STUDIUM UNIVERSITATIS BERGOMENSIS
UNIVERSITY OF BERGAMO

Doctor of Philosophy in
ANALYTICS FOR ECONOMICS AND BUSINESS
XXIX Cycle

The Value of the Right Distribution for the Newsvendor Problem and a bike–sharing problem

---

Co–ordinator:
Marida Bertocchi †
Luca Bertazzi

Supervisors:
Francesca Maggioni
Luca Bertazzi

Candidate:
Matteo Cagnolari
Candidate ID: 1031618

To my three mothers.

Giovanna

Monica

Concetta

Declaration of Autorship

I, Matteo Cagnolari, declare that this thesis titled, The Value of the Right Distribution for the Newsvendor Problem and a bike–sharing problem, and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

Acknowledgements

First of all I would like to thank Prof. Francesca Maggioni and Prof. Luca Bertazzi for time and patience they have dedicated to me, for the method they taught me and for the many many things I have learnt from them.
I thank all my beloved ones, my mother Giovanna, my ount Monica and my grandmother Concetta for having allowed me to come so far, even if they are not able to understand what I did.
I thank Matteo for the having supported me in the hardest time, for being at my side, always, for his patience and for his understanding.
I will never forget.

Matteo

Preface

Real–world problems are always affected by uncertainty. Every day we are asked to make decisions with respect to a future that has to be yet known. We cannot completely know or understand the consequences of our choices, but if we are able to identify a sufficient number of possible realizations of the future, then we can understand what kind of decisions to make today without being fully unarmed against the upcoming uncertainty. This is the typical approach adopted in Stochastic Programming [34], [11].

In this thesis, we introduce the Value of the Right Distribution, a new concept in Stochastic Programming which allows us to quantify the error when the problem is solved by assuming a probability distribution while another one realizes instead. In order to show how it applies, we first study a cost–based variant of the Newsvendor Problem considering a set of possible probability distributions for the stochastic demand. Then, we use it alongside other concepts to study a real case of bike–sharing, a kind of public service that is becoming more and more popular in metropolitan regions.

In Chapter 1, a variant of the classical Newsvendor problem is studied. This variant has application in supply chain management and in the supply of public services, such as bike–sharing and car–sharing services. For this problem, a two–stage stochastic programming model is formulated and closed–form and approximate expressions of optimal solutions are obtained under different probability distributions of the demand. A computational study shows, in a systematic way with respect to the unit costs, how the expected cost and the classical Value of the Stochastic Solution (VSS) [10] vary when the probability distribution varies. Finally, a worst–case analysis is carried out, showing the maximum increase in the expected cost that can be obtained guessing a probability distribution different than the right one.

In Chapter 2, stochastic optimization methods for a bike-sharing problem are investigated. The problem considered is the one of a bike–sharing service provider who needs to manage a fleet of bicycles and a set of bike–stations with fixed capacity in order to serve the rental stochastic demand over space and time. First, it is shown that this bike–sharing problem is a variant of the Newsvendor problem with transshipment; then, two–stage and multi–stage models for one-way and two-way rental systems are proposed. In the one–way problem, models differ according to the inclusion of the user–transshipment, service–transshipment and service–level satisfaction. A computational study based on real data of the one-way bike–sharing system of the city of Bergamo, La BiGi, is provided showing the differences in optimal inventory levels, under different assumptions for the probability distribution. The Value of the Right Distribution introduced in the previous chapter and known measures from the literature are computed to investigate the quality of the expected value solutions under several probability distributions. Finally, the in–sample stability and the sensitivity analysis over stock-out cost, time-waste cost and transshipment cost are provided.

STUDIUM UNIVERSITATIS BERGOMENSIS
UNIVERSITY OF BERGAMO

Abstract

The Value of the Right Distribution for the Newsvendor Problem and a bike–sharing problem

Matteo Cagnolari

In this thesis, we introduce the new concept of Value of the Right Distribution, which measures the importance in Stochastic Programming of knowing the right probability distribution of the stochastic demand. We also introduce the new concepts of Recourse Penalty Bound and Maximum Recourse Penalty bound, which measure respectively the error bound and the worst–case performance bound given a certain mismatch between two probability distributions. In order to show how they apply, we study a cost–based variant of the Newsvendor problem. Moreover, we obtain closed–form and approximate expressions for the optimal quantity to order depending on the probability distribution assumed for the stochastic demand. Then, we use this new concepts to investigate bike–sharing problems. Two–stage and multi–stage stochastic optimization models are proposed. Finally, numerical results are provided.

Symbols and Notations

In this section we include a description of major symbols and notations used in all the chapters of the current work. Additional notation may be needed within specific chapter/sections and it is explained when used. To the greatest extent possible, we have attempted to keep unique meanings for each item. In those cases where an item has additional uses, they should be clear from context.

Symbol	Definition
$+$	Superscript indicates the positive part of a real (i.e., $a^+ = \max(a, 0)$) or unrestricted variable (e.g., $y = y^+ y^-, y^+ \geq 0, y^- \geq 0$) and its objective coefficients (e.g., q^+), subscript as non-negative values in a set (e.g., $\mathbb{R}^+$) or the right–limit ($F^+(t) = lim_{s \to t} F(s)$)
$*$	Indicates an optimal value or solution (e.g., x^*)
A	First–stage matrix (e.g., $Ax = b$), also used to indicate an event or subset, $A \in \mathcal{A} \subset \Omega$
$\mathcal{A}$	Collection of subsets
b	First-stage right-hand side (e.g., $Ax = b$)
B	Matrix, basis submatrix, or index set of a basis
$\mathcal{E}$	Exponential distribution
$\mathbb{E}$	Mathematical expectation operator
f	Probability density/mass function
F	Cumulative distribution function
g	Function (usually in constraints ($g(x)$ or $g_j(x)$))
h	Right–hand side in second–stage ($Wy = h - Tx$), also $h^t(\omega)$ in multistage problems
H	Number of stages (horizon) in multistage problems
i	Subscript index of functions (f_i) or vector elements (x_i)
I	Identity matrix or index set ($i \in I$) or vector elements (x_i)
j	Subscript index of functions (g_j) or vector elements (x_j)
$\mathcal{L}$	Log–normal distribution
m	Number of constraints (m_1, m_2)
n	Numberof variables (n_1, n_2)
N	Set
$\mathcal{N}$	Normal distribution
p	Probability of a random element (e.g., $p_s = p[\boldsymbol{\xi} = \xi_s]$)
q	Second–stage objective vector ($q^T y$)
Q	Second–stage (multistage) value function with random argument ($Q(x, \boldsymbol{\xi})$ or $Q^t(x^t, \xi^t)$)

Symbol	Definition
$\mathcal{Q}$	Second–stage (multistage) expected value
	(recourse) function ($\mathcal{Q}(x)$ or $\mathcal{Q}^t(x^t)$)
$\mathbb{R}$	Real numbers
s	Scenario index
S	Scenario Set
t	Superscript stage or period index for multistage
	programs ($t = 1, \ldots, H$)
T	Technology matrix, the transpose of a
	matrix or vector, total number of time periods
$\mathcal{U}$	Uniform distribution
W	Recourse matrix
x	First–stage decision vector or multistage decision vector (x^t)
$\mathcal{X}$	feasible set of x
y	Second–stage decision vector
z	Objective value (e.g., $\min z$)
$\mathbb{Z}$	Integers
ε	Weight in a linear combination (e.g $\varepsilon \in [0, 1]$)
ζ	Parameter of the Log–normal distribution
η	Parameter of the Log–normal distribution
$\boldsymbol{\theta}$	Random vector with realization θ
$\bar{\boldsymbol{\theta}}$	Vector of the expected values of all origin–destination pairs
$\bar{\theta}$	Vector of the expected values for each bike–stations
λ	Dual multiplier, parameter in a convex combination
$\boldsymbol{\xi}$	Random vector with realization ξ
Ξ	Support of the random vector $\boldsymbol{\xi}$
σ	Dual multiplier, standard deviation or σ–field
Σ	Summation
ϕ	Function, density distribution of a standard normal
Φ	Function, cumulative distribution of a standard normal
ψ	Newly obtained second–stage (multistage) value function
Ψ	Newly obtained second–stage (multistage) expected value function

Contents

List of Figures

List of Tables

Chapter 1

The Value of the Right Distribution in Stochastic Programming

1.1 Introduction

Uncertainty characterizes the nature of our reality: every day we are asked to make decisions to solve problems where some of the parameters are unknown. Modeling a problem with uncertainty in an appropriate way is critical to have good solutions. Stochastic Programming (SP) and Robust Optimization (RO) are two modeling approaches adopted alternatively to face problems with uncertainties in data, both in single period and multi-period decision–making processes. In the former, uncertainties are modeled as random variables with known probability distributions and the user has informations about the form of the distribution (see for example Birge and Louveaux (2011) [11], Ruszczynski and Shapiro (2003) [52]), King and Wallace (2012) [34]). When such information is not available or hard to obtain, RO allows the user to address the uncertain nature of the problem without making specific assumptions on probability distributions by assuming that the uncertain parameters belongs to a deterministic uncertainty set. This is the general distinction between the approaches of Stochastic Programming and Robust Optimization. In RO uncertainties are usually modeled as random variables with distributions that are unknown to the modeler, but are constrained to lie within a known support. The original problem is reformulated as a deterministic convex program by means of a min–max approach which guarantees the feasibility and optimality of the solution against all instances of the parameters within the uncertainty set (Ben-Tal and Nemirovski (1998) [5], Ben-Tal and Nemirovski (1999) [6], Ben-Tal and Nemirovski (2000) [7], Ben-Tal et al. (2004) [4] and (2009) [3], Bertsimas et al. (2011) [8], Dupacova (1998) [19]). The deterministic program is therefore "robust" against perturbations in the model parameters.

Each approach has a main drawback: SP requires *specificity* in order to know exactly the true form of the probability distribution describing the uncertainty, RO is *conservative* and the optimal solution may have a strong dependence on the chosen uncertainty set. The min–max stochastic programming approach aims to bridge the gap between the two: the optimal decisions are sought for the worst–case probability distributions within a family of possible probability distributions, defined by certain properties such as their support and moments (see for example Goh and Sim (2010)[24]).

We now enlight other drawbacks about the SP and RO approaches in order to better specify on the goals of this chapter:

- concerning the SP approach, it is not considered the possibility that one of the pillar assumption of the stochastic model (*i.e.* the assumption of the probability distribution for the stochastic process) is false. The use of a stochastic approach instead of another relies on the fact that one probability distribution is chosen to describe the historical data, given that the future will depend on the past. But what if this assumption is wrong? How large is the error due to our mistake? Which is the worst–case having the largest error? For example, in Bertsimas and Thiele (2006) [9] it is shown that, if the assumed probability distribution is in fact different from the actual distribution, the optimal solution may perform poorly (in their experiment it is assumed that the probability distribution has identical first and second moments to the actual one, while we will also address the case where the latters are different);

- concerning the RO approach, it is too much conservative and it does not consider the information carried by the probability distributions. Does exist a way to capture the importance of the ambiguity?

To this extent, in this chapter we introduce the new concepts of *Value of the Right Distribution* (VRD), *Recourse Penalty Bound* (RPB) and *Maximum Recourse Penalty Bound* (MRPB).

In SP, the probability distribution is often selected on the basis of a given time series. Doing that, it is implicitly assumed that the future will be similar to the past. So, it can happen that the selected probability distribution is not a good estimate of the future. To this purpose, we are interested in comparing what happens when a SP model is solved by assuming a probability distribution with respect to solving the same model by assuming a different one. In the following,

we will refer the first distribution as *guessed* distribution and the second as *right* distribution. The Value of the Right Distribution measures the error when a probability distribution is assumed in a SP model but another one realizes instead. It is the difference in the cost of the optimal solution obtained assuming a guessed probability distribution with respect to the cost of the optimal solution with the probability distribution that actually realizes. The cost of both solutions is computed by using the right probability distribution. The Recourse Penalty Bound deepens the analysis for those pairs of probability distributions considered problematic or highly risky. The Maximum Recourse Penalty Bound is, at last, a worst–case analysis that measures the maximum error or loss deriving from a pairs of probability distributions.
For the sake of simplicity, we will compute the above mentioned measures using the following probability distributions: Uniform, Exponenial, Normal and Log–normal. To investigate such measures we use as example the *Newsvendor Problem*, which is among the most studied problems by the stochastic programming community. The main motivation behind the choice of it, it is not to understand something new about it but use it as an example to introduce the before mentioned new concepts.
Our variant of the Newsvendor Problem is referred to as the *Cost–based Newsvendor Problem*, in which our aim is to determine the order quantity of a single item with stochastic demand over a single period. In the classical Newsvendor Problem (Birge and Louveaux (2011) [11]), a newsvendor goes to a publisher every morning and buys a certain quantity of a newspaper at a given unit purchase price. This number is usually bounded above by some limit, representing either the newsvendors purchase power or a limit set by the publisher to each newsvendor. The newsvendor then walks along the streets to sell as many newspapers as possible at a given unit selling price. Any unsold newspaper can be returned to the publisher at a given unit return price, less than the purchase price. Demand for newspapers is described by a random variable. The newsvendor cannot return to the publisher during the day to buy more newspapers. The problem is to determine how many newspapers to buy every morning to max-

imize the expected profit. In the *Cost–based Newsvendor Problem* we assume that the selling price is equal to 0 and that the unsold newspapers cannot be returned to the publisher. The objective function is to minimize the expected total cost.

This variant has several practical applications. For example, the newsvendor is an intermediate node in a supply chain or is providing a service, such as bike–sharing or car–sharing services, where the fair paid by the user is often zero for some months if the service is new or symbolic (cents of euros per minute) otherwise. In these cases, the focus of the decision–maker is not on maximizing the profit as in the classical *Newsvendor problem*, but on minimizing the cost of the service while ensuring a certain service level. If the service level is measured in terms of stock-out cost, an optimal solution can be obtained by minimizing the sum of the cost of the service and of the stock–out cost. For this problem, we formulate a two–stage stochastic programming model, we first provide optimal solutions in closed form for several probability distributions and we computationally compare in a systematic way the optimal cost and the classical *Value of the Stochastic Solution* (VSS) (Birge (1982) [10]). These closed form functions provide the means to analyze both the optimal solutions and associated costs according to the specific characteristics defining the problem.

As last consideration, we apply the newly developed methodologies to the case study considered in the second chapter.

This chapter is organized as follows. In Section 1.2 we define the Value of the Right Distribution and in Section 1.3 we study a cost–based variant of the Newsvendor Problem in order to show how the VRD applies. In Section 1.4 we provide guidelines for the use of the VRD and RPB. Section 1.5 concludes the chapter.

1.2 The Value of the Right Distribution

A critical assumption adopted in the majority of newsvendor models is that the demand probability distribution is known. However, such assumption is rarely justifiable because, upon reality, it is at best be possible to estimate a few low–order moments or to identify the range and some basic modality and symmetry properties of the demand distribution (Hanasusanto et al. (2015) [26]). In such cases there exist multiple distributions that describe the stochastic demand according to the available information, thus the decision maker cannot figure out which of them is the true one. If the newsvendor model is solved chosing arbitrarily one of the possible distributions, the decision maker may obtain a solution that performs poorly under the true demand distribution. Such situation may rise also when a probability distribution is assumed to be the true one for the demand in a stochastic programming model but such assumption turns out to be false. Our focus is then on those class of problems in which the support and/or few low–order moments can be identified and the probability demand distribution is unknown among possible candidates.

In this section, we introduce the concept of *Value of the Right Distribution* which measure the error when a certain probability distribution is assumed for the stochastic demand but another one realizes instead. Such measure can be adopted in two ways:

- a priori, when we are in presence of multiple probability distributions which may describe the stochastic demand and we are interested in evaluating the possible losses deriving from all the mismatches between the possible distributions, finding at the same time which mismatch is more critical;

- a posteriori, to measure the error when a certain probability distribution, which reaveals to be false, has been already assumed for the stochastic demand and the obtained solution from solving the model has been already adopted.

The Value of the Right Distribution applies not only when two or more probability distributions may be mismatched, but also when the estimate of the type of the distribution is correct but its parameters may be wrong. We are refering in particular to the estimate of the standard deviation. For example, consider the case where it is assumed normality for the stochastic demand with an estimated standard deviation which turns out to be biased and thus wrong: the probability distribution is the same, but the second moment is different. Alongside the Value of the Right Distribution we also provide the Recourse Penalty Bound and Maximum Recourse Penalty Bound which deepen the analysis for those mismatches that are critical. More details are provided in Section 1.3.4.
The Value of the Right Distribution may be investigated when either a stochastic programming approach or a distributionally robust approach are adopted:

- in the case of stochastic programming, we are interested in a priori or posteriori analysis. In the former we doubt about the true form of the distribution or on the estimate of its moments, in the latter we observe that the assumed distribution or estimation of the moments turns out to be wrong;

- in the case of distributionally robust optimization, we measure the value of the ambiguity that exists among each possible pair of the distributions defining the ambiguity set. We thus obtain more information from the ambiguity se while seeking at the solution that guarantees the best performance in the worst–case.

Such situations may be observed in the following real situations:

- *New services*: we can look at services which can be similar to already existing services in the same city or identical in other cities, but we will hardly obtain a good estimate of the parameters of the distribution, low–order moments or even the true form of the distribution fitting for our case study. Such situation is the case where the demand probability distribution may be known but the support and/or low–order moments are difficult to be properly estimated. A priori and/or posteriori analysis may fit for such situation;

- *New single products*: consider for instance a new game for consoles and personal computers which is going to be released worldwide in few months. The form of the distribution may be guessed by analyzing the hystorical data of games of similar gender sold by the same company. A first estimate of the parameters and low–order moments of the distribution may instead be given by using the so called *pre–order* option, which gives to the company a measure of the future sales;

- *Failure of automatized processes*: failures are typical of automatized processes, in particular for clothing and food industries which involves a massive production. Failures may be described equally well by different probability distributions given an estimate of the support and low–order moments.

We now formally introduce the class of problems and methodologies affected by uncertainty and we show to which of them the concept of VRD can be applied.
Consider the following generic optimization problems

$$\min_{x}\{c^T x | Ax = b\}, \tag{1.1}$$

where c, A b are known data of sizes $n \times 1$, $m \times n$ and $m \times 1$, respectively. The tuple (c, A, b) represents the numerical values of the entries. For this kind of problems, the knowledge of (c, A, b) is assumed to be complete. However, when facing real–world applications, in the majority of cases the knowledge of (c, A, b) is incomplete due to the uncertainty characterizing the problem, and when a mathematical optimization model is posed, the uncertainty has to be considered if it is proved to be important for the decision model.
The most difficult part is establishing a decision model such that the uncertainty is described in an appropriate way without losing information or making wrong

assumption about it.

According to Ellsberg (Ellsberg (1961) [20]), we distinguish between:

- *uncertainty problem*, if the probability model is fully known, but the realizations of the random variables are unknown. For example, when a sample of past data is available, we are able to define a probability model: we describe the uncertainty as a random variable or a stochastic process by identifying its probability distribution using statistical methods of model selection and parameter estimation. Adding the uncertainty in (1.1), these problems can be described with a *stochastic formulation*:

$$\min_{x}\{c^T x + \mathbb{E}_{\boldsymbol{\xi}}[\mathrm{Q}(x, \boldsymbol{\xi})]|Ax = b\}, \tag{1.2}$$

  where $\boldsymbol{\xi} \in \Xi \subset \mathbb{Z}^+$ represents the random vector formed by the coefficient of the second–stage problem data $\boldsymbol{q}$, $\boldsymbol{h}$ and by the technology matrix $\boldsymbol{T}$ and recourse matrix $\boldsymbol{W}$. Boldface is used to represent the fact that the considered data are uncertain. $\mathbb{E}_{\boldsymbol{\xi}}$ denotes the mathematical expectation with respect to $\boldsymbol{\xi}$ and $\mathrm{Q}(x, \boldsymbol{\xi}) = \min\{\boldsymbol{q}^T \boldsymbol{y}|\boldsymbol{W}\boldsymbol{y} = \boldsymbol{h} - \boldsymbol{T}x, y \geq 0\}$, where y is the second–stage variable.

- *ambiguity problem*, if the probability model itself is unknown. For example, when there is not just one probability model, but a set of models $\mathcal{P}$ (the ambiguity set), which are all possible descriptions of the reality. We extend the baseline problem (1.2) to the ambiguity problem with a *distributionally robust formulation*:

$$\min_{x} \max_{\mathbb{P} \in \mathcal{P}}\{c^T x + \mathbb{E}_{\mathbb{P}}[\mathrm{Q}(x, \boldsymbol{\xi})]|Ax = b\}, \tag{1.3}$$

  where $\mathbb{E}_{\mathbb{P}}$ denotes the mathematical expectation with the respect to $\mathbb{P}$. (1.3) is of min–max type. An optimal solution to (1.3) is called a *distributionally robust solution*.

One last case is *robust optimization* (Ben–Tal and Nemirovski (2009) [3]), which defines the uncertainty as a collection of programs of a common structure with the data (c, A, b) varying in a given uncertainty set $\mathcal{U}$. These programs can be described with a *robust formulation*:

$$\{\min_x \{c^T x | Ax = b\} : (c, A, b) \in \mathcal{U}\}. \tag{1.4}$$

An optimal solution to (1.4) aims at immunizing against the uncertainty characterizing the problem while remaining feasible whatever the realization of the data within $\mathcal{U}$. For this reasons, a solution of this kind is called *robust feasible.* The best possible robust feasible solution is the one that solves the following optimization problem:

$$\min_x \{\max c^T x | Ax = b \;\; \forall (c, A, b) \in \mathcal{U}\}. \tag{1.5}$$

(1.5) is of worst–case type.

Robust optimization ignores the informations given by the scenarios and by the probability distributions the random parameters may have. On the contrary, stochastic optimization makes the assumption that the parameter $\boldsymbol{\xi}$ follow a probability distribution $\mathbb{P}$ such that the functions $\mathrm{Q}(x, \boldsymbol{\xi})$ become random variables (Pflug and Pichler (2014) [47].

Figure 1.1 shows the concepts introduced so far, describing as well when a certain approach has to be used.

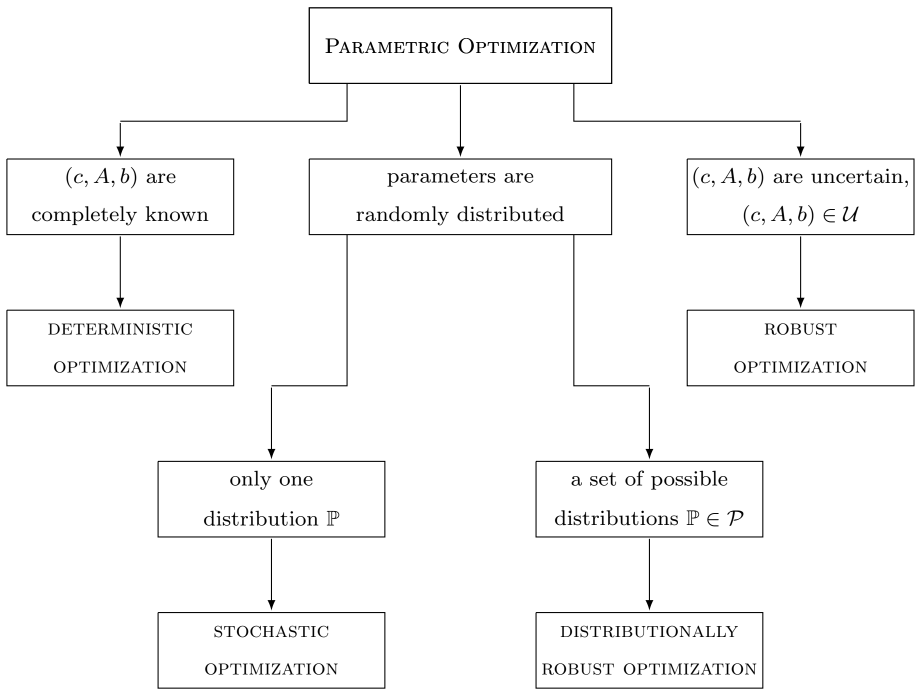


Figure 1.1: Optimization approach to be used depending on the parameters of the problem.

Depending on the problem parameters, we understand what kind of optimization approach has to be adopted. Thus, the concept of *robustness* is used in different ways. For example, minimizing $x \mapsto \mathrm{Q}(x, \xi)$ for a fixed ξ is a deterministic optimization problem, minimizing $x \mapsto \max\{\mathrm{Q}(x, \boldsymbol{\xi}) : \boldsymbol{\xi} \in \Xi\}$ is a robust optimization problem while minimizing $\mathrm{Q}(x, \boldsymbol{\xi})$ in x with $\boldsymbol{\xi}$ random variable is a stochastic optimization problem. The ambiguity problem (1.3), however, does not specify a prior and therefore it has the structure of a distributionally robust stochastic problem. It is a combination of a robust and a stochastic problem (Zackova (1966) [65]).

In our work, we do not consider the case where (c, A, b) are completely known, nor the case where they vary in a given uncertainty set $\mathcal{U}$. We are not interested in deterministic nor in robust optimization. We focus only in those cases where the parameters are randomly distributed (see Figure 1.1) while facing a situation of ambiguity. Our aim is to measure the error deriving from a wrong assumption of the probability distribution in stochastic programming and to investigate the value of the ambiguity in presence of multiple probability distributions.

We now formally describe the concept of Value of the Right Distribution. Consider a stochastic program having random variable $\boldsymbol{\xi}$. Let $\mathcal{G}$ be the *guessed probability distribution* of $\boldsymbol{\xi}$ and $\mathcal{R}$ be the *right probability distribution* of $\boldsymbol{\xi}$.

The *Recourse Problem assuming the guessed distribution* $\mathcal{G}$ ($RP_{\mathcal{G}}$) is defined as follows:

$$RP_{\mathcal{G}} = \min_{x \in \mathcal{X}} z_{\mathcal{G}}(x, \boldsymbol{\xi}), \tag{1.6}$$

where $\mathcal{X}$ is the set of feasible decisions x and $z_{\mathcal{G}}(x, \boldsymbol{\xi})$ is the expected cost computed assuming a guessed distribution $\mathcal{G}$. We denote with $x^*_{\mathcal{G}}$ the corresponding optimal solution.

Let the *Out–of–distribution value* (OD) be the expected cost of $x^*_{\mathcal{G}}$ computed on the basis of the right distribution $\mathcal{R}$:

$$OD = z_{\mathcal{R}}(x^*_{\mathcal{G}}, \boldsymbol{\xi}), \tag{1.7}$$

where $z_{\mathcal{R}}(x, \boldsymbol{\xi})$ is the expected cost computed by assuming the right distribution $\mathcal{R}$. This value is referred to as the .

The *Recourse problem assuming the right distribution* $\mathcal{R}$ is defined as follows:

$$RP_{\mathcal{R}} = \min_{x \in \mathcal{X}} z_{\mathcal{R}}(x, \boldsymbol{\xi}). \tag{1.8}$$

We denote with $x^*_{\mathcal{R}}$ the corresponding optimal solution.

Definition 1.2.1. *The Value of the Right Distribution* (VRD) *is*

$$VRD = OD - RP_{\mathcal{R}}.$$

Note that VRD is always non–negative.

1.3 A cost–based variant of the Newsvendor Problem

In this Section, we study a cost–based variant of the Newsvendor Problem as example in order to apply the concepts of VRD, RPB and MRP. In Subsection 1.3.1, 1.3.2 and 1.3.3 the problem description, literature review and model formulations are provided, respectively. Then we introduce in Subsection 1.3.4 the concepts of Recourse Penalty Bound and Maximum Recourse Penalty Bound. At last, Subsection 1.3.5 shows numerical results.

1.3.1 Problem Description

A supplier replenishes a retailer facing the stochastic demand of a single item over a single period. The retailer purchases the item at a given unit procurement cost and tackles unit holding cost for the leftover quantity (surplus) and unit stock-out cost for the shortfall quantity depending, respectively, on the positive or negative inventory level in which she will incur after demand realization. In order to avoid trivialities, we assume that the unit stock–out cost is strictly greater than the unit procurement cost. The order quantity can be any nonnegative real number. The delivery is assumed to be instantaneous (lead–time equal to zero). Backlogging is not allowed. The sequence of the operations is the following: the order quantity is computed, the ordered units are shipped and received by the retailer and, at last, the demand that occurs is satisfied. The aim is to determine the order quantity that minimizes the expected total cost, given by the sum of the procurement cost, the holding cost for the leftover quantity and the stock–out cost for the shortfall quantity.

1.3.2 Literature Review

The *Newsvendor problem* is one of the simplest stochastic inventory problems, involving a one–time purchase decision and a stochastic sales outcome. If we rethink it as an investment problem, it can be interpreted as the simplest stochastic version of the point–in, point–out investment problem of Jevons (Stanley (2013) [59]). Even if the *Newsvendor problem* has a simple structure, human decision–makers systematically make non–optimal decisions (Schweitzer et al. (2000) [57]). This problem is sometimes also called *Single Period Problem* (SPP). A well–known result is proposed in Porteus (1990) [48]: the decision maker has to determine the order quantity by matching the so–called *critical ratio* to the uncertain demand.

We structure our literature review into papers that identify the uncertainty in customer demands, distinguishing between papers that prefer a stochastic or a robust approach, providing as well the case of distributionally robust optimization. For general comprehension, we also mention some examples where demand is multi–modal and where a multi–period formulation is adopted. For comprehensive reviews of the *Newsvendor problem*, we refer to Khouja and Moutaz (1999) [32] and Qin et al. (2011) [49].

In Summerfield and Dror (2012) [60] two–stage decentralized inventory problems are proposed by using a unifying framework defined as a taxonomy multilevel graph. This allows to model and link different problems of competing retailers who independently procure inventory in response to uncertain demands. It is given to the retailers the possibility to coordinate inventory transshipment to satisfy shortage with overage based on profit sharing agreements.

In Rossi et al. (2014) [51] it is introduced novel strategy to address the issue of demand estimation which analytically combines confidence interval analysis and inventory optimisation in single-item single-period stochastic inventory optimisation problems. In their numerical experiments, they assume for the demand to have a binomial, Poisson, or exponential probability distributions. In Kaki et al. (2015) [29] the impact of supply uncertainty on newsvendor decisions is studied. In addition to stochastic demand, stochastic supply yield is introduced. A closed–form solution for a specific copula–based dependence structure is provided for the optimal order quantity, when the demand and supply uncertainties are interdependent. Then, it is shown how dependence impacts the newsvendors decision, profit and risk level. Experimental results show how difficult newsvendor decisions under supply uncertainty are for human subjects, showing that, under low–profit conditions, subjects are able to incorporate supply uncertainty quite well in their decisions, while under high–profit conditions, the deviation from the optimum is much more significant.

In a stochastic programming framework it would be interestind to consider other probability distributions along the ones considered in the papers mentioned above and to derive close–form and approximate expressions for the optimal order quantity. In a single–period single–item newsvendor problem, such forms would provide a quick computation of the optimal solution to be compared assuming different probability distributions.

In Carizzosa et al. (2016) [12] a setting which combines temporal dependence and tractable robust optimization for the *Newsvendor problem* is proposed: the demand is modeled as a time series which follows an autoregressive process and then a robust distribution–free autoregressive method to maximize the worst–case revenue is given. A closed–form expression for the optimal solution is provided for same cases, while for other cases the problem is numerically solved. Then, the optimal robust solution is compared with the solutions obtained in three versions of the classic approach, in which either the demand distribution is unknown, and autocorrelation is neglected, or it is assumed to follow an auto regressive process with normal error terms. Extension to multiperiod and multiproduct models are also provided and discussed.

In Lin and Ng (2011) [36] a robust model to determine the optimal order quantity in a multi–item single period problem is provided. The model is a minimax regret multi–market newsvendor model, where demands are only known to be bounded within some given intervals. A linear solution method for the classical version of the problem is proposed and an approximation solution algorithm, based on integer programming for the capacitated version of the problem, is implemented. Then, the performance of the proposed minimax regret model is compared with typical average–case and worst-case models, showing that the former outperformed the latter in terms of risk–related criteria and mean profit.

In Wu et al. (2014) [64] a risk–averse version of the newsvendor problem with

quantity and price competitions is proposed in order to obtain optimal quantity and pricing decisions under the Conditional Value–at–Risk (CVaR) criteria. For quantity competition, two demand splitting rules are considered (demand allocation and demand reallocation), while for price competition, both additive and multiplicative demand are considered. It is shown that for certain competitive environments and for demand reallocation, competition does not result in profit losses, even if it always leads to overstocking, by avoiding/reducing overstocking under the risk–neutral criterion. Moreover, it is showed that the order quantity, sale price, and the expected profit decrease in the degree of risk aversion, both high price sensitivity and competition intensity force decision makers to lower their prices and high price sensitivity always reduces the order quantity while competition can have the opposite effect.
In Khanra et al (2014) [31] a sensitivity analysis is provided, justified by the fact that, in such problem as the *Newsvendor problem*, the quality of decisions in inventory management models depends on the accuracy of the estimated stochastic parameters. A lower bound of the cost deviation for symmetric unimodal demand distributions has been first identified. Then, it is shown that conditions for symmetry/skewness of cost deviation are closely linked with symmetry/skewness of the demand density function and that the *Newsvendor prolem* is sensitive to suboptimal ordering decisions. Moreover, the *Newsvendor problem* resulted to be more sensitive than the classical *Economic Order Quantity* model. The most influential parameter for the optimal order quantity resulted to be the expected value of the stochastic demand.

As proposed in the above mentioned references, it is typically prefered to assume for the stochastic demand to follow a unimodal or normal distribution since unimodality and normality lend themselves to easier mathematical manipulations.
The first distributionally robust newsvendor model has been proposed in Scarf (1959) [54] in which it is assumed that only the first two–moments of the univariate demand distribution are known. The analytical expression of the optimal order quantity is then derived. This model was extended in Gallego and Moon

[23],but still only first and second–order moments were assumed to be known. In Perakis and Roels (2008) [46] are derived the optimal order quantities that minimize the newsvendor's maximum opportunity cost from choosing a particular demand distribution within the ambiguity set, mitigating the conservativeness of the worst–case approach. In Natarajan (2008) [45] is then introduced the asymmetry into the robust newsvendor problem by using mean, variance and semivariance to design the ambiguity set.
However, in Vaagen and Wallace (2008) [63] it is demonstrated that unimodality is inadequate for some classes of product whose demand is affected by certain characteristics, such as seasonality and fashion trends, because the assumption of unimodality leads to solutions with unfavourable risk–reward tradeoffs.
In Hanasusanto et al. (2015) [26] are investigated risk–averse multi–dimensional newsvendor model in which unimodality and normality are not justifiable since the demands of the products taken into consideration for their model were strongly correlated and subject to fashion trends that were not fully understood at the time when the orders were placed. They assumed that demand distribution is known to be multimodal and they proposed a distributionally robust optimization formulation which admited an efficient numerical solution in quadratic decision rules.

The newsvendor problem has been also extended to the multi–period formulation. In Kim et al. (2015) [33] it is proposed a multi–stage stochasic programming model with integer recourse decisions to determine the optimal inventory control with a non–stationary demand. In Behret and Kahraman (2010) [2] it is proposed a multi–period newsvendor problem with fuzzy demand and pre–season extension for innovative products. They determined the best order period and the optimal order quantity that minimizes the fuzzy expected total cost. At last, in Azad et al (2016) [1] the single–period nesvendor problem is extended into a multi–period and time–dependent newsvendor problem with price–dependent stochastic demand.

1.3.3 Model Formulations

In this subsection we propose model formulations for a cost–based variant of the Newsvendor Problem. Let us first introduce the following parameters holding for all the models:

Deterministic parameters

- $c \in \mathbb{R}^+$: procurement cost per unit of item purchased from the supplier, where $\mathbb{R}^+$ denotes the set of the positive real numbers;
- $h \in \mathbb{R}^+$: holding cost per unit of positive inventory level after demand realization;
- $v \in \mathbb{R}^+$: stock–out cost per unit of unmet demand after its realization, $v > c$;

Stochastic parameters
Let $(\Xi, \mathcal{A}, p)$ be a probability space with Ξ set of outcomes, $\sigma - algebra\ \mathcal{A}$, probability p and $\xi \in \Xi$ a particular outcome representing the stochastic demand. We define $\boldsymbol{\xi} \in \Xi \subset \mathbb{Z}^+$ the uncertain future demand.

This subsection is organized as follows. In Paragraph 1.3.3.A we provide a two–stage stochastic programming formulation while in Paragraph 1.3.3.B we provide a distributionally robust optimization formulation. In both paragraphs, we consider both continuous and discrete cases.

1.3.3.A A two–stage stochastic programming formulation

Let $\mathcal{S}$ be set of scenarios, $\mathcal{S} = \{1, \ldots, S\}$. We denote with ξ_s the realization of the stochastic process $\boldsymbol{\xi}$ in the scenario s, $s \in \mathcal{S}$.
The problem can be formulated as a two–stage stochastic linear program with recourse. Let us introduce the first–stage and the second–stage decision variables:

- x: first–stage non–negative decision variable corresponding to the order quantity. We denote with x^* the optimal order quantity. This decision must be taken before the realization of the stochastic demand $\boldsymbol{\xi}$;
- $I^+(\boldsymbol{\xi})$ and $I^-(\boldsymbol{\xi})$: second–stage non–negative decision variables representing, respectively, the positive inventory level (surplus) and the unmet demand (shortage), given the realization of the demand.

The two–stage stochastic linear program can be formulated as follows:

$$\begin{aligned} \min \quad & z(x, \boldsymbol{\xi}) = cx + \mathcal{Q}(x) \\ & x \geq 0, \end{aligned} \tag{1.9}$$

where

$$\mathcal{Q}(x) = \mathbb{E}_{\boldsymbol{\xi}}[\mathrm{Q}(x, \boldsymbol{\xi})], \tag{1.10}$$

and

$$\mathrm{Q}(x, \boldsymbol{\xi}) \quad = \min \quad hI^+(\boldsymbol{\xi}) + vI^-(\boldsymbol{\xi}) \tag{1.11}$$

$$\text{s.t.} \quad I^-(\boldsymbol{\xi}) - I^+(\boldsymbol{\xi}) = \boldsymbol{\xi} - x, \tag{1.12}$$

$$I^+(\boldsymbol{\xi}), I^-(\boldsymbol{\xi}) \geq 0. \tag{1.13}$$

The objective function (1.9) minimizes the expected total cost corresponding to the non–negative order quantity x and to the recourse function $\mathcal{Q}(x)$. Constraint (1.12) defines the inventory level and (1.13) are the nonnegative constraints for the second-stage variables. We note that the stochastic program (1.9)–(1.13) has *simple recourse*.

Second–stage variables mutually exclude each other, since when one of the two is strictly positive the other is zero. It is possible to capture that with the following notation:

- $I^+(\boldsymbol{\xi}) = \max(x - \boldsymbol{\xi}, 0) = (x - \boldsymbol{\xi})^+$,
- $I^-(\boldsymbol{\xi}) = \max(\boldsymbol{\xi} - x, 0) = (\boldsymbol{\xi} - x)^+$.

By substitution, constraint (1.12) is rewritten as

$$x = \boldsymbol{\xi} - (\boldsymbol{\xi} - x)^+ + (x - \boldsymbol{\xi})^+. \tag{1.14}$$

The following properties for $\mathrm{Q}(x, \boldsymbol{\xi})$ hold true:

Proposition 1.3.1. *(a) The recourse function $Q(x, \boldsymbol{\xi})$ is a piecewise linear convex function in $\boldsymbol{\xi}$.*

(b) The recourse function $Q(x, \boldsymbol{\xi})$ is a piecewise linear convex function in x $\forall x \geq 0$ and s.t. $Q(x, \boldsymbol{\xi}) < +\infty$.

Proof. Let $d = \boldsymbol{\xi} - x^*$, $\boldsymbol{y} = (\boldsymbol{y}^+, \boldsymbol{y}^-) = (I^+(\boldsymbol{\xi}), I^-(\boldsymbol{\xi}))$ be the second-stage solution vector, $\boldsymbol{q} = (\boldsymbol{q}^+, \boldsymbol{q}^-) = (h, s)$ be the vector of the coefficient parameters of the second-stage variables and $\boldsymbol{W} = [\mathbf{1} - \mathbf{1}]$ be the recourse matrix.
To show convexity in (a) and (b) we need to prove that $f(d) = \min\{\boldsymbol{q}^T \boldsymbol{y} | \boldsymbol{W}^T \boldsymbol{y} = d\}$ is a convex function in d.
We consider two different values, d_1 and d_2, and a convex combination of the two, $d_t = td_1 + (1-t)d_2$, $t \in [0, 1]$.
Let $\boldsymbol{y}_1^*$ and $\boldsymbol{y}_2^*$ be two optimal solutions of $f(d) = \min\{\boldsymbol{q}^T \boldsymbol{y} | \boldsymbol{W}^T \boldsymbol{y} = d\}$ for $d = d_1$ and $d = d_2$, respectively. Then, $t\boldsymbol{y}_1^* + (1-t)\boldsymbol{y}_2^*$ is a feasible solution of the problem $f(d_t) = \min\{\boldsymbol{q}^T \boldsymbol{y} | \boldsymbol{W}^T \boldsymbol{y} = d_t\}$. Now let $\boldsymbol{y}_t^*$ be an optimal solution of the problem $f(d_t)$. Thus, we have:

$$\begin{aligned} f(d_t) = f(td_1 + (1-t)d_2) = \boldsymbol{q}^T \boldsymbol{y}_t^* \leq \boldsymbol{q}^T (t\boldsymbol{y}_1^* + (1-t)\boldsymbol{y}_2^*) &= \\ = t\boldsymbol{q}^T \boldsymbol{y}_1^* + (1-t)\boldsymbol{q}^T \boldsymbol{y}_2^* &= \\ = tf(d_1) + (1-t)f(d_2)&. \end{aligned}$$

This proves convexity of f in d, as required.
Piecewise linearity follows from the existence of finitely many different optimal bases for the second-stage program. □

We now study the expected total cost function (1.9). Since the first term is deterministic, we can compute its expected value with respect to $\boldsymbol{\xi}$ without loss of generality. We obtain:

$$z(x, \boldsymbol{\xi}) = c\mathbb{E}_{\boldsymbol{\xi}}[x] + h\mathbb{E}_{\boldsymbol{\xi}}[(x - \boldsymbol{\xi})^{+}] + v\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x)^{+}]. \tag{1.15}$$

Note that (1.9) is equivalent to (1.15). We now substitute (1.14) in (1.15). We obtain:

$$\begin{aligned} z(x, \boldsymbol{\xi}) &= c\mathbb{E}_{\boldsymbol{\xi}}[\boldsymbol{\xi} + (x - \boldsymbol{\xi})^{+} - (\boldsymbol{\xi} - x)^{+}] + h\mathbb{E}_{\boldsymbol{\xi}}[(x - \boldsymbol{\xi})^{+}] + v\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x)^{+}] = \\ &= c\bar{\xi} + c\mathbb{E}_{\boldsymbol{\xi}}[(x - \boldsymbol{\xi})^{+}] - c\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x)^{+}] + h\mathbb{E}_{\boldsymbol{\xi}}[(x - \boldsymbol{\xi})^{+}] + v\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x)^{+}] = \\ &= c\bar{\xi} + (c + h)\mathbb{E}_{\boldsymbol{\xi}}[(x - \boldsymbol{\xi})^{+}] + (v - c)\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x)^{+}]. \end{aligned} \tag{1.16}$$

Thus, we can rewrite the two-stage stochastic linear program (1.9)–(1.13) as follows:

$$\begin{aligned} \min \quad & z(x, \boldsymbol{\xi}) = c\bar{\xi} + \min \mathit{\Psi}(x) \\ & x \geq 0, \end{aligned} \tag{1.17}$$

where

$$\mathit{\Psi}(x) = \mathbb{E}_{\boldsymbol{\xi}}[\Psi(x, \boldsymbol{\xi})], \tag{1.18}$$

and

$$\Psi(x, \boldsymbol{\xi}) = (c + h)(x - \boldsymbol{\xi})^{+} + (v - c)(\boldsymbol{\xi} - x)^{+}. \tag{1.19}$$

If we compare $\mathrm{Q}(x, \boldsymbol{\xi})$ and $\Psi(x, \boldsymbol{\xi})$, we can observe that, by replacing the definition of the first–stage variable into the stochastic program (1.9)–(1.13), we obtain a new definition of the recourse cost function in (1.18) which takes into consideration all the cost parameters of the problem c, h and v. This leads to the following observations:

- the definition obtained in (1.19) clarifies the reason why we assume that the stock–out cost parameter v is strictly greater than the procurement cost c;
- we have two definitions of the expected total cost function:

$$z(x, \boldsymbol{\xi}) = cx + \mathcal{Q}(x),$$

and

$$z(x, \boldsymbol{\xi}) = c\bar{\xi} + \Psi(x).$$

Moreover, minimizing $c\bar{\xi} + \Psi(x)$ is equivalent to minimize $\Psi(x)$, as $c\bar{\xi}$ is constant.

We now show that it is possible to compute the expected surplus in terms of the expected shortage. From (1.14) we define the surplus as:

$$(x - \boldsymbol{\xi})^+ = x - \boldsymbol{\xi} + (\boldsymbol{\xi} - x)^+.$$

Taking the expectation on both sides, we obtain:

$$\mathbb{E}_{\boldsymbol{\xi}}(x - \boldsymbol{\xi})^+ = x - \bar{\xi} + \mathbb{E}_{\boldsymbol{\xi}}(\boldsymbol{\xi} - x)^+. \tag{1.20}$$

If we replace (1.20) in (1.18), we have:

$$\Psi(x) = (c + h)(x - \bar{\xi}) + (v + h)\mathbb{E}_{\boldsymbol{\xi}}(\boldsymbol{\xi} - x)^+. \tag{1.21}$$

The case with continuous probability distributions

We now study in more detail the two–stage stochastic linear program (1.17)–(1.19) assuming any continuous probability distribution for the stochastic demand $\boldsymbol{\xi}$. Then, we analyze the Uniform, Exponential, Normal and Log–normal probability distributions to understand how the optimal solution, expected surplus, expected shortage and expected total cost functions depend on which probability distribution we assume for the stochastic demand.

Assuming any continuous probability distribution, the model can be analytically solved as follows. Let $F_{\boldsymbol{\xi}}(x)$ be the cumulative distribution function (*cdf*) of the random variable $\boldsymbol{\xi}$ evaluated at x. By construction, $\Psi(x)$ can be computed as:

$$\begin{aligned}\Psi(x) = (c + h)\int_0^x (x - \boldsymbol{\xi})dF(\boldsymbol{\xi}) + (v - c)\int_x^{+\infty} (\boldsymbol{\xi} - x)dF(\boldsymbol{\xi}) = \\ = (c + h)x\int_0^x dF(\boldsymbol{\xi}) - (c + h)\int_0^x \boldsymbol{\xi} dF(\boldsymbol{\xi}) \\ + (v - c)\int_x^{+\infty} \boldsymbol{\xi} dF(\boldsymbol{\xi}) - (v - c)x\int_x^{+\infty} dF(\boldsymbol{\xi}).\end{aligned} \tag{1.22}$$

Since

$\int_0^x dF(\boldsymbol{\xi}) = F_{\boldsymbol{\xi}}(x)$ and integrating by part $\int_0^x \boldsymbol{\xi} dF(\boldsymbol{\xi})$ we have $xF_{\boldsymbol{\xi}}(x) - \int_0^x F(\boldsymbol{\xi})d\boldsymbol{\xi}$,

then:

$$\int_x^{+\infty} dF(\boldsymbol{\xi}) = 1 - F_{\boldsymbol{\xi}}(x) \text{ and } \int_x^{+\infty} \boldsymbol{\xi} dF(\boldsymbol{\xi}) = 1 - xF_{\boldsymbol{\xi}}(x) + \int_0^x F(\boldsymbol{\xi})d(\boldsymbol{\xi}).$$

By replacing it in (1.22), it follows that:

$$\begin{aligned}
\Psi(x) &= (c+h)xF_{\boldsymbol{\xi}}(x) - (c+h)[xF_{\boldsymbol{\xi}}(x) - \int_0^x F(\boldsymbol{\xi})d\boldsymbol{\xi}] + \\
&+ (v-c)[1 - xF_{\boldsymbol{\xi}}(x) + \int_0^x F(\boldsymbol{\xi})d\boldsymbol{\xi}] - (v-c)x[1 - F_{\boldsymbol{\xi}}(x)] = \\
&= v - c - x(v-c) + (c+h)\int_0^x F(\boldsymbol{\xi})d\boldsymbol{\xi} + (v-c)\int_0^x F(\boldsymbol{\xi})d\boldsymbol{\xi} = \\
&= v - c + x(c-v) + (h+v)\int_0^x F(\boldsymbol{\xi})d\boldsymbol{\xi}. \qquad (1.23)
\end{aligned}$$

Taking the first derivative of (1.23) with respect to x we obtain:

$$\Psi'(x) = c - v + (h+v)F_{\boldsymbol{\xi}}(x). \qquad (1.24)$$

Setting $\Psi'(x) = 0$, we obtain:

$$F_{\boldsymbol{\xi}}(x^*) = \frac{v-c}{v+h}, \qquad (1.25)$$

and therefore the optimal solution is $x^* = F_{\boldsymbol{\xi}}^{-1}(\frac{v-c}{v+h})$. If we denote with α the retailer's cost ratio $\frac{v-c}{v+h}$, then $x^* = F_{\boldsymbol{\xi}}^{-1}(\alpha)$. Therefore, the optimal order quantity x^* depends on the retailer's cost ratio α and on the assumed probability distribution.

Let us now compute x^* in closed–form for some continuous probability distributions, as a function of the retailer's cost ratio α.

Uniform distribution The Uniform distribution describes a random variable in which all values in the support $[a, b]$ are equally probable. The probability density function *pdf* and the cumulative distribution function *cdf* are:

$$f(\boldsymbol{\xi}; a, b) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq \boldsymbol{\xi} \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F(\boldsymbol{\xi}; a, b) = \begin{cases} \frac{\boldsymbol{\xi}-a}{b-a} & \text{if } a \leq \boldsymbol{\xi} \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Since the expected value is $\bar{\xi} = \frac{a+b}{2}$, then $b = 2\bar{\xi} - a$.

Let us now compute the optimal order quantity x^* in closed form. Since $F_{\boldsymbol{\xi}}(x) : \Re \to [0, 1]$ is continuous from the right and stricly increasing in the interval $[0, 1]$, then it is invertible. By applying the *Inverse transform method*, since $F_{\boldsymbol{\xi}}(x) = \frac{x-a}{b-a}$ for any $x \in [a, b]$, it follows that $\alpha = \frac{x-a}{b-a}$. Therefore,

$$\begin{aligned} x^* &= a + \alpha(b - a) = \\ &= a - \alpha a + \alpha b = \\ &= a - \alpha a + \alpha(2\bar{\xi} - a) = \\ &= a(1 - 2\alpha) + \alpha 2\bar{\xi}. \end{aligned} \tag{1.26}$$

Let us now explicitly define the recourse function (1.18) in this case. The expected surplus is:

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\xi}}[(x^* - \boldsymbol{\xi})^+] &= \int_a^x (x^* - \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\boldsymbol{\xi} = \\ &= \int_a^x (x^* - \boldsymbol{\xi}) \frac{1}{b-a} d\boldsymbol{\xi} = \\ &= \lim_{\boldsymbol{\xi} \to x^{*-}} F(\boldsymbol{\xi}) - \lim_{\boldsymbol{\xi} \to a^+} F(\boldsymbol{\xi}) = \\ &= \lim_{\boldsymbol{\xi} \to x^{*-}} \frac{1}{b-a}(x^* \boldsymbol{\xi} - \frac{\boldsymbol{\xi}^2}{2}) - \lim_{\boldsymbol{\xi} \to a^+} \frac{1}{b-a}(x^* \boldsymbol{\xi} - \frac{\boldsymbol{\xi}^2}{2}) = \\ &= \frac{x^{*2}}{2(b-a)} - \frac{1}{b-a}(a x^* - \frac{a^2}{2}) = \\ &= \frac{(x^* - a)^2}{2(b-a)}, \end{aligned} \tag{1.27}$$

while the expected shortage is:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi}-x^*)^+] &= \int_x^b (\boldsymbol{\xi}-x^*)f(\boldsymbol{\xi})d\boldsymbol{\xi} = \\
&= \int_x^b (\boldsymbol{\xi}-x^*)\frac{1}{b-a}d\boldsymbol{\xi} = \\
&= \lim_{\boldsymbol{\xi}\to b^-} F^{tr}(\boldsymbol{\xi}) - \lim_{\boldsymbol{\xi}\to x^{*+}} F^{tr}(\boldsymbol{\xi}) = \\
&= \lim_{\boldsymbol{\xi}\to b^-} \frac{1}{b-a}(\frac{\boldsymbol{\xi}^2}{2}-x^*\boldsymbol{\xi}) - \lim_{\boldsymbol{\xi}\to x^{*+}} \frac{1}{b-a}(\frac{\boldsymbol{\xi}^2}{2}-x^*\boldsymbol{\xi}) \\
&= \frac{b^2-2bx^*}{2(b-a)} - \frac{1}{b-a}(\frac{{x^*}^2}{2}-{x^*}^2)) \\
&= \frac{(b-x^*)^2}{2(b-a)}.
\end{aligned} \tag{1.28}$$

Let us define the expected surplus (1.27) in terms of the expected shortage (1.28) and use (1.21) as definition of the recourse cost function. Then, the recourse cost for x^* equal to the value computed in (1.26) is:

$$\begin{aligned}
\Psi(x^*) &= (c+h)(a(1-2\alpha)+\alpha 2\bar{\xi}-\bar{\xi}) + (v+h)\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi}-x)^+] = \\
&= (c+h)(a(1-2\alpha)+\bar{\xi}(2\alpha-1)) + (v+h)\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi}-x)^+] = \\
&= (c+h)(\bar{\xi}-a)(2\alpha-1) + (v+h)\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi}-x)^+].
\end{aligned} \tag{1.29}$$

Therefore, replacing (1.29) in (1.17), we have:

$$\begin{aligned}
z(x^*,\boldsymbol{\xi}) &= c\bar{\xi} + (c+h)(\bar{\xi}-a)(2\alpha-1) + (v+h)\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi}-x)^+] = \\
&= ac + ah - h\bar{\xi} + 2\alpha(c+h)(\bar{\xi}-a) + (v+h)\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi}-x)^+] = \\
&= ac + (\bar{\xi}-a)[2\alpha(c+h)-h] + (v+h)\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi}-x)^+].
\end{aligned} \tag{1.30}$$

Exponential distribution The (*pdf*) of an Exponential distribution is

$$f(\boldsymbol{\xi};\lambda) = \begin{cases} \lambda e^{-\lambda\boldsymbol{\xi}} & \text{if } \boldsymbol{\xi} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and its *cdf* is

$$F(\boldsymbol{\xi};\lambda) = \begin{cases} 1-e^{-\lambda\boldsymbol{\xi}} & \text{if } \boldsymbol{\xi} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is the rate parameter. We choose $\lambda^{-1} = \bar{\xi}$, *i.e.* $\lambda = \bar{\xi}^{-1}$, and we formally describe the *pdf* and the *cdf* by using the alternative parametrization. We obtain:

$$f(\boldsymbol{\xi};\bar{\xi}) = \begin{cases} \bar{\xi}^{-1}e^{-\bar{\xi}^{-1}\boldsymbol{\xi}} & \text{if } \boldsymbol{\xi} \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F(\boldsymbol{\xi};\bar{\xi}) = \begin{cases} 1 - e^{-\bar{\xi}^{-1}\boldsymbol{\xi}} & \text{if } \boldsymbol{\xi} \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The distribution is supported on the interval $[0,\infty)$. Since the stochastic demand in our problem has a finite support $[a,b]$, the *pdf* must be truncated. A truncated *pdf* $f^{tr}(\boldsymbol{\xi})$ and a truncated *cdf* $F^{tr}(\boldsymbol{\xi})$ for the Exponential distribution in the range $[a,b]$ are as follows:

$$f^{tr}(\boldsymbol{\xi};\bar{\xi},a,b) = \begin{cases} \bar{\xi}^{-1}e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}(1 - e^{-\bar{\xi}^{-1}b})^{-1} & \text{if } a \leq \boldsymbol{\xi} \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F^{tr}(\boldsymbol{\xi};\bar{\xi},a,b) = \begin{cases} (1 - e^{-\bar{\xi}^{-1}\boldsymbol{\xi}})(1 - e^{-\bar{\xi}^{-1}b})^{-1} & \text{if } a \leq \boldsymbol{\xi} \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Since $F^{tr}(x) : \Re \to [0,1]$ is continuous from the right and stricly increasing, then it is invertible. Therefore, $x^* = (F^{tr})^{-1}(\alpha)$. We use the *Inverse transform method* to get an explicit expression for x^* in a closed form. We start by studying the truncated *pdf* of the Exponential distribution. We obtain:

$$f^{tr}(x) = \bar{\xi}^{-1}e^{-\bar{\xi}^{-1}x}(1 - e^{-\bar{\xi}^{-1}b})^{-1}.$$

Then, we study the truncated *cdf*. We obtain:

$$\begin{aligned} F^{tr}(x) &= \int_0^x \bar{\xi}^{-1}e^{-\bar{\xi}^{-1}t}(1 - e^{-\bar{\xi}^{-1}b})^{-1}dt = \\ &= (1 - e^{-\bar{\xi}^{-1}b})^{-1}\int_0^x \bar{\xi}^{-1}e^{-\bar{\xi}^{-1}t}dt = \\ &= (1 - e^{-\bar{\xi}^{-1}b})^{-1}F(x). \end{aligned}$$

By setting $F^{tr}(x) = \alpha$, we have

$$\begin{aligned}
\alpha &= (1 - e^{-\bar{\xi}^{-1}b})^{-1}(1 - e^{-\bar{\xi}^{-1}x}) \\
e^{-\bar{\xi}^{-1}x} &= 1 - \alpha(1 - e^{-\bar{\xi}^{-1}b}) \\
x^* &= -\bar{\xi}\ln(1 - \alpha k), \qquad (1.31)
\end{aligned}$$

where

$$k = 1 - e^{-\bar{\xi}^{-1}b}.$$

We now want to compute the expected surplus and expected shortage in this case. The expected surplus is:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\xi}}[(x^* - \boldsymbol{\xi})^+] &= \int_a^{x^*} (x^* - \boldsymbol{\xi}) f^{tr}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \\
&= \int_a^{x^*} (x^* - \boldsymbol{\xi}) \frac{\bar{\xi}^{-1} e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}}{k} d\boldsymbol{\xi} = \\
&= \lim_{\boldsymbol{\xi} \to x^{*-}} F^{tr}(\boldsymbol{\xi}) - \lim_{\boldsymbol{\xi} \to a^+} F^{tr}(\boldsymbol{\xi}) = \\
&= \lim_{\boldsymbol{\xi} \to x^{*-}} \frac{1}{\bar{\xi}^{-1}k}[\bar{\xi} e^{-\bar{\xi}^{-1}\boldsymbol{\xi}} - \bar{\xi} e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}(x^* - \boldsymbol{\xi})] + \\
&\quad - \lim_{\boldsymbol{\xi} \to a^+} \frac{1}{\bar{\xi}^{-1}k}[\bar{\xi} e^{-\bar{\xi}^{-1}\boldsymbol{\xi}} - \bar{\xi} e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}(x^* - \boldsymbol{\xi})] \\
&= \frac{1}{\bar{\xi}^{-1}k}[e^{-\bar{\xi}^{-1}x^*} - e^{-\bar{\xi}^{-1}a}(\bar{\xi}^{-1}a - \bar{\xi}^{-1}x^* - 1)] = \\
&= \frac{1}{\bar{\xi}^{-1}k}[e^{-\bar{\xi}^{-1}x^*} + e^{-\bar{\xi}^{-1}a}(\bar{\xi}^{-1}(x^* - a) - 1)], \qquad (1.32)
\end{aligned}$$

and the expected shortage is:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi}-x^*)^+] &= \int_{x^*}^{b}(\boldsymbol{\xi}-x^*)f^{tr}(\boldsymbol{\xi})d\boldsymbol{\xi} = \\
&= \int_{x^*}^{b}(\boldsymbol{\xi}-x^*)\frac{\bar{\xi}^{-1}e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}}{k}d\boldsymbol{\xi} = \\
&= \lim_{\boldsymbol{\xi}\to b^-}F^{tr}(\boldsymbol{\xi}) - \lim_{\boldsymbol{\xi}\to x^{*+}}F^{tr}(\boldsymbol{\xi}) = \\
&= \lim_{\boldsymbol{\xi}\to b^-}\frac{1}{\bar{\xi}^{-1}k}[-\bar{\xi}e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}(\boldsymbol{\xi}-x^*) - \bar{\xi}e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}]+ \\
&\quad - \lim_{\boldsymbol{\xi}\to x^{*+}}\frac{1}{\bar{\xi}^{-1}k}[-\bar{\xi}e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}(\boldsymbol{\xi}-x^*) - \bar{\xi}e^{-\bar{\xi}^{-1}\boldsymbol{\xi}}] \\
&= \frac{1}{\bar{\xi}^{-1}k}[e^{-\bar{\xi}^{-1}b}(\bar{\xi}^{-1}x^* - \bar{\xi}^{-1}b - 1)] - (-\frac{1}{\bar{\xi}^{-1}k}e^{-\bar{\xi}^{-1}x^*}) = \\
&= \frac{1}{\bar{\xi}^{-1}k}[e^{-\bar{\xi}^{-1}b}(\bar{\xi}^{-1}(x^*-b) - 1) + e^{-\bar{\xi}^{-1}x^*}]. \qquad (1.33)
\end{aligned}$$

Normal distribution The *pdf* of a Normal distribution is

$$f(\boldsymbol{\xi};\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{\boldsymbol{\xi}-\bar{\xi}}{\sigma})^2},$$

and the *cdf* is

$$F(\boldsymbol{\xi};\mu,\sigma) = \frac{1}{2}[1 + erf(\frac{\boldsymbol{\xi}-\bar{\xi}}{\sigma\sqrt{2}})],$$

where $\bar{\xi}$ is the expectation of the Normal distribution, σ is the standard deviation and $erf(\bullet)$ is the error function.

The distribution is supported on $\mathbb{R}$. Since the stochastic demand in our problem has a finite support $[a,b]$, the *pdf* must be truncated. A truncated *pdf* $f^{tr}(\boldsymbol{\xi})$ and a truncated *cdf* $F^{tr}(\boldsymbol{\xi})$ for the Normal distribution in the range $[a,b]$ are as follows:

$$f^{tr}(\boldsymbol{\xi};\bar{\xi},\sigma,a,b) = \begin{cases} \phi(\frac{\boldsymbol{\xi}-\bar{\xi}}{\sigma})(\Phi(\frac{b-\bar{\xi}}{\sigma}) - \Phi(\frac{a-\bar{\xi}}{\sigma}))^{-1} & \text{if } a \le \boldsymbol{\xi} \le b, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F^{tr}(\boldsymbol{\xi};\bar{\xi},\sigma,a,b) = \begin{cases} (\Phi(\frac{\boldsymbol{\xi}-\bar{\xi}}{\sigma}) - \Phi(\frac{a-\bar{\xi}}{\sigma}))(\Phi(\frac{b-\bar{\xi}}{\sigma}) - \Phi(\frac{a-\bar{\xi}}{\sigma}))^{-1} & \text{if } a \le \boldsymbol{\xi} \le b, \\ 0 & \text{otherwise.} \end{cases}$$

If $\boldsymbol{\xi}$ is normally distributed, then $\boldsymbol{\xi} = \bar{\xi} + \sigma Z$ where Z is the standard normal random variable. Let $\Phi(z) = p[Z \leq z]$ be the *cdf* of the standard random variable. Since $\alpha = p[\boldsymbol{\xi} < x^*] = \Phi(z_\alpha)$ and $\Phi^{-1}(\alpha) = z_\alpha$, it follows that:

$$x^* = \bar{\xi} + \sigma z_\alpha. \tag{1.34}$$

The Normal distribution does not have a closed–form inverse and therefore the *Inverse transform method* cannot be applied. We use an approximation method for the standard normal distribution proposed in Schmeiser (1978) [55], from which we define:

$$z_\alpha = \frac{\alpha^{0.135} - (1-\alpha)^{0.135}}{0.1975} \tag{1.35}$$

for $0.0013499 \leq \alpha \leq 0.9986501$ which matches the true normal distribution with one digit after decimal point. We obtain an approximated expression for x^*:

$$x^* \approx \bar{\xi} + \frac{\sigma\alpha^{0.135} - (1-\alpha)^{0.135}}{0.1975} \tag{1.36}$$

We now want to compute the expected surplus and expected shortage in this case. The expected surplus is:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\xi}}[(x^* - \boldsymbol{\xi})^+] &= \mathbb{E}_{\boldsymbol{\xi}}[(\bar{\xi} + \sigma z_\alpha - \boldsymbol{\xi})^+] = \\
&= \sigma \mathbb{E}_{\boldsymbol{\xi}}[(z_\alpha - Z)^+] = \\
&= \sigma \int_{-\infty}^{z_\alpha} (z_\alpha - Z)\phi(Z)dZ = \\
&= \sigma[z_\alpha \int_{-\infty}^{z_\alpha} d\Phi(Z) - \int_{-\infty}^{z_\alpha} Z d\Phi(Z)] = \\
&= \sigma[z_\alpha \Phi(z_\alpha) - \int_{-\infty}^{z_\alpha} Z d\Phi(Z)] = \\
&= \sigma[\alpha z_\alpha + \phi(z_\alpha)],
\end{aligned} \tag{1.37}$$

and the expected shortage is:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x^*)^+] &= \mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - \bar{\xi} - \sigma z_\alpha)] = \\
&= \sigma \mathbb{E}_{\boldsymbol{\xi}}[(Z - z_\alpha)^+] = \\
&= \sigma \int_{z_\alpha}^{+\infty} (Z - z_\alpha)\phi(Z) dZ = \\
&= \sigma[\int_{z_\alpha}^{+\infty} Z d\Phi(Z) - z_\alpha \int_{z_\alpha}^{+\infty} d\Phi(Z)] = \\
&= \sigma[\int_{z_\alpha}^{+\infty} Z d\Phi(Z) - z_\alpha(1 - \Phi(z_\alpha))] = \\
&= \sigma[\phi(z_\alpha) - z_\alpha(1 - \alpha)]. \qquad (1.38)
\end{aligned}$$

We now define the expected surplus (1.37) in terms of the expected shortage (1.38) and we use (1.20) as definition of the recourse cost function. We compute the recourse cost in x^* by using (1.34). We obtain:

$$\begin{aligned}
\Psi(x^*) &= (c + h)(x^* - \bar{\xi}) + (v + h)\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x^*)^+] = \\
&= (c + h)\sigma z_\alpha + (v + h)\sigma[\phi(z_\alpha) - z_\alpha(1 - \alpha)] = \\
&= (v + h)\sigma\phi(z_\alpha). \qquad (1.39)
\end{aligned}$$

Then, we replace (1.39) in (1.17). We obtain:

$$z(x^*, \boldsymbol{\xi}) = c\bar{\xi} + (v + h)\sigma\phi(z_\alpha). \qquad (1.40)$$

Log–normal distribution The *pdf* of a Log–normal distribution is

$$f(\boldsymbol{\xi}; \mu, \sigma) = \frac{1}{\boldsymbol{\xi}\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\ln \boldsymbol{\xi} - \bar{\xi}}{\sigma})^2},$$

and the *cdf* is

$$F(\boldsymbol{\xi}; \mu, \sigma) = \frac{1}{2} erfc(-\frac{\ln \boldsymbol{\xi} - \bar{\xi}}{\sigma\sqrt{2}})],$$

where $\bar{\xi}$ is the expectation of the Log–normal distribution, σ is the standard deviation and $erfc(\bullet)$ is the complementary error function.
The distribution is supported on $(0, \infty)$. Since the stochastic demand in our problem has a finite support $[a, b]$, the *pdf* must be truncated. A truncated *pdf*

$f^{tr}(\boldsymbol{\xi})$ and a truncated *cdf* $F^{tr}(\boldsymbol{\xi})$ for the Log–normal distribution in the range $\boldsymbol{\xi} \in [a, b]$ are given as follows:

$$f^{tr}(\boldsymbol{\xi}; \bar{\xi}, \sigma, a, b) = \begin{cases} f(\boldsymbol{\xi}; \bar{\xi}, \sigma)(F(b; \bar{\xi}, \sigma) - F(a; \bar{\xi}, \sigma))^{-1} & \text{if } a \leq \boldsymbol{\xi} \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F^{tr}(\boldsymbol{\xi}; \bar{\xi}, \sigma, a, b) = \begin{cases} \frac{F(\boldsymbol{\xi}; \bar{\xi}, \sigma) - F(a; \bar{\xi}, \sigma)}{F(b; \bar{\xi}, \sigma) - F(a; \bar{\xi}, \sigma)} & \text{if } a \leq \boldsymbol{\xi} \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

If $\boldsymbol{\xi}$ is lognormally distributed with parameters ζ and η, then $\ln(\boldsymbol{\xi})$ is normally distributed with mean ζ and variance η^2.
Since $\ln(\boldsymbol{\xi}) = \zeta + \eta Z$, then $\boldsymbol{\xi} = e^{\zeta + \eta Z}$ and it follows that:

$$x^* = e^{\zeta + \eta z_\alpha}. \tag{1.41}$$

Since $F_{\boldsymbol{\xi}}(x^*) = p[\boldsymbol{\xi} \leq x^*]$, then $p[\boldsymbol{\xi} \leq x^*] = p[\ln(\boldsymbol{\xi}) \leq ln(x^*)] = p[\zeta + \eta Z \leq \zeta + \eta z_\alpha] = p[Z \leq z_\alpha] = \Phi(z_\alpha) = \alpha$. We obtain that $\bar{\xi} = e^{\zeta + \frac{1}{2}\eta^2}$ and $\sigma^2 = \bar{\xi}^2(e^{\eta^2} - 1)$. The Log-normal distribution does not have a closed–form inverse and therefore the *Inverse transform method* cannot be applied. We can derive an approximate expression for x^* using (1.35). We obtain:

$$x^* \approx e^{\zeta + \eta \frac{\alpha^{0.135} - (1-\alpha)^{0.135}}{0.1975}}. \tag{1.42}$$

We now want to compute the expected surplus and expected shortage in this case. In Gallego (1993) [23], the expected surplus is defined as:

$$\mathbb{E}_{\boldsymbol{\xi}}[(x^* - \boldsymbol{\xi})^+] = \bar{\xi}(\Phi(\eta - z_\alpha)) - x^* \Phi(-z_\alpha), \tag{1.43}$$

where

$$\Phi(-z_\alpha) = \frac{c + h}{v + h}, \tag{1.44}$$

and the expected shortage is:

$$\mathbb{E}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x^*)^+] = \bar{\xi}(\Phi(\eta - z_\alpha) - 1) + x^*(1 - \Phi(-z_\alpha)). \tag{1.45}$$

The case with discrete probability distributions

When a real–world problem needs to be numerically solved, the assumption of the discrete distribution is mostly agreed. We study the two–stage stochastic linear program (1.17)–(1.19) assuming the discretized Uniform, Exponential, Normal and Log–normal probability distributions to understand how the optimal solution, the expected surplus, the expected shortage and the expected total cost functions are depending on the probability distribution we assume for the stochastic demand.

When the stochastic demand $\boldsymbol{\xi}$ has a very large number of possible realizations, the standard approach is to give a representation of the random vector's distribution trought the uses of scenarios, say $s = 1, 2, \ldots, S$, each representing a possible realization ξ_s of the stochastic demand $\boldsymbol{\xi}$ with the corresponding probability mass p_s.

The two-stage stochastic linear program (1.17)–(1.19) is reformulated as the following Linear Programming problem:

$$\min z(x, \boldsymbol{\xi}) = c\bar{\xi} + \min_x \mathit{\Psi}(x) \tag{1.46}$$
$$x \geq 0,$$

where

$$\mathit{\Psi}(x) = \sum_{s=1}^{S} p_s \Psi(x, \xi_s), \tag{1.47}$$

and

$$\Psi(x, \xi_s) = (c + h)(x - \xi_s)^+ + (v - c)(\xi_s - x)^+. \tag{1.48}$$

Thus, the recourse function can be written as:

$$\mathit{\Psi}(x) = (c + h) \sum_{s=1}^{S} p_s (x - \xi_s)^+ + (v - c) \sum_{s=1}^{S} p_s (\xi_s - x)^+, \tag{1.49}$$

where $\sum_{s=1}^{S} p_s (x - \xi_s)^+$ and $\sum_{s=1}^{S} p_s (\xi_s - x)^+$ are the expected surplus and expected shortage, respectively.

Given any discrete probability distribution, the definition of the expected surplus is:

$$\sum_{s=1}^{S} p_s(x-\xi_s)^+ = \sum_{S:\xi_s<x} (F(\xi_s)-F(\xi_{s-1}))(x-\xi_s) =$$
$$= \sum_{k=0}^{x-a}(1-F(x+k)), \tag{1.50}$$

and the expected shortage is:

$$\sum_{s=1}^{S} p_s(\xi_s-x)^+ = \sum_{S:\xi_s>x} (F(\xi_s)-F(\xi_{s-1}))(\xi_s-x) =$$
$$= \sum_{k=0}^{b-x-a} (1-F(x+k)). \tag{1.51}$$

Discretized Uniform distribution If we replace the discretized *cdf* of the Uniform distribution in (1.50), the expected surplus becomes:

$$\sum_{k=0}^{x-a}(1-F(x+k)) = \sum_{k=0}^{x-a}(1-\frac{x+k-a}{b-a}),$$

and using the property of the arithmetic progression, we obtain:

$$(x-a)(1-\frac{x-a}{b-a}) - \frac{x^2}{2(b-a)} = \frac{(x-a)^2}{2(b-a)}. \tag{1.52}$$

which is the same definition obtained in (1.19) in the continuous case. We apply the same for (1.51). The expected shortage becomes:

$$\sum_{k=0}^{b-a-x} (1-F(x+k)) = \sum_{k=0}^{b-a-x} (1-\frac{x+k-a}{b-a})$$
$$= (b-a-x)(1-\frac{x-a}{b-a}) - \frac{(b-a-x)(b-a-x)}{2(b-a)}$$
$$= \frac{(b-x)^2}{2(b-a)}. \tag{1.53}$$

which is the same definition obtained in (1.20) in the continuous case. We define the expected surplus in terms of the expected shortage using (1.20) and we use (1.21) as definition of the recourse cost function. We compute the recourse cost in x^*. We obtain:

$$\Psi(x^*) = (c+h)(\bar{\xi}-a)(2\alpha-1) + (v+h)\frac{(b-x)^2}{2(b-a)}. \tag{1.54}$$

Then, we substitute (58) into (9). We obtain:

$$z(x^*, \boldsymbol{\xi}) = ac + (\bar{\xi} - a)[2\alpha(c+h) - h] + (v+h)\frac{(b-x)^2}{2(b-a)}. \tag{1.55}$$

Discretized Exponential distribution If we replace the discretized *cdf* of the Exponential distribution in (1.50), the expected surplus becomes:

$$\sum_{k=0}^{x-a}(1 - F(x+k)) = \sum_{k=0}^{x-a}(1 - \frac{1 - \bar{\xi}^{-1}e^{-\bar{\xi}^{-1}(x+k)}}{k}) =$$
$$= \sum_{k=0}^{x-a} -\frac{1}{k}\sum_{k=0}^{x-a} + \bar{\xi}^{-1}\sum_{k=0}^{x-a} e^{-\bar{\xi}^{-1}(x+k)},$$

and using the property of the arithmetic and geometric progressions, from Birge and Louveaux (2011) [11], we have:

$$\sum_{s=1}^{S} p_s(x - \xi_s)^+ = \sum_{k=0}^{\infty} e^{-\lambda(x+k)} = \frac{e^{-\lambda x}}{1 - e^{-\lambda}}, \tag{1.56}$$

while

$$\sum_{s=1}^{S} p_s(\xi_s - x)^+ = \lfloor x \rfloor + 1 - e^{-\lambda(x - \lfloor x \rfloor)}\sum_{k=0}^{\lfloor x \rfloor} e^{-\lambda k} = \tag{1.57}$$
$$= \lfloor x \rfloor + 1 - \left(\frac{e^{-\lambda(x - \lfloor x \rfloor)} - e^{-\lambda(x+1)}}{1 - e^{-\lambda}}\right) \tag{1.58}$$

Note that the definitions of the expected surplus and expected shortage in the discrete case are different from the definitions obtained in the continuous case.

1.3.3.B A distributionally robust formulation

In order to complete our analysis, in this section we show a *Distributionally robust formulation* for the two–stage stochastic linear program (1.17)–(1.19). In Scarf (1958) [54] it is derived for the first time a min–max order formula for the distribution–free risk–neutral newsvendor problem, also addressed as "Scarf's ordering rule". It provides a closed–form expression of the order quantity that maximizes the worst–case expected profit associated with the demand of a single product when only the mean and the variance, rather then the full distribution itself, are known. Then Gallego and Moon (1993) [23] extended his work by proposing a study for the distribution–free newsvendor problem and a more compact proof for the optimality of the Scarf's ordering rule. In our study, this corresponds to minimize the maximum expected total cost for all the demand distributions or equivalently to find the order quantity that minimizes the expected total cost against the worst possible distribution of the demand with mean $\bar{\xi}$ and variance σ^2.

Let $\mathcal{P}$ be the ambiguity set, a set of many different distributions $\mathbb{P}$, $\mathbb{P} \in \mathcal{P}$, that are consinstent with the available information. Then, if we order a certain quantity x and only the mean and standard deviation of the demand distribution are known, we obtain:

$$\min_{x} \max_{\mathbb{P}\in\mathcal{P}} z(x, \boldsymbol{\xi}) = cx + h\mathbb{E}_{\mathbb{P}}[(x - \boldsymbol{\xi})^{+}] + v\mathbb{E}_{\mathbb{P}}[(\boldsymbol{\xi} - x)^{+}] \tag{1.59}$$

s.t.

$$\mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}] = \bar{\xi} \tag{1.60}$$

$$\mathbb{E}_{\mathbb{P}}[(\boldsymbol{\xi} - \bar{\xi})^2] = \sigma^2 \tag{1.61}$$

$$\boldsymbol{\xi} \in \Xi. \tag{1.62}$$

For the sake of completeness, we provide the continuous and discrete formulation of (1.59)–(1.62). The continuous formulation is:

$$\min_{x} \max_{\mathbb{P}\in\mathcal{P}} [cx + h\int_0^x (x - \boldsymbol{\xi}) d\mathbb{P}(\boldsymbol{\xi}) + v\int_x^\infty (\boldsymbol{\xi} - x) d\mathbb{P}(\boldsymbol{\xi})] \tag{1.63}$$

$$\text{s.t.} \quad \int_\Xi d\mathbb{P}(\boldsymbol{\xi}) = 1 \tag{1.64}$$

$$\int_\Xi \boldsymbol{\xi} d\mathbb{P}(\boldsymbol{\xi}) = \bar{\xi} \tag{1.65}$$

$$\int_\Xi [(\boldsymbol{\xi} - \bar{\xi})^2] d\mathbb{P}(\boldsymbol{\xi}) = \sigma^2. \tag{1.66}$$

We discretize (1.63)–(1.66). We study the support Ξ of the stochastic demand. It can be defined as

$$\Xi = \cup_{i=1}^{I} \Xi_i,$$

from which it follows that

$$p_i = \int_{\Xi_i} d\mathbb{P}(\boldsymbol{\xi}),$$

and

$$\xi_i = \frac{1}{p_i} \int_{\Xi_i} \boldsymbol{\xi} d\mathbb{P}(\boldsymbol{\xi}).$$

We use p_i and ξ_i to approximate the min–max problem. We obtain:

$$\min_x \max_{\mathbb{P} \in \mathcal{P}} [cx + h \sum_{I:\xi_i < x} p_i(x - \xi_i) + v \sum_{I:\xi_i > x} p_i(\xi_i - x)] \tag{1.67}$$

$$\text{s.t.} \quad p_i \geq 0 \qquad \forall i = 1, \dots, I \tag{1.68}$$

$$\sum_{i=1}^{I} p_i = 1 \tag{1.69}$$

$$\sum_{i=1}^{I} \xi_i p_i = \bar{\xi} \tag{1.70}$$

$$\sum_{i=1}^{I} [(\xi_i - \bar{\xi})^2] p_i = \sigma^2 \tag{1.71}$$

$$\xi_i \in \Xi_i \qquad \forall i = 1, \dots, I. \tag{1.72}$$

We want to determine for (1.59) the stockage policy that minimizes the maximum cost that would occure, considering all distributions with the given mean and standard deviation.

From paragraph 1.3.3.A, the expected total cost function

$$z(x, \boldsymbol{\xi}) = cx + h\mathbb{E}[(x - \boldsymbol{\xi})^+] + v\mathbb{E}[(\boldsymbol{\xi} - x)^+],$$

where $[(x - \boldsymbol{\xi})^+] = x - \boldsymbol{\xi} + \mathbb{E}[(\boldsymbol{\xi} - x)^+]$, can be rewritten as:

$$\begin{aligned} z(x, \boldsymbol{\xi}) &= cx + h\mathbb{E}[x - \boldsymbol{\xi} + (\boldsymbol{\xi} - x)^+] + v\mathbb{E}[(\boldsymbol{\xi} - x)^+] = \\ &= cx + hx - h\bar{\xi} + h\mathbb{E}[(\boldsymbol{\xi} - x)^+] + v\mathbb{E}[(\boldsymbol{\xi} - x)^+] = \\ &= cx + h(x - \bar{\xi}) + (h + v)\mathbb{E}[(\boldsymbol{\xi} - x)^+]. \end{aligned} \tag{1.73}$$

In Gallego and Moon (1993) [23] it is showed that the expected shortage can be defined in function of $\bar{\xi}$ and σ with the following inequality:

$$\mathbb{E}[(\boldsymbol{\xi} - x)^+] \leq \frac{[\sigma^2 + (x - \bar{\xi})^2]^{\frac{1}{2}} - (x - \bar{\xi})}{2} \tag{1.74}$$

Since we are minimizing the expected total cost in the worst case and in (1.73) the only remaining variable is the expected shortage, we are looking for the case where the expected shortage is the largest, that is to consider:

$$\mathbb{E}[(\boldsymbol{\xi} - x)^+] = \frac{[\sigma^2 + (x - \bar{\xi})^2]^{\frac{1}{2}} - (x - \bar{\xi})}{2}.$$

So, problem (1.59) is reduced to minimizing

$$z(x, \boldsymbol{\xi}) = cx + h(x - \bar{\xi}) + \frac{(h + v)}{2}[(\sigma^2 + (x - \bar{\xi})^2)^{\frac{1}{2}} - (x - \bar{\xi})] \tag{1.75}$$

We compute the first derivative with respect to x and we set it to 0. We obtain that the optimal quantity to order is:

$$x^* = \bar{\xi} + \sigma \frac{(2c + h - v)\sqrt{-(c + h)(c - v)}}{2(c + h)(c - v)} \tag{1.76}$$

Note that $-(c+h)(c-v)$ is always positive since $v > c$. (1.76) minimizes (1.59) against the worst distribution.

1.3.4 The Recourse Penalty Bound and Maximum Recourse Penalty Bound

We now focus on the recourse function $\Psi(x)$, given that $c\bar{\xi}$ is constant. Let $x^{\alpha}_{\mathcal{R}}$ be the optimal solution of the *Recourse problem assuming the right distribution* $\mathcal{R}$ when the retailer's cost ratio is α. We define $\Psi_{\mathcal{R}}(x^{\alpha}_{\mathcal{R}})$ to be the value of the recourse function evaluated in $x^{\alpha}_{\mathcal{R}}$ by using the right distribution $\mathcal{R}$. Let $x^{\alpha}_{\mathcal{G}}$ be the optimal solution of the *Recourse problem assuming a guessed distribution* $\mathcal{G}$ when the retailer's cost ratio is α. We define $\Psi_{\mathcal{R}}(x^{\alpha}_{\mathcal{G}})$ to be the value of the recourse function evaluated in $x^{\alpha}_{\mathcal{G}}$ by using the right distribution $\mathcal{R}$.

Definition 1.3.2.

The Recourse Penalty Bound RPB is:

$$RPB = \frac{\Psi_{\mathcal{R}}(x^{\alpha}_{\mathcal{G}})}{\Psi_{\mathcal{R}}(x^{\alpha}_{\mathcal{R}})} - 1.$$

Definition 1.3.3. *The Maximum Recourse Penalty Bound ($MRPB$) is:*

$$MRPB = \max_{0<\alpha<1} RPB.$$

MRPB provides the worst–case performance bound for the recourse penalty cost.

The RPB states the relative value associated to the error regarding the assumed distribution (i.e., the general concept of the VRD) as a function of the value of the retailer's cost ratio (i.e., α). The MRPB measures the maximum value RPB obtained over the interval $0 < \alpha < 1$. These measures help the decision–maker to determine how much a wrong assumption on the probability distribution for the stochastic process can prejudicate the performance of the model and which mismatch among the possible probability distributions is more penalizing for a certain α, thus the cost structure of the problem, or in the worst case.

Let us consider the Exponential distribution to be the guessed distribution and the Uniform distribution to be the right distribution. We provide the following theorem.

Theorem 1.3.4. *Let the Exponential distribution $\mathcal{E}$ be the guessed distribution and the Uniform distribution $\mathcal{U}$ be the right distribution. Then,*

$$RPB(\alpha) = \frac{1}{4}\frac{\delta(\alpha)^2}{\alpha(1-\alpha)} + \frac{1+\delta(\alpha)}{1-\alpha} - 1.$$

Proof. Consider first the Uniform distribution. The expected surplus is:

$$\mathbb{E}^{\mathcal{U}}_{\boldsymbol{\xi}}[(x^{\alpha}_{\mathcal{U}} - \boldsymbol{\xi})^+] = \frac{(x-a)^2}{2(b-a)^2} = \frac{x^2}{4\bar{\xi}} = \bar{\xi}\alpha^2,$$

and the expected shortage is:

$$\mathbb{E}^{\mathcal{U}}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x^{\alpha}_{\mathcal{U}})^+] = \frac{(b-x)^2}{2(b-a)} = \frac{(2\bar{\xi}-x)^2}{4\bar{\xi}} = \bar{\xi}(1-\alpha)^2.$$

Thus, the recourse function is:

$$\begin{aligned}
\Psi_{\mathcal{U}}(x^{\alpha}_{\mathcal{U}}) &= (c+h)\bar{\xi}\alpha^2 + (v-c)\bar{\xi}(1-\alpha)^2 = \\
&= (c+h)\bar{\xi}\alpha^2 + (v-c)\bar{\xi}(1-2\alpha+\alpha^2) = \\
&= (s+h)\bar{\xi}\alpha^2 + (v-c)\bar{\xi}(1-2\alpha) = \\
&= \bar{\xi}[(s+h)\alpha^2 + (v-c)(1-2\alpha)].
\end{aligned}$$

Since $\alpha = \frac{v-c}{v+h}$, then:

$$\begin{aligned}
\Psi_{\mathcal{U}}(x^{\alpha}_{\mathcal{U}}) &= \bar{\xi}[(v+h)\frac{(v-c)^2}{(v+h)^2} + (v-c)(1-2\frac{v-c}{v+h})] = \\
&= \bar{\xi}[\frac{(v-c)^2}{(v+h)} + s - c - 2\frac{(v-c)^2}{(v+h)}] = \\
&= \bar{\xi}\frac{-(v-c)^2 + (v-c)(v+h)}{v+h} = \\
&= \bar{\xi}(v-c)\frac{c+h}{v+h}.
\end{aligned}$$

Let us now consider the Exponential distribution. Since $x^{\alpha}_{\mathcal{E}} = -\bar{\xi}\ln(1-k\alpha)$ with $k = 1-e^{-2}$, let be

$$x^{\alpha}_{\mathcal{E}} = -\bar{\xi}\delta(\alpha),$$

with $\delta(\alpha) = \ln((1-\alpha)+\alpha e^{-2})$.

The expected surplus computed by using the right distribution U is:

$$\mathbb{E}^{\mathcal{U}}_{\boldsymbol{\xi}}[(x^{\alpha}_{\mathcal{E}} - \boldsymbol{\xi})^+] = \frac{(-\bar{\xi}\delta(\alpha))^2}{4\bar{\xi}} = \frac{1}{4}\bar{\xi}\delta(\alpha)^2,$$

and the expected shortage computed by using the right distribution U is:

$$\mathbb{E}^{\mathcal{U}}_{\boldsymbol{\xi}}[(\boldsymbol{\xi} - x^{\alpha}_{\mathcal{E}})^{+}] = \frac{(2\bar{\xi} + \bar{\xi}\delta(\alpha))^2}{4\bar{\xi}} = \frac{1}{4}\bar{\xi}(2 + \delta(\alpha))^2 = \frac{1}{4}\bar{\xi}\delta(\alpha)^2 + \bar{\xi}(1 + \delta(\alpha)).$$

Thus, the recourse function computed by using the right distribution U is:

$$\begin{aligned}\Psi_{\mathcal{U}}(x^{\alpha}_{\mathcal{E}}) &= \frac{1}{4}(c+h)\bar{\xi}\delta(\alpha)^2 + \frac{1}{4}(v-c)\bar{\xi}\delta(\alpha)^2 + (v-c)\bar{\xi}(1+\delta(\alpha)) = \\ &= \frac{1}{4}(v+h)\bar{\xi}\delta(\alpha)^2 + (v-c)\bar{\xi}(1+\delta(\alpha)) \\ &= \bar{\xi}[\frac{1}{4}(v+h)\delta(\alpha)^2 + (v-c)(1+\delta(\alpha))].\end{aligned}$$

Therefore,

$$RPB(\alpha) = \frac{\Psi_{\mathcal{U}}(x^{\alpha}_{\mathcal{E}})}{\Psi_{\mathcal{U}}(x^{\alpha}_{\mathcal{U}})} - 1 = \frac{\frac{1}{4}(v+h)\delta(\alpha)^2 + (v-c)(1+\delta(\alpha))}{(v-c)\frac{c+h}{v+h}} =$$

$$= \frac{1}{4}\frac{(v+h)^2\delta(\alpha)^2}{(v-c)(c+h)} + \frac{(v+h)(1+\delta(\alpha))}{(v+h)} - 1 = \frac{1}{4}\frac{\delta(\alpha)^2}{\alpha(1-\alpha)} + \frac{1+\delta(\alpha)}{1-\alpha} - 1.$$

□

Figure 1.2 shows the plot of the function $RPB(\alpha)$ for $0 < \alpha < 1$.

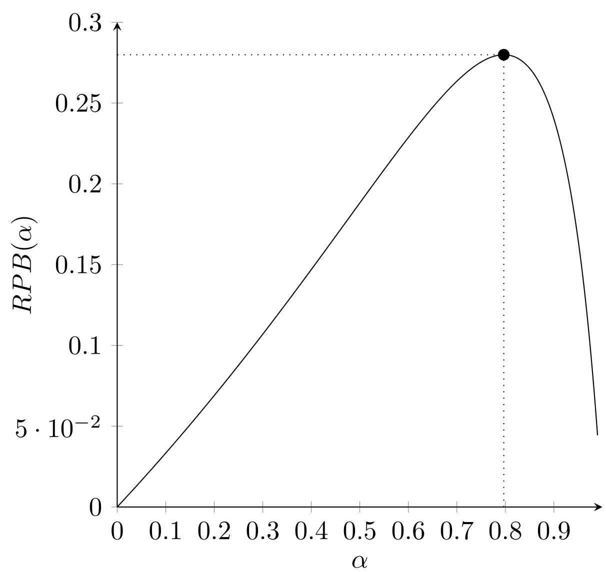


Figure 1.2: $RPB(\alpha)$

In order to compute MRPB, we numerically solve with *Mathematica* the following non-linear optimization problem:

$$\max \frac{1}{4}\frac{\delta(\alpha)^2}{\alpha(1-\alpha)} + \frac{1+\delta(\alpha)}{1-\alpha} - 1 \tag{1.77}$$

s.t.

$$0 < \alpha < 1 \tag{1.78}$$

The optimal solution of this model is $\alpha^* = 0.796$. Therefore, $MRPB \approx 28\%$.

1.3.5 Numerical Results

In this subsection we propose numerical results of the two–stage and distributionally robust models proposed in the previous subsection. This subsection is organized as follows. In Paragraph 1.3.4.A the parameter definitions are provided, in Paragrapgh 1.3.4.B we provide a description of the computational experiments carried out and of the obtained results. In Paragraph 1.3.4.C we propose the results of the in–sample stability. In Paragraph 1.3.4.D we measure the value of the stochastic solution under different demand distributions and different values of the retailer's cost ratio. In Paragraph 1.3.4.E we measure the Value of the Right Distribution and we introduce and describe the Deviation Test. In Paragraph 1.3.4.F we provide the distributionally robust solution. At last, we provide and comment the distributionally robust solution.

1.3.4.A. Parameter definitions

We set the problem parameters as follows:

- procurement cost $c = 4$;
- holding cost $h = 1$;
- stock–out cost v, $v = 4 + 0.2\tau$, where $\tau = 0, 1, \ldots, 1000$. These values allow to compute $\alpha \in (0, 1)$ as shown in Figure 1.3. Note that α is an increasing function in v.

- minimum value of the stochastic demand $a = 0$;
- maximum value of the stochastic demand $b = 200$;
- expected value of the stochastic demand $\bar{\xi} = 100$.

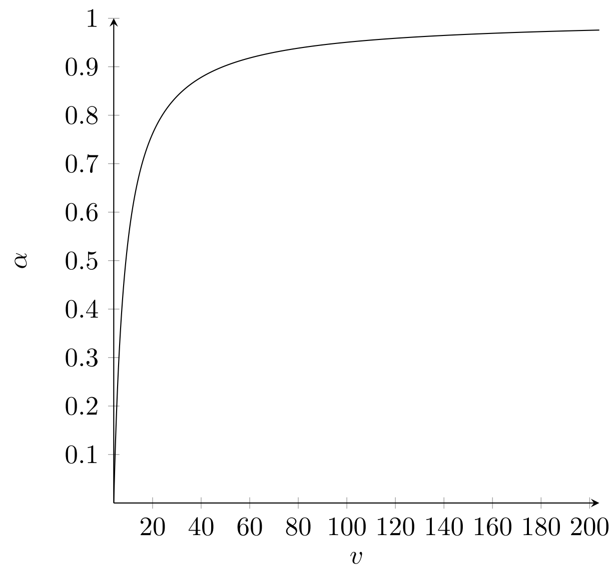


Figure 1.3: Graphical representation of α as function of v.

Let $\mathcal{U}$ be the Uniform distribution, $\mathcal{E}$ be the Exponential distribution, $\mathcal{N}$ be the Normal distribution and $\mathcal{L}$ be the Log–normal distribution. The corresponding parameters are the following:

- standard deviation of the Uniform distribution, $\sigma_{\mathcal{U}} = \sqrt{\frac{1}{12}(b-a)^2} = 57.74$;
- rate parameter and standard deviation of the Exponential distribution, $\lambda = \bar{\xi}^{-1} = 0.01$ and $\sigma_{\mathcal{E}} = \sqrt{\lambda^{-2}} = \bar{\xi} = 100$, respectively;
- standard deviation of the Normal distribution, $\sigma_{\mathcal{N}} = \sqrt{20} = 4.47$;
- location and scale paramaters of the Log-normal distribution, $\zeta = \ln(\bar{\xi}) -$

$\frac{1}{2}\ln(\mathrm{cv}^2+1) = 4.60417$ and $\eta = \sqrt{\ln(\mathrm{cv}^2+1)} = 0.0447$, respectively, where $\mathrm{cv} = \sigma\bar{\xi}^{-1}$ is the coefficient of variation.

Figure 1.4 and Figure 1.5 show the comparison of the *pdf*s $f(\boldsymbol{\xi})$ and *cdf*s $F(\boldsymbol{\xi})$ of the probability distributions, respectively. Note that the stochastic demand is defined on the same support $\boldsymbol{\xi} \in [0, 200]$ with same expected value $\bar{\xi} = 100$ for all distributions.

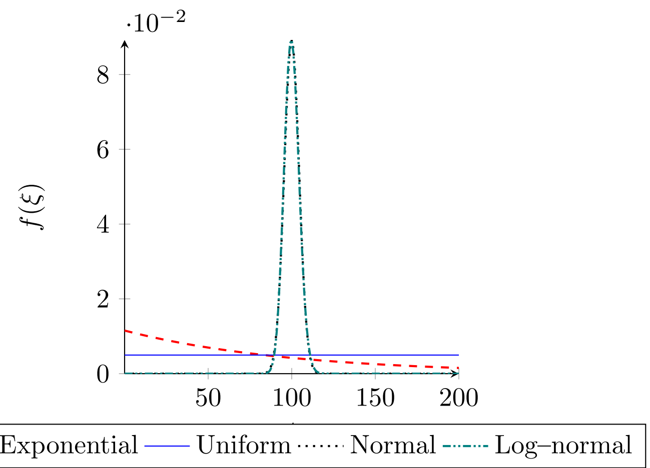


Figure 1.4: Probability density functions of the stochastic demand $\boldsymbol{\xi}$.

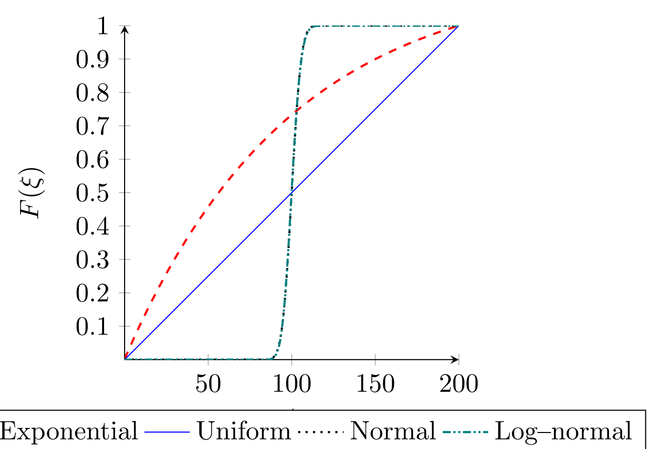


Figure 1.5: Cumulative distribution functions of the stochastic demand $\boldsymbol{\xi}$.

The *pdf*s and *cdf*s of the Normal and Log–normal distributions are similar (almost identical): this is a particular case due to the standard deviation to which the distributions are initially set. Uniform, Exponential, Normal and Log–normal distributions are truncated as shown in Paragraph 1.3.3.A.

1.3.4.B Computational experiments

Our aim is to understand how the assumption of a probability distribution and the estimated variance affect the optimal order quantity to order, the expected holding cost, the expected stock–out cost and the recourse cost functions. The following three experiments are carried out:

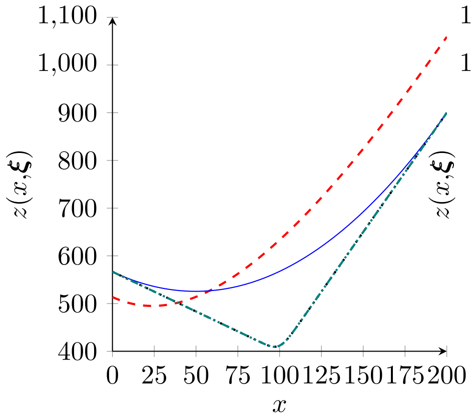
1. we assume that it is unknown how to compute the optimal order quantity x^*. The aim of the experiment is to understand how the order quantity x affects the expected total cost;

2. we assume to know how to compute the optimal order quantity x^*, given our analysis in Subsection 1.3.3. The aim of the experiment is to compute the corresponding expected holding cost, expected stock–out cost and recourse cost depending on the retailer's cost ratio α;

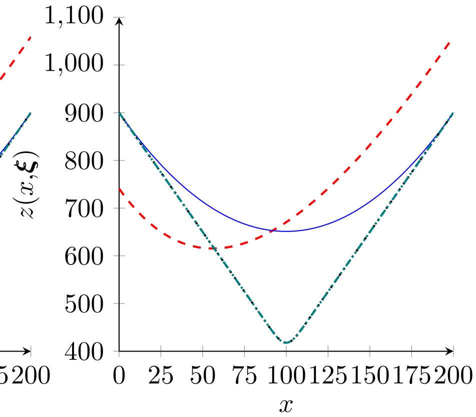
3. we assume that the standard deviations of the Normal and Log–normal distributions are equal to the standard deviation of the Uniform and Exponential distribution, respectively. The aim is to compare the optimal order quantity x^* when the mean and the standard deviation are the same for all distributions.

Experiment 1 shows the importance of knowing hot to compute exactly the optimal order quantity, Experiment 2 shows the key role of the standard deviation while Experiment 3 shows the importance of knowing the right distribution and the true standard deviation.

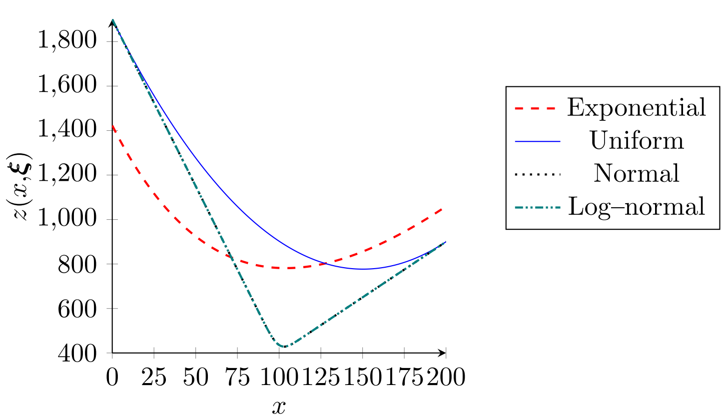
Experiment 1 In this experiment, we assume that it is unknown how to compute the optimal order quantity x^*. Our aim is to understand how the expected total cost are affected by the order quantity x. Since we may obtain different expected total costs depending on $\alpha \in (0, 1)$, which is given by an increasing value of v (see the previous paragraph and Figure 1.3), we arbitrarly choose three values of the retailer's cost ratio: $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$. Figures 1.6a, 1.6b and 1.6c show the expected total cost over the feasible set of x, assuming for the demand the Uniform, Exponential, Normal and Log–normal probability distributions, respectively, for fixed values of α.



(a) Expected total cost for $\alpha = 0.25$.



(b) Expected total cost for $\alpha = 0.5$.



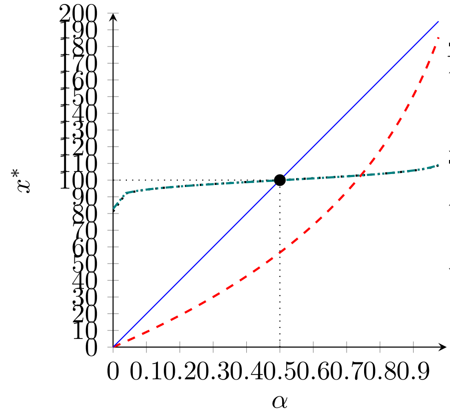
(c) Expected total cost for $\alpha = 0.75$.

Figure 1.6: Expected total cost.

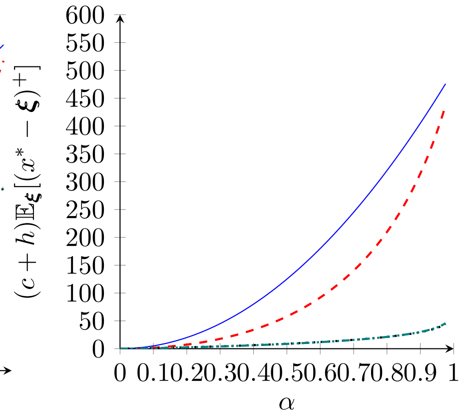
If one does not know how to compute the optimal order quantity and the unit stock–out cost is low, thus the shortage is not costly (see Figure 1.6a), a good judgement would lead to buy a low inventory level, possibly below the mean. For this reason, we observe that as we order more, the expected total cost increases. The opposite situation can be observed in Figure 1.6c, where the unit stock–out cost is high and the shortage is costly. Figure 1.6b is the average case, where the surplus and the shortage are equally paid when the Uniform, Normal and Log–normal distributions realize since they are symmetric around the mean.

From Experiment 1 we show the importance of knowing how to compute exactly the optimal order quantity and that even a good judgment may be misleading.

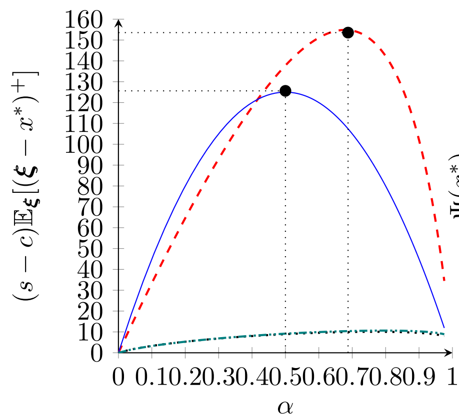
Experiment 2 In this experiment, we assume that it is known how to compute the optimal order quantity x^* depending on the probability distribution assumed for the stochastic demand $\boldsymbol{\xi}$. The aim of this experiment is to compute the optimal order quantity and the corresponding expected holding cost, expected shortage cost and recourse cost for $\alpha \in (0, 1)$. We set for each istance a different value of v, increased of 0.2 each time. We solve 200 instances and we obtain the following graphs: Figure 1.7a shows the optimal order quantity, Figure 1.7b shows the expected holding cost function, Figure 1.7c shows the expected shortage cost function, while Figure 1.7d shows the recourse cost functions.



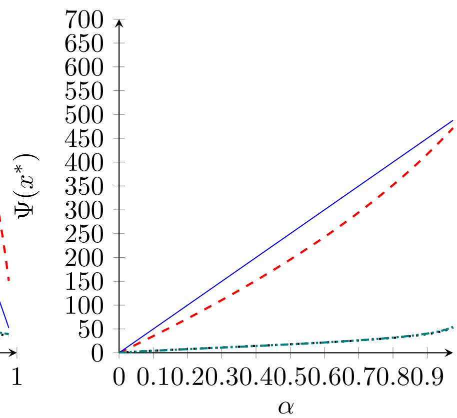
(a) Optimal values of x^*.



(b) Expected holding cost functions given x^*.



(c) Expected stock–out cost given x^*.



(d) Recourse cost functions given x^*.

Figure 1.7: Optimal order quantity, expected holding cost, expected stock–out cost and recourse cost functions.

In Figure 1.7a we can observe that to a larger standard deviation corresponds a wider range of variation of the optimal order quantity in order to edge against the greater uncertainty in the stochastic demand. This is the case under the assumption of the Uniform and Exponential distributions. On the contrary, under the assumption of the Normal and Log–normal distributions, for which the standard deviation is small, we order around the expected demand in the entire domain of α.

In Figure 1.7b, the Exponential distribution has the largest standard deviation but it assigns higher mass probabilities to the first half of possible outcomes of the stochastic demand, *i.e.* $\sum_{\xi=1}^{100} p_\xi > \sum_{\xi=100}^{200} p_\xi$, making a surplus more likely to realize than assuming a Uniform distribution. For this reason, we observe the highest expected holding cost under the assumption of the Uniform distribution.

In Figure 1.7c, we observe the highest expected shortage costs assuming the Exponential distribution due to its highest standard deviation. Note also that the shapes of the two functions assuming the Normal and Log–normal distributions are different from the other two: this is due to the smaller standard deviation.

At last, in Figure 1.7d we obtain the highest expected recourse costs assuming the Uniform distribution, observing that as α tends to 1, the recourse costs under the assumption of the Uniform and Exponential distributions converges to 500.

We omit the expected total costs since they are simply shifted upwards by the value $c\bar{\xi} = 400$ (see (1.17)).

Note that when $v = 2c + h$, $\alpha = \frac{1}{2}$. When the probability density function is symmetric, we observe the following:

- if $v < 2c + h$, then $\alpha \in (0, \frac{1}{2})$ and $x^* < \bar{\xi}$;
- if $v = 2c + h$, then $\alpha = \frac{1}{2}$ and $x^* = \bar{\xi}$;
- if $v > 2c + h$, then $\alpha \in (\frac{1}{2}, 1)$ and $x^* > \bar{\xi}$.

We now compare the expected total cost obtained using x^* with the expected total cost obtained using $x = \bar{\xi}$ in Figures 1.6a – 1.6c to prove the goodness of the choice of x^* instead of $\bar{\xi}$. We are comparing the solution of the recourse problem using the optimal solution with the solution of the expected value problem using the expected value solution. Table 1.1 show the results for $\alpha = 0.25$, Table 1.2

for $\alpha = 0.5$ and Table 1.3 for $\alpha = 0.75$.

	$\mathcal{E}$	$\mathcal{U}$	$\mathcal{N}$	$\mathcal{L}$
$z(x = \bar{\xi}, \boldsymbol{\xi})$	633.39	567.5	411.84	411.84
$z(x^*, \boldsymbol{\xi})$	495.07	525.62	409.44	409.3
savings (%)	27.94	7.97	0.59	0.64

Table 1.1: Comparison between expected total costs with $\alpha = 0.25$ under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$.

	$\mathcal{E}$	$\mathcal{U}$	$\mathcal{N}$	$\mathcal{L}$
$z(x = \bar{\xi}, \boldsymbol{\xi})$	670.58	651.24	417.77	417.79
$z(x^*, \boldsymbol{\xi})$	615.46	651.24	417.77	417.79
savings (%)	8.96	0	0	0

Table 1.2: Comparison between expected total costs with $\alpha = 0.5$ under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$.

	$\mathcal{E}$	$\mathcal{U}$	$\mathcal{N}$	$\mathcal{L}$
$z(x = \bar{\xi}, \boldsymbol{\xi})$	782.13	902.49	435.53	435.51
$z(x^*, \boldsymbol{\xi})$	781.51	776.87	428.31	428.75
savings (%)	0.08	16.17	1.69	1.58

Table 1.3: Comparison between expected total costs with $\alpha = 0.75$ under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$.

From Tables 1.1–1.3 we observe that:

- for $\alpha = 0.25$, the choice of the optimal order quantity x^* determines lower expected total cost instead of using the expected value $\bar{\xi}$. Under the Normal and Log–normal distributions we observe a tiny difference because of the lower standard deviation of the two, ad the optimal order quantity is close to the solutions of experiment 1;

- for $\alpha = 0.5$, the choice of x^* determines lower expected total cost under the assumption of the Exponential distribution only, while for the others it is indifferent since the optimal solution of the stochastic it the expected value solution, *i.e.* $x^* = \bar{\xi} = 100$;

- for $\alpha = 0.75$, the choice of x^* determines lower expected total cost instead of using $\bar{\xi}$.

We have shown how the standard deviation characterizes the shape of expected holding cost, expected shortage cost and recourse cost functions, the range of variation of the optimal order quantity and the worst–distribution for each class of cost (holding, stock–out and recourse). In the last experiment, we stress this fact by setting to higher values the standard deviation and the scale parameter of the Normal and Log–normal distributions, respectively.

Experiment 3 In this last experiment, we set the standard deviation of the Normal and Log–normal distributions equal to the ones of the Uniform and Exponential distributions, respectively. Our aim is to compare the results obtained for the optimal order quantity and cost functions when the standard deviations of the distributions are similar, showing the key role of the standard deviation. The new parameters of the distributions are the following:

- standard deviation of the Normal distribution $\mathcal{N}_1$, $\sigma_{N1} = \sigma_{\mathcal{U}} = 57.74$;

- scale parameter of the Log–normal distribution $\mathcal{L}_1$, $\eta_1 = 0.536$;

- standard deviation of the Normal distribution $\mathcal{N}_2$, $\sigma_{N2} = \sigma_{\mathcal{E}} = 100$;

- scale parameter of the Log–normal distribution $\mathcal{L}_2$, $\eta_2 = 0.833$;

The parameters of the Uniform and Exponential distributions remain the same. Figure 1.8 shows as example the shapes of the probability density funtions of $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{L}_1$ and $\mathcal{L}_2$.

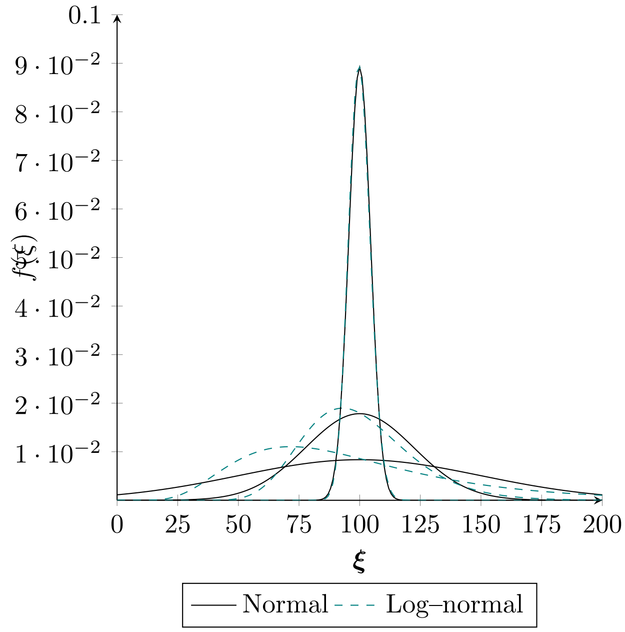


Figure 1.8: Normal and Log–normal *pdf*s for increasing values of the standard deviation.

From Figure 1.8 we observe that as the standard deviation increases, the skewness of the Log–normal distribution increases and the median shift towards the left side. As consequence, it is reasonable to expect similar results under the assumption of the Exponential and Log–normal distributions and of the Uniform and Normal distributions.

Figures 1.9, 1.10, 1.11 and 1.12 show respectively the optimal order quantity x^*, the corresponding expected holding cost, expected stock–out cost and recourse cost for $\alpha \in (0,1)$. On the left side of the graphs the standard deviation is $\sigma = 58$, while on the right side it is $\sigma = 100$.

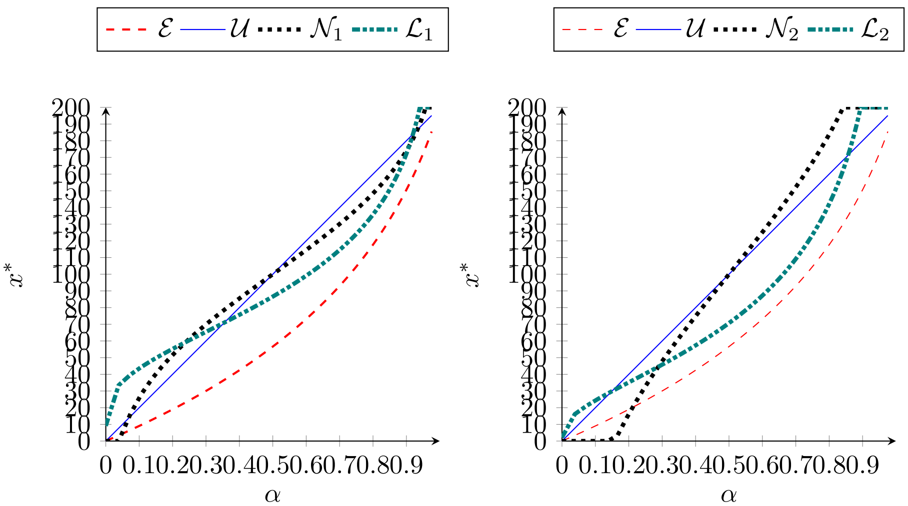


Figure 1.9: Optimal values of x^* under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normals $\mathcal{N}_1, \mathcal{N}_2$, and Log–normals $\mathcal{L}_1, \mathcal{L}_2$.

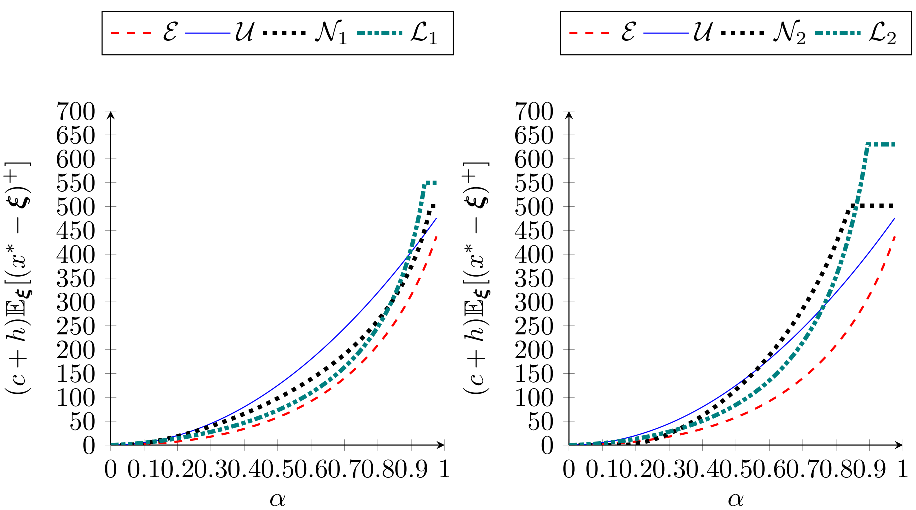


Figure 1.10: Expected holding cost functions given x^* under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normals $\mathcal{N}_1, \mathcal{N}_2$, and Log–normals $\mathcal{L}_1, \mathcal{L}_2$.

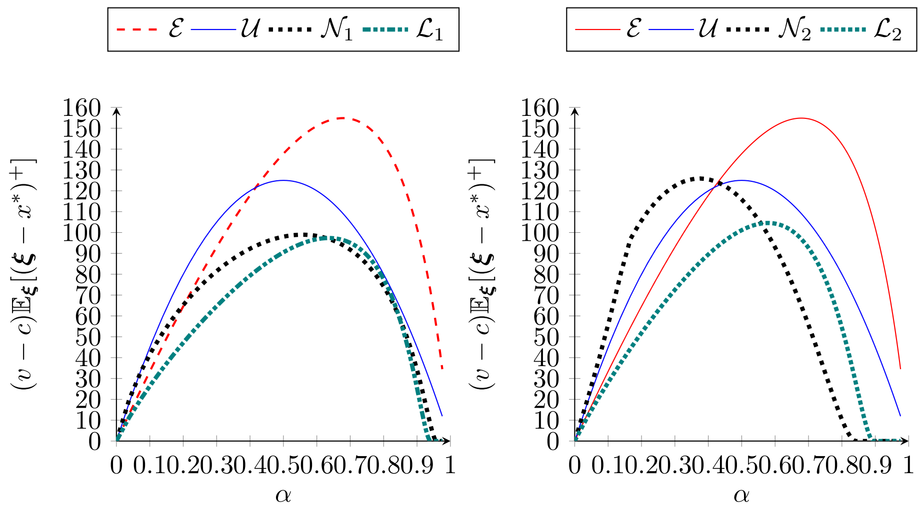


Figure 1.11: Expected shortage cost functions given x^* under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normals $\mathcal{N}_1, \mathcal{N}_2$, and Log–normals $\mathcal{L}_1, \mathcal{L}_2$.

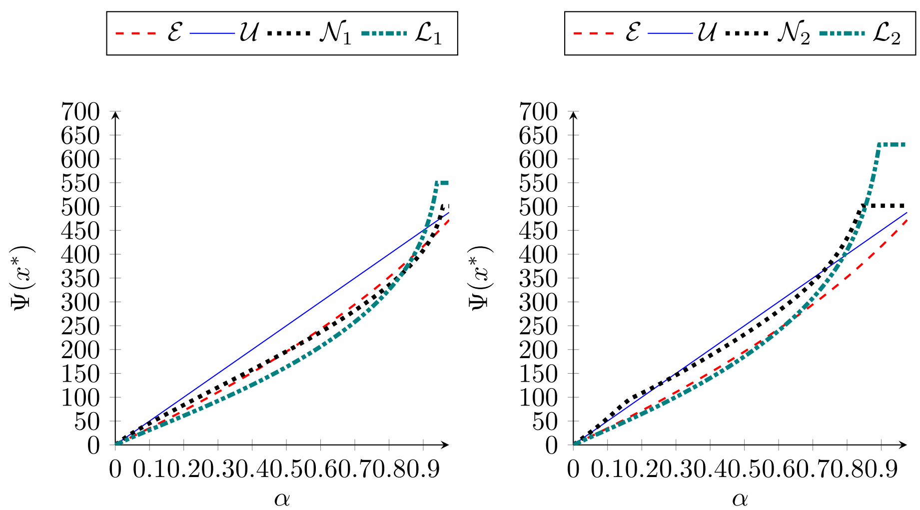


Figure 1.12: Recourse cost functions given x^* under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normals $\mathcal{N}_1, \mathcal{N}_2$, and Log–normals $\mathcal{L}_1, \mathcal{L}_2$.

In Figure 1.9 we observe that $x^* \in [0, 200]$ as α increases from 0 to 1 for all the distributions: the greater the standard deviation is, the wider is the range of values that x^* can assume in order to hedge against the greater uncertainty. From this figure we understand the importance of knowing the true distribution of the stochastic demand, since we observe different values depending on the assumed, or guessed, distribution.

In Figure 1.10 we observe that under the assumption of the Uniform distribution we still obtain the highest expected holding cost when $\sigma = 58$, except for particullary large α, which are really unrealistic to be obtained. The assumption of the Exponential distributions, on th contrary, provides the lowest since it assigns greater mass probabilities to low values of the demand.

From Figure 1.11 we observe that the assumption of the Exponential distribution still provides the highest expected shortage cost in both cases.

From Figure 1.12 we observe that as α tends to 1, the recourse costs are determined by the expected holding costs since the expected stock–out costs tend to be 0.

With this last experiment we have shown that we may obtain very different results for the optimal quantity to order (and the related costs) depending on the assumed distribution, the estimated standard deviation, or both. We ask how large is the error, and the loss, if one mismatches the right distribution with a wrong one within a set of possible distributions, or if the standard deviation is badly estimated.

1.3.4.C In–sample stability

In this section, we perform the *in–sample* stability of the scenario–based two–stage stochastic linear program (1.46)–(1.48), which is one of the two stability requirements for testing a scenario generation method.

Figure 1.13 shows the optimal value of the expected total cost computed each time considering a different number of scenarios with retailer's cost ratio $\alpha = 0.6$.

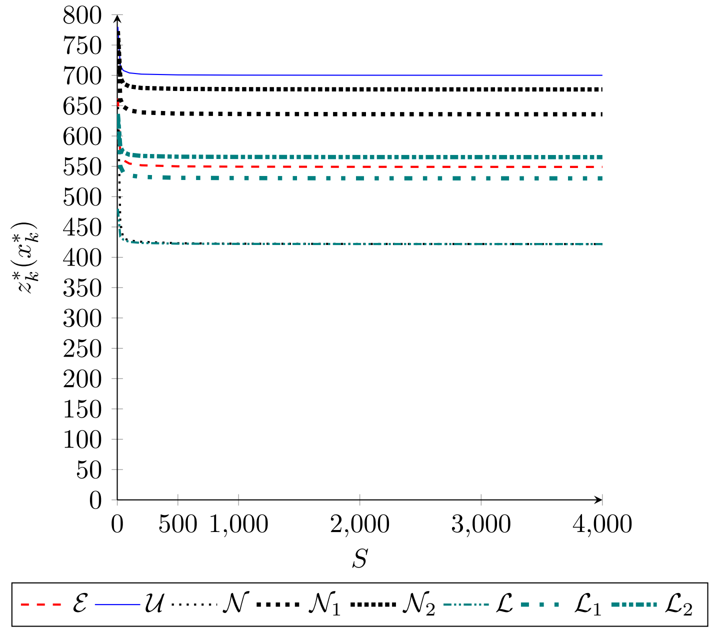


Figure 1.13: Optimal value of the objective function for a certain scenario tree under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normals $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2$, and Log–normals $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$.

Note that considering more than 500 scenarios leads to any sensible reduction of the variation in the expected total cost.

1.3.4.D The Value of the Stochastic Solution

In this section, we compute the *Value of the Stochastic Solution* (Birge (1982), [10]), which we recall to be the gain from solving the stochastic model instead of its deterministic counterpart.
In order to obtain a numerical comparison under the assumption of different retailer's cost ratio α and different probability distributions for $\boldsymbol{\xi}$, we now solve both the RP and EEV for $\alpha = 0, 0.1, 0.2, \ldots, \approx 1$, assuming each time for the stochastic demand $\boldsymbol{\xi}$ a different probability distribution among those considered in this study. Then, we compute the *Value of the Stochastic Solution* and we

express it as:

$$VSS(\%) = \frac{EEV - RP}{RP} * 100,$$

which is the marginal gain from solving the scenario–based two–stage stochastic linear program instead of considering the the expected value solution.

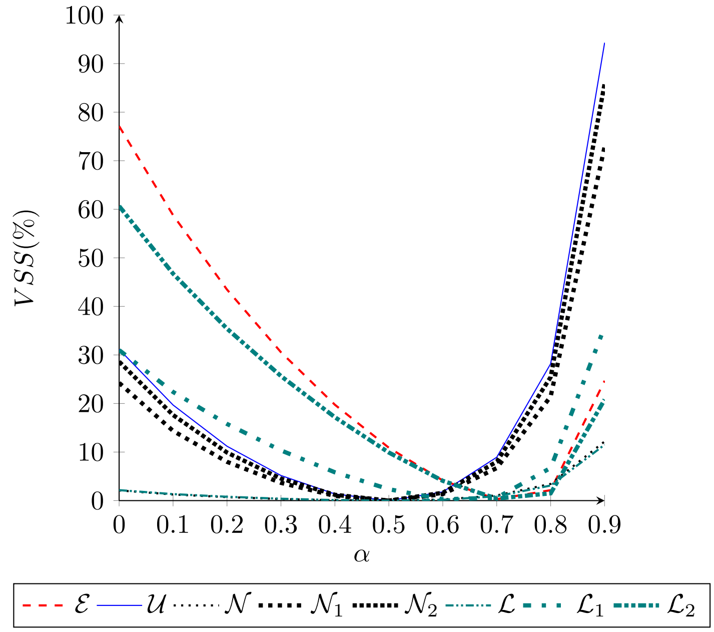


Figure 1.14: Value of the Stochastic Solution under assumption of Exponential $\mathcal{E}$, Uniform $\mathcal{U}$, Normals $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2$, and Log–normals $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$.

The higher the standard deviation is, the larger is the gain from solving the stochastic model instead of the deterministic one. Moreover, from Figure 1.14 we observe that the VSS is strictly monotonically non-increasing in $\alpha \in (0, F(\bar{x}(\bar{\xi}))]$ and strictly monotonically non-decreasing in $\alpha \in [F(\bar{x}(\bar{\xi})), 1)$.
The obtained results show that solving the scenario–based two–stage stochastic linear program (1.46)–(1.48) is always advantageous than considering the expected value problem, except when $F^{-1}(\alpha) = x^* = \bar{x}(\bar{\xi})$.

1.3.4.E The VRD and the Deviation Test

In this section, we compute the VRD for all the possible distribution pairs (right vs. guessed) of the probability distributions considered in this chapter, given a fixed value of the retailer's cost ratio α. We want to collect information from the distributions and their standard deviations using the VRD, which provides the opportunity cost values and allows us to investigate how each distribution is able to hedge the risk associated with errors occurring in the formulation of the model.

We assign a priori the right distribution for the stochastic programming model and we compute the optimal order quantity and the associated expected total cost. Then we choose a guessed distribution to compute the optimal order quantity and we use such solution to evaluate the expected total cost assuming the right distribution. We apply such approach for all possible right–guessed distribution pairs and we construct a table reporting all the VRD. Note that such table can be constructed for any problem and probability distributions.

Table 1.4 reports the optimal order quantities $x^*_{\mathcal{R}}$ while Table 1.5 shows the VRD(%) values assuming one by one as right the Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normals $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ and Log–normals $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ distributions for $\alpha = 0.9$.

	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{L}$	$\mathcal{L}_1$	$\mathcal{L}_2$
$x^*_{\mathcal{R}}$	177	200	126	170	200	126	189	194

Table 1.4: Optimal order quantity under assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normals $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2$, and Log–normals $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ distributions, respectively.

		Guessed distribution $\mathcal{G}$							
		$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{N}_1$	$\mathcal{N}_2$	$\mathcal{L}$	$\mathcal{L}_1$	$\mathcal{L}_2$
	$\mathcal{U}$	-	7.39	37.79	0.66	7.39	37.79	2.26	4.28
	$\mathcal{E}$	4.34	-	33.5	6.75	0	33.5	1.66	0.83
Right	$\mathcal{N}$	35.61	55.46	-	29.57	55.46	0	45.97	50.28
distribution	$\mathcal{N}_1$	0.4	7.09	30.4	-	7.09	30.4	3.3	4.96
$\mathcal{R}$	$\mathcal{N}_2$	11.17	0	68.94	16.51	-	68.94	4.26	2.06
	$\mathcal{L}$	28.25	46.52	0	22.79	46.52	-	37.63	41.63
	$\mathcal{L}_1$	1.07	0.08	33.47	2.81	0.08	33.47	-	0
	$\mathcal{L}_2$	1.38	0	23.58	2.73	0	23.58	0.23	-

Table 1.5: Value of the Right distribution (%) assuming respectively as right the Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normals $\mathcal{N}, \mathcal{N}_1, \mathcal{N}_2$, and Log–normals $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ distributions.

From Table 1.5 we observe that for the same probability distribution considered right, *e.g.* the Normal distribution $\mathcal{N}_2$, we obtain both very small and very large values for the VRD. This can be explained as follows, recalling that $\mathcal{N}_2$ is a Normal with the standard deviation of the Exponential distribution.
We observe that when $\mathcal{N}_2$ is mismatched with $\mathcal{E}$ the VRD is 0 because under the assumption of both the distributions the optimal order quantity is the same, $x^*_{\mathcal{R}} = 200$ (see Table 1.4). However, this case does not give us any particular information if we look at $x^*_{\mathcal{R}}$ because we are evaluating an objective function in the same value, even if the optimal order quantities are obtained from different distributions with different standard deviations.
Now consider $\mathcal{L}_1$ and $\mathcal{L}_2$: the standard deviations, compared to the one of $\mathcal{N}_2$, are different in the first case ($\sigma_{\mathcal{N}_2} = 100$, $\sigma_{\mathcal{L}_1} = 57.74$) and identical in the second ($\sigma_{\mathcal{N}_2} = \sigma_{\mathcal{L}_2} = 100$). The gap of the two VRD, 2.20, is very small compared to the larger difference of the two standard deviations, $\sigma_{\mathcal{L}_1} = 57.74$ and $\sigma_{\mathcal{L}_2} = 100$: one should expect a larger gap between the two VRD, as it happens when mismatching $\mathcal{N}_2$ with $\mathcal{U}$ ($\sigma_{\mathcal{N}_2} = 100$, $\sigma_{\mathcal{U}} = 57.74$) and $\mathcal{N}_2$ with $\mathcal{E}$ ($\sigma_{\mathcal{N}_2} = \sigma_{\mathcal{E}} = 100$). The reason of the small gap between the VRD of $\mathcal{L}_1$ and $\mathcal{L}_2$ lies in the similar shape of the tails of the two distributions compared to

the distribution of $\mathcal{N}_2$ (see Figure 1.8, recalling that $\alpha = 0.9$). At first sight, it seems that the factor of dominance for the values of the VRD are the distributions. However, this is a particular case (with $\alpha = 0.9$), and the shape a probability distribution is given by the scale parameter, which is the standard deviation. Also, the standard deviation determines the range of variation of the optimal order quantities, which can be similar for close standard deviations, or even identical, as in the case mentioned above with $x^*_{\mathcal{R}} = 200$ for $\mathcal{N}_2$ and $\mathcal{E}$.
The informations gathered from the VRD matrix lead us to believe that the factor of dominace for the VRD is indeed the standard deviation. However, this needs to be numerically proved, and for this purpose we propose the folling test, addressed as Deviation Test.

Deviation Test We now provide a numerical experiment in order to test how the VRD changes as the standard deviation increases considering different probability distributions. Our aim is to understand if the VRD is high for some mismatches or because of the standard deviations of the distributions.
The initial setting for the deterministic paremeters is:

- procurement cost $c = 4$;
- holding cost $h = 1$;
- stock–out cost $v = 50$;

We use a large value for v in order to obtain perceptible values of the right distributions. We consider the Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions and we use the following setting in order to create several probability models:

- we recall that demand $\boldsymbol{\xi}$ is defined on the finite support $\boldsymbol{\xi} \in [0, 200]$ with expected value $\bar{\xi} = 100$;
- the standard deviation of the Normal distribution is setted to five different values: 20% of $\bar{\xi}$, 40% of $\bar{\xi}$, 60% of $\bar{\xi}$, 80% of $\bar{\xi}$, $\bar{\xi}$;

- scale and location paramaters of the Lognormal distribution are computed respectively with the formulas provided in Section 1.6.1;
- the standard deviation assuming the Uniform distribution is fixed, $\sigma_{\mathcal{U}} = \sqrt{\frac{1}{12}(b-a)^2} = 57.73$;
- the standard deviation assuming the Exponential distribution is $\sigma_{\mathcal{E}} = \lambda_{ij} = \bar{\xi}^{-1} = 100$;

Note that the standard deviations of the Uniform and Exponential distributions are fixed, given the set of data. Then, we choose the right probability distributions and we compare it with the possible guesses in order to compute the VRD(%), even with the same distribution but with different standard deviations. Table 1.6 reports the obtained results. Note that, concerning the standard deviation, on the left column we report the "true" values assumed for the probability distributions, while in the row on top we report the values used for the experiment.

			$\mathcal{G}$											
			$\mathcal{N}$					$\mathcal{L}$					$\mathcal{U}$	$\mathcal{E}$
		σ(%)	0.2	0.4	0.6	0.8	1	0.2	0.4	0.6	0.8	1	0.6	1
$\mathcal{R}$	$\mathcal{N}$	0.2	0	14.2	35.8	55.7	55.7	0	11.7	28	41	51.3	34.9	55.7
		0.58	28.8	5	0.1	3.8	3.8	28.8	6.5	0.3	0.5	2.6	0	3.8
		1	71.1	35.1	11.1	0	0	71.1	38.7	18.2	7.3	1.8	11.8	0
	$\mathcal{L}$	0.2	0	9.3	28.2	46.5	46.5	0	7.4	21.2	32.9	42.4	27.5	46.5
		0.58	14.7	1.8	0.1	2.9	2.9	14.7	2.7	0	0.5	2.1	0.1	2.3
		1	24	9.2	1.5	0	0	23.9	10.6	3.5	0.7	0	1.7	0
	$\mathcal{U}$	0.58	41.4	10	0.04	6.7	6.7	41.4	12.5	0.9	0.7	4.1	0	6.7
	$\mathcal{E}$	1	33.5	15.2	4.3	0	0	33.5	16.8	7.5	2.9	0.7	4.6	0

Table 1.6: VRD(%) values obtained from the Deviation Test.

From Table 1.6 we understand that two mismatched are relevant for the VRD:

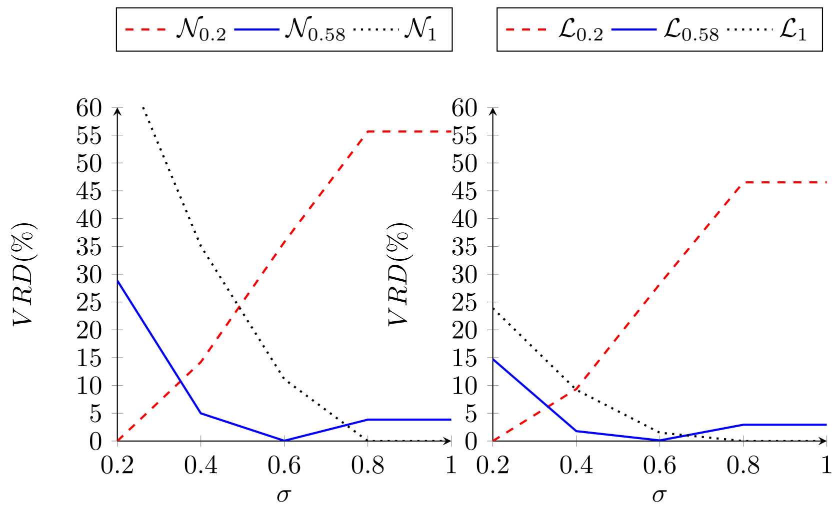
1. the right and the guessed distribution are the same, but the standard deviation is different.

Observe the case where the right distribution is $\mathcal{N}$ with $\sigma = 0.58$ and the guessed distribution is another Normal with different standard deviation: when the distance between the standard deviations is approximatively 0, the VRD is neglectable. As the distance of the two increases, the VRD increases. This case may realize when the guessed distribution is right, but the standard deviation has been badly estimated. Thus the VRD provides a measure of the error in terms of loss in the objective function;

2. the right and the guessed distribution are different.

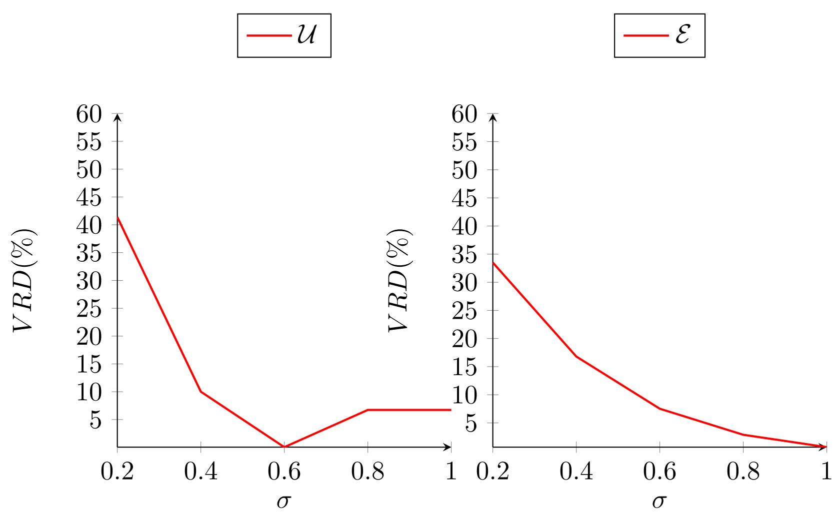
   Observe the case where the right distribution is $\mathcal{E}$ and the guessed distribution is $\mathcal{L}$: the VRD is neglectable when the standard deviations are the same (as also the optimal order quantity), while it increases as the standard deviation of the Log–normal distribution differs from the standard deviation of the Exponential. This case may realize when the guessed distribution is wrong, by the standard deviation han been correctly estimated.

We understand that the issue, or the factor of dominace for the VRD, is the standard deviation. Figures 1.15 and 1.16 show a graphic representation for some mistmatches. For simplicity of notations, as example, $\mathcal{N}_{0.2}$ stands for the VRD(%) values a Normal distribution with standard deviation $\sigma = 0.2 * \bar{\xi}$.



(a) $\mathcal{R} = \mathcal{N}$, $\mathcal{G} = \mathcal{N}$. (b) $\mathcal{R} = \mathcal{L}$, $\mathcal{G} = \mathcal{N}$.

Figure 1.15: Graphical representation of the VRD(%) functions under assumption of the Normal $\mathcal{N}$ (a) and Log–normal $\mathcal{L}$ (b) distributions.



(a) $\mathcal{R} = \mathcal{U}$, $\mathcal{G} = \mathcal{N}$. (b) $\mathcal{R} = \mathcal{E}$, $\mathcal{G} = \mathcal{L}$.

Figure 1.16: Graphical representation of the VRD(%) functions under assumption of the Uniform $\mathcal{U}$ (a) and Exponential $\mathcal{E}$ (b) distributions.

When mismatching two distributions, the Value of the Right Distribution is large if the standard deviation of the right distribution is different from the standard deviation of the guessued distribution, which can also be the same. This is equivalent to say that the parameter is badly estimated and different from its true value, and the VRD measures the loss derived from an error of estimation.

1.3.4.F The distributionally robust solution

In this paragraph we compute the optimal distributionally robust solution by means of (1.76), proposed in Paragraph 1.3.3.B. We first show in Figure 1.17 how the optimal solution changes in $\alpha \in (0, 1)$ for $\sigma = 20$, $\sigma = 58$ and $\sigma = 100$.

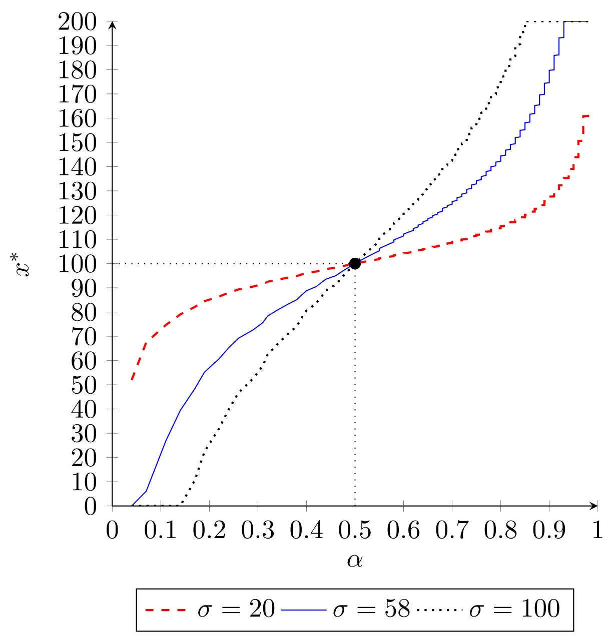


Figure 1.17: Distributionally robust optimal solution depending on α for different values of σ.

From Figure 1.17 we observe that when the shortage is not costly ($\alpha \in (0, 0.15]$, as the standard deviation increases, the optimal order quantity reduces.

The reason is that, while facing a greater uncertainty, we prefer to do nothing since the possible stock–out cost will be low. Note however, as the standard deviation increases, the inventory level of the optimal order quantity increases faster in order to hedge against the greater uncertainty. For this reason, we observe that when $\sigma = 58$ and $\sigma = 100$, the choice of the optimal order quantity cover all the possibile solutions. Moreover:

- when $\alpha < \frac{1}{2}$, $x^* < \bar{\xi}$;
- when $\alpha = \frac{1}{2}$, $x^* = \bar{\xi}$;
- when $\alpha > \frac{1}{2}$, $x^* > \bar{\xi}$;

Figure 1.17 also enlights the importance of assigning a good estimate of the standard deviation in order to solve the distributionally robust optimization model since we obtain very different solutions depending of the chosen σ.
We now compare the distributionally robust solutions obtained for $v = 50$, which corresponds to $\alpha = 0.9$, reported in Table 1.7 with the ones obtained in Table 1.4 by solving the stochastic programming model under assumption of all the four considered distributions.

	$\sigma = 20$	$\sigma = 58$	$\sigma = 100$
x^*	127	178	200

Table 1.7: Optimal order quantities for $\alpha = 0.9$ for the distributionally robust formulation assuming different values of the standard deviation.

Comparing Tables 1.4 and 1.7, we understand that when $\sigma = 20$, the minimization of the expected total cost in the worst–case is obtained under the Normal or Log–normal distributions (which are almost identical when the standard deviation is low), when $\sigma = 58$ under the Uniform disitribution and when $\sigma = 200$ under the Exponential distribution.

1.4 Guidelines for the use of VRD and RPB

The Value of the Right Distribution and the Recourse Penalty Bound can be used to understand to measure the error (loss) in case a guessed probability distribution assumed for the stochastic process is wrong against the right one, investigating at the same time if the ambiguity is important.
Here there are now proposed the guidelines of how such measures can be applied in practice to study the original problem.

- step 1: compute the VRD values. In order to compute the VRD, one has to assume a priori the right distribution, which is contained in a subset of possible distributions. Then a table reporting the VRD is constructed for all possible distribution pairs (guessed vs. right): it provides the opportunity cost values for all possible distribution pairs. It allows a decision maker to investigate how each distribution is able to hedge the risk associated with errors occurring in the formulation of the model;

- step 2: compute the RPB and MRPB values. If the VRD values are large for all or some distribution pairs, then the RPB and MRPB are computed for those pairs identified as problematic or highly risky as a means to deepen the analysis applied on the problem.

1.5 Conclusions

In this chapter, we have introduced the new concepts of Value of the Right Distribution, Recourse Penalty Bound and Maximum Recourse Penalty Bound. In order to show how they apply, we have proposed a cost–based variant of the Newsvendor Problem considering a set of possible probability distributions for the stochastic demand. We have studied the case where one of the pillar assumption of stochastic programming is false, which is that the probability distribution assumed for the stochastic process in a stochastic programming model is wrong against the true one, and we have shown how to measure the error in the objective function when such situation realizes (the concept of VRD). Then, to deepen the analysis for those cases in which the error is large, we have shown how to compute the loss in the recourse function depending on the parameters of the problems (the retailer's cost ratio) and a particular mismatch between two probability distributions (the concept of RPB). At last, we have shown how large the maximum error is in the worst–case (the concept of MRPB). We have understood that the magnitude of the loss in the objective function is mainly determined by the error in the estimates of the true standard deviation and not by a particular mismatch between two probability distributions. At last, the Value of the Right Distribution can also be applied in distributionally robust optimization to measure the value of the ambiguity.

Moreover, close–form and approximate expressions have been derived to compute the optimal order quantity depending on the assumed demand probability distribution. The stochastic programming and distributionally robust approaches have been adopted to solve the problem and from the comparison of the solutions it was possible to understand in the distributionally robust formulation under which probability distribution we obtain the minimization of the expected total cost in the worst–case.

Chapter 2

Two–stage and Multi–stage Stochastic Programming Models for Bike–Sharing Problems

2.1 Introduction and Literature Review

This study investigates one–way bike–sharing systems. In these systems, users arrive at a station, pick up a bike, use it for a while in a short–term rental, and then return it to another station of their choice.

The first bike–sharing system was installed in Amsterdam in 1965, and since then such systems have increased their popularity as a new transportation mode in a large number of cities worldwide, counting more than 1000 operating systems and more than 300 systems planned, under construction or close to be implemented (DeMaio and Meddin (2014) [16]). Bike–sharing systems contribute towards obtaining a more sustainable mobility and decreasing traffic and pollution caused by car transportation.

Bike–stations are strategically located in a geographical region or city, with the aim to encouraging the user community to participate bike–sharing programs in order to discourage the use of vehicles of property. The benefits are identified in Shaheen et al. (2010) [58] in flexible mobility, emission reductions, physical activity gains, reduced congestion and fuel use, individual financial savings and support for multimodal transport connections. For these reasons and for the economic nature and operative structure, for which a detailed description will be given in further sections, the bike–sharing services are often provided by a private provider funded by a public authority.

Users that wish to enjoy a bike–sharing program are generally required to provide credit card details, which act both as a deposit, as well as payment for registration and usage fees.

In Fishman et al. (2013) [21] the existing literature concerning bike–sharing programs is extensively analyzed, identifying sustainability challenges, bike–sharing schemes (e.g., one–way, two–way), mode substitution and impacts, usage rates, user motivation, preference and purpose, safety concepts and rebalancing.

In bike–sharing systems, bicycles are sometimes concentrated in some areas of the city while there is sometimes a limited fleet availability: in the first case, users that reach their final destination may find a bike–station with unavailable capacity (and thus they are forced to move forward another bike–station with available capacity in order to return the bicycle), while in the second case users may not find available bicycles to rent. In order to avoid the inefficiency created by full and/or empty stations, the bicycle fleet is redistributed in the system through rebalancing.

The rebalancing is performed by capacitated vehicles and it is normally required at the end of the day, when the system is closed. It has a key role for the optimization of the service and requires an operator in order to move the bicycles across the system from a bike–station to another. In Yang et al. (2011) [62] the rebalancing is described as a major problem and, in order to reduce its impact in bike–sharing systems, it is suggested to offer rewards for those that ride bikes against the flow. In Fishman et al. (2013) [21] is reported that this strategy is employed by a number of bike–share programs, including Capital Bike Share in Washington, DC (Capital Bike Share, 2011), although the effectiveness of this strategy is limited (Virginia Tech, 2012).

In literature, the terms rebalancing and repositioning are used as synonyms.

The Bike–sharing Rebalancing Problem (BRP) is the problem in which a fleet of capacitated vehicles is employed in order to re-distribute the bicycles with the objective of minimizing the total cost.

In Dell'Amico et al. (2014) [15], the BRP is viewed as a special one–commodity pickup–and–delivery capacitated vehicle routing problem. They present four mixed integer linear programming formulations of this problem. In Dell'Amico et al (2016) [14], the BRP is approached with a destroy–and–repair metaheuristic algorithm, which makes use of a new effective constructive heuristic and of several local search procedures.

In Gaspero et al. (2016) [17], the BRP is studied by using of Constraint Programming (CP): they first introduce two different CP models, including two custom branching strategies that focus on the most promising routes, and then they incorporate both models in a Large Neighbourhood Search (LNS) approach that is adapted to the respective CP model. At last, they perform an experimental evaluation of the adopted approaches on three different benchmark sets of instances derived from real-world bike–sharing systems. They finally prove that the pure CP approach outperforms the state-of-the-art MILP on large instances and that the LNS approach is competitive with other existing approaches.

Another form of the BRP is the Static Balancing Repositioning Problem (SBRP), in which repositioning is performed when the demand in the system is negligible, usually at night.

In Chemla et al. (2013) [13], several algorithms are presented to solve the SBRP, which is proved to be NP–hard. A branch–and–cut algorithm for solving a relaxation of the SBRP is proposed and an upper bound of the optimal solution is obtained by a tabu search when the sequence of the visited vertices is given. At last, it is proven that building a feasible solution of the problem by using the one obtained by the relaxation is an NP–hard problem, but an optimal solution can be often found by a tabu search initialized with the optimal solution of the relaxation often shows that it is the optimal one.

Ho and Szeto (2014) [27] propose an iterated tabu search heuristic to solve the SBRP by selecting a subset of stations to visit, sequencing them, and determining the pick-up/drop–off quantities under the various operational constraints with the objective to minimize the total penalties incurred at all the stations.

In Forma et al. (2015) [22], it is proposed a 3–step mathematical programming based heuristic for the SBRP. In the first step, stations are clustered according to geographic as well as inventory bicycles considerations using a specialized saving heuristic. In the second step, the repositioning vehicles are routed through the clusters, while tentative inventory decisions are made for each individual station, and in the third step the original repositioning problem is solved. The last two steps are formulated as Mixed Integer Linear Programs.

In Li et al. (2016) [35], a new SBRP in which multiple types of bicycles are considered is introduced. The problem is formulated as a mixed–integer linear programming problem to minimize the total cost, which consists of the route travel cost, penalties for unmet demand, and penalties associated with the substitution and occupancy strategies. A combined hybrid genetic algorithm is proposed to solve this problem: (i) a modified version of a hybrid genetic search with adaptive diversity control to determine routing decisions, and (ii) a greedy heuristic to determine the loading and unloading instructions at each visited station, the substitution and occupancy strategies.

In literature, rebalancing is sometimes addressed as a form of transshipment. The transshipment is the shipment of goods to an intermediate destination: in a bike–sharing system, it is the reallocation of bicycles from bike–stations with inventory levels above a certain threshold to bike–stations with inventory levels belove a certain threshold. In our work, rebalancing is a form of transshipment.

In Dong and Rudi (2004) [18] and Zhang (2005) [66] (the latter extends the contributions proposed in the former) it is proved that when the transshipment is considered, the optimal solution depends on the unit transshipment cost (i.e., on the future rebalancing costs) alongside with the other paramters of the problem.

We now enlight the state of the art concerning the determination of the optimal inventory (bicycle) levels for a bike–sharing system.

In Schuijbroek et al. (2016) [56], it is stated that finding the optimal inventory levels is intractable. They use the service level requirements of each bike–stations in order to design near–optimal vehicle routes to rebalance the inventory levels. To do that, they propose a cluster–first route–second heuristic, in which a polynomial–size Clustering Problem simultaneously considers the service level feasibility and approximates routing costs, showing that the heuristic outperforms a pure mixed–integer programming formulation and a constraint programming approach.

In Sayarshad et al. (2012) [53], a mathematical model to optimize a bike–sharing system is proposed by determining the minimum required bike fleet size that minimizes simultaneously unmet demand, unutilized bikes and the transport of empty bikes between rental stations to meet demand. The optimal inventory level for each bike–station is not considered and the unmet demand is compensated with the transport of empty bicycles. The problem is handled with a deterministic approach.

In Raviv and Kolka (2013) [50], an inventory model for the management of bike rental stations is introduced and a numerical solution method used to solve it. In order to benchmark the proposed method, the bike–sharing system has been simulated with three different settings of initial inventories to measure a user dissatisfaction function. The inventory level of each bike–station is not optimally determined solving a model and it is stated that it should be reviewed by the operator whenever demand patterns or station capacity change. However, it is known that human decision–makers systematically make non–optimal decisions [32].

In Lu (2016) [37], a mathematical programming model, the bike fleet allocation (BFA) model is formulated to determine the number of bicycles deployed at each station on each day of the week in a bike-sharing system. A time–space network is first constructed to describe time–dependent bike flows in the system and then a fleet allocation model, that considers average historical demand and fixed fleet size, is formulated based on the time–space network.

The literature proposed so far, can be summarized in three groups of contribution:

1. **approaches to face the rebalancing problem**: Dell'Amico et al. (2014,2016) [15, 14], Gaspero et al. (2016) [17], Chemla et al. (2013) [13], Ho and Szeto (2014) [27], Forma et al. (2015) [22] and Li et al. (2016) [35]. In their works, the authors use the system *as it is* at the end of the service as input for the developed rebalacing procedures: they measure inventory levels but they no further investigate on why and how those levels have been obtained, nor what it the optimal quantity to place at the beginning of the service in each bike–stations. Indeed, they just need to know what is the inventory level to be reached for a certain bike–station in order to apply the rebalacing procedure (*e.g.*, scheduling, BRP, SBRP);

2. **solutions to determine the size of the bicycle fleet**: Schuijbroek et al. (2016) [56], Sayarshad et al. (2012) [53], Raviv and Kolka (2013) [50]. In their works, rebalacing is not considered. Lu (2016) [37] is interested in computing the number of bicycles to place in each bike–station by

describing time–dependent bike flows thought time–space networks, but none of them propose a stochastic mathematical program to determine the optimal inventory levels of each bike–station considering the repositioning at the same time;

3. **optimal solution in case of transshipment**: Dong and Rudi (2004) [18] and Zhang (2005) [66], from which we understand that in a bike–sharing system with transshipment, the latter must be considered in order to compute the optimal inventory level of each bike–station. If not so, the solution will be non–optimal.

To the best of our knowledge, the two problems, determining the number of bicycles for each bike–station (thus the fleet size) and the rebalacing problem, are considered disjoinly. Moreover, stochastic mathematical programs have never been proposed to solve them. The contribution of this chapter is to propose two–stage and multi–stage stochastic optimization models which consider at the same time both the problem of determining the optimal inventory levels of each bike–station and the rebalancing problem. The optimal size of the fleet is consequently derived. We consider the rebalancing part of bike–sharing problems as a recursive action in order to determine the optimal inventory levels. The proposed optimization models are non time–dependent bike flows (differently from [37]) and the unmet demand is lost (differently from [53]). In this work, we consider the bike–sharing problem as a variant of the newsvendor problem with transshipment and we investigate the role of the stochasticity through the tools proposed in the previous chapter.
At last, we adress some research questions to be answered at the end of this chapter:

1. does the bike–sharing problem need to be studied by means of mathematical formulation in order to achieve a good optimal solution? This question is justified by the fact that upon reality we may observe cases, even for bike–sharing, where a mathematical decision model is replaced with the human judgement. For some bike–sharing services, the common

(human) choice is to place a set of bike–stations and full their capacity. For such decision makers, the problem is solved. How bad this solution can perform?

2. does an optimal solution exists for the bike–sharing problem which consider as well the rebalacing problem?

3. is it possible to decompose the main problem in a class of subproblems of lower complexity, one for each bike–station, where the aim is to determine the inventory level of the single bike–station? If so, in which cases?

4. is the transshipment necessary in order to optimize the service level and thus to increase the customer satisfaction?

5. how the parameters of the bike–sharing problem affect the optimal solution?

6. how this study can be helpful to a good manager?

This Chapter is organized as follows: Section 2.2 describes the bike–sharing problem, Section 2.3 provides two–stage and multi–stage model formulations, Section 2.4 describes the case study analyzed while in Section 2.5 computational results are discussed. Finally, Section 2.6 concludes the Chapter with managerial insights and suggestions for future research.

2.2 Problem Description

We consider the problem faced by a bike–sharing service provider (hereafter referred as provider) who needs to manage a fleet of bicycles and a set of bike–stations with fixed capacity in order to serve the rental stochastic demand over space and time. The stochastic demand changes over space and time. The bike–sharing system is a one–way rental system, which allows the user to pick and return the rented bike at different bike–stations. The provider places the bicycle at a given unit procurement cost and tackles unit stock–out cost for the shortfall quantity (shortage) depending on the bike-park level (positive or negative) in which she will incur after demand realization. The number of bicycles that cannot be returned to a bike–station, due to the lack of vacant locks, define the overflow, which is paid by the provider as a unit time–waste cost. Moreover, she tackles unit transshipment cost after the end of the rentals when the bike–station inventory levels are rebalanced.

Each bike rental demand is defined by an origin–destination pair, where the destination is unknown to the provider.

A stochastic demand to each origin–destination pair is assigned.

The rent must start at the user–defined time period or it will be lost. A lost rent determines a shortage for the provider and a reduction of the service level for the user. The shortage realizes when a rental demand arises in a certain bike-station but no bicycles is available for the rent: the user quits the service and looks towards an alternative transportation mode. A shortage causes a cost increase, a contraction of the service level and a reduction of the likelihood of receiving future rental requests from users whose demand could not be satisfied at first.

The rent must end at the user–defined time period. The number of bicycles that cannot be left in a bike–station with saturated capacity at the end of the rent determines an overflow and the user cannot quit the service until the bicycles is redirected and positioned by the user itself in the nearest bike–station with available capacity. An overflow causes a waste of time for the user and a cost.

The number of bicycles to place at the beginning of the service in each bike–station can be any nonnegative integer number, the initial positioning is assumed to be instantaneous (lead time equal to zero) and backlogging is not allowed.
The optimal solution to this problem defines the optimal number of bicycles to place in each bike–station and thus the optimal size of the bicycle fleet.
The sequence of the operations is the following: the number of bicycle to place at the beginning of the service in each bike–station is computed and placed, stochastic demands realize, the available bicycles are rented, the users drive the bicycles toward their destinations, a new composition of the fleet over the entire set of bike–stations is obtained and, at last, the provider reallocates the bicycles over the bike–stations in order to match the desired bicycle quantity to be placed for the new incoming stochastic demands or to balance the possible future overflow.
The aim is to determine the number of bicycles to place in each bike–stations for each given time period that minimizes the expected total cost over the operational time horizon, where the expected total cost is given by the sum of the bicycles procurement costs plus the stock-out costs for the unmet demands, the time–waste costs for the overflows and the transshipment costs for having adjusted bike–stations inventory levels.
In the literature (see [18],[66]), transshipment is performed at some cost before satisfying the demand when it is already revealed, in order to offset the shortages of some with the surplus of others. In a bike–sharing problem, it is not possible to offset the shortage of a bike–station with the surplus of another one, since the user instantly quit the service when there are no bicycles available for the rental. In this study, the transshipment is intended as a rebalancing (recursive) action to balance the inventory levels of the bike–stations at the end of the service when shortages and surpluses are realized.

2.3 Model Formulation

In the classical newsvendor problem (Birge and Louveaux (2011) [11]) a newsvendor goes to a publisher every morning and buys a certain quantity of newspapers at a given unit purchase price. This number is usually bounded above by some limit, representing either the newsvendors purchase power or a limit set by the publisher to each newsvendor. The newsvendor then walks along the streets to sell as many newspapers as possible at a given unit selling price. Any unsold newspaper can be returned to the publisher at a given unit return price, less than the purchase price. Demand for newspapers varies over days and is described by a random variable. It is assumed here that the newsvendor cannot return to the publisher during the day to buy more newspapers. The problem is to compute how many newspapers to buy every morning.

We start by studying the actors and the structure of the newsvendor problem in order to find a syllogism with the bike–sharing problem. Suppose a bike–sharing system with a single bike–station operating each day.

In the newsvendor problem, we have a newsvendor who is supplied by a publisher for a certain number of newspapers that cannot exceed the newsvendor's capacity. In the bike–sharing problem, we have a bike–station which is supplied by a provider for a certain number of bicycles that cannot exceed the bike–station capacity.

In the newsvendor problem, the newsvendor walks around the street in order to satisfy customer demands. In the bike–sharing problem, users walk around the streets in order to satisfy their rental demands at the bike–station. The stochastic demand clearly changes over time in both problems.

In the newsvendor problem, the newsvendor cannot return to the publisher during the day to buy more newspapers. In the bike–sharing problem, the provider cannot instantly supply the bike–station if a rental demand arises and no bicycles are available in order to satisfy it. The rental demand is lost.

In the newsvendor problem, the optimal number of newspapers to be purchased by the newsvendor is computed. In the bike–sharing problem, we are willing to determine the optimal number of bicycles to place at the bike–station.

In the newsvendor problem, the newspapers are bought at a purchase price, sold at a selling price, returned at a return price. In the bike–sharing problem, the bicycles are placed at a procurement cost and any possible shortage of bicycles or of vacant locks are paid at certain penalty costs.

We can state the following similarities for the three actors of both problems:

- newspaper $\equiv$ bicycles, the main actor, for which the optimal inventory level has to be determined;
- newsvendor $\equiv$ bike–station, the actor that plays as host for newspapers/bicycles;
- publisher $\equiv$ provider, the actor who supplies the host with newspaper/bicycles.

Our first aim is to show that the bike–sharing problem can be written as a cost–based inventory problem (see Chapter 1) and since the cost–based inventory problem is a variant of the newsvendor problem, then the bike–sharing problem is a variant of the newsvendor problem.

2.3.1 Two–stage stochastic optimization models for the bike–sharing problem

In this section, we formally provide a two–stage model formulation following the sequence of operations described in Section 2.2.

Let the followings holding true for all the models proposed in this section:

Sets

- $\mathcal{B}$: set of bike–stations, $\mathcal{B} = \{1, \ldots, B\}$;
- $\mathcal{S}$: set of scenarios, $\mathcal{S} = \{1, \ldots, S\}$;

Deterministic parameters

- $c_i \in \mathbb{R}^+$: procurement cost per bicycle placed in bike–station $i \in \mathcal{B}$;
- $v_i \in \mathbb{R}^+$: stock–out cost per bicycle in bike–station $i \in \mathcal{B}$;
- $w_i \in \mathbb{R}^+$: time–waste cost per bicycle due to overflow in bike–station $i \in \mathcal{B}$;
- $t_{ij} \in \mathbb{R}^+$: unit transshipment cost per bicycle transshiped from bike–station i to bike–station j, $i, j \in \mathcal{B}$;
- $f \in \mathbb{R}^+$: fixed transshipment cost;
- $b_{is}^{end} \in \mathbb{Z}^+$: the desired quantity of bicycles to have in bike–station $i \in \mathcal{B}$ in scenario $s \in \mathcal{S}$ at the end of the service;
- $k_i \in \mathbb{Z}^+$: capacity of bike–station $i \in \mathcal{B}$.

Stochastic parameters Let $(\Xi, \mathcal{A}, p)$ be a probability space with Ξ set of outcomes, $\sigma - algebra$ $\mathcal{A}$, probability p and $\xi \in \Xi$ a particular outcome representing the rental demand on each origin–destination pair of bike–stations. We define:

- $\xi_{ijs} \in \Xi \subset \mathbb{Z}^+$: stochastic rental demand from bike–station i to bike–station j in scenario s, $i, j \in \mathcal{B}$, $s \in \mathcal{S}$.

We now introduce first–stage and second–stage variables. The first–stage variables are:

- x_i: number of bicycles to be assigned to bike–station $i \in \mathcal{B}$ at the beginning of the service. We denote with x_i^* the optimal number of bicycles to place at bike–station i and with $\boldsymbol{x} = [x_1, \ldots, x_B]^T$ the first-stage decision vector over all bike–stations. The decision must be taken before the realizations of the random events ξ_{ijs}.

After the placement of the bicycles, the stochastic demands ξ_{ijs} occur on each origin–destination pair $i, j \in \mathcal{B}$ and the minimum between the available and requested bicycles is rented. Then, the possible surplus or shortage follows immediately in each bike–station.
The second–stage decision variables are:

- β_{ijs}: number of rented bicycles from bike–station i to bike–station j in scenario s;

- $I^{+}_{is} = \max(x_i - \sum_{j=1}^{B} \beta_{ijs}, 0) = (x_i - \sum_{j=1}^{B} \beta_{ijs})^{+}$: surplus of bicycles that realizes in bike–station i in scenario s. Note that the surplus does not involve any cost for the provider;

- $I^{-}_{ijs} = \max(\beta_{ijs} - x_i, 0) = (\beta_{ijs} - x_i)^{+}$: shortage of bicycles that realizes in each origin–destination pair i, j in scenario s.

Figure 2.1 shows the sequence of operations realized at the current time for a scenario s.

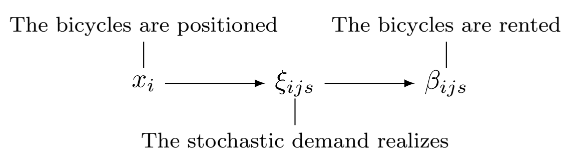


Figure 2.1: Sequence of operations for the two–stage case.

We study both the cases of a two–way and one–way bike–sharing systems.

1.3.1.A Two–way bike–sharing system

Suppose that the bike–sharing system is *two–way*: users must start and terminate the rent at the same bike–station. Our case study, as the greatest majority of the existing bike–sharing systems, is *one–way*, but a two–way bike–sharing system may exist nevertheless. Moreover, suppose that the service is executed once and the transshipment option is not available: the bicycles are rented and when they are dropped off the service ends. At the end of the day, the number of rented bicycles is equal to the number of returned bicycles, thus an overflow may not occur.

Let $z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi})$ be the expected total cost function over the set of bike–stations. The problem can be formulated as a Two–Stage Integer Stochastic Program as follows:

$$\min_{\boldsymbol{x}} z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=1}^{B} c_i x_i + \sum_{s=1}^{S} p_s \sum_{i=1}^{B} \sum_{j=1}^{B} v_i I_{ijs}^{-} \tag{2.1}$$

$$\text{s.t.} \qquad I_{is}^{+} - \sum_{j=1}^{B} I_{ijs}^{-} = x_i - \sum_{j=1}^{B} \xi_{ijs}, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.2}$$

$$x_i, I_{is}^{+}, I_{ijs}^{-} \in \mathbb{Z}^{+}, \qquad \forall i, j \in \mathcal{B}, \forall s \in \mathcal{S}. \tag{2.3}$$

The objective function (2.1) is the minimization of the expected total cost given by the sum over the set of bike–stations of the procurement cost for the positioned bicycles and the sum over the set of scenarios and set of bike–stations of the stock–out cost for shortage. Constraints (2.2) ensure the balance between the surplus and shortage. Constraints (2.3) define the integrality of first–stage and second-stage variables.

Comparing (2.1)–(2.3) with (1)–(5) in Chapter 1, we can note that the two models are equivalent if we consider a single bike–station for (2.1)–(2.3) or multiple retailers for (1)–(5). Thus, a bike–sharing problem can be considered as a variant of the newsvendor problem. The only difference between (2.1)–(2.3) and (1)–(5) lies in the fact that in the bike–sharing problem the surplus is not costly, thus omitted from the objective function (2.1).

We have observed that a bike–sharing problem, even if in its more general form, can be written as a cost–based inventory problem, and since the cost–based inventory problem is a variant of the newsvendor problem, then the bike–sharing problem is also a variant of the newsvendor problem.

For a two–way bike–sharing problem as modelled in (2.1)–(2.3), we state the following Lemma:

Lemma 2.3.1. *Let $\xi_{is} = \sum_{j=1}^{B} \xi_{ijs}$, $I_{is}^{-} = \sum_{j=1}^{B} I_{ijs}^{-}$ and $z_i(\boldsymbol{x}, \boldsymbol{\xi})$ be respectively the rental demand, expected shortage and expected total cost of the bike–station*

$i \in \mathcal{B}$. *Then,*

$$z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=1}^{B} z_i(\boldsymbol{x}, \boldsymbol{\xi}).$$

Proof. First–stage variable x_i is deterministic and therefore can be moved inside the expectation without loss of generality. Then we substitute x_i with its definition.

$$\begin{aligned}
z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi}) &= \sum_{s=1}^{S} p_s \sum_{i=1}^{B} [c_i x_i + v_i \sum_{j=1}^{B} I_{ijs}^{-}] = \\
&= \sum_{s=1}^{S} p_s \sum_{i=1}^{B} \Big[c_i [\xi_{ijs} + I_{is}^{+} - \sum_{j=1}^{B} I_{ijs}^{-}] + v_i \sum_{j=1}^{B} I_{ijs}^{-} \Big] = \\
&= \sum_{s=1}^{S} p_s \sum_{i=1}^{B} \Big[c_i [\xi_{is} + I_{is}^{+} - I_{is}^{-}] + v_i I_{is}^{-} \Big] = \\
&= \sum_{i=1}^{B} \Big[[c_i \bar{\xi}_i] + \sum_{s=1}^{S} p_s [c_i I_{is}^{+} + (v_i - c_i) I_{is}^{-}] \Big] = \\
&= \sum_{i=1}^{B} z_i(\boldsymbol{x}, \boldsymbol{\xi}),
\end{aligned}$$

where

$$z_i(\boldsymbol{x}, \boldsymbol{\xi}) = c_i \bar{\xi}_i + \sum_{s=1}^{S} p_s [c_i I_{is}^{+} + (v_i - c_i) I_{is}^{-}].$$

□

Lemma 2.3.1 shows that in a two–way bike–sharing system, when the stations are independent, the original problem can be decomposed in sub–problems, one for each bike–station included in $\mathcal{B}$, that can be solved individually.

1.3.1.B One–way bike–sharing system with user–transshipment

Suppose now a one–way bike–sharing system, where users may start and terminate the rent at different bike–stations.

When the rented bicycles reach their initial destination, they can be dropped off if the bike station has available capacity. If not, the rented bicycles exceeding the bike–station capacity have to be redirected to the nearest station with available capacity in order to be dropped off. Thus, an overflow may occur.

We introduce the following second–stage variables:

- ρ_{ijs}: number of redirected bicycles from bike–station i to bike–station j in scenario s, $i,j \in \mathcal{B}, s \in \mathcal{S}$;
- $O^+_{is} = \max(k_i - x_i + \sum_{j=1}^{B} \beta_{ijs} - \sum_{j=1}^{B} \beta_{jis}, 0) = (k_i - x_i + \sum_{j=1}^{B} \beta_{ijs} - \sum_{j=1}^{B} \beta_{jis})^+$: residual capacity in bike–station i in scenario s, $i,j \in \mathcal{B}, s \in \mathcal{S}$;
- $O^-_{is} = \max(\sum_{j=1}^{B} \beta_{jis} - k_i + x_i - \sum_{j=1}^{B} \beta_{ijs}, 0) = (\sum_{j=1}^{B} \beta_{jis} - k_i + x_i - \sum_{j=1}^{B} \beta_{ijs})^+$: overflow in bike–station i in scenario s, $i,j \in \mathcal{B}, s \in \mathcal{S}$.

Note that a particular kind of transshipment (named user–transshipment) realizes at the unit time–waste cost per redirected bicycle instead of a unit transshipment cost. Thus, the transshipment fixed cost is not charged since the transshipment option is not adopted even if a repositioning occurs. A graphical description of repositioning for a generic scenario s is provided in Figure 2.2.

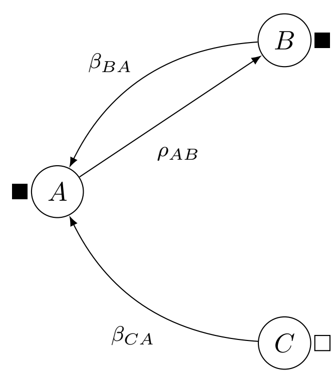


Figure 2.2: Redirected quantity ρ_{AC} from A to C of the overflow O^-_A that realizes from B to A for a generic scenario s. For simplicity of notation, the index s has been dropped.

Suppose that two users start the rent at the same time from B and C, respectively, and drive their bicycles towards A. The user that started from B arrives first and fulls the bike–station capacity in A, represented with a black square. Thus, the user arriving from C is forced to drive towards B, where the bike–station has available capacity, in order to drop–off the bicycle and quit the service.

If one extends the example proposed in Figure 2.2 to a real case, how does the renter know where to go in order to redirect himself? This is a technical issue, depending on the software (app for mobiles) provided alongside the bike–sharing service. There are two possibile cases:

- the app is dynamic and it gets constantly updated. The renter uses the app to check if the bike–station depicted as destination has an available lock. He then moves towards it and in the meanwhile it gets fully locked. Then he has to check again on the app where is the closest bike–station with an available lock in order to redirect himself;

- the app is static and it does not provide real–time informations about bike–stations. The renter moves towards its destination only to discover once he has arrived that all locks are unavailable. He then redirect himself "blindly" towards another bike–station.

Figure 2.3 shows the updated sequence of operations realized for a generic scenario s in a one–way bike–sharing system.

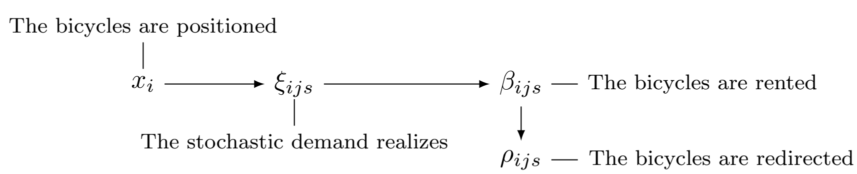


Figure 2.3: Sequence of operations for a scenario s in a one–way bike–sharing system.

The problem can be formulated as a Two–Stage Integer Stochastic Program as follows:

$$\textbf{P1}. \quad \min_{\boldsymbol{x}} z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=1}^{B} c_i x_i + \sum_{s=1}^{S} p_s \sum_{i=1}^{B} [\sum_{j=1}^{B} v_i I_{ijs}^- + w_i O_{is}^-] \tag{2.4}$$

$$\text{s.t.} \quad x_i \leq k_i, \qquad \forall i \in \mathcal{B}, \tag{2.5}$$

$$\beta_{ijs} = \xi_{ijs} - I_{ijs}^-, \qquad \forall i, j \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.6}$$

$$I_{is}^+ - \sum_{j=1}^{B} I_{ijs}^- = x_i - \sum_{j=1}^{B} \xi_{ijs}, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.7}$$

$$O_{is}^+ - O_{is}^- = k_i - x_i + \sum_{j=1}^{B} \beta_{ijs} - \sum_{j=1}^{B} \beta_{jis}, \ \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.8}$$

$$x_i, \beta_{ijs}, I_{is}^+, I_{ijs}^-, O_{is}^+, O_{is}^- \in \mathbb{Z}^+, \qquad \forall i, j \in \mathcal{B}, \forall s \in \mathcal{S}. \tag{2.9}$$

The objective function (2.4) is the minimization of the expected total cost given by the sum over the set of bike–stations of the procurement cost for the positioned bicycles and the sum over the set of scenarios and set of bike–stations of the stock–out cost for the shortage plus the time–waste cost for the overflow. Constraints (2.5) enforce the positioned bicycle quantities to be less then or equal to the bike–station capacity. Constraints (2.6) define the rented quantities given by the difference between the stochastic demand and the shortage. Constraints (2.7) ensure the balance between the surplus and shortage while constraints (2.8) ensure the balance between the residual quantity and the overflow. Constraints (2.9) define the integrality of first–stage and second-stage variables.

We now consider the service level (SL), which is a measure for the performance of the bike–sharing system and thus of the bicycle inventory levels. Generally, service level measures the performance of a system and it fixes the percentage to which goals (e.g. fill rate, customer satisfaction, rental demands) should be achieved. A typical use of service level constraints can be observed in Moon and Choi (1994) [44], Jha and Shanker (2009) [28] and Taleizadeh et al (2010) [61].

Let $SL \in [0,1]$ be the desired service level and let $\varepsilon_1, \varepsilon_2 \in [0,1]$ two weights such that $\varepsilon_1 + \varepsilon_2 = 1$. Let $O^-_{max} = \sum_{i=1}^{B} x_i - k_j$ be the maximum possible overflow, which is the overflow in the worst-case scenario ($i.e$ all the rented bicycles have the same destination). We implement the SL in (2.4)–(2.9) and we obtain the following Two–Stage Integer Stochastic Program (P2).

$$\textbf{P2}. \quad \min_{\boldsymbol{x}} z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=1}^{B} c_i x_i + \sum_{s=1}^{S} p_s \sum_{i=1}^{B} [\sum_{j=1}^{B} v_i I^-_{ijs} + w_i O^-_{is}] \tag{2.10}$$

$$\text{s.t.} \quad x_i \leq k_i, \qquad \forall i \in \mathcal{B}, \tag{2.11}$$

$$\beta_{ijs} = \xi_{ijs} - I^-_{ijs}, \qquad \forall i, j \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.12}$$

$$I^+_{is} - \sum_{j=1}^{B} I^-_{ijs} = x_i - \sum_{j=1}^{B} \xi_{ijs}, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.13}$$

$$O^+_{is} - O^-_{is} = k_i - x_i + \sum_{j=1}^{B} \beta_{ijs} - \sum_{j=1}^{B} \beta_{jis}, \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.14}$$

$$\varepsilon_1 \sum_{j=1}^{B} \frac{\beta_{ijs}}{\xi_{ijs}} - \varepsilon_2 \frac{O^-_{is}}{O^-_{max}} \geq SL, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.15}$$

$$x_i, \beta_{ijs}, I^+_{is}, I^-_{ijs}, O^+_{is}, O^-_{is}, O^-_{max}, \in \mathbb{Z}^+, \qquad \forall i, j \in \mathcal{B}, \forall s \in \mathcal{S}. \tag{2.16}$$

Constraints (2.15) define the service level greater or equal to a certain value SL for each bike–station and in each scenario, given by the weighted sum of the rented quantities over the stochastic demand minus the overflow over the maximum overflow.

When the redirected bicycles reach their final destination and are dropped off, a new composition of the fleet is obtained over the set of bike–stations. The provider may be willing to adopt the transshipment option in order to match the desired bicycle quantities in each bike–station that have to be in place at the end of the service.
We introduce the following second–stage variables:

- τ_{ijs}: number of transshiped bicycles from bike–station i to bike–station j in scenario s, $\forall i, j \in \mathcal{B}, \forall s \in \mathcal{S}$;
- $T^+_{is} = \max(k_i - O^+_{is} + \sum_{j=1}^{B} \rho_{jis} - b^{end}_{is}, 0) = (k_i - b^{end}_{is} - O^+_{is} + \sum_{j=1}^{B} \rho_{jis})^+$: exceed of bicycles in bike–station i in scenario s, $\forall i \in \mathcal{B}, \forall s \in \mathcal{S}$;
- $T^-_{is} = \max(b^{end}_{is} - k_i + O^+_{is} - \sum_{j=1}^{B} \rho_{jis}, 0) = (b^{end}_{is} - k_i + O^+_{is} - \sum_{j=1}^{B} \rho_{jis})^+$: failure of bicycles in bike–station i in scenario s, $\forall i \in \mathcal{B}, \forall s \in \mathcal{S}$;
- y_s: binary variable representing the activation of the transshipment option in scenario s, $\forall s \in \mathcal{S}$.

$$y_s = \begin{cases} 1 & \text{if } \sum_{i=1}^{B} s_i T^-_{is} > f, \\ 0 & \text{if } \sum_{i=1}^{B} s_i T^-_{is} < f. \end{cases}$$

The activation of the transshipment option is explained as follows. According to Figure 2.3, after the bicycles have been redirected to the final bike–station with available capacity, and immediantly before the possibility of the transshipment, the bicycles in each bike–station and for each scenario are:

- T^+_{is} if $k_i - O^+_{is} + \sum_{j=1}^{B} \rho_{jis} > b^{end}_{is}$, (exceedings);
- T^-_{is} if $k_i - O^+_{is} + \sum_{j=1}^{B} \rho_{jis} < b^{end}_{is}$, (failures).

The provider is willing to understand if it is less costly to perform the transshipment in order to match at the end of the service the stationed quantities with the desired quantities or to do nothing by accepting as it is the state of the system. In the latter case, it means that the bike–stations where T^-_{is} realizes are going to face the new incoming stochastic demand with a quantity of bicycles lesser than the desired (optimal) quantity. Thus, the probability of a shortage increses. It follows that:

- if $\sum_{i=1}^{B} s_i T^-_{is} > f$, ($i.e$ if doing nothing is more costly than performing the transshipment at the fixed cost f), then the transshipment option is activated and the transshipment is performed;

– if $\sum_{i=1}^{B} s_i T_{is}^{-} < f$, ($i.e$ if doing nothing is less costly than performing the transshipment at the fixed cost f), then the transshipment option is not activated and the transshipment is not performed.

Figure 2.5 shows the whole sequence of operations realized at the end of the service for a scenario s.

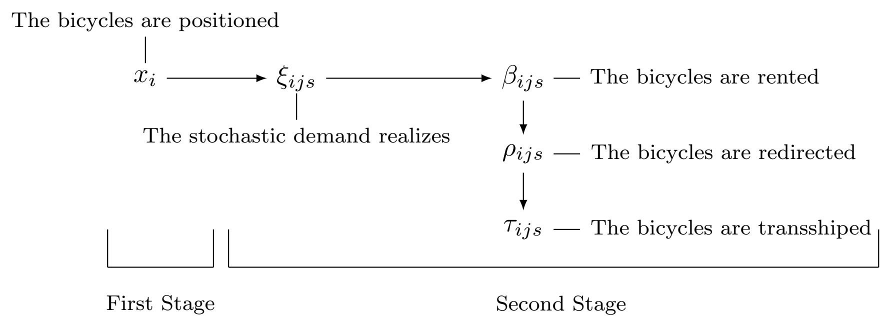


Figure 2.4: Sequence of operations at the end of the service for a scenario s.

Suppose that the provider is interested in performing the transshipment if the stationed bicycle quantities do not match the desired (optimal) quantities to be place at the end of the service and if it is convenient. The problem can be formulated as the following Two–Stage Integer Stochastic Program.

$$\textbf{P3}.\ \min_{\boldsymbol{x}} z^{\mathcal{B}}(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{i=1}^{B} c_i x_i + \sum_{s=1}^{S} p_s \Big[\sum_{i=1}^{B} [v_i \sum_{j=1}^{B} I_{ijs}^{-} + w_i O_{is}^{-}] + f y_s \Big] \tag{2.17}$$

$$\text{s.t.} \quad x_i \leq k_i, \quad \forall i \in \mathcal{B}, \tag{2.18}$$

$$\beta_{ijs} = \xi_{ijs} - I_{ijs}^{-}, \quad \forall i, j \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.19}$$

$$I_{is}^{+} - \sum_{j=1}^{B} I_{ijs}^{-} = x_i - \sum_{j=1}^{B} \xi_{ijs}, \quad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.20}$$

$$O_{is}^{+} - O_{is}^{-} = k_i - x_i + \sum_{j=1}^{B} \beta_{ijs} - \sum_{j=1}^{B} \beta_{jis}, \quad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.21}$$

$$\sum_{j=1}^{B} \rho_{ijs} = O_{is}^{-}, \quad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.22}$$

$$\sum_{j=1}^{B} \rho_{jis} \leq O_{is}^{+}, \quad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.23}$$

$$T_{is}^{+} - T_{is}^{-} = k_i - O_{is}^{+} + \sum_{j=1}^{B} \rho_{jis} - x_i, \quad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \tag{2.24}$$

$$\sum_{i=1}^{B} v_i T_{is}^{-} - f - M y_s \leq 0, \quad \forall s \in \mathcal{S}, \tag{2.25}$$

$$y_s \in \{0, 1\}, \quad \forall s \in \mathcal{S}, \tag{2.26}$$

$$x_i, \beta_{ijs}, \rho_{ijs}, I_{is}^{+}, I_{ijs}^{-}, O_{is}^{+}, O_{is}^{-}, T_{is}^{+}, T_{is}^{-} \in \mathbb{Z}^{+}, \forall i, j \in \mathcal{B}, \forall s \in \mathcal{S}. \tag{2.27}$$

The objective function (2.17) is the minimization of the expected total cost given by the sum over the set of bike–stations of the procurement cost for the positioned bicycle quantities and the sum over the set of scenarios and set of bike–stations of the stock–out cost for the shortage and the failure plus the time–waste cost for the overflow plus the fixed transshipment cost if the transshipment option is activated. Constraints (2.18) enforce the positioned bicycle quantities to be lesser or equal to the bike–station capacity. Constraints (2.19) define the rented quantities given by the difference between the stochastic demand and the shortage. Constraints (2.20) ensure the balance between surplus and shortage while constraints (2.21) ensure the balance between residual quantity and the overflow. Constraints (2.22) define the sum of all the redirected bicycle quantities from a bike–station equal to its overflow while constraints (2.23) guarantee that the sum of all the redirected bicycle quantities to a bike–station cannot exceed its residual capacity. Constraints (2.24) ensure the balance between exceeding and failure. Note that b_{is}^{end} has been replaced with x_i: we want to have at the end of the service the number of bicycles placed at the beginning before demand realization, in each scenario and for each bike–station. Constraints (2.25) define the activation of the transshipment option using the big–M method [25]. At last, constraint (2.26) and (2.27) define the integrality of first–stage and second–stage variables.

Figure 2.5 shows how the constraints of the program P3 are structured according to the sequence of operations.

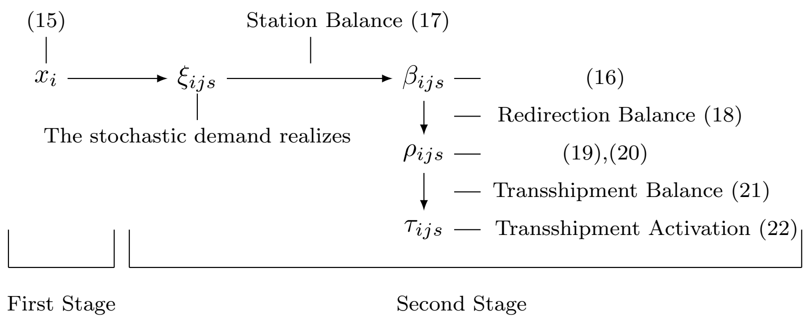


Figure 2.5: Sequence of operations at the end of the service for a generic scenario s.

The three sets of balancing constraints, (2.20), (2.21) and (2.24), allow the program to manage jointly all the phases of the problem in the same stage without losing information. Moreover, they also define the second–stage variables β_{ijs}, ρ_{ijs} and τ_{ijs}.

We study the objective function (2.17). Note that

$$x_i = \sum_{j=1}^{B} \xi_{ijs} + I_{is}^{+} - I_{is}^{-} = \xi_{is} + I_{is}^{+} - I_{is}^{-}, \tag{2.28}$$

and

$$\sum_{j=1}^{B} \beta_{ijs} = \xi_{is} - I_{is}^{-}. \tag{2.29}$$

Since

$$O_{is}^{+} = T_{is}^{-} - T_{is}^{+} + k_i + \sum_{j=1}^{B} \rho_{jis} - x_i,$$

and

$$O_{is}^{-} = O_{is}^{+} - k_i + x_i - \sum_{j=1}^{B} \beta_{ijs} + \sum_{j=1}^{B} \beta_{jis},$$

then

$$O_{is}^- = T_{is}^- - T_{is}^+ - \xi_{is} + I_{is}^- + \sum_{j=1}^{B}(\beta_{jis} + \rho_{jis}). \tag{2.30}$$

Substituting (2.28), (2.29) and (2.30) in (2.17), the objective function becomes:

$$\begin{aligned}
z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi}) &= \sum_{s=1}^{S} p_s \Big[\sum_{i=1}^{B} [c_i x_i + v_i I_{is}^- + w_i O_{is}^-] + f y_s \Big] = \\
&= \sum_{s=1}^{S} p_s \Big[\sum_{i=1}^{B} [c_i(\xi_{is} + I_{is}^+ - I_{is}^-) + v_i I_{is}^- + \\
&+ w_i(T_{is}^- - T_{is}^+ - \xi_{is} + I_{is}^- + \sum_{j=1}^{B}(\beta_{jis} + \rho_{jis}))] + f y_s \Big] = \\
&= \sum_{i=1}^{B}(c_i - w_i)\bar{\xi}_i + \sum_{s=1}^{S} p_s \sum_{i=1}^{B} \Big[c_i I_{is}^+ + (v_i + w_i - c_i) I_{is}^- + \\
&+ w_i(T_{is}^- - T_{is}^+ + \sum_{j=1}^{B}(\beta_{jis} + \rho_{jis})) \Big] + \bar{f}.
\end{aligned} \tag{2.31}$$

In (2.31) the first term is deterministic, the second term is the newly obtained recourse function and $\bar{f}$ is the *expected fixed transshipment cost*, which is the transshipment cost that will realize on average depending only by the possible scenarios.

We propose another formulation for the bike–sharing problem with transshipment where the transshipment always realizes each day at the end of the service, which is expected to be more tractable from a computational point of view since it does not make use of binary variables.

Suppose that the provider is always willing to perform the transshipment at a unit transshipment cost t_{ij} if the stationed bicycle quantities do not match the desired (optimal) quantities to be placed at the end of the service. The problem can be formulated as a Two–Stage Integer Stochastic Program.

$$\textbf{P4}.\ \min_{\boldsymbol{x}} z^{\mathcal{B}}(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{i=1}^{B} c_i x_i + \sum_{s=1}^{S} p_s \sum_{i=1}^{B}[v_i \sum_{j=1}^{B} I_{ijs}^{-} + wO_{is}^{-} + \sum_{j=1}^{B} t_{ij}\tau_{ijs}] \quad (2.32)$$

$$\text{s.t.} \quad x_i \leq k_i, \qquad \forall i \in \mathcal{B}, \quad (2.33)$$

$$\beta_{ijs} = \xi_{ijs} - I_{ijs}^{-}, \qquad \forall i,j \in \mathcal{B}, \forall s \in \mathcal{S}, \quad (2.34)$$

$$I_{is}^{+} - \sum_{j=1}^{B} I_{ijs}^{-} = x_i - \sum_{j=1}^{B} \xi_{ijs}, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \quad (2.35)$$

$$O_{is}^{+} - O_{is}^{-} = k_i - x_i + \sum_{j=1}^{B} \beta_{ijs} - \sum_{j=1}^{B} \beta_{jis}, \quad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \quad (2.36)$$

$$\sum_{j=1}^{B} \rho_{ijs} = O_{is}^{-}, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \quad (2.37)$$

$$\sum_{j=1}^{B} \rho_{jis} \leq O_{is}^{+}, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \quad (2.38)$$

$$T_{is}^{+} - T_{is}^{-} = k_i - O_{is}^{+} + \sum_{j=1}^{B} \rho_{jis} - x_i, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \quad (2.39)$$

$$\sum_{j=1}^{B} \tau_{ijs} = T_{is}^{+}, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \quad (2.40)$$

$$\sum_{j=1}^{B} \tau_{jis} = T_{is}^{-}, \qquad \forall i \in \mathcal{B}, \forall s \in \mathcal{S}, \quad (2.41)$$

$$x_i, \beta_{ijs}, \rho_{ijs}, I_{is}^{+}, I_{ijs}^{-}, O_{is}^{+}, O_{is}^{-}, T_{is}^{+}, T_{is}^{-} \in \mathbb{Z}^{+}, \forall i,j \in \mathcal{B}, \forall s \in \mathcal{S}. \quad (2.42)$$

The objective function (2.32) is the minimization of the expected total cost given by the sum over the set of bike–stations of the procurement cost for the positioned bicycle quantities and the sum over the set of scenarios and set of bike–stations of the stock–out cost for the shortage plus the time–waste cost for the overflow plus the transshipment cost for the transshipped bicycle quantities. Constraints (2.33) – (2.39) are identical to program P3. Constraints (2.40) define the sum of all the transshiped bicycle quantities from a bike–station equal to its exceed, while constraints (2.41) guarantee that the sum of all the transshiped

bicycle quantities to a bike–station is equal to its failure. At last, constraints (2.42) define the integrality of first–stage and second–stage variables.
Similarly to (2.17), we study the objective function (2.32) substituting (2.28), (2.29) and (2.30). It becomes:

$$
\begin{aligned}
z^{\mathcal{B}}(\boldsymbol{x},\boldsymbol{\xi}) &= \sum_{s=1}^{S} p_s \Big[\sum_{i=1}^{B} [c_i x_i + v_i I_{is}^{-} + w_i O_{is}^{-}] + \sum_{j=1}^{B} t_{ij}\tau_{ijs} \Big] = \\
&= \sum_{s=1}^{S} p_s \sum_{i=1}^{B} \Big[c_i(\xi_{is} + I_{is}^{+} - I_{is}^{-}) + v_i I_{is}^{-} + \\
&+ w_i[T_{is}^{-} - T_{is}^{+} - \xi_{is} + I_{is}^{-} + \sum_{j=1}^{B}(\beta_{jis} + \rho_{jis})] + \sum_{j=1}^{B} t_{ij}\tau_{ijs} \Big] = \\
&= \sum_{i=1}^{B}(c_i - w_i)\bar{\xi}_i + \sum_{s=1}^{S} p_s \sum_{i=1}^{B} \Big[c_i I_{is}^{+} + (v_i + w_i - c_i) I_{is}^{-} + \\
&+ w_i[T_{is}^{-} - T_{is}^{+} + \sum_{j=1}^{B}(\beta_{jis} + \rho_{jis})] + \sum_{j=1}^{B} t_{ij}\tau_{ijs} \Big]. \qquad (2.43)
\end{aligned}
$$

In (2.43) first term is deterministic while second term is the newly obtained recourse function. Comparing (2.43) with (2.32), as for (2.31), bike–stations result to be dependent since the optimal number of bicycle to place in bike–station i depends also on the stochastic demands of bike–station j.

The bike–sharing problems described in programs P3 and P4 are variants of the newsvendor problem where the recourse action is the transshipment.
From Dong and Rudi [18] we understand that in a bike–sharing problem with transshipment the optimal inventory level for each bike–station is obtained from the so called *critical fractile*, given by a combination of the cost parameters c_i, v_i and w_i for program P3 and c_i, v_i, w_i and t_{ij} for program P4. It follows that in program P4 the optimal inventory levels depends also on the unit transshipment cost t_{ij}.
Finally, we can conclude that the optimal inventory levels in program P3 and

P4 depends of the transshipped quantities (program P3) and of the unit transshipped cost (program P4), thus for both programs the transshipment component cannot be consider asunder from the more generic bike–sharing problem in order to determine the optimal inventory levels.
Moreover, from Zhang [66], we understand that the optimal inventory level of each bike-station with transshipment depends on the multivariate demand distribution through the marginal distributions and the distribution of the sum of the demands. Thus, the *critical fractile* is expression of the marginal distributions and of the distribution of the sum of the demands.

We no further investigates the objective functions of programs P3 and P4 in order to obtain a critical fractile for the optimal quantity to order since it has no use for the computational analysis in Section 2.5 and its been extesively analyzed in Zhang [66].

2.3.2 Multi–stage stochastic optimization models for a one–way bike–sharing system

In this section, we formulate a multi–stage stochastic model. The aim of the model is the same, that is to determine, for each bike–station, the optimal number of bicycles to place in order to minimize the system expected total cost taking into account of the dynamic nature of rental demands over the day.

The major difference between the two–stage and multi–stage models is that in the latter, we have to model the structure of the underlying scenario tree. This can be done in two ways: we can either model the problem by scenarios and then add a set of so–called *non–anticipativity constraints*, or write the model in terms of nodes of the scenario tree and describe the tree structure by giving to each node (except the root of the tree) a pointer to its parent, i.e. the node immediately preceding it (Maggioni et al. [41]). We need the following notation, in addition to the one introduced in Section 2.3.1:

- $\mathcal{N} := \{n : n = 0, \ldots, N\}$: ordered set of nodes of the scenario tree structure;

- $0 \in \mathcal{N}$: the root of the tree, which represents the time instant for which we want to determine the number of bicycles to place;
- $\mathcal{F} := \{n : n = N - F + 1, \ldots, N\} \subset \mathcal{N}$: the set of the leaves of the tree, that is the nodes in the last stage of the model; since the *number of scenarios* S is equal to the number of leaves, we get $S = F = |\mathcal{F}|$;
- pa(n): the parent of node $n \in \mathcal{N} \setminus \{0\}$;
- x_{i0}: total number of bicycles to assign in bike–station $i \in \mathcal{B}$ at the beginnig of the service $0 \in \mathcal{N}$;
- p_n: the probability of node $n \in \mathcal{N}$; we have considered the leaves (and therefore the scenarios) to be equiprobable, so that

$$p_n = \begin{cases} \frac{1}{|\mathcal{F}|} & \text{if } \ n \in \mathcal{F}, \\ \sum_{m \in \mathcal{N} \setminus \{0\}, pa(m)=n} p_m & \text{if } \ n \in \mathcal{N} \setminus \mathcal{F}. \end{cases}$$

In order to obtain multi–stage model formulations for a one–way bike–sharing problem, the daily rental demand of each origin–destination pair into three rental demands, respectively for morning, afternoon and evening periods. The recourse realizes at the end of the last stage, only at night, when the service is not available for the user community. It consists in rebalancing the bicycle fleet through the transshipment of bicycles from bike–stations with an excess to bike–stations with a failure. We believe that it is realistic and reasonable to admit the recourse only at night because it is coherent with the reasons and politics that justify the bike–sharing service, such as reducing pollution and congestion. Transshipment could be operated during the day at the end of each period, but it woud consist in deploying three times a fleet of vehicles which contribute to increment the daily pollution and congestion, which is clearly against the idea of bike–sharing.

The multi–stage model corresponding to program P3 can be then formulated as follows:

$$\min_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=1}^{B} c_i x_{i0} + \sum_{n=1}^{N} p_n \Big[\sum_{i=1}^{B} [v_i \sum_{j=1}^{B} I^-_{ijn} + w_i O^-_{in}] + f y_n \Big] \tag{2.44}$$

$$\text{s.t.} \quad x_{i0} \leq k_i, \qquad \forall i \in \mathcal{B}, \tag{2.45}$$

$$\beta_{ijn} = \xi_{ijn} - I^-_{ijn}, \qquad \forall i, j \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, \tag{2.46}$$

$$I^+_{in} - \sum_{j=1}^{B} I^-_{ijn} = x_{i0} - \sum_{j=1}^{B} \xi_{ijn}, \qquad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, \tag{2.47}$$

$$O^+_{in} - O^-_{in} = k_i - x_{i0} + \sum_{j=1}^{B} \beta_{ijn} - \sum_{j=1}^{B} \beta_{jin}, \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, \tag{2.48}$$

$$\sum_{j=1}^{B} \rho_{ijn} = O^-_{in}, \qquad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, \tag{2.49}$$

$$\sum_{j=1}^{B} \rho_{jin} \leq O^+_{in}, \qquad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, \tag{2.50}$$

$$T^+_{in} - T^-_{in} = k_i - O^+_{in} + \sum_{j=1}^{B} \rho_{jin} - x_{i0}, \qquad \forall i \in \mathcal{B}, \forall n \in \mathcal{F}, \tag{2.51}$$

$$\sum_{i=1}^{B} v_i T^-_{in} - f - M y_n \leq 0, \qquad \forall i \in \mathcal{B}, \forall n \in \mathcal{F}, \tag{2.52}$$

$$y_n \in \{0, 1\}, \qquad \forall n \in \mathcal{F}, \tag{2.53}$$

$$x_{i0} \in \mathbb{Z}^+, \qquad \forall i \in \mathcal{B}, \tag{2.54}$$

$$\beta_{ijn}, \rho_{ijn}, \tau_{ijn},$$

$$I^+_{in}, I^-_{ijn}, O^+_{in}, O^-_{in}, T^+_{in}, T^-_{in} \in \mathbb{Z}^+, \qquad \forall i, j \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}. \tag{2.55}$$

The objective function (2.44) is the minimization of the expected total cost given by the sum over the set of bike–stations of the procurement cost for the positioned bicycles at the beginning of the day and the sum over the set of nodes and set of bike–stations of the stock–out cost for the shortage and the failure plus the time–waste cost for the overflow plus the fixed transshipment cost if the transshipment option is activated. Constraints (2.45) enforce the positioned bicycle quantities at the beginning of the day to be lesser or equal to the bike–station capacity. Constraints (2.46) define the rented quantities given by the difference between the stochastic demand and the shortage. Constraints (2.47) ensure the balance between surplus and shortage, while constraints (2.48) ensure the balance between residual quantity and the overflow. Constraints (2.49) define the sum of all the redirected bicycle quantities from a bike–station equal to its overflow while constraints (2.50) guarantee that the sum of all the redirected bicycle quantities to a bike–station cannot exceed its residual capacity. Constraints (2.51) ensure the balance between exceeding and failure. Constraints (2.52) define the activation at the end of the day of the transshipment option using the big–M method [25]. Finally, (2.53) – (2.55) define the integrality of the first–stage and recourse variables.

The multi–stage model corresponding to program P4 can be then formulated as follows:

$$\min_{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{i=1}^{B} c_i x_{i0} + \sum_{n=1}^{N} p_n \Big[\sum_{i=1}^{B} [v_i \sum_{j=1}^{B} I^-_{ijn} + w_i O^-_{in} + \sum_{j=1}^{B} t_{ij}\tau_{ijn}] \Big] \quad (2.56)$$

$$\text{s.t.} \quad x_{i0} \le k_i, \quad \forall i \in \mathcal{B}, \; (2.57)$$

$$\beta_{ijn} = \xi_{ijn} - I^-_{ijn}, \quad \forall i,j \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, (2.58)$$

$$I^+_{in} - \sum_{j=1}^{B} I^-_{ijn} = x_{i0} - \sum_{j=1}^{B} \xi_{ijn}, \quad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, (2.59)$$

$$O^+_{in} - O^-_{in} = k_i - x_{i0} + \sum_{j=1}^{B} \beta_{ijn} - \sum_{j=1}^{B} \beta_{jin}, \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, (2.60)$$

$$\sum_{j=1}^{B} \rho_{ijn} = O^-_{in}, \quad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, \; (2.61)$$

$$\sum_{j=1}^{B} \rho_{jin} \le O^+_{in}, \quad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, (2.62)$$

$$T^+_{in} - T^-_{in} = k_i - O^+_{in} + \sum_{j=1}^{B} \rho_{jin} - x_{i0}, \quad \forall i \in \mathcal{B}, \forall n \in \mathcal{F}, (2.63)$$

$$\sum_{j=1}^{B} \tau_{ijn} = T^+_{in}, \quad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, (2.64)$$

$$\sum_{j=1}^{B} \tau_{jin} = T^-_{in}, \quad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}, (2.65)$$

$$x_{i0} \in \mathbb{Z}^+, \quad \forall i \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \mathcal{F} \quad (2.66)$$

$$\beta_{ijn}, \rho_{ijn}, \tau_{ijn},$$

$$I^+_{in}, I^-_{ijn}, O^+_{in}, O^-_{in}, T^+_{in}, T^-_{in} \in \mathbb{Z}^+, \quad \forall i,j \in \mathcal{B}, \forall n \in \mathcal{N} \setminus \{0\}. (2.67)$$

The objective function (2.56) is the minimization of the expected total cost given by the sum over the set of bike–stations of the procurement cost for the positioned bicycles at the beginning of the service and the sum over the set of scenarios and set of bike–stations of the stock–out cost for the shortage plus the time–waste cost for the overflow plus the transshipment cost for the transshipped bicycles. Constraints (2.57) – (2.63) are identical to the ones in program P3. Constraints (2.64) define the sum of all the transshipped bicycles from a bike–station equal to its exceed, while constraints (2.65) guarantee that the sum of all the transshiped bicycles to a bike–station is equal to its failure, both at the end of the service. Finally, constraints (2.66) and (2.67) define the integrality of the first–stage and recourse variables, respectively.

2.4 The Case Study: the bike–sharing service 'La BiGi'

In this section, we present the case study used for testing program P4, both in its two–stage and multi–stage model formulations proposed in Section 2.3.
In 2013, July 10, the City Council of Bergamo approved the Traffic Urban Plan, known as *Piano Urbano del Traffico* (PUT).
The PUT, among many other topics about the revaluation of the territory and the growth of new inland activities, promotes the bicycle mobility, identified as an effective and sustainable solution to the needs of mobility in urban areas and peri-urban of Bergamo.
In line with the strategic directions of network development, already defined by the Territory Govern Plan, known as *Piano di Governo del Territorio*, new operational guidelines are defined for the interconnection and completion of the existing cycle network, the 'hinge' auctions, which are activated progressively in the period of the Plan, and safety. These guidelines will form the basis for the development in the management phase and for the scheduling of the activities.
The phases of the development of the bike–sharing network 'La BiGi' are defined through a greater spreading of the stations (up to double the current network) with the aim of progressively expanding the coverage and the capillarity of the service in the city center and in the main corridors of urban mobility. Moreover, concerning the car parkings in public or private facility with less then 150 units for which a fee must be paid, the Municipal Administration reserves the right to ask for the activation of one or up to five bike–stations (if the number of parkings exceeds the 150 units), which costs have to be charched up to the bike–sharing service provider.
The phase of implementation of the the bike–sharing was focused in addition to improving the infrastructure, to the upgrade of the bike–sharing "La BiGi", according to two phases of implementation: to improve the existing network by doubling the number of bike–stations with the aim to expanding progressively

the coverage of the service.
In particular, the installation of two new bike–stations is planned in the vicinity of Tramway Bergamo-Albino. The bike–stations have been built in the end of 2013 as part of a project funded by the Ministry for the Environment, Land and Sea. In order to support the expansion and maintenance of the bike–sharing network, politics of coverage of the investments capable to help the network economic sustainability will have to be promoted.
At the current time, September 2016, 22 bike–stations are operative. Figure 2.6 shows where the bike–stations are located on the city jurysdiction of Bergamo.

Figure 2.6: Map of the bike–station locations provided by 'La Bigi'.

The aim of our study is to determine the optimal size of the bicycle fleet given by the optimal number of bicycles to place in each bike–station, minimizing the expected total cost of the bike–sharing service, "La BiGi".

2.5 Numerical Results

In this section, we present numerical results from the case described in Section 2.4, based on historical data from period May 2013 to December 2015 reported in the Appendix in Tables 2.23 – 2.54.

This section is organized as follows: in Subsection 2.5.1 we analyse the real data and how scenarios are generated; in Subsection 2.5.2 we investigate and comment computational times, in–sample stability and optimal inventory levels, under different assumption for the probability distribution. In Subsection 2.5.3 we analyze the quality of the Expected Value Solution by computing several well-known measures in Stochastic Programming and the Value of the Right Distribution (VRD). In Subsection 2.5.4 we perform a sensititity analysis on the cost parameters of the problem, while in Subsection 2.5.5 we test a set of instances.

2.5.1 Data Analysis, Scenario Generation and Instances Parameters

Bike–sharing service in Bergamo, named "La Bigi", started in May 2013 with 18 bike–stations. 22 stations are currently operative but not all of them were activated at the same time. In February 2015 the station of Auchan was opened, in September Bianzana and San Fermo and, at last, in December Ospedale Papa Giovanni. Table 2.1 reports labels assigned to each bike–station.

Bike–Station	Label
Alpini	A
Auchan	B
Battisti	C
Bianzana	D
Borgo Palazzo	E
Cavour	F
Coghetti	G
Don Bosco	H
Maj	I
Matteotti	J
Oberdan	K
Ospedale Papa Giovanni	L
Paleocapa	M
Palma Il Vecchio	N
Pirovano	O
Rezzara	P
San Fermo	Q
Sant'anna	R
Santo Spirito	S
Sant'Orsola	T
Tironi	U
Viale Emanuele	V

Table 2.1: Bike–stations labeling.

We have at our disposal monthly usage data given by total rentals between each pair of bike–stations for each month from May 2013 up to December 2015. We use them as real data. We cannot know if a bicycle moved from A to C initially had B as destination and then it was forced to be redirected. This is

a technical issue which depends by the software provided alongside the bike–sharing service: it could be solved by an app for mobiles which queries the user for knowning the initial and final destinations in case of redirection.
Since the provider of the bike–sharing system is interested in daily data, we estimate daily demands in order to test program P4. We divide the monthly rentals of each origin–destination pair by the number of day in a mounth, obtaining an average daily rental (*e.g.*, $\xi_{ij}^{day} = \frac{\xi_{ij}^{month}}{D}$, where D is the number of days in a month and ξ_{ij}^{day} and ξ_{ij}^{month} are daily and monthly rentals for couple $i, j \in \mathcal{B}$, respectively).
Bike-sharing service in Bergamo is relatively young and during year 2015 4 bike–stations were added: no data are available for these bike–stations in the years 2013–2014. Moreover, due to the limited size of the available historical data, time–series models cannot be used to generate scenarios of demand.
We consider winter and summer demands. For the first, we collect the samples of the following months: October, November and December 2013, January, February, March, October, November, December 2014 and January, February, March, October, November, December 2015 for a total of 15 scenarios. For the second, we collect the samples of the following months: May, June, July, August, September 2013, April, May, June, July, August, September 2014 and April, May, June, July, August, September 2015 for a total of 17 scenarios.
From the set of historical data, the following parameters are estimated for the winter demand:

- minimum $m_{ij} = \min(\xi_{ij}^{Oct2013}, \xi_{ij}^{Nov2013}, \ldots, \xi_{ij}^{Dec2015})$ (see Table 2.55 in Appendix);
- maximum $M_{ij} = \max(\xi_{ij}^{Oct2013}, \xi_{ij}^{Nov2013}, \ldots, \xi_{ij}^{Dec2015})$ (see Table 2.56 in Appendix);
- expected value $\bar{\xi}_{ij} = (\xi_{ij}^{Oct2013}, \xi_{ij}^{Nov2013}, \ldots, \xi_{ij}^{Dec2015}) \div S$ (see Table 2.58 in Appendix);
- standard deviation $\sigma_{ij} = \sqrt{\left[(\xi_{ij}^{Oct2013} - \bar{\xi}_{ij})^2 + \cdots + (\xi_{ij}^{Dec2015} - \bar{\xi}_{ij})^2\right] \div S - 1}$

(see Table 2.61 in Appendix);

Note that the rental demand for each origin–destination pair under the assumption of each possible distribution is defined on the same finite support and has the same expected value, i.e. $\boldsymbol{\xi}_{ij} \in [m, M]$ and $\mathbb{E}[\boldsymbol{\xi}_{ij}] = \bar{\xi}_{ij}, \forall i, j \in N$ and for all probability distributions of $\boldsymbol{\xi}_{ij}$.
In Subsection 2.5.3 we will compute the Value of the Right Distribution for the case study proposed in this chapter. We will use the same probability distributions considered in Chapter 1 because they are among the most studied and it will result easier for the reader to understand the numerical results since the same matches for the probability models have already been tested. However, for the sake of completeness, we report that in Dell'Amico et al. (2014) [15], for the case study they proposed about the bike–sharing of Reggio Emilia, the Normal distribution is depicted as the fittest, given their time series.
The four distributions for each possible origin–destination pair rental demands are:

- Uniform distribution;
- Exponential distribution;
- Normal distribution;
- Log–normal distribution;

for which we estimate the following parameters:

- standard deviation of the Uniform distribution, $\sigma^{\mathcal{U}}_{ij} = \sqrt{\frac{1}{12}(M_{ij} - m_{ij})^2}$ (see Table 2.62 in Appendix);
- standard deviation of the Exponential distribution, $\sigma^{\mathcal{E}}_{ij} = \lambda_{ij} = \bar{\xi}^{-1}_{ij}$ (see Table 2.63 in Appendix);
- location parameter of the Log–normal distribution, $\zeta_{ij} = \ln(\bar{\xi}_{ij}) - \frac{1}{2}\ln(1 + \text{cv}^2_{ij})$ (see Table 2.59 in Appendix);

- scale parameter of the Log–normal distribution, $\eta_{ij} = \ln(1 + \text{cv}_{ij}^2)^{\frac{1}{2}}$ (see Table 2.60 in Appendix);

We create four probability models, one for each possibile probability distribution considered in this experiment, and we generate the corresponding scenario tree with increasing scenario cardinality up to 1000. A similar approach has been adopted also for summer demand.
In our computational experiment we use the following values:

- procurement cost $c_i = c_j = 2$;
- stock–out cost $v_i = v_j = 4$;
- time–waste cost $w_i = w_j = 8$;
- uni transshipment cost $t_{ij} = t_{ji} = 1$;
- bike–station capacity $k_i = k_j = 30$;

Note that results reported in the following sections refer to the winter demand. It will be specified when a comparison with the summer demand is proposed.

2.5.2 Computational time, in–sample stability and optimal inventory levels for a two–stage formulation

In this Section we provide the computational times required to solve program P4, study the in–sample stability (Kaut and Wallace (2003) [30]) and describe the optimal quantity to place in each bike–station. We compute well-known stochastic measures in Stochastic Programming, we perform a sensitivity analysis on the cost parameters of the problem and, finally, we test a set of instances.

Computational Time All computations were performed on a machine Intel(R) Core(TM) i7 CPU @ 2.67GHz, overclockable up to 3.2 GHz with 12G RAM.

Table 2.2 and Figure 2.7 show the average computational time over 20 istances required to solve program P4 for an increasing number of scenarios $S = 100, \ldots, 1000$.

Number of Scenarios S	100	200	300	400	500	600	700	800	900	1000
Comp. Time (secs)	8	17	32	85	135	192	275	355	461	597

Table 2.2: Computational time required to solve program P4.

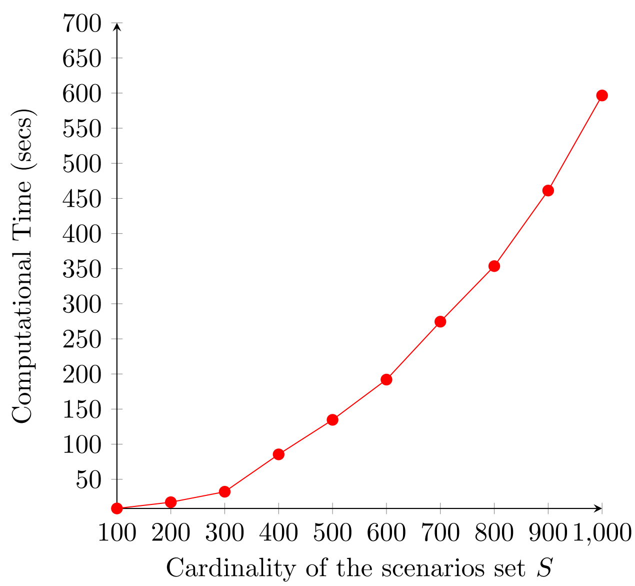


Figure 2.7: Computational time required to solve program P4.

Note that the largest instance considered ($S = 1000$) required on average approximatively 10 minutes.

In–sample stability We denote with $z^{\mathcal{B}}_{\mathcal{U}}(\boldsymbol{x}, \boldsymbol{\xi})$ the expected total cost obtained under the assumption of the Uniform distribution, with $z^{\mathcal{B}}_{\mathcal{E}}(\boldsymbol{x}, \boldsymbol{\xi})$ the one obtained under the assumption of the Exponential distribution, with $z^{\mathcal{B}}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{\xi})$ the one obtained under the assumption of the Normal distribution and with $z^{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{\xi})$ the one obtained under the assumption of the Log–normal distribution.

Table 2.3 and Figure 2.8 show the obtained results.

Scenario Trees	1	2	3	4	5	6	7	8	9	10
Number of Scenarios S	100	200	300	400	500	600	700	800	900	1000
$z^{\mathcal{B}}_{\mathcal{U}}(\boldsymbol{x}, \boldsymbol{\xi})$	488	484	484	483	486	484	484	485	485	484
$z^{\mathcal{B}}_{\mathcal{E}}(\boldsymbol{x}, \boldsymbol{\xi})$	1079	1070	1072	1073	1076	1079	1083	1076	1080	1076
$z^{\mathcal{B}}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{\xi})$	497	496	496	496	497	495	497	496	497	497
$z^{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{\xi})$	484	489	490	489	488	489	489	488	488	487

Table 2.3: In–sample stability for the expected total cost.

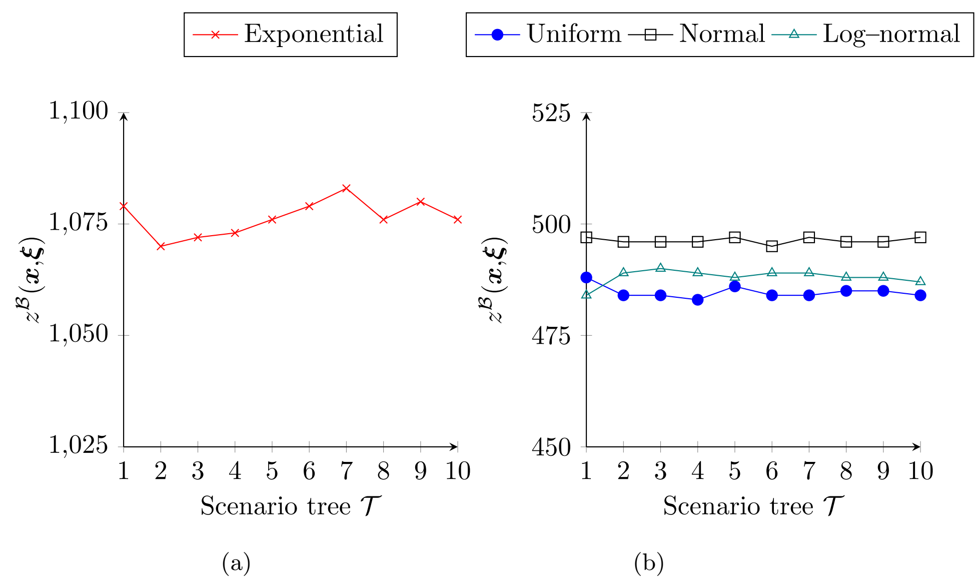


Figure 2.8: In–sample stability for the expected total cost under assumption of Exponential distribution (a) and Uniform, Normal and Log–normal distributions (b).

From Table 2.3 and Figure 2.8 we can observe that the stability performs well after 500 scenarios for the considered probability distributions. The assumption of the Exponential distribution provides the highest expected total costs, while the assumption of the Uniform distribution provides the lowest. This is justified by the previous considerations about standard deviations of Uniform and Exponential distributions proposed in Section 2.5.1 (see Tables 2.62 and 2.63 in the Appendix): using the estimated parameters obtained from real data analysis, the scenario generation produces rental demands with highest variability across the bike–sharing system assuming an Exponential distribution, while the lowest variability is obtained assuming a Uniform distribution. In the following section, we investigate how this fact also affects the choice of the optimal inventory levels.

Optimal quantity to place We now study the optimal number of bicycles x^* to place in each bike–station under different probability distributions. We denote with $x^*_{\mathcal{U}}$ the optimal quantity obtained under the assumption of the Uniform distribution, with $x^*_{\mathcal{E}}$ the optimal quantity obtained under the assumption of the Exponential distribution, with $x^*_{\mathcal{N}}$ the optimal quantity obtained under the assumption of the Normal distribution and with $x^*_{\mathcal{L}}$ the optimal quantity obtained under the assumption of the Log–normal distribution.

Table 2.4 and Figure 2.9 show the obtained results.

Station i	A	B	C	D	E	F	G	H	I	J	K
$x^*_{\mathcal{U}}$	30	0	2	3	3	11	28	3	6	28	11
$x^*_{\mathcal{E}}$	15	3	10	6	14	26	14	18	18	23	20
$x^*_{\mathcal{N}}$	30	0	2	3	3	12	28	3	6	28	11
$x^*_{\mathcal{L}}$	30	0	2	3	3	11	28	3	6	28	11

Station i	L	M	N	O	P	Q	R	S	T	U	V
$x^*_{\mathcal{U}}$	0	2	5	7	16	2	9	12	6	13	6
$x^*_{\mathcal{E}}$	0	8	14	19	23	9	23	27	22	25	18
$x^*_{\mathcal{N}}$	0	2	5	7	16	3	9	12	6	13	6
$x^*_{\mathcal{L}}$	0	2	5	7	16	2	8	12	5	13	5

Table 2.4: Optimal number of bicycles to place x^* in each bike–station i.

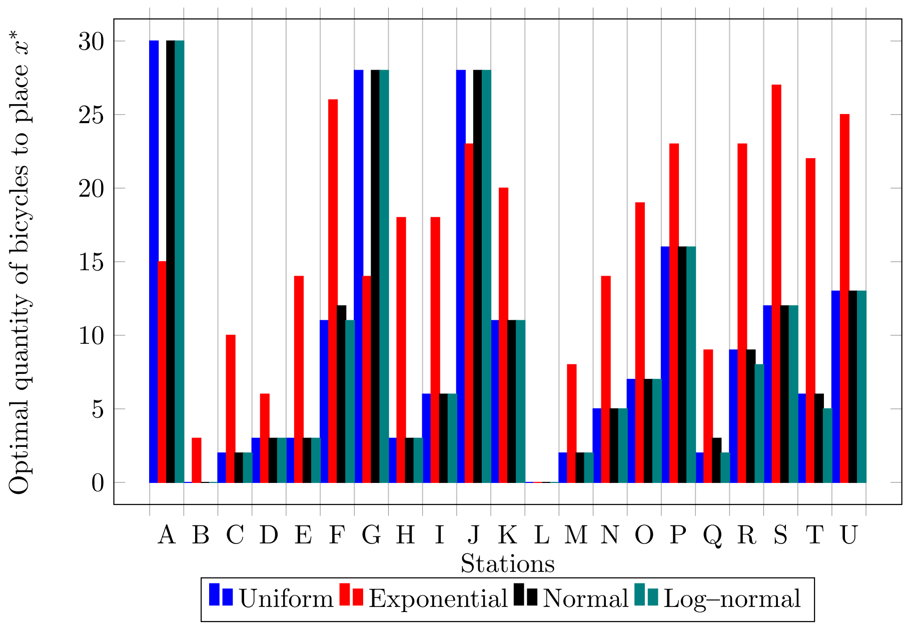


Figure 2.9: Optimal number of bicycles to place in each bike–station i for the winter demand.

From Table 2.4 and Figure 2.9 we note that the optimal number of bicycles to place is similar under the assumption of the Uniform, Normal and Log–normal distributions, while under the assumption of the Exponential distribution we have a different behaviour: when under the formers we place more, under the latter we place less, and viceversa. This is due to the standard deviation obtained assuming an Exponential distribution (compare Tables 2.61 – 2.63 in Appendix): as the expected value of the stochastic demand decreses, the standard deviation of the Exponential distribution increases (recalling that $\sigma_{ij}^{\mathcal{E}} = \lambda_{ij} = \bar{\xi}_{ij}^{-1}$). When the expected value $\bar{\xi}_{ij}$ is close to 0, the standard deviation of the Exponential distribution is sensibly large compared to the ones obtained the other three distributions. Note that the optimal inventory level of station L is 0 for all the distributions because it is the last activated station (Ospedale Papa Giovanni, December 2015): we have no historical available data, except for December, and demand is low.

The results proposed in Table 2.4 and Figure 2.9 use the origin and destination sets which are given from the current bike–sharing system in Bergamo. We use the network, and the usage data, as they are, without further investigating on why some bike–stations have been located in a particular area. This is not our concern. Our concern is to determine the optimal number to place in each bike–stations accordingly to the usage data we have. If for a bike–stations the optimal number of bicycles is 0 or approximatively 0, this can be due by the following reasons:

- demand has been badly estimated during the strategic phase, when bike–stations had to be located on the urban territory;
- the available usage data are not enough to determine correctly the optimal number of bicycles to place (the case of station L);
- Bergamo is a clustered city, with a small but highly dense–populated city center with satellite low populated areas around it. Clearly, demands in the satellites are sensibly small compared to the ones in the city center, and it may results, expecially during the summer or winter peaks, that a small number of bicycles is required in the satellite stations in order to satisfy demands (stations B, D, E, H, L, Q).

Figure 2.10 shows the optimal number of bicycles that is placed on average across the bike–sharing system under the assumption of each distribution, *i.e.*

$$\breve{x}^* = \frac{\sum_{i=1}^{B} x_i^*}{B}$$

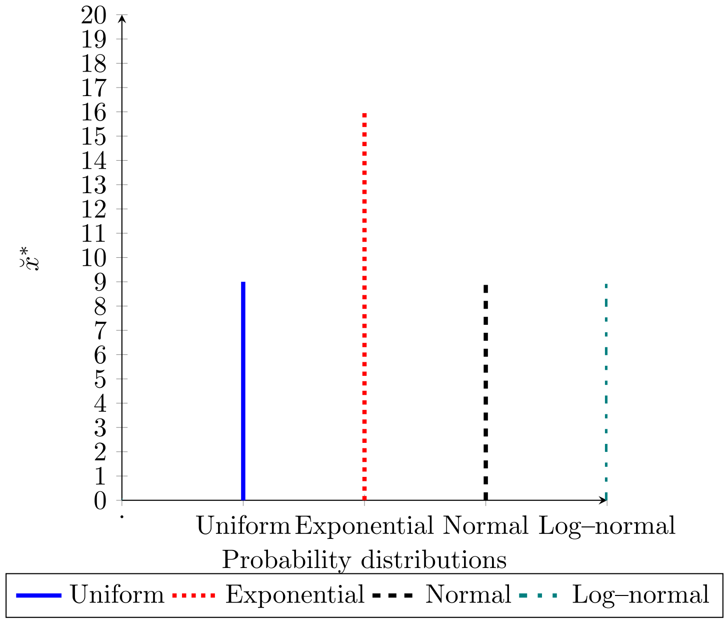


Figure 2.10: Average placed quantity $\breve{x}^* = \frac{\sum_{i=1}^{B} x_i^*}{B}$.

We observe that under the assumption of the Uniform, Normal and Log–normal distributions the average number of bicycles is 9, while under the assumption of the Exponential distribution the model is more prudential placing more bicycles. Once again, this result is justify by the different standard deviation we get assuming the four considered distributions. Note that, independently of the assumed distribution among Uniform, Normal and Log–normal, we place on average the same quantity and we expect to be not penalized heavyly if we mismatch among Uniform, Normal and Log–normal distributions.

We now consider the summer demand. Table 2.5 shows the expected total costs considering 500 scenarios while Table 2.6 and Figure 2.11 give the optimal number of bicycles to place.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
$z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi})$	492	1164	500	490

Table 2.5: Expected total cost under under assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$.

Station i	A	B	C	D	E	F	G	H	I	J	K
$x^*_{\mathcal{U}}$	27	1	2	1	2	12	26	4	6	27	12
$x^*_{\mathcal{E}}$	18	17	6	5	16	26	25	19	24	24	21
$x^*_{\mathcal{N}}$	27	1	2	1	3	12	26	4	6	27	12
$x^*_{\mathcal{L}}$	27	1	2	1	2	12	26	4	6	27	11

Station i	L	M	N	O	P	Q	R	S	T	U	V
$x^*_{\mathcal{U}}$	0	2	7	8	17	0	8	12	11	11	6
$x^*_{\mathcal{E}}$	0	19	22	21	24	2	24	20	26	22	21
$x^*_{\mathcal{N}}$	0	2	7	8	17	0	8	13	11	11	6
$x^*_{\mathcal{L}}$	0	2	6	8	17	0	8	13	10	11	6

Table 2.6: Optimal number of bicycles to place x^* in each bike–station i.

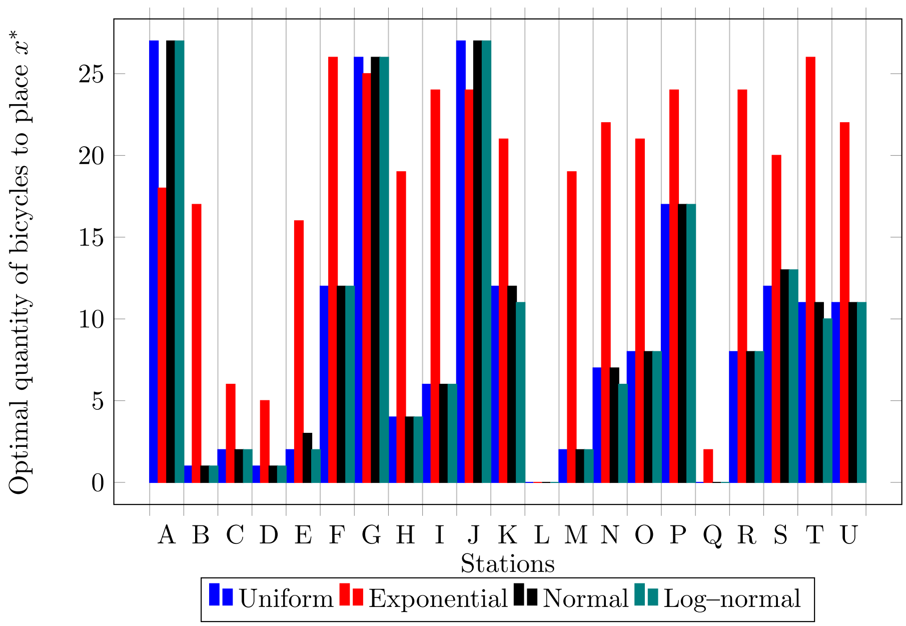


Figure 2.11: Optimal number of bicycles to place x^* in each bike–station i for the summer demand.

From Tables 2.3 and 2.5 we observe that expected total costs of winter and summer demands are similar, as also for the optimal number of bicycles to place (Tables 2.4 and 2.6 and Figures 2.9 and 2.11). This results could be justified by the compensation between warm months (June, July and August) and cold months (December, january and Fabruary) for which other transportation modes are preferred. Moreover, during the remaining months of the year, the bike–sharing service is used without particoular difference among warm and cold season. For this reason, one could further consider a single demand, without discriminating between summer and winter.

2.5.3 Stochastic measures in Stochastic Programming

In this section we provide numerical results of well–known stochastic measures in Stochastic Programming (Maggioni et al. (2014) [39], Maggioni and Pflug (2016) [42]).

We start from the Expected Result of using the Rxpected Value Solution (EEV) (Birge and Louveaux (2011) [11]). In order to compute the EEV, we first consider the following problem where all the stochastic rental demands ξ_{ijs} are replaced with their expected value $\bar{\xi}_{ij}$:

$$EV = \min_{\boldsymbol{x}} z^{\mathcal{N}}(\boldsymbol{x}, \bar{\boldsymbol{\xi}}),$$

from which we obtain an optimal first–stage solution $\bar{x}_i(\bar{\xi}_{ij}), \forall i$, called the *Expected Value Solution*. Table 2.7 shows the expected number of bicycles placed in each bike–station.

Station i	A	B	C	D	E	F	G	H	I	J	K
$\bar{x}_i(\bar{\xi})$	21	0	1	2	1	2	22	1	0	19	4

Station i	L	M	N	O	P	Q	R	S	T	U	V
$\bar{x}_i(\bar{\xi})$	0	1	2	2	10	0	3	5	1	6	2

Table 2.7: Expected Value Solution $\bar{x}_i(\bar{\xi})$.

We now insert the first–stage expected value solution in the original problem (the recourse problem RP) by assuming the 4 possible probability distributions (Uniform, Exponential, Normal and Log–normal) for the stochastic rental demands (Maggioni et al. (2014) [38]). We obtain:

$$EEV_{\mathcal{U}} = \min_{\boldsymbol{x}} \mathbb{E}_{\boldsymbol{\xi}}[z^{\mathcal{B}}_{\mathcal{U}}(\bar{x}_i(\bar{\xi}), \boldsymbol{\xi})],$$
$$EEV_{\mathcal{E}} = \min_{\boldsymbol{x}} \mathbb{E}_{\boldsymbol{\xi}}[z^{\mathcal{B}}_{\mathcal{E}}(\bar{x}_i(\bar{\xi}), \boldsymbol{\xi})],$$
$$EEV_{\mathcal{N}} = \min_{\boldsymbol{x}} \mathbb{E}_{\boldsymbol{\xi}}[z^{\mathcal{B}}_{\mathcal{N}}(\bar{x}_i(\bar{\xi}), \boldsymbol{\xi})],$$
$$EEV_{\mathcal{L}} = \min_{\boldsymbol{x}} \mathbb{E}_{\boldsymbol{\xi}}[z^{\mathcal{B}}_{\mathcal{L}}(\bar{x}_i(\bar{\xi}), \boldsymbol{\xi})].$$

Table 2.8 shows the expected values for using the expected value solutions under the assumption of the Uniform, Exponential, Normal and Log–normal distributions.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
EEV	631	1482	641	628

Table 2.8: Expected results for using the Expected Value Solution under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

We now compute the Value of the Stochastic Solution (VSS) (Birge (1982) [10]), which is the gain from solving the two–stage integer stochastic program:

$$VSS = EEV - RP.$$

Table 2.9 shows the Values of the Stochastic Solutions obtained under the four distributions and the corresponding penalty (%), computed as $\frac{VSS}{RP} \times 100$.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
VSS	145	406	144	140
%	29.8	37.7	29	28.7

Table 2.9: Value of the Stochastic Solution under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

From Table 2.9 we note that adopting the Expected Value Solution we have the worst performance with the Exponential distribution. This result is again explained by the standard deviation (compare Table 2.63 in Appendix) we get assuming the Exponential distribution.

Let us now measure the *Wait–and–See* solution (WS) (Birge and Louveaux (2011) [11]) obtained by solving a sequence of deterministic problems obtained by relaxing the non–anticipativity constraints and taking the expectation:

$$WS = \mathbb{E}_{\boldsymbol{\xi}}[\min_{\boldsymbol{x}} z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi})].$$

Table 2.10 shows the Wait–and–See solution under the assumption of the four distributions.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
WS	449	931	454	443

Table 2.10: Wait–and–See solutions under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

The difference between the optimal values of the here–and–now solution and the wait–and–see solution

$$EVPI = RP - WS,$$

is referred to as the *Expected Value of Perfect Information*. Table 2.11 reports this value for the four probability distributions and the corresponding gain (%), computed as $\frac{WS}{RP} \times 100$.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
EVPI	37	145	43	45
%	7.6	13.4	8.7	9.2

Table 2.11: Expected Value of Perfect Information under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

From Table 2.11 we understand that knowing in advance the information about the future realization of the demand is valuable for all the considered distributions, expecially in the case of the Exponential due to its greater variability described by the higher standard deviation. However, for this case study, at the moment, it is not possible to have in advance the information about the future rental demands. This is a technical issue: the software provided alongside the bike–sharing service does not queries the users about their future movements. From Table 2.7 we can observe that in the Expected Value Solution some first-stage variables are set to zero. Let $\mathcal{J}$ be the set of indices for which the components of the expected value solution $\bar{x}_i(\bar{\xi})$ are at zero. Then let $\hat{x}$ be the solution of:

$$\min_x \mathbb{E}_{\boldsymbol{\xi}}[z^{\mathcal{B}}(x, \boldsymbol{\xi})],$$
$$\text{s.t.} \quad x_j = \bar{x}_j(\bar{\xi}), \qquad j \in \mathcal{J}.$$

We then compute the *Expected Skeleton Solution Value* (Maggioni and Wallace (2012) [43])

$$ESSV = \mathbb{E}_{\boldsymbol{\xi}}[z^{\mathcal{B}}(\hat{x}, \boldsymbol{\xi})].$$

Table 2.12 show it under the assumption of the Uniform, Normal, Log–normal and Exponential distributions.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
ESSV	502	1132	511	504

Table 2.12: Expected Skeleton Solution Value under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

We now compare results obtained in Table 2.12 with the here-and-now solution and we measure the *Loss of Using the Skeleton Solution*:

$$LUSS = ESSV - RP.$$

Table 2.13 reports the obtained results and shows the penalty (%), computed as $\frac{LUSS}{RP} \times 100$, for having used the skeleton solution.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
LUSS	16	56	14	66
%	3.3	5.2	2.8	3.3

Table 2.13: Loss Using the Skeleton Solution under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

From Table 2.13 we observe that the LUSS is positive under the four considered distributions but not large. We try to investigate the reason of this behaviour. The deterministic model produced the right non–zero variables $\hat{x}_i$,

but set to be 0 some variables that are positive in the here–and–now solution (compare stations B,I,L,Q in Table 2.4 and Table 2.7). Since in the here–and–now solution the optimal inventory level, depending on the assumed distribution, is close to 0 or 0 for stations B, L and Q, the resulting LUSS is not large, but still positive. This could justify a better understanding of the Expected Value Solution and the possible links between the deterministic and stochastic ones by using the *Generalized Loss Using the Skeleton Solution* (*GLUSS*) (Maggioni et al. (2015) [40]): a measure of the badness/goodness of deterministic solution based on the information brought by the reduced costs of the continuous relaxation of the deterministic one.
We now consider the Expected Value Solution $\bar{x}_i(\bar{\xi})$, $\forall i$, as a starting point (input) to (2.32) of program P4 and we compare, in terms of objective functions, to (2.32) without such input. This is equivalent to adding in program P4 constraints $x_i \geq \bar{x}_i(\bar{\xi}), \forall i$ and hence solve the following problem with solution $\tilde{x}$:

$$\begin{aligned} &\min_x \mathbb{E}_{\boldsymbol{\xi}}[z^{\mathcal{B}}(x, \boldsymbol{\xi})], \\ &\text{s.t.} \;\; x_i \geq \bar{x}_i(\bar{\xi}), \qquad \forall i. \end{aligned}$$

We then compute the *Expected Input Value* (Maggioni and Wallace (2012) [43])

$$EIV = \mathbb{E}_{\boldsymbol{\xi}}[z^{\mathcal{B}}(\tilde{x}, \boldsymbol{\xi})].$$

Table 2.14 shows the Expected Input Values under the four considered distributions.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
EIV	486	1088	497	489

Table 2.14: Expected Input Value under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

We now compare results obtained in Table 2.14 with the stochastic solution proposed in Table 2.3, measuring the *Loss of Upgrading the Deterministic*

Solution:

$$LUDS = EIV - RP.$$

Table 2.15 reports the obtained results and shows the penalty (%), computed as $\frac{LUDS}{RP} \times 100$, for having upgraded the deterministic solution.

Distribution	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
LUDS	0	12	0	1
%	0	1	0	0

Table 2.15: Loss of Upgrading the Skeleton Solution under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

From Table 2.15 we observe that the Expected Value Solution is upgradeable to become good in the stochastic setting for all the four considered distributions, expecially under the assumption of the Uniform, Exponential and Log–normal distributions.

Recall now the concepts of *guessed* and *right* distributions and the associated measures defined in Chapter 1. We now insert an optimal solution $\boldsymbol{x}^*_{\mathcal{G}}$, obtained assuming a guessed distribution, inside the recourse problem assuming the right distribution ($\text{RP}_{\mathcal{R}}$). We do it for the four probability distributions (*e.g.*, the Uniform distribution is considered the right one and we compute the expected total cost by using the optimal solutions obtained assuming the Exponential, Normal and Log–normal distributions, respectively). Then, we compute the Out–of–Distribution value (OD), which is the value for having fixed the optimal first–stage solution obtained assuming a wrong distribution in the recourse problem with the right distribution. Table 2.16 reports the obtained Out-of–Distribution values.

	$\mathcal{G}$			
OD	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
$(\mathcal{R}=\mathcal{U})$	-	877	486	487
$(\mathcal{R}=\mathcal{E})$	1348	-	1344	1355
$(\mathcal{R}=\mathcal{N})$	495	877	-	496
$(\mathcal{R}=\mathcal{L})$	489	879	490	-

Table 2.16: Out–of–Distribution values under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

We then compute the *Value of the Right Distribution* (VRD),

$$VRD = OD - RP_{\mathcal{R}},$$

which is the loss from solving the two–stage integer stochastic program assuming a wrong distribution and a measure of the importance of ambiguity in stochastic optimization.
Table 2.17 reports the obtained Values of the Right Distribution.

	$\mathcal{G}$			
VRD	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
$(\mathcal{R}=\mathcal{U})$	-	391	0	1
$(\mathcal{R}=\mathcal{E})$	272	-	268	279
$(\mathcal{R}=\mathcal{N})$	0	380	-	0
$(\mathcal{R}=\mathcal{L})$	1	391	2	-

Table 2.17: Value of the Right Distribution under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

From Table 2.17 we observe that the ambiguity plays an important role only when the Exponential distribution is considered as possible candidate to be both right and wrong: the penalties for having assumed the Exponential as right distribution or the penalty for having assumed right another distribution while the right one was the Exponential, are very large. The VRD is negligible for all the other possible mismatches between distributions. These results are justified with the following reasons:

1. the optimal first–stage solutions obtained assuming the Uniform, Normal and Log–normal distributions are similar (see Table 2.4);
2. the average number of placed bicycles in the bike–sharing system is the same assuming the Uniform, Normal and Log–normal distributions;
3. the standard deviation of the Exponential distributions is sensibly large with respect to the ones of the other three distributions;
4. the Deviation Test proposed in Section 1.6.5. of Chapter 1 shows that the issue is not the Exponential distribution itself but its standard deviation which is very far from the standard deviations of the other probability distributions (which are similar).

The uncertainty of the future, described by the standard deviation of the assumed distributions for the stochastic process, plays a crucial role for determining the Value of the Right Distribution.

2.5.4 Sensitivity Analysis

In this section we perform a sensitivity analysis on the stock–out cost v, the time–waste cost w and the transshipment cost t to understand how the expected total cost function is sensible to a variation in cost parameters. As before, we consider different probability distributions.

Stock–out cost The initial setting of the instance is the following:

- $c = 2$;
- $s = 5$;
- $w = 25$;
- $t = 1$.

We perform a sensitivity analysis on $s = 5, \ldots, 55$. We then compute the expected total cost function for each instance. Figure 2.12 shows the obtained results.

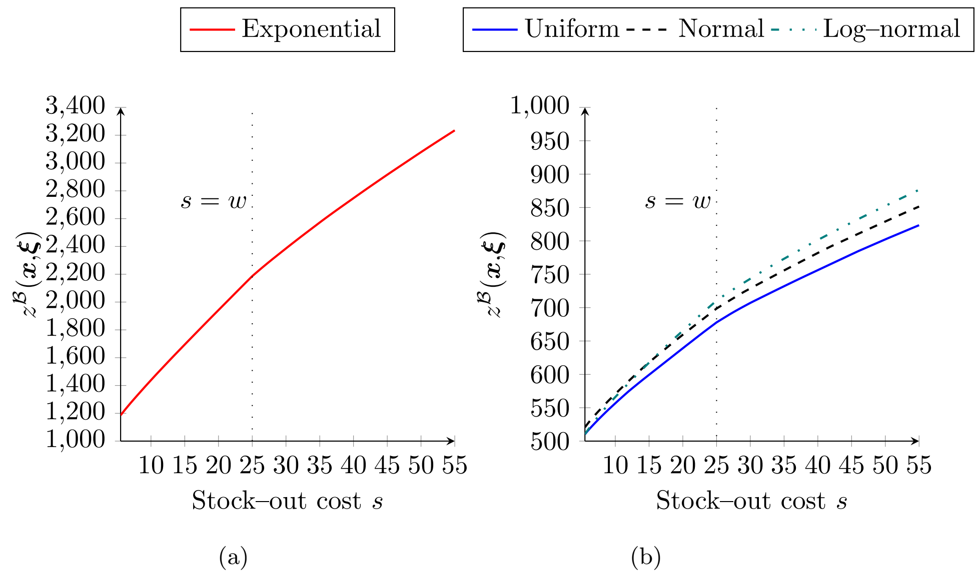


Figure 2.12: Sensitivity analysis of expected total cost versus stock–out cost $s = 5, \ldots, 55$. (a) refers to the Exponential distribution while (b) refers to the Uniform, Normal and Log–normal distributions.

From Figure 2.12 we can observe that when $s = w$, the slopes of the expected total cost functions, under the assumption of the four considered distributions, change. Note that when $s = w$, expected shortage and expected overflow are equally paid. Expected total cost are monotonitically increasing.

Time–waste cost The initial setting of the instance is the following:

- $c = 2$;
- $s = 25$;
- $w = 5$;
- $t = 1$.

We perform a sensitivity analysis on $w = 5, \ldots, 55$. We then compute the expected total cost function for each instance. Figure 2.13 shows the obtained results.

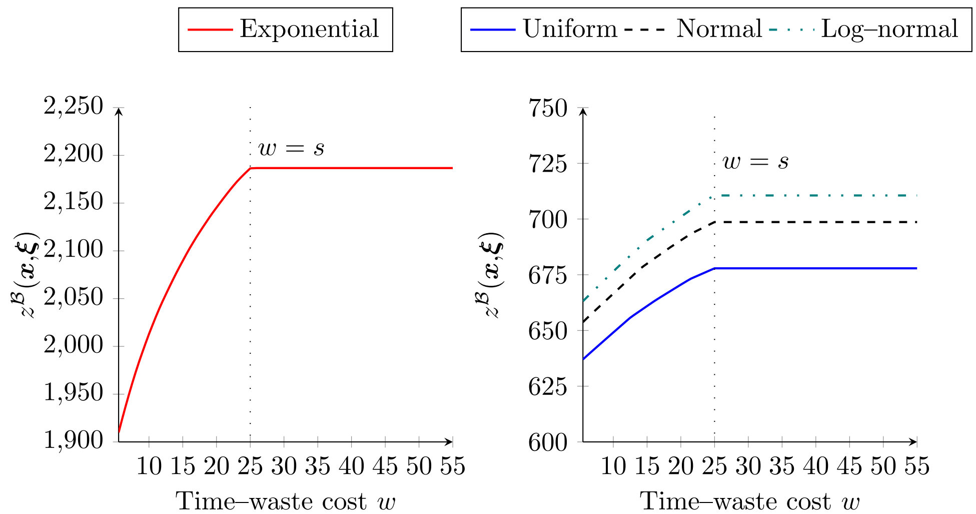


Figure 2.13: Sensitivity analysis of expected total cost versus time–waste cost $w = 5, \ldots, 55$. (a) refers to the Exponential distribution while (b) refers to the Uniform, Normal and Log–normal distributions.

From Figure 2.13 we can observe that for $w \geq s$, the expected total cost function, under the four considered distributions, become constant. This result can be explained as follows. Suppose an extreme case, for some $w \gg s$, such that a single unit of overflows costs more than full shortages of all stations, *i.e.* $I^-_{is} = \xi_{is}$, $\forall i \in \mathcal{B}$ and $\forall s \in \mathcal{S}$: the optimal first–stage decision is to leave the bike–stations empty, in order to reduce the probability of an overflow to 0. We understand that the optimal solution and the resulting expected total cost depend also on the link between rental demands and available capacities.

Transshipment Cost We test two different initial settings to measure the sensitivity of the expected total cost function. The initial setting of the first instance is the following:

- $c = 2$;
- $s = 5$;
- $w = 25$;

- $t = 1$.

We perform a sensitivity analysis on $t = 1, \dots, 51$. We then compute the expected total cost function for each instance. Figure 2.14 shows the obtained results.

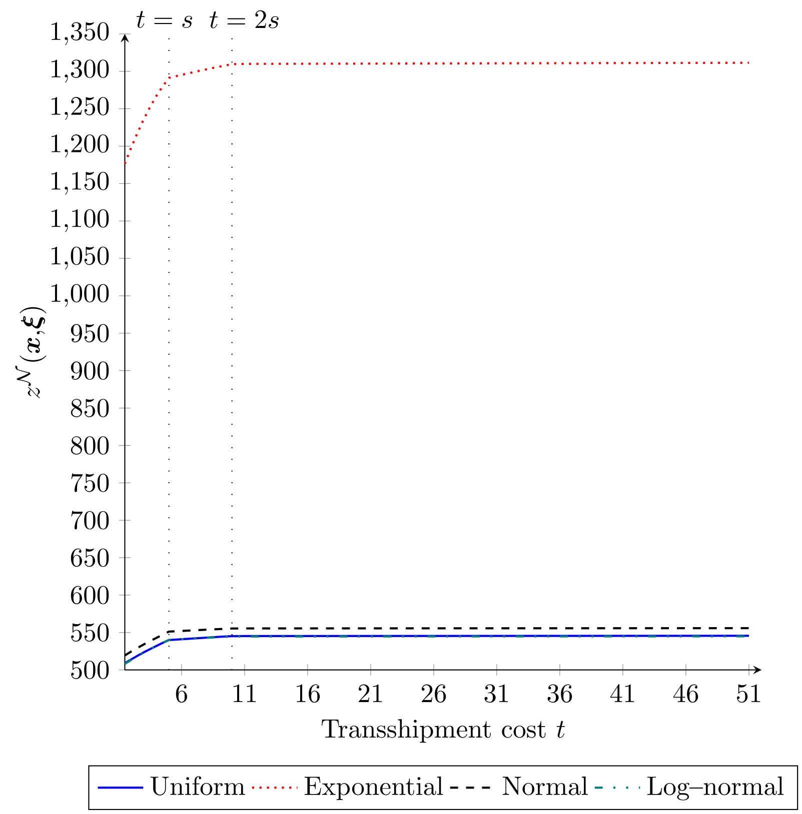


Figure 2.14: Sensitivity analysis of expected total cost versus transshipment cost $t = 1, \dots, 51$ with $s = 25$ and $w = 5$.

From Figure 2.14 we observe that the slope of the expected total cost function changes initially at $t = s$ and then at $t = 2s$ under the assumption of the four considered distributions. For $t \geq 2s$, the expected total cost is constant. This result can be explained similarly to the time–waste cost. Suppose an extreme case, for some $t \gg s+w$, such that a single unit of transshipment costs more than shortages plus overflows, or full shortages, or full overflows (this strictly depends on the values of the cost parameters): the optimal first–stage decision is to leave the bike–stations empty, in order to reduce the probability of transshipment to 0 (note that we tested our instances using program P4, were the transshipment is mandatory. If it is not, cleary an optimal decision is to perform not the transshipment).

We understand that the optimal solution and the resulting expected total cost also depend on the distributions the rental demands across the bike–sharing system.

The initial setting of the second instance is the following:

- $c = 2$;
- $s = 25$;
- $w = 5$;
- $t = 1$.

We perform a sensitivity analysis on $t = 1, \ldots, 51$. We then compute the expected total cost function for each instance. Figure 2.15 shows the obtained results.

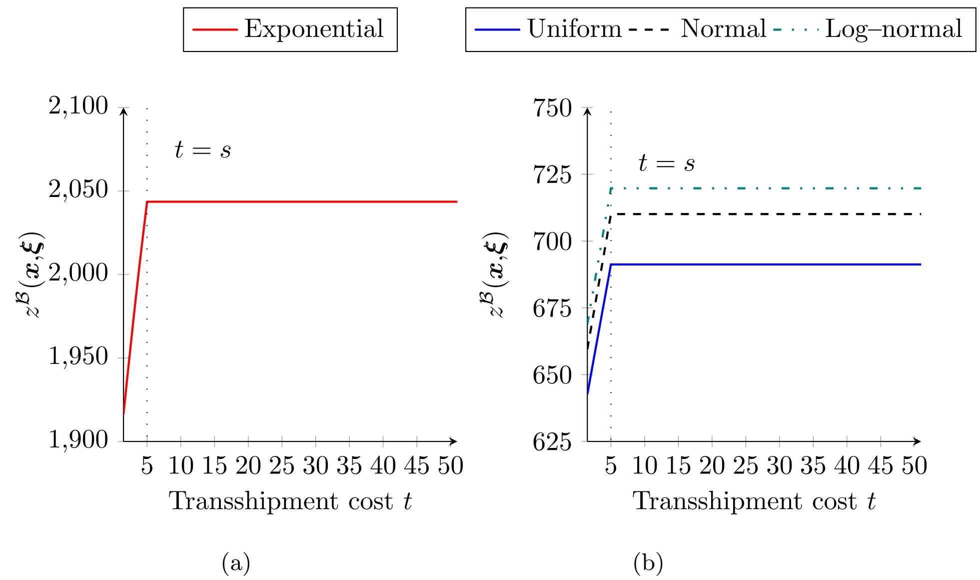


Figure 2.15: Sensitivity analysis of expected total cost versus transshipment cost $t = 1, \ldots, 51$ with $s = 5$ and $w = 25$. (a) refers to the Exponential distribution, while (b) refers to the Uniform, Normal and Log–normal distributions.

From Figure 2.15 we can observe that for $t \geq s$, the expected total cost function, under the four considered distributions, becomes constant. This result can be explained as follows. Suppose an extreme case, for some $t \gg s$, such that a single unit of transshipment costs more than shortages: the optimal first–stage decision would be to leave the bike–stations empty, in order to reduce the probability of transshipment to 0. In our experiment this is not due to the same reason of the previous one, since the cost parameter under investigation is the same. However, we understand something more: the transshipment is linked *only* with the shortage and *not* with the overflow. This result is justified as follows. The overflow is redirected by user itself: it is a form of user–transshipment which realizes independently of the value to which the cost parameter of the transshipment is setted. On the contrary, transshipment has the goal to balance the system at the end of the service accordingly to the optimal quantities needed for the next one in order to reduce the shortage of the next service.

2.5.5 Testing other instances

It is impossible to investigate every combination in the values of the cost parameters of the problem. In this section, we test 8 instances given by a combination of the cost parameters c, s, w and t. We identify each instance with a number from 1 to 8 and we set the values for the parameters of the first instance with $c = s = w = t = 5$.

Tables 2.18 reports the results obtained for the expected total costs under the Uniform, Exponential, Normal and Log–normal distributions. Table 2.19 reports the optimal number of bicycles to place in bike–stations A, B, J and P depending on the assumed distributions. We have chosen to report 4 bike–stations over 22, selecting those with the highest rentals demands, thus with the highest inventory levels.

$z^{\mathcal{B}}(\boldsymbol{x}, \boldsymbol{\xi})$					
Label	Parameters	$\mathcal{U}$	$\mathcal{E}$	$\mathcal{N}$	$\mathcal{L}$
1	$c = s = w = t$	1057	2015	1068	1036
2	$c = w = t, s = 5c$	1400	3544	1436	1424
3	$c = s = t, w = 5c$	1057	2015	1068	1037
4	$c = s = w, t = 5c$	1057	2015	1068	1037
5	$s = 2c, w = 3c, t = \frac{1}{5}c$	1207	2637	1228	1198
6	$s = 3c, w = 2c, t = \frac{1}{5}c$	1270	2953	1298	1274
7	$s = 2t, w = 3t, c = \frac{1}{5}t$	382	1215	395	396
8	$s = 3t, w = 2t, c = \frac{1}{5}t$	412	1398	425	432

Table 2.18: Expected total cost for a given combination of cost parameters under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

	x^*															
	$\mathcal{U}$				$\mathcal{E}$				$\mathcal{N}$				$\mathcal{L}$			
instance	A	G	J	P	A	G	J	P	A	G	J	P	A	G	J	P
1	18	17	17	8	4	3	6	6	16	17	14	9	4	4	3	4
2	30	30	30	18	19	20	30	30	30	30	30	19	30	30	30	18
3	10	11	12	9	4	3	6	4	8	6	6	4	10	13	14	9
4	12	15	15	9	4	3	6	6	11	13	14	9	12	18	16	9
5	30	28	29	17	16	15	24	25	30	28	28	17	30	28	28	16
6	30	28	30	18	18	17	28	28	30	29	29	18	30	29	29	17
7	30	29	30	19	19	18	28	28	30	30	29	19	30	29	29	18
8	30	30	30	19	21	22	30	30	30	30	30	20	30	30	30	19

Table 2.19: Optimal number of bicycles to place in bike–stations A,G, J and P for a given combination of cost parameters under the assumption of Uniform $\mathcal{U}$, Exponential $\mathcal{E}$, Normal $\mathcal{N}$ and Log–normal $\mathcal{L}$ probability distributions.

From Table 2.18 and 2.19 we observe that the optimal first–stage decisions and the corresponding expected total cost vary depending on the relationship between the cost parameters of the problem. The obtained results enforce the fact that the optimal inventory levels are dependent from the ratio between c, s, w and t, with this analysis we also understand that $\boldsymbol{x}^*$ also depends on the distribution of the demands across the set of bike–stations and of their capacity. In particular, in Table 2.19 for instance 3, the optimal invetory levels to the minimum realization of the demand in station A, G, J and P according to the scenarios on which program P4 is solved, while instance 8 sets them to the maximum realization (or available capacity).

2.5.6 The multi–stage case

In this Section we study the in–sample stability (Kaut and Wallace (2003) [30]) and we describe the optimal quantity to place in each bike–station of the multi–stage formulation of program P4. We split each day (24 hours) in three time periods, defined as follows:

- time period 1: from 6 am to 12 am;

- time period 2: from 12 am to 6 pm;
- time period 3: from 6pm to 00 am.

The time period that goes from 00 am to 6 am is reserved to transshipment operations. Even if time is not explicitly included in the model (due to the node formulation), we denote by T the number of stages and introduce a time index $t = 0, 1, \ldots, T$, with $t = 0$ corresponding to the root of the scenario tree.

In–sample stability We tested the program on different scenario trees, for an increasing number of scenarios, constructed as shown in Table 2.20. The branching at each node is constant and corresponds to the number of nodes of the second stage for each scenario tree.

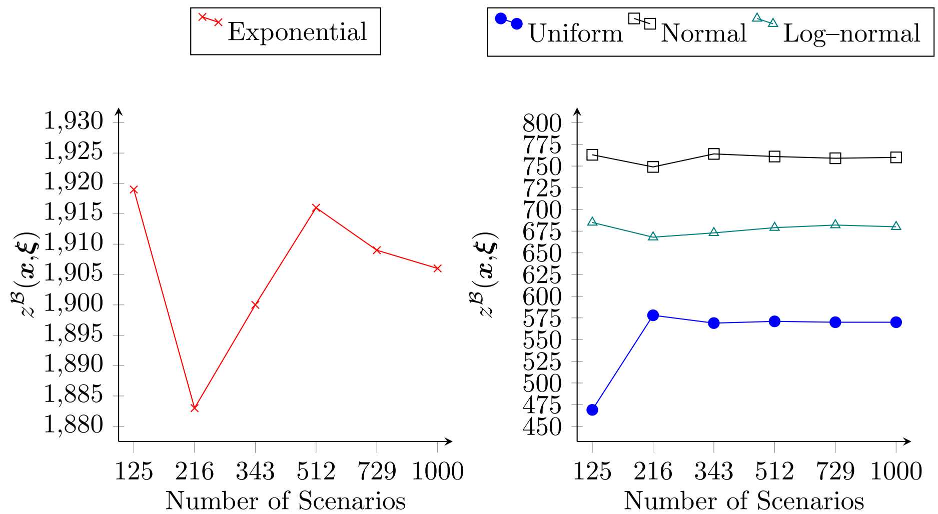
Scenario Trees	1	2	3	4	5	6
Nodes of the second stage	5	6	7	8	9	10
Total nodes	155	258	399	589	819	1110
Leafs (scenarios)	125	216	343	512	729	1000

Table 2.20: Structure of the scenario trees adopted for testing the multi–stage formulation of program P4.

Table 2.21 and Figure 2.16 show the obtained results.

Scenario Trees	1	2	3	4	5	6
$z^{\mathcal{B}}_{\mathcal{U}}(\boldsymbol{x}, \boldsymbol{\xi})$	469	578	569	571	570	570
$z^{\mathcal{B}}_{\mathcal{E}}(\boldsymbol{x}, \boldsymbol{\xi})$	1919	1883	1900	1916	1909	1906
$z^{\mathcal{B}}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{\xi})$	763	749	764	761	759	760
$z^{\mathcal{B}}_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{\xi})$	685	668	673	679	682	680

Table 2.21: In–sample stability for the expected total cost.



(a) (b)

Figure 2.16: In–sample stability for the expected total cost the under assumption of Exponential distribution (a) and under assumption of the Uniform, Normal and Log–normal distributions (b) probability distributions.

From Table 2.3 and Figure 2.8 we can observe that in–sample stability performs better in the two–stage formulation, but the expected total costs appear to be stable even in the multi–stage formulation. Under the assumption of the Exponential distribution a larger number of scenarios should be considered to get in–sample stability. We set the scenario tree $\mathcal{T}^* = 4$, which corresponds to 512 scenarios. The assumption of the Exponential distribution provides again the highest expected total costs, while the assumption of the Uniform distribution provides the lowest. This behaviour is justified by the standard deviations of the distributions as for the two–stage model formulation (see Tables 2.61 – 2.63 in Appendix).

Optimal quantity to place We now study the optimal number of bicycles x^* to place at the beginning of the service in each bike–station, which corresponds to node 0 (the root of the scenario tree) under different assumption on the demand probability distribution. Table 2.22 and Figure 2.17 show the obtained results.

Station i	A	B	C	D	E	F	G	H	I	J	K
$x^*_{\mathcal{U}}$	10	0	1	1	1	2	8	1	2	8	2
$x^*_{\mathcal{E}}$	12	0	1	2	1	4	4	1	3	14	3
$x^*_{\mathcal{N}}$	11	0	1	1	2	3	10	2	1	9	3
$x^*_{\mathcal{L}}$	10	0	1	2	2	2	8	1	2	7	2

Station i	L	M	N	O	P	Q	R	S	T	U	V
$x^*_{\mathcal{U}}$	0	1	1	1	5	0	2	3	1	3	1
$x^*_{\mathcal{E}}$	0	2	3	3	4	0	2	4	3	4	2
$x^*_{\mathcal{N}}$	0	0	1	1	5	1	2	4	2	4	1
$x^*_{\mathcal{L}}$	0	1	1	1	3	0	1	2	1	2	0

Table 2.22: Optimal number of bicycles to place x^* in each bike–station i.

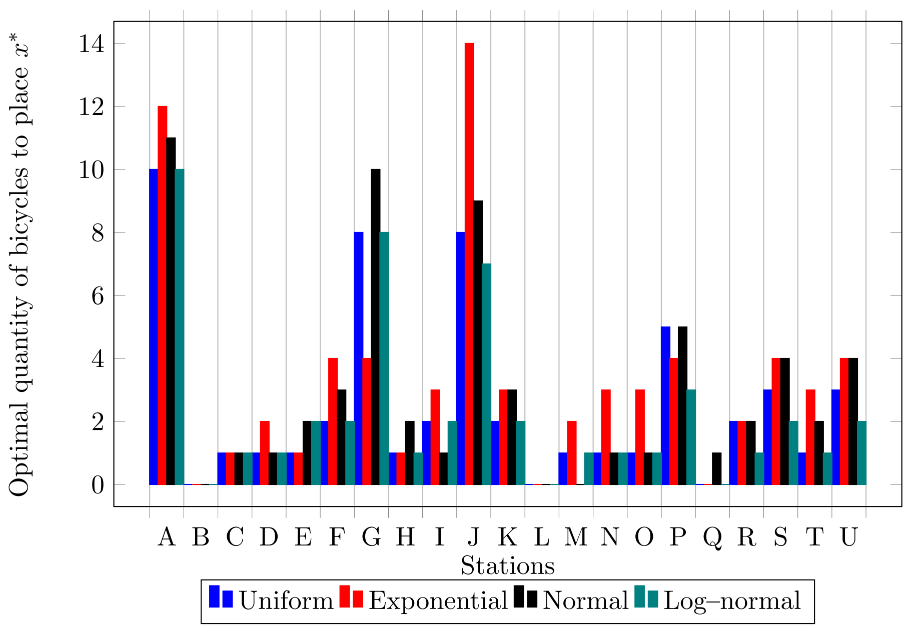


Figure 2.17: Optimal number of bicycles to place x^* in each bike–station i in the multi–stage case.

From Table 2.4 and Figure 2.9 we observe that the optimal quantity to place for all bike–stations are lower than the ones of the two–stage model formulation proposed in Table 2.22 and Figure 2.17. This result in justied as follows: the optimal solution of the multi–stage formulation is less myopic compared to the one of the two–stage in the sense that bicycles assigned at the beginning of the service can be rented in the three time periods. The program considers that a single bicycle can be used multiple times. For example, a user rent a bicycles at 09:00 am from station B to get to station C at 09:15 am. Then, a new user rent the same bicycle at 09:20 to get to station F at 09:50. In the afternoon, another user rent the same bicycles to ge to station G at 16:00. Thus, the inventory levels are reduced according to the ability of the multi–stage formulation to consider not only the possible realizations of 'tomorrow', but also of the subsequent futures and the dynamic nature of the fleet. We observe at last that the optimal inventory levels obtained under the assumption of the Exponential distribution are sized equal to the ones obtained assuming the other distributions, differently from the two–stage formulation.

2.6 Conclusions

In Chapter 2 we have proposed two–stage and multi–stage stochastic formulations for a bike–sharing problem and studied the real case of the bike–sharing system in Bergamo 'La BiGi'. Optimal solutions for the two–stage and multi–stage models have been provided under different assumptions of probability distributions for the rental demand for each origin–destination pair. It is given to the provider the possibility to rebalance the inventory level in each bike–station by transshipping a certain number of bicycles from a bike–station to another according to the optimal inventory levels required at a bike–stations. The obtained results lead to the following managerial insights:

1. **Why a recourse action?** A recourse action, identified in the transshipment of bicycles between bike–stations, is required in order to match the optimal inventory levels at the beginning of the next service or in a certain period of the day. In our case study, the recourse action is taken at night after the realizations of the rental demands and when the service is not available for the user community. However, a recourse action could be taken during the service at the end of a certain period of the day (e.g., noon or end of the working time), by using a multi–stage stochastic formulation where the daily rental demand is decomposed according to certain periods (e.g. morning, afternoon, evening). In our case study, the recourse is mandatory for the optimization of the service in order to edge against the fluctation of the inventory level of each bike–station which can lead to a level far from the optimal number of required bicycles. In such case, if no action is taken, even after a small interval of time after the beginning of the service, a bike–station may come up in a shortage (if the inventory level is far below its optimum) or an overflow (if the inventory level is far below).

2. **Two–stage vs. Multi–stage.** A two–stage stochastic formulation consider the entire day as a single period, only one optimal inventory level is

determined for each bike–station and the optimal number of bicycles to have at the end of the service must be the same to have at the beginning of the next day. The model does not consider inventory levels during the day, increasing the probability to incure in a shortage of bicycles at some bike–stations. A multi–stage stochastic formulation is proposed by splitting the running day into three periods. As result, more than one optimal inventory levels to have at the end of each period are determined for each bike–station. In our case–study, the recourse action is available only at the end of the service, but the provider could adopt it more than once during the day in order to match the optimal inventory levels to have at the end of each time period. Moreover, a multi–stage formulation catches the movement of the bicycles fleet among the bike–stations in different instants of time during the day, considering in the determination of the optimal inventory levels the incoming and outcoming bicycles.

3. **One or more recourse actions?** Bike–sharing services, or more generally vehicle–sharing services, have been introduced in the city life in order to reduce pollution and traffic congestion, becoming through time part of the culture of our society. In our case study, when a recourse action is adopted, a fleet of vehicles is deployed at night on the city streets to realize the transshipment, picking up some bicycles at those bike–station which inventory levels are above their optimum in order to compensate the shortages of others. When a recourse action is taken multiple times during the day, multiple times a fleet of vehicles is deployed on the city streets already congestioned by the daily traffic. Moreover, the pollution produced by the vehicles concurs to increase the overall city pollution, resulting both against the original goals for which bike–sharing system have been introduced. Due to this facts, the recourse action should be adoped only once at night, in order to not interfere with the city congestion during the working day.

In the introduction of this chapter we have addressed some research questions.

At the end of this study, we are able to provide them an answer.

1. the bike–sharing problem is not trivial and it must be studied by means of mathematical modelling formulations in order to achieve a good optimal solution. From the case study and the numerical results we understand that the human judgment may lead to critical inefficiency, very low user sastifaction level and high total cost. The number of bicycles to assign to each bike–station cannot be decided arbitrarely, nor be identical for each of them: each bike–station requires a specific number of bicycles determined by a mathematical program;

2. an optimal solution for the bike–sharing problem with rebalancing can be determined by means of a stochastic programming approach;

3. we have shown that it is possible to decompose the main problem in a class of subproblems only when the bike–stations are independent from each other;

4. the transshipment is mandatory in order to increase the service level: it reallocates the bicycles fleet at the end of the service according to the optimal number of bicycles to have at the beginning of the service of the next day. It reduces the probability of a shortage and, as consequence, it increases the customer level satisfaction;

5. we have shown that the parameters of the problem, and the ratio between them, play a key roles in determining the optimal solutions. The sensitivity analysis carried out in the numerical results explains that, for certain values or relationship among the parameters, the optimal solution does not change and the expected total cost is constant;

6. this study helps to manage the problem of the bike–sharing in many ways. A manager can determine the optimal number of bicycles for each bike–station, she can observe how the optimal solution changes if a parameters is setted to a particular value (*e.g.*, in order to reduce the shortage or

to allow it to be higher, to admit a greater overflow or to decrease it, to reduce the number of transshiped bicycles at the end of the service or to increase it). At last, by means of the newly introduced concept of the Value of the Right Distribution, a manager can measure the loss in the expected total cost when another distribution realizes instead of the one assumed for the demand.

The numerical results proposed in Chapter 2 validate the utility of the Value of the Right distribution proposed in Chapter 1, being addressed as a tool to measure a priori the errors when mismatching the right distribution or the true standard deviation. It is also helpfull to understand the importance of the ambiguity between the considered probability distributions. Moreover, the Deviation Test proposed in Chapter 1 confirm our conclusions about the standard deviation being the real issue, and not the mismatching between probability distributions.

However, the VRD is only applied on instances where all the stochastic parameters are formulated using a single type of distribution and the principal moments of the demand distributions (i.e., means and standard deviations) are assumed known. From a higher point of view, multiple types of distributions may be present in a given instance and the moments associated to the distributions of the stochastic parameters may not be known with certainty. We address such cases as possible avenues for future research.

Appendix

In this appendix we collect real data of rental demads of bike–sharing system 'LaBigi' and the estimated parameters.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.63	0.00	0.47	0.00	0.07	0.93	2.73	0.40	0.43	0.33	2.10	0.00	0.03	0.80	0.23	0.73	0.00	1.40	2.33	0.33	2.40	0.97
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.50	0.00	0.17	0.00	0.00	0.33	0.07	0.00	0.00	0.13	0.00	0.00	0.00	0.13	0.00	0.10	0.00	0.00	0.10	0.00	0.00	0.17
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.07	0.00	0.00	0.00	0.03	0.30	0.03	0.00	0.13	0.03	0.00	0.00	0.00	0.03	0.00	0.03	0.00	0.03	0.13	0.17	0.00	0.03
F	0.93	0.00	0.37	0.00	0.10	1.23	0.77	0.50	0.33	0.17	1.20	0.00	0.00	0.10	0.33	1.03	0.00	0.47	0.47	0.17	0.93	0.17
G	3.03	0.00	0.00	0.00	0.00	0.87	1.50	0.03	0.37	4.20	0.20	0.00	0.00	0.07	0.00	2.57	0.00	0.30	0.50	0.80	0.80	0.63
H	0.03	0.00	0.00	0.00	0.03	0.43	0.17	0.17	0.00	1.37	0.00	0.00	0.03	0.00	0.00	0.03	0.00	0.00	0.03	0.57	0.20	0.60
I	0.47	0.00	0.17	0.00	0.20	0.07	0.23	0.00	0.23	0.33	0.10	0.00	0.07	0.17	0.47	0.07	0.00	0.10	0.03	0.13	0.00	0.03
J	0.70	0.00	0.23	0.00	0.10	0.27	4.27	1.60	0.53	0.67	0.63	0.00	0.07	0.60	0.17	0.57	0.00	0.47	1.20	0.10	0.57	0.23
K	1.53	0.00	0.10	0.00	0.00	2.13	0.40	0.00	0.07	0.77	0.27	0.00	0.00	0.20	0.03	0.27	0.00	0.10	0.37	0.13	0.07	0.13
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.00	0.00	0.00	0.00	0.00	0.00	0.07	0.07	0.00	0.07	0.03	0.00	0.13	0.00	0.00	0.13	0.00	0.00	0.03	0.00	0.10	0.07
N	1.17	0.00	0.00	0.00	0.03	0.10	0.03	0.07	0.07	0.77	0.10	0.00	0.00	0.13	0.00	0.33	0.00	0.03	0.33	0.10	0.10	0.30
O	0.40	0.00	0.03	0.00	0.00	0.40	0.10	0.00	0.27	0.10	0.03	0.00	0.00	0.00	0.07	0.03	0.00	0.07	0.03	0.07	0.00	0.10
P	0.67	0.00	0.00	0.00	0.03	0.50	2.50	0.27	0.17	0.20	0.27	0.00	0.03	0.33	0.07	0.87	0.00	0.13	1.33	0.23	0.57	0.23
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	1.27	0.00	0.00	0.00	0.00	0.80	0.17	0.03	0.13	0.23	0.03	0.00	0.00	0.37	0.13	0.30	0.00	0.47	0.57	0.17	0.27	0.20
S	2.33	0.00	0.17	0.00	0.13	0.40	0.40	0.10	0.00	1.43	0.87	0.00	0.13	0.20	0.20	0.67	0.00	0.47	0.70	0.73	0.33	0.27
T	0.30	0.00	0.03	0.00	0.00	0.10	1.27	0.43	0.07	0.07	0.10	0.00	0.07	0.13	0.07	0.13	0.00	0.03	0.30	1.00	0.20	0.10
U	2.50	0.00	0.00	0.00	0.00	0.23	0.73	0.17	0.00	0.57	0.10	0.00	0.00	0.00	0.00	0.37	0.00	0.17	0.60	0.07	0.27	0.13
V	0.63	0.00	0.03	0.00	0.03	0.00	1.17	0.50	0.00	0.10	0.10	0.00	0.10	0.33	0.00	0.17	0.00	0.37	0.23	0.10	0.17	0.20

Table 2.23: May 2013: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.33	0.00	0.70	0.00	0.17	1.07	3.67	0.10	0.47	0.73	2.57	0.00	0.10	0.77	0.57	1.20	0.00	1.67	2.73	0.50	2.30	1.00
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.33	0.00	0.00	0.00	0.00	0.83	0.17	0.00	0.00	0.30	0.03	0.00	0.00	0.07	0.00	0.17	0.00	0.00	0.23	0.17	0.07	0.03
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.17	0.00	0.00	0.00	0.00	0.10	0.03	0.00	0.53	0.23	0.03	0.00	0.00	0.03	0.00	0.10	0.00	0.07	0.07	0.17	0.13	0.03
F	0.90	0.00	0.37	0.00	0.07	0.80	1.50	0.50	0.33	0.27	1.73	0.00	0.00	0.10	0.57	0.73	0.00	0.73	0.57	0.27	1.67	0.23
G	4.40	0.00	0.07	0.00	0.03	1.27	0.87	0.27	0.37	5.90	0.47	0.00	0.13	0.17	0.03	3.70	0.00	0.37	0.43	1.37	0.80	0.73
H	0.10	0.00	0.00	0.00	0.00	0.60	0.43	0.30	0.00	1.10	0.03	0.00	0.10	0.07	0.03	0.27	0.00	0.00	0.00	0.40	0.10	0.70
I	0.93	0.00	0.07	0.00	0.60	0.20	0.57	0.07	0.43	0.73	0.07	0.00	0.07	0.03	0.40	0.30	0.00	0.33	0.17	0.17	0.03	0.27
J	1.03	0.00	0.37	0.00	0.03	0.20	6.30	2.17	0.63	0.70	1.23	0.00	0.17	1.07	0.30	0.77	0.00	0.43	1.53	0.17	1.03	0.20
K	2.07	0.00	0.10	0.00	0.07	2.37	0.70	0.07	0.10	0.83	0.33	0.00	0.03	0.03	0.03	0.67	0.00	0.13	0.70	0.27	0.23	0.07
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.00	0.00	0.00	0.00	0.17	0.07	0.00	0.03	0.00	0.00	0.03	0.10	0.07	0.13	0.00	0.07	0.10	0.10	0.07	0.10
N	1.03	0.00	0.03	0.00	0.00	0.13	0.17	0.00	0.10	0.57	0.10	0.00	0.03	0.20	0.23	0.30	0.00	0.10	0.50	0.33	0.13	0.27
O	0.97	0.00	0.07	0.00	0.03	0.33	0.03	0.03	0.63	0.13	0.07	0.00	0.00	0.13	0.07	0.27	0.00	0.00	0.03	0.13	0.00	0.03
P	0.83	0.00	0.10	0.00	0.03	0.83	4.43	0.43	0.13	0.53	0.77	0.00	0.10	0.37	0.40	0.97	0.00	0.10	1.07	0.07	0.77	0.47
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	1.87	0.00	0.03	0.00	0.03	0.90	0.13	0.03	0.30	0.60	0.17	0.00	0.10	0.27	0.20	0.30	0.00	0.67	0.43	0.37	0.17	0.27
S	2.43	0.00	0.07	0.00	0.20	0.50	0.60	0.00	0.20	1.23	1.43	0.00	0.20	0.33	0.17	0.90	0.00	0.23	0.47	0.73	0.90	0.60
T	0.37	0.00	0.17	0.00	0.00	0.20	1.80	0.50	0.23	0.03	0.13	0.00	0.10	0.20	0.13	0.20	0.00	0.17	0.63	0.87	0.27	0.10
U	1.97	0.00	0.10	0.00	0.10	0.80	0.77	0.07	0.10	0.37	0.23	0.00	0.17	0.17	0.07	1.40	0.00	0.20	0.70	0.20	0.37	0.57
V	0.63	0.00	0.03	0.00	0.03	0.17	1.23	0.43	0.47	0.10	0.10	0.00	0.10	0.13	0.07	0.40	0.00	0.40	0.30	0.20	0.80	0.37

Table 2.24: June 2013: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.73	0.00	0.63	0.00	0.03	1.10	4.17	0.10	1.03	1.27	3.33	0.00	0.10	1.40	1.30	1.67	0.00	2.40	2.80	0.43	3.47	1.37
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.37	0.00	0.13	0.00	0.00	0.47	0.17	0.00	0.00	0.40	0.10	0.00	0.00	0.00	0.00	0.10	0.00	0.10	0.13	0.17	0.13	0.17
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.30	0.00	0.00	0.00	0.03	0.23	0.03	0.00	0.33	0.17	0.00	0.00	0.00	0.03	0.10	0.03	0.00	0.13	0.23	0.07	0.00	0.10
F	1.40	0.00	0.23	0.00	0.10	0.43	1.50	1.17	0.53	0.53	1.40	0.00	0.03	0.40	0.53	1.17	0.00	2.03	0.60	0.37	1.37	0.13
G	5.67	0.00	0.13	0.00	0.00	1.60	1.87	0.37	0.63	7.87	0.23	0.00	0.20	0.37	0.17	4.00	0.00	0.20	0.43	1.20	1.00	1.40
H	0.13	0.00	0.00	0.00	0.00	1.37	0.43	0.23	0.00	1.67	0.03	0.00	0.30	0.10	0.07	0.50	0.00	0.03	0.17	0.50	0.23	0.90
I	0.93	0.00	0.00	0.00	0.47	0.47	0.67	0.13	1.03	0.63	0.13	0.00	0.03	0.03	1.73	0.30	0.00	0.70	0.23	0.57	0.07	0.43
J	0.87	0.00	0.43	0.00	0.30	0.30	6.90	2.63	0.77	1.37	1.23	0.00	0.23	1.37	0.47	0.90	0.00	0.70	1.60	0.40	1.50	0.50
K	2.87	0.00	0.13	0.00	0.00	1.73	0.60	0.07	0.20	1.10	0.63	0.00	0.07	0.13	0.07	0.33	0.00	0.07	0.73	0.47	0.37	0.57
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.00	0.00	0.00	0.10	0.27	0.30	0.00	0.10	0.07	0.00	0.23	0.10	0.13	0.50	0.00	0.00	0.17	0.67	0.23	0.40
N	1.60	0.00	0.27	0.00	0.00	0.43	0.33	0.07	0.10	0.77	0.37	0.00	0.13	0.50	0.10	0.53	0.00	0.27	0.57	0.30	0.60	0.33
O	1.33	0.00	0.00	0.00	0.03	0.53	0.17	0.07	1.60	0.13	0.17	0.00	0.10	0.03	0.20	0.23	0.00	0.57	0.30	0.00	0.07	0.20
P	0.90	0.00	0.07	0.00	0.10	0.90	4.90	0.70	0.53	1.03	0.57	0.00	0.60	0.63	0.07	2.43	0.00	0.53	1.43	0.13	1.50	1.57
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.47	0.00	0.13	0.00	0.03	1.53	0.50	0.07	0.80	1.00	0.17	0.00	0.17	0.50	0.73	0.33	0.00	0.47	0.77	0.33	0.20	0.20
S	2.57	0.00	0.23	0.00	0.20	0.47	0.93	0.10	0.10	1.23	1.17	0.00	0.13	0.47	0.50	0.90	0.00	0.93	1.77	1.33	0.50	0.43
T	0.33	0.00	0.10	0.00	0.03	0.33	1.43	0.63	0.43	0.07	0.33	0.00	0.80	0.20	0.07	0.33	0.00	0.37	1.00	1.37	0.33	0.17
U	2.90	0.00	0.00	0.00	0.07	0.90	0.77	0.23	0.07	1.20	0.27	0.00	0.33	0.43	0.13	2.07	0.00	0.27	0.67	0.23	1.23	0.77
V	0.97	0.00	0.10	0.00	0.00	0.17	1.87	0.53	0.63	0.37	0.37	0.00	0.17	0.63	0.10	1.43	0.00	0.33	0.40	0.27	1.20	0.33

Table 2.25: July 2013: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.57	0.00	1.73	0.00	0.70	1.57	4.20	0.33	1.33	1.30	2.87	0.00	0.07	2.30	1.90	1.37	0.00	2.87	3.50	0.47	3.63	1.50
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.77	0.00	0.03	0.00	0.03	0.57	0.13	0.03	0.00	0.37	0.07	0.00	0.13	0.03	0.00	0.10	0.00	0.00	0.03	0.13	0.20	0.17
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.67	0.00	0.00	0.00	0.03	0.03	0.23	0.03	0.47	0.17	0.07	0.00	0.00	0.00	0.00	0.10	0.00	0.10	0.20	0.07	0.00	0.07
F	2.23	0.00	0.10	0.00	0.20	0.57	1.07	0.80	0.73	0.53	1.40	0.00	0.07	0.20	0.47	1.07	0.00	1.93	0.70	0.87	1.87	0.13
G	5.00	0.00	0.07	0.00	0.10	1.57	1.33	0.13	0.73	10.00	0.70	0.00	0.47	0.37	0.90	5.37	0.00	0.00	0.57	2.17	1.30	0.93
H	0.10	0.00	0.00	0.00	0.00	0.90	0.37	0.13	0.00	2.13	0.23	0.00	0.23	0.10	0.10	0.27	0.00	0.17	0.03	0.10	0.23	0.60
I	1.57	0.00	0.03	0.00	0.03	0.50	0.77	0.07	0.83	0.73	0.00	0.00	0.13	0.07	1.63	0.27	0.00	0.70	0.13	0.17	0.27	1.17
J	1.27	0.00	0.20	0.00	0.23	0.23	9.57	3.00	0.70	1.10	2.10	0.00	0.33	0.87	0.43	1.17	0.00	0.90	2.33	0.30	1.03	0.57
K	3.87	0.00	0.13	0.00	0.13	1.10	0.93	0.17	0.00	1.43	0.63	0.00	0.00	0.13	0.20	0.33	0.00	0.30	0.63	0.40	0.43	1.13
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.00	0.00	0.00	0.10	0.30	0.13	0.10	0.57	0.10	0.00	0.30	0.33	0.07	0.43	0.00	0.23	0.00	0.57	0.33	0.03
N	1.17	0.00	0.17	0.00	0.00	0.37	0.20	0.17	0.10	0.80	0.30	0.00	0.40	0.53	0.10	1.20	0.00	0.33	0.17	0.37	0.33	0.10
O	1.70	0.00	0.03	0.00	0.00	0.37	0.60	0.00	1.63	0.50	0.20	0.00	0.07	0.00	0.50	0.17	0.00	0.43	0.37	0.37	0.13	0.87
P	1.60	0.00	0.07	0.00	0.00	1.43	5.17	0.37	0.30	1.17	0.70	0.00	0.43	0.80	0.20	2.53	0.00	0.60	1.23	0.30	1.30	1.10
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.27	0.00	0.07	0.00	0.00	1.47	0.27	0.10	0.70	1.03	0.23	0.00	0.20	0.53	0.43	0.73	0.00	0.83	0.77	0.53	0.13	0.20
S	2.93	0.00	0.13	0.00	0.23	0.40	1.13	0.10	0.10	1.97	0.97	0.00	0.13	0.13	0.30	1.20	0.00	0.60	1.17	1.13	0.47	0.17
T	0.50	0.00	0.13	0.00	0.00	0.37	2.00	0.43	0.23	0.13	0.63	0.00	0.57	0.47	0.63	0.13	0.00	0.53	0.90	1.63	0.60	0.23
U	3.40	0.00	0.00	0.00	0.00	1.00	1.23	0.17	0.33	0.90	0.20	0.00	0.40	0.43	0.10	1.70	0.00	0.10	0.50	0.67	0.70	0.60
V	1.87	0.00	0.03	0.00	0.07	0.33	1.73	0.43	1.37	0.23	0.50	0.00	0.10	0.33	0.63	0.73	0.00	0.13	0.20	0.30	0.80	0.37

Table 2.26: August 2013: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.57	0.00	0.33	0.00	0.50	0.40	1.93	0.53	0.30	0.70	2.30	0.00	0.07	0.90	0.87	0.50	0.00	1.37	1.03	0.20	2.10	0.90
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.23	0.00	0.07	0.00	0.00	0.17	0.07	0.00	0.03	0.33	0.03	0.00	0.00	0.10	0.00	0.00	0.00	0.07	0.00	0.07	0.00	0.10
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.60	0.00	0.03	0.00	0.23	0.13	0.03	0.03	0.17	0.10	0.00	0.00	0.07	0.03	0.03	0.03	0.00	0.03	0.43	0.00	0.00	0.07
F	0.63	0.00	0.00	0.00	0.13	0.33	0.60	0.93	0.33	0.43	1.10	0.00	0.17	0.23	0.13	0.53	0.00	0.83	0.57	0.47	0.70	0.07
G	1.73	0.00	0.20	0.00	0.00	0.97	1.20	0.07	0.37	5.97	0.33	0.00	0.20	0.10	0.00	2.17	0.00	0.03	0.07	0.70	0.50	0.63
H	0.20	0.00	0.00	0.00	0.00	0.83	0.17	0.13	0.00	1.63	0.03	0.00	0.10	0.10	0.03	0.30	0.00	0.07	0.00	0.33	0.07	0.83
I	0.40	0.00	0.00	0.00	0.30	0.10	0.30	0.03	0.30	0.77	0.13	0.00	0.07	0.07	1.40	0.33	0.00	0.10	0.17	0.20	0.07	0.37
J	0.80	0.00	0.30	0.00	0.17	0.23	5.57	1.60	0.50	1.03	1.23	0.00	0.20	0.60	0.63	1.47	0.00	0.57	1.33	0.17	1.47	0.20
K	2.20	0.00	0.03	0.00	0.03	1.60	0.40	0.07	0.17	1.37	0.43	0.00	0.00	0.10	0.10	0.47	0.00	0.27	0.17	0.13	0.33	0.17
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.00	0.00	0.00	0.00	0.07	0.03	0.07	0.13	0.07	0.27	0.00	0.00	0.13	0.13	0.00	0.07	0.00	0.13	0.00	0.07	0.27	0.10
N	0.67	0.00	0.03	0.00	0.00	0.07	0.53	0.00	0.07	0.37	0.07	0.00	0.03	0.50	0.03	0.43	0.00	0.30	0.27	0.27	0.07	0.10
O	0.57	0.00	0.00	0.00	0.00	0.33	0.00	0.00	1.33	0.37	0.20	0.00	0.03	0.00	0.20	0.03	0.00	0.03	0.20	0.33	0.00	0.27
P	0.60	0.00	0.03	0.00	0.00	0.80	2.60	0.67	0.13	0.63	0.53	0.00	0.23	0.50	0.00	1.43	0.00	0.33	0.60	0.30	1.00	0.60
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	1.30	0.00	0.00	0.00	0.03	0.53	0.10	0.07	0.10	0.17	0.40	0.00	0.17	0.20	0.13	1.27	0.00	0.60	0.27	0.47	0.10	0.10
S	0.80	0.00	0.10	0.00	0.43	0.50	0.17	0.07	0.07	1.43	0.40	0.00	0.07	0.03	0.20	0.50	0.00	0.17	0.87	1.03	0.30	0.10
T	0.23	0.00	0.03	0.00	0.00	0.17	0.77	0.30	0.07	0.13	0.23	0.00	0.17	0.23	0.30	0.13	0.00	0.97	0.73	1.03	0.03	0.07
U	1.93	0.00	0.00	0.00	0.00	0.63	0.47	0.17	0.10	1.30	0.27	0.00	0.03	0.07	0.00	1.07	0.00	0.00	0.23	0.03	0.23	0.63
V	0.90	0.00	0.03	0.00	0.00	0.10	0.87	0.63	0.57	0.30	0.30	0.00	0.03	0.27	0.13	0.43	0.00	0.10	0.30	0.07	0.47	0.20

Table 2.27: September 2013: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.90	0.00	0.87	0.00	0.70	0.97	5.03	0.27	0.87	2.00	4.57	0.00	0.07	2.17	1.10	2.20	0.00	3.63	3.40	1.13	4.87	1.83
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	1.13	0.00	0.07	0.00	0.03	0.33	0.10	0.07	0.13	0.50	0.03	0.00	0.00	0.00	0.03	0.07	0.00	0.07	0.10	0.03	0.10	0.07
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.33	0.00	0.00	0.00	0.67	0.37	0.03	0.03	0.50	0.23	0.07	0.00	0.07	0.03	0.03	0.13	0.00	0.13	0.37	0.10	0.10	0.20
F	1.03	0.00	0.27	0.00	0.13	0.40	1.50	0.57	0.50	0.20	1.33	0.00	0.10	0.33	0.30	0.83	0.00	0.83	0.67	0.47	1.43	0.30
G	5.33	0.00	0.07	0.00	0.07	1.17	1.40	0.33	1.07	6.67	0.30	0.00	0.30	0.37	0.10	4.17	0.00	0.13	0.30	1.07	0.40	1.30
H	0.23	0.00	0.03	0.00	0.00	1.47	0.27	0.20	0.00	1.33	0.03	0.00	0.20	0.03	0.00	0.20	0.00	0.00	0.00	0.37	0.30	0.23
I	0.53	0.00	0.17	0.00	0.73	0.23	1.20	0.13	0.77	0.87	0.37	0.00	0.13	0.03	1.50	0.43	0.00	0.30	0.20	0.30	0.77	0.17
J	1.83	0.00	0.43	0.00	0.37	0.30	8.07	2.33	0.93	1.33	1.40	0.00	0.30	1.60	0.43	1.23	0.00	1.97	1.90	0.23	2.30	0.30
K	5.03	0.00	0.13	0.00	0.07	1.83	0.80	0.07	0.20	1.40	0.40	0.00	0.10	0.20	0.37	0.33	0.00	0.20	0.80	0.27	0.70	0.40
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.00	0.00	0.00	0.03	0.17	0.23	0.07	0.27	0.13	0.00	0.20	0.30	0.00	0.43	0.00	0.07	0.07	0.00	0.30	0.13
N	1.73	0.00	0.00	0.00	0.00	0.43	0.10	0.00	0.03	1.10	0.33	0.00	0.27	0.97	0.10	0.67	0.00	0.17	0.30	0.73	0.17	0.23
O	1.50	0.00	0.07	0.00	0.00	0.37	0.03	0.00	1.17	0.27	0.33	0.00	0.00	0.03	0.33	0.07	0.00	0.23	0.40	0.40	0.00	0.00
P	1.40	0.00	0.13	0.00	0.13	0.57	5.37	0.43	1.03	1.17	0.57	0.00	0.43	0.73	0.10	1.20	0.00	0.27	0.87	0.20	2.50	0.77
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.93	0.00	0.07	0.00	0.07	0.57	0.13	0.00	0.20	1.57	0.67	0.00	0.10	0.77	0.57	1.00	0.00	0.53	0.53	0.07	0.47	0.17
S	2.87	0.00	0.17	0.00	0.23	0.17	0.47	0.10	0.20	2.07	1.40	0.00	0.10	0.47	0.20	1.27	0.00	0.70	1.40	0.83	0.77	0.37
T	0.80	0.00	0.00	0.00	0.03	0.33	1.00	0.67	0.33	0.13	0.23	0.00	0.10	0.80	0.17	0.17	0.00	0.20	0.67	0.57	0.27	0.10
U	4.93	0.00	0.07	0.00	0.03	0.47	0.67	0.13	0.57	1.70	0.67	0.00	0.07	0.10	0.00	2.50	0.00	0.23	0.97	0.27	0.80	0.77
V	1.00	0.00	0.20	0.00	0.07	0.03	0.97	0.83	0.20	0.20	0.43	0.00	0.30	0.40	0.33	0.63	0.00	0.10	0.23	0.27	0.57	0.23

Table 2.28: October 2013: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.27	0.00	1.43	0.00	0.17	0.83	3.90	0.13	0.30	1.53	3.47	0.00	0.03	1.93	0.73	1.50	0.00	1.97	2.30	0.93	3.93	1.37
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	1.57	0.00	0.13	0.00	0.00	0.37	0.20	0.03	0.00	0.40	0.00	0.00	0.00	0.00	0.00	0.13	0.00	0.00	0.00	0.00	0.10	0.00
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.30	0.00	0.03	0.00	0.10	0.13	0.03	0.03	0.57	0.23	0.07	0.00	0.00	0.00	0.07	0.00	0.00	0.10	0.37	0.07	0.00	0.03
F	0.63	0.00	0.20	0.00	0.03	0.63	0.90	0.53	0.10	0.13	1.33	0.00	0.00	0.30	0.87	0.80	0.00	0.70	0.27	0.10	1.37	0.03
G	4.53	0.00	0.10	0.00	0.07	0.57	0.93	0.10	0.87	6.37	0.30	0.00	0.23	0.37	0.07	5.23	0.00	0.17	0.27	0.83	0.37	1.03
H	0.20	0.00	0.00	0.00	0.00	1.13	0.20	0.07	0.07	1.10	0.07	0.00	0.17	0.00	0.03	0.13	0.00	0.03	0.03	0.13	0.17	0.70
I	0.50	0.00	0.10	0.00	0.40	0.23	1.10	0.03	0.80	0.50	0.67	0.00	0.03	0.10	1.43	0.27	0.00	0.17	0.17	0.23	0.37	0.00
J	1.27	0.00	0.30	0.00	0.13	0.33	6.57	2.03	0.43	0.70	0.80	0.00	0.30	1.00	0.23	0.70	0.00	1.00	0.90	0.20	2.33	0.17
K	3.27	0.00	0.07	0.00	0.00	1.93	0.40	0.03	0.37	0.77	0.27	0.00	0.00	0.17	0.17	0.17	0.00	0.13	0.63	0.30	0.97	0.30
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.00	0.00	0.00	0.00	0.03	0.03	0.10	0.10	0.07	0.13	0.00	0.00	0.20	0.10	0.03	0.23	0.00	0.00	0.00	0.10	0.13	0.03
N	1.63	0.00	0.00	0.00	0.00	0.17	0.17	0.07	0.13	0.97	0.27	0.00	0.03	0.43	0.00	0.50	0.00	0.03	0.40	0.33	0.10	0.20
O	0.87	0.00	0.03	0.00	0.07	0.77	0.07	0.00	0.87	0.23	0.10	0.00	0.00	0.07	0.10	0.13	0.00	0.37	0.23	0.40	0.07	0.27
P	1.07	0.00	0.03	0.00	0.07	0.40	6.03	0.43	0.83	0.63	0.30	0.00	0.10	0.43	0.13	1.07	0.00	0.23	1.17	0.03	2.27	0.87
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.47	0.00	0.17	0.00	0.03	0.53	0.10	0.03	0.33	1.03	0.30	0.00	0.07	0.03	0.67	0.57	0.00	0.37	0.30	0.27	0.17	0.07
S	2.50	0.00	0.10	0.00	0.20	0.27	0.47	0.03	0.23	1.43	0.90	0.00	0.07	0.27	0.23	0.87	0.00	0.33	0.80	0.37	0.60	0.20
T	0.70	0.00	0.03	0.00	0.03	0.13	0.63	0.20	0.37	0.10	0.30	0.00	0.03	0.47	0.20	0.10	0.00	0.20	0.33	0.33	0.33	0.17
U	3.23	0.00	0.03	0.00	0.03	0.67	0.60	0.17	0.50	1.57	0.40	0.00	0.03	0.13	0.00	2.57	0.00	0.10	0.67	0.17	0.50	1.07
V	0.73	0.00	0.10	0.00	0.03	0.03	0.97	1.00	0.10	0.13	0.17	0.00	0.07	0.33	0.20	0.57	0.00	0.03	0.17	0.10	0.73	0.17

Table 2.29: November 2013: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.50	0.00	0.93	0.00	0.07	0.43	3.20	0.23	0.50	1.17	3.10	0.00	0.03	1.53	0.37	1.27	0.00	1.60	1.57	0.80	2.50	0.70
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.73	0.00	0.07	0.00	0.00	0.27	0.07	0.03	0.07	0.37	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.07	0.07	0.03	0.00
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.10	0.00	0.00	0.00	0.27	0.17	0.03	0.03	0.33	0.30	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.13	0.47	0.07	0.00	0.03
F	0.23	0.00	0.13	0.00	0.07	0.33	0.63	0.33	0.10	0.23	1.13	0.00	0.03	0.20	0.57	0.23	0.00	0.67	0.23	0.03	0.60	0.07
G	3.93	0.00	0.07	0.00	0.00	0.33	0.63	0.17	0.57	6.50	0.50	0.00	0.03	0.20	0.00	4.27	0.00	0.07	0.20	0.73	0.23	0.93
H	0.17	0.00	0.00	0.00	0.00	1.20	0.50	0.03	0.10	0.60	0.03	0.00	0.23	0.03	0.10	0.10	0.00	0.03	0.03	0.10	0.27	0.23
I	0.33	0.00	0.00	0.00	0.20	0.17	0.87	0.07	0.43	0.53	0.17	0.00	0.03	0.03	0.87	0.63	0.00	0.13	0.07	0.17	0.53	0.23
J	1.40	0.00	0.13	0.00	0.00	0.10	6.87	1.37	0.63	1.10	0.77	0.00	0.07	0.97	0.57	0.80	0.00	1.07	1.00	0.00	1.67	0.13
K	3.00	0.00	0.10	0.00	0.07	0.97	0.53	0.00	0.03	0.93	0.07	0.00	0.07	0.03	0.23	0.40	0.00	0.13	0.30	0.17	0.33	0.17
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.00	0.00	0.00	0.03	0.20	0.13	0.03	0.00	0.00	0.00	0.03	0.10	0.00	0.13	0.00	0.13	0.03	0.03	0.13	0.00
N	1.37	0.00	0.03	0.00	0.00	0.13	0.30	0.07	0.10	1.03	0.07	0.00	0.00	0.40	0.03	0.13	0.00	0.03	0.27	0.30	0.03	0.13
O	0.70	0.00	0.03	0.00	0.03	0.53	0.00	0.07	0.60	0.50	0.20	0.00	0.00	0.00	0.03	0.03	0.00	0.50	0.23	0.27	0.07	0.20
P	0.97	0.00	0.10	0.00	0.00	0.30	4.70	0.07	0.57	0.90	0.37	0.00	0.20	0.33	0.10	0.93	0.00	0.17	0.90	0.00	1.80	0.20
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	1.90	0.00	0.00	0.00	0.00	0.63	0.07	0.00	0.10	1.07	0.13	0.00	0.07	0.07	0.67	0.43	0.00	0.43	0.23	0.23	0.20	0.20
S	2.00	0.00	0.07	0.00	0.27	0.23	0.43	0.00	0.20	1.03	0.57	0.00	0.07	0.30	0.07	0.57	0.00	0.20	0.50	0.33	0.57	0.20
T	0.63	0.00	0.00	0.00	0.03	0.10	0.57	0.00	0.30	0.07	0.17	0.00	0.03	0.53	0.10	0.00	0.00	0.40	0.30	0.20	0.17	0.07
U	2.10	0.00	0.07	0.00	0.00	0.37	0.37	0.20	0.53	1.37	0.23	0.00	0.10	0.10	0.03	2.40	0.00	0.03	0.40	0.23	0.53	0.63
V	0.57	0.00	0.00	0.00	0.03	0.03	0.67	0.80	0.07	0.10	0.07	0.00	0.07	0.30	0.37	0.17	0.00	0.07	0.20	0.03	0.53	0.17

Table 2.30: December 2013: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.27	0.00	1.43	0.00	0.03	0.50	4.07	0.07	0.30	0.97	2.77	0.00	0.10	1.87	0.70	1.27	0.00	2.00	1.63	0.97	2.13	0.87
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	1.13	0.00	0.00	0.00	0.00	0.13	0.27	0.00	0.00	0.27	0.03	0.00	0.03	0.00	0.00	0.03	0.00	0.00	0.03	0.00	0.03	0.13
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.10	0.00	0.00	0.00	0.13	0.13	0.00	0.00	0.13	0.27	0.03	0.00	0.00	0.00	0.03	0.03	0.00	0.03	0.33	0.10	0.07	0.03
F	0.73	0.00	0.17	0.00	0.00	0.17	1.17	0.40	0.03	0.10	1.00	0.00	0.10	0.13	0.37	0.43	0.00	0.20	0.17	0.07	0.43	0.00
G	4.93	0.00	0.10	0.00	0.03	0.40	0.47	0.23	0.67	7.73	0.30	0.00	0.03	0.13	0.03	4.17	0.00	0.20	0.13	0.53	0.17	0.67
H	0.10	0.00	0.03	0.00	0.00	1.07	0.53	0.20	0.03	0.63	0.03	0.00	0.07	0.07	0.10	0.00	0.00	0.07	0.03	0.13	0.13	0.50
I	0.37	0.00	0.00	0.00	0.10	0.07	0.77	0.03	0.27	0.67	0.20	0.00	0.00	0.03	1.03	0.23	0.00	0.10	0.03	0.10	0.27	0.10
J	1.07	0.00	0.40	0.00	0.03	0.10	7.17	1.53	0.50	0.77	0.77	0.00	0.07	0.77	0.23	0.60	0.00	0.83	1.00	0.07	1.13	0.17
K	3.13	0.00	0.03	0.00	0.00	0.97	0.23	0.03	0.07	1.03	0.13	0.00	0.00	0.03	0.07	0.13	0.00	0.17	0.27	0.03	0.10	0.13
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.00	0.00	0.00	0.10	0.07	0.10	0.00	0.07	0.00	0.00	0.07	0.10	0.03	0.13	0.00	0.03	0.00	0.07	0.20	0.00
N	1.43	0.00	0.03	0.00	0.00	0.07	0.13	0.07	0.13	1.03	0.07	0.00	0.10	0.33	0.03	0.17	0.00	0.07	0.07	0.33	0.27	0.03
O	0.87	0.00	0.07	0.00	0.00	0.43	0.03	0.03	0.67	0.57	0.10	0.00	0.00	0.00	0.07	0.10	0.00	0.23	0.47	0.27	0.03	0.10
P	0.67	0.00	0.03	0.00	0.03	0.23	4.57	0.10	0.33	0.63	0.33	0.00	0.10	0.30	0.13	0.83	0.00	0.20	0.97	0.03	1.03	0.27
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.60	0.00	0.00	0.00	0.03	0.17	0.10	0.03	0.00	0.43	0.17	0.00	0.13	0.03	0.43	0.33	0.00	0.40	0.23	0.17	0.10	0.10
S	1.40	0.00	0.00	0.00	0.13	0.13	0.27	0.07	0.03	1.30	0.53	0.00	0.00	0.33	0.33	0.67	0.00	0.23	0.67	0.17	0.43	0.13
T	0.73	0.00	0.00	0.00	0.03	0.03	0.70	0.13	0.47	0.03	0.07	0.00	0.03	0.40	0.17	0.10	0.00	0.07	0.23	0.10	0.10	0.00
U	1.43	0.00	0.03	0.00	0.03	0.60	0.17	0.13	0.40	0.87	0.17	0.00	0.07	0.33	0.00	1.23	0.00	0.13	0.63	0.07	0.13	0.47
V	0.60	0.00	0.00	0.00	0.00	0.00	0.93	0.80	0.17	0.03	0.07	0.00	0.00	0.30	0.60	0.10	0.00	0.03	0.07	0.03	0.53	0.13

Table 2.31: January 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.43	0.00	0.87	0.00	0.03	0.43	2.97	0.07	0.13	0.77	3.43	0.00	0.03	1.33	0.77	1.03	0.00	1.77	2.27	0.97	2.17	1.23
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.87	0.00	0.07	0.00	0.00	0.20	0.20	0.03	0.03	0.27	0.03	0.00	0.00	0.00	0.03	0.07	0.00	0.03	0.00	0.07	0.10	0.13
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.13	0.00	0.00	0.00	0.03	0.37	0.00	0.00	0.17	0.37	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.20	0.07	0.03	0.03
F	0.37	0.00	0.20	0.00	0.17	0.20	1.03	0.33	0.10	0.10	0.80	0.00	0.00	0.07	0.67	0.27	0.00	0.67	0.30	0.07	0.57	0.07
G	4.37	0.00	0.10	0.00	0.00	0.47	0.73	0.13	0.77	8.63	0.40	0.00	0.13	0.20	0.00	3.87	0.00	0.13	0.30	0.73	0.40	0.43
H	0.03	0.00	0.03	0.00	0.00	0.97	0.37	0.07	0.03	0.60	0.07	0.00	0.10	0.00	0.03	0.07	0.00	0.00	0.00	0.03	0.33	0.37
I	0.13	0.00	0.13	0.00	0.00	0.13	1.07	0.03	0.63	0.63	0.20	0.00	0.00	0.03	0.60	0.20	0.00	0.13	0.10	0.23	0.27	0.03
J	0.47	0.00	0.07	0.00	0.17	0.10	8.23	1.23	0.50	1.07	0.60	0.00	0.07	0.47	0.67	0.33	0.00	0.83	0.80	0.17	1.17	0.13
K	3.77	0.00	0.00	0.00	0.03	0.73	0.57	0.00	0.10	0.83	0.27	0.00	0.00	0.07	0.13	0.03	0.00	0.10	0.40	0.07	0.23	0.23
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.00	0.00	0.07	0.00	0.00	0.00	0.27	0.03	0.03	0.00	0.00	0.00	0.10	0.43	0.00	0.20	0.00	0.07	0.00	0.10	0.23	0.03
N	0.97	0.00	0.00	0.00	0.00	0.13	0.03	0.03	0.20	0.73	0.00	0.00	0.13	0.20	0.07	0.23	0.00	0.13	0.07	0.50	0.03	0.03
O	1.13	0.00	0.03	0.00	0.00	0.67	0.07	0.00	0.43	0.67	0.20	0.00	0.00	0.10	0.07	0.13	0.00	0.30	0.33	0.53	0.03	0.17
P	0.43	0.00	0.00	0.00	0.07	0.17	4.27	0.17	0.20	0.30	0.27	0.00	0.20	0.17	0.10	0.73	0.00	0.17	0.50	0.03	1.37	0.87
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	1.80	0.00	0.00	0.00	0.03	0.37	0.13	0.00	0.07	0.70	0.20	0.00	0.13	0.13	0.53	0.43	0.00	0.20	0.33	0.17	0.40	0.17
S	2.17	0.00	0.07	0.00	0.37	0.20	0.27	0.07	0.03	1.10	0.20	0.00	0.00	0.13	0.17	0.40	0.00	0.20	0.43	0.07	0.23	0.10
T	0.87	0.00	0.00	0.00	0.00	0.20	0.70	0.03	0.40	0.10	0.13	0.00	0.20	0.50	0.13	0.07	0.00	0.20	0.17	0.10	0.00	0.07
U	1.57	0.00	0.10	0.00	0.00	0.47	0.47	0.20	0.10	0.83	0.40	0.00	0.17	0.17	0.20	1.47	0.00	0.30	0.50	0.03	0.23	0.53
V	0.57	0.00	0.10	0.00	0.00	0.03	0.90	0.57	0.00	0.03	0.07	0.00	0.03	0.13	0.37	0.63	0.00	0.07	0.10	0.03	0.60	0.07

Table 2.32: February 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.87	0.00	0.70	0.00	0.23	0.93	4.33	0.20	0.90	1.33	4.77	0.00	0.27	1.93	1.07	1.63	0.00	2.97	2.80	1.23	3.20	1.30
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.93	0.00	0.07	0.00	0.00	0.23	0.10	0.00	0.10	0.17	0.00	0.00	0.03	0.00	0.00	0.10	0.00	0.10	0.07	0.00	0.03	0.00
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.27	0.00	0.00	0.00	0.20	0.70	0.07	0.07	0.43	0.40	0.00	0.00	0.10	0.00	0.03	0.03	0.00	0.10	0.20	0.10	0.00	0.10
F	1.17	0.00	0.10	0.00	0.20	0.47	1.70	0.50	0.63	0.20	1.50	0.00	0.23	0.17	0.93	0.37	0.00	1.10	0.47	0.47	1.37	0.40
G	5.77	0.00	0.00	0.00	0.00	0.67	0.87	0.23	1.17	9.80	0.50	0.00	0.23	0.30	0.17	5.30	0.00	0.27	0.20	1.30	0.53	0.97
H	0.23	0.00	0.00	0.00	0.00	1.40	0.53	0.07	0.00	1.17	0.03	0.00	0.07	0.13	0.00	0.10	0.00	0.07	0.00	0.57	0.27	0.23
I	0.70	0.00	0.03	0.00	0.47	0.23	1.53	0.03	1.17	0.83	0.50	0.00	0.03	0.07	1.23	0.27	0.00	0.43	0.10	0.53	0.73	0.67
J	1.33	0.00	0.20	0.00	0.13	0.20	9.80	2.30	0.83	2.23	1.17	0.00	0.30	1.20	0.87	0.90	0.00	1.57	1.67	0.53	1.77	0.37
K	4.30	0.00	0.03	0.00	0.00	2.10	0.37	0.03	0.33	1.37	0.27	0.00	0.03	0.07	0.20	0.37	0.00	0.13	0.73	0.10	0.90	0.60
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.07	0.00	0.00	0.00	0.00	0.03	0.47	0.13	0.13	0.37	0.10	0.00	0.03	0.17	0.07	0.20	0.00	0.30	0.03	0.17	0.63	0.17
N	1.60	0.00	0.00	0.00	0.00	0.23	0.33	0.00	0.10	1.00	0.00	0.00	0.20	0.33	0.10	0.27	0.00	0.33	0.20	0.43	0.30	0.17
O	1.37	0.00	0.03	0.00	0.17	1.17	0.23	0.00	0.97	0.93	0.17	0.00	0.13	0.33	0.30	0.47	0.00	0.27	0.23	0.60	0.07	0.33
P	1.00	0.00	0.00	0.00	0.00	0.57	6.20	0.27	0.40	0.73	0.37	0.00	0.23	0.40	0.23	0.90	0.00	0.37	0.67	0.13	0.87	0.43
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.60	0.00	0.03	0.00	0.07	1.07	0.23	0.03	0.20	1.17	0.30	0.00	0.37	0.17	0.40	0.53	0.00	0.33	0.50	0.43	0.47	0.17
S	2.13	0.00	0.07	0.00	0.63	0.60	0.60	0.03	0.10	2.27	0.77	0.00	0.13	0.13	0.40	0.77	0.00	0.30	0.63	0.43	0.17	0.13
T	0.70	0.00	0.07	0.00	0.03	0.27	1.50	0.63	0.80	0.17	0.20	0.00	0.23	0.57	0.17	0.17	0.00	0.53	0.37	0.67	0.33	0.07
U	3.30	0.00	0.03	0.00	0.03	0.57	0.37	0.23	0.63	2.07	0.83	0.00	0.37	0.30	0.13	1.07	0.00	0.33	0.47	0.17	0.87	1.07
V	1.43	0.00	0.07	0.00	0.00	0.17	1.10	0.80	0.30	0.27	0.30	0.00	0.10	0.37	0.53	0.30	0.00	0.10	0.23	0.30	0.97	0.27

Table 2.33: March 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.73	0.00	0.40	0.00	0.13	0.97	4.03	0.33	0.93	1.40	4.03	0.00	0.07	2.00	1.23	1.60	0.00	2.50	2.67	1.17	2.07	1.37
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.63	0.00	0.00	0.00	0.00	0.20	0.10	0.03	0.03	0.37	0.10	0.00	0.00	0.20	0.07	0.10	0.00	0.00	0.03	0.10	0.10	0.10
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.37	0.00	0.00	0.00	0.17	0.37	0.00	0.00	0.30	0.20	0.13	0.00	0.03	0.00	0.17	0.07	0.00	0.00	0.33	0.07	0.13	0.03
F	1.00	0.00	0.07	0.00	0.20	0.77	1.87	0.67	0.43	0.43	1.17	0.00	0.33	0.17	1.03	0.73	0.00	0.90	0.53	0.50	1.50	0.03
G	5.87	0.00	0.27	0.00	0.00	0.90	1.50	0.23	1.03	9.73	0.63	0.00	0.47	0.50	0.27	6.27	0.00	0.40	0.30	1.60	0.97	1.03
H	0.37	0.00	0.03	0.00	0.00	1.30	0.50	0.17	0.10	1.80	0.10	0.00	0.03	0.00	0.03	0.27	0.00	0.03	0.07	0.73	0.50	0.07
I	0.70	0.00	0.03	0.00	0.30	0.30	0.90	0.07	0.80	0.90	0.20	0.00	0.17	0.10	1.67	0.37	0.00	0.37	0.07	1.03	0.70	0.50
J	1.50	0.00	0.30	0.00	0.13	0.60	10.87	3.13	0.97	2.13	1.13	0.00	0.50	1.07	0.67	1.20	0.00	1.03	1.80	0.20	2.90	0.40
K	3.63	0.00	0.03	0.00	0.07	1.80	0.60	0.03	0.33	1.30	0.63	0.00	0.13	0.13	0.33	0.40	0.00	0.13	0.70	0.30	1.30	0.30
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.03	0.00	0.03	0.03	0.63	0.10	0.10	0.20	0.20	0.00	0.17	0.07	0.10	0.27	0.00	0.17	0.00	0.37	0.30	0.13
N	1.13	0.00	0.07	0.00	0.00	0.27	0.23	0.07	0.07	1.17	0.30	0.00	0.07	0.37	0.07	0.33	0.00	0.13	0.30	0.73	0.47	0.13
O	1.43	0.00	0.00	0.00	0.00	1.00	0.43	0.03	1.30	0.47	0.17	0.00	0.17	0.03	0.20	0.27	0.00	0.30	0.47	0.50	0.27	0.37
P	1.07	0.00	0.07	0.00	0.03	0.87	7.23	0.27	0.67	1.03	0.60	0.00	0.13	0.77	0.13	2.10	0.00	0.50	0.63	0.43	1.50	0.90
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.53	0.00	0.07	0.00	0.07	0.90	0.17	0.20	0.13	1.17	0.13	0.00	0.23	0.23	0.57	1.03	0.00	0.63	0.20	0.23	0.60	0.17
S	2.37	0.00	0.10	0.00	0.53	0.57	0.47	0.10	0.17	1.67	1.13	0.00	0.07	0.10	0.53	0.63	0.00	0.30	0.53	0.67	0.47	0.30
T	0.80	0.00	0.10	0.00	0.10	0.13	1.73	0.80	1.30	0.33	0.17	0.00	0.30	0.53	0.20	0.20	0.00	0.63	0.60	0.70	0.57	0.17
U	2.23	0.00	0.03	0.00	0.13	1.03	0.97	0.37	0.43	2.07	1.03	0.00	0.17	0.23	0.17	2.03	0.00	0.57	0.33	0.57	0.80	1.13
V	1.20	0.00	0.07	0.00	0.00	0.10	1.37	0.50	0.13	0.40	0.43	0.00	0.23	0.53	0.30	0.50	0.00	0.13	0.17	0.13	0.80	0.23

Table 2.34: April 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.87	0.00	0.60	0.00	0.17	0.97	4.80	0.53	0.73	1.33	5.23	0.00	0.10	1.87	1.23	2.10	0.00	1.77	4.50	0.63	2.60	1.67
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.80	0.00	0.20	0.00	0.00	0.40	0.10	0.07	0.07	0.20	0.10	0.00	0.03	0.03	0.00	0.10	0.00	0.07	0.13	0.07	0.37	0.17
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.67	0.00	0.03	0.00	0.00	0.27	0.13	0.07	0.37	0.33	0.07	0.00	0.03	0.07	0.10	0.27	0.00	0.07	0.30	0.20	0.23	0.17
F	0.83	0.00	0.30	0.00	0.43	0.57	2.13	0.80	0.37	0.53	1.97	0.00	0.10	0.43	0.57	1.50	0.00	1.00	0.77	0.50	1.30	0.47
G	6.60	0.00	0.20	0.00	0.00	1.73	1.73	0.33	0.50	12.43	0.40	0.00	1.13	0.33	0.13	8.00	0.00	0.30	0.23	1.53	0.93	1.27
H	0.37	0.00	0.00	0.00	0.00	1.57	0.70	0.23	0.03	1.50	0.07	0.00	0.03	0.10	0.03	0.20	0.00	0.00	0.10	1.00	0.20	0.23
I	0.60	0.00	0.23	0.00	0.43	0.17	0.33	0.07	1.10	0.80	0.17	0.00	0.17	0.10	1.33	0.80	0.00	0.30	0.07	0.67	0.63	0.53
J	1.73	0.00	0.13	0.00	0.07	0.37	12.37	2.73	0.90	2.63	1.53	0.00	0.27	1.83	0.93	1.77	0.00	1.53	1.77	0.47	2.23	0.30
K	4.67	0.00	0.07	0.00	0.03	2.87	0.83	0.03	0.17	2.27	0.97	0.00	0.40	0.37	0.20	0.83	0.00	0.37	1.23	0.13	1.27	0.37
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.07	0.00	0.00	0.00	0.07	0.10	0.93	0.03	0.10	0.37	0.23	0.00	0.23	0.30	0.03	0.47	0.00	0.13	0.03	0.27	0.50	0.33
N	1.33	0.00	0.07	0.00	0.00	0.27	0.37	0.27	0.07	1.40	0.90	0.00	0.20	0.97	0.10	0.50	0.00	0.23	0.57	0.63	0.37	0.40
O	1.27	0.00	0.00	0.00	0.07	0.43	0.20	0.00	1.40	1.00	0.13	0.00	0.03	0.33	0.43	0.33	0.00	0.27	0.23	0.63	0.13	0.27
P	1.97	0.00	0.03	0.00	0.10	1.13	8.10	0.50	1.00	1.43	0.80	0.00	0.47	1.03	0.57	2.97	0.00	0.23	1.13	0.23	2.40	0.77
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.20	0.00	0.03	0.00	0.07	1.07	0.17	0.07	0.30	1.13	0.37	0.00	0.23	0.17	0.40	0.37	0.00	0.63	0.43	0.60	0.60	0.27
S	3.97	0.00	0.20	0.00	0.77	0.37	0.50	0.07	0.03	2.20	1.43	0.00	0.10	0.17	0.23	1.07	0.00	0.50	1.17	0.63	0.63	0.37
T	0.93	0.00	0.20	0.00	0.07	0.37	1.60	0.90	0.67	0.20	0.37	0.00	0.57	0.80	0.60	0.57	0.00	0.40	0.57	0.60	0.60	0.13
U	2.50	0.00	0.00	0.00	0.17	1.10	0.77	0.20	0.53	1.97	1.03	0.00	0.07	0.50	0.30	2.30	0.00	0.73	0.37	0.43	0.83	0.83
V	1.43	0.00	0.23	0.00	0.00	0.13	1.77	0.57	0.27	0.40	0.20	0.00	0.40	0.37	0.23	0.63	0.00	0.37	0.23	0.47	0.87	0.47

Table 2.35: May 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	1.10	0.00	1.27	0.00	0.23	0.83	3.43	0.33	0.87	2.00	4.10	0.00	0.10	1.50	1.33	2.40	0.00	2.63	4.03	0.67	2.07	1.73
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.83	0.00	0.10	0.00	0.03	0.67	0.20	0.03	0.07	0.43	0.07	0.00	0.20	0.17	0.10	0.13	0.00	0.10	0.07	0.20	0.27	0.00
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.60	0.00	0.03	0.00	0.10	0.30	0.13	0.00	0.57	0.13	0.13	0.00	0.00	0.03	0.07	0.37	0.00	0.13	0.20	0.07	0.13	0.03
F	0.93	0.00	0.50	0.00	0.17	0.73	1.80	0.47	0.33	0.23	1.60	0.00	0.13	0.40	0.57	0.70	0.00	0.87	0.53	0.37	1.07	0.17
G	4.13	0.00	0.17	0.00	0.00	1.37	1.23	0.50	0.53	10.47	0.60	0.00	1.47	0.33	0.27	5.63	0.00	0.20	0.47	1.30	0.80	0.83
H	0.40	0.00	0.00	0.00	0.03	1.13	0.77	0.27	0.03	1.27	0.13	0.00	0.03	0.27	0.10	0.30	0.00	0.07	0.10	0.47	0.07	0.13
I	1.00	0.00	0.07	0.00	0.73	0.23	0.70	0.00	0.50	0.80	0.20	0.00	0.17	0.13	1.03	0.87	0.00	0.37	0.13	0.73	0.57	0.17
J	1.97	0.00	0.53	0.00	0.17	0.33	9.27	2.40	1.03	1.63	1.13	0.00	0.30	1.57	0.97	1.67	0.00	1.10	1.87	0.27	1.67	0.23
K	4.30	0.00	0.13	0.00	0.07	1.70	0.50	0.10	0.13	0.97	0.50	0.00	0.30	0.53	0.47	0.50	0.00	0.30	0.50	0.27	1.03	0.37
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.10	0.00	0.10	0.00	0.13	0.07	0.80	0.03	0.03	0.20	0.43	0.00	0.47	0.10	0.10	0.43	0.00	0.17	0.07	0.30	0.43	0.17
N	1.53	0.00	0.07	0.00	0.03	0.27	0.23	0.23	0.10	2.13	0.30	0.00	0.13	0.33	0.07	1.00	0.00	0.13	0.57	0.60	0.23	0.57
O	1.37	0.00	0.00	0.00	0.00	0.60	0.27	0.10	0.93	0.80	0.53	0.00	0.13	0.23	0.27	0.20	0.00	0.47	0.73	0.20	0.07	0.13
P	1.53	0.00	0.17	0.00	0.13	0.80	7.30	0.43	1.87	0.90	0.63	0.00	0.20	1.00	0.37	1.50	0.00	0.43	0.67	0.37	1.30	0.60
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	3.67	0.00	0.17	0.00	0.03	0.53	0.13	0.07	0.20	1.20	0.23	0.00	0.20	0.27	0.47	0.40	0.00	0.57	0.53	0.30	0.57	0.03
S	3.63	0.00	0.23	0.00	0.77	0.33	0.43	0.20	0.10	1.53	0.97	0.00	0.03	0.43	0.57	1.03	0.00	0.57	1.00	0.67	0.37	0.13
T	0.80	0.00	0.17	0.00	0.00	0.40	1.90	0.27	0.37	0.07	0.47	0.00	0.40	0.63	0.30	0.23	0.00	0.63	0.47	1.03	0.47	0.23
U	1.83	0.00	0.20	0.00	0.07	0.53	0.63	0.20	0.67	1.33	1.00	0.00	0.13	0.27	0.17	1.47	0.00	0.40	0.40	0.30	1.00	1.00
V	0.87	0.00	0.03	0.00	0.00	0.13	1.73	0.30	0.17	0.13	0.20	0.00	0.27	0.40	0.23	0.43	0.00	0.03	0.10	0.40	1.10	0.27

Table 2.36: June 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	1.13	0.00	0.90	0.00	0.20	1.00	3.10	0.27	0.47	1.53	4.83	0.00	0.10	1.83	1.13	2.53	0.00	2.03	4.63	0.87	2.63	1.00
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	1.00	0.00	0.70	0.00	0.00	0.77	0.20	0.00	0.07	0.73	0.10	0.00	0.03	0.07	0.10	0.27	0.00	0.03	0.10	0.23	0.10	0.23
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.20	0.00	0.03	0.00	0.13	0.43	0.03	0.00	0.77	0.17	0.07	0.00	0.07	0.03	0.13	0.07	0.00	0.07	0.20	0.00	0.13	0.13
F	1.37	0.00	0.67	0.00	0.30	0.43	1.40	0.57	0.40	0.17	1.50	0.00	0.13	0.27	0.63	1.10	0.00	0.77	0.50	0.47	0.57	0.20
G	3.57	0.00	0.27	0.00	0.03	1.07	1.90	0.47	0.37	13.37	0.37	0.00	1.80	0.23	0.13	7.17	0.00	0.23	0.73	2.10	0.77	0.80
H	0.33	0.00	0.00	0.00	0.00	0.87	0.83	0.27	0.10	1.50	0.10	0.00	0.23	0.27	0.10	0.47	0.00	0.10	0.03	0.43	0.17	0.17
I	0.57	0.00	0.13	0.00	0.50	0.13	0.63	0.07	0.37	0.53	0.17	0.00	0.43	0.20	1.47	0.53	0.00	0.17	0.00	0.40	0.57	0.30
J	1.37	0.00	0.70	0.00	0.10	0.47	12.07	2.40	0.70	2.30	1.37	0.00	0.40	0.80	0.77	1.40	0.00	1.43	1.40	0.23	1.80	0.23
K	5.27	0.00	0.07	0.00	0.03	1.73	0.60	0.10	0.10	1.07	0.80	0.00	0.43	0.07	0.53	0.40	0.00	0.07	0.63	0.23	0.93	0.27
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.13	0.00	0.10	0.00	0.00	0.13	1.23	0.23	0.17	0.17	0.03	0.00	0.50	0.20	0.20	0.20	0.00	0.13	0.23	0.17	0.43	0.30
N	1.70	0.00	0.13	0.00	0.03	0.37	0.13	0.17	0.07	1.13	0.03	0.00	0.30	0.50	0.20	0.63	0.00	0.10	0.47	0.63	0.13	0.17
O	1.40	0.00	0.03	0.00	0.07	0.87	0.30	0.00	0.77	0.60	0.60	0.00	0.20	0.23	0.13	0.27	0.00	0.63	0.83	0.17	0.07	0.43
P	1.57	0.00	0.13	0.00	0.07	1.03	7.77	0.37	1.57	1.03	0.50	0.00	0.13	0.77	0.17	3.10	0.00	0.53	0.87	0.27	1.43	0.50
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	3.00	0.00	0.07	0.00	0.00	0.73	0.27	0.10	0.10	0.90	0.23	0.00	0.10	0.57	0.30	0.63	0.00	0.37	0.73	0.27	0.30	0.07
S	3.37	0.00	0.20	0.00	0.57	0.37	0.83	0.03	0.07	2.10	1.07	0.00	0.17	0.23	0.77	1.23	0.00	0.67	0.77	1.13	0.37	0.30
T	0.93	0.00	0.07	0.00	0.03	0.37	2.23	0.40	0.33	0.37	0.40	0.00	0.10	0.87	0.10	0.27	0.00	0.37	0.73	1.37	0.50	0.13
U	1.87	0.00	0.10	0.00	0.03	0.33	0.47	0.13	0.60	2.07	0.60	0.00	0.27	0.10	0.10	1.33	0.00	0.43	0.20	0.63	0.87	1.03
V	1.07	0.00	0.17	0.00	0.03	0.13	1.10	0.27	0.10	0.23	0.27	0.00	0.27	0.30	0.17	0.83	0.00	0.07	0.17	0.27	0.80	0.30

Table 2.37: July 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.73	0.00	0.23	0.00	0.10	0.27	1.63	0.53	0.43	0.67	2.77	0.00	0.10	0.63	0.60	0.60	0.00	1.13	1.90	1.03	1.40	0.53
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.40	0.00	0.37	0.00	0.03	0.17	0.23	0.00	0.03	0.33	0.03	0.00	0.33	0.03	0.00	0.23	0.00	0.07	0.07	0.07	0.00	0.03
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.10	0.00	0.00	0.00	0.10	0.07	0.03	0.00	0.13	0.10	0.03	0.00	0.10	0.03	0.07	0.00	0.00	0.00	0.13	0.00	0.27	0.57
F	0.70	0.00	0.23	0.00	0.00	0.50	0.93	0.33	0.23	0.13	1.43	0.00	0.10	0.20	0.37	0.53	0.00	0.80	0.50	0.20	0.33	0.03
G	1.43	0.00	0.13	0.00	0.03	0.83	1.20	0.47	0.27	7.43	0.37	0.00	0.87	0.13	0.07	4.60	0.00	0.20	0.27	0.83	0.50	0.90
H	0.30	0.00	0.00	0.00	0.07	0.63	0.57	0.53	0.00	1.07	0.07	0.00	0.20	0.03	0.00	0.20	0.00	0.17	0.10	0.23	0.17	0.10
I	0.33	0.00	0.00	0.00	0.00	0.27	0.40	0.07	0.27	0.40	0.07	0.00	0.07	0.13	0.67	0.47	0.00	0.20	0.00	0.30	0.07	0.13
J	1.30	0.00	0.50	0.00	0.10	0.23	8.23	1.47	0.40	0.63	0.97	0.00	0.23	0.57	0.60	0.77	0.00	0.83	0.87	0.17	1.20	0.30
K	2.63	0.00	0.03	0.00	0.03	1.40	0.70	0.03	0.03	0.83	0.77	0.00	0.43	0.03	0.33	0.40	0.00	0.33	0.37	0.23	0.53	0.20
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.23	0.00	0.10	0.17	0.73	0.17	0.10	0.10	0.47	0.00	0.13	0.03	0.00	0.20	0.00	0.10	0.00	0.00	0.07	0.20
N	0.37	0.00	0.00	0.00	0.00	0.33	0.23	0.03	0.03	0.37	0.03	0.00	0.10	0.27	0.00	0.30	0.00	0.07	0.40	0.23	0.10	0.13
O	0.73	0.00	0.03	0.00	0.07	0.27	0.07	0.03	0.40	0.30	0.37	0.00	0.00	0.00	0.27	0.20	0.00	0.27	0.23	0.17	0.03	0.27
P	0.63	0.00	0.07	0.00	0.03	0.70	5.30	0.27	0.70	0.60	0.67	0.00	0.27	0.27	0.13	2.10	0.00	0.47	0.70	0.07	0.70	0.40
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	1.80	0.00	0.10	0.00	0.00	0.80	0.07	0.00	0.03	0.63	0.23	0.00	0.20	0.13	0.17	0.50	0.00	0.77	0.30	0.13	0.10	0.13
S	1.20	0.00	0.27	0.00	0.23	0.30	0.40	0.17	0.00	1.03	0.60	0.00	0.07	0.77	0.37	0.63	0.00	0.33	0.83	0.33	0.27	0.20
T	0.97	0.00	0.03	0.00	0.00	0.07	1.20	0.17	0.03	0.30	0.17	0.00	0.17	0.20	0.13	0.13	0.00	0.00	0.27	0.63	0.23	0.00
U	0.97	0.00	0.03	0.00	0.13	0.27	0.27	0.10	0.13	1.43	0.50	0.00	0.13	0.20	0.03	0.73	0.00	0.03	0.37	0.13	0.37	0.37
V	0.33	0.00	0.03	0.00	0.20	0.10	0.67	0.13	0.13	0.13	0.10	0.00	0.10	0.37	0.30	0.47	0.00	0.07	0.10	0.07	0.43	0.30

Table 2.38: August 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.80	0.00	1.07	0.00	0.17	0.83	5.97	0.13	1.40	1.60	5.37	0.00	0.23	1.67	1.77	1.80	0.00	3.03	5.27	1.53	3.43	1.33
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	1.03	0.00	0.37	0.00	0.00	0.77	0.17	0.07	0.07	0.67	0.10	0.00	0.07	0.03	0.03	0.20	0.00	0.07	0.17	0.27	0.13	0.23
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.27	0.00	0.00	0.00	0.13	0.53	0.03	0.03	0.13	0.53	0.10	0.00	0.03	0.10	0.13	0.10	0.00	0.00	0.17	0.00	0.07	0.50
F	1.60	0.00	0.63	0.00	0.13	1.10	2.00	0.53	0.47	0.27	2.27	0.00	0.20	0.33	0.77	0.80	0.00	1.20	0.80	0.63	0.87	0.13
G	5.17	0.00	0.30	0.00	0.07	1.33	1.57	0.30	0.30	13.53	0.63	0.00	0.97	0.43	0.40	7.37	0.00	0.10	0.43	1.30	1.23	1.73
H	0.10	0.00	0.10	0.00	0.00	0.93	0.43	0.17	0.17	0.73	0.00	0.00	0.20	0.40	0.10	0.77	0.00	0.10	0.20	0.67	0.37	0.07
I	1.27	0.00	0.07	0.00	0.13	0.53	0.87	0.47	1.10	0.47	0.13	0.00	0.17	0.17	1.03	0.83	0.00	0.30	0.07	0.87	0.70	0.37
J	1.60	0.00	0.43	0.00	0.33	0.47	11.77	1.87	1.10	2.03	1.33	0.00	0.47	1.83	0.87	1.60	0.00	2.00	1.70	0.43	1.87	0.37
K	4.57	0.00	0.27	0.00	0.10	2.87	0.77	0.07	0.07	0.67	0.93	0.00	0.07	0.20	0.70	0.60	0.00	0.33	1.33	0.27	1.03	0.87
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.10	0.00	0.00	0.13	0.83	0.30	0.07	0.13	0.07	0.00	0.10	0.33	0.13	0.30	0.00	0.10	0.07	0.27	0.47	0.20
N	2.23	0.00	0.07	0.00	0.00	0.67	0.33	0.23	0.13	1.10	0.13	0.00	0.07	0.47	0.37	0.90	0.00	0.37	0.80	0.47	0.10	0.73
O	1.80	0.00	0.07	0.00	0.07	0.90	0.57	0.07	1.00	0.77	0.53	0.00	0.03	0.40	0.63	0.47	0.00	0.60	0.87	0.20	0.17	0.23
P	1.87	0.00	0.20	0.00	0.03	0.93	7.70	0.23	1.83	0.93	0.73	0.00	0.37	0.77	0.23	2.17	0.00	0.40	1.10	0.20	1.10	0.60
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	4.03	0.00	0.07	0.00	0.00	0.97	0.30	0.07	0.27	1.70	0.33	0.00	0.43	0.60	0.67	0.50	0.00	0.70	0.60	0.23	0.43	0.23
S	5.13	0.00	0.17	0.00	0.57	0.80	0.63	0.30	0.07	1.87	1.33	0.00	0.10	0.60	0.67	1.07	0.00	0.63	1.53	0.73	1.27	0.43
T	1.77	0.00	0.10	0.00	0.00	0.17	2.43	0.57	0.27	0.23	0.30	0.00	0.17	0.50	0.17	0.27	0.00	0.43	0.47	0.67	0.73	0.10
U	2.67	0.00	0.10	0.00	0.10	0.40	0.93	0.37	0.90	2.07	1.07	0.00	0.33	0.33	0.27	1.27	0.00	0.67	1.37	0.43	1.13	1.60
V	1.33	0.00	0.40	0.00	0.37	0.10	1.63	0.37	0.17	0.40	0.37	0.00	0.17	1.10	0.13	0.73	0.00	0.07	0.27	0.23	1.40	0.50

Table 2.39: September 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.87	0.00	1.17	0.00	0.67	1.50	5.37	0.20	1.60	1.67	5.00	0.00	0.07	1.77	1.00	3.00	0.00	2.53	5.97	1.40	3.37	1.03
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	1.27	0.00	0.67	0.00	0.03	0.93	0.20	0.07	0.07	0.60	0.00	0.00	0.03	0.00	0.23	0.17	0.00	0.07	0.27	0.10	0.10	0.40
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.63	0.00	0.00	0.00	0.10	0.50	0.03	0.03	0.03	0.83	0.13	0.00	0.03	0.03	0.03	0.03	0.00	0.13	0.13	0.10	0.03	0.47
F	1.60	0.00	0.83	0.00	0.30	1.03	2.27	0.50	0.50	0.17	2.53	0.00	0.10	0.63	1.33	0.93	0.00	1.03	0.77	0.83	0.90	0.10
G	5.20	0.00	0.27	0.00	0.00	1.43	2.03	0.20	0.50	13.67	0.40	0.00	1.13	0.47	0.10	6.33	0.00	0.37	0.40	1.50	1.20	1.63
H	0.13	0.00	0.07	0.00	0.03	0.97	0.37	0.17	0.10	1.47	0.13	0.00	0.40	0.17	0.13	0.37	0.00	0.13	0.07	0.60	0.27	0.20
I	1.57	0.00	0.13	0.00	0.13	0.30	0.87	0.30	1.50	0.90	0.30	0.00	0.27	0.10	0.77	1.10	0.00	0.27	0.03	0.57	0.60	0.37
J	1.47	0.00	0.80	0.00	0.70	0.30	12.73	2.47	1.27	1.70	1.47	0.00	0.37	1.63	0.53	1.83	0.00	1.87	2.20	0.37	1.23	0.43
K	5.03	0.00	0.03	0.00	0.03	2.87	0.77	0.10	0.17	0.90	0.53	0.00	0.07	0.17	0.73	0.23	0.00	0.20	0.97	0.40	1.27	0.57
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.10	0.00	0.00	0.00	0.00	0.07	0.97	0.43	0.13	0.07	0.17	0.00	0.13	0.03	0.03	0.20	0.00	0.07	0.20	0.13	0.53	0.10
N	1.57	0.00	0.00	0.00	0.00	0.97	0.23	0.03	0.03	1.20	0.17	0.00	0.07	0.63	0.27	0.93	0.00	0.07	0.37	0.47	0.27	0.73
O	1.40	0.00	0.03	0.00	0.13	1.57	0.20	0.00	1.00	0.37	0.63	0.00	0.00	0.17	0.23	0.30	0.00	0.63	0.30	0.27	0.07	0.10
P	2.13	0.00	0.23	0.00	0.10	0.87	7.17	0.30	2.30	1.30	0.60	0.00	0.13	0.80	0.07	1.67	0.00	0.43	0.83	0.40	0.80	0.80
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	3.47	0.00	0.10	0.00	0.00	1.07	0.30	0.03	0.30	1.83	0.50	0.00	0.07	0.47	0.60	0.40	0.00	0.53	0.77	0.23	0.37	0.27
S	4.87	0.00	0.37	0.00	0.53	0.83	0.93	0.30	0.07	1.80	0.83	0.00	0.13	0.30	0.47	0.80	0.00	1.37	1.17	1.17	1.23	0.53
T	1.30	0.00	0.10	0.00	0.03	0.63	2.37	0.67	0.33	0.23	0.67	0.00	0.10	0.90	0.13	0.27	0.00	0.40	0.97	1.10	0.60	0.17
U	2.47	0.00	0.13	0.00	0.03	0.50	1.03	0.50	0.50	1.53	1.07	0.00	0.43	0.23	0.20	0.90	0.00	0.50	1.23	0.37	1.30	1.17
V	1.20	0.00	0.47	0.00	0.27	0.10	2.00	0.27	0.13	0.23	0.37	0.00	0.20	0.90	0.27	0.63	0.00	0.23	0.33	0.33	1.13	0.40

Table 2.40: October 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.33	0.00	0.63	0.00	0.63	0.50	2.93	0.17	0.43	0.93	3.90	0.00	0.00	1.27	0.40	1.07	0.00	1.73	3.33	0.67	1.70	0.73
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.73	0.00	0.30	0.00	0.00	0.37	0.20	0.07	0.00	0.37	0.03	0.00	0.00	0.00	0.13	0.03	0.00	0.00	0.27	0.03	0.03	0.10
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.60	0.00	0.00	0.00	0.13	0.53	0.07	0.00	0.03	0.67	0.10	0.00	0.00	0.00	0.00	0.20	0.00	0.10	0.33	0.13	0.00	0.20
F	0.97	0.00	0.47	0.00	0.23	0.17	1.57	0.40	0.10	0.13	1.07	0.00	0.07	0.43	0.67	0.57	0.00	0.47	0.30	0.13	0.47	0.00
G	2.90	0.00	0.07	0.00	0.00	1.17	1.00	0.20	0.07	7.87	0.27	0.00	0.30	0.13	0.07	3.83	0.00	0.10	0.17	0.47	0.30	0.97
H	0.27	0.00	0.20	0.00	0.00	0.43	0.27	0.10	0.03	0.60	0.00	0.00	0.20	0.17	0.07	0.00	0.00	0.07	0.20	0.23	0.33	0.07
I	0.47	0.00	0.00	0.00	0.07	0.27	0.10	0.00	0.27	0.23	0.07	0.00	0.07	0.03	0.40	0.30	0.00	0.13	0.03	0.53	0.17	0.10
J	0.90	0.00	0.27	0.00	0.63	0.13	8.53	1.30	0.40	0.77	0.73	0.00	0.03	1.10	0.30	0.60	0.00	1.20	1.00	0.23	1.37	0.13
K	2.77	0.00	0.03	0.00	0.07	2.00	0.47	0.07	0.10	0.73	0.17	0.00	0.07	0.13	0.57	0.10	0.00	0.13	0.40	0.20	0.73	0.10
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.00	0.00	0.00	0.00	0.00	0.20	0.30	0.13	0.03	0.07	0.03	0.00	0.03	0.07	0.03	0.03	0.00	0.07	0.10	0.03	0.00	0.00
N	1.17	0.00	0.03	0.00	0.03	0.63	0.07	0.17	0.00	1.07	0.03	0.00	0.03	0.27	0.10	0.47	0.00	0.20	0.07	0.00	0.23	0.30
O	0.97	0.00	0.00	0.00	0.00	0.57	0.20	0.00	0.20	0.23	0.27	0.00	0.00	0.07	0.03	0.23	0.00	0.13	0.40	0.50	0.00	0.07
P	0.80	0.00	0.13	0.00	0.07	0.60	3.97	0.17	0.43	0.57	0.27	0.00	0.03	0.30	0.03	0.97	0.00	0.17	0.23	0.13	0.73	0.53
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.27	0.00	0.00	0.00	0.00	0.37	0.07	0.13	0.17	0.90	0.10	0.00	0.03	0.40	0.20	0.23	0.00	0.27	0.47	0.23	0.27	0.10
S	3.57	0.00	0.23	0.00	0.47	0.43	0.20	0.40	0.03	0.80	0.33	0.00	0.07	0.03	0.33	0.37	0.00	0.70	0.97	0.50	0.93	0.17
T	0.70	0.00	0.07	0.00	0.07	0.23	0.80	0.23	0.53	0.17	0.33	0.00	0.03	0.17	0.23	0.17	0.00	0.20	0.27	0.50	0.30	0.07
U	1.27	0.00	0.03	0.00	0.00	0.30	0.53	0.03	0.20	1.47	0.63	0.00	0.13	0.23	0.00	0.63	0.00	0.30	0.63	0.13	0.50	0.90
V	0.57	0.00	0.17	0.00	0.07	0.00	1.50	0.10	0.13	0.13	0.03	0.00	0.07	0.30	0.07	0.10	0.00	0.03	0.07	0.07	0.83	0.10

Table 2.41: November 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.53	0.00	0.93	0.00	0.40	0.27	2.53	0.07	0.23	0.97	2.87	0.00	0.00	0.97	0.77	1.27	0.00	1.67	2.87	0.40	1.23	0.50
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	0.70	0.00	0.20	0.00	0.00	0.23	0.17	0.00	0.00	0.50	0.00	0.00	0.03	0.00	0.00	0.17	0.00	0.03	0.03	0.03	0.07	0.13
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.80	0.00	0.00	0.00	0.23	0.50	0.00	0.03	0.03	0.50	0.03	0.00	0.00	0.00	0.00	0.07	0.00	0.03	0.10	0.07	0.00	0.00
F	0.73	0.00	0.13	0.00	0.53	0.23	1.57	0.40	0.17	0.10	1.07	0.00	0.07	0.33	0.50	0.50	0.00	0.40	0.43	0.30	0.17	0.07
G	2.53	0.00	0.07	0.00	0.00	1.23	0.73	0.10	0.40	8.23	0.07	0.00	0.60	0.03	0.00	3.13	0.00	0.13	0.20	0.43	0.37	0.33
H	0.20	0.00	0.13	0.00	0.00	0.37	0.33	0.27	0.03	1.17	0.07	0.00	0.17	0.27	0.00	0.03	0.00	0.23	0.03	0.03	0.17	0.07
I	0.30	0.00	0.00	0.00	0.03	0.07	0.20	0.00	0.20	0.20	0.03	0.00	0.10	0.00	0.27	0.20	0.00	0.13	0.03	0.57	0.17	0.17
J	0.80	0.00	0.30	0.00	0.43	0.10	8.43	1.33	0.30	0.60	1.03	0.00	0.13	0.93	0.30	0.70	0.00	0.83	0.90	0.13	1.43	0.07
K	2.07	0.00	0.07	0.00	0.10	1.60	0.30	0.00	0.07	0.70	0.07	0.00	0.00	0.13	0.57	0.17	0.00	0.17	0.57	0.37	0.70	0.27
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.00	0.00	0.00	0.03	0.57	0.23	0.00	0.03	0.10	0.00	0.07	0.07	0.03	0.07	0.00	0.00	0.10	0.00	0.17	0.03
N	0.77	0.00	0.00	0.00	0.00	0.33	0.00	0.37	0.03	1.03	0.17	0.00	0.10	0.30	0.27	0.27	0.00	0.13	0.07	0.20	0.10	0.23
O	0.87	0.00	0.10	0.00	0.03	0.40	0.00	0.00	0.13	0.30	0.40	0.00	0.07	0.13	0.07	0.13	0.00	0.30	0.47	0.57	0.03	0.07
P	0.43	0.00	0.10	0.00	0.13	0.47	3.73	0.20	0.23	0.40	0.33	0.00	0.03	0.27	0.20	1.17	0.00	0.20	0.77	0.03	0.53	0.50
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.67	0.00	0.03	0.00	0.10	0.33	0.17	0.10	0.13	0.63	0.07	0.00	0.00	0.27	0.10	0.17	0.00	0.50	0.23	0.33	0.33	0.27
S	2.80	0.00	0.20	0.00	0.30	0.23	0.23	0.20	0.13	0.80	0.63	0.00	0.07	0.17	0.37	0.50	0.00	0.47	0.90	0.40	0.33	0.33
T	0.50	0.00	0.07	0.00	0.10	0.10	0.80	0.23	0.40	0.13	0.27	0.00	0.00	0.07	0.17	0.17	0.00	0.27	0.40	0.57	0.17	0.07
U	1.37	0.00	0.10	0.00	0.00	0.27	0.47	0.23	0.13	1.20	0.50	0.00	0.03	0.13	0.00	0.37	0.00	0.33	0.43	0.13	0.20	1.17
V	0.53	0.00	0.17	0.00	0.00	0.07	0.70	0.13	0.03	0.07	0.33	0.00	0.07	0.30	0.00	0.47	0.00	0.20	0.10	0.17	0.90	0.17

Table 2.42: December 2014: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.27	0.00	0.90	0.00	0.60	0.27	3.00	0.10	0.43	0.60	3.47	0.00	0.00	0.87	0.20	1.40	0.00	1.50	3.87	0.60	2.13	0.63
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
C	1.07	0.00	0.23	0.00	0.00	0.20	0.17	0.00	0.00	0.33	0.13	0.00	0.00	0.10	0.00	0.03	0.00	0.07	0.03	0.00	0.00	0.13
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.87	0.00	0.00	0.00	0.03	0.53	0.03	0.00	0.07	0.73	0.10	0.00	0.00	0.03	0.03	0.07	0.00	0.03	0.27	0.10	0.03	0.00
F	0.60	0.00	0.33	0.00	0.57	0.27	1.77	0.37	0.13	0.17	1.27	0.00	0.03	0.73	0.73	0.67	0.00	0.50	0.43	0.20	0.73	0.07
G	3.30	0.00	0.10	0.00	0.07	1.07	0.77	0.07	0.03	9.10	0.13	0.00	0.13	0.07	0.10	3.33	0.00	0.13	0.47	0.63	0.37	0.23
H	0.10	0.00	0.00	0.00	0.00	0.63	0.37	0.13	0.00	1.43	0.00	0.00	0.10	0.20	0.07	0.00	0.00	0.10	0.10	0.10	0.33	0.30
I	0.30	0.00	0.03	0.00	0.00	0.07	0.03	0.07	0.10	0.27	0.07	0.00	0.03	0.00	0.33	0.20	0.00	0.00	0.13	0.53	0.37	0.17
J	0.57	0.00	0.43	0.00	0.83	0.37	8.67	1.40	0.63	0.47	0.73	0.00	0.07	1.27	0.37	0.70	0.00	0.97	1.20	0.20	0.93	0.13
K	2.77	0.00	0.20	0.00	0.00	1.43	0.43	0.07	0.07	0.87	0.37	0.00	0.03	0.07	0.60	0.20	0.00	0.23	0.57	0.00	0.80	0.10
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.00	0.03	0.00	0.00	0.07	0.13	0.17	0.00	0.13	0.00	0.00	0.07	0.00	0.00	0.03	0.00	0.13	0.03	0.00	0.10	0.03
N	0.97	0.00	0.00	0.00	0.00	1.23	0.07	0.37	0.00	1.10	0.00	0.00	0.00	0.17	0.30	0.33	0.00	0.13	0.10	0.13	0.07	0.30
O	0.87	0.00	0.03	0.00	0.00	0.47	0.03	0.03	0.23	0.50	0.40	0.00	0.00	0.13	0.03	0.23	0.00	0.13	0.63	0.73	0.07	0.07
P	0.70	0.00	0.00	0.00	0.07	0.63	5.13	0.23	0.10	0.67	0.27	0.00	0.00	0.30	0.07	1.03	0.00	0.17	0.47	0.00	0.97	0.53
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.33	0.00	0.03	0.00	0.00	0.70	0.17	0.10	0.10	0.83	0.13	0.00	0.23	0.27	0.23	0.07	0.00	0.33	0.20	0.30	0.17	0.07
S	3.13	0.00	0.13	0.00	0.33	0.30	0.73	0.07	0.00	1.70	0.50	0.00	0.07	0.17	0.57	0.83	0.00	0.60	0.50	0.43	0.73	0.47
T	0.33	0.00	0.03	0.00	0.00	0.17	0.73	0.30	0.33	0.13	0.17	0.00	0.00	0.23	0.13	0.07	0.00	0.27	0.57	0.57	0.17	0.07
U	1.80	0.00	0.03	0.00	0.03	0.43	0.50	0.03	0.20	1.10	1.00	0.00	0.03	0.20	0.13	0.47	0.00	0.20	1.07	0.10	0.40	0.97
V	0.60	0.00	0.03	0.00	0.00	0.00	0.47	0.17	0.00	0.13	0.33	0.00	0.03	0.57	0.23	0.77	0.00	0.07	0.07	0.00	1.03	0.10

Table 2.43: January 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.57	0.00	1.33	0.00	0.57	0.40	2.20	0.20	0.40	0.73	3.70	0.00	0.03	1.27	0.60	1.40	0.00	2.03	3.90	0.77	2.80	0.67
B	0.00	0.00	0.00	0.00	0.00	0.03	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.00	0.00
C	0.97	0.00	0.30	0.00	0.00	0.70	0.13	0.00	0.00	0.30	0.07	0.00	0.00	0.03	0.03	0.00	0.00	0.10	0.13	0.03	0.07	0.23
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.60	0.00	0.00	0.00	0.03	0.63	0.00	0.00	0.10	1.20	0.00	0.00	0.00	0.07	0.00	0.07	0.00	0.00	0.20	0.13	0.00	0.03
F	0.40	0.00	0.43	0.00	0.53	0.37	1.93	0.20	0.03	0.30	1.03	0.00	0.00	0.73	0.83	0.37	0.00	0.50	0.33	0.27	0.87	0.00
G	2.80	0.10	0.03	0.00	0.00	1.57	0.67	0.17	0.07	8.60	0.17	0.00	0.07	0.10	0.00	3.10	0.00	0.20	0.47	0.90	0.30	0.20
H	0.10	0.00	0.00	0.00	0.00	0.47	0.33	0.20	0.00	1.10	0.00	0.00	0.10	0.33	0.00	0.03	0.00	0.07	0.07	0.13	0.30	0.13
I	0.27	0.00	0.07	0.00	0.03	0.07	0.07	0.03	0.00	0.23	0.17	0.00	0.00	0.07	0.50	0.10	0.00	0.20	0.03	0.27	0.63	0.17
J	0.80	0.00	0.27	0.00	0.93	0.10	9.17	1.93	0.47	0.90	0.70	0.00	0.03	0.80	0.40	0.57	0.00	0.93	1.17	0.23	1.43	0.20
K	3.00	0.00	0.17	0.00	0.10	1.40	0.53	0.00	0.07	0.33	0.50	0.00	0.00	0.00	0.50	0.40	0.00	0.20	0.27	0.03	0.77	0.33
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.10	0.00	0.00	0.07	0.03	0.03	0.03	0.00	0.07	0.00	0.03	0.33	0.03
N	1.23	0.00	0.00	0.00	0.03	1.17	0.17	0.27	0.00	1.17	0.03	0.00	0.00	0.07	0.10	0.20	0.00	0.00	0.03	0.13	0.17	0.37
O	1.13	0.00	0.00	0.00	0.07	0.60	0.07	0.03	0.50	0.33	0.20	0.00	0.00	0.03	0.20	0.13	0.00	0.13	0.70	0.50	0.03	0.03
P	0.87	0.00	0.13	0.00	0.00	0.53	4.17	0.07	0.07	0.20	0.17	0.00	0.13	0.37	0.13	0.97	0.00	0.10	0.37	0.10	0.53	0.60
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.13	0.00	0.00	0.00	0.07	0.57	0.23	0.03	0.03	0.60	0.17	0.00	0.20	0.30	0.13	0.13	0.00	0.10	0.50	0.23	0.43	0.03
S	4.00	0.00	0.27	0.00	0.20	0.30	0.50	0.23	0.00	1.27	0.63	0.00	0.00	0.13	0.83	0.47	0.00	0.50	0.57	0.33	0.53	0.07
T	0.67	0.00	0.03	0.00	0.00	0.17	1.00	0.10	0.43	0.10	0.13	0.00	0.07	0.23	0.13	0.17	0.00	0.13	0.37	0.50	0.27	0.00
U	2.77	0.00	0.00	0.00	0.00	0.30	0.47	0.20	0.33	2.00	0.63	0.00	0.10	0.20	0.03	0.43	0.00	0.47	0.63	0.20	0.40	0.93
V	0.63	0.00	0.10	0.00	0.00	0.00	0.27	0.17	0.07	0.13	0.50	0.00	0.00	0.37	0.10	0.40	0.00	0.00	0.10	0.20	1.17	0.07

Table 2.44: February 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.90	0.40	0.97	0.00	0.47	1.13	4.40	0.03	0.87	1.43	4.27	0.00	0.07	1.87	0.87	1.70	0.00	3.70	5.30	0.57	3.53	1.17
B	0.13	0.07	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.27	0.03	0.00	0.00	0.03	0.03	0.30	0.00	0.03	0.00	0.00	0.07	0.00
C	0.80	0.00	0.13	0.00	0.00	0.50	0.07	0.07	0.00	0.40	0.07	0.00	0.00	0.07	0.00	0.27	0.00	0.03	0.10	0.03	0.03	0.43
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.73	0.00	0.03	0.00	0.13	0.70	0.03	0.00	0.10	1.43	0.10	0.00	0.03	0.03	0.03	0.07	0.00	0.40	0.60	0.13	0.00	0.07
F	0.70	0.43	0.30	0.00	0.83	0.37	3.03	0.40	0.33	0.60	1.67	0.00	0.00	1.03	1.43	0.73	0.00	0.70	0.47	0.57	0.60	0.13
G	5.00	0.07	0.20	0.00	0.00	2.83	1.27	0.20	0.07	9.63	0.47	0.00	0.17	0.23	0.10	5.03	0.00	0.10	1.30	1.47	0.67	0.43
H	0.23	0.03	0.00	0.00	0.00	0.53	0.47	0.33	0.00	1.83	0.03	0.00	0.17	0.53	0.03	0.13	0.00	0.00	0.00	0.07	0.63	0.17
I	0.53	0.00	0.07	0.00	0.30	0.17	0.10	0.07	0.17	0.70	0.10	0.00	0.23	0.03	0.87	0.27	0.00	0.40	0.10	0.50	0.47	0.07
J	1.40	0.43	0.30	0.00	0.77	0.20	10.40	2.90	1.10	0.73	1.40	0.00	0.20	1.50	1.20	0.83	0.00	1.07	1.60	0.20	1.53	0.30
K	4.37	0.07	0.17	0.00	0.00	2.40	0.57	0.07	0.13	1.17	0.53	0.00	0.03	0.17	0.70	0.53	0.00	0.53	0.50	0.17	1.37	0.20
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.00	0.03	0.00	0.00	0.03	0.07	0.33	0.07	0.03	0.03	0.00	0.00	0.10	0.03	0.00	0.10	0.00	0.13	0.00	0.00	0.13	0.10
N	1.57	0.07	0.10	0.00	0.00	0.97	0.10	0.50	0.03	1.33	0.20	0.00	0.03	0.23	0.20	0.60	0.00	0.20	0.20	0.27	0.13	0.70
O	1.60	0.03	0.03	0.00	0.10	0.77	0.27	0.00	0.67	1.13	0.63	0.00	0.00	0.10	0.50	0.30	0.00	0.33	0.93	0.47	0.10	0.23
P	1.10	0.43	0.10	0.00	0.03	0.67	5.93	0.30	0.40	0.77	0.50	0.00	0.13	0.20	0.17	1.57	0.00	0.37	1.03	0.07	1.63	0.30
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	4.17	0.00	0.10	0.00	0.00	1.13	0.23	0.03	0.07	0.60	0.50	0.00	0.30	0.27	0.50	0.43	0.00	0.40	0.83	0.33	0.83	0.17
S	5.00	0.03	0.07	0.00	0.60	0.40	0.90	0.33	0.03	2.03	0.70	0.00	0.10	0.23	1.03	1.07	0.00	0.97	0.67	1.13	0.93	0.37
T	0.83	0.07	0.03	0.00	0.07	0.17	1.50	0.30	0.73	0.20	0.40	0.00	0.00	0.23	0.13	0.30	0.00	0.07	1.07	0.50	0.40	0.10
U	3.20	0.13	0.03	0.00	0.03	0.87	0.73	0.43	0.37	2.33	0.83	0.00	0.37	0.40	0.03	0.87	0.00	0.57	0.83	0.30	0.80	1.37
V	0.93	0.00	0.33	0.00	0.03	0.10	0.33	0.17	0.07	0.30	0.90	0.00	0.07	0.50	0.27	0.33	0.00	0.20	0.20	0.10	1.17	0.20

Table 2.45: March 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.87	0.67	0.57	0.00	0.53	1.20	4.10	0.30	0.73	1.30	4.40	0.00	0.07	2.40	1.37	1.07	0.00	2.83	5.27	0.80	3.57	1.73
B	0.23	0.00	0.23	0.00	0.03	0.03	0.13	0.07	0.00	0.20	0.10	0.00	0.03	0.00	0.10	0.87	0.00	0.03	0.00	0.03	0.10	0.03
C	0.70	0.30	0.43	0.00	0.10	0.67	0.07	0.00	0.03	0.50	0.13	0.00	0.00	0.03	0.03	0.17	0.00	0.13	0.07	0.03	0.07	0.40
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.77	0.00	0.07	0.00	0.03	0.83	0.10	0.00	0.53	0.97	0.23	0.00	0.33	0.00	0.13	0.13	0.00	0.20	0.50	0.13	0.03	0.23
F	1.97	0.27	0.53	0.00	0.67	0.40	2.37	0.50	0.30	0.23	1.73	0.00	0.03	1.00	0.63	0.40	0.00	0.47	0.67	0.40	0.47	0.17
G	5.30	0.23	0.10	0.00	0.03	2.13	1.53	0.17	0.20	9.77	0.67	0.00	0.33	0.03	0.13	4.93	0.00	0.23	0.60	1.60	0.43	0.90
H	0.40	0.13	0.00	0.00	0.00	0.17	0.50	0.07	0.00	1.63	0.10	0.00	0.13	0.23	0.17	0.20	0.00	0.10	0.13	0.13	0.33	0.17
I	0.63	0.00	0.13	0.00	0.13	0.23	0.40	0.10	0.00	0.87	0.07	0.00	0.17	0.17	0.87	0.43	0.00	0.33	0.10	0.63	0.17	0.13
J	1.43	0.60	0.30	0.00	0.80	0.30	10.93	2.00	1.03	1.47	2.10	0.00	0.20	1.67	1.60	0.73	0.00	0.93	1.20	0.43	2.03	0.37
K	4.87	0.30	0.10	0.00	0.10	2.43	0.70	0.03	0.17	1.40	0.57	0.00	0.13	0.23	0.77	0.53	0.00	0.53	0.77	0.23	1.10	0.63
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.07	0.03	0.03	0.00	0.40	0.07	0.50	0.13	0.07	0.07	0.03	0.00	0.27	0.03	0.13	0.17	0.00	0.20	0.00	0.03	0.27	0.03
N	1.80	0.00	0.10	0.00	0.00	1.20	0.23	0.10	0.17	2.20	0.17	0.00	0.03	0.37	0.07	0.57	0.00	0.17	0.20	0.33	0.73	0.93
O	1.50	0.10	0.00	0.00	0.00	0.53	0.20	0.13	1.03	1.40	0.60	0.00	0.07	0.07	0.50	0.30	0.00	0.20	0.77	0.23	0.00	0.10
P	0.77	0.93	0.27	0.00	0.13	0.60	5.87	0.50	0.50	0.90	0.47	0.00	0.10	0.73	0.03	1.60	0.00	0.13	0.80	0.13	1.03	0.97
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.67	0.00	0.23	0.00	0.00	0.73	0.50	0.07	0.13	0.80	0.53	0.00	0.60	0.33	0.67	0.23	0.00	0.53	0.67	0.20	0.37	0.03
S	4.23	0.00	0.17	0.00	0.47	0.40	0.73	0.17	0.03	1.33	0.93	0.00	0.07	0.27	1.17	1.23	0.00	1.00	0.80	0.77	0.57	0.37
T	0.63	0.17	0.10	0.00	0.03	0.13	2.33	0.17	0.63	0.13	0.33	0.00	0.00	0.40	0.20	0.17	0.00	0.20	0.57	0.93	0.13	0.07
U	2.73	0.03	0.07	0.00	0.03	0.77	0.53	0.23	0.03	2.07	0.80	0.00	0.23	1.13	0.03	1.13	0.00	0.40	0.37	0.20	1.00	1.17
V	0.83	0.03	0.30	0.00	0.17	0.03	1.07	0.27	0.03	0.30	0.93	0.00	0.07	1.03	0.17	1.10	0.00	0.07	0.23	0.10	1.37	0.10

Table 2.46: April 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.57	0.97	1.07	0.00	0.80	1.53	3.50	0.53	1.03	2.07	4.33	0.00	0.23	2.00	1.63	1.30	0.00	2.13	5.33	1.03	4.20	1.60
B	0.30	0.03	0.03	0.00	0.00	0.00	0.40	0.00	0.00	0.30	0.73	0.00	0.00	0.07	0.03	0.70	0.00	0.03	0.07	0.10	0.33	0.00
C	0.90	0.00	0.60	0.00	0.00	0.83	0.23	0.00	0.13	0.97	0.17	0.00	0.00	0.03	0.00	0.13	0.00	0.07	0.17	0.13	0.03	0.07
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	1.17	0.00	0.00	0.00	0.13	0.60	0.00	0.00	0.57	0.97	0.27	0.00	0.50	0.00	0.17	0.33	0.00	0.13	0.40	0.10	0.03	0.23
F	2.27	0.50	0.60	0.00	0.60	0.57	2.23	0.40	0.53	0.40	2.47	0.00	0.00	0.40	1.17	0.70	0.00	1.17	0.90	0.57	0.43	0.10
G	4.50	0.50	0.17	0.00	0.07	1.97	1.33	0.47	0.30	10.10	0.50	0.00	0.40	0.00	0.20	4.47	0.00	0.27	0.97	1.67	0.37	0.67
H	0.23	0.03	0.00	0.00	0.00	0.20	0.77	0.33	0.03	1.80	0.17	0.00	0.30	0.53	0.07	0.30	0.00	0.13	0.07	0.20	0.43	0.27
I	0.73	0.00	0.17	0.00	0.27	0.23	0.37	0.17	0.33	0.87	0.20	0.00	0.57	0.10	0.83	0.37	0.00	0.23	0.13	0.50	0.13	0.10
J	1.40	0.37	0.60	0.00	0.57	0.20	9.07	1.80	0.73	1.07	1.77	0.00	0.33	1.40	1.37	1.13	0.00	1.37	2.03	0.23	1.60	0.20
K	5.33	0.67	0.10	0.00	0.10	3.07	0.67	0.13	0.07	1.30	1.23	0.00	0.03	0.40	0.67	0.70	0.00	0.67	0.63	0.63	1.30	0.63
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.07	0.00	0.00	0.00	0.50	0.00	0.50	0.27	0.13	0.20	0.07	0.00	0.10	0.13	0.30	0.30	0.00	0.10	0.13	0.07	0.33	0.10
N	1.87	0.13	0.03	0.00	0.03	0.83	0.33	0.37	0.03	1.47	0.43	0.00	0.00	0.30	0.43	0.83	0.00	0.23	0.40	0.33	0.37	1.03
O	1.47	0.03	0.07	0.00	0.13	1.03	0.20	0.00	1.00	1.50	0.93	0.00	0.37	0.33	0.33	0.33	0.00	0.07	0.67	0.33	0.13	0.17
P	1.17	1.00	0.03	0.00	0.33	0.77	6.40	0.60	0.57	0.57	0.50	0.00	0.10	0.83	0.27	1.17	0.00	0.70	1.30	0.23	0.97	0.87
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.67	0.07	0.10	0.00	0.17	1.23	0.23	0.10	0.20	0.93	0.47	0.00	0.50	0.63	0.53	0.57	0.00	0.63	0.43	0.30	0.30	0.13
S	4.43	0.03	0.40	0.00	0.57	0.33	0.60	0.03	0.00	2.47	1.27	0.00	0.07	0.50	0.83	1.63	0.00	1.17	1.37	0.87	0.77	0.23
T	1.13	0.33	0.03	0.00	0.00	0.33	2.07	0.37	0.53	0.10	0.60	0.00	0.03	0.43	0.03	0.17	0.00	0.20	0.80	0.90	0.23	0.13
U	3.10	0.27	0.03	0.00	0.00	0.77	0.33	0.33	0.10	1.57	0.90	0.00	0.30	0.50	0.33	1.10	0.00	0.23	1.23	0.07	1.00	1.40
V	1.20	0.03	0.13	0.00	0.10	0.13	0.83	0.33	0.17	0.30	0.70	0.00	0.10	0.93	0.20	0.67	0.00	0.17	0.30	0.17	1.43	0.10

Table 2.47: May 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.77	0.90	0.97	0.00	0.27	1.23	4.50	0.47	1.13	2.40	2.23	0.00	0.07	2.10	1.13	0.97	0.00	2.27	4.33	0.63	2.90	0.97
B	0.43	0.00	0.03	0.00	0.00	0.00	0.03	0.03	0.00	0.47	0.13	0.00	0.03	0.50	0.00	0.43	0.00	0.00	0.03	0.07	0.23	0.07
C	0.80	0.07	0.57	0.00	0.03	0.87	0.10	0.00	0.00	1.43	0.30	0.00	0.00	0.07	0.10	0.17	0.00	0.13	0.10	0.17	0.10	0.13
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.57	0.00	0.10	0.00	0.13	0.47	0.23	0.00	0.07	0.77	0.23	0.00	0.37	0.00	0.03	0.07	0.00	0.20	0.60	0.03	0.13	0.03
F	1.53	0.20	0.67	0.00	0.40	0.37	2.27	0.47	0.23	0.37	1.57	0.00	0.10	0.23	0.90	1.00	0.00	0.80	0.67	0.30	0.70	0.13
G	4.33	0.07	0.10	0.00	0.30	1.93	1.30	0.07	0.60	10.03	0.43	0.00	0.93	0.20	0.30	4.83	0.00	0.30	0.53	1.73	0.20	0.40
H	0.17	0.03	0.00	0.00	0.00	0.03	0.33	0.23	0.03	1.43	0.00	0.00	0.10	0.33	0.00	0.30	0.00	0.10	0.03	0.07	0.13	0.27
I	0.83	0.00	0.07	0.00	0.10	0.20	0.40	0.03	0.43	0.37	0.20	0.00	0.23	0.10	0.90	0.53	0.00	0.20	0.07	0.23	0.43	0.00
J	1.63	0.53	0.70	0.00	0.43	0.27	10.17	1.70	0.63	1.23	1.70	0.00	0.47	1.57	1.20	1.17	0.00	1.33	2.47	0.37	1.03	0.10
K	3.37	0.20	0.17	0.00	0.03	1.87	0.47	0.07	0.13	0.87	0.83	0.00	0.00	0.27	1.43	0.57	0.00	0.67	0.90	0.60	0.93	0.53
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.07	0.00	0.03	0.00	0.57	0.03	0.63	0.13	0.07	0.23	0.07	0.00	0.07	0.20	0.13	0.17	0.00	0.30	0.10	0.07	0.43	0.00
N	2.00	0.73	0.13	0.00	0.07	0.27	0.17	0.33	0.17	1.50	0.20	0.00	0.13	0.37	0.17	0.93	0.00	0.20	0.20	0.40	0.33	0.73
O	0.77	0.03	0.03	0.00	0.10	1.13	0.20	0.00	1.13	1.60	1.17	0.00	0.20	0.03	0.40	0.10	0.00	0.37	0.63	0.33	0.13	0.10
P	1.13	0.53	0.23	0.00	0.03	0.80	5.53	0.60	0.70	0.77	1.07	0.00	0.07	0.40	0.10	1.57	0.00	0.60	0.80	0.17	0.97	0.60
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	2.53	0.00	0.10	0.00	0.03	1.03	0.17	0.03	0.13	0.83	0.63	0.00	0.47	0.53	0.23	0.50	0.00	0.70	0.80	0.30	0.33	0.07
S	3.63	0.10	0.30	0.00	0.50	0.30	0.60	0.07	0.03	2.73	1.20	0.00	0.10	0.23	0.77	1.67	0.00	0.57	0.63	0.67	0.77	0.23
T	0.83	0.17	0.07	0.00	0.00	0.20	2.27	0.03	0.20	0.10	0.57	0.00	0.13	0.50	0.27	0.13	0.00	0.20	0.67	0.43	0.30	0.13
U	2.27	0.20	0.03	0.00	0.00	0.83	0.23	0.13	0.27	1.43	0.67	0.00	0.37	0.30	0.27	0.80	0.00	0.20	0.73	0.30	0.90	1.00
V	0.50	0.00	0.07	0.00	0.03	0.03	0.43	0.27	0.03	0.23	0.60	0.00	0.17	1.00	0.30	0.40	0.00	0.10	0.20	0.20	0.67	0.33

Table 2.48: June 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.17	0.13	0.57	0.00	0.03	0.60	1.63	0.20	0.60	0.90	1.13	0.00	0.03	0.67	0.47	0.27	0.00	0.77	1.20	0.23	1.60	0.60
B	0.03	0.07	0.00	0.00	0.00	0.00	0.10	0.00	0.00	0.20	0.00	0.00	0.00	0.20	0.00	0.13	0.00	0.00	0.00	0.03	0.03	0.13
C	0.33	0.00	0.20	0.00	0.00	0.27	0.03	0.00	0.03	0.53	0.07	0.00	0.00	0.00	0.03	0.03	0.00	0.07	0.10	0.00	0.00	0.00
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.43	0.00	0.00	0.00	0.00	0.33	0.03	0.00	0.10	0.57	0.10	0.00	0.37	0.03	0.00	0.00	0.00	0.07	0.13	0.00	0.00	0.00
F	1.00	0.07	0.23	0.00	0.23	0.30	0.43	0.10	0.20	0.07	0.60	0.00	0.00	0.10	0.37	0.37	0.00	0.23	0.23	0.10	0.13	0.00
G	2.40	0.07	0.00	0.00	0.07	0.37	0.40	0.00	0.17	4.07	0.10	0.00	0.17	0.13	0.03	2.60	0.00	0.00	0.20	0.57	0.10	0.00
H	0.07	0.00	0.00	0.00	0.00	0.07	0.23	0.17	0.00	0.70	0.00	0.00	0.03	0.07	0.00	0.10	0.00	0.00	0.07	0.00	0.00	0.13
I	0.53	0.00	0.03	0.00	0.00	0.00	0.10	0.03	0.03	0.37	0.07	0.00	0.07	0.17	0.37	0.10	0.00	0.07	0.03	0.07	0.13	0.10
J	0.83	0.13	0.23	0.00	0.47	0.23	3.47	0.77	0.27	0.77	0.50	0.00	0.10	0.57	0.33	0.60	0.00	0.73	0.90	0.07	0.57	0.30
K	1.23	0.00	0.00	0.00	0.00	0.77	0.13	0.03	0.03	0.53	0.20	0.00	0.00	0.03	0.57	0.10	0.00	0.30	0.47	0.20	0.40	0.37
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.00	0.00	0.00	0.00	0.43	0.03	0.23	0.03	0.00	0.10	0.00	0.00	0.07	0.07	0.00	0.03	0.00	0.07	0.00	0.00	0.13	0.00
N	0.77	0.13	0.03	0.00	0.00	0.03	0.00	0.27	0.13	0.53	0.03	0.00	0.03	0.03	0.13	0.10	0.00	0.00	0.07	0.23	0.03	0.13
O	0.53	0.00	0.03	0.00	0.07	0.43	0.13	0.00	0.50	0.53	0.47	0.00	0.03	0.20	0.23	0.07	0.00	0.17	0.07	0.00	0.10	0.00
P	0.07	0.27	0.03	0.00	0.00	0.10	2.70	0.10	0.07	0.37	0.33	0.00	0.07	0.10	0.10	0.50	0.00	0.17	0.40	0.10	0.13	0.27
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	0.93	0.00	0.00	0.00	0.00	0.23	0.03	0.03	0.03	0.57	0.10	0.00	0.07	0.23	0.33	0.10	0.00	0.30	0.33	0.07	0.07	0.10
S	0.90	0.00	0.03	0.00	0.20	0.10	0.20	0.03	0.03	1.03	0.93	0.00	0.07	0.00	0.23	0.33	0.00	0.23	0.37	0.27	0.33	0.10
T	0.27	0.03	0.03	0.00	0.00	0.07	0.77	0.07	0.03	0.07	0.20	0.00	0.00	0.20	0.03	0.10	0.00	0.10	0.27	0.43	0.13	0.07
U	1.20	0.00	0.00	0.00	0.00	0.07	0.07	0.07	0.03	0.40	0.17	0.00	0.10	0.03	0.17	0.13	0.00	0.03	0.27	0.20	0.30	0.23
V	0.27	0.13	0.00	0.00	0.00	0.00	0.13	0.40	0.07	0.03	0.33	0.00	0.03	0.17	0.10	0.40	0.00	0.07	0.07	0.10	0.13	0.03

Table 2.49: July 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.23	0.47	0.30	0.00	0.00	0.87	1.37	0.23	0.37	1.00	1.00	0.00	0.10	0.43	1.00	0.43	0.00	0.20	1.50	1.23	1.37	0.47
B	0.37	0.03	0.00	0.00	0.00	0.00	0.37	0.00	0.00	0.03	0.00	0.00	0.13	0.03	0.03	0.17	0.00	0.00	0.23	0.00	0.23	0.00
C	0.30	0.00	0.20	0.00	0.03	0.37	0.17	0.00	0.03	0.37	0.03	0.00	0.03	0.00	0.00	0.03	0.00	0.03	0.00	0.10	0.07	0.10
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
E	0.27	0.00	0.07	0.00	0.33	0.43	0.03	0.00	0.10	0.80	0.23	0.00	0.17	0.07	0.13	0.20	0.00	0.07	0.20	0.20	0.00	0.03
F	0.97	0.40	0.20	0.00	0.37	0.40	0.97	0.53	0.20	0.07	1.40	0.00	0.10	0.27	0.67	0.47	0.00	0.20	0.67	0.50	0.37	0.03
G	1.57	0.23	0.17	0.00	0.00	0.87	1.03	0.10	0.10	7.07	0.30	0.00	0.57	0.27	0.03	3.50	0.00	0.77	0.13	0.30	0.33	0.27
H	0.27	0.03	0.00	0.00	0.07	0.47	0.20	0.23	0.03	1.40	0.10	0.00	0.03	0.07	0.00	0.07	0.00	0.27	0.03	0.00	0.03	0.03
I	0.37	0.03	0.00	0.00	0.17	0.07	0.07	0.13	0.07	0.27	0.10	0.00	0.07	0.03	0.70	0.30	0.00	0.10	0.20	0.07	0.03	0.03
J	1.00	0.20	0.50	0.00	0.87	0.20	6.80	1.20	0.43	1.47	1.37	0.00	0.23	0.63	1.07	1.30	0.00	0.53	1.43	1.93	1.63	0.17
K	0.57	0.00	0.13	0.00	0.07	1.80	0.27	0.10	0.03	1.10	0.90	0.00	0.13	0.00	0.47	0.40	0.00	0.20	0.37	0.80	0.70	0.40
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.10	0.07	0.07	0.00	0.30	0.10	0.73	0.07	0.03	0.07	0.10	0.00	0.03	0.17	0.07	0.30	0.00	0.00	0.07	0.20	0.13	0.07
N	0.67	0.10	0.03	0.00	0.00	0.07	0.33	0.03	0.07	0.63	0.07	0.00	0.07	0.33	0.03	0.93	0.00	0.10	0.13	0.80	0.03	0.43
O	0.63	0.00	0.00	0.00	0.17	0.70	0.07	0.00	0.63	1.07	0.67	0.00	0.03	0.17	2.30	0.13	0.00	0.03	0.37	0.30	0.23	0.07
P	0.67	0.37	0.13	0.00	0.20	0.10	3.83	0.50	0.27	0.80	0.33	0.00	0.17	0.73	0.30	0.93	0.00	0.13	0.57	1.07	0.30	0.33
Q	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
R	0.27	0.07	0.10	0.00	0.03	0.07	1.13	0.43	0.00	0.07	0.27	0.00	0.13	0.13	0.17	0.10	0.00	1.10	0.30	0.37	0.20	0.07
S	1.77	0.13	0.03	0.00	0.20	1.03	0.07	0.10	0.20	1.03	0.33	0.00	0.13	0.07	0.63	0.40	0.00	0.17	0.90	0.60	0.03	0.17
T	1.10	0.07	0.13	0.00	0.37	0.20	0.47	0.07	0.10	1.50	0.87	0.00	0.17	0.67	0.60	1.13	0.00	0.37	0.60	0.50	0.43	0.27
U	0.80	0.27	0.03	0.00	0.00	0.33	0.27	0.00	0.00	1.17	0.70	0.00	0.27	0.10	0.23	0.30	0.00	0.27	0.03	0.43	0.03	0.30
V	0.23	0.00	0.03	0.00	0.07	0.00	0.20	0.03	0.00	0.13	0.37	0.00	0.03	0.57	0.13	0.23	0.00	0.03	0.13	0.23	0.23	0.50

Table 2.50: August 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.57	0.30	0.87	0.27	0.80	1.07	3.87	0.17	2.00	2.53	4.80	0.00	0.20	1.43	2.50	1.27	0.13	0.73	2.67	3.90	3.03	1.63
B	0.40	0.07	0.00	0.00	0.00	0.03	0.47	0.00	0.00	0.33	0.03	0.00	0.17	0.00	0.00	0.40	0.00	0.03	0.13	0.17	0.80	0.10
C	1.13	0.00	0.60	0.00	0.00	0.50	0.13	0.00	0.07	0.40	0.17	0.00	0.07	0.07	0.17	0.03	0.00	0.10	0.17	0.10	0.03	0.33
D	0.27	0.00	0.00	0.10	0.03	0.10	0.03	0.00	0.00	0.60	0.17	0.00	0.10	0.03	0.00	0.07	0.03	0.13	0.03	0.07	0.00	0.23
E	0.80	0.00	0.00	0.03	0.10	1.17	0.17	0.03	0.13	1.37	0.20	0.00	0.47	0.00	0.17	0.27	0.03	0.17	0.37	0.30	0.23	0.13
F	1.27	0.60	0.47	0.13	0.80	0.57	2.23	0.57	0.37	0.37	2.17	0.00	0.10	0.63	1.00	1.20	0.03	0.20	1.07	0.70	0.40	0.10
G	5.93	0.47	0.10	0.00	0.23	2.13	1.13	0.20	0.30	12.97	0.67	0.00	1.60	0.27	0.37	6.43	0.00	1.20	0.20	1.03	0.93	0.40
H	0.20	0.00	0.07	0.03	0.00	0.63	0.07	0.17	0.00	1.57	0.10	0.00	0.07	0.23	0.10	0.17	0.03	0.40	0.00	0.10	0.20	0.07
I	1.43	0.07	0.17	0.00	0.03	0.33	0.20	0.10	0.17	0.90	0.23	0.00	0.20	0.03	0.83	0.40	0.03	0.53	0.20	0.20	0.30	0.20
J	2.80	0.60	0.53	0.57	1.23	0.33	13.40	1.93	0.87	1.33	1.73	0.00	0.33	2.13	2.07	1.13	0.20	0.33	1.70	2.73	2.50	0.33
K	4.47	0.10	0.07	0.20	0.03	2.40	1.10	0.20	0.20	0.83	1.17	0.00	0.33	0.43	0.60	0.47	0.07	0.60	0.57	0.93	0.93	0.63
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.07	0.17	0.13	0.20	0.40	0.07	2.17	0.10	0.03	0.27	0.17	0.00	0.73	0.10	0.13	0.20	0.00	0.13	0.17	0.13	0.67	0.10
N	1.57	0.13	0.07	0.00	0.00	0.60	0.10	0.33	0.20	1.53	0.37	0.00	0.13	0.37	0.17	0.53	0.00	0.43	0.30	1.20	0.40	0.17
O	1.97	0.00	0.13	0.03	0.20	1.67	0.23	0.07	0.93	2.33	0.63	0.00	0.03	0.20	1.60	0.70	0.00	0.20	1.20	0.57	0.40	0.23
P	1.00	0.87	0.10	0.07	0.23	0.93	6.90	0.63	0.40	0.80	0.77	0.00	0.37	0.63	0.27	1.50	0.00	0.30	1.17	1.27	1.67	0.47
Q	0.20	0.00	0.00	0.03	0.03	0.03	0.00	0.03	0.03	0.27	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.03	0.03	0.13	0.00	0.00
R	0.73	0.07	0.00	0.10	0.00	0.33	1.77	0.20	0.50	0.23	0.37	0.00	0.20	0.50	0.17	0.30	0.00	1.10	0.17	0.90	0.63	0.13
S	2.73	0.07	0.17	0.07	0.10	1.13	0.17	0.00	0.20	1.17	0.53	0.00	0.73	0.20	1.40	1.03	0.07	0.37	1.10	0.93	0.77	0.20
T	3.30	0.10	0.23	0.13	0.57	0.47	1.03	0.10	0.10	2.97	1.23	0.00	0.23	0.87	0.77	1.33	0.20	1.07	0.60	0.80	1.27	0.43
U	2.17	1.07	0.07	0.03	0.13	0.37	0.73	0.33	0.40	1.83	0.60	0.00	0.37	0.27	0.50	1.87	0.03	1.00	1.00	1.27	0.37	1.27
V	1.03	0.03	0.20	0.17	0.00	0.07	0.87	0.10	0.03	0.17	0.70	0.00	0.03	0.47	0.57	0.40	0.00	0.03	0.23	0.10	1.53	0.37

Table 2.51: September 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.53	0.50	1.17	0.27	0.97	0.87	4.60	0.03	1.37	3.00	5.83	0.00	0.13	1.00	2.93	1.70	0.17	2.77	3.67	1.03	3.83	2.43
B	0.17	0.43	0.07	0.00	0.03	0.03	0.27	0.00	0.00	0.03	0.00	0.00	0.13	0.00	0.00	0.63	0.00	0.07	0.00	0.00	0.40	0.00
C	2.00	0.00	0.57	0.13	0.00	0.73	0.17	0.00	0.07	0.37	0.07	0.00	0.03	0.13	0.13	0.10	0.00	0.10	0.40	0.17	0.30	0.40
D	0.50	0.00	0.17	0.07	0.10	0.30	0.20	0.03	0.07	2.87	0.37	0.00	0.20	0.07	0.10	0.07	0.13	0.10	0.23	0.17	0.07	0.93
E	1.10	0.03	0.10	0.00	0.33	2.00	0.10	0.00	0.20	1.13	0.07	0.00	0.43	0.07	0.30	0.17	0.40	0.13	0.33	0.20	0.20	0.13
F	0.87	0.37	0.87	0.37	1.30	0.70	2.40	0.50	0.27	0.27	2.27	0.00	0.33	1.10	1.07	1.20	0.40	1.03	0.60	0.20	0.77	0.03
G	5.47	0.33	0.30	0.17	0.03	2.10	0.93	0.27	0.00	13.40	0.53	0.00	1.50	0.20	0.17	6.63	0.00	0.20	0.90	1.37	1.13	0.37
H	0.10	0.00	0.03	0.07	0.00	0.60	0.37	0.10	0.00	1.10	0.13	0.00	0.43	0.17	0.00	0.07	0.00	0.00	0.00	0.03	0.03	0.43
I	1.50	0.00	0.17	0.00	0.10	0.10	0.07	0.13	0.17	0.70	0.10	0.00	0.10	0.07	0.83	0.23	0.00	0.33	0.20	0.37	0.83	0.13
J	2.97	0.47	0.80	2.17	0.77	0.37	11.73	1.47	1.20	0.93	1.30	0.00	0.47	1.30	1.67	1.73	0.60	1.87	2.03	0.60	1.93	0.23
K	5.17	0.00	0.13	0.30	0.07	2.33	0.80	0.23	0.03	1.10	0.83	0.00	0.27	0.30	0.53	0.63	0.03	0.40	0.37	1.07	1.43	0.47
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.03	0.17	0.00	0.33	0.27	0.30	2.13	0.20	0.10	0.20	0.03	0.00	0.73	0.27	0.10	0.17	0.00	0.10	0.03	0.17	0.70	0.13
N	1.90	0.07	0.03	0.07	0.03	0.83	0.27	0.33	0.07	1.47	0.20	0.00	0.23	0.30	0.20	0.77	0.00	0.10	0.43	0.80	0.23	0.13
O	2.20	0.03	0.20	0.03	0.07	0.97	0.20	0.03	0.90	2.00	0.73	0.00	0.03	0.13	1.07	0.63	0.37	0.93	0.87	0.43	0.40	0.20
P	1.40	0.60	0.20	0.23	0.03	1.00	7.50	0.43	0.30	1.10	0.60	0.00	0.30	0.87	0.47	2.20	0.00	1.30	1.60	0.30	1.97	0.50
Q	0.60	0.07	0.03	0.17	0.13	0.27	0.03	0.03	0.03	0.13	0.07	0.00	0.00	0.00	0.47	0.03	0.20	0.90	0.10	0.00	0.00	0.07
R	3.17	0.23	0.13	0.07	0.07	1.07	0.27	0.00	0.03	1.67	0.40	0.00	0.30	0.10	0.63	1.17	0.67	1.10	1.30	0.07	1.23	0.20
S	3.73	0.03	0.30	0.40	0.77	0.47	0.83	0.00	0.30	2.03	1.30	0.00	0.13	0.40	0.83	1.47	0.30	0.57	0.73	0.63	1.17	0.17
T	1.03	0.07	0.10	0.30	0.07	0.37	2.33	0.13	0.40	0.13	0.87	0.00	0.10	0.63	0.27	0.40	0.03	0.43	0.47	0.63	0.37	0.10
U	2.17	0.37	0.23	0.07	0.07	0.80	0.70	0.43	1.17	1.90	1.03	0.00	0.60	0.30	0.57	2.40	0.07	0.93	0.77	0.77	0.47	1.43
V	1.00	0.03	0.30	0.73	0.07	0.10	0.77	0.10	0.27	0.10	0.43	0.00	0.07	0.63	0.17	0.80	0.03	0.37	0.37	0.23	1.73	0.17

Table 2.52: October 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.30	0.60	0.97	0.27	0.77	1.03	5.37	0.23	1.80	2.47	6.67	0.00	0.07	1.53	2.43	2.40	0.20	2.63	3.50	0.83	3.57	1.80
B	0.00	0.07	0.07	0.10	0.00	0.20	0.23	0.00	0.07	0.07	0.07	0.00	0.17	0.03	0.00	0.37	0.03	0.10	0.03	0.13	0.37	0.03
C	1.50	0.00	0.57	0.07	0.00	0.87	0.20	0.07	0.13	0.10	0.10	0.00	0.17	0.07	0.13	0.23	0.27	0.17	0.07	0.13	0.17	0.50
D	0.20	0.10	0.13	0.03	0.10	0.20	0.13	0.00	0.07	1.30	0.57	0.00	0.07	0.07	0.10	0.07	0.17	0.07	0.10	0.03	0.00	0.43
E	1.03	0.00	0.00	0.07	0.10	1.10	0.17	0.07	0.10	1.27	0.03	0.00	0.50	0.07	0.40	0.20	0.17	0.07	0.60	0.17	0.03	0.10
F	1.03	0.40	0.80	0.27	0.67	0.80	2.20	0.53	0.20	0.60	1.73	0.00	0.27	1.07	0.83	0.83	0.57	1.03	0.67	0.30	1.10	0.13
G	6.80	0.13	0.33	0.17	0.00	1.63	0.90	0.13	0.00	10.87	0.37	0.00	0.70	0.07	0.27	7.00	0.00	0.27	1.07	2.00	0.90	0.97
H	0.10	0.00	0.00	0.10	0.03	0.47	0.47	0.03	0.03	1.37	0.17	0.00	0.30	0.17	0.07	0.20	0.03	0.07	0.10	0.03	0.30	0.30
I	1.30	0.07	0.27	0.03	0.03	0.07	0.07	0.00	0.23	0.77	0.20	0.00	0.30	0.10	1.10	0.13	0.07	0.33	0.13	0.20	1.37	0.10
J	2.87	0.53	0.97	0.90	1.33	0.43	10.60	1.90	0.90	0.97	1.33	0.00	0.43	1.83	1.63	1.10	0.83	2.53	2.33	0.73	2.93	0.00
K	5.60	0.03	0.00	0.43	0.03	2.57	0.90	0.10	0.07	1.03	0.50	0.00	0.43	0.37	0.73	0.83	0.10	0.50	0.50	1.30	1.60	0.70
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.10	0.23	0.00	0.07	0.17	0.53	1.50	0.33	0.07	0.17	0.03	0.00	0.43	0.27	0.17	0.33	0.03	0.03	0.07	0.00	1.10	0.07
N	1.73	0.07	0.07	0.10	0.00	0.80	0.30	0.27	0.13	1.67	0.30	0.00	0.23	0.40	0.07	0.90	0.03	0.07	0.13	0.97	0.13	0.20
O	1.67	0.00	0.10	0.07	0.30	1.27	0.23	0.03	1.13	2.27	1.07	0.00	0.03	0.03	0.83	0.67	0.20	0.60	0.70	0.30	0.23	0.20
P	1.57	0.53	0.13	0.13	0.07	0.73	8.33	0.37	0.13	0.93	0.87	0.00	0.37	0.77	0.63	2.13	0.03	1.17	1.27	0.53	1.30	0.30
Q	0.60	0.07	0.23	0.13	0.23	1.10	0.00	0.03	0.03	0.73	0.03	0.00	0.03	0.00	0.20	0.03	0.10	0.80	0.17	0.00	0.00	0.07
R	3.10	0.07	0.27	0.13	0.07	1.17	0.17	0.03	0.27	1.97	0.37	0.00	0.07	0.07	0.83	0.97	0.67	0.53	0.83	0.37	1.23	0.50
S	3.33	0.10	0.47	0.23	0.50	0.40	0.77	0.03	0.13	2.20	1.20	0.00	0.13	0.33	0.73	1.90	0.73	0.33	0.73	0.77	1.17	0.43
T	0.90	0.03	0.07	0.03	0.03	0.37	3.03	0.03	0.07	0.33	1.03	0.00	0.13	0.90	0.20	0.20	0.10	0.23	0.90	0.87	0.13	0.00
U	1.80	0.37	0.30	0.00	0.00	0.27	0.43	0.50	1.67	2.63	1.27	0.00	1.00	0.10	0.23	1.33	0.03	0.93	1.10	0.77	0.43	1.90
V	1.07	0.07	0.13	0.67	0.10	0.07	1.10	0.27	0.17	0.17	0.60	0.00	0.23	0.70	0.30	0.40	0.03	0.47	0.57	0.13	1.37	0.13

Table 2.53: November 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.27	0.40	0.43	0.20	0.63	1.03	4.17	0.10	0.97	2.30	4.33	0.07	0.07	1.40	2.27	1.27	0.77	2.00	3.23	0.53	2.17	0.87
B	0.03	0.07	0.07	0.10	0.00	0.23	0.13	0.00	0.20	0.03	0.00	0.00	0.00	0.13	0.00	0.20	0.00	0.07	0.07	0.07	0.20	0.00
C	1.10	0.00	0.33	0.13	0.03	0.80	0.20	0.03	0.10	0.13	0.00	0.00	0.03	0.03	0.07	0.20	0.20	0.03	0.27	0.07	0.13	0.37
D	0.23	0.07	0.13	0.07	0.00	0.13	0.03	0.00	0.30	1.83	0.80	0.00	0.03	0.03	0.03	0.03	0.17	0.07	0.27	0.13	0.00	0.13
E	0.90	0.03	0.03	0.00	0.00	1.37	0.13	0.03	0.10	1.37	0.17	0.00	0.33	0.03	0.20	0.27	0.03	0.13	0.43	0.23	0.00	0.10
F	0.90	0.10	0.73	0.10	0.87	0.87	2.17	0.43	0.43	0.30	2.37	0.00	0.20	0.73	1.63	0.47	0.53	0.87	0.67	0.33	0.67	0.17
G	5.27	0.23	0.27	0.00	0.07	1.27	0.80	0.03	0.03	9.97	0.57	0.10	0.97	0.40	0.10	7.87	0.00	0.20	0.70	2.37	0.67	0.70
H	0.20	0.00	0.03	0.00	0.03	0.13	0.10	0.00	0.03	1.40	0.03	0.00	0.10	0.13	0.00	0.23	0.03	0.03	0.07	0.17	0.07	0.03
I	1.17	0.10	0.13	0.27	0.03	0.27	0.03	0.07	0.10	0.33	0.33	0.00	0.30	0.03	0.50	0.33	0.03	0.17	0.20	0.13	1.17	0.17
J	2.43	0.17	0.60	1.03	0.47	0.17	9.77	1.07	0.80	0.83	1.47	0.00	0.43	1.00	1.70	1.00	1.13	2.17	1.70	0.17	1.83	0.07
K	3.93	0.03	0.03	0.60	0.00	2.87	0.70	0.17	0.40	1.07	0.37	0.00	0.13	0.23	0.63	0.53	0.07	0.60	0.93	0.30	1.13	0.57
L	0.00	0.03	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.03
M	0.03	0.07	0.03	0.07	0.30	0.33	1.20	0.10	0.10	0.03	0.13	0.00	0.00	0.10	0.07	0.10	0.00	0.00	0.07	0.03	0.57	0.07
N	1.20	0.03	0.10	0.00	0.00	0.63	0.20	0.17	0.00	1.13	0.07	0.00	0.20	0.17	0.07	0.70	0.00	0.23	0.23	0.53	0.17	0.37
O	1.33	0.00	0.03	0.00	0.10	1.43	0.27	0.07	0.90	1.87	0.80	0.00	0.07	0.07	0.33	0.60	0.07	0.33	0.83	0.30	0.50	0.30
P	0.93	0.30	0.30	0.07	0.00	0.60	6.90	0.63	0.17	0.60	0.63	0.00	0.23	0.83	0.43	1.40	0.10	1.00	1.33	0.13	0.80	0.93
Q	1.17	0.03	0.23	0.17	0.03	1.23	0.03	0.03	0.07	0.33	0.03	0.00	0.00	0.00	0.07	0.03	0.00	0.07	0.10	0.07	0.00	0.07
R	2.27	0.03	0.10	0.20	0.07	0.80	0.13	0.03	0.23	1.90	0.43	0.00	0.10	0.13	0.47	0.57	0.17	0.83	0.80	0.33	0.73	0.33
S	2.33	0.03	0.53	0.23	0.40	0.50	1.10	0.17	0.07	1.60	1.20	0.00	0.20	0.23	0.37	1.57	0.43	0.30	0.73	0.97	0.77	0.07
T	0.67	0.07	0.07	0.07	0.00	0.30	1.70	0.43	0.13	0.20	0.70	0.00	0.07	0.43	0.33	0.23	0.03	0.27	0.50	1.13	0.60	0.07
U	1.23	0.30	0.10	0.07	0.07	0.37	0.47	0.07	1.17	2.07	0.80	0.00	0.40	0.17	0.23	0.63	0.00	0.47	0.57	0.60	0.27	1.47
V	0.67	0.00	0.33	0.23	0.03	0.13	0.83	0.03	0.23	0.37	0.37	0.03	0.10	0.23	0.23	0.63	0.10	0.13	0.07	0.23	1.37	0.23

Table 2.54: December 2015: average daily rentals for each origin–destination pair.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.27	0.00	0.43	0.20	0.03	0.27	2.20	0.03	0.13	0.60	2.77	0.07	0.00	0.87	0.20	1.03	0.17	1.50	1.57	0.40	1.23	0.50
B	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.00	0.00
C	0.70	0.00	0.00	0.07	0.00	0.13	0.07	0.00	0.00	0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
D	0.20	0.00	0.13	0.03	0.00	0.13	0.03	0.00	0.07	1.30	0.37	0.00	0.03	0.03	0.03	0.03	0.13	0.07	0.10	0.03	0.00	0.13
E	0.10	0.00	0.00	0.00	0.00	0.13	0.00	0.00	0.03	0.23	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.10	0.07	0.00	0.00
F	0.23	0.00	0.10	0.10	0.00	0.17	0.63	0.20	0.03	0.10	0.80	0.00	0.00	0.07	0.30	0.23	0.40	0.20	0.17	0.03	0.17	0.00
G	2.53	0.07	0.00	0.00	0.00	0.33	0.47	0.03	0.00	6.37	0.07	0.10	0.03	0.03	0.00	3.10	0.00	0.07	0.13	0.43	0.17	0.20
H	0.03	0.00	0.00	0.00	0.00	0.13	0.10	0.00	0.00	0.60	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.03	0.03
I	0.13	0.00	0.00	0.00	0.00	0.07	0.03	0.00	0.00	0.20	0.03	0.00	0.00	0.00	0.27	0.10	0.00	0.00	0.03	0.10	0.17	0.00
J	0.47	0.00	0.07	0.90	0.00	0.10	6.57	1.07	0.30	0.47	0.60	0.00	0.03	0.47	0.23	0.33	0.60	0.83	0.80	0.00	0.93	0.00
K	2.07	0.00	0.00	0.30	0.00	0.73	0.23	0.00	0.03	0.33	0.07	0.00	0.00	0.00	0.07	0.03	0.03	0.10	0.27	0.00	0.10	0.10
L	0.00	0.03	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.03
M	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.00	0.00
N	0.77	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.73	0.00	0.00	0.00	0.07	0.00	0.13	0.00	0.00	0.03	0.00	0.03	0.03
O	0.70	0.00	0.00	0.00	0.00	0.37	0.00	0.00	0.13	0.23	0.10	0.00	0.00	0.00	0.03	0.03	0.07	0.13	0.23	0.27	0.00	0.00
P	0.43	0.00	0.00	0.07	0.00	0.17	3.73	0.07	0.07	0.20	0.17	0.00	0.00	0.17	0.03	0.73	0.00	0.10	0.23	0.00	0.53	0.20
Q	0.60	0.03	0.03	0.13	0.03	0.27	0.00	0.03	0.03	0.13	0.03	0.00	0.00	0.00	0.07	0.03	0.00	0.07	0.10	0.00	0.00	0.07
R	1.80	0.00	0.00	0.07	0.00	0.17	0.07	0.00	0.00	0.43	0.07	0.00	0.00	0.03	0.10	0.07	0.17	0.10	0.20	0.07	0.10	0.03
S	1.40	0.00	0.00	0.23	0.13	0.13	0.20	0.00	0.00	0.80	0.20	0.00	0.00	0.03	0.07	0.37	0.30	0.20	0.43	0.07	0.17	0.07
T	0.33	0.00	0.00	0.03	0.00	0.03	0.57	0.00	0.07	0.03	0.07	0.00	0.00	0.07	0.10	0.00	0.03	0.07	0.17	0.10	0.00	0.00
U	1.23	0.00	0.00	0.00	0.00	0.27	0.17	0.03	0.10	0.83	0.17	0.00	0.03	0.10	0.00	0.37	0.00	0.03	0.40	0.03	0.13	0.47
V	0.53	0.00	0.00	0.23	0.00	0.00	0.27	0.03	0.00	0.03	0.03	0.03	0.00	0.13	0.00	0.10	0.03	0.00	0.07	0.00	0.53	0.07

Table 2.55: Minimum m of the Uniform distribution.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.90	0.60	1.43	0.27	0.97	1.50	5.37	0.27	1.80	3.00	6.67	0.07	0.27	2.17	2.93	3.00	0.77	3.70	5.97	1.40	4.87	2.43
B	0.17	0.43	0.07	0.10	0.03	0.23	0.27	0.00	0.20	0.27	0.07	0.00	0.17	0.13	0.03	0.63	0.03	0.10	0.07	0.13	0.40	0.03
C	2.00	0.00	0.67	0.13	0.03	0.93	0.27	0.07	0.13	0.60	0.13	0.00	0.17	0.13	0.23	0.27	0.27	0.17	0.40	0.17	0.30	0.50
D	0.50	0.10	0.17	0.07	0.10	0.30	0.20	0.03	0.30	2.87	0.80	0.00	0.20	0.07	0.10	0.07	0.17	0.10	0.27	0.17	0.07	0.93
E	1.10	0.03	0.10	0.07	0.67	2.00	0.17	0.07	0.57	1.43	0.17	0.00	0.50	0.07	0.40	0.27	0.40	0.40	0.60	0.23	0.20	0.47
F	1.60	0.43	0.87	0.37	1.30	1.03	3.03	0.57	0.63	0.60	2.53	0.00	0.33	1.10	1.63	1.20	0.57	1.10	0.77	0.83	1.43	0.40
G	6.80	0.33	0.33	0.17	0.07	2.83	2.03	0.33	1.17	13.67	0.57	0.10	1.50	0.47	0.27	7.87	0.00	0.37	1.30	2.37	1.20	1.63
H	0.27	0.03	0.20	0.10	0.03	1.47	0.53	0.33	0.10	1.83	0.17	0.00	0.43	0.53	0.13	0.37	0.03	0.23	0.20	0.60	0.63	0.70
I	1.57	0.10	0.27	0.27	0.73	0.30	1.53	0.30	1.50	0.90	0.67	0.00	0.30	0.10	1.50	1.10	0.07	0.43	0.20	0.57	1.37	0.67
J	2.97	0.53	0.97	2.17	1.33	0.43	12.73	2.90	1.27	2.23	1.47	0.00	0.47	1.83	1.70	1.83	1.13	2.53	2.33	0.73	2.93	0.43
K	5.60	0.07	0.20	0.60	0.10	2.87	0.90	0.23	0.40	1.40	0.83	0.00	0.43	0.37	0.73	0.83	0.10	0.60	0.97	1.30	1.60	0.70
L	0.00	0.03	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.03
M	0.10	0.23	0.07	0.33	0.30	0.53	2.13	0.43	0.13	0.37	0.17	0.00	0.73	0.43	0.17	0.43	0.03	0.30	0.20	0.17	1.10	0.17
N	1.90	0.07	0.10	0.10	0.03	1.23	0.33	0.50	0.20	1.67	0.33	0.00	0.27	0.97	0.30	0.93	0.03	0.33	0.43	0.97	0.30	0.73
O	2.20	0.03	0.20	0.07	0.30	1.57	0.27	0.07	1.17	2.27	1.07	0.00	0.13	0.33	1.07	0.67	0.37	0.93	0.93	0.73	0.50	0.33
P	2.13	0.60	0.30	0.23	0.13	1.00	8.33	0.63	2.30	1.30	0.87	0.00	0.43	0.87	0.63	2.20	0.10	1.30	1.60	0.53	2.50	0.93
Q	1.17	0.07	0.23	0.17	0.23	1.23	0.03	0.03	0.07	0.73	0.07	0.00	0.03	0.00	0.47	0.03	0.20	0.90	0.17	0.07	0.00	0.07
R	4.17	0.23	0.27	0.20	0.10	1.17	0.30	0.13	0.33	1.97	0.67	0.00	0.37	0.77	0.83	1.17	0.67	1.10	1.30	0.43	1.23	0.50
S	5.00	0.10	0.53	0.40	0.77	0.83	1.10	0.40	0.30	2.27	1.40	0.00	0.20	0.47	1.03	1.90	0.73	1.37	1.40	1.17	1.23	0.53
T	1.30	0.07	0.10	0.30	0.10	0.63	3.03	0.67	0.80	0.33	1.03	0.00	0.23	0.90	0.33	0.40	0.10	0.53	1.07	1.13	0.60	0.17
U	4.93	0.37	0.30	0.07	0.07	0.87	1.03	0.50	1.67	2.63	1.27	0.00	1.00	0.40	0.57	2.57	0.07	0.93	1.23	0.77	1.30	1.90
V	1.43	0.07	0.47	0.73	0.27	0.17	2.00	1.00	0.30	0.37	0.90	0.03	0.30	0.90	0.60	0.80	0.10	0.47	0.57	0.33	1.73	0.40

Table 2.56: Maximum M of the Uniform distribution.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	1.71	3.33	1.07	4.29	2	1.13	0.26	6.67	1.03	0.56	0.21	15	7.5	0.66	0.64	0.5	2.14	0.38	0.27	1.11	0.33	0.68
B	12	4.62	30	20	60	8.57	6.67	-	10	7.5	30	-	12	15	60	3	60	20	30	15	5	60
C	0.74	-	3	10	60	1.88	6	30	15	2.86	15	-	12	15	8.57	7.5	7.5	12	5	12	6.67	4
D	2.86	20	6.67	20	20	4.62	8.57	60	5.45	0.48	1.71	-	8.57	20	15	20	6.67	12	5.45	10	30	1.88
E	1.67	60	20	30	3	0.94	12	30	3.33	1.2	12	-	4	30	5	7.5	4.62	5	2.86	6.67	10	4.29
F	1.09	4.62	2.07	4.29	1.54	1.67	0.55	2.61	3	2.86	0.6	-	6	1.71	1.03	1.4	2.07	1.54	2.14	2.31	1.25	5
G	0.21	5	6	12	30	0.63	0.8	5.45	1.71	0.1	3.16	10	1.3	4	7.5	0.18	-	4.62	1.4	0.71	1.46	1.09
H	6.67	60	10	20	60	1.25	3.16	6	20	0.82	12	-	4	3.75	15	5.45	60	8.57	10	3.16	3	2.73
I	1.18	20	7.5	7.5	2.73	5.45	1.28	6.67	1.33	1.82	2.86	-	6.67	20	1.13	1.67	30	4.62	8.57	3	1.3	3
J	0.58	3.75	1.94	0.65	1.5	3.75	0.1	0.5	1.28	0.74	0.97	-	4	0.87	1.03	0.92	1.15	0.59	0.64	2.73	0.52	4.62
K	0.26	30	10	2.22	20	0.56	1.76	8.57	4.62	1.15	2.22	-	4.62	5.45	2.5	2.31	15	2.86	1.62	1.54	1.18	2.5
L	-	30	-	-	-	-	15	-	-	-	-	-	-	-	-	-	-	-	30	-	-	30
M	20	8.57	30	5	6.67	3.75	0.94	4.62	15	5.45	12	-	2.73	4.62	12	4.29	60	6.67	10	12	1.82	12
N	0.75	30	20	20	60	1.54	6	4	10	0.83	6	-	7.5	1.94	6.67	1.88	60	6	4.29	2.07	6	2.61
O	0.69	60	10	30	6.67	1.03	7.5	30	1.54	0.8	1.71	-	15	6	1.82	2.86	4.62	1.88	1.71	2	4	6
P	0.78	3.33	6.67	6.67	15	1.71	0.17	2.86	0.85	1.33	1.94	-	4.62	1.94	3	0.68	20	1.43	1.09	3.75	0.66	1.76
Q	1.13	20	7.5	6.67	7.5	1.33	60	30	20	2.31	20	-	60	-	3.75	30	10	2.07	7.5	30	-	15
R	0.34	8.57	7.5	7.5	20	1.5	5.45	15	6	0.83	2.73	-	5.45	2.5	2.14	1.62	2.4	1.67	1.33	4	1.5	3.75
S	0.31	20	3.75	3.16	2.22	2.07	1.54	5	6.67	0.65	1.25	-	10	4	1.82	0.88	1.94	1.28	1.09	1.62	1.43	3.33
T	1.22	30	20	6	20	3	0.56	3	2.31	5.45	1.82	-	8.57	2.07	4.62	5	15	3.33	1.62	1.62	3.33	12
U	0.32	5.45	6.67	30	30	1.76	1.67	3.75	1.13	0.58	1.4	-	1.94	4	3.53	0.68	30	2.07	1.22	2.5	1.4	0.85
V	1.02	30	4.29	2.07	7.5	12	0.88	1.94	6.67	5	2.14	30	6.67	1.94	3.33	2.22	15	4.29	3.16	6	0.88	4.29

Table 2.57: Rate parameter λ of the Exponential distribution.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.58	0.30	0.93	0.23	0.50	0.88	3.78	0.15	0.97	1.80	4.72	0.07	0.13	1.52	1.57	2.02	0.47	2.60	3.77	0.90	3.05	1.47
B	0.08	0.22	0.03	0.05	0.02	0.12	0.15	0.00	0.10	0.13	0.03	0.00	0.08	0.07	0.02	0.33	0.02	0.05	0.03	0.07	0.20	0.02
C	1.35	0.00	0.33	0.10	0.02	0.53	0.17	0.03	0.07	0.35	0.07	0.00	0.08	0.07	0.12	0.13	0.13	0.08	0.20	0.08	0.15	0.25
D	0.35	0.05	0.15	0.05	0.05	0.22	0.12	0.02	0.18	2.08	0.58	0.00	0.12	0.05	0.07	0.05	0.15	0.08	0.18	0.10	0.03	0.53
E	0.60	0.02	0.05	0.03	0.33	1.07	0.08	0.03	0.30	0.83	0.08	0.00	0.25	0.03	0.20	0.13	0.22	0.20	0.35	0.15	0.10	0.23
F	0.92	0.22	0.48	0.23	0.65	0.60	1.83	0.38	0.33	0.35	1.67	0.00	0.17	0.58	0.97	0.72	0.48	0.65	0.47	0.43	0.80	0.20
G	4.67	0.20	0.17	0.08	0.03	1.58	1.25	0.18	0.58	10.02	0.32	0.10	0.77	0.25	0.13	5.48	0.00	0.22	0.72	1.40	0.68	0.92
H	0.15	0.02	0.10	0.05	0.02	0.80	0.32	0.17	0.05	1.22	0.08	0.00	0.25	0.27	0.07	0.18	0.02	0.12	0.10	0.32	0.33	0.37
I	0.85	0.05	0.13	0.13	0.37	0.18	0.78	0.15	0.75	0.55	0.35	0.00	0.15	0.05	0.88	0.60	0.03	0.22	0.12	0.33	0.77	0.33
J	1.72	0.27	0.52	1.53	0.67	0.27	9.65	1.98	0.78	1.35	1.03	0.00	0.25	1.15	0.97	1.08	0.87	1.68	1.57	0.37	1.93	0.22
K	3.83	0.03	0.10	0.45	0.05	1.80	0.57	0.12	0.22	0.87	0.45	0.00	0.22	0.18	0.40	0.43	0.07	0.35	0.62	0.65	0.85	0.40
L	0.00	0.03	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.03
M	0.05	0.12	0.03	0.20	0.15	0.27	1.07	0.22	0.07	0.18	0.08	0.00	0.37	0.22	0.08	0.23	0.02	0.15	0.10	0.08	0.55	0.08
N	1.33	0.03	0.05	0.05	0.02	0.65	0.17	0.25	0.10	1.20	0.17	0.00	0.13	0.52	0.15	0.53	0.02	0.17	0.23	0.48	0.17	0.38
O	1.45	0.02	0.10	0.03	0.15	0.97	0.13	0.03	0.65	1.25	0.58	0.00	0.07	0.17	0.55	0.35	0.22	0.53	0.58	0.50	0.25	0.17
P	1.28	0.30	0.15	0.15	0.07	0.58	6.03	0.35	1.18	0.75	0.52	0.00	0.22	0.52	0.33	1.47	0.05	0.70	0.92	0.27	1.52	0.57
Q	0.88	0.05	0.13	0.15	0.13	0.75	0.02	0.03	0.05	0.43	0.05	0.00	0.02	0.00	0.27	0.03	0.10	0.48	0.13	0.03	0.00	0.07
R	2.98	0.12	0.13	0.13	0.05	0.67	0.18	0.07	0.17	1.20	0.37	0.00	0.18	0.40	0.47	0.62	0.42	0.60	0.75	0.25	0.67	0.27
S	3.20	0.05	0.27	0.32	0.45	0.48	0.65	0.20	0.15	1.53	0.80	0.00	0.10	0.25	0.55	1.13	0.52	0.78	0.92	0.62	0.70	0.30
T	0.82	0.03	0.05	0.17	0.05	0.33	1.80	0.33	0.43	0.18	0.55	0.00	0.12	0.48	0.22	0.20	0.07	0.30	0.62	0.62	0.30	0.08
U	3.08	0.18	0.15	0.03	0.03	0.57	0.60	0.27	0.88	1.73	0.72	0.00	0.52	0.25	0.28	1.47	0.03	0.48	0.82	0.40	0.72	1.18
V	0.98	0.03	0.23	0.48	0.13	0.08	1.13	0.52	0.15	0.20	0.47	0.03	0.15	0.52	0.30	0.45	0.07	0.23	0.32	0.17	1.13	0.23

Table 2.58: Expected value $\bar{\xi}$ of the Normal distribution.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	-0.6	-1.3	-0.1	-1.5	-0.8	-0.2	1.3	-2.0	-0.2	0.5	1.5	-3.1	-2.2	0.4	0.3	0.6	-1.0	0.9	1.3	-0.2	1.1	0.3
B	-2.6	-1.6	-3.6	-3.5	-4.2	-2.3	-2.0	-7.1	-2.4	-2.1	-3.5	-7.1	-2.6	-2.8	-4.2	-1.2	-4.6	-3.1	-3.5	-2.8	-1.7	-4.2
C	0.2	-7.1	-1.3	-2.4	-4.5	-0.8	-1.8	-3.7	-3.0	-1.1	-2.9	-7.3	-2.8	-3.0	-2.4	-2.2	-2.4	-2.7	-1.8	-2.7	-2.1	-1.6
D	-1.2	-3.4	-1.9	-3.1	-3.5	-1.6	-2.4	-4.6	-1.9	0.7	-0.6	-7.4	-2.4	-3.1	-2.9	-3.1	-1.9	-2.5	-1.8	-2.5	-3.9	-0.9
E	-0.7	-4.2	-3.3	-3.9	-1.3	-0.1	-2.7	-3.6	-1.4	-0.3	-2.6	-7.3	-1.7	-3.7	-1.9	-2.2	-1.8	-1.8	-1.1	-2.0	-2.6	-1.7
F	-0.2	-1.6	-0.9	-1.6	-0.6	-0.6	0.5	-1.0	-1.3	-1.2	0.5	-7.3	-2.0	-0.7	-0.1	-0.4	-0.7	-0.5	-0.8	-1.0	-0.3	-1.8
G	1.5	-1.7	-2.0	-2.9	-3.7	0.3	0.1	-1.8	-0.8	2.3	-1.3	-2.6	-0.5	-1.5	-2.2	1.7	-7.4	-1.6	-0.5	0.2	-0.5	-0.2
H	-2.0	-4.2	-2.6	-3.4	-4.5	-0.3	-1.2	-1.9	-3.2	0.1	-2.7	-7.3	-1.5	-1.5	-2.9	-1.9	-4.6	-2.4	-2.5	-1.4	-1.2	-1.1
I	-0.3	-3.1	-2.2	-2.4	-1.2	-1.8	-0.5	-2.1	-0.5	-0.7	-1.2	-7.3	-2.1	-3.2	-0.2	-0.7	-3.7	-1.7	-2.3	-1.2	-0.4	-1.3
J	0.4	-1.4	-0.8	0.3	-0.6	-1.4	2.2	0.6	-0.3	0.2	0.0	-7.3	-1.6	0.1	-0.2	0.0	-0.2	0.5	0.4	-1.2	0.6	-1.7
K	1.3	-3.5	-2.5	-0.9	-3.2	0.5	-0.6	-2.4	-1.7	-0.2	-0.9	-7.3	-1.8	-1.9	-1.1	-1.0	-2.8	-1.2	-0.6	-0.7	-0.3	-1.0
L	-7.3	-3.5	-7.3	-7.4	-7.3	-7.3	-3.1	-7.3	-7.3	-7.3	-7.3	-7.3	-7.3	-3.7	-7.3	-7.3	-7.4	-7.3	-3.7	-7.3	-7.3	-3.7
M	-3.2	-2.2	-3.7	-1.9	-2.2	-1.6	-0.2	-1.7	-2.9	-1.9	-2.8	-7.3	-1.3	-1.7	-2.7	-1.6	-4.6	-2.1	-2.5	-2.7	-0.8	-2.7
N	0.3	-3.5	-3.3	-3.4	-4.5	-0.6	-2.0	-1.6	-2.5	0.2	-2.0	-7.3	-2.2	-0.8	-2.1	-0.7	-4.6	-1.9	-1.6	-0.9	-1.9	-1.1
O	0.3	-4.2	-2.5	-3.7	-2.1	-0.1	-2.2	-3.7	-0.6	0.0	-0.7	-7.3	-3.0	-2.0	-0.8	-1.2	-1.7	-0.8	-0.6	-0.7	-1.6	-2.0
P	0.2	-1.3	-2.1	-2.0	-2.9	-0.6	1.8	-1.2	-0.1	-0.4	-0.7	-7.3	-1.7	-0.8	-1.3	0.3	-3.4	-0.6	-0.2	-1.5	0.3	-0.7
Q	-0.2	-3.1	-2.3	-1.9	-2.2	-0.5	-4.6	-3.4	-3.1	-1.0	-3.1	-7.4	-4.6	-7.4	-1.6	-3.4	-2.6	-1.1	-2.1	-3.9	-7.4	-2.7
R	1.1	-2.3	-2.2	-2.1	-3.2	-0.5	-1.8	-2.9	-2.0	0.1	-1.1	-7.3	-1.9	-1.1	-0.9	-0.6	-1.1	-0.6	-0.4	-1.5	-0.6	-1.5
S	1.1	-3.1	-1.5	-1.2	-0.9	-0.8	-0.5	-1.8	-2.1	0.4	-0.3	-7.3	-2.5	-1.5	-0.7	0.0	-0.7	-0.4	-0.1	-0.6	-0.5	-1.3
T	-0.2	-3.5	-3.2	-2.1	-3.2	-1.2	0.5	-1.3	-0.9	-1.8	-0.8	-7.3	-2.4	-0.9	-1.6	-1.7	-2.9	-1.3	-0.6	-0.6	-1.3	-2.6
U	1.0	-1.8	-2.1	-3.9	-3.6	-0.6	-0.6	-1.5	-0.3	0.5	-0.4	-7.3	-0.9	-1.5	-1.5	0.2	-3.7	-0.9	-0.3	-1.1	-0.4	0.1
V	-0.1	-3.5	-1.6	-0.9	-2.3	-2.7	0.0	-0.9	-2.1	-1.7	-0.9	-3.7	-2.1	-0.7	-1.3	-0.9	-2.9	-1.7	-1.3	-2.0	0.1	-1.5

Table 2.59: Location parameter ζ of the Lognormal distribution.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.42	0.42	0.30	0.17	0.55	0.43	0.27	0.50	0.54	0.42	0.26	0.83	0.66	0.26	0.56	0.33	0.67	0.29	0.33	0.31	0.32	0.42
B	0.49	0.47	0.55	0.96	0.55	0.47	0.38	0.55	0.51	0.46	0.50	0.55	0.52	0.47	0.55	0.35	0.96	0.39	0.50	0.50	0.45	0.55
C	0.32	0.55	0.62	0.39	0.85	0.50	0.34	0.76	0.73	0.38	0.68	0.85	0.77	0.73	0.73	0.61	0.87	0.62	0.65	0.65	0.60	0.65
D	0.46	1.82	0.14	0.39	0.96	0.37	0.65	0.96	0.69	0.37	0.36	0.96	0.69	0.39	0.56	0.39	0.14	0.24	0.47	0.64	0.96	0.68
E	0.53	1.43	0.79	0.96	0.65	0.57	0.66	0.70	0.63	0.51	0.57	0.85	0.79	0.73	0.76	0.65	0.75	0.63	0.42	0.39	0.76	0.69
F	0.39	0.47	0.56	0.54	0.61	0.48	0.34	0.28	0.59	0.52	0.33	0.85	0.67	0.56	0.40	0.39	0.19	0.41	0.39	0.57	0.44	0.67
G	0.26	0.30	0.60	0.96	0.81	0.47	0.38	0.42	0.68	0.24	0.46	0.83	0.66	0.52	0.62	0.29	0.96	0.39	0.57	0.45	0.49	0.47
H	0.44	0.55	0.75	0.85	0.85	0.48	0.40	0.56	0.70	0.31	0.64	0.85	0.49	0.61	0.68	0.63	0.96	0.65	0.68	0.66	0.46	0.53
I	0.56	0.51	0.63	0.91	0.70	0.46	0.67	0.68	0.64	0.43	0.57	0.85	0.70	0.60	0.43	0.58	0.83	0.53	0.53	0.49	0.49	0.62
J	0.46	0.45	0.51	0.45	0.57	0.46	0.19	0.29	0.38	0.41	0.31	0.85	0.60	0.31	0.56	0.40	0.30	0.37	0.34	0.57	0.31	0.53
K	0.27	0.44	0.62	0.33	0.70	0.37	0.34	0.65	0.60	0.32	0.49	0.85	0.74	0.56	0.55	0.53	0.47	0.51	0.38	0.69	0.49	0.49
L	0.85	0.50	0.85	0.96	0.85	0.85	0.83	0.85	0.85	0.85	0.85	0.85	0.85	0.83	0.85	0.85	0.96	0.85	0.83	0.85	0.85	0.83
M	0.67	0.43	0.81	0.71	0.81	0.70	0.68	0.55	0.65	0.63	0.73	0.85	0.71	0.62	0.68	0.54	0.96	0.63	0.68	0.70	0.60	0.62
N	0.25	0.50	0.75	0.85	0.85	0.57	0.58	0.63	0.64	0.19	0.65	0.85	0.66	0.51	0.60	0.49	0.96	0.55	0.55	0.54	0.49	0.56
O	0.31	0.55	0.63	0.83	0.69	0.42	0.68	0.73	0.49	0.60	0.54	0.85	0.77	0.62	0.67	0.59	0.63	0.50	0.42	0.30	0.72	0.57
P	0.39	0.43	0.60	0.52	0.64	0.37	0.24	0.48	0.68	0.40	0.39	0.85	0.56	0.47	0.61	0.34	0.85	0.63	0.40	0.68	0.43	0.41
Q	0.38	0.39	0.78	0.14	0.67	0.65	0.96	0.00	0.39	0.64	0.39	0.96	0.96	0.96	0.68	0.00	0.83	0.82	0.30	0.96	0.96	0.00
R	0.24	0.48	0.68	0.47	0.64	0.47	0.39	0.65	0.58	0.42	0.49	0.85	0.59	0.60	0.44	0.53	0.66	0.45	0.48	0.40	0.56	0.51
S	0.32	0.39	0.59	0.31	0.41	0.44	0.43	0.65	0.63	0.32	0.43	0.85	0.55	0.44	0.50	0.44	0.42	0.51	0.33	0.52	0.46	0.51
T	0.28	0.50	0.64	0.77	0.60	0.50	0.49	0.64	0.42	0.42	0.58	0.85	0.64	0.51	0.33	0.50	0.56	0.45	0.46	0.48	0.53	0.57
U	0.41	0.47	0.63	0.96	0.67	0.35	0.34	0.57	0.57	0.30	0.43	0.85	0.66	0.40	0.71	0.53	0.83	0.55	0.33	0.61	0.48	0.33
V	0.33	0.50	0.60	0.54	0.74	0.64	0.42	0.63	0.59	0.50	0.54	0.83	0.63	0.42	0.51	0.48	0.56	0.63	0.56	0.60	0.34	0.44

Table 2.60: Scale parameter η of the Lognormal distribution.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.26	0.13	0.29	0.04	0.30	0.40	1.02	0.08	0.57	0.79	1.25	0.07	0.10	0.40	0.95	0.69	0.35	0.78	1.29	0.28	1.00	0.64
B	0.04	0.11	0.02	0.06	0.01	0.06	0.06	0.00	0.05	0.07	0.02	0.00	0.05	0.03	0.01	0.12	0.02	0.02	0.02	0.04	0.09	0.01
C	0.45	0.00	0.23	0.04	0.02	0.29	0.06	0.03	0.06	0.14	0.05	0.00	0.07	0.06	0.10	0.09	0.14	0.06	0.15	0.06	0.10	0.18
D	0.17	0.05	0.02	0.02	0.06	0.08	0.08	0.02	0.14	0.80	0.22	0.00	0.09	0.02	0.04	0.02	0.02	0.02	0.09	0.07	0.04	0.41
E	0.34	0.01	0.05	0.04	0.24	0.66	0.06	0.03	0.21	0.46	0.05	0.00	0.23	0.03	0.18	0.10	0.19	0.14	0.15	0.06	0.09	0.18
F	0.37	0.11	0.29	0.14	0.44	0.31	0.65	0.11	0.21	0.20	0.57	0.00	0.13	0.36	0.40	0.29	0.09	0.28	0.19	0.27	0.37	0.15
G	1.23	0.06	0.11	0.10	0.03	0.79	0.50	0.08	0.45	2.40	0.16	0.10	0.57	0.14	0.09	1.61	0.00	0.09	0.44	0.66	0.35	0.45
H	0.07	0.01	0.09	0.05	0.02	0.41	0.13	0.10	0.04	0.39	0.06	0.00	0.13	0.18	0.05	0.13	0.02	0.09	0.08	0.23	0.16	0.21
I	0.51	0.03	0.09	0.15	0.29	0.09	0.59	0.12	0.53	0.25	0.21	0.00	0.12	0.03	0.39	0.38	0.03	0.12	0.07	0.17	0.40	0.23
J	0.83	0.13	0.28	0.73	0.42	0.13	1.84	0.58	0.31	0.58	0.33	0.00	0.16	0.37	0.59	0.46	0.27	0.64	0.54	0.23	0.60	0.12
K	1.07	0.02	0.07	0.15	0.04	0.68	0.20	0.09	0.14	0.28	0.23	0.00	0.18	0.11	0.24	0.24	0.03	0.19	0.24	0.51	0.44	0.21
L	0.00	0.02	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.00	0.00	0.03	0.00	0.00	0.03
M	0.04	0.05	0.03	0.16	0.14	0.21	0.81	0.13	0.05	0.13	0.07	0.00	0.30	0.15	0.06	0.14	0.02	0.11	0.08	0.07	0.36	0.06
N	0.33	0.02	0.04	0.05	0.02	0.40	0.11	0.18	0.07	0.23	0.12	0.00	0.10	0.28	0.10	0.28	0.02	0.10	0.14	0.28	0.09	0.23
O	0.46	0.01	0.07	0.03	0.12	0.43	0.10	0.03	0.34	0.83	0.34	0.00	0.06	0.11	0.42	0.23	0.15	0.28	0.25	0.16	0.21	0.10
P	0.52	0.13	0.10	0.08	0.05	0.23	1.48	0.18	0.91	0.31	0.21	0.00	0.13	0.26	0.22	0.51	0.05	0.49	0.39	0.20	0.68	0.24
Q	0.35	0.02	0.12	0.02	0.10	0.54	0.02	0.00	0.02	0.31	0.02	0.00	0.02	0.00	0.21	0.00	0.10	0.47	0.04	0.04	0.00	0.00
R	0.71	0.06	0.10	0.07	0.04	0.33	0.07	0.05	0.11	0.53	0.19	0.00	0.12	0.26	0.22	0.35	0.31	0.29	0.38	0.10	0.40	0.15
S	1.05	0.02	0.17	0.10	0.19	0.22	0.29	0.15	0.10	0.50	0.36	0.00	0.06	0.12	0.30	0.52	0.22	0.42	0.31	0.34	0.34	0.16
T	0.23	0.02	0.04	0.15	0.03	0.18	0.93	0.24	0.19	0.08	0.35	0.00	0.08	0.26	0.07	0.11	0.04	0.14	0.30	0.32	0.17	0.05
U	1.30	0.09	0.11	0.04	0.03	0.21	0.21	0.16	0.55	0.54	0.32	0.00	0.38	0.10	0.23	0.83	0.03	0.29	0.28	0.27	0.37	0.41
V	0.33	0.02	0.15	0.28	0.11	0.06	0.50	0.36	0.10	0.11	0.27	0.03	0.10	0.22	0.16	0.23	0.04	0.16	0.19	0.11	0.39	0.11

Table 2.61: Standard deviation σ of the Normal distribution.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	0.14	0.13	0.25	0.01	0.23	0.33	1.36	0.04	0.51	0.88	1.92	0.00	0.05	0.36	1.08	0.65	0.13	0.77	2.35	0.25	1.71	0.63
B	0.03	0.09	0.01	0.02	0.01	0.04	0.04	0.00	0.04	0.05	0.01	0.00	0.03	0.02	0.01	0.13	0.01	0.02	0.01	0.02	0.08	0.01
C	0.36	0.00	0.15	0.01	0.01	0.19	0.04	0.01	0.02	0.10	0.02	0.00	0.03	0.02	0.04	0.05	0.05	0.03	0.08	0.03	0.06	0.10
D	0.06	0.02	0.01	0.01	0.02	0.03	0.03	0.01	0.04	0.47	0.09	0.00	0.03	0.01	0.01	0.01	0.01	0.01	0.03	0.02	0.01	0.19
E	0.25	0.01	0.02	0.01	0.15	0.60	0.03	0.01	0.11	0.32	0.03	0.00	0.10	0.01	0.08	0.05	0.07	0.08	0.10	0.03	0.04	0.10
F	0.38	0.09	0.18	0.05	0.36	0.21	0.88	0.07	0.13	0.10	0.54	0.00	0.06	0.26	0.37	0.24	0.03	0.22	0.13	0.19	0.34	0.08
G	2.23	0.05	0.06	0.03	0.01	0.94	0.47	0.06	0.31	5.66	0.10	0.00	0.42	0.09	0.05	2.69	0.00	0.06	0.31	0.63	0.26	0.41
H	0.04	0.01	0.04	0.02	0.01	0.37	0.09	0.06	0.02	0.33	0.03	0.00	0.07	0.11	0.02	0.07	0.01	0.04	0.04	0.12	0.13	0.15
I	0.41	0.02	0.05	0.05	0.17	0.04	0.44	0.06	0.44	0.16	0.14	0.00	0.06	0.02	0.33	0.25	0.01	0.09	0.03	0.10	0.32	0.15
J	0.94	0.11	0.22	0.34	0.37	0.06	4.20	0.59	0.24	0.55	0.21	0.00	0.09	0.38	0.42	0.44	0.11	0.52	0.45	0.17	0.67	0.09
K	1.63	0.01	0.04	0.06	0.02	0.73	0.15	0.04	0.07	0.27	0.18	0.00	0.09	0.07	0.15	0.19	0.01	0.10	0.16	0.36	0.44	0.13
L	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M	0.02	0.04	0.01	0.05	0.06	0.11	0.73	0.09	0.02	0.07	0.03	0.00	0.17	0.09	0.03	0.08	0.01	0.06	0.04	0.03	0.28	0.03
N	0.30	0.01	0.02	0.02	0.01	0.31	0.06	0.10	0.04	0.23	0.06	0.00	0.05	0.22	0.06	0.19	0.01	0.06	0.08	0.24	0.05	0.16
O	0.44	0.01	0.04	0.01	0.06	0.32	0.05	0.01	0.26	0.68	0.24	0.00	0.02	0.06	0.26	0.14	0.06	0.19	0.16	0.10	0.10	0.06
P	0.52	0.13	0.06	0.03	0.02	0.20	2.53	0.12	0.79	0.28	0.16	0.00	0.09	0.16	0.13	0.42	0.02	0.32	0.38	0.11	0.65	0.17
Q	0.12	0.01	0.04	0.01	0.04	0.24	0.01	0.00	0.01	0.13	0.01	0.00	0.01	0.00	0.08	0.00	0.04	0.20	0.01	0.01	0.00	0.00
R	0.86	0.04	0.05	0.02	0.02	0.25	0.04	0.02	0.06	0.45	0.13	0.00	0.07	0.17	0.17	0.28	0.10	0.25	0.28	0.07	0.30	0.10
S	1.68	0.02	0.11	0.03	0.14	0.16	0.22	0.08	0.06	0.42	0.32	0.00	0.04	0.09	0.24	0.45	0.09	0.31	0.24	0.28	0.27	0.10
T	0.24	0.01	0.02	0.05	0.02	0.13	0.92	0.15	0.17	0.06	0.24	0.00	0.04	0.20	0.04	0.08	0.01	0.10	0.22	0.26	0.13	0.03
U	1.76	0.07	0.06	0.01	0.01	0.13	0.21	0.10	0.47	0.57	0.28	0.00	0.24	0.06	0.12	0.77	0.01	0.22	0.20	0.17	0.31	0.41
V	0.22	0.01	0.10	0.10	0.05	0.03	0.54	0.24	0.06	0.06	0.21	0.00	0.06	0.18	0.13	0.16	0.01	0.10	0.10	0.06	0.32	0.06

Table 2.62: Standard deviation σ of the Uniform distribution.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
A	1.71	3.33	1.07	4.29	2	1.13	0.26	6.67	1.03	0.56	0.21	15	7.5	0.66	0.64	0.5	2.14	0.38	0.27	1.11	0.33	0.68
B	12	4.62	30	20	60	8.57	6.67	-	10	7.5	30	-	12	15	60	3	60	20	30	15	5	60
C	0.74	-	3	10	60	1.88	6	30	15	2.86	15	-	12	15	8.57	7.5	7.5	12	5	12	6.67	4
D	2.86	20	6.67	20	20	4.62	8.57	60	5.45	0.48	1.71	-	8.57	20	15	20	6.67	12	5.45	10	30	1.88
E	1.67	60	20	30	3	0.94	12	30	3.33	1.2	12	-	4	30	5	7.5	4.62	5	2.86	6.67	10	4.29
F	1.09	4.62	2.07	4.29	1.54	1.67	0.55	2.61	3	2.86	0.6	-	6	1.71	1.03	1.4	2.07	1.54	2.14	2.31	1.25	5
G	0.21	5	6	12	30	0.63	0.8	5.45	1.71	0.1	3.16	10	1.3	4	7.5	0.18	-	4.62	1.4	0.71	1.46	1.09
H	6.67	60	10	20	60	1.25	3.16	6	20	0.82	12	-	4	3.75	15	5.45	60	8.57	10	3.16	3	2.73
I	1.18	20	7.5	7.5	2.73	5.45	1.28	6.67	1.33	1.82	2.86	-	6.67	20	1.13	1.67	30	4.62	8.57	3	1.3	3
J	0.58	3.75	1.94	0.65	1.5	3.75	0.1	0.5	1.28	0.74	0.97	-	4	0.87	1.03	0.92	1.15	0.59	0.64	2.73	0.52	4.62
K	0.26	30	10	2.22	20	0.56	1.76	8.57	4.62	1.15	2.22	-	4.62	5.45	2.5	2.31	15	2.86	1.62	1.54	1.18	2.5
L	-	30	-	-	-	-	15	-	-	-	-	-	-	-	-	-	-	-	30	-	-	30
M	20	8.57	30	5	6.67	3.75	0.94	4.62	15	5.45	12	-	2.73	4.62	12	4.29	60	6.67	10	12	1.82	12
N	0.75	30	20	20	60	1.54	6	4	10	0.83	6	-	7.5	1.94	6.67	1.88	60	6	4.29	2.07	6	2.61
O	0.69	60	10	30	6.67	1.03	7.5	30	1.54	0.8	1.71	-	15	6	1.82	2.86	4.62	1.88	1.71	2	4	6
P	0.78	3.33	6.67	6.67	15	1.71	0.17	2.86	0.85	1.33	1.94	-	4.62	1.94	3	0.68	20	1.43	1.09	3.75	0.66	1.76
Q	1.13	20	7.5	6.67	7.5	1.33	60	30	20	2.31	20	-	60	-	3.75	30	10	2.07	7.5	30	-	15
R	0.34	8.57	7.5	7.5	20	1.5	5.45	15	6	0.83	2.73	-	5.45	2.5	2.14	1.62	2.4	1.67	1.33	4	1.5	3.75
S	0.31	20	3.75	3.16	2.22	2.07	1.54	5	6.67	0.65	1.25	-	10	4	1.82	0.88	1.94	1.28	1.09	1.62	1.43	3.33
T	1.22	30	20	6	20	3	0.56	3	2.31	5.45	1.82	-	8.57	2.07	4.62	5	15	3.33	1.62	1.62	3.33	12
U	0.32	5.45	6.67	30	30	1.76	1.67	3.75	1.13	0.58	1.4	-	1.94	4	3.53	0.68	30	2.07	1.22	2.5	1.4	0.85
V	1.02	30	4.29	2.07	7.5	12	0.88	1.94	6.67	5	2.14	30	6.67	1.94	3.33	2.22	15	4.29	3.16	6	0.88	4.29

Table 2.63: Standard deviation σ of the Exponential distribution.

Bibliography

[1] Reza Azad Gholami, Leif Kristoffer Sandal, and Jan Uboe. Channel coordination in a multi-period newsvendor model with dynamic, price-dependent stochastic demand. 2016.

[2] Hülya Behret and Cengiz Kahraman. A multi-period newsvendor problem with pre-season extension under fuzzy demand. *Journal of Business Economics and Management*, 11(4):613–629, 2010.

[3] Aharon Ben-Tal, Laurent El Ghaoui, and Arkadi Nemirovski. *Robust optimization*. Princeton University Press, 2009.

[4] Aharon Ben-Tal, Alexander Goryashko, Elana Guslitzer, and Arkadi Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2004.

[5] Aharon Ben-Tal and Arkadi Nemirovski. Robust convex optimization. *Mathematics of operations research*, 23(4):769–805, 1998.

[6] Aharon Ben-Tal and Arkadi Nemirovski. Robust solutions of uncertain linear programs. *Operations research letters*, 25(1):1–13, 1999.

[7] Aharon Ben-Tal and Arkadi Nemirovski. Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical programming*, 88(3):411–424, 2000.

[8] Dimitris Bertsimas, David B Brown, and Constantine Caramanis. Theory and applications of robust optimization. *SIAM review*, 53(3):464–501, 2011.

[9] Dimitris Bertsimas and Aurélie Thiele. A robust optimization approach to inventory theory. *Operations Research*, 54(1):150–168, 2006.

[10] John R Birge. The value of the stochastic solution in stochastic linear programs with fixed recourse. *Mathematical programming*, 24(1):314–325, 1982.

[11] John R Birge and Francois Louveaux. *Introduction to stochastic programming*. Springer Science & Business Media, 2011.

[12] Emilio Carrizosa, Alba V Olivares-Nadal, and Pepa Ramírez-Cobo. Robust newsvendor problem with autoregressive demand. *Computers & Operations Research*, 68:123–133, 2016.

[13] Daniel Chemla, Frédéric Meunier, and Roberto Wolfler Calvo. Bike sharing systems: Solving the static rebalancing problem. *Discrete Optimization*, 10(2):120–146, 2013.

[14] Mauro Dell, Manuel Iori, Stefano Novellani, Thomas Stützle, et al. A destroy and repair algorithm for the bike sharing rebalancing problem. *Computers & Operations Research*, 71:149–162, 2016.

[15] Mauro Dell'Amico, Eleni Hadjicostantinou, Manuel Iori, and Stefano Novellani. The bike sharing rebalancing problem: Mathematical formulations and benchmark instances. *Omega*, 45:7–19, 2014.

[16] Paul DeMaio and R Meddin. The bike sharing world map. *The Bike-Sharing Blog (Updated April 19, 2013), httpWbike-sharing. blogspot. com/.[Accessed April 30, 2013.]*, 2014.

[17] Luca Di Gaspero, Andrea Rendl, and Tommaso Urli. Balancing bike sharing systems with constraint programming. *Constraints*, 21(2):318–348, 2016.

[18] Lingxiu Dong and Nils Rudi. Who benefits from transshipment? exogenous vs. endogenous wholesale prices. *Management Science*, 50(5):645–657, 2004.

[19] Jitka Dupačová. Reflections on robust optimization. In *Stochastic Programming Methods and Technical Applications*, pages 111–127. Springer, 1998.

[20] Daniel Ellsberg. Risk, ambiguity, and the savage axioms. *The quarterly journal of economics*, pages 643–669, 1961.

[21] Elliot Fishman, Simon Washington, and Narelle Haworth. Bike share: a synthesis of the literature. *Transport reviews*, 33(2):148–165, 2013.

[22] Iris A Forma, Tal Raviv, and Michal Tzur. A 3-step math heuristic for the static repositioning problem in bike-sharing systems. *Transportation research part B: methodological*, 71:230–247, 2015.

[23] Guillermo Gallego and Ilkyeong Moon. The distribution free newsboy problem: review and extensions. *Journal of the Operational Research Society*, 44(8):825–834, 1993.

[24] Joel Goh and Melvyn Sim. Distributionally robust optimization and its tractable approximations. *Operations research*, 58(4-part-1):902–917, 2010.

[25] Igor Griva, Stephen G Nash, and Ariela Sofer. *Linear and nonlinear optimization*. Siam, 2009.

[26] Grani A Hanasusanto, Daniel Kuhn, Stein W Wallace, and Steve Zymler. Distributionally robust multi-item newsvendor problems with multimodal demand distributions. *Mathematical Programming*, 152(1-2):1–32, 2015.

[27] Sin C Ho and WY Szeto. Solving a static repositioning problem in bike-sharing systems using iterated tabu search. *Transportation Research Part E: Logistics and Transportation Review*, 69:180–198, 2014.

[28] JK Jha and Kripa Shanker. Two-echelon supply chain inventory model with controllable lead time and service level constraint. *Computers & Industrial Engineering*, 57(3):1096–1104, 2009.

[29] Anssi Käki, Juuso Liesiö, Ahti Salo, and Srinivas Talluri. Newsvendor decisions under supply uncertainty. *International Journal of Production Research*, 53(5):1544–1560, 2015.

[30] Michal Kaut and Stein W Wallace. Evaluation of scenario-generation methods for stochastic programming. 2003.

[31] Avijit Khanra, Chetan Soman, and Tathagata Bandyopadhyay. Sensitivity analysis of the newsvendor model. *European Journal of Operational Research*, 239(2):403–412, 2014.

[32] Moutaz Khouja. The single-period (news-vendor) problem: literature review and suggestions for future research. *Omega*, 27(5):537–553, 1999.

[33] Gitae Kim, Kan Wu, and Edward Huang. Optimal inventory control in a multi-period newsvendor problem with non-stationary demand. *Advanced Engineering Informatics*, 29(1):139–145, 2015.

[34] Alan J King and Stein W Wallace. *Modeling with stochastic programming*. Springer Science & Business Media, 2012.

[35] Yanfeng Li, WY Szeto, Jiancheng Long, and CS Shui. A multiple type bike repositioning problem. *Transportation Research Part B: Methodological*, 90:263–278, 2016.

[36] Jun Lin and Tsan Sheng Ng. Robust multi-market newsvendor models with interval demand data. *European Journal of Operational Research*, 212(2):361–373, 2011.

[37] Chung-Cheng Lu. Robust multi-period fleet allocation models for bike-sharing systems. *Networks and Spatial Economics*, 16(1):61–82, 2016.

[38] Francesca Maggioni, Elisabetta Allevi, and Marida Bertocchi. Bounds in multistage linear stochastic programming. *Journal of Optimization Theory and Applications*, 163(1):200–229, 2014.

[39] Francesca Maggioni, Elisabetta Allevi, and Marida Bertocchi. Monotonic bounds in multistage mixed-integer linear stochastic programming: theoretical and numerical results, 2014.

[40] Francesca Maggioni, Teodor Gabriel Crainic, Guido Perboli, and Walter Rei. *The Generalized Skeleton Solution: A New Measure of the Quality of the Deterministic Solution in Stochastic Programming.* 2015.

[41] Francesca Maggioni, Michal Kaut, and Luca Bertazzi. Stochastic optimization models for a single-sink transportation problem. *Computational Management Science*, 6(2):251–267, 2009.

[42] Francesca Maggioni and Georg Ch Pflug. Bounds and approximations for multistage stochastic programs. *SIAM Journal on Optimization*, 26(1):831–855, 2016.

[43] Francesca Maggioni and Stein W Wallace. Analyzing the quality of the expected value solution in stochastic programming. *Annals of Operations Research*, 200(1):37–54, 2012.

[44] Ilkyeong Moon and Sangjin Choi. The distribution free continuous review inventory system with a service level constraint. *Computers & industrial engineering*, 27(1-4):209–212, 1994.

[45] Karthik Natarajan, Melvyn Sim, and Joline Uichanco. Asymmetry and ambiguity in newsvendor models. *submitted to Operations Research*, 2008.

[46] Georgia Perakis and Guillaume Roels. Regret in the newsvendor model with partial information. *Operations Research*, 56(1):188–203, 2008.

[47] Georg Pflug and Alois Pichler. *Multistage Stochastic Optimization.* Springer, 2014.

[48] Evan L Porteus. Stochastic inventory theory. *Handbooks in Operations Research and Management Science*, 2:605–652, 1990.

[49] Yan Qin, Ruoxuan Wang, Asoo J Vakharia, Yuwen Chen, and Michelle MH Seref. The newsvendor problem: Review and directions for future research. *European Journal of Operational Research*, 213(2):361–374, 2011.

[50] Tal Raviv and Ofer Kolka. Optimal inventory management of a bike-sharing station. *IIE Transactions*, 45(10):1077–1093, 2013.

[51] Roberto Rossi, Steven Prestwich, S Armagan Tarim, and Brahim Hnich. Confidence-based optimisation for the newsvendor problem under binomial, poisson and exponential demand. *European Journal of Operational Research*, 239(3):674–684, 2014.

[52] Andrzej P Ruszczynski and Alexander Shapiro. *Stochastic programming*, volume 10. Elsevier Amsterdam, 2003.

[53] Hamidreza Sayarshad, Sepideh Tavassoli, and Fang Zhao. A multi-periodic optimization formulation for bike planning and bike utilization. *Applied Mathematical Modelling*, 36(10):4944–4951, 2012.

[54] Herbert Scarf, KJ Arrow, and S Karlin. A min-max solution of an inventory problem. *Studies in the mathematical theory of inventory and production*, 10:201–209, 1958.

[55] Bruce W Schmeiser. Approximations to the inverse cumulative normal function for use on hand calculators. Technical report, DTIC Document, 1978.

[56] Jasper Schuijbroek, Robert Hampshire, and Willem-Jan van Hoeve. Inventory rebalancing and vehicle routing in bike sharing systems. 2013.

[57] Maurice E Schweitzer and Gérard P Cachon. Decision bias in the newsvendor problem with a known demand distribution: Experimental evidence. *Management Science*, 46(3):404–420, 2000.

[58] Susan Shaheen, Stacey Guzman, and Hua Zhang. Bikesharing in europe, the americas, and asia: past, present, and future. *Transportation Research*

Record: Journal of the Transportation Research Board, (2143):159–167, 2010.

[59] Jevons William Stanley. *Theory of Political Economy*. Hardpress Limited, 2013.

[60] Nichalin S Summerfield and Moshe Dror. Stochastic programming for decentralized newsvendor with transshipment. *International Journal of Production Economics*, 137(2):292–303, 2012.

[61] Ata Allah Taleizadeh, Seyed Taghi Akhavan Niaki, and Amir Abbas Najafi. Multiproduct single-machine production system with stochastic scrapped production rate, partial backordering and service level constraint. *Journal of Computational and Applied Mathematics*, 233(8):1834–1849, 2010.

[62] Yang Tang, Haixiao Pan, and Qing Shen. Bike-sharing systems in beijing, shanghai, and hangzhou and their impact on travel behavior. In *Transportation Research Board 90th Annual Meeting*, number 11-3862, 2011.

[63] Hajnalka Vaagen and Stein W Wallace. Product variety arising from hedging in the fashion supply chains. *International Journal of Production Economics*, 114(2):431–455, 2008.

[64] Meng Wu, Stuart X Zhu, and Ruud H Teunter. A risk-averse competitive newsvendor problem under the cvar criterion. *International Journal of Production Economics*, 156:13–23, 2014.

[65] Jitka Žáčková. On minimax solutions of stochastic linear programming problems. *Časopis pro pěstování matematiky*, 91(4):423–430, 1966.

[66] Jun Zhang. Transshipment and its impact on supply chain members performance. *Management Science*, 51(10):1534–1539, 2005.