions focusing particularly on their role in capturing key aspects of plausible opinion aggregation. Further, we outline approaches to analyzing the relationship of opinion aggregation and voting power and thereby contribute to our understanding of major components that determine the outcome of voting.

Keywords: *plausible opinion aggregation, monotonicity, no-show paradox*

1. Introduction

Rule of the people, by the people and for the people was the expressed goal of institutional design of Abraham Lincoln. But how can we determine that this goal has been achieved? It would seem that the first aspect – rule of the people – has been achieved when an overwhelming majority of the population accepts the decisions of the rulers as binding. The second aspect – the rule by the people – can be regarded as achieved when the views of the people are reflected in the decisions of the rulers. The third aspect, in turn, refers to the degree to which the decisions of the rulers serve the interest of the

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people. Clearly, all three aspects deal with matters of degree and are thus inherently vague. This is particularly true of the second and third aspects. Our focus in this paper is on the second aspect: to what extent are the views of the population reflected in the decisions that are regarded as collectively binding? We shall gloss over several things that play a role in political decision making in modern democracies such as the fact that binding decisions are typically made by elected representatives rather than the population at large. Our primary interest is more theoretical, viz. how to ascertain that the views of the voters are reasonably well reflected in the decisions made by the voters themselves, i.e. without mediation of representatives. What kind of correspondence between collective decisions and voter opinions would be desirable? What is the role of voting power in the selection of the outcome and how do we measure such power?

Obviously, different choice making rules establish somewhat different correspondences between voter opinions and collective decisions. Of the desiderata one could impose on collective choice rules those pertaining to responsiveness seem particularly important: collective decisions should be responsive to changes in voter opinions. To put it in another way: unresponsive rules cannot be viewed as compatible with the government by the people or for the people, for that matter. As examples of obviously unresponsive rules one could mention dictatorial rules as well as "trivial ones". The former always result in outcomes preferred by one individual, regardless of the opinions of others, while trivial ones exclude some outcomes, no matter what the distribution of opinions of the voters might be. Apart from these kinds of completely unresponsive rules, there are others that respond to changes in voter preferences in counterintuitive ways. For example, a rule might specify as the collective choice an alternative that is ranked worst by the largest number of voters. In what follows, we shall discuss some properties that capture various aspects of responsiveness. It turns out that there are, indeed, several such notions. Responsiveness can also be called into question if an increase in vote shares results in a decrease of a priori voting power. We discuss measures of voting power that indicate such results. Our contribution is thus of mainly a conceptual and theoretical nature. We aim to provide conceptual clarification to issues pertaining to how decision making systems respond to the opinions of their members. Related issues have been discussed in depth and with nice results by Gambarelli (see e.g., [20–22].)

This article is organized as follows. The next section deals with the principal notion of the responsiveness of voting procedures, viz. monotonicity, i.e. the requirement that additional support does not render winners into non-winners. Section 3 discusses a related desideratum, viz. that of invulnerability to the no-show paradox. This paradox occurs whenever a group of individuals is better off by not voting at all than by voting according to their preferences, *ceteris paribus*. Thereafter a particularly striking example of the no-show paradox is touched upon: the P-BOT paradox. Section 5 discusses a notion that is often confused with the concept of monotonicity just defined, viz. that of Maskin monotonicity. The latter has a crucial role in the theory of mechanism design, but is largely absent in the theory of voting. We then move on to the monotonicity of measures of voting power, a topic that has been of particular interest to Gianfranco Gambarelli. We also discuss how monotonicity can be achieved and, finally, suggest some future lines of research.

2. Monotonicity in opinion aggregation

Informally stated, monotonicity says that additional support, *ceteris paribus*, never turns winners into non-winners. The *ceteris paribus* clause is very important here. Slightly more formally, let $R = (R_1, ..., R_n)$ be an n-person profile of complete and transitive preference relations. Denote by P_i the asymmetric part of the relation. We consider a preference profile $R = (R_1, ..., R_n)$, a set of alternatives A and x in A so that $F(A, R) = x$, i.e. rule F when applied to profile R over the set of alternatives A results in x being elected. Consider now any $R' = (R_1', ..., R_n')$ such that for all y, z in A: $(x \neq y, x \neq z)$ and for all i in N: yR_iz if and only if $yR'z$, while for at least one i in N and w in A: wP_ix but $xP_i'w$. Now, F is monotonic if under the preceding conditions $F(A, R') = x$.

Many collective choice rules are, indeed, monotonic, for example, the plurality rule which gives each voter one vote. The winner is the alternative that has been voted for by more voters than any other alternative. Suppose that this system is implemented so that each voter submits his/her full ranking of alternatives and the system then singles out the first ranked ones, whereupon the alternative ranked first by more voters than any other is declared the winner. It is easy to see that improving the winner's position in at least one voter's ranking without making any other changes in the preferences cannot make it a non-winner in the changed profile, provided that the same rule is applied. Improving a winning alternative's position *vis-a-vis* some other alternative cannot bring about a new winner, since after such a change the former winner has at least the same number of first positions as before, while all other alternatives have at most the same number of first positions as before the change.

Even though monotonicity is an intuitively quite plausible property, there are relatively commonly used rules that do not satisfy it. To wit, plurality runoff and Hare's system are well-known and widely used methods that are non-monotonic (see e.g., [42]). Typically, non-monotonic methods involve several stages of computing the result. For example, according to plurality runoff one first examines whether there is a candidate that is ranked first by more than half of the voters. If there is, then this candidate is the winner. Otherwise, a real or fictitious second round is conducted. In this round, only the two front-runners from the first round are presented, whereupon the one that gets more votes is elected. According to Hare's system there are potentially several rounds of restricting the set of candidates, but the criterion of winning is basically the same as in plurality runoff: the winner has the support of at least half of the electorate. Another concept that pertains to responsiveness of invulnerability to the no-show paradox.

3. Invulnerability to the no-show paradox

A no-show paradox occurs whenever an alternative, say *x*, is elected in a given profile, but *y* is elected if the original profile is enlarged by a group of voters that prefer *x* to y, *ceteris paribus*. So, in essence this paradox occurs when a group of identically-minded voters is better off by abstaining than by voting according to their preferences. The change in outcomes may be large or small, it may also pertain to outcomes that are ranked lowly in the abstainers' preferences. The essential point is that abstaining, *ceteris paribus*, brings about a better outcome for the abstainers than voting according to their preferences.

This is the now established view of the no-show paradox. In fact, the original notion of the no-show paradox, as expressed by Fishburn and Brams, says something quite different: *The addition of identical ballots with candidate x ranked last may change the winner from some other candidate to x* [17]. In this original version, the no-show paradox pertains to situations where, in order to avoid the worst outcome, a voter is better off by not voting at all. This original notion of the no-show paradox is similar in spirit to the currently established view, but different in focusing specifically on avoiding the worst possible outcome rather than improving the outcome in more general terms. It seems that the now established definition appeared for the first time in [39] and from there spread throughout the social choice community. We shall here adopt the established view.

Table 1. The no-show paradox and plurality runoff

26 voters	47 voters	2 voters	25voters
A	B	B	C
B	C	C	A
C	A	A	B

Table 1 illustrates this paradox in the context of the plurality runoff system. We have three alternatives (A, B, C) and 100 voters. The latter are divided into four groups of voters, each with identical preferences over the alternatives. The preferences of each group are indicated by listing the alternatives in the order of preference from top to bottom. Thus, the group consisting of 26 voters has A as their first, B as their second and C as their lowest-ranked alternative. Assuming that all voters reveal their preferences in voting, there will be a runoff between A and B, whereupon A wins in the second round. Suppose now that the group of 47 voters indicated in the second column decide not to vote at all, *ceteris paribus*. It follows that there will be a runoff between A and C in which the latter defeats the former, i.e. C wins. The outcome is thus better for the abstainers than the one resulting from their voting according to their preferences[4].

[4]Of course, the 49 voters in the two middle columns may force the outcome C by voting strategically for C in the first round or – better yet – by convincing the A-supporters to vote for B in the first round

A more dramatic and at the same time more important version of the no-show paradox occurs when the outcome resulting from abstaining – again *ceteris paribus* – is ranked first in the preferences of the abstainers. This version is known as the strong no--show paradox [46]. Table 2 demonstrates that Black's procedure may lead to the strong version of the no-show paradox. Black's procedure is a hybrid of Condorcet's winner criterion and the Borda count: if a Condorcet winner exists, it will be the winner, otherwise the Borda winner is elected. In Table 2, alternative D is the Condorcet winner and is, thus, elected by Black. However, if the right-most voter abstains, there is no Condorcet winner any longer. Thus, the Borda winner E becomes the Black winner. E is the first-ranked alternative of the abstainer [44][5]. Hence, by abstaining this voter brings about his first-ranked alternative, while by voting according to his preferences (i.e. sincere voting) he brings about a worse outcome for himself[6].

Under specific circumstances, vulnerability to the strong no-show paradox confronts a voter with a rather bizarre choice: either to reveal his preferences and settle for an outcome that is not his best or to abstain in order to secure his best outcome. One could, with some justification, argue that procedures vulnerable to the strong paradox fly in the face of the basic rationale of elections: to disclose popular opinion about the alternatives.

Table 2. Black's procedure
and the strong no-show paradox

1 voter	1 voter	1 voter	1 voter	1 voter
D	E	C	D	E
E	A	D	E	B
A	C	E	B	A
B	B	A	C	D
C	D	B	A	C

The best-known results on no-show paradoxes are due to Moulin [39] and Pérez [45, 46]. Moulin proved a theorem stating the incompatibility of the Condorcet principle and invulnerability to the no-show paradox. The Condorcet principle states that if there is a Condorcet winner alternative in a profile, then this alternative, and only this one,

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whereupon B would win. It is also conceivable that the A-supporters may convince the C-supporters (all of them) to vote for A in the first round to avoid B being elected, and so on. To succeed all these stratagems require complete information about the profile and perfect coordination of balloting. It should be emphasized, though, that the no-show paradox is not about strategic voting, but about bizarre voting outcomes under the assumption that the voters do what the voting systems expect them to do: reveal their opinions about the alternatives at hand.

[5]A much earlier report, [47], provides a similar example showing that Black's procedure violates a criterion called voter adaptability. This criterion is almost the same as invulnerability to no-show paradox.

[6]For a comprehensive overview of similar paradoxes, see [13].

should be elected. Rules that always satisfy the Condorcet principle are known as Condorcet extensions. Pérez subsequently strengthened Moulin's result by showing that, apart from a couple of exceptions, all Condorcet extensions are vulnerable to the strong no-show paradox. The exceptions are the maximin rule and Young's procedure. So, examples such as Table 2 can be found for nearly all rules that satisfy the Condorcet principle.

The twin paradox is Moulin's finding [39]. It is similar in spirit to the no-show paradox and turns out to characterize largely the same class of rules. Denote by N a set of n voters. Suppose that in a given profile, the rule F results in alternative x being chosen. Let us now add an $(n + 1)$th voter j with an identical ranking over the alternatives as that of some voter, say i in N. Denote the augmented profile of $n + 1$ voters by N^+. The twin paradox occurs if the choice resulting from applying F to N^+ is y and, moreover, x is strictly preferred to y by voters i and j. In other words, the twin paradox occurs when a voter (or a group with identical preference rankings) is better off with a smaller number of identically-minded partners than with a larger number of them.

Table 3. Nanson's rule and the twin paradox

5 voters	5 voters	6 voters	1 voter	2 voters
A	B	C	C	C
B	C	A	B	B
D	D	D	A	D
C	A	B	D	A

In Table 3, Nanson's rule results in B [44]. Here we have an instance of the twin paradox: if the right-most group of 2 voters lost one of its members, C would win. If this voter returns, B wins. Clearly the twin is not welcome by the remaining member of the right-most group. In fact, there would be an incentive for this group to coordinate so that only one of the members vote in order to secure the best possible outcome for the group.

4. Invulnerability to P-BOT paradoxes

To clarify the distinctions between various types of voting paradoxes, Felsenthal and Tideman introduce the concepts of P-TOP and P-BOT paradoxes [16]. The former class consists of those that have been called strong no-show paradoxes above. To quote the authors: *According to this paradox ... if a candidate, say candidate x, has been elected initially, then it is possible that another candidate, y, will be elected if, ceteris paribus, additional voters whose top-ranked candidate is x join the electorate*.

The latter class exhibits equally, if not more, bizarre behavior. To wit, an instance of the P-BOT paradox occurs *if one of the candidates, say candidate c, who has not been elected originally, may be elected if, ceteris paribus, the electorate is increased as a result of additional voters whose bottom-ranked candidate is c join the electorate* … [16]. Systems vulnerable to P-BOT paradoxes present a voter with an obvious dilemma: should one express one's preferences by voting and risk helping one's lowest-ranked candidate to be elected, or should one abstain thereby contributing to the election of some more "tolerable" candidate. It should be added, though, that vulnerability to a paradox does not mean that this kind of dilemma would be faced by a voter in all or even in a majority of elections. Nonetheless the occurrence of a P-BOT paradox is certainly an unpleasant surprise for a group that joins the original electorate only to find that its lowest-ranked candidate got elected as a result of the group's activity.

Vulnerability to the P-BOT paradox is in fact quite common among voting procedures [15]. Table 4 illustrates P-BOT paradox in terms of Kemeny's method. This method – it will be recalled – decomposes any preference profile into a set of pairwise rankings of alternatives. These are the compared with the pairwise rankings of each logically possible strict preference ranking. Thus, e.g., if the alternatives are A, B and C, one of the possible rankings, viz. $A \succ B \succ$ C, is decomposed into the following ordered pairs: (A≻B), (A ≻ C), (B ≻ C). One then counts the support in the profile for each of these pairwise rankings, support here meaning the number of voters having each particular ordered pair in their preference relations. Finally, the Kemeny winner is the ranking that has the largest support in the profile.

Table 4. Kemeny's method
is vulnerable to the P-BOT paradox

5 voters	3 voters	3 voters
D	A	A
B	D	D
C	C	B
A	B	C

In the profile illustrated by Table 4, A is the strong Condorcet winner and – since Kemeny's method is a Condorcet extension – is the Kemeny winner as well. Suppose now that a group of 4 voters with the unanimous ranking B ≻ C ≻ A ≻ D joins the electorate. Computing the support for all 24 rankings yields D ≻ B ≻ C ≻A as the Kemeny ranking in the expanded profile. Thus, A was the winner in the original 11 voter profile, but D – the lowest-ranked alternative of the 4 new entrants – becomes the winner in the expanded one.

5. Maskin monotonicity

In the preceding definitions, the *ceteris paribus* clause is to be taken seriously. The importance of this statement becomes obvious when one turns to another concept of monotonicity that plays a prominent role in mechanism design, viz. Maskin monotonicity [38]. It is similar to the concept of monotonicity outlined above, but dispenses altogether with the *ceteris paribus* clause. More specifically Maskin monotonicity is the following property of a procedure. Assume that alternative x wins in profile Q over the set of alternatives A. Also suppose that profile Q' is formed over A so that for any alternative y in A, x is preferred to y by at least as many voters in Q' as in Q. Maskin monotonicity requires that x is still chosen in Q'. The important point here is that no restrictions are imposed on those pairs of alternatives in Q' that do not include x. It is easy to show that Maskin monotonicity is not satisfied by typical voting procedures [43]. Consider the plurality procedure and Table 5 [43].

Table 5. The plurality rule
is not Maskin monotonic

2 voters	1 voter	1 voter	1 voter
A	B	C	D
B	C	B	C
C	A	A	B
D	D	D	A

Call the profile illustrated by Table 5 profile Q. Clearly alternative A wins in Q. Suppose now that the profile is transformed into Q' so that A is ranked at least as high by all voters and strictly higher by at least one voter. To wit, in Q' we keep the rankings of all alternatives the same for the two left-most voters who rank A first, we lift A ahead of C in the next ranking of P', we interchange the ranks of B and C for the next voter in Q' and lift B ahead of both C and D in the right-most ranking in Q′. These changes keep the position of A the same or higher in Q' than in Q for all voters. However, as a result of these changes, A is no longer the plurality winner in Q'; it is B. Hence the plurality rule does not satisfy Maskin monotonicity.

Obviously Maskin monotonicity is a very demanding property. The explanation is simple: by excluding *ceteris paribus* clauses it allows all kinds of transformations that include the preference rankings of other alternatives than the original winner. In a way the fact that voting systems do not satisfy Maskin monotonicity is another way of saying that they do not satisfy the independence of irrelevant alternatives from Arrow's theorem: whether x is collectively preferred to y is typically dependent, not just on their mutual rankings, but also on how they are related to other alternatives. The relationship of Maskin monotonicity with independence of irrelevant alternatives and other properties is exhaustively covered by [1] and [2].

6. Is voting power monotonic?

The paradoxical results above are the result of combining voting rules with specific distributions of voting weights and particular preferences of the voters. It is not always obvious which of the three ingredients produces results which are considered paradoxical because of some violation of monotonicity. Intuition says that all three are responsible for such paradoxes. However, paradoxes of non-monotonicity also appear when we analyze the game form of voting games, i.e., if we abstract from preferences. Power indices are standard instruments used to illustrate these issues and discuss their implications and properties.

There are two, quite different, models of monotonicity relevant in applying power indices and discriminating between various measures[7]. One discusses the relationship between a given vote distribution $w = (w_1, ..., w_n)$ and a power distribution $\pi = (\pi_1, ..., \pi_n)$. If $w_i < w_j$ implies $\pi_i \leq \pi_j$, for all w and π, then the power measure is monotonic, i.e., satisfies local monotonicity (LM). The other concept of monotonicity compares two vote distributions w and w' and asks whether $\pi_i \leq \pi'_i$ if $w_i < w'_i$ holds for all voters. The distribution of w' can be seen to represent *changes* compared to w. There is a series of paradoxes, including the paradox of redistribution which demonstrates that all the standard power indices, i.e., Shapley–Shubik index, Banzhaf index, etc., fail to satisfy this *comparative monotonicity* requirement. However, the Shapley–Shubik index and the Banzhaf index satisfy LM. On the other hand, power indices based on minimum winning coalitions, the Deegan–Packel index and the public good index (PGI), violate both LM and comparative monotonicity. Do we have to conclude that voting power is not monotonically responsive to changes in the vote distribution? Of course, the answer depends on the power measure we choose to apply.

The Shapley–Shubik index satisfies global monotonicity. The (normalized) Banzhaf index does not, as demonstrated by [48]. Global monotonicity compares two voting games $v = (d; w)$ and $v' = (d; w')$, where d represents the decision rule, and requires for each pair of voters i and j in N that $\pi_i \leq \pi'_i$ if $w_i < w'_i$ and $w_j > w'_j$, given that all the other voting weights in w and w' are identical.

[7]It has been suggested to speak of an "index" if the values of a power measure add up to one. We will not follow this convention. However, the values of the measures discussed in this paper add up to one – either through standardization or by their very nature. The latter holds for the Shapley–Shubik index.

A notorious example to illustrate that PGI and the Deegan–Packel index violate LM is the voting game $v^\circ = (51; 35, 20, 15, 15, 15)$. The corresponding PGI is $h(v^\circ) = (4/15, 2/15, 3/15,3/15,3/15)$, while the corresponding Deegan–Packel index is $\rho(v^\circ) = (18/60, 9/60, 11/60, 11/60, 11/60)$. In both cases, the second group of voters has less a priori voting power but more votes than the groups with a smaller number of votes.

In what follows, we restrict ourselves to weighted voting games, like v° above, fully described by the decision rule ("quota") d and the vote distribution w. This corresponds to the presentation of the voting paradoxes above. Note, alternatively, that voting games can also be described by their sets of minimum winning coalitions (MWC). A winning coalition S is a MWC if $S\backslash\{i\}$ is a losing one, for all $i \in S$. Obviously, all the players of a MWC coalition have a swing position, i.e., they are decisive (also called crucial) to this coalition winning. The PGI of player i, h_i, is the number of MWCs that have i as a member divided by the number of all the swing positions the players have in all MWCs of the game. If m_i is the number of MWCs that have i as a member then i's PGI value is

$$h_i = \frac{m_i}{\sum_{i \in N} m_i} \tag{1}$$

The underlying assumption is that collective decisions are about public goods and, in principle, non-exclusion and non-rivalry in consumption apply. Everybody has access to enjoy the public good or to suffer from it. The members of a MWC are decisive for a particular public good: they get what they want – otherwise they would not vote for the particular good (e.g., a "policy") that the coalition represents. Others either free-ride or suffer – in any case, they have no power in relation to this particular good[8].

The PGI has been identified with solidarity (within the members of a MWC)[9], and yet the values h_i look like shares. However, this is due to standardization: their sum is 1 (for a non-standardized version of the PGI, called the public value, see [26]).

The Deegan–Packel index considers the number of players (e.g., "parties") a MWC coalition has and divides the value of the coalition S, standardized as $v(S) = 1$, by the cardinality $|S| = s$[10]. This measure considers the value of a coalition to be a private good that is equally shared among the members of a coalition which is a minimum winning one, in order to keep shares large. In general, this does not correspond to the aggregation of preferences and the determination of a collective outcome. Therefore, this index seems to be of minor interest for the discussion in this paper.

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[8]See [23]. For an axiomatization of the PGI, see [34] and [40]. [9] provides a valuable summary of the PGI literature and the history of this measure.

[9]See [19].

[10]For details, see [12]. For a discussion of this measure, taking a priori unions into account, see [5].

Given that N is the set of all players of game v, player i is a swing player if $S \subset N$ is a winning coalition and $S \setminus \{i\}$ is a losing coalition. The normalized Banzhaf index of player i counts the number of coalitions S that have i as a swing player. In Eq. (2), this number is c_i. For normalization this number is divided by the total number of swing positions that characterize the game v. A formal definition of the normalized Banzhaf index is

$$\beta_i = \frac{c_i}{\sum_{i \in N} c_i} \tag{2}$$

Felsenthal and Machover [14] conjecture that this measure represents *I*-power, capturing the impact of player i on the voting outcome (see [51] for a critical discussion). It should be relevant for the discussion of voting paradoxes.

With explicit reference to PGI, Bertini et al. [9] introduced the public help index (PHI) which has the same mathematical structure as Eqs. (1) and (2). However, it is based on the number of winning coalitions which have player i as a member, irrespective of whether i has a swing in a particular coalition S or not. This measure satisfies LM and global monotonicity. However, does it measure power? It does not refer to swing players, but to membership. Even dummy players get assigned values. Yet, a dummy player i in winning coalition S cannot reject a particular public good, generated by coalition S, if i does not like this good, but even suffers from it. A dummy player i can neither "exclude" him/herself, as the coalition offers a public good, nor can i veto a public good if it is "a bad" to him or her. In the case of PGI, those who are members of a MWC can prevent the availability of the corresponding good simply by leaving the coalition or by voting "no".

The PHI invites us to consider preferences which decide whether a public good is "privately" good or bad. It measures the potential for success which implies making a comparison of outcome and preferences. The latter is the approach proposed by this paper.

We will not go into the details of the definition of the Shapley–Shubik index here but point out that it is based on counting swing players, without restricting to MWCs, and permutations, i.e., orderings of players. There is an immediate interpretation of a permutation as expressing the ideological closeness of the players. This invites us to consider preferences in addition to power relations defined by winning coalitions.

The application of power indices is motivated by the widely shared "hypothesis" that the vote distribution is a poor proxy for a priori voting power. If this is the case, does it make sense to evaluate a power measure by means of a property that refers to the vote distribution as suggested by LM? Of course, our intuition supports LM. However, if we could trust our intuition, do we need these highly sophisticated power measures at all?[11] If we take the vote distribution w as a proxy for the power distribution then we have no problems with the concepts of monotonicity defined above. However, in the case of the voting game $v = (51; 49, 48, 3)$, this could be difficult to justify with respect to both political experiences and data.

7. How to achieve monotonicity?

Holler and Napel argued that the PGI shows non-monotonicity with respect to the vote distribution (and this confirms that the measure does not satisfy LM) if the game is not decisive and no winning coalition may exist, as in the above weighted voting game $v° = (51; 35, 20, 15, 15, 15)$, or if it improper, i.e., two winning coalition may exist the same time [27, 28]. However, there are voting games that are proper and decisive and still LM is not satisfied for the PGI. And yet, a violation of LM suggests that perhaps we should worry about the design of the decision situation. The fact that the PGI shows such violations, while other power measures do not, could be considered an asset and not a pathology. But what is the reason for PGI violating LM? How can we cure a violation of LM?

The more popular power measures, i.e., the Shapley–Shubik index or the Banzhaf index satisfy LM and thus do not indicate any particularity if the game is neither decisive nor proper. Yet, these measures also show a violation of LM if we consider a priori unions and the equal probability of permutations and coalitions, respectively, does no longer apply[12]. This suggests that a deviation of the equal probability of coalitions causes a violation of LM. Note since the PGI considers MWC only, this is formally equivalent to put a zero weight on coalitions that have surplus players. Is this the ("technical") reason why the PGI may show non-monotonicity?

Instead of accepting the violation of LM, we may ask which decision situations guarantee monotonic results for the PGI. An answer to this question may help to design adequate voting bodies. Obviously, the PGI satisfies LM for unanimity games, dictator games and symmetric games. The latter are games that give equal power to each voter; in fact, unanimity games are a subset of symmetric games. Note that for these types of games the PGI is identical with the normalized Banzhaf index.

Holler et al. analyse alternative constraints on the number of players and other properties of the decision situations [33]. For example, it is obvious that LM will not be violated by any of the known power measures, including PGI, if there are n voters and $n - 2$ voters are dummies. It is, however, less obvious that LM is also satisfied for the

[11]See [25] and [29] for this argument.

[12]See [3] for examples of voting games with a priori unions and [6] as well as [5] for a discussion. Closely related are voting games with incompatible players, see [5] for the PGI and [7] for Banzhaf index.

PGI if one constrains the set of games so that there are only $n - 4$ dummies, i.e., voting games with 4 players or less satisfy LM.

This illustrates the approach that has been labelled constrained monotonicity [33]. We can think of many set of constraints that guarantee LM for PGI. A hypothesis that needs further research is that the PGI does not show a violation of LM if the voting game is decisive and proper and the number of decision makers is lower than 6. The idea of restricting the set of games such that LM applies for PGI has been further elaborated in [4] in the form of weighted monotonicity of power. The core of this concept is to give the various MWC weights such that the modified PGI satisfies monotonicity. Then we get PGI-monotonicity. These considerations are relevant for all power indices if we drop the equal probability assumption and, for example, take the possibility of a priori unions into account.

Following the approach chosen in [33], Freixas and Kurz [19] distil a subset of (weighted) voting games which guarantee LM for the PGI and the Deegan–Packel index. They show that LM is satisfied for this measure if the voting game is uniform, i.e., if all MWC have the same number of members. This is a sufficient condition, but it is not necessary as Freixas and Kurz demonstrate using the game example $v^* = (3; 2, 1, 1, 1)$. This game is not uniform but satisfies LM (the game is proper and decisive). There are three MWC of two members and one with three members. The PGI values are $h(v^*) = (3/9, 2/9, 2/9, 2/9)$. They are identical to the values of the normalized Banzhaf index for this voting game. As already mentioned, a core result of [33] is that the PGI satisfies LM is the number of voters is four or less – so uniformity is not needed to guarantee LM for voting game v^*.

Widgrén [51] proved the following linear relationship that relates the normalized Banzhaf index and the PGI[13].

$$\beta_i = (1 - \gamma)h_i + \gamma\varepsilon_i \tag{3}$$

where

$$\varepsilon_i = \frac{\bar{c}_i}{\sum_{i \in N} \bar{c}_i} \qquad \text{and} \qquad \gamma = \frac{\sum_{i \in N} \bar{c}_i}{\sum_{i \in N} c_i}$$

Here, c_i represents the number of coalitions that contain player i as a swing player and $\bar{c}_i$ represents the number of coalitions which have a swing player i but are not minimum winning. Loosely speaking, the coalitions represented by $\bar{c}_i$ are the source of the

[13]Widgrén [51] uses the symbols θ_i for the PGI and C_i for the set of coalitions that contain i as a swing player. Correspondingly, c_i is the number of elements of C_i.

difference between the normalized Banzhaf index, β_i, and the PGI, h_i. Can we identify the corresponding factors in Eq. (3) as the cause for the violation of LM that characterizes the PGI, but not the Banzhaf? Can we see from the properties of this factor whether the PGI will indicate a violation for a particular game, or not? These questions have not been answered so far, but it is immediate from Eq. (3) that the PGI satisfies LM for unanimity games, dictator games and symmetric games. For these games $\gamma = 0$ and the PGI equals the normalized Banzhaf index (which satisfies LM for all voting games).

Freixas and Kurz introduce a class of new indices that result out of convex combinations of PGI and Banzhaf index choosing weights such that LM is satisfied [19]. The resulting indices are more solidary than the Banzhaf index and perhaps a better *yardstick for doing a fair division of a public good*. It turns out that there are *costs* [of obtaining LM inasmuch as] *the achievable new indices satisfying LM are closer to the Banzhaf index than to the Public Good Index* [19]. If we reconsider that LM is implied by the dominance relation and thus by the desirability relation, proposed in [18] to define *reasonable power measures*, then it seems that there is not much space to take care of solidarity. However, this discussion needs, first of all, definitions of solidarity, fairness, etc. Still, the class of new indices, elaborated in [19] prepares the ground and raises relevant questions.

The approach of [19] is somewhat related to project of strict proportional representation through randomizing over several decision rules applied to a particular vote distribution such that the weighted power indices are identical to the seat distribution (see [24, 8, 49] for this approach[14]). Then, LM is of course guaranteed. However, in this exercise the convex combination implied by the randomization was over indices of one kind, only. Combinations of indices of different proveniences were not considered and no indication of more or less solidarity could be derived.

We conclude this section with two questions that we could not answer to our satisfaction: How is the Freixas–Kurz model [19] of combining Banzhaf index and PGI related to Widgrén's Eq. (3) above? Do the weights, proposed by Freixas and Kurz, specify PGI-monotonicity?

8. Non-monotonicity of voting and of voting power. A research project

If we abstract from preferences and focus on a priori voting power in the case of the no-show paradox, illustrated in Table 1, then we see that the power of the group of 47 voters carries over to the group of 2 voters. The latter has no power if the 47 voters

[14]Related, [35] presents an "average representation of games" such that the resulting power distributions are proportional to the distribution of weights. Again, LM is guaranteed.

vote while the group of 47 voters have a power of 1/3 if measured by the standard power indices. However, if the 47 voters abstain then the power of the group of 2 voters is 1/3 while the power of the group of 47 voters is zero. Since the members of the group of 2 voters have identical preferences as the members of the group of 47 voters, the group of 2 voters is an adequate representation of the group of 47 voters.

Of course, we get the same result, including the no-show paradox, as long as the active representatives of the preference ordering B ≻ C ≻A are not numerous enough to make B an alternative in the second round competing with A. Given the votes and preferences for the groups of 26 voters and 25 voters, this is the case when the active representatives of the preferences have less than 25 votes. On the other hand, the active representatives have to number more than 1 vote so that alternative C gets a majority of votes in the second round when competing with A. Thus Table 1 represents an extreme case highlighting a paradoxical effect of no-show.

There is a non-monotonicity in this example that relates votes, power and preferences. By forgoing the casting of votes and thus reducing the voting power to zero, the group of 47 voters gets a better outcome, captured by the winning alternative, than by making use of its votes and therefore exercising voting power. However, abstaining is also making use of voting power as obviously it has an impact on the outcome and corresponds to the agents will[15]. Note that in the discussion of the paradoxes in Chapters 2–4, coalition formation was not considered: sincere voting was assumed and voters voted according to their preferences. However, the no-show paradox proposes that all the members of a group of voters abstain, i.e., behave like a bloc. Moreover, the paradox relies on that the members of another group, characterized by the same preference ordering as the abstaining group, vote. The coordination between the two groups is not discussed. It seems implicit that the two groups behave like a coalition, splitting votes, in order to avoid there least preferred alternative.

However, if we generalize this assumption then we enter the realm of power indices. If so, we can discuss whether the no-show paradox corresponds to the paradox of the quarreling members. This paradox is based on the assumption that coalitions are impossible that contain members that have a "quarrel". The paradox prevails if the sum of power values of two quarreling members X and Y is larger than the power value of the union of X and Y, i.e., $\pi(X) + \pi(Y) > \pi(X \cup Y)$[16]. This result can be observed for any standard measure of voting power.

But are we allowed to add up the power values? If the preferences of A and B are identical, this seems to be justifiable – as in the case of the groups of 47 voters and 2 voters in preference profile given in Table 1. We conclude: The potential to coordinate

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[15]This relates to Max Weber's definition of power: *In general, we understand by "power" the chance of a man or of a number of men to realize their own will in a communal action even against the resistance of others who are participating in the action* [50].

[16]For the paradox of quarreling members, see [36] and [11].

votes within a bloc generates extra voting power. The case of the no-show paradox illustrated in Table 1 gives substance to the paradox of quarreling members.

If there is coordination such that the splitting up is successful, then of course we could also think that, given a quota of 51 votes in the first round, the members of the group of 47 voters vote strategically. Instead of abstaining, they could transfer 24 and 23 of their votes for A and C, respectively, in the first round, supporting a run-off of A and C in the second round. In the second round all 47 members could vote for C, further augmenting the voting share of the winning alternative C.

An abstention of 47 votes in a committee of 100 does not look good. On the other hand, one might argue that the voting of 24 members of the 47 group in favor of alternative A does not seem to be realistic. Nor is the splitting of votes of voters with preference orderings $B \succ C \succ A$ convincing. Moreover, the basic assumption of the no-show paradox is that the other group of voters do not abstain and vote sincerely. Is this realistic?

Such peculiarities could be observed when Urho Kekkonen was elected to be President in Finland in 1956. In the second round (out of a three-round run-off voting procedure), a sufficiently large share of representatives of the socialist-communist party SKDL casted their votes for their least preferred candidate K.A. Fagerholm and the others voted for Kekkonen, just to make sure that the Condorcet winner J.K. Paasikivi will not participate in the decisive run-off voting round. The other parties voted for their most preferred candidates. As a result Kekkonen was elected in the third round. He and all other candidates would have been defeated by Paasikivi in pairwise comparison [37].

Kekkonen was president of Finland for the next 25 years. The no-show paradox in Table 1 and the case of Kekkonen's election clearly demonstrate that we cannot rely on preferences only, if we want to forecast the outcome of voting. In the two cases, "intention" matters, but also the pure numbers of votes and the voting procedure given by the quota and the sequence of rounds. Power indices are taking care of the latter.

Recent party politics and coalition formation in the German Bundestag can also serve as a nice illustration of the interrelationship between a priori voting power, determined by the seat shares and the decision rules, and party preferences, representing political ideologies. In the election of September 22, 2013, the CDU/CSU lost its partner FDP from the previous government as the Liberals' vote share was below the 5% threshold necessary for entering the Bundestag. The forming of a coalition government was cumbersome, but also necessary because even the 311 seats of the CDU/CSU were not sufficient for a majority government as the total number of seats were 631. The second strongest faction was formed by Social Democrats (SPD) which controlled 193 seats, while the Green Party and the left-wing party "Die Linke" were only represented by 63 and 64 seats, respectively. Given the seat distribution (311, 193, 63, 64) the set of MWC is obvious, but some of them were not feasible because of *unsurmountable differences in the party preferences*. The CDU/CSU bargained with the Green Party, but even this coalition turned out to be infeasible because of the diversity of political opinions. Alternatively, a red-red-green coalition of the SPD, Die Linke, and Green Party was discussed in the public, but less so among politicians, because the SPD leadership made clear that the party's political position does not allow to cooperate with Die Linke in a coalition. In the end, the Große Koalition of CDU/CSU and SPD was formed.

Already the day after there were party members of the CDU/CSU and the Greens that suggested that the two parties revise their policy positions so that they can form coalitions in the future. Also to prepare for this, the two parties formed coalition governments in several regions, i.e., Bundesländer. The argument for this inclination for convergence is based on numbers, i.e., on a priori voting power. On the one hand, the CDU/CSU wants to have an alternative to the SPD as coalition partner, on the other, it is not so sure that the incompatibility of SPD and Die Linke is of permanence. There could be a day when a red-red-green coalition will be feasible. And then the CDU/CSU must be in a position to make the Green Party an offer that this party cannot reject, but which is also consistent with its own perspective.

These considerations were driven by numbers. This should not be surprising since the power to govern is encapsulated in number of seats and majority quota. Now the considerations have to be revised, and the revision already started, because it is expected that a new player entered the playfield, the right-wing party AfD, with an expected share of 10–12%. Moreover, perhaps even the Liberals might come back to the Bundestag, outrunning the 5% threshold. Both possibilities will have immediate numerical effects, but of course the more exciting dimension of coalitions is embedded in party preferences.

To sum up: Taking care of the preferences of the voters gives a justification for treating group of voters as bloc that it interacts with other blocs of voters forming winning coalitions. The power of such a bloc can then be measured by one of the indices, depending which index seems more appropriate for the underlying problem: is it the division of a cake (or political spoils) or on the specification of a public good? A related question is: Can power index theories contribute to the analysis of the aggregation of preferences? Note this question does not imply that we should consider preferences when measuring voting power[17]. But the units that enter the voting power analysis (factions of party representatives, voting blocs, etc.) are the result of the preferences of their members. Nevertheless, this does not necessarily mean that the internal aggregation problem is solved and the units vote in accordance to these preferences.

More specifically, we can ask whether there is a relationship between the paradoxes in the aggregation of preferences and the non-monotonicities we find in the application

[17]This refers to extensive discussion that is summarized in [41] and [10] which reflect the two controversial views on the *possibility and impossibility of a preference-based power index*. See also [30] for further details.

of power indices. Again, this not to say that we have to consider preferences when measuring power. But we cannot speak of the success of a vote without considering preferences, and success, the matching of outcome and preferences, is what most voters want.

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POWER ON DIGRAPHS

It is assumed that relations between n players are represented by a directed graph or digraph. Such a digraph is called invariant if there is a link (arc) between any two players between whom there is also a directed path. We characterize a class of power indices for invariant digraphs based on four axioms: Null player, Constant sum, Anonymity, and the Transfer property. This class is determined by $2n - 2$ parameters. By considering additional conditions about the effect of adding a directed link between two players, we single out three different, one-parameter families of power indices, reflecting several well-known indices from the literature: the Copeland score, β- and apex type indices.

Keywords: *digraph, power index, transfer property, link addition*

1. Introduction

We consider situations where the relations between n players are reflected by a directed graph or digraph. There are several interpretations possible. A directed link (or arc) from player i to player j may reflect that player i controls player j, for instance i is an investor who has the majority of the shares in firm j. With this interpretation, our paper is related to Gambarelli and Owen [7], where the players are firms or investment companies. It is also a special case of the approach by Hu and Shapley [10, 11] and of the mutual control structures of Karos and Peters [12]. Somewhat related, our model

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can also be seen as a permission structure as introduced by Gilles et al. [9]. A directed link from player i to player j then reflects the fact that player j needs the permission of player i in order to cooperate with other players. Still another interpretation of such a graph is as an information structure: player j can get information from player i only if there is a directed link from player i to player j. These are applications where the links are between decision making agents. Other types of applications may have arcs representing the results of matches played among teams in a sports competition, or the structure of links between internet pages.

A digraph will be called invariant if for any two players who are connected via a directed path, there is also a direct link. For instance, in the control interpretation, if player i controls player j and j controls k, then player j controls k indirectly; in an invariant digraph there will be a direct link between i and k. The term invariance refers to the fact that such indirect control relations are already included – alternatively, such digraphs are often called transitive.

Our aim in this paper is to study power indices, reflecting the power of players as a consequence of their positions in the digraph. We impose four axioms on a power index: Null player, Constant sum, Anonymity, and the Transfer property. The Transfer property (first introduced in a different format by Dubey [3]) is an additivity condition in the spirit of [14]. It is the main tool to ensure that a power index is completely determined by its values on digraphs of an elementary form. In such elementary digraphs, there is a set M of players who each have a link to one and the same player j. Due to the Anonymity condition, only the cardinality of M matters, but it makes a difference whether or not player j is in M: this is why we arrive at a class of power indices with $2(n-1)$ degrees of freedom (parameters)[4]. The Null player axiom ensures that players outside M (other than possibly j) have zero power. Combined with the Constant sum axiom, we obtain that the sum of the players' powers is always 0. This is in contrast with classical power indices (e.g., [15], see also [6, 8, 1]), where power is between 0 and 1, and sums to 1. In our approach, it is natural to allow negative power. A null player is a player who has neither incoming nor outgoing links in the digraph, and it is natural to assign power 0 to such a player. In turn then, it is equally natural to allow negative power for a player who has only incoming links.

An invariant digraph is a special case of an invariant mutual control structure studied by Karos and Peters [12]. Our main result here (Theorem 4.4) is related to the main result in [12], but the axioms imposed in the present paper are weaker, since they apply to power indices on a considerably smaller class of mutual control structures. Therefore, compared to that result, Theorem 4.4 requires a new and different, though less complicated, proof.

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[4]More precisely, by the Null player and Constant sum axioms the case where $M = \{j\}$ is determined, so that by Anonymity we have $n-1$ cases with $j \in M$, and $n-1$ cases with $j \notin M$.

The Constant sum axiom and the possibility of negative power are two features by which the indices in this paper distinguish themselves from digraph power measures such as the outdegree and β-measure, axiomatized by van den Brink and Gilles [18], which both satisfy a normalization that implies that the sum of the powers of the players depends on the digraph. We find three subclasses, related to the Copeland score, β-measure and apex-measure, by imposing three different additional conditions with respect to link addition[5]. First, we impose the condition that adding a link from player *i* to player *j* does not change the power of the players who already had a link to player *j*. This may make sense, for instance, in a situation where player *j* has to do tasks for players who have a link to *j* – assuming there is no capacity constraint on what player *j* can do. Imposing this condition (called Link addition 1) on top of the four basic conditions above singles out a one-parameter family of power indices closely related to the Copeland score [2] from social choice theory: the power of a player is proportional to the number of players to whom he has a link minus the number of players who have a link to him.

The next condition requires that adding a link does not change the power of player *j* to whom this link is incoming. This implies that the players that already had a link to *j* now have to share the power they had with the newcomer. Also, this condition (Link addition 2) singles out a one-parameter family of power indices, this time closely related to the concept of a β-measure, as in [17, 18].

The final condition that we consider (Link addition 3) requires that adding a link from player *i* to player *j* equally increases or reduces the power of *i* and *j*. Again, we obtain a one-parameter family of power indices, with the property that incoming links have no effect on a player's power. These power indices are similar to the apex-type power index in [16].

The organization of the paper is as follows. Section 2 introduces invariant digraphs, Section 3 the main axioms for a power index, Section 4 the main characterization result, and Section 5 the refinements based on Link addition axioms. Section 6 concludes.

2. Preliminaries

For a set A we denote by $P(A)$ the set of all subsets of *A*, and by $P_0(A)$ the set of all nonempty subsets of *A*. By $|A|$ we denote the number of elements of *A*.

[5]The effect of link addition was introduced in communication graph games by Myerson [13], who introduced the axiom of fairness, stating that deleting an undirected communication link between two players has the same effect on their payoffs. Together with so-called component efficiency this characterizes a Shapley type solution, later referred to as the Myerson value.

Let $N = \{1, ..., n\}$ with $n \geq 2$ denote the set of players. Elements of $P(N)$ are called coalitions. A directed graph or digraph on N is a map $C: P(N) \to P(N)$ satisfying $C(\emptyset) = \emptyset$ and $C(S) = \cup_{i \in S} C(i)$ for every $S \in P_0(N)$. Hence, a digraph C is completely determined by its values $C(i)$ for $i \in N$. The graphical interpretation of a digraph C is, indeed, that there is a link from $i \in N$ to $j \in N$ if and only if $j \in C(i)$[6]. As mentioned in the Introduction, various interpretations with respect to applications are possible. The set $C(i)$ can be interpreted as the set of players controlled by player i (cf. [10–12]), or the set of players who need permission from i [9]; or the set of players to whom player i can communicate, etc. The reason for defining a digraph not just for singletons lies in the axiomatic approach to power indices presented later on. Observe that a digraph C is trivially monotonic: if $S, T \in P(N)$ and $S \subseteq T$ then $C(S) \subseteq C(T)$.

For a digraph C, a directed path from $i \in N$ to $j \in N$ is a sequence $(i_1, i_2), (i_2, i_3), ..., (i_k, i_{k+1}) \in N \times N$ for some $k \in \mathbb{N}$ such that $i_{\ell+1} \in C(i_\ell)$ for each $\ell = 1, ..., k$, $i = i_1$, and $j = i_{k+1}$. A digraph C is invariant if for all $i, j \in N$, if there is a directed path from i to j, then $j \in C(i)$[7]. The expression “invariant” refers to the fact that in an invariant digraph adding a link between players having a directed path connecting them, does not change the digraph.

Note that an arbitrary digraph can be made invariant by simply adding links for every pair of players between whom there exists a directed path. The following observation, the easy proof of which is omitted, will be useful in the sequel.

Lemma 2.1. Let C be a digraph. Then C is invariant if and only if $C(i \cup C(i)) = C(i)$ for all $i \in N$.

We denote the set of all digraphs (on N) by $\mathcal{D}$ and the set of all invariant digraphs by $\mathcal{D}^*$.

In the sequel, the following observations will be relevant. For $C, D \in \mathcal{D}$ define $C \cup D$ and $C \cap D$ by $(C \cup D)(S) = C(S) \cup D(S)$ and $(C \cap D)(S) = C(S) \cap D(S)$ for all $S \in P(N)$. Then $C \cup D \in \mathcal{D}$, but even if $C, D \in \mathcal{D}^*$ then it does not necessarily follow that $C \cup D \in \mathcal{D}^*$. See the following example.

Example 2.2. Let $N = \{1, 2, 3\}$ and let $C, D \in \mathcal{D}^*$ be defined by $C(1) = 2$, $D(2) = 3$, and $C(i) = \emptyset$ and $D(i) = \emptyset$ in the remaining cases[8]. Then $(C \cup D)(1) = 2$

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[6]Usually, what we call players are called nodes or vertices in a digraph. Because of the applications we have in mind, we refer to them as players.

[7]In the literature, these digraphs are usually referred to as transitive digraphs.

[8]Here and elsewhere we often omit set braces if confusion is unlikely.

and $(C \cup D)(2) = 3$ but $3 \notin (C \cup D)(1)$, so that $C \cup D \notin \mathcal{D}^*$. (See also Remark 2.9 in [12].)

Further, if $C, D \in \mathcal{D}^*$, then it does not necessarily follow that $C \cap D \in \mathcal{D}$, but if $C \cap D \in \mathcal{D}$, then also $C \cap D \in \mathcal{D}^*$. The latter is straightforward to prove by using Lemma 2.1; for the former see the following example.

Example 2.3. Let $n \geq 4$ and let $C(1) = 3$, $C(2) = 4$, $D(1) = 4$, $D(2) = 3$, and $C(i) = D(i) = \emptyset$ for all $i \neq 1, 2$. Then $C, D \in \mathcal{D}^*$, but $C \cap D \notin \mathcal{D}$, since $(C \cap D)(i) = \emptyset$ for all i, but $(C \cap D)(12) = 34$.

Example 2.3 in fact shows that our definition of the intersection $C \cap D$ of two digraphs cannot be restricted to taking intersections for singleton coalitions only and then extending to arbitrary coalitions by taking unions.

3. Axioms for a power index

We consider power indices for invariant digraphs. A power index is a map $\phi : \mathcal{D}^* \to \mathcal{R}^N$. We propose four basic axioms, in the spirit of the standard axioms of the Shapley [14] value for cooperative games. We will then prove that these four axioms characterize a large, $2(n-1)$-parameter family of power indices.

Player $i \in N$ is *a null player* in $C \in \mathcal{D}^*$ if $C(i) = \emptyset$ and $i \notin C(N)$: i.e. player i is an isolated node in the digraph. The digraph in which every player is a null player, is denoted by O, i.e., $O(S) = \emptyset$ for all $S \subseteq N$. This corresponds to a graph without any links.

The first axiom requires that null players have zero power.

Null Player (NP). $\phi_i(C) = 0$ for every null player i in C, for every $C \in \mathcal{D}^*$.

The second axiom states that the sum of the powers of all the players is the same for any digraph.

Constant sum (CS). $\sum_{i \in N} \phi_i(C) = \sum_{i \in N} \phi_i(D)$ for all $C, D \in \mathcal{D}^*$.

The combination of these two axioms has as a simple consequence that the powers of all the players in a digraph always add up to zero.

Lemma 3.1. Let ϕ be a power index satisfying NP and CS. Then $\sum_{i \in N} \phi_i(C) = 0$ for every $C \in \mathcal{D}^*$.

Proof. By NP, $\phi_i(O) = 0$ for every $i \in N$. Hence, by CS, $\sum_{i\in N}\phi_i(C) = \sum_{i\in N}\phi_i(O) = 0$ for every $C \in \mathcal{D}^*$. □

Usually in the literature, power indices take values between 0 and 1 and sum up to 1 (e.g., [15]). In our case, we allow the possibility that power can be negative. Typically, for instance, player j with $C(j) = \varnothing$, but $j \in C(i)$ for some other player i, seems to have less power than a null player, so that we may wish to assign negative power to such a player. A null player in a simple game does not add any power to any coalition. However, in a simple game players cannot be subordinate to other players, so that a null player is then a player with minimal power. In our model, this is different, and there can be players who have less power than a "null player". The expression null player suits its definition above well, whether in its quantitative meaning of "zero", or in its qualitative meaning of "without value or consequence, amounting to nothing".

Let $\pi : N \to N$ be a permutation. Then for $C \in \mathcal{D}^*$ we define $\pi C \in \mathcal{D}^*$ by

$$(\pi C)(S) = \pi(C(\pi^{-1}(S)))$$

The next axiom is standard[9].

Anonymity (AN). $\phi_{\pi(i)}(\pi C) = \phi_i(C)$ for every player $i \in N$, every permutation π of N, and every $C \in \mathcal{D}^*$.

The final axiom replaces the usual additivity condition known from the Shapley value. For cooperative games, it was first introduced by Dubey [3] in the format (1) presented in the next section. See, further, [4, 5].

Transfer property (TP). $\phi(C) - \phi(C') = \phi(D) - \phi(D')$ for all $C, C', D, D' \in \mathcal{D}^*$ such that $C' \subseteq C$, $D' \subseteq D$, and $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for every $S \subseteq N$.

An alternative and stronger version of the Transfer property would be obtained by imposing the condition $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ only for singletons, i.e., replacing it by $C(i) \setminus C'(i) = D(i) \setminus D'(i)$ for every $i \in N$. This, however, is too strong for our purposes: it would have the same effect as adding the Link addition 1 condition, see Section 5, and it would exclude other power indices, for instance those characterized in that section.

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[9]Nevertheless, it may be interesting to investigate the consequences of dropping this condition and assuming that there may be asymmetries between the players apart from those resulting from the digraphs.

4. Characterization of power indices satisfying NP, CS, AN, and TP

The main result of this section is Theorem 4.4, which is a characterization of all power indices satisfying Null player, Constant sum, Anonymity, and the Transfer property. We start out by proving a consequence of the Transfer property.

Lemma 4.1. Let ϕ be a power index satisfying TP. Then

$$\phi(C \cap D) + \phi(C \cup D) = \phi(C) + \phi(D) \tag{1}$$

for all $C, D \in \mathcal{D}^*$ with $C \cup D, C \cap D \in \mathcal{D}^*$.

Proof. Let $C, D \in \mathcal{D}^*$ with $C \cup D, C \cap D \in \mathcal{D}^*$. Clearly,

$$(C(S) \cup D(S)) \setminus C(S) = D(S) \setminus (C(S) \cap D(S))$$

for all $S \subseteq N$. Hence by TP, $\phi(C \cup D) - \phi(C) = \phi(D) - \phi(C \cap D)$, implying (1). □

For $C \in \mathcal{D}^*$ and $j \in N$ we define the digraph C_j by

$$C_j(i) = \begin{cases} \{j\} & \text{if } \; j \in C(i) \\ \varnothing & \text{otherwise} \end{cases}$$

It is easy to see that $C_j \in \mathcal{D}^*$. Observe that in a graphical representation of C_j, the graph has only links pointing to *j*, namely from those players *i* with $j \in C(i)$.

Lemma 4.2. Let ϕ be a power index satisfying TP and NP, and let $C \in \mathcal{D}^*$. Then $\phi(C) = \sum_{j \in N} \phi(C_j)$.

Proof. We first show that $\cup_{k \in T} C_k \in \mathcal{D}^*$ for every $T \in P_0(N)$. Let $S, T \in P_0(N)$, then

$$\bigcup_{k \in T} C_k(S) = \bigcup_{k \in T} \bigcup_{j \in S} C_k(j) = \bigcup_{j \in S} \bigcup_{k \in T} C_k(j)$$

which implies that $\cup_{k \in T} C_k \in \mathcal{D}$. Let $i \in N$ and $j \in \cup_{k \in T} C_k(i \cup (\cup_{h \in T} C_h(i)))$, then there exists an $\ell \in T$ such that $j \in C_\ell(i \cup (\cup_{h \in T} C_h(i)))$. In turn, since $C_\ell \in \mathcal{D}$, this implies that $j \in C_\ell(i)$ or there exists an $m \in T$ such that $j \in C_m(i)$. Hence, $j \in \cup_{k\in T} C_k(i)$, so $\cup_{k\in T} C_k(i \cup (\cup_{h\in T} C_h(i))) \subseteq \cup_{k\in T} C_k(i)$. The converse inclusion follows by monotonicity. Hence $\cup_{k\in T} C_k \in \mathcal{D}^*$ by Lemma 2.1.

Next, for every $T \in P_0(N)$ and $k \notin T$, we have $C_k \cap (\cup_{\ell\in T} C_\ell) = O \in \mathcal{D}^*$ (the empty digraph). By NP, $\phi(O) = 0 \in \mathcal{R}^n$. Also, $C = \cup_{i\in N} C_i$.

By the preceding arguments and by repeatedly applying (1), we obtain

$$\phi(C) = \phi(C_1 \cup (\cup_{i=2,\dots,n} C_i)) = \phi(C_1) + \phi(\cup_{i=2,\dots,n} C_i) - \phi(C_1 \cap (\cup_{i=2,\dots,n} C_i))$$
$$= \phi(C_1) + \phi(\cup_{i=2,\dots,n} C_i) = \sum_{j\in N} \phi(C_j),$$

which concludes the proof of the lemma. □

By Lemma 4.2 we may concentrate on digraphs of the form C_j. More generally, for $M \in P_0(N)$ and $j \in N$ the digraph $U_{M,j}$ is defined by

$$U_{M,j}(i) = \begin{cases} \{j\} & \text{if } i \in M \\ \varnothing & \text{otherwise} \end{cases} \tag{2}$$

Clearly, $U_{M,j}$ is invariant.

Lemma 4.3. Let the power index φ on $\mathcal{D}^*$ satisfy NP, CS, and AN. Then there exist $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_n \in \mathcal{R}$, with $\alpha_0 = 0$, $\beta_1 = 0$ such that for every $M \in P(N)$ and $j \in N$, with $m = |M|$:

(a) if $j \notin M$, then for every $i \in N$

$$\phi_i(U_{M,j}) = \begin{cases} 0 & \text{if } i \notin M \cup j \\ \alpha_m / m & \text{if } i \in M \\ -\alpha_m & \text{if } i = j \end{cases}$$

(b) if $j \in M$, then for every $i \in N$

$$\phi_i(U_{M,j}) = \begin{cases} 0 & \text{if } \notin M \\ \beta_m / m & \text{if } i \in M \backslash j \\ \beta_m / m - \beta_m & \text{if } i = j \end{cases}$$

Proof. Straightforward from the axioms. □

Observe that for $C \in D^*$ and $j \in N$, we have $C_j = U_{M_j^C, j}$ where $M_j^C = \{i \in N : j \in C(i)\}$

From Lemmas 4.2 and 4.3 we obtain our main result.

Theorem 4.4. A power index ϕ on D^* satisfies NP, CS, AN, and TP if and only if there exist $\alpha_0, \alpha_1, ..., \alpha_{n-1}, \beta_1, ..., \beta_n \in R$, with $\alpha_0 = 0$ and $\beta_1 = 0$, such that for each $C \in D^*$ we have $\phi(C) = \sum_{j \in N} \phi(C_j)$, with each $\phi(C_j) = \phi(U_{M_j^C, j})$ defined as in (a) and (b) of Lemma 4.3.

Proof. The only-if direction follows from Lemmas 4.2 and 4.3. For the if-direction, let $\alpha_0, \alpha_1, ..., \alpha_{n-1}, \beta_1, ..., \beta_n \in R$, with $\alpha_0 = 0$ and $\beta_1 = 0$, and with $\phi(C) = \sum_{j \in N} \phi(C_j)$, where each $\phi(C_j) = \phi(U_{M_j^C, j})$ is defined as in (a) and (b) of Lemma 4.3. It is obvious that φ satisfies AN and CS. For NP, note that *i* is a null player in $C \in D^*$ if and only if $C_i = O$ and $i \notin M_j^C$ for all $j \in N$. This implies that $\varphi_i(C_j) = 0$ for all $j \in N$ and therefore $\phi_i(C) = 0$.

For TP, let $C', C, D', D \in$ (HTML translation failed) D^* such that $C' \subseteq C$, $D' \subseteq D$, and $C(S) \setminus C'(S) = D(S) \setminus D'(S)$ for every $S \subseteq N$. Then it is straight-forward to show that $C'_j \subseteq C_j$, $D'_j \subseteq D_j$, and $C_j \backslash C'_j = D_j \backslash D'_j$ for all $j \in N$. Hence, to show TP, it is sufficient to show that

$$\varphi(C_j) - \varphi(C'_j) = \varphi(D_j) - \varphi(D'_j) \quad (*)$$

for all $j \in N$. Now fix *j* and observe that $C'_j \subseteq C_j$ and $D'_j \subseteq D_j$ imply $M_j^{C'} \subseteq M_j^C$ and $M_j^{D'} \subseteq M_j^D$. If $M_j^{C'} = M_j^C$, then $C'_j = C_j$ and therefore $D'_j = D_j$, so that (*) follows. Otherwise, both $M_j^{C'} \subset M_j^C$ and $M_j^{D'} \subset M_j^D$ (where $\subset$ denotes strict in-clusion). Then, for $S = N \setminus M_j^{C'}$ we have $C_j(S) \setminus C'_j(S) = \{j\} = D_j(S) \setminus D'_j(S)$. This implies $S \cap M_j^{D'} = \emptyset$, hence $M_j^{D'} \subseteq M_j^{C'}$. Similarly, $M_j^{C'} \subseteq M_j^{D'}$, hence $M_j^{D'} = M_j^{C'}$. This implies that also $M_j^D = M_j^C$, so $C_j = D_j$ and $C'_j = D'_j$, and (*) follows again. □

In the concluding Section 6 we provide examples to show that the four axioms in Theorem 4.4 are logically independent. In the next section we consider specific subclasses of the family of power indices satisfying the conditions of Theorem 4.4, following from the possible effects of adding links in a digraph. Here, we suggest a scaling condition on a power index ϕ by which we can single out a unique member of this family[10].

Scaling (SC). For all $C \in D^*$

• If $j \in N$, $C(j) = \varnothing$, and $j \in C(k)$ for some $k \in N$, then $\phi_j(C) = -1$.

• If $i, j \in N$, $C(i) = C(j)$, $j \in C(k)$ for some $k \in N$, $i \notin C(k)$ for all $k \in N$, then $\phi_j(C) = \phi_i(C) - 1$.

The first bullet point fixes the power of a player without any outgoing link but with at least one incoming link to -1. The second point fixes the difference in power between two players with the same outgoing links to 1, if one player has no incoming links but the other player has at least one incoming link. We leave the proof of the following corollary to the reader. Denote a power index ϕ as in Theorem 4.4 by $\phi^{\alpha,\beta}$, where $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in R^{n-1}$ and $\beta = (\beta_2, ..., \beta_n) \in R^{n-1}$.

Corollary 4.5. A power index ϕ on D^* satisfies NP, CS, AN, TP, and SC if and only if $\phi = \phi^{\alpha,\beta}$ with $\alpha = \beta = (1, ..., 1)$.

Clearly, in the digraph O, which has no arcs at all, every power index $\phi^{\alpha,\beta}$ assigns 0 to every player. We conclude with a consideration of the following question: for which digraphs do all players have zero power?

Remark 4.6. Suppose that every component of a digraph $C \in D^*$ is a complete graph on the nodes in that component, i.e., for all i, j in the same component there is a link from i to *j*, as well as a link from j to *i*; in particular, there is a link from i to *i*. Then it can be deduced from the description in Lemma 4.3 that $\phi^{\alpha,\beta}(C) = (0, ..., 0)$ for every power index satisfying the conditions in Theorem 4.4. Examples of such digraphs are the complete digraph on all nodes (players), which is the invariant digraph associated with, for instance, a circular digraph; and the digraph without any links, where all the players are isolated and therefore null players.

5. Adding links: Copeland, β- and apex type indices

In this section we consider the class of power indices characterized in Theorem 4.4 and study refinements following from the effect of adding a directed link (arc) between

[10]This axiom is similar to the Controlled player axiom in [12].

player i and player j. Let Φ denote the class of all power indices satisfying NP, CS, AN, and TP. As above, a generic element of Φ is denoted as $\phi^{\alpha,\beta}$, where $\alpha = (\alpha_1, ..., \alpha_{n-1}) \in R^{n-1}$ and $\beta = (\beta_2, ..., \beta_n) \in R^{n-1}$.

The first axiom says that if we add an additional link to player j from player *i*, then this should not change the power of the players who already have a link to *j*. In the parlance of control or permission: if player j becomes additionally controlled by some player *i*, then this should not change the power of the players who were already controlling *j*.

Recall that for a digraph $C \in D^*$ and player *j*, M_j^C is the set of players with a link to *j*, i.e., $M_j^C = \{i \in N : j \in C(i)\}$.

Link addition 1 (LA1). $\phi_h(C) = \phi_h(C')$ for all $C, C' \in D^*$, $j \in N$, and $h \in M_j^C \setminus \{j\}$, such that there is $i \in N$ with $i \notin M_j^C$, $C'(i) = C(i) \cup \{j\}$ and $C'(\ell) = C(\ell)$ for all $\ell \in N \setminus \{i\}$.

Theorem 5.1. Let $\phi = \phi^{\alpha,\beta} \in \Phi$. Then ϕ satisfies LA1 if and only if there exists a $c \in R$ such that $\alpha_k = kc$ for all $k = 1, ..., n-1$ and $\beta_k = kc$ for all $k = 2, ..., n$.

Thus, under LA1 we obtain a one-parameter family of power indices of the form

$$\phi_i^c(C) = \sum_{j \in C(i)} c - |M_i^C| c = c\left(|C(i)| - |M_i^C|\right) \tag{3}$$

where $c \in R$. For instance, for $c = 1$ and using the terminology of control, the power of player i is equal to the number of players controlled by player i minus the number of players controlling player *i*, which yields the Copeland score in social choice theory [2][11, 12]

Proof. For the if-direction let $c \in R$ and let ϕ^c be as in (3). We show that ϕ^c satisfies LA1. Let C, C', h, j, i be as in the statement of the axiom. Then $h \neq i$ and therefore $\phi_h^c(C) = c(|C(h)| - |M_h^C|) = c(|C'(h)| - |M_h^{C'}|) = \phi_h^c(C')$.

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[11]In the social choice theory, for a given preference profile the Copeland score of an alternative *x* is the number of alternatives beaten by *x* minus the number of alternatives beating *x* in pairwise comparison.

[12]An alternative way of characterizing the class $\{\phi^C : c \in R\}$ is by strengthening the transfer property in the way indicated at the end of Section 3.

For the only-if direction, let $\phi = \phi^{\alpha,\beta} \in \Phi$ satisfy LA1. First suppose $M \in P_0(N)$ and $i, j \in N \setminus M$ with $i \neq j$ and $|M| = k$ for some $k \in \{1, ..., n-2\}$. Then for $h \in M$, by LA1 with $U_{M,j}$ in the role of C and $U_{M \cup \{i\}, j}$ in the role of C', $\phi_h(U_{M,j}) = \phi_h(U_{M\cup\{i\},j})$, hence $\alpha_k / k = \alpha_{k+1}/(k+1)$. Let $c = \alpha_1$, then this implies that $\alpha_k = kc$ for all $k = 1, ..., n-1$. Now suppose $M' \in P_0(N)$ with $j \notin M'$ and $|M'| = k \in \{1, ..., n-1\}$. For $h \in M'$, again by LA1, $\phi_h(U_{M',j}) = \phi_h(U_{M'\cup\{j\},j})$, hence $\alpha_k / k = \beta_{k+1}/(k+1)$. This implies $\beta_k = kc$ for all $k = 2, ..., n-1$. □

The following axiom requires that it is player j whose power does not change if an additional link is added from some player i to j. In terms of control: if a player j becomes additionally controlled by some player i then this should not change the power of player j.

Link addition 2 (LA2). $\phi_j(C) = \phi_j(C')$ for all $C, C' \in D^*$, $j \in N$ with $M_j^C \setminus \{j\} \neq \emptyset$, and $i \in N$ with $i \notin M_j^C$, $C'(i) = C(i) \cup \{j\}$ and $C'(\ell) = C(\ell)$ for all $\ell \in N \setminus \{i\}$.

Theorem 5.2. Let $\phi = \phi^{\alpha,\beta} \in \Phi$. Then ϕ satisfies LA2 if and only if there exists a $c \in R$ such that $\alpha_k = c$ for all $k = 1, ..., n-1$ and $\beta_k = \dfrac{k}{k-1} c$ for all $k = 2, ..., n$.

The power indices characterized in Theorem 5.2 take the form

$$\bar{\phi}_i^c(C) = \sum_{j \in C(i) \setminus \{i\}} \frac{c}{|M_j^C \setminus \{j\}|} - c 1_{\{M_i^C \neq \emptyset\}} \qquad (4)$$

where $1_{\{P\}} = 1$ if statement P is true and $1_{\{P\}} = 0$ otherwise. A power index $\bar{\phi}^c$ is similar to the idea of the β-measure as in van den Brink and Gilles [18] or its reflexive variant in van den Brink and Borm [17]: if player i has a link to player j, then he equally shares the amount of power c with the other players linked to j, except possibly j[13]. The difference is that player j loses c in power.

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[13]The β measure is introduced for irreflexive digraphs (i.e. $i \notin C(i)$ for all $i \in N$) and assigns to player i in irreflexive digraph C the score $\beta_i(C) = \sum_{j \in C(i)} 1/\left|M_j^C\right|$. Its reflexive variant assigns the score $\beta_i^{\text{refl}}(C) = \sum_{j \in C(i) \cup \{i\}} 1/|M_j^C|\left(+1\right)$. The difference between the two lies in whether the set of loops is taken into account or not. So, the reflexive variant equals the β measure of the reflexive digraph obtained by adding all loops $(i \in C(i)$ for all $i \in N)$.

Proof. For the if-direction let $c \in R$ and $\bar{\phi}^c$ be defined as in (4). We show that $\bar{\phi}^c$ satisfies LA2. Let C, C', j, i be as in the statement of the axiom, then

$$\bar{\phi}_j^c(C) = \sum_{\ell \in C(j)\setminus\{j\}} \frac{c}{|M_\ell^C \setminus \{\ell\}|} - c = \sum_{\ell \in C'(j)\setminus\{j\}} \frac{c}{|M_\ell^{C'} \setminus \{\ell\}|} - c = \bar{\phi}_j^c(C')$$

as is straightforward to verify, both for the case $i \neq j$ and for the case $i = j$.

For the only-if direction, let $\phi = \phi^{\alpha,\beta} \in \Phi$ satisfy LA2. First suppose $M \in P_0(N)$ and $i, j \in N \setminus M$ with $i \neq j$ and $|M| = k$ for some $k \in \{1, ..., n-2\}$. Then by LA2, with $U_{M,j}$ in the role of C and $U_{M\cup\{i\},j}$ in the role of C', $\phi_j(U_{M,j}) = \phi_j(U_{M\cup\{i\},j})$, hence $\alpha_k = \alpha_{k+1}$. Let $c = \alpha_1$, then this implies that $\alpha_k = c$ for all $k = 1, ..., n-1$. Now suppose $M' \in P_0(N)$ with $j \notin M'$ and $|M'| = k \in \{1, ..., n-1\}$. Again by LA2, $\phi_j(U_{M',j}) = \phi_j(U_{M'\cup\{j\},j})$, hence by the previous argument, $c = \frac{\beta_{k+1}}{k+1} - \beta_{k+1} = \frac{-k}{k+1}\beta_{k+1}$, and thus $\beta_{k+1} = \frac{k+1}{k}c$, which implies that $\beta_k = \frac{k}{k-1}c$ for all $k = 2, \ldots, n-1$. □

The final axiom we consider says that if we add a link from player *i* to player *j* then both have the same gain or loss in power.

Link addition 3 (LA3). $\phi_i(C') - \phi_i(C) = \phi_j(C') - \phi_j(C)$ for all $C, C' \in D^*$, $j \in N$ with $M_j^C \neq \emptyset$, and $i \in N$ with $i \notin M_j^C$, $C'(i) = C(i) \cup \{j\}$ and $C'(\ell) = C(\ell)$ for all $\ell \in N \setminus \{i\}$.

Theorem 5.3. Let $\phi = \phi^{\alpha,\beta} \in \Phi$. Then ϕ satisfies LA3 if and only if there exists a $c \in R$ such that $\alpha_k = \frac{2}{k+1}c$ for all $k = 1, ..., n-1$, and $\beta_k = 0$ for all $k = 2, ..., n$.

The power indices characterized in Theorem 5.3 take the form

$$\tilde{\phi}_i^c(C) = \sum_{j \in C(i)\setminus C(j)} \frac{\alpha_{|M_j^C|}}{|M_j^C|} - \alpha_{|M_i^C|} 1_{\{M_i^C \neq \emptyset,\ i \notin C(i)\}} \qquad (5)$$

with α_k as in Theorem 5.3. In terms of control, according to a power index $\tilde{\phi}^c$, if player *j* controls himself, then no player, including player *j*, derives (positive or negative) power from controlling *j*. Further, the (negative, if $c > 0$) power from being controlled decreases and converges to zero as the number of controlling players increases. We note that $\tilde{\phi}^c$ is related to the apex power index in [16][14].

Proof. For the if-direction let $c \in R$ and $\tilde{\phi}^c$ be defined as in (5). We show that $\tilde{\phi}^c$ satisfies LA3. Let C, C', j, i be as in the statement of the axiom, and $k = |M_j^C| \in \{1, ..., n-1\}$. First, if $j \notin M_j^C$, $i \neq j$, then

$$\tilde{\phi}_i^c(C') - \tilde{\phi}_i^c(C) = \frac{\alpha_{k+1}}{k+1} = \frac{2c}{(k+1)(k+2)}$$

and

$$\tilde{\phi}_j^c(C') - \tilde{\phi}_j^c(C) = -\alpha_{k+1} + \alpha_k = \frac{2c}{(k+1)(k+2)}$$

so that LA3 holds. Second, if $j \in M_j^C$, $i \neq j$, then

$$\tilde{\phi}_i^c(C') - \tilde{\phi}_i^c(C) = \frac{\beta_{k+1}}{k+1} = 0$$

and

$$\tilde{\phi}_j^c(C') - \tilde{\phi}_j^c(C) = \frac{\beta_{k+1}}{k+1} - \beta_{k+1} - (\frac{\beta_k}{k} - \beta_k) = 0$$

Thus LA3 also holds here.

For the only-if direction, let $\phi = \phi^{\alpha,\beta} \in \Phi$ satisfy LA3. First suppose $M \in P_0(N)$ and $i, j \in N \setminus M$ with $i \neq j$ and $|M| = k$ for some $k \in \{1, ..., n-2\}$. Then $\phi_j(U_{M\cup\{i\},j}) - \phi_j(U_{M,j}) = -\alpha_{k+1} + \alpha_k$, whereas $\phi_i(U_{M\cup\{i\},j}) - \phi_i(U_{M,j}) = \alpha_{k+1}/(k+1)$. By LA3 with $U_{M,j}$ in the role of C and $U_{M\cup\{i\},j}$ in the role of C', this implies $\alpha_k = \frac{2}{k+1}c$ for all $k = 1, ..., n-1$ with $\alpha_1 = c$.

[14]The apex-measure is introduced for irreflexive digraphs and assigns to player i in an irreflexive digraph C the score $a_i(C) = \sum_{j\in C(i)} 2/\left(|M_j^C|(|M_j^C|+1)\right) + \left(\left(|M_i^C| - 1/\left(M_i^C|+1\right)\right)\right) 1_{\{M_i^C \neq \emptyset\}}$. It can be defined for reflexive digraphs in an obvious way.

Now suppose $M' \in P_0(N)$ with $j \in M'$, $i \notin M'$, and $|M'| = k \in \{1, \dots, n-1\}$. Then $\phi_j(U_{M' \cup \{i\}, j}) - \phi_j(U_{M', j}) = -k\beta_{k+1}/(k+1) + (k-1)\beta_k/k$, and $\phi_i(U_{M' \cup \{i\}, j}) - \phi_i(U_{M', j})$ $= \beta_{k+1}/(k+1)$. By LA3, this implies $\beta_{k+1} = ((k-1)/k)\beta_k$ for all $k \in \{1, ..., n-1\}$. Since $\beta_1 = 0$, it follows that $\beta_k = 0$ for all $k = 2, ..., n$. □

Remark 5.4. If we weaken LA3 to condition LA3′ by strengthening the premiss that $M_j^C \neq \emptyset$ to $M_j^C \setminus \{j\} \neq \emptyset$ as in LA1 and LA2, then we obtain a two-parameter family: there exists a $c \in R$ such that $\alpha_k = \frac{2}{k+1}c$ for all $k = 1, ..., n-1$, and $d \in R$ such that $\beta_k = \frac{1}{k-1}d$ for all $k = 2, ..., n$. We omit the proof of this claim.

6. Concluding remarks

We provide a summary and further relations with the literature, and finally show that the axioms in Theorem 4.4 are independent.

6.1. Summary and further relations with the literature

We axiomatized a class of power indices for invariant digraphs using the axioms of Constant sum, Anonymity, Null player and the Transfer property, inspired by Karos and Peters [12], see Theorem 4.4. By adding one of the three considered link addition axioms, in each case we obtained a subclass of indices related to the Copeland score, β-measure and apex-measure, respectively. One main difference related to the last two measures is that these satisfy a normalization condition where the total sum of the powers of all the players depends on the digraph, whereas for the indices defined in this paper, this sum is always zero. As a consequence, in the present paper players can have negative power, which is not possible according to the β-measure and apex-measure.

The β- and apex-measures also satisfy the Anonymity and Null player axioms, and the β-measure even satisfies a stronger version of NP, in which every player who has no outgoing arcs has power zero. As we saw above, the Transfer and Null player properties imply that we can find the power values for a digraph by adding up the power values over special elementary digraphs, one associated with each player, where such a digraph consists of all the links going into this player (see Lemma 4.2). For digraphs this is also implied by the property of Additivity over Independent partitions used by van den Brink and Gilles [18] to axiomatize the outdegree and β-measure, which requires that the power index for a digraph is the sum of the powers in a partition of the digraph such that every player has a positive indegree in at most one subgraph in the partition.

The Copeland score for digraphs assigns to each player the difference between its outdegree and indegree. Therefore, the power indices satisfying LA1 characterized in Theorem 5.1 can be seen as multiples of the Copeland score.

The power indices characterized by adding LA2 (see Theorem 5.2) are similar to multiples of the β-measure, with the exception that players who are dominated but do not dominate have negative power, while they have zero power according to the β-measure (and even positive power according to the reflexive β-measure). This corresponds well with the interpretation of mutual control.

The power indices characterized by adding LA3 (see Theorem 5.3) are similar to multiples of the apex-measure, with the exception that players who are dominated but do not dominate have negative power, while they have positive power according to the apex-measure. Similar to the apex-measure, according to these indices, a player who is dominated obtains a greater power if he becomes dominated by more predecessors. This reflects the concept that control over a player decreases when more players formally have control over this player, and therefore its level of self-control increases but will never become positive.

We illustrate the different classes of power indices we considered here by an example.

Example 6.1. For $N = \{1, ..., n\}$ and $k \in N \setminus \{n\}$, consider the structures $C_n^k \in D^*$ given by $C_n^k(i) = \{n\}$ for all $i \in \{1, ..., k\}$, and $C_n^k(i) = \varnothing$ otherwise. In Table 1, we give the powers of the successor n and predecessors $i \in M_n^C$ for various numbers of predecessors.

Table 1. Power indices for Example 6.1

	1	2	3	4	...	k
$\phi_i^c,\ i \in M_n^C$	c	c	c	c	...	c
ϕ_n^c	$-c$	$-2c$	$-3c$	$-4c$	...	$-kc$
$\overline{\phi}_i^c,\ i \in M_n^C$	c	$\frac{c}{2}$	$\frac{c}{3}$	$\frac{c}{4}$	...	$\frac{c}{k}$
$\overline{\phi}_n^c$	$-c$	$-c$	$-c$	$-c$	...	$-c$
$\tilde{\phi}_i^c,\ i \in M_n^C$	c	$\frac{c}{3}$	$\frac{c}{6}$	$\frac{c}{10}$	...	$\frac{2c}{k(k+1)}$
$\tilde{\phi}_n^c$	$-c$	$-\frac{2c}{3}$	$-\frac{c}{2}$	$-\frac{2c}{5}$	...	$-\frac{2c}{k+1}$

Finally, we remark that for an arbitrary digraph $C \in \mathcal{D}$ one could define $\phi(C) = \phi(C^*)$, where C^* is the invariant extension of C obtained as explained in Section 2, namely by adding links for each pair of players between whom there exists a directed path.

6.2. Independence of the axioms in Theorem 4.4

By providing four examples, we show that the axioms in the main characterization, Theorem 4.4, are logically independent.

Not NP, but CS, AN, TP. For $U_{M,j} \in \mathcal{D}^*$ with $M = \{j\}$ define $\phi_i(U_{M,j}) = 1/(n-1)$ for all $i \in N \backslash \{j\}$ and $\phi_j(U_{M,j}) = -1$. In all other cases, define $\phi(U_{M,j})$ as in Lemma 4.3. Extend to the whole of $\mathcal{D}^*$ by taking sums as in Theorem 4.4.

Not CS, but NP, AN, TP. For $U_{M,j} \in \mathcal{D}^*$ with $M = \{j\}$ define $\phi_i(U_{M,j}) = 0$ for all $i \in N \backslash \{j\}$ and $\phi_j(U_{M,j}) = 1$. In all other cases, define $\phi(U_{M,j})$ as in Lemma 4.3. Extend to the whole of $\mathcal{D}^*$ by taking sums as in Theorem 4.4.

Not AN, but CS, NP, TP. For $U_{M,j} \in \mathcal{D}^*$ with $|M| = 2$ and $j \notin M$ (hence $n \geq 3$) define $\phi_i(U_{M,j}) = 0$ for all $i \notin M \cup \{j\}$, also, for $i \in M$ when $i > k$ and $M = \{i, k\}$, $\phi_k(U_{M,j}) = 1$, and $\phi_j(U_{M,j}) = -1$. In all other cases, define $\phi(U_{M,j})$ as in Lemma 4.3. Extend to the whole of $\mathcal{D}^*$ by taking sums as in Theorem 4.4.

Not TP, but AN, CS, NP. For $C \in \mathcal{D}^*$ define

$$n_1 = |\{i \in N : C(i) \neq \emptyset,\ C_i = O\}|,\ n_2 = |\{i \in N : C(i) = \emptyset,\ C_i \neq O\}|$$

If $n_1, n_2 > 0$, then define

$$\phi_i(C) = \begin{cases} 1/n_1 & \text{if } C(i) \neq \emptyset,\ C_i = O \\ -1/n_2 & \text{if } C(i) = \emptyset,\ C_i \neq O \\ 0 & \text{otherwise} \end{cases}$$

and define $\phi(C) = (0, ..., 0)$ in all other cases. To see that ϕ does not satisfy TP, let $n = 4$ and consider $C, D \in D^*$ with $C(1) = 2$, $D(3) = 4$, and $C(S), D(S) = \varnothing$ in all other cases. Then $\phi(C \cup D) + \phi(C \cap D) = (1/2, -1/2, 1/2, -1/2)$, whereas $\phi(C) = (1, -1, 0, 0)$ and $\phi(D) = (0, 0, 1, -1)$, thus violating (1) and therefore TP.

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AN APPROACH FROM BANKRUPTCY RULES APPLIED TO THE APPORTIONMENT PROBLEM IN PROPORTIONAL ELECTORAL SYSTEMS

(Discrete) bankruptcy problems associated with apportionment problems have been defined. The authors studied which allocations for apportionment problems have been obtained when (discrete) bankruptcy rules were applied to the associated bankruptcy problems. They have shown that the (discrete) constrained equal losses rule coincides with the greatest remainder method for apportionment problems. Furthermore, new properties related to governability have been proposed for apportionment methods. Finally, several apportionment methods satisfying governability properties have been applied to the case of the Spanish Elections in 2015.

Keywords: *apportionment rules, bankruptcy rules, standard of comparisons, proportional electoral systems, governability*

1. Introduction

Many situations of social and economic interest fall under the umbrella of allocation issues [9], including apportionment problems and bankruptcy problems, on which one can find an extensive literature. The apportionment problem consists of determining how to divide a given (nonnegative) integer number among a set of individuals according to their respective sizes. In electoral systems with proportional representation, the apportionment problem arises in two situations: (i) in the allotment of seats to constitu-

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encies, if any, and (ii) in the allocation of seats to the political parties within each constituency. There are many apportionment methods, but none satisfies all desirable basic properties (see [5, 22]). Nevertheless, electoral systems do not only involve the distribution of seats, but also other questions must be taken into account. Horowitz considered six aims for analyzing or designing an electoral system [20], but not all of them are mutually compatible. Two of them are proportionality of seats to votes and durable governments. These are not compatible and, hence, a balance between them is necessary.

The bankruptcy problem consists of determining how to divide a given estate (perfectly divisible) among a set of creditors according their claims on the estate. There are also many bankruptcy rules satisfying various natural and reasonable properties. Bankruptcy rules are characterized using different subsets of these properties [34, 35]. In the literature, we can also find bankruptcy problems with indivisibilities, so-called discrete bankruptcy problems (see [28, 29, 17–19, 23, 24, 10, 11] among others). Therefore, they could be approached in a similar way to apportionment problems. In this paper, we do the converse, we approach apportionment problems from the point of view of (discrete) bankruptcy problems. As far as we know, this approach is new. With this purpose in mind, for each apportionment problem we define a bankruptcy problem associated with it; we can thereby apply bankruptcy rules and obtain allotments for the corresponding apportionment problem. We show that the discrete constrained equal losses rule (DCEL) provides the same allocation as the greatest remainder method. Similarly, the M-up method for bankruptcy rules, associated with a proper standard of comparison will again correspond to the greatest remainder method. With this aim, we introduce several properties related to governability and the corresponding up-methods are defined. In addition, we also compare the performance of these up-methods with the greatest remainder method and the d'Hondt method for the case study of the Spanish elections in 2015.

The rest of the paper is organized as follows. Section 2 contains basic concepts on apportionment problems and bankruptcy problems. In Section 3, we introduce bankruptcy problems associated with apportionment problems. We show that the discrete constrained equal losses rule coincides with the greatest remainder method. We also introduce properties related to governability and define new apportionment rules based on bankruptcy rules. In Section 4 we apply the apportionment rules defined in Section 3 to the case study of the Spanish elections in 2015. Section 5 concludes.

2. Preliminaries. Apportionment problems and bankruptcy problems

In this section, some notation and basic concepts related to apportionment problems and bankruptcy problems will be introduced.

Let $\mathfrak{I}$ be the set of all nonnegative integer numbers {0, 1, 2, 3, ...}. Given $N = \{1, 2, 3, ..., n\}$, V^N is the set of all nonnegative integer n-vectors $v = (v_1, v_2, ..., v_n)$. Let $\mathfrak{R}_+^N$ be the set of all nonnegative n-vectors $d = (d_1, d_2, ..., d_n)$.

2.1. Apportionment problems

Apportionment problems concern allocating available resources in integral parts to a number of individuals according to a set of nonnegative integers, each representing the size of one individual. An example in politics is the allocation of seats in a legislature among political parties or constituencies according to the number of votes or inhabitants, respectively. The main goal in an apportionment problem is to find an allocation as proportional as possible to the set of nonnegative integers associated to the claimants. The statement of the problem is very simple, but its solution is not easy and there are several different methods for solving it (for details on apportionment problems, see [5] and [22]).

An apportionment problem is defined by a 3-tuple (N, v, h), where N is a finite set of individuals (for example, political parties), $v \in V^N$ is the vector of their sizes (for example, the number of votes) and $h \in \mathfrak{I}$ is the amount to be distributed (for example, the number of seats).

An apportionment method A is a function from $V^N \times \mathfrak{I}$ to V^N such that

$$A((v_1, v_2, ..., v_n); h) \subset V^N$$

and for each $(a_1, a_2, ..., a_n) \in A((v_1, v_2, ..., v_n); h)$, $\sum_{i=1}^{n} a_i = h.$

Therefore, when the apportionment method does not provide a unique allocation vector, another criterion must be considered to select just one allocation, for example, some kind of priority.

Given a set of individuals N, a vector $v = (v_1, v_2, ..., v_n) \in V^N$ and $h \in \mathfrak{I}$, three allotments can be defined for each individual $i \in N$:

- Exact allotment: $q_i = \dfrac{v_i}{\sum_{j=1}^{n} v_j} h$.
- Lower allotment: $\lfloor q_i \rfloor$ is the integer part of q_i.
- Upper allotment: $\lceil q_i \rceil = \lfloor q_i \rfloor + 1$, if q_i is not an integer number and $\lceil q_i \rceil = \lfloor q_i \rfloor$, if q_i is an integer number.

In general, two different types of apportionment methods can be distinguished: quotient/quota methods and divisor methods, but other methods can be found in the literature, for example, minimax methods [13, 14].

Using quotient methods, a fixed quota C is chosen for distributing h and the size of each electorate is divided by C. Hence, the number of units allocated to each electorate is given by $[v_i/C]$, and when $\sum_{i=1}^{n} \lfloor v_i/C \rfloor$ does not sum to h, then some criterion is used to adapt the allocation to h. This quota C measures the minimum size required to have a right to one unit. Depending on the selected quota and the tie breaking rule, different quotient methods are obtained.

The method of greatest remainders (GR) (also known as Hamilton's method) is the quotient method using the so-called natural quota[3] or Hare quota[4] given by $H = \sum_{j=1}^{n} v_j/h$ and when $\sum_{i=1}^{n} \lfloor q_i \rfloor$ sums to less than h, the priority or deservingness criterion used to assign one extra unit to adapt the allocation to h is the greatest remainder $q_i - \lfloor q_i \rfloor$.

Using divisor methods, the h units are allocated in h steps one by one, in each step the size of each electorate is divided by the divisor $d(x)$, which is a function of the number of units x allocated to that electorate in the previous steps, and the current unit is allocated to the individual with the highest quotient. In the initial step, all individuals have previously received 0 units. Depending on the divisor function used, different divisor methods are obtained. Some well-known examples are the following:

• D'Hondt method (also known as the greatest divisors method or Jefferson's method): $d(x) = x+1,\ x = 0,\ 1, 2, \ldots$

• Sainte–Laguë method (also known as the major fractions method or Webster's method): $d(x) = 2x+1,\ x = 0,\ 1,\ 2, \ldots$

• The geometric mean method (also known as the equal proportions method or the Hill–Huntington method): $d(x) = \sqrt{(x+1)(x+2)},\ x = 0,\ 1,\ 2, \ldots$

There are several properties that may be desirable for an apportionment method. In the paper by Balinski and Ramirez [2] scale-invariance, exactness and anonymity are considered as the three most fundamental properties:

• Scale invariance: for every $\lambda > 0$, $A((\lambda v_1, \lambda v_2, \ldots, \lambda v_n); h) = A((v_1, v_2, \ldots, v_n); h)$.

[3]Note that the exact allotment is given by $q_i \equiv v_i/H$.

[4]Another common quota used in quotient methods is the Droop quota given by $D = \left\lfloor 1 + \sum_{j=1}^{n} v_j/(h+1) \right\rfloor$.

• Exactness: If $\sum_{j=1}^{n} v_j = h,$ then $A((v_1, v_2, ..., v_n); h) = \{(v_1, v_2, ..., v_n)\}$.

• Anonymity: If π is a permutation of N, then $A((v_{\pi(1)}, v_{\pi(2)}, ..., v_{\pi(n)}); h)$ $= \pi(A((v_1, v_2, ..., v_n); h))$.

These three properties are satisfied by all divisor methods and by the method of greatest remainders. However, there are more properties which may be desirable for apportionment problems:

• Balancedness: if $a \in A((v_1, v_2, ..., v_n); h),$ then $v_i = v_j$ implies $|a_i - a_j| \le 1.$

• Monotonicity: If $a \in A((v_1, v_2, ..., v_n); h)$, then there exists $a' \in A((v_1, v_2, ..., v_n);$ $h+1)$ such that $a' \ge a.$

• Responsiveness: If $a \in A((v_1, v_2, ..., v_n); h)$, then $v_i > v_j$ implies $a_i \ge a_j$.

• Lower allotment: $\forall a \in A((v_1, v_2, ..., v_n); h),\ a_i \ge \lfloor q_i \rfloor,\ \forall i \in N.$

• Upper allotment: $\forall a \in A((v_1, v_2, ..., v_n); h),\ a_i \le \lceil q_i \rceil,\ \forall i \in N.$

The greatest remainder method does not satisfy monotonicity (the Alabama paradox) and the divisor methods do not satisfy, in general, the allotment properties. Using these properties and other properties, the different apportionment methods can be characterized. For example, divisor methods are characterized by Balinski and Young [4, 5].

2.2. Bankruptcy problems

Bankruptcy problems concern allocating available resources (estate) to a number of individuals (claimants) according to their demands/claims when all these demands add up to more than the available resources. In a classical bankruptcy situation, the available resources are perfectly divisible, but over the last years the case of indivisible resources has also been studied (see, for example [28, 29, 17–19, 23, 24, 16, 10, 11, 6]). The main goal in a bankruptcy problem is to find an allocation which is as fair as possible, taking into account the demands of the claimants. Many solutions have been proposed depending on the principle(s) of fairness used (see, for a survey on bankruptcy problems, [34, 35]).

A bankruptcy problem is defined by a 3-tuple (N, d, E), where N is the finite set of claimants, $d \in \Re_+^N$ is the vector of their demands/claims and $E \in \Re_+$ is the amount of available resources (perfectly divisible), such that $\sum_{i=1}^{n} d_i \ge E,$ i.e., there are not enough resources to fully satisfy the demands of the claimants.

A bankruptcy rule B is a function from $\Re_+^N \times \Re_+$ to $\Re_+^N$ such that

$$B((d_1, d_2, ..., d_n); E) \in \Re_+^N$$

$$\sum_{i=1}^{n} d_i \geq E \text{ and } \sum_{i=1}^{n} B_i((d_1, d_2, ..., d_n); E) = E$$

There are many different bankruptcy rules, but in this paper we only consider three of them: the proportional rule, the constrained equal awards (CEA) rule and the constrained equal losses (CEL) rule.

- The proportional rule (P): $P((d_1, d_2, ..., d_n); E) = \frac{E}{\sum_{i=1}^{n} d_i}(d_1, d_2, ..., d_n)$.
- The constrained equal awards rule (CEA): CEA$((d_1, d_2, ..., d_n); E) = (\min\{d_1, \lambda\}, \min\{d_2, \lambda\}, ..., \min\{d_n, \lambda\})$, where $\lambda > 0$ is chosen so that $\sum_{i=1}^{n} \min\{d_i, \lambda\} = E$.
- The constrained equal losses rule (CEL): CEL$((d_1, d_2, ..., d_n); E) = (\max\{0, d_1 - \lambda\}, \max\{0, d_2 - \lambda\}, ..., \max\{0, d_n - \lambda\})$, where $\lambda > 0$ is chosen so that $\sum_{i=1}^{n} \max\{d_i - \lambda\} = E$.

There are several properties that are very natural for a bankruptcy rule and some of them are completely analogous to those for apportionment rules.

- Homogeneity: for every $\lambda > 0$, $B((\lambda d_1, \lambda d_2, ..., \lambda d_n); \lambda E) = \lambda B((d_1, d_2, ..., d_n); E)$.
- Efficiency: $\sum_{j=1}^{n} B_j((d_1, d_2, ..., d_n); E) = E$.
- Anonymity: If π is a permutation of N, then $B((d_{\pi(1)}, d_{\pi(2)}, ..., d_{\pi(n)}); E) = \pi(B((d_1, d_2, ..., d_n); E))$.
- Equal treatment of equals: If $d_i = d_j$, then $B_i(d; E) = B_j(d; E)$.
- Monotonicity: If $\sum_{i=1}^{n} d_j \geq E' > E$, then $B((d_1, d_2, ..., d_n); E') \geq B((d_1, d_2, ..., d_n); E')$.
- Order-preservation: If $d_i > d_j$, then $B_i(d; E) \geq B_j(d; E)$ and $d_i - B_i(d; E) \geq d_j - B_j(d; E)$.
- Respect of minimal rights: $B_i(d; E) \geq \max\{E - \sum_{j:j\neq i} d_j, 0\}$, for all i.
- Boundedness of claims: $B_i(d; E) \leq d_i$, for all i.

An excellent survey on axiomatic analysis of bankruptcy rules is given by Thomson [34, 35].

3. The apportionment problem as a bankruptcy problem

As was pointed out in Section 2.2, over the last years bankruptcy problems with indivisibilities have also been studied. Thus, if we consider bankruptcy situations in which the demands are nonnegative integer numbers and the estate consists of a nonnegative integer number of indivisible units, then the definitions of the discrete bankruptcy problem and discrete bankruptcy rule are similar to the classical ones:

A discrete bankruptcy problem is defined by a 3-tuple (N, d, E), where N is the finite set of claimants, $d \in V^N$ is the vector of their demands/claims and $E \in \Im$ is the amount of available resources, such that $\sum_{i=1}^{n} d_i \geq E$.

A discrete bankruptcy rule B is a function from $V^N \times \Im$ to V^N given by

$$B((d_1, d_2, ..., d_n); E) \subset V^N$$

such that for each $(b_1, b_2, ..., b_n) \in B((d_1, d_2, ..., d_n); E)$, $\sum_{i=1}^{n} b_i = E.$

We should highlight that the only difference between the definition of the bankruptcy problem and the definition of the discrete bankruptcy problem is that in the former the demands and the estate are non-negative real numbers and in the latter they are non-negative integers. Considering the rules, in addition to the difference between using real or integer numbers, bankruptcy rules are single valued, while discrete bankruptcy rules are set valued. Therefore, when a discrete bankruptcy rule does not provide a unique allocation vector, another criterion must be considered to select a single allocation, for example, some kind of priority or deservingness.

Note that both the apportionment problem and the discrete bankruptcy problem are very similar, the only difference is the condition $\sum_{i=1}^{n} d_i \geq E.$ Bankruptcy problems with indivisibilities are usually sorted out using methods of assigning priority. The simplest method of assigning priority is to consider a priority order defined over the set of claimants to break possible ties for the very last units to be allocated. Another alternative is to use the idea of a standard of comparison [36]. A standard of comparison is a strict binary relation ρ defined over all agent–claim pairs such that $(i, x+1)\rho(i, x)$ for all $x \in \Im$, where $(i, x)\rho(j, y)$ means that agent i with claim x units has priority over agent j with claim y units. A standard of comparison is called monotonic if $(i, x+1)\rho(j, x)$ for all claimants i, j, and for all $x \in \Im$ [19]. Likewise, Herrero and Martínez [19] illustrate how to define a monotonic standard of comparison which coincides with the d'Hondt method used in apportionment problems. In general, any method of apportionment could be used as a discrete bankruptcy rule with a suitable standard of comparison.

Next we do the converse, that is, we will use discrete bankruptcy rules for solving apportionment problems. To do this, first we must define the discrete bankruptcy problem associated with a given apportionment problem.

Definition 1. Given an apportionment problem (N, v, h), we define an associated discrete bankruptcy problem (N, d, E) as follows:

- N is the set of claimants,
- $d_i = \lfloor q_i \rfloor + 1$, for all $i \in N$,
- $E = h$.

It is obvious that the problem (N, d, E) is a bankruptcy problem because $\sum_{r \in N} d_i \geq E$. Now we can apply any discrete bankruptcy rule to this problem and obtain an allocation for the apportionment problem (N, v, h). One of these rules is the discrete constrained equal losses rule (DCEL), which is applied in two steps. Firstly, the CEL rule is applied to the problem as if it was a bankruptcy problem without indivisibilities and each claimant is assigned the corresponding integer part. In the second step, an extra unit is allocated to each claimant whose allocation according to CEL is not integer, following a priority order. For this discrete rule, Giménez-Gómez and Vilella [16] prove that the DCEL rule is the recursive discrete P-rights rule[5] with P = {WOP}, where WOP is the property of weak order preservation, which is given by:

- If $d_i > d_j$, then $B_i(d; E) \geq B_j(d; E)$ and $d_i - B_i(d; E) \geq d_j - B_j(d; E)$.
- If $d_i = d_j$, then $\left| B_i(d; E) - B_j(d; E) \right| \leq 1$.

Note that condition (*i*) is related to the property of responsiveness and (*ii*) is related to the property of being balanced for apportionment problems.

Theorem 2. Let (N, v, h) be an apportionment problem and (N, d, E) the associated discrete bankruptcy problem. Let σ be the priority rule defined over the set N as follows:

$$i \sigma j \Leftrightarrow q_i - \lfloor q_i \rfloor > q_j - \lfloor q_j \rfloor$$

Thus DCEL(d; E) = GR(v; h), provided that the tie breaking rule used is the same for both methods.

Proof. We will first prove that $\lfloor CEL_i(d; E) \rfloor = \lfloor q_i \rfloor$ for all $i \in N$. Indeed, we have the following:

[5]See [15] for details on recursive P-rights processes.

$$\sum_{i\in N} d_i - E = \sum_{i\in N} \lfloor q_i \rfloor + n - h \le \sum_{i\in N} q_i + n - h = n \qquad (1)$$

where $N = \{1, 2, ..., n\}$. This implies that

$$e = \frac{\sum_{i\in N} d_i - E}{n} \le 1$$

Hence, taking into account that $d_i \ge 1$ for all $i \in N$, we can subtract e from each d_i for all $i \in N$, and all these differences are nonnegative. Thus, we have

$$\sum_{i\in N} \max\{0, d_i - e\} = \sum_{i\in N} (d_i - e) = E$$

Therefore, by definition of the CEL rule and the structure of the claims, we obtain that $\text{CEL}_i(d; E) = \lfloor q_i \rfloor + 1 - e$, and this implies that $\lfloor \text{CEL}_i(d; E) \rfloor = \lfloor q_i \rfloor$. This completes the first step of the DCEL rule.

For the second step, we know that $\sum_{i\in N} d_i - E \le n$, therefore each claimant will receive at most one extra unit. Applying the priority rule σ, the extra units will go to the claimants with the greatest remainders $q_i - \lfloor q_i \rfloor$. Thus $\text{DCEL}_i(d; E) = \lfloor q_i \rfloor + 1$ for the claimants with highest priorities (greatest remainders) and $\text{DCEL}_i(d; E) = \lfloor q_i \rfloor$ for the claimants with lowest priorities.

Note that if $\lfloor q_i \rfloor$ is an integer, then agent i will not receive an extra unit. Indeed, let us consider the set $I = \{i \in N : q_i \in \Im\} \subset N$ and $|I|$ its cardinality, then we have

$$E - \sum_{i\in N} \lfloor q_i \rfloor = \sum_{i\in N} q_i - \sum_{i\in N} \lfloor q_i \rfloor = \sum_{i\notin I} (q_i - \lfloor q_i \rfloor) \le n - |I|$$

Thus, the number of extra units to be allocated in the second step is less than or equal to $n - |I|$, therefore the agents in I will not receive any extra unit, because they are the agents with the lowest priorities according to σ.

If we apply the GR method to the apportionment problem, then we obtain the same results, therefore DCEL(d; E) = GR(v; h).

Finally, in the case of a tie we could use the same tiebreak rule in both the DCEL rule and the GR method and the result holds. If no tiebreak rule is applied, then we can conclude that the allocation sets defined by the DCEL rule and the GR method coincide. This completes the proof.

In the following example, we show that the DCEA rule does not coincide, in general, with the GR method for apportionment problems.

Example 3. Let us consider the apportionment problem (N, v, h), where N = {1, 2, 3, 4, 5}, v = (1000, 500, 300, 150, 50) and h = 8. In this case, the exact allotment vector is q = (4, 2, 1.2, 0.6, 0.2) and GR(v; h) = (4, 2, 1, 1, 0). The associated bankruptcy problem is given by N = {1, 2, 3, 4, 5}, d = (5, 3, 2, 1, 1) and E = 8. CEA(d; E) = (2, 2, 2, 1, 1) = DCEA(d; E). However, CEL(d; E) = (4.2, 2.2, 1.2, 0.2, 0.2) and DCEL(d; E) = (4 + 0, 2 + 0, 1 + 0, 0 + 1, 0 + 0) = (4, 2, 1, 1, 0) = GR(v; h), as expected.

Remark 4. In Example 3, if we take $d_i = \lceil q_i \rceil$, then d = (4, 2, 2, 1, 1). This implies CEL(d; E) = (3.6, 1.6, 1.6, 0.6, 0.6) and DCEL(d; E) = (3 + 0, 1 + 0, 1 + 1, 0 + 1, 0 + 1) = (3, 1, 2, 1, 1) $\neq$ GR(v, h). Therefore, we have to consider a new priority rule, for example, $i \sigma j \Leftrightarrow q_i - \lfloor \mathrm{CEL}_i(d;\, E) \rfloor > q_j - \lfloor \mathrm{CEL}_j(d;\, E) \rfloor$, and then DCEL($d$; E) = (3 + 1, 1 + 1, 1 + 0, 0 + 1, 0 + 0) = (4, 2, 1, 1, 0) = GR(v, h).

M-up methods and M-down methods are other discrete bankruptcy rules [19]. These methods are applied unit by unit according to a monotonic standard of comparison. In each step, one unit is allocated (M-up methods) or subtracted (M-down methods) to the agent corresponding to the agent-claim pair with the highest priority and this agent-claim pair is removed for the next step. Herrero and Martínez [19] show that any M-down method can be interpreted as a discrete version of the constrained equal awards rule. This rule makes it advantageous for parties to split and this is not a desirable property for apportionment problems in electoral systems. In fact, rules which make it advantageous for parties to merge are preferred, because there is an incentive to form coalitions before and not after the election. For this reason, in this paper we will focus on M-up methods (see [8] for details on manipulations involving coalitions in bankruptcy problems).

Given an apportionment problem (N, v, h) and the associated discrete bankruptcy problem (N, d, E), we consider the following monotonic standard of comparison ρ^* defined over all agent-claim pairs:

- $(i,\, x)\rho^*(j,\, y) \Leftrightarrow x > y$, for all $i, j \in N$ and $x, y \in \Im$.
- $(i,\, x)\rho^*(j,\, x) \Leftrightarrow q_i - \lfloor q_i \rfloor > q_j - \lfloor q_j \rfloor$, for all $i, j \in N$ such that $i \neq j$, and $x \in \Im$.

Theorem 5. Let (N, v, h) be an apportionment problem and (N, d, E) the associated discrete bankruptcy problem. Let U^{ρ^*} denote the solution of the bankruptcy problem according to the M-up method with the monotonic standard of comparison ρ^*. Then, $U^{\rho^*}(d;\, E)$ = GR(v; h), provided that the tie breaking rule is the same for both methods.

Proof. For each $i \in N = \{1, 2, ..., n\}$, there are d_i pairs $(i, 1), (i, 2), ..., (i, d_i)$. By Eq. (1), we know that after $\sum_{i \in N} \lfloor q_i \rfloor$ steps the U^{ρ^*} method has allocated exactly $\lfloor q_i \rfloor$ units to agent i, for all $i \in N$. At this point, the remaining pairs are $(1, 1), (2, 1), ..., (n, 1)$ and the remaining units are allocated to the agents with the highest priority according to criterion (ii) in the definition of ρ^* above. These agents are those with the greatest remainders, so $U^{\rho^*}(d; E) = \text{GR}(v; h)$.

Finally, in the case of a tie we could use the same tiebreak rule for both the U^{ρ^*} method and the GR method and the result holds. If no tiebreak rule is applied, then the sets of allocations defined by the U^{ρ^*} method and the GR method coincide. This completes the proof.

Given an apportionment problem (N, v, h) and the associated discrete bankruptcy problem (N, d, E), we can consider the following monotonic standard of comparison ρ^g defined over all agent-claim pairs:

- $(i, x)\rho^g(j, y) \Leftrightarrow x > y$, for all $i, j \in N$ and $x, y \in \Im$.
- $(i, x)\rho^g(j, x) \Leftrightarrow v_i > v_j$, for all $i, j \in N$ such that $i \neq j$, and $x \in \Im$.

Condition (i) of the monotonic standard of comparison ρ^g is the same as for ρ^*, but condition (ii) favors agents with the largest sizes (number of votes) instead of agents with the greatest remainders. Therefore, the apportionment obtained by applying U^{ρ^g} does not satisfy the upper allotment property, in general, but it exceeds the upper allotment by at most one unit. Nevertheless, in electoral systems, this shortcoming could be seen as an interesting quality in terms of governability because it could make the formation of durable government coalitions easier. For example, in [20] six goals for an electoral system are established, one of them is durable governments. Related to this goal, we introduce the following property for apportionment methods:

- Governability. If $a \in A((v_1, v_2, ..., v_n); h)$, then $v_i > v_j$ implies $a_i - \lfloor q_i \rfloor \geq a_j - \lfloor q_j \rfloor$.

Example 6. Let us consider again the apportionment problem (N, v, h), where $N = \{1, 2, 3, 4, 5\}$, $v = (1000, 500, 300, 150, 50)$ and $h = 8$. The associated discrete bankruptcy problem is given by $N = \{1, 2, 3, 4, 5\}$, $d = (5, 3, 2, 1, 1)$ and $E = 8$. To calculate $\rho^g(d; E)$, we consider the chain of the first 8 pairs with highest priorities:

$$(1, 5)\rho^g(1, 4)\rho^g(1, 3)\rho^g(2, 3)\rho^g(1, 2)\rho^g(2, 2)\rho^g(3, 2)\rho^g(1, 1).$$

Therefore, $U^{\rho^g}(d; E) = (5, 2, 1, 0, 0)$.

It is easy to check that the apportionment obtained by applying U^{ρ^g} satisfies the property of governability, but the GR method does not. A stronger version of governability can be stated in the following terms

- Strong governability. If $a \in A((v_1, v_2, ..., v_n); h)$ and $v_i > v_j$, then

$$\begin{cases} a_i - \lfloor q_i \rfloor > a_j - \lfloor q_j \rfloor & \text{if } a_j - \lfloor q_j \rfloor > 0 \\ a_i - \lfloor q_i \rfloor \geq a_j - \lfloor q_j \rfloor & \text{if } a_j - \lfloor q_j \rfloor = 0 \end{cases}$$

This property means that no agent with fewer votes can obtain a better or equal extra allocation than another with more votes, with respect to their lower bounds.

The extreme case which favors the most popular party while respecting the lower bounds is the following:

- All for the winner If $v_i = \max\{v_1, v_2, ..., v_n\}$, there exists $a \in A(v; h)$ such that

$$a_i = \lfloor q_i \rfloor + \left(h - \sum_{j \in N} \lfloor q_j \rfloor \right)$$

In all cases, the lower bound is respected, if we consider a bankruptcy situation in which the vector of lower bounds plays the role of a reference point. Additionally, each agent has an associated upper bound, which represents the maximum number of units that can be allocated to her, so that the upper bounds add up to more than the available resource units. This situation resembles bankruptcy problems with references [31, 32]. Therefore, we could also define a bankruptcy problem with references associated with an apportionment problem, including a priority rule or standard of comparison for allocating all the units exceeding the sum of the reference point.

To finish this section, we consider the definition of the bankruptcy problem associated with the apportionment problem given in Remark 4, i.e., $d_i = \lceil q_i \rceil$. Given an apportionment problem (N, v, h) and the associated discrete bankruptcy problem (N, d, E) with $d_i = \lceil q_i \rceil$ for all $i \in N$, we consider the following (non-monotonic) standard of comparison ρ^{BY} defined over all agent-claim pairs:

$$(i, x)\rho^{BY}(j, y) \Leftrightarrow \frac{v_i}{x} > \frac{v_j}{y} \text{ for all } i, j \in N \text{ and } x, y \in \Im$$

The $U^{\rho^{BY}}$ method is very similar (or equivalent) to the quota method for apportionment problems [3]. It is a variant of the d'Hondt method, but respecting the upper bound.

In fact, if we set $d_i = h$ for all $i \in N$ in the associated discrete bankruptcy problem, then we obtain the d'Hondt method.

4. Case study. Spanish elections in 2015

The discussion on the need to reform an electoral system recurrently arises in democratic countries after elections, because there are different groups which consider the final result does not correspond with the reality of the society. In the case of Spain, several proposals for reforming the Spanish electoral system can be found in the literature (see, for example, [26]). Two common complaints have traditionally been made (i) the non-proportional distribution of seats in a parliament regarding the number of votes obtained by each party and (ii) the number of voters represented by a seat. However, perfect proportionality is not possible and there are several methods for measuring the proportionality or disproportionality of an electoral system (see, for example, [21, 12]). In particular, in electoral systems aimed at proportional representation, there are two main variants, national lists and constituency lists, as well as combinations of both (see, for an example of bi-apportionment by Maier et al. [25]). It is usually claimed that the method of apportionment used to allocate first the seats among the constituencies and then, within each constituency, the seats among the parties is responsible for the resulting disproportionality (for example, in papers by Sánchez-Soriano [23] and Curiel [7], the Spanish Electoral System and the Electoral System of Surinam, respectively, have been analyzed from a mathematical point of view). However, this is not the only controversial aspect of an electoral system. For example, in [20], Mercik analyzes the veto power of the President of Poland. In fact, the goals of an electoral system are more than just proportionality. In the paper by Horowitz [20], six goals for an electoral system have been established: proportionality of seats to votes, accountability to constituents, durable governments, victory of the Condorcet winner, interethnic and interreligious conciliation, and office holding by minorities.

In this section, we analyze the distribution of seats in the Spanish Congress after the Spanish elections in 2015. The Congress consists of 350 members. To elect the members of the Congress, the country is divided into 52 constituencies, 50 provinces and 2 autonomous cities. The seats are allocated to the constituencies in proportion to their populations using the greatest remainder method. Because the populations of the constituencies are quite different, the constituencies are quite asymmetric in terms of the number of seats. For each constituency, the political parties present ordered lists of their candidates for election and the seats are allocated to political parties using the d'Hondt method. We will use the results of the Spanish elections in 2015 to check the effect of using several of the methods of apportionment introduced in Section 3 with regard to the property of governability. The goal is to analyze how to avoid directly giving seats to the winning party in elections, which favors governability, as is done in the Italian and Greek electoral systems, among others.

When we use quotient methods, we must make a decision on how to address the minimum thresholds required to obtain a seat. In particular, what should be done when some parties do not obtain enough votes to reach the threshold, because, depending on the decision, the quotas of the other political parties can vary. In our analysis, we assume that there is no minimum threshold and we will use the following methods of apportionment: d'Hondt, the greatest remainders (GR), the Up method with a monotonic standard of comparison satisfying governability (U^{ρ^g}), the Up method with a standard of comparison satisfying strong governability ($U^{\rho^{sg}}$) and the Up method with a standard of comparison satisfying "all for the winner" (U^{ρ^w}).

Table 1. Distribution of seats in the Spanish Congress for five apportionment rules 52 constituencies (50 provinces and 2 autonomous cities)

Political party	Exact allotment	Present (d'Hondt)	GR	U^{ρ^g}	$U^{\rho^{sg}}$	U^{ρ^w}
PP	101.22	123	103	125	162	181
PSOE	77.56	90	86	95	78	66
Podemos	44.74	42	49	44	32	30
Ciudadanos	49.16	40	52	28	21	21
En Comú Podem	13.01	12	11	13	14	16
Compromís–Podemos	9.42	9	8	10	9	7
ERC	8.42	9	8	10	9	6
DIL	7.93	8	7	8	11	13
En Marea	5.74	6	6	7	5	2
PNV	4.23	6	4	6	6	5
IU	12.96	2	9	1	1	1
Bildu	3.06	2	3	3	2	2
CC	1.14	1	2	–		
UDC	0.91	–	1	–	–	–
MÉS	0.47	–	1	–	–	–
Total seats	–	350	350	350	350	350

In Table 1, we observe that the effect of the all-for-the-winner method (W) is similar to that of elections in uninominal constituencies. When the constituencies have few seats (less than 5) to be elected, all (or almost all) of the seats go to the winner and there are several parties with more than 15% of the votes. Consequently, this method of apportionment reaches the desired effect of facilitating the formation of a government. The winner obtains an absolute majority of the seats, but proportionality is drastically reduced. Furthermore, we observe that the method of apportionment satisfying governability performs similarly to the d'Hondt method, only Ciudadanos is clearly harmed in favor of the other parties. Therefore, in this scenario, the method of apportionment satisfying governability does not get the desired effect of facilitating governance, but the method of apportionment satisfying strong governability does. Of course, three relatively popular parties, PSOE, Podemos and Ciudadanos, are clearly harmed in favor of the winning party, PP, but a coalition between PP and Ciudadanos would be sufficient to form a government. Therefore, this method of apportionment would improve the possibilities of forming a stable government. As it is well-known, the greatest remainder method produces a greater dispersion in the distribution of seats among political parties. Finally, as for the number of political parties represented in the Congress, all of these methods except for the GR method give almost the same range of ideologies.

Table 2. Distribution of seats in the Spanish Congress for five apportionment rules
19 constituencies (17 regions and 2 autonomous cities)

Political party	Exact allotment	Present (d'Hondt)	GR	U^{ρ^g}	$U^{\rho^{sg}}$	U^{ρ^w}
PP	101.22	114	104	113	127	136
PSOE	77.56	82	79	87	81	77
Podemos	44.74	49	48	47	44	41
Ciudadanos	49.16	46	49	44	40	40
En Comú Podem	13.01	13	12	12	14	15
Compromís–Podemos	9.42	8	8	9	8	8
ERC	8.42	8	8	8	8	7
DIL	7.93	7	7	8	7	7
En Marea	5.74	6	6	6	6	5
PNV	4.23	5	4	5	5	4
IU	12.96	7	15	6	6	6
Bildu	3.06	3	4	3	2	2
CC	1.14	1	1	1	1	1
Pacma	3.08	–	2	–	–	–
Nós–Candidatura Galega	0.99	1	1	1	1	1
UDC	0.91	–	1	–	–	–
MÉS	0.47	–	1	–	–	–
Total seats	–	350	350	350	350	350

In Table 2, the number of seats in each constituency (region) has been calculated as the sum of the number of seats in the provinces belonging to it. For d'Hondt, the apportionment satisfying governability and the greatest remainder methods, we can derive the distribution of seats are similar to those in Table 1. In this scenario, with fewer constituencies, we observe that the effect of the properties of strong governability and all for the winner declines significantly with respect to the scenario with more constituencies. Nonetheless, the winning party still receives a more than proportional share of the seats, without affecting too much the proportionality of the distribution of seats in the Congress. However, this bonus (13 or 23 seats, respectively) would have a positive effect on the possibilities of forming a government, because a coalition between PP and Ciudadanos would have a total of 167 seats (close to the 176 seats required for an absolute majority) using the Up-method with the strong governability property and 176 seats (an absolute majority) using the all for the winner property. Regarding the number of political parties represented in the Congress, all of the methods except for the GR method give the same range of ideologies.

In Table 3, we observe that all of the analyzed methods of apportionment work similarly, the only difference concerns the number of political parties represented in the Congress.

Table 3. Distribution of seats in the Spanish Congress for five apportionment rules based on the national vote with no constituencies

Political party	Exact allotment	d'Hondt	GR	U^{ρ^g}	$U^{\rho^{sg}}$	U^{ρ^w}
PP	101.22	104	101	102	106	113
PSOE	77.56	80	78	78	81	77
Podemos	44.74	46	45	45	45	44
Ciudadanos	49.16	50	49	50	51	49
En Comú Podem	13.01	13	13	14	13	13
Compromís-Podemos	9.42	9	9	10	9	9
ERC	8.42	8	8	9	8	8
DIL	7.93	8	8	8	7	7
En Marea	5.74	5	6	6	5	5
PNV	4.23	4	4	5	4	4
IU	12.96	13	13	13	12	12
Bildu	3.06	3	3	4	3	3
CC	1.14	1	1	1	1	1
Pacma	3.08	3	3	3	3	3
UPyD	2.17	2	2	2	2	2
Nós-Candidatura Galega	0.99	1	1	–	–	–
UDC	0.91	–	1	–	–	–
Vox	0.81	–	1	–	–	–
Recortes 0–Grupo Verde	0.68	–	1	–	–	–
MÉS	0.47	–	1	–	–	–
PCPE	0.44	–	1	–	–	–
Geroa Bai	0.43	–	1	–	–	–
Total seats	–	350	350	350	350	350

In view of these results, we can conclude that the effect of the governability properties decreases when the size of constituencies increases (or, equivalently, the number of constituencies decreases).

5. Conclusions

We have analyzed apportionment problems from the perspective of (discrete) bankruptcy problems. As far as we know, this approach is new. For this purpose, we have defined bankruptcy problems associated with apportionment problems. One important decision in the definition of the bankruptcy problem is how to determine the appropriate claims of the agents. We have defined the claim of each agent as her lower allotment plus one, but other criteria can be followed, for example based on upper allotments. Another alternative is to directly use the exact allotments of the agents involved in the apportionment problem as claims in the associated bankruptcy problem. In this case, bankruptcy problems with an integer estate and non-integer claims are obtained [10, 11]. But even bankruptcy problems with references could be associated with apportionment problems where the references are the lower allotments and the utopian claims can range from the upper allotments to the total number of units to be distributed [31, 32].

We have also shown that the discrete constrained equal losses rule applied to the associated discrete bankruptcy problem coincides with the greatest remainder method applied to the corresponding apportionment problem. But we have also used various standards of comparisons and the associated Up-methods to define new apportionment rules with new properties related to governability. These properties try to catch the idea that political parties with more votes should benefit in the allotment of seats, in order to facilitate, somehow, the formation of governments. Therefore, the link between apportionment problems and bankruptcy problems could be used to analyze both and further research can be done in this direction.

Finally, we have applied Up-methods with different properties of governability to the case of the Spanish election in 2015 and compared these methods with the greatest remainder and d'Hondt methods. We have observed that the effect of the governability properties decreases when the size of constituencies increases (or, equivalently, the number of constituencies decreases).

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