Optimal Asset-Liability Management for Defined Benefit Pension Fund Under Stochastic Correlation

Ph.D. candidate: Mohammad Mehdi Hosseinzadeh

Supervisor: Prof. Giorgio Consigli

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

UNIVERSITÁ DEGLI STUDI DI BERGAMO

Ph.D. School in Economics, Applied Mathematics and Operational Research

May 2017

Declaration of Authorship

I, Mohammad Mehdi Hosseinzadeh, declare that this thesis titled, *Optimal Asset-Liability Management for Defined Benefit Pension Fund Under Stochastic Correlation* and the work presented are my own.

May, 2017

Abstract

We consider a second pillar pension fund problem relying on a multi-stage stochastic asset-liability management (ALM) model which is specified with an asset universe including money-market, fixed-income, inflation-linked bond as well as equity and commodity. The current value of liability is determined under the assumptions of constant pension fund future pension payments and their current market value (current fund obligation) under assumption of constant pension fund population by discounting all future pension payments. Pension payments are random and determined by the evolution of the population and by inflation. Over a long term horizon discount rates will also fluctuate and derive the evaluation of the fund liabilities. The pension manager will seek an optimal investment strategy to fund all liabilities and generate the surplus.

We present an extension of a scenario tree generation procedure to include stochastic correlations among asset classes and test whether, as claimed by several authors, such extension is effective during crises periods, when correlation clustering is commonly claimed to affect the markets and reduce significantly the effectiveness of portfolio diversification. We test the sensitivity of the first-stage implementable decision to alternative assumptions on the returns' correlations and their impact on the portfolio terminal distribution during a crisis period.

The funding ratio (FR) is the ratio of the portfolio assets to the liabilities. A pension funds, primary aim is to assess the FR at every decision stage over time. The pension fund's manager wishes to have sufficient liquidity and to control interest and inflation rate risks with a minimum return guarantee. Asset returns are defined with respect to a risk exposure captured by the concept of risk capital, recently introduced in modern pension systems and which is becoming a standard in Institutional ALM and in particular in pension fund ALM. In this thesis the elements of a real-world case problem are discuss and results presented over a 10-year horizon with the pension fund economic and financial constraints.

Focusing on a period, between 2009-2011, of increasing markets' volatility, we analyze the effectiveness of a long-term, discrete dynamic investment strategy under an assumption of stochastic correlation. The method relies on the definition of a probability space generated through Monte Carlo simulation and the implementation of a scenario generation scheme with a Dynamic Conditional Correlation (DCC) model. We consider a defied benefit (DB) pension fund problem: under a DB scheme benefits are defined in terms of percentage of last year salaries. The liability of pension fund is also called defined benefit obligation (DBO) under such assumption. stressed funding condition will arises when assets value decreases and liability value increase. The analysis of pension funds market perspectives is strictly related with evolution of the funding ratio. The collected evidence supports the inclusion of stochastic correlation between asset returns during the recent European financial crisis. Over a three year backtesting period which includes the 2009-2011 sovereign crisis, the introduced extension is shown to generate an effective hedge to positive risk premium.

Acknowledgments

I would like to express my sincere gratitude to all kinds of helps from the people who supported me through my Ph.D. studies, and in particular:

To my supervisor Prof. Giorgio Consigli for his insightful supervision and commitment.

To the coordinator of the doctoral school Prof. Marida Bertocchi, for her advice and educated views.

To Prof. Vittorio Moriggia for useful comments and discussions.

To my parents, for their love and constant support.

Contents

Notation and Key Definitions

The following standard notation and key definitions are used throughout the thesis:

$t \in \mathcal{T}, t = \{t_0, t_1, ..., t_n\}$	planning horizon time partition: time increments from time 0 to the 10 year model horizon
T	end of the decision horizon
ω	random outcome (uncertainty)
$(\Omega, \mathbb{F}, \mathbb{P})$	probability space with the triple: Ω the sample space, $\mathbb{F}$ is σ-algebra on the space and $\mathbb{P}$ is the probability measure
$\mathbb{E}$	is used for expectation and $\mathbb{E}_{\mathbb{F}_t}$ is $\mathbb{E}(\cdot\|\mathbb{F}_t)$ on $\mathbb{F}_t$ will denote conditional expectation with respect to the σ-algebra at time t
$\mathcal{N}_t$	set of nodes at stage t
n^-	ancestor of n
$a(n)$	set of ancestors:$\{n^-, n^{--}, ..., n_0\}$
$c(n)$	children nodes-subtree originating from n
t_n	time associated with node n
$x^+_{i,n}$	amount bought of asset i in node n
$x^-_{i,n}$	amount sold of asset i in node n
$x_{i,h,n}$	amount held of $i - th$ asset in node n bought in node h

$x^-_{i,h,n}$	amount sold of $i-th$ asset in node n bought in node h
$x_{i,n}$	amount held of asset i in node n
$\overset{0}{x}$	initial asset position
L_n	pension payment at node n
L_n^{NET}	net pension payment at node n (difference between pension benefits and any contributions)
Λ_n	defined benefit obligation (DBO) at node n
Φ_n	funding ratio at node n
$v_{i,n}$	value of asset i in node n
$r_{i,n}$	price return of asset i in node n
p_n	probability in node n
$\mathcal{I}$	asset set
ζ^j	statistical risk factors
e_n^r	standard random variables at node n
$c_{j,r}$	Choleski elements of the correlation matrix
$c_{i,j,t}$	elements of the dynamic correlation matrix
Δt	time increment between two nodes
σ^j	constant volatility
h_n	shortfall with respect to target at node n
$\widetilde{W}^j$	target wealth at stage j
W_n	portfolio wealth at node n
l_i/u_i	lower/upper bounds of the asset i
p_s	survival probability for pensioners
b_a	annual pension benefit payment

List of Tables

List of Figures

List of Abbreviations

ALM	Asset-Liability Management
BIS	Bank for International Settlements
BM	Benefit Multiplier
CALM	computer-aided asset liability management
DB	Defined Benefit
DBO	Defined Benefit Obligation
DC	Defined Contribution
DCC	Dynamic Stochastic Correlation
DEO	Duration Enhancing Overlay
DFA	Dynamic Financial Analysis
DS	Data-Stream
DSP	Dynamic Stochastic Programming
FAS	Final Average Salary
GDP	Gross Domestic Product
H&N	Here & Now
IMF	International Monetary Fund
LDI	Liability-Driven Investing
MC	Monte-Carlo
MSP	Multi-Stage Stochastic Programming

OECD	Organization for Economic Co-operation and Development
PAYG	Pay-As-You-Go
P&C	Property & Casualty
PF	Pension Fund
SP	Stochastic Programming
TR	Total Return

Introduction

The review of literature demonstrates that over the past several decades, financial markets have gone through a soared fluctuation. In particular, the recent period of sovereign crisis in Europe (2009-2011) has been affected by the previous conditions in financial markets. The European market was suffering from an unprecedented reduction in interest rates, increasing default risk of selected sovereign borrowers and increasing correlation and systematic risks in the Euro zone. In this situation, the financial institutions such as pension funds, which had long-term horizon plans, were faced with downside potential of the financial markets in many countries. Two most important risks taken by pension funds are investment (underfunded) and longevity risk. During such periods of market fluctuation, the adoption of dynamic portfolio optimization strategies for securities exposed to market risk may play a crucial role.

Recent European sovereign criss, has changed in the relationship among different markets. Hence, correlations changing among different kind of asset classes has become a major task for investors. By focusing on market prices, we propose a numerical approach to incorporate scenario generation with dynamic conditional correlation (DCC) in a discrete dynamic stochastic programming problem and test its effectiveness during the recent crisis. DCC method has been used in the econometric and financial problems. We cite some of the most relevant contributions [55, 56, 118, 69, 128, 58, 41].

Depending on the specific structure, asset prices are assumed to be driven by risk factors and correlated benchmark indexes. Therefore, Multistage Stochastic Programming (MSP) methods provide a powerful paradigm for decision making under such uncertainty. In financial planning problems, MSP has been used in several areas as follows: asset-liability management [27, 49, 144, 9], financial engineering applications [82, 144, 29, 122, 53], large financial institutions problem [19, 103, 47, 143, 28, 99] and dynamic portfolio problems [27, 119, 143].

The purpose of the current thesis is to assess the underperformance of ALM models during the financial crisis and how traditionally long-term institutional investors such as pension fund must cope with the shrinking of their shorter term budget of risk embedded in regulatory changes, as well as containing their drawdown during phases of systematic risk such a worst case scenario. During the crisis period financial institutions such as pension funds were faced with downside potential of the financial markets. A study by [60] revealed that 90% of the private-sector defined benefit pension systems in the UK and US were underfunded. The reason was mostly bad modeling or the absence of modeling, together with difficult conditions in the financial markets.

Taking right intermediate decisions for DB pension funds that have long-term investment horizons and control the liquidity shortfall risk of pension funds over the short-term horizons will be important to meet long-term targets. Since managing long-term assets and controlling the pension funds' risks are relative to the market evidence in the long-run, their assessment requires long-term simulation and optimization modeling ability. However at the same time, we have to control the risk of market drop risk and get the right decision for our portfolio allocation in short and medium-term horizons. During the investment horizon, investors may be faced with drawdown or systematic risk of market. In this case, they have to be able to balance the returns of portfolio against the discounted liability and also, long-term investors should improve short-term risk management.

Motivation

In the current research, an effort is made to assess the underperformance of ALM models during the recent market crisis and the way the traditionally long term institutional investors must cope with the shrinking of their shorter term budget of risk embedded in regulatory changes, as well as containing their drawdown and liquidity shortfall during phases of systematic risk. In this case, they have to be able to balance the returns of portfolio against the discounted liability and also, long-term investors should improve intermediate term risk management. Nowadays also an increase in cross-asset correlations is a main source of uncertainty for investors. Construction of an optimal portfolio with a set of constraints requires a forecast of the covariance matrix of the returns. If the correlations and volatilities are changing dynamically, then the fund hedge ratio should be adjusted to account for the most recent information over long horizon.

To cope with this problem, we introduce the time depending stochastic correlation into the scenario generation model over the long horizon which allows changes in volatilities and correlations among a set of underlying assets. This approach can identify possible changes in correlation and managing the risk in a more volatile and correlate investment universe. Therefore, we challenge the optimal dynamic DB pension funds ALM problem under stochastic correlation with multi-critical targets together with financial and regulatory constraints and computational procedures that can help us analyze the pension fund's investment strategy under different target combination and worst case scenario as well.

Aims and Contributions

Pension funds are institutional investors whose key objective is to generate retirement income through effective ALM. In this thesis we consider the following research objectives:

- Analyze the impact on dynamic ALM strategies of alternative assumption on asset correlations and trade-off between return targets.
- Develop a decision tool based on MSP to study the evolution of the PF funding ratio under stressed market conditions.
- Extend currently available scenario generation method to incorporate stochastic dynamics of asset returns correlations.
- Analyze optimal MSP-based hedging policies in presence of stochastic correlations.

The following may regarded as main contributions of the research work:

- The definition of an optimal ALM strategy during the recent (2009-2011) crisis in the EU zone in presence of correlation clustering.
- Development of a scenario generation method based on a dynamic conditional correlation model for asset returns.

- The benchmarking of alternative optimization approaches (e.g. static vs multi-period, under constant or stochastic correlation) over 2009-2011 with out-of sample results.

- Scenario based analysis of pension fund funding conditions under alternative market assumptions and derivation of an optimal targets convex combination.

Structure of the Thesis

In the present thesis, the analysis is based on a MSP optimization approach which integrates an uncertainty scenario generation model of DB pension fund and investment risk with a realistic representation of the ALM problem. In the empirical part of the thesis, a dynamic ALM model for the DB pension funds will be applied to a real-world case study. The case study will be designed for a large institutional investors in order to identify the optimal asset allocation over certain planning horizon with inclusion of capital and liability constraints generated by a DB pension fund.

Following the introductory remarks, the thesis proceeds as follows. In chapter 1, the institutional ALM with specific features on pension funds, methodology and modeling issues are presented. Chapter 2 deals with long-term financial risk and markets' instability with investment universe, stochastic risk factors, two layer asset simulation structure and decision making. Chapter 3 discus the dynamic decision approach for institutions with dynamic stochastic programming framework and scenario generation. Chapter 4 presents a mathematical modeling for optimal DB pension fund management with stochastic correlation and asset returns simulation. Chapter 5 offers an empirical work and case study on DB pension fund and the related results. Finally, chapter 6 concluding remarks.

Chapter 1

Institutional Asset Liability Management

Financial activities by financial institutions continually require decisions made about the resources and allocation of the resources on different uses. The ever increasing inclination of financial institution to concentration on rates of asset return and liability and risk control by the institutions has made it necessary for the managers of institutions to use ALM knowledge in their responsibility. In order to assist the managers of financial institutions, treasurers, chiefs of bank branches, ALM experts, ALM combines the new techniques employed for profiteering and, risk management of trade institutions and makes them practical.

Institutional investors such as pension funds, insurers and sovereign wealth funds, due to the longer term nature of their liabilities, represent a potentially major source of long-term financing for illiquid assets. Over the last decade, these investors have been looking for new sources of long-term, inflation protected returns. Asset allocation trends observed in recent years show a gradual globalization of portfolios with an increased interest in emerging markets and diversification into new asset classes [113].

Investment management is more than asset management. As a matter of fact, investments relate with the art of managing the whole balance-sheet of financial institutions where not only assets, but also liabilities and capital play a role Figure 1.1.

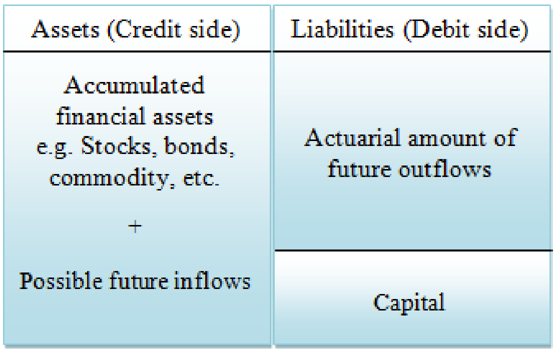


Figure 1.1: Stylized balance-sheet structure

Assets have to be financed by liabilities, and capital is needed to run the market safely. In particular, ALM is the branch of investment theory emphasizing the importance of the interaction between assets and liabilities thus representing a core activity for many-long term institutional investors, especially pension fund.

The aim of this thesis is to use this knowledge in order to find *the best possible strategy* for managing the mismatch between assets and liabilities for the financial institutions. The asset must be invested over time to achieve favorable return subject to various uncertainties, policy and legal constraints, taxes and other requirements do not diversify properly across markets or across time, particularly in relation to their liability commitments. There are many motivations for studying asset liability management, including: 1) the results may be useful to set guidelines for institutions and individual investors concerning their asset allocation mixes, the models integrate various decisions over time with the constraints, preference and uncertainties inherent in the investment problem and 2) the models consider temporal dependence of asset and liability commitments, path dependent preferences, short and long term trade-offs and provide for realistic measurement of risks and their trade-off with investment returns considering the effects of taxes, transaction cost and other problem features [145].

There are numerous application areas for ALM including pension fund [13, 145, 103, 108, 99, 14, 72, 47, 43, 42, 60, 85, 143, 144, 62, 9, 98, 3, 15], insurance company [18, 143, 144] for the Japanese Yasuda-Kasai Insurance company, [64, 40, 100, 101, 106, 28, 98, 11, 32] for the P&C insurance through stochastic programming approaches and banks [87, 38, 98, 10]. In the present research the main focus is on the defined benefit pension fund investors.

1.1 Literature Review

Asset-liability management is a term whose meaning has evolved and enjoyed remarkable popularity in recent years. From its origins as an actuarial and cashflow matching technique, ALM has grown into a conceptual framework for financial management and a professional activity in its own right. In a world governed by financial markets and physical commodities, it is vital to analyze objectively the economic effects of price movements on balance sheets, earnings growth and enterprise value. Its use began in insurance companies and banks and has now extended to most financial institutions and corporations. ALM has been defined by American Society of Actuaries (SOA) (2003) as follows:

"ALM is the practice of managing a business so that decisions and actions taken with respect to assets and liabilities are coordinated. ALM can be defined as the ongoing process of formulating, implementing, monitoring and revising strategies related to assets and liabilities to achieve an organization's financial objectives, given the organization's risk tolerances and other constraints. ALM is relevant to, and critical for, the sound management of the finances of any organization that invests to meet its future cash flow needs and capital requirements."

The sphere of ALM applications has gained a momentum over the past decade with software developments that have benefited several sectors of the financial industry dedicated to wealth management [47, 121, 146, 137, 98].

Mathematical models of asset liability management have been extensively studied by practitioners as well as operations research experts. The literature starts from the static portfolio optimization model through mean-variance techniques [92, 93] and developing complex mathematical models [18, 87] to improve decision making. In a model known as Towers Perrin model [103], downside risk was used as a risk measure which quantifies the probability and extent of the shortfall when one exists. Also in a chapter book to Stochastic Optimization Methods in Finance and Energy edited by Bertocchi et al. [9], Dempster et al. proposed a model for funds design to support investment products which yield a minimum guaranteed return. [19] on the other hand analyzed the performance of fixed-mix strategy employing mean-variance frontier with dynamic multi-period programming strategy based on costs in the objective function.

[87] developed a model for the Vancouver city saving union. This model took into account many features including simultaneous consideration of assets and liabilities to meet accounting principles and match the liquidity of assets and liabilities, transaction costs, uncertainty in deposit withdrawals and legal and policy constraints. The proposed model allows constraint vi-

olations through penalty costs in the objective function. The performance of this model has been compared with that of [16] stochastic decision tree approach, which lacked flexibility in the capital loss constraints and category limit constraints and which limited the choice of its solutions. The simulations demonstrated higher risk adjusted returns for the Kusy-Ziemba model.

The Russell Yasuda Kassi model is another model proposed by [18] to handle the financial planning for a Japanese Insurance company. Through the model, the different legal and policy constraints, multiple accounts and multistage planning horizon were successfully handled. They compared the performance of the model against the old methodology followed by the firm, mean-variance analysis, and found the returns generated by the allocations form the model were substantially higher than the mean-variance allocation. [27] developed a computer-aided asset liability management (CALM) model which is a generic model for the integrated dynamic management of financial assets and liabilities. With this model, they handled uncertainties pertaining to both the assets and liability. [48] developed an ALM model and compared it with the static decision making framework. He used binary variables to explicitly model the probability of underfunding at intermediate point as well as the horizon. The dynamic nature of the model allowed the decision making to react to the latest economic developments and chose a trade-off between long term consequences and short term gains. In comparison to the static models, the ALM model resulted in lower funding cost as well as lower probability of underfunding. [51] developed a mixed integer stochastic programming model using variable as in [48]. They extended the use of binary variables to model conditional constraints in order to allow for investment in derivatives when the normal asset return fall dramatically so as to prevent the portfolio form from further decline.

[141] discussed asset liability management under benchmark and mean-variance criterion in a jump diffusion market in which the Lévy process was used to describe the dynamics of risky asset's price and liability. [88] investigated on mean-variance asset liability management with endogenous liabilities under multi-period setting. [23] studied the mean-variance asset liability management considering cointegrated assets and insurance liabilities. [139] solved analytically the continuous-time mean-variance asset liability management with endogenous liabilities. [89] focused on multi-period portfolio optimization for asset-liability management with bankrupt control. [139] considered uncontrolled cash flow into account and solved the multi-period asset liability management model by adopting the dynamic programming approach. [90] considered dynamic portfolio technique of risky assets under uncertain exit time and stochastic market and this work can be extended to the asset liability management area with modification in terms of liability. [39] extend stochastic control methodology to addressing an ALM problem with a classical solution under two different sets of assumptions and jump-diffusion uncertainty on both assets and liabilities.

A lot of work has been carried out to study the application of ALM in pension fund management. For instance, [95, 96] has attempted to employ the ALM problem in a continuous-time framework, and has extended the intertemporal selection analysis to account for the presence of liability in the asset allocation policy. He has applied the optimal portfolio selection approach to the pension funds in 1990. [129] specifically aimed at asset allocation and retirement decisions for a pension fund. [85] proposed a stochastic programming based model for Dutch pension fund. He constructed a tree scenario by random sampling, adjusted random sampling and tree fitting approach. [61] further studied the relative performance of stochastic programming model versus fixed models by conducting out of sample tests. They found that the relative dominance of the stochastic programming (SP) model reduced under these conditions. This was attributed to the SP models having an advantage of optimization to future scenarios which were lost when conducting out of sample tests. They underscored the importance of the scenario generation procedure in developing a SP ALM model. [70] focused on the management of a pension fund under mortality risk and financial risk. [73] developed an ALM model for a Finnish pension Company. [54] also conducted a similar study for a model developed for a Czech pension fund. [123] attempt to the case of a time-varying opportunity set of the pension's asset portfolio. [91] investigated the mean-variance optimization problem for a single cohort of workers in an accumulation phase of a defined benefit pension scheme.

As [78] states, the aim of pension fund asset management is to provide funding for the pension liabilities, but a pension fund sponsor has also a secondary goal that is the achievement of an *earnings spread* (i.e. the positive gap between assets and liabilities). By the same token, [21] argues that traditional efficient frontier method is not capable enough to maximize the return of a pension fund. He proposes instead matching pension assets against pension liabilities, saying: "match the assets and the liabilities and go to bed". [107] also suggest that any attempt to manage pension plan risk must consider both asset valuation and the risk of interest rate decreases. [126] suggest that pension fund managers should avoid severe underperformances and asset-liability mismatches every year, in order to follow an appropriate ALM. The Kodak pension plan [115], for example, implemented an established ALM system for pensions in 1999, protecting its surplus over the subsequent recession. The situation repeated itself during the 2008 crash when most pension plan funding ratios dropped further. Again, systematic risk management via

ALM models would have largely protected the pension plans. Another example of successful application in asset liability management using stochastic programming model is Tower Perrin-Tillinghast which has been discussed in [103]. Several institutional applications in asset-liability management are provided in [145, 146, 144, 9].

Advocates of conservative investments, known as liability-driven investing (LDI) have proposed a portfolio tilted to fixed income securities, similar to the portfolio of an insurance company [99]. According to [99], any model with a fixed correlation and single-period structure, such as the traditional Markowitz model, will be unlikely to provide much diversification benefits since the market's behavior during a crash is very different from the behavior during normal times. [99] suggest that an efficient approach for dealing with pension-surplus protection is to implement dynamic strategies involving long-term government bonds (or strips during crises). They termed this strategy as "DEO" which stand for duration enhancing overlay. [70] investigated on optimal contribution rate of a stochastic defined benefit pension fund with a stochastic mortality which is modeled by a jump process. [138] focused on asset allocation problem for defined contribution pension funds with stochastic income and mortality risk under a multi-period mean-variance framework. [83] considered optimal savings management for individuals with defined contribution pension plans. [97] addressed integrated risk management for U.S. defined benefit pensions with models and metrics. He focused on the impact of plan underfunding for the operation of the pension benefit guaranty corporation.

1.2 Pension Funds

Pension fund is a financial institution to provide the pension benefits of retired employees. Pension funds collect retirement savings from workers and their employers, and invest this money in a wide range of assets. They manage the money of up to millions of individuals and they have major impact on the stability of financial markets through their investment behavior.

According to [77], national pension systems are typically represented by a *"multi-pillar"* structure, with the sources of retirement income derived from a mixture of government, employment, and individual savings. They have divided pensions into three pillars, based primarily on the source of savings:

- State pension
- Occupational pension
- Private pension

The first pillar is public and is financed through government pay-as-you-go (PAYG) pension structure where current contributions pay for current benefits, and managed by public institutions. The contributions are divided between employer and employee and they are a fixed percentage of earned wages. The second pillar is funded through employer and organized at the workplace and is fully funded pension fund and most often PAYG structure as well. This pillar can be separated into defined benefit (DB) and defined contribution (DC) or hybrid plans. The third pillar, private savings plans and products for individuals, often tax-advantaged saving. In the current research, the focus is on the second pillar pension system.

1.2.1 Occupational Pension Plans

Occupational pension benefits can be generally divided into two different schemes of the retirement saving built up during occupation: DB and DC plans. In a DB plan, employer guarantees to provide specific pension benefits to the employee related to individual salary and length of employment. Pension fund managers have to make detailed calculations on how much they will be required to pay out to pensioners. In the DB plan, two systems are distinguished: a system based on the final salary and a system based on the average salary. The second way that pension funds can be organized is through a DC scheme. In this arrangement, the employee is primarily responsible for saving for his/her own retirement. In this scheme, the employee sets aside a certain percentage of his/her salary in a tax-deferred individual account, matched in part or in full by the employer's contribution, which also goes into the individual account. When employee retires, the level of pension benefit is based on his/her individual account balance and is no longer the employer's responsibility. The most common DC plans are 401(k) plans in the private sector and 403(b) plans in the public and nonprofit sectors [117].

There are several advantages and disadvantages of DB plans for a plan participant. In fact, DB and DC are completely opposite plan types, the advantages of a DB plan tend to be the disadvantages of a DC plan and vice versa. The major advantage of DB is that the post-retirements' benefits are fixed in advance by the sponsor and contributions. There are three aspects to this certainty. First, employees know the amount of annual pension benefits. Second, in most cases, an annual pension benefit is also indexed for inflation. Therefore, employees do not have to worry about losing the purchasing power of their pension benefit to inflation in the many years he/she will continue to live after retirement. Third, the retiree in a DB plan does not have to be worried about the financial implication of longevity. The employee is guaranteed pension benefits for as long as he lives. When a DB plan has enough members, it can use average mortality for long-term financial planning purposes. In this way, a member who lives longer than an average life span does not have to worry that the plan will run out of financial resources to fund his longer life span, as it will be balanced out by other members with life spans shorter than average. In other words, each employee has a guaranteed lifetime annuity. DB plans are definitely preferred by workers [117].

As [117] states, the main disadvantage of a DB plan is the funding risk for the plan's government sponsor. The funding risk refers to both short-term volatility in pension contribution and long-term uncertainty in required funding to meet future pension obligation. This funding risk results from several assumptions built into the calculation of costs related to funding pension benefits. The first and most important assumption is the assumed rate of return on the investment. The second risk with regard to the rate of return is whether it can be realized in the long run. Because the assumed rate of return is long-term in nature and is based on historical rates of return, there is always the potential that this assumed rate of return might not be realized in the future.

1.2.2 Pension Benefit Design

To understand the pension plan management, we need to know what pension benefits are and how they are determined. Every type of pension provides the employee with an income after some event has happened. As aforementioned, DB and DC plans are two extremely different ways to determine benefits. In this section, we briefly discuss some important types to define the benefit for the DB plans [117].

- **Normal service benefit**

It is called also normal retirement benefit because the benefit can be received only when the employee reaches a normal retirement age. The normal retirement benefit is defined by the following formula:

$$\textit{Final average salary} \times \textit{Years of credited service} \times \textit{Benefit multiplier}$$

Final average salary (FAS) is the average salary over the last few years prior to a member's retirement or termination of employment. Years of credited service are the number of years the retiree or the terminated employee has worked for one particular employer. Benefit multiplier (BM) means the percentage of final average salary the retiree can replace in his/her annual retirement benefit for each year of service. While these three factors appear straightforward, there are many variations of these factors in actual pension benefit design (For more details see [117]).

In order to receive the normal retirement benefits, there are three primary ways to determine the requirement in the public sector: age, years of service, or a combination of both.

Age: If the requirement for normal retirement is based on age, then the employee has to reach a certain age.

Year of service: If it is under year of service, an employee can retire at any age and collect normal retirement benefit as long as she has worked for a certain number of years.

Under combination of age and years of service: Under combination of age and years of service, an employee can retire with normal benefit as long as her age and years of service add up to a certain number. This type of rule provides more flexibility for employees in meeting the requirement for normal pension benefits.

- **Early retirement benefit**

In addition to normal retirement benefits, some pension plans also allow for early retirement benefits, which is offered to those who retire before they meet the requirement for normal benefit.

- **Post-employment benefit adjustment**

For employees, it is important to know not only what the pension benefit is at the time of retirement, but also whether such pension benefits will be adjusted for cost of living in the future. Without such adjustment, inflation will gradually erode the purchasing power of pension benefit over time. Because an average retiree is expected to live for many years after retirement, such adjustment is a significant benefit related to the normal or early retirement benefit for the employee.

1.2.3 Aims and Interests in Pension Funds

The key aim of a pension fund is to ensure that it is able to fulfill its promises to the members over a long-time horizon. The promises consist of the pension payments derived from the wealth which pensioners have saved up and which active members are currently accumulating. Therefore, the pension fund faces the uncertainty of not knowing the total amount it will have to pay to any particular pensioner between retirement and death. Since the pension fund relies on the financial markets in order to accumulate and grow the savings for retirement it is also subject to movements on the financial markets. These movements may not be in favour of the investment targets. These uncertainties combined with the very long-time horizon make the goal stated above a highly difficult one to achieve (Figure 1.2).

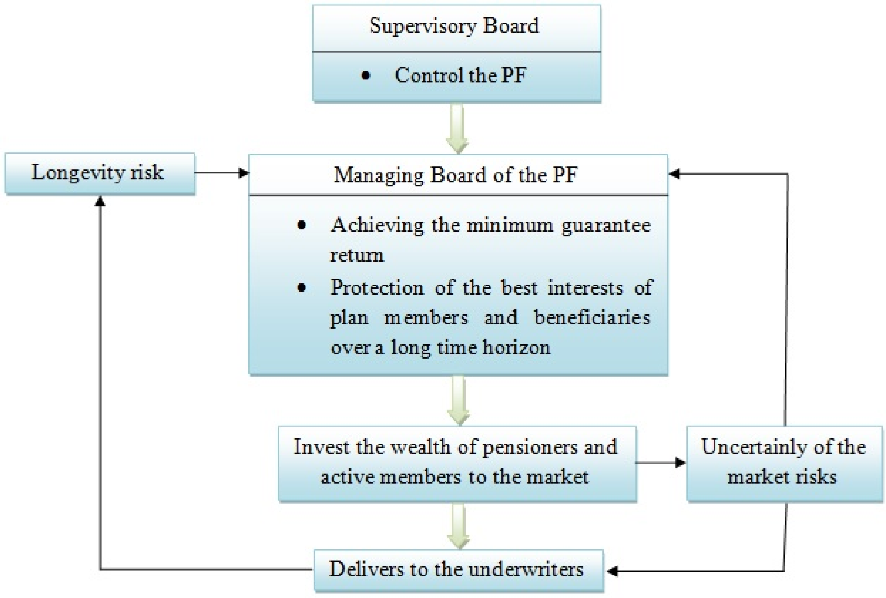


Figure 1.2: General structure of a pension funds' manager's view

The perfect pension fund would then also reach this goal in a transparent and systematic way, ideally also reflecting the individual's interests and distributing surplus wealth in a manner that every stakeholder benefits equally. Today, also more and more moral issues with regard to the actual investments are raised. Pension funds want to ensure that their wealth is only invested into morally high standing, long term oriented sustainable investment opportunities.

In a pension system where the young active members' and the retired passive members' wealth is managed as a fully funded pension fund, several interests need to be achieved.

Every party involved in the pension fund has particular interests: the pensioners and the active members shortly before retirement are interested in a secure and stable pension which is best achieved by a low risk, conservative investment strategy. The younger active members are interested in highest possible returns in order to augment their future pensions. The sponsoring companies' interests, finally, lie in minimizing the need of paying supplemental funds into the pension fund. This could be the case if the pension fund is in a financial distress situation and cannot fully cover its promised payments.

The task of the pension fund manager is to accomplish the goals of all parties involved, while observing the legal requirements, achieving a minimal guaranteed return, and with the additional difficulties of uncertain market returns, liquidity needs, and demographic trends.

1.3 Methodology & Modeling Issues

A pension fund ALM problem belongs to the general class of financial planing problems under uncertainty with the following main features:

- a long-term objective;
- several financial and regulatory constraints;
- a complex set of risk drivers affecting the investment portfolio overall risk exposure.

To tackle this decision problem, consistently with mainstream modeling approaches, we can adopt either: a discrete modeling framework [105, 27, 47, 9] or a continuous framework [127, 95].

In the present research, the analysis is based on a DSP optimization approach with discrete time and space setting [145, 143, 144, 9] which integrates an uncertainty scenario generation model of DB pension fund and investment risk with a realistic representation of the ALM problem. This work deals with relevant criteria for optimal DB pension funds portfolio selection when assets, liabilities, capital efficiency and possibility of portfolio's shortfall with respect to the fund's targets simultaneously considered.

In the empirical part of the thesis a dynamic ALM model for a large institutional investor has been applied to a case study. The case study is designed for a DB pension fund in order to identify the optimal asset allocation over certain planning horizon with inclusion of capital and liability constraints.

Some of the most important modeling issues faced by a pension fund manager in the determination of the optimal asset allocations over time to the product maturity have been listed by [47] as follows:

- Stochastic nature of asset returns and liabilities

  Both the future asset return and the liability streams are unknown. Liabilities, in particular, are determined by actuarial events and have to be matched by the assets. Thus each allocation decision will have to take into account the liabilities level which, in turn, is directly linked to the contribution policy requested by the fund.

- Long-term investment horizons

  The typical investment horizon is very long. This means that the fund's portfolio needs to be rebalanced many times, through a sequence of *"buy, hold and sell"* decisions. Therefore, dynamic stochastic optimization techniques are needed to take explicitly into account the on-going rebalancing of the asset-mix.

- Risk of under-funding

  There is a very important requirement to monitor and manage the probability of under-funding for both individual clients and the fund, that is the confidence level with which the pension fund will be able to meet its targets without resort to its guarantor.

- Inflation risk

  Inflation is one of the most important risks in pension funds. In fact, both assets and liabilities of the fund can be impacted by unexpected inflation shocks over horizon. Assets may suffer because an increasing of inflation rate normally triggers an interest rate up-shift therefore negatively hitting the fixed-income nominal assets. Liabilities can also be under pressure since future salary of pensioners and active members are inflation sensitive.

- Management constraints

  The management of a pension fund is also dictated by a number of solvency requirements which are put in place by the appropriate regulating authorities. These constraints greatly affect the suggested allocation and must always be considered. Moreover, since the fund's portfolio must be actively managed, the markets' bid-ask spreads, taxes, policies and other frictions must also be modeled.

The uncertain variables in the modeling need to be approximated by a scenario tree with a finite number of states at each time. Important practical issues such as transaction costs, multiple state variables, market incompleteness due to uncertainty in liability streams that is not spanned by existing securities, taxes and trading limits, regulatory restrictions and corporate policy requirements can be handled within the stochastic programming framework.

Figure 1.3 illustrates the processes, models and other requirements to construct dynamic ALM strategic with periodic portfolio rebalancing. It should be noted that knowledge of several independent highly technical disciplines is required for dynamic ALM strategic in addition to professional domain knowledge. Corresponding to the Figure 1.3 shows the system design which describes the separate - largely automated and implementation part - tasks which must be undertaken to obtain recommended strategic decisions once statistical, optimization and implementation part have been specified. Each of the blocks of the following figure will be treated in detail in a subsequent chapters of the thesis.

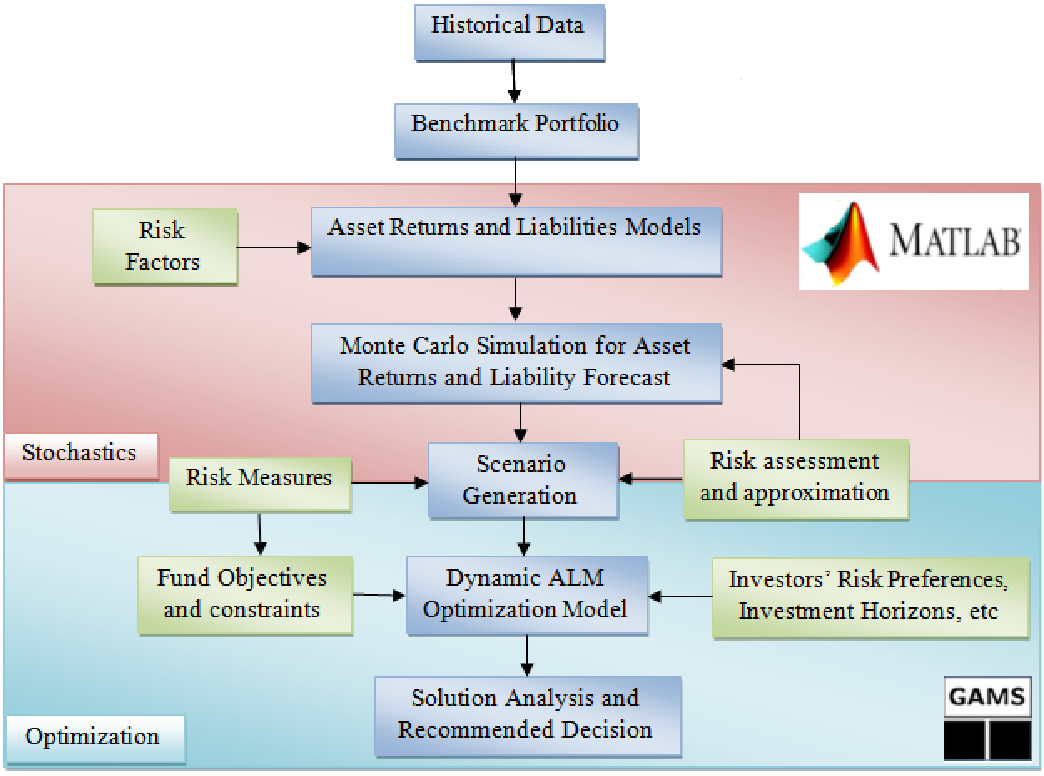


Figure 1.3: Dynamic stochastic optimization for financial planing

Chapter 2

Long-Term Financial Risk and Markets' Instability

The recent markets' instabilities have started since financial markets crisis in United States which broke out in 2007 and swept away rapidly across major financial institutions of United States and Europe. Having spread to giant financial institutions, the instabilities of the markets turned into a pervasive crisis in financial markets and culminated in a dramatic decline of stocks prices and fall of stocks.

According to [62], after the first financial crisis between 2000 and 2003, a second financial crisis within less than ten years during 2007/2008 swept away the financial institutions such as pension funds that have long-term investment horizon across the world. This financial crisis amplified the shock waves that the previous crisis had sent to pension funds through-out the world. For many pension funds, the current crisis was far worse than the last one. In order to prevent deficits in pension funds in the aftermath of the crisis, many countries implemented some new regulations and applied new management risk tools, neither of which was efficient. For the long term financial planning risk integration is essential because multiple factors affect asset-liability portfolios at the same time. Without considering integration, the measures will be incorrect and, subsequently, so also the portfolio management.

Here in the current chapter, the main objective is to cover and introduce the investment universe of pension funds, stochastic framework of assets return for the given asset classes, impact of market instability on pension finds' asset and liability and long-term risk control in a dynamic setting.

2.1 Investment Universe

The financial institutions such as pension funds may invest in several asset classes. In the analysis, we consider a long-term asset allocation for the pension fund based on the investment universe including bank account, money-market with a risk-free short-term interest rate and risk bearing investments such as government bonds, equity and commodities. Bonds are considered as riskless assets while stock and commodity markets are risky assets, whereas the money-market is considered a risk-free investment. For each such investment opportunity a dedicated statistical model has been developed to generate future price and return scenarios describing the uncertainty the investment manager is facing over time and must be input to the optimization problem.

Table 2.1 illustrates the set of investment classes, benchmark indexes and the associated risk factors relevant in this study. We have five investment classes plus bank account and relevant benchmark indexes with associated risk factors.

Table 2.1: Market benchmarks and relative risk factors

Investment Class	**Benchmark**	**Risk factors**
Money market	Euribor 12 month	Inflation rate, economic cycle, 12 month short rate
European Gov. bond	JPM Global Emu	Euro stock, interest rate, economic cycle
	JPM Global ex-EMU	Euro stock, inflation rate
Infl. Linked bond	Barclays Infl. Linked	Euro stock, interest rate, inflation rate
Equity	MSCI Europe Index	Stock risk premium, interest rate
Commodity	Dow Jones Comm.	Euro stock, economic cycle, euro bond market

2.1.1 Historical Data

We use delegate total return (TR) indices for the asset classes. A total return index includes the reinvestment of dividends in the case of stock markets and the gains or losses of the price variation in the case of bond markets, respectively. All of the mentioned data sets in this thesis have been collected at a quarterly frequency through the Data-Stream (DS) source. Furthermore, for all data sets, the time-specified of past observed market data starts from January 1999 and ends in December 2015.

2.2 Stochastic Models for Asset Returns

In the financial planning, the major decision of an investor regarding the portfolio is to choose the allocation between the different asset classes. Hence, one of the key elements in a dynamic ALM scheme is a model for the movement of assets price and return in the market. Furthermore, for the optimization modeling and scenario generation, we need to know about behavior of our portfolio over time horizon. Since our portfolio is changing over time and is exposed to some stochastic factors, we introduce assets return models in discrete-time with the modelling framework consists of stochastic differential equations [29].

According to the Figure 2.1, we consider two layers structure, at the first layer we drive the risk factors for each investment opportunity, then in the second layer compute the equity risk premium and returns of each asset class based on first layer of the model have been formulated. Risk factors essentially

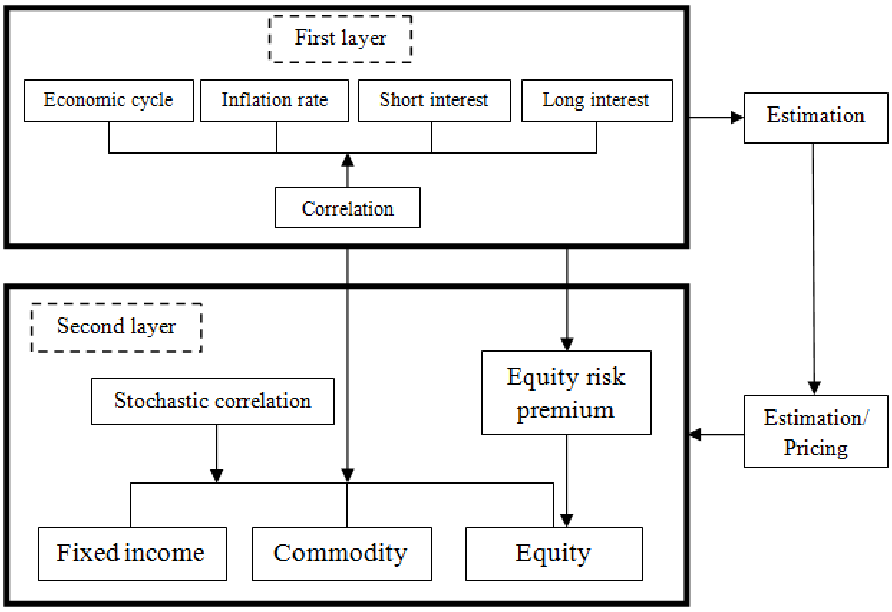


Figure 2.1: Structure of statistical risk factors

affect in the long-term financial position of the portfolio return and need to be consider in the dynamic stochastic optimization modeling. Therefore, for the given investment universe, we need to identify associated risk factors relevant to the set of investment opportunities [105, 27, 37, 5, 47, 42, 29].

Table 2.1 includes the associated risk factors of each investment opportunities indexes relevant in this study. We consider inflation rate and 10 years Euro interest rate in the risk factor modeling. Inflation and interest rate are two critically important risk factors for the pension funds. An inflation rate process is also considered to derive inflation-adjusted pension payments over the time horizon. In fact, both assets and liabilities can be influenced by unexpected inflation shocks over the decision horizon.

Apart from inflation and interest rate in the Euro area the following core risk factors are considered: economic cycle (Gross domestic product (GDP)), the MSCI Euro equity benchmark and Euro bond market.

We refer here to the risk process as the random process of the financial factors embodying the risk sources of the problem [37, 29]. We regard scenario approximation as an efficient statistical procedure to approximate in a discrete framework the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and has been discussed more in details in chapter 4. From the statistical modeling viewpoint, first layer risk factors ζ_n^j and then second layer asset price return $r_{i,n}$ are computed with multivariate Gaussian return model with autoregression and exogenous variables. Generic stochastic difference equation for the risk factors ζ_n^j for all $n \in \mathcal{N}_t$ are as follow:

$$\zeta_n^j = \mu_n^j + \sigma^j \sqrt{\Delta t} \sum_{r \leq j} c_{j,r} e_n^r \tag{2.1}$$

In equation (2.1) vector μ_n^j gives stochastic drift of the risk factors at node n and Δt defines the time increment between nodes n^- and n. Correlation is introduced directly on the realizations e_n^r of four standard normal variables through the Choleski elements $c_{j,r}$ of the correlation matrix with normal distribution $N(0,1)$ and illustrated in Table 4.2.

Furthermore, the general multivariate Gaussian return formulation of asset price returns $r_{i,n}$ and evaluation of assets value $v_{i,n}$ for all $n \in N_t$ are considered as follow:

$$r_{i,n} = \mu_{i,n} + \sigma_{i,t_n} \sqrt{\Delta t} \sum_r c_{i,r,t_n} e_n^r \tag{2.2}$$

$$v_{i,n} = v_{i,n^-} (1 + r_{i,n}) \tag{2.3}$$

The dedicated stochastic models to each risk factor and asset benchmark index with more details have been described in chapter 4.

2.3 Market Instability

The periods of worldwide Lehman crisis and also sovereign crisis in Europe (2009-2011) have affected previous condition in financial markets. In particular European market was suffering from unprecedented reduction of interest rates, increasing default risk of selected sovereign borrowers and increasing correlation and systematic risks in the Euro zone. Therefore, pension funds both DB and DC plans were hit hard by the market instability [4, 136, 111]. According to a chapter book on Asset-Liability Management for Financial Institutions edited by [134], the causes of the crisis spawned factors that impacted asset-liability portfolios and resulted in risk management failures. The crisis was not caused by one factor alone, and it was not relegated to one geographical location. The crisis was a systemic worldwide failure. The complex financial system had reached a point of instability and several triggers caused the system to fail, resulting in extreme draw-downs. Most existing risk management systems were incapable of handling the resulting factors individually and especially simultaneously.

Considering the importance of the crisis and its aftermaths in the pension funds the present section discusses the key sources of susceptibility for pension funds during the recent financial crisis. How pension funds' assets and liabilities were effected during the market instability and why this can be controlled in dynamic stochastic framework better than another model are discuss as follow [4, 136, 111, 86, 2].

2.3.1 Asset & Liability Shocks

The market instability and the ensuing economic and financial crisis have had a major impact on pension funds' assets. According to the [4, 136] assets' shocks during the recent market instability and financial crisis have reduced the value of assets accumulated to finance retirement by around 20-25% on average. However, there is a large variability among countries, ranging from positive but small returns in some countries to falls of over 30% in Ireland and the United States (see Figure 2.2). This variability is explained in part by differences in portfolio compositions, as well as the regulatory environment [114]. Moreover, the increase in unemployment originating from the current economic conditions will reduce the amount of pensions' savings, which will negatively affect the future retirement incomes.

As stated in [4], the crisis is also causing a shift in asset allocation patterns, with investors moving into more conservative investments. Such moves risk locking in portfolio losses and could also reduce the potential of funds to generate retirement incomes in future. The fall in the value of assets accumulated for retirement affects the solvency of pension plan sponsors and the funding levels of plans providing DB pension funds. The funding levels of pension funds providing DB pensions have fallen well below 90% in most OECD countries. As a result, the value of their assets fails to cover their

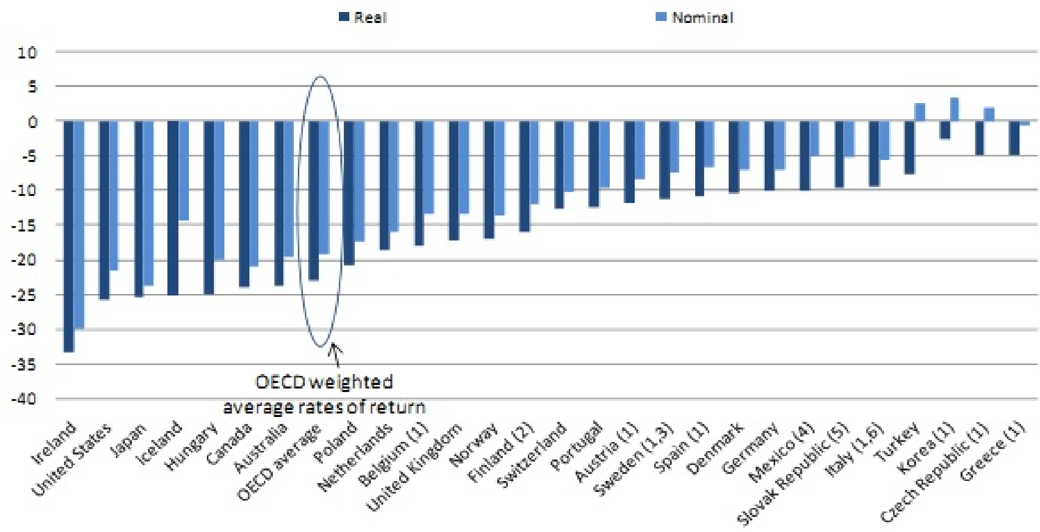


Figure 2.2: Pension Fund Returns (Jan-Oct 2008) in Selected OECD Countries (Source: OECD Global Pension Statistics).

pension liabilities.

[136] reports the changes in the value of liabilities for accounting purposes for pension plans of corporations in major equity indices as a function of the changes in the relevant yield curve. In general, accounting liabilities have dropped in October 2008 as credit markets froze and spread on corporate debt increased. However, they have rapidly increased since then due to a sharp drop in interest rates for all yield curves. For instance, short term U.K. yields decreased by 100-150 bps between August and December 2008 while long term yields decreased by 40-60 bps. Other shocks are channeled through the income statement of DB pension plans. In general, pension plan income will be affected by a slowdown in the economy. For instance, rising unemployment will translate in lower contribution income for the plan which, in turn, will affect the funding status of a DB plan. Equivalently, the rate of return on assets is likely to decrease during a recession, also affecting the funding status of a DB plan. In what to come, how to control of these market

shocks in dynamic ALM framework has been discussed in more details.

2.3.2 Managing Impact of Market Shocks

As aforementioned, the unexpected market shocks need to be considered and managed in the ALM modeling. In order to measure and handle the market instability that impacted pension funds' asset-liability portfolios, the following factors are essential to be considered in the framework of the ALM models:

- Dynamic
- Stochastic
- Risk integrate
- Economic and financial theories
- Scenario tree analysis

During the crisis period, most of the firms used static portfolio optimization framework, such as Markowitz mean-variance allocation for the modeling, which are short-sighted and when rolled forward lead to radical portfolio rebalancing unless severely constrained by the portfolio manager's intuition. In practice, fund allocations are wealth dependent and face time-varying investment opportunities, path dependent returns due to cash inflows and outflows, transactions costs and time or state dependent volatilities. Hence all conditions necessary for a sequence of myopic static model allocations to be dynamically optimal are violated. By contrast, the dynamic stochastic programming models incorporated in the system described on Figure 1.3 automatically hedge current portfolio allocations against future uncertainties in asset returns and liabilities over a longer horizon, leading to more robust decisions and previews of possible future problems and benefits [47]. By the same token, [86] argues that the dynamic allows portfolio decisions to be revisited in the future as financial and economic conditions change.

Also, stochastic in the model allows the prices, rates, and other factors calculated in the future uncertainty structure to change over time and level and capture extremes and fat tails. These are essential components in calculating interest rates, credit spreads, prices, etc. Hence these are required in ALM modeling and managing portfolios in the financial crisis, with the repricing of instruments based on the uncertainty structure.

Risk integration is essential, because multiple factors affect asset-liability portfolios at the same time. Without considering integration, the measures will be incorrect and subsequently, also the portfolio management strategy.

Economic and financial theories, including new ones, address one of the most important recommendations made by Joseph Stiglitz:

" We need new theories as well as some of the old ones and models need to be constructed that have dynamic, stochastic, and general equilibrium, and which include systemic risks."

The last but not least, the factor required to be considered in the dynamic ALM modeling is the scenario tree generation. The scenario tree describes the uncertainty structure or stochastics and dynamics of the factors considered. The tree will value the uncertainty in interest rates, exchange rates, equity prices, liquidity requirements, nonperforming assets, contingent liabilities, capital position, etc. The tree maps out the movements of the factors considered, along with their dependencies. The scenario tree must be built efficiently. It must be small enough to solve, yet big enough to capture the stochastics with a good collection of fat-tail scenarios.

All in all, these factors which was mention and disused about them in this section are necessary to consider in the ALM framework.

2.4 Long-Term Risk Control in a Dynamic Setting

Investment performance is enhanced by capturing rebalancing gains when a portfolio is modified. Rebalancing a portfolio can be considered as an option to be exercised when adding value to the investors' performance. Moreover, in the dynamic setting there is information related to both expected positive and negative cash-flows which can not be used in a traditional static approach.

There are several reasons for applying the dynamic setting framework to controlling the risks rather than traditional static approach. Some reasons such as time consistency and inflation and liquidity risks can not be managed without a solid dynamic structure and deserve special attention and must be analyzed in depth. Therefore, in the present section the effort is to set-up an optimal financial planning approach in dynamic framework to control the investment risks. To this end, an optimal time decomposition [45] and inflation and liquidity risks have been considered in a dynamic setting framework over the time span.

2.4.1 Time Consistency

As aforementioned, time consistency is a main property for the dynamic setting risk control. The possibility of extending the time period is one of the key element and the most relevant advantage of dynamic optimization approach. The idea is to have a time-consistent optimal portfolio strategy of an investor with an exposed liability stream over long time horizon for a pension fund and several intermediate steps (after six months, one year, two years and so on) where the portfolio rebalancing is allowed. This structure is contain a dynamic optimization setting since the optimal investment decision at starting point which called *Here & Now* (H&N) decision, already takes into account the possibility to adjust the decision several times along the way depending on the potential market and target evolution. Moreover, the adoption of a dynamic setting approach allows both the extension of the decision time horizon and a more accurate short-term modeling of pension fund variables.

The time span extension of the current thesis is based on a discrete time and space setting framework [105, 27, 47, 143, 144, 9]. The time set indexed by $t = 0, 1, ..., T - 1$ corresponds to times at which the funds' portfolio needs to be rebalanced, through a sequence of *buy, hold and sell* decisions and T is the end of the planning horizon in which no decision is made.

2.4.2 Inflation and Liquidity Risks for Pension Funds

Dynamic setting framework should be preferred to a static approach due to other two critically important risk-factors, namely inflation and liquidity shortfall risk.

For the inflation modeling dynamic approach allows simulating inflation dynamic and modeling real assets class consistently both over time and across asset classes. Inflation is a key risk factor for the optimization modeling. Both assets and liabilities can be impacted by unexpected inflation shocks. Assets can suffer because an inflation increase normally triggers an interest rate up-shift thus negatively hitting the fixed-income nominal assets. Liability can also be under pressure since future pensioners are inflation sensitive.

For these reasons it is of paramount importance to properly and consistently model inflation risk when projecting both assets and liabilities over time [50].

Liquidity risk is also a key issue for asset allocation decision. It is useful to understand how much liquidity risk can be born by the specific pension fund portfolio depending on the duration of its liabilities, the expected future business and the various regulatory requirements. Also, attaching a certain degree of liquidity to each and every investible asset class can be useful to measure the aggregate liquidity of the portfolio and to eventually impose bounds on the optimization [50].

Chapter 3

Dynamic Decision Approach

Dynamic decision approach is interdependent decision making that takes place in an environment that changes over time due to the previous actions of the decision maker and the events over time. Dynamic decision modeling is a very challenging task. The multitude of problems, the domain-specificity, the uncertainty, and the temporal nature of the underlying phenomena all contribute to the intricacy of the dynamic decision modeling process. The formulation of dynamic decision models for financial applications generally require the definition of a risk-reward objective and financial stochastic models to represent the uncertainty underlying the decision problem. Dynamic decisions under uncertainty are very common in financial planning and financial engineering problems. Most practical decision problems involve uncertainty and the solution of the optimization problem and the quality of the resulting strategy will depend critically on the adopted financial model and its consistency with observed market dynamics.

Based on Bellman's equations and on the well behaved properties of the risk neutral formulation, several models have been developed for different applications such as asset and liability management, portfolio selection, risk measures and etc. Indeed, some important works, for instance [18, 27, 109, 143, 9, 131, 29, 67, 68, 120, 124, 32, 26] contained efficient algorithms to solve these problems.

What follows in the present chapter is to introduce the limits of static approaches versus dynamic framework, decision criteria, dynamic stochastic optimization models, scenario generation and tree expansion.

3.1 Limits of Static Approaches

The most well known and likely the most widely used method for the static approach is the [92] mean-variance model. This approach can be easily implemented in a spreadsheet but the standard implementation of the mean-variance model is static (one-period) and thus fails to capture the multi-period nature of the financial problem. Single-period models are generally unable to respond in an appropriate manner to protect the investor's wealth or surplus. Another contributing factor has been the lack of attention to liquidity concerns. Single-period static portfolio models do not properly address transaction costs or liquidity constraints. Markowitz's mean-variance analysis has been extended to incorporate multiple periods and market frictions, [133, 75, 20] but at the cost of greatly increased complexity.

The static approach is not compatible to be applied for the financial institutions for a number of reasons. First of all, the available information in the input phase about projected cash-flows for both assets and liabilities can not be fully utilized. Second, the idea of buying the optimal portfolio and holding it until the end of a pre-defined time period dose not reflect the managerial options concretely available to the investment manager who, in reality, can rebalance the fund's portfolio if and when necessary. Third, the output of the static optimization approach is quite poor since it does not provide with any clue about the likelihood to meet the true targets over time horizon [50].

The traditional static approach is not able to consider the dynamic nature of the liabilities and can not cope with the flow of capital in and out of the scheme in the form of contributions and benefit payments of the pension fund. Inflation and Liquidity are two critically important risk factors for long-term optimization planning of a pension fund and should be explicitly and consistently modeled in dynamic manner. Last but not least, the time consistency in dynamic framework and the related possibility of portfolio rebalancing over time makes to optimization framework more realistic and effective than traditional static approach.

3.2 Decision Criteria

A fund manager (or managing board) has a double objectives: firstly to manage the investment strategies of the fund and secondly to take into account the guarantees given to all investors. Guarantee for all participants of the fund must be ensured with a high probability by right investment strategies. However this task is not straightforward, as it requires right decisions over long-term financial planning and dealing with a stochastic liability and market uncertainly.

As already mentioned, a complex financial planning for a pension fund can not be managed based on simplistic target structure. Different performance indicators, potentially in trade-off with each other, are typically required to capture the management decision process over the horizon. Dynamic optimization approach is capable to meet these aims with simultaneous optimization of different targets consist of performance measures which can be risk adjusted at different holding periods over an extended time span. Since there is not any single perfect indicator to measure performance over horizon, the particular interest of this thesis is to have three different targets for the short, medium and long-term investment horizon.

Hence, the performance indicators adopted in this study are the short-term target of one year, medium-term target of three years and long-term target of ten years for the fund. Therefore, from a mathematical viewpoint, we propose in section 4.3.3.2 a MSP which includes a multi-criteria objective function with several financial and regulatory constraints. The proposed objective function is a similar approach that proposed in [28] for property and casualty (P&C) insurance.

3.3 Dynamic Stochastic Optimization

In the current research, dynamic stochastic optimization (also known as dynamic stochastic programming) approach used to select allocations that are optimal with respect to fund liabilities and suitable measures of underfunding risk for a pension fund management [47, 43, 44, 42]. Dynamic stochastic optimization involves simulating economic factors, asset returns and liabilities forward over a number of scenarios. In order to overcome the portfolio risks, a dynamic stochastic optimization approach can be implemented following some main steps. Inputs such as assets, liabilities and constraints are set over different stages thus allowing a realistic representation of the problem. Moreover, asset dynamic is modeled through statistical simulations taking into account various scenarios and their respective implications in term of investment strategy.

Dynamic stochastic optimization models cover stochastic processes in which time plays an essential role [125, 94, 81]. The emphasis is often on Markov decision processes (also known as probabilistic dynamic programming) and the optimization of stochastic models [137, 130]. In [53] extensions of time-consistent risk measures are studied within multi-stage stochastic programs, and a stochastic dual dynamic programming approach is proposed for their solution. [84] combines model of multi-stage stochastic programming with a stochastic control framework.

[87] developed a multi-stage stochastic linear programming model for ALM. Their model includes the uncertainties of institutional, legal, financial, and bank-related policies. They demonstrate that the ALM model developed, is theoretically and operationally superior to deterministic programming model (e.g. mean variance, [92]). Some other notable financial planning applications can be found in [109, 110].

DSP approaches are increasingly being adopted to address different types of financial planning problems: from the classical ALM problem [27, 49, 144], large enterprise-wide risk management problems [143], to financial engineering applications and complex portfolio management problems [82, 144, 9], modeling the assets and liabilities of insurance products with guarantees [33, 36, 74, 34, 47, 43, 42, 45].

Several authors highlighted the advantages of multi-stage dynamic stochastic programming in asset and liability modeling (see for example [100, 47, 9]).

According to [142], at each decision date of the DSP the portfolio manager needs to assess the current state of the economy (i.e. interest rates and market prices), he also needs to assess future fluctuations in interest rates, market prices and cash flows at possible states of the economy needs to be incorporated into a investment decision of buying and selling securities, and short-term borrowing or lending. At the next decision date the portfolio manager is faced with new information and possible future states that need to be incorporated into the new investment decision.

The stochastic programming model specifies a sequence of investment decisions at each of the discrete trading times. At each decision period in the scenario tree, the investment decision is made given the current state of the portfolio and a set of possible scenarios at successor states. Thus the current portfolio composition depends on the previous decisions and the realized scenarios in the interim. The model will determine an optimal decision at each state in the scenario tree, given the information available at that state. Given that there is a multitude of succeeding future states of the economy, the optimal decision will not depend on clairvoyance, but should anticipate the future states of the economy [142].

Strategic ALM for the pension fund requires the dynamic formulation of portfolio rebalancing decisions together with appropriate risk management in terms of a dynamic stochastic optimization problem. A problem of pension fund management is dynamic due to the achievement of intermediate targets as well as changing employments conditions over time (work versus retirement, etc.) and the time distribution of liabilities and income variations (salary growth, etc.) forces a dynamic representation of the problem. The sequence of actions taken in face of uncertainty and their random consequences need to be taken into account within a given time frame. Furthermore, the problem is stochastic because of the effectiveness of any adopted strategy and the achievement of the targets do depend on a sequence of random events, such as the evolution of the random processes modeling a set of relevant financial markets. The long-term nature of the decision problem, furthermore, imposes a specific effort in the development of the model of uncertainty [25].

Uncertainty is modeled over time using scenarios that approximate the future. High-performance workstations and PCs are used to enable exact and approximate algorithms to determine robust decisions that hedge against future uncertainty. Then as the uncertainty becomes known period by period, recourse decisions responding to the new information can be made [137]. In what follows, we introduce mathematical formulation of stochastic programming with two-stage and multi-stage problems.

3.3.1 Two-Stage Stochastic Problem Formulation

The two-stage stochastic linear program is the problem of finding [132]:

$$\begin{aligned} &min_{x\in\mathbb{R}^n} C^T x + \mathbb{E}[Q(x,\omega)] \\ &s.t.\ \ Ax = b, x \geq 0 \end{aligned} \tag{3.1}$$

In problem (3.1), $Q(x,\omega)$ is the optimal value of the second stage problem

$$\begin{aligned} &min_{y\in\mathbb{R}^m}\ \ q^T y \\ &s.t.\ \ Tx + Wy = h, x \geq 0 \end{aligned} \tag{3.2}$$

Here $\omega := (q, h, T, W)$ are the data of the second stage problem and some or all elements of vector ω are as random. Also, the expectation operator at the first stage problem (3.1) is taken with respect to the probability distribution of ω. In a general way two-stage stochastic programming problems can be written in the following form

$$min_{x\in X}\{f(x) := \mathbb{E}[F(x,\omega)]\} \tag{3.3}$$

where $F(x,\omega)$ is the optimal value of the second stage problem

$$min_{y\in g(x,\omega)} g(x,y,\omega) \tag{3.4}$$

Here $X \subset \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}$ and $g : \mathbb{R}^n \times \Omega \to \mathbb{R}^m$ is a multi-function. In particular, the linear two-stage problem (3.1) and (3.2) can be formulated in the above form with

$$g(x, y, \omega) := C^T x + q^T(\omega) y$$

$$g(x, \omega) := \{y : T(\omega)x + W(\omega)y = h(\omega), y \geq 0\}$$

3.3.2 Two-Stage Scenario Based Formulation

Following [132], we introduce concept of nonanticipativity into two-stage problems structure. Consider the first stage problem (3.3) and assume that the number of scenarios is finite $\Omega = \{\omega_1, \omega_2, ..., \omega_S\}$ with respective to (positive) probabilities $p_1, p_1, ..., p_S$. We relax the first stage problem by replacing vector x with S vectors, one for each scenario $x_1, ..., x_S$. Then we have the following relaxation of problem (3.3):

$$\begin{aligned} &min_{x_1,...,x_S} \sum_{s=1}^{S} p_s F(x_s, \omega_s) \\ &s.t. \;\; x_s \in X, \;\; s = 1, ..., S \end{aligned} \tag{3.5}$$

Such problem can be split into S smaller problems for each scenarios:

$$min_{x_s \in X} F(x_s, \omega_s), \;\; s = 1, ..., S \tag{3.6}$$

The optimal values of problem (3.6) with sum of weighted parameter p_s for $s = 1, ..., S$ is equal to the optimal value of problem (3.5). For instance, in the case of two-stage linear program, relaxation leads to solving S smaller problems

$$\begin{aligned} &min_{x_s \geq 0, y_s \geq 0} \;\; C^T x_s + q_s^T y_s \\ &s.t. \;\; A x_s = b \\ &\qquad T_s x_s + W_s y_s = h_s \end{aligned} \tag{3.7}$$

First stage decision variables x_s in (3.5) are now allowed to depend on a realization of the random data at the second stage. Hence, problem (3.5) is not suitable for modeling a two stage decision process. This can be fixed by introducing the nonanticipativity constraint

$$(x_1, x_1, ..., x_S) \in \mathfrak{I} \tag{3.8}$$

where $\mathfrak{I} := \{x = (x_1, ..., x_S) : x_1 = ... = x_S\}$ is a linear subspace of the nS-dimensional vector space $\chi := \mathbb{R}^n \times ... \times \mathbb{R}^n$. Due to the constraint (3.8), all realizations $x_s, s = 1, ..., S$, of the first stage decision vector are equal to each other and they do not depend on the realization of the random data. The constraint (3.8) can be written in different forms, which can be convenient in various situations. Problem (3.5) together with the nonanticipativity constraint (3.8) can be formulated as follow:

$$\begin{aligned} min_{x_1,...,x_s} & \sum_{s=1}^{S} p_s F(x_s, \omega_s) \\ s.t.\ \ & x_1 = ... = x_s \\ & x_s = \sum_{i=1}^{S} p_i x_i \\ & x_s \in X, \quad s = 1, ..., S \end{aligned} \tag{3.9}$$

Such non-anticipativity constraints are especially important in multi-stage modeling. The two-stage stochastic programming models can be naturally extended to a multi-stage setting. We discuss formulation of such decision processes in next section for a multi-stage recourse dynamic portfolio management problem.

3.3.3 Multi-Stage Recourse Formulation of Dynamic Portfolio Management

We consider a stochastic programming problem in the form of a multi-stage stochastic programming in discrete time stage [59, 27, 132]. In the multi-stage setting, the uncertain data is revealed gradually over time and our decisions should be adapted to this process. The decision process has the form:

$$decision(x_1) \rightsquigarrow observation(\omega_2) \rightsquigarrow decision(x_2) \rightsquigarrow ...$$
$$... \rightsquigarrow observation(\omega_T) \rightsquigarrow decision(x_T)$$

The general instance of a multi-stage stochastic programming with recourse has been formulated as follows:

$$min_{x_1 \in \chi_1}\{f_1(x_1) + \mathbb{E}_{\omega_2}[min_{x_2 \in \chi_2}(f_2(x_2) + ... + \mathbb{E}_{\omega_T}[min_{x_T \in \chi_T} f_T(x_T)])]\} \tag{3.10}$$

The separable objective function (3.10) is defined by the period functionals $f_t(x_t)$ for $t = 1, 2, ..., T$ and carries a nested structure consistent with the underlying tree process while the coefficient tree process depends on ω and will determine the feasibility region of the problem. The multistage problem is linear if the objective functions and the constraint functions are linear. In a general formulation for the linear multi-stage case of problem (3.10) we have

$$\begin{aligned} & f_t(x_t) := c_t^T x_t & t = 2, ..., T \\ & \chi_1 := \{x_1 : A_1 x_1 = b_1, \;\; x_1 \geq 0\}, & \\ & \chi_2 := \{x_2 : B_2(\omega)x_1 + A_2(\omega)x_2 = b_2(\omega), \;\; x_2 \geq 0\} \;\; a.s. & \\ & \chi_t := \{x_t : \{B_t(\omega)x_{t-1} + A_t(\omega)x_t = b_t(\omega) \;\; x_t \geq 0\} \;\; a.s. & t = 2, ..., T \end{aligned} \tag{3.11}$$

In (3.12) the first row is the linear case of objective function while the second row for $t = 1$ defined as deterministic constraints on the first-stage decision x_1, and for $t = 2, 3, ..., T$ constraints defined as stochastic form regions for the recourse decisions $x_2, x_3, ..., x_T$. Also, in (3.10) $\mathbb{E}_{\omega_t}$ for $t = 2, ..., T$ is the conditional expectation with respect to the filtration $\mathbb{F}_1 := \{0, \Omega\} \subset \mathbb{F}_2 \subset ... \subset \mathbb{F}_T := \mathbb{F}$, where $\mathbb{F}_t := \sigma\{\omega, t\}$ is the σ-field generated by the history ω^t of the data process ω for $t = 2, ..., T$ and $\mathbb{P}$ is a probability measure on this space. In the multi-stage stochastic problem the *here-and-now* decision is taken under full uncertainty and provides an optimal hedging portfolio strategy with respect to the future possible scenarios. As aforementioned, every decision of the model (3.10) together with (3.12) have a random impact on the following stages modeled through the technology matrices B, while recourse actions are associated with the A matrices.

In multi-stage problem formulation the optimal policy or decision process $\overset{0}{x} := (\overset{0}{x}_1, \overset{0}{x}_2, .., \overset{0}{x}_T)$ has been dependence on the realizations of the vector data process $\omega := (\omega_2, ..., \omega_T)$ in $(\Omega, \mathbb{F}, \mathbb{P})$, with the sample space defined as $\Omega := \Omega_2 \times \Omega_3 \times ... \times \Omega_T$.

Problem (3.10) may be given a more compact dynamic programming representation which takes advantage of the structure exhibited by the set of constraints. For each $t = 1, ..., T - 1$, we have

$$\begin{gathered} min_{x_t}[f_t(x_t) + v_{t+1}(\omega^t, x^t)] \\ s.t.\ \ B_t x_{t-1} + A_t x_t = b_t \end{gathered} \tag{3.12}$$

where v_{t+1} expresses the optimal expected cost for the stages from $t + 1$ to T, given the decision history $x^t := (x_1, ..., x_t)$ and the realized history of the random process $\omega^t := (\omega_1, ..., \omega_t)$. Specifically,

$$v_{t+1}(\omega^t, x^t) := \mathbb{E}_{\omega_{t+1}}[min_{x_{t+1}}(f_{t+1}(x_{t+1}) + ... + \mathbb{E}_{\omega_t} min_{x_T} f_T(x_T))] \tag{3.13}$$

where the minimizations are taken subject to the appropriate financial and regulatory constraints.

In real applications, linear combinations of risk-reward measures are becoming increasingly popular. Risk can either be specified with respect to a benchmark measure or with respect to a real-valued risk function. In problem (3.14) a penalty coefficient is introduced to the risk measure

$$\begin{gathered} max_{x \in X} \sum_t \mathbb{E}_{\omega_t}[R(x_t, \omega_t, t) - \gamma\rho(x_t, \omega_t, t)] \\ s.t. \;\; g_t(x_t, \omega_t, t) \;\; a.s. \end{gathered} \tag{3.14}$$

For $t = 1, 2, ..., T$ suppose $W_t(\omega, x)$ is a portfolio managed wealth process and $\widetilde{W}_t(\omega, x)$ stands for a wealth process leading to a relative optimization problem

$$min_{x \in X}\{\sum_t \mathbb{E}_{\omega_t}[f(W_t, \widetilde{W}_t)] | g_t(x_t, \omega_t, t) \in Q \;\; a.s.\} \tag{3.15}$$

Correspondingly, at the end of each time period and on the basis of the current information, a portfolio manager adopts an optimal decision in the face of the uncertainty that (s)he is now facing. This decision needs to be feasible with respect to the constraints induced by the future values of the random data process and is influenced by the current composition of the portfolio.

3.3.4 Multi-Stage Scenario Based Formulation

A scenario-based formulation is a standard solution technique for a discrete time state. Suppose that we have a multi-stage formulation with finite number of scenarios, say S, for the problem data. Assuming that the finite number of realizations of the stochastic process, $\omega_1, ..., \omega_T$, is concentrated on a finite number of points, denoted by $\omega_1, ..., \omega_s$. This allows for a derivation of a deterministic equivalent formulation on the form

$$
\begin{aligned}
&min \sum_{s=1}^{S} p_s [f_1(x_1^s) + f_2(x_2^s) + f_3(x_3^s) + ... + f_T(x_T^s)] \\
&s.t. \;\; A_1 x_1 && = b_1 \\
&\quad B_2^s x_1 + A_2^s x_2^s && = b_2 \;\; a.s. \\
&\qquad B_3^s x_2 + A_3^s x_3^s && = b_3 \;\; a.s. \\
&\qquad\qquad \ddots \\
&\qquad\qquad\qquad B_T^s x_{T-1} + A_T^s x_T^s && = b_T \;\; a.s. \\
&\quad l_t \le x_t^s \le u_t \;\; a.s. \;\; t = 0, 1, ..., T, \quad s = 1, ..., S
\end{aligned}
\tag{3.16}
$$

where p^s is associated probability of each scenario s and $x^s = (x_1^s, x_2^s, ..., x_T^s)$ denotes the corresponding sequence of decisions. In the problem (3.16), all parts of the decision vector are allowed to depend on all parts of the random data. However, the decision x_t at stage t, should be allowed to depend on the data observed up to stage t. However, for the first stage, the decision should be independent of possible realizations of the data process. At stage $t = 1, ..., T$ the scenarios that have the same history $\omega_{[t]}$ cannot be distinguished, so in order to correct this problem non-anticipativity constraints are included

$$x_t^s = x_t^l \;\; \forall s, l \;\; \text{for which} \;\; \omega_{[t]}^s = \omega_{[t]}^l \quad t = 0, 1, ..., T \tag{3.17}$$

Problem (3.16) together with the nonanticipativity constraints (3.17) are equivalent to the original formulation (3.10).

3.4 Scenario Generation

In models of decision making under uncertainty, we are often faced with the problem of representing the uncertainties in a form suitable for quantitative models. The uncertainties are expressed in terms of multivariate continuous distributions, or a discrete distribution [76]. [8] argue that during the scenario tree generation, we do not forecast the future state of a random variable but try do generate a finite set of realistic possible scenarios.

As [42] state, scenario generation translates a set of continuous stochastic dynamics into a discrete event tree structure (see Figure 4.1). The scenarios are arranged in a tree expansion structure and decisions are made at points where the tree branches. Each of these decisions is optimal with respect to the all the simulated evolution of the asset returns and liabilities that could occur after the decision point. At the inception time (root node), *"here & now"* (H&N) decision captures information in the history of the variables up to that point

The procedure is based on generating a tree over the entire planning horizon. In the tree scenarios are always randomly generated relying on the MC simulator by moving stage by stage and within each stage node by node. Each node in the tree corresponds to a different state of the vector process, whose elements are risk factors, asset returns and liabilities while a scenario is a complete path from the root node to the leaves.

We consider the MC approach to generate the scenario and all scenarios are equally weighted. This means that we consider the probability of any particular one to inversely proportional to the total number of scenarios at that point in time.

3.4.1 Tree Expansion

To build the tree, we need to define the expanded tree branching factors, setting the number of childs for each node (Figure 4.1).

The tree expansion method relies on relatively simple ideas from an economic viewpoint, which however prove rather effective in capturing the financial risk underlying the problem. Hence, Figure 4.1 illustrates the scenario tree structure with decision stages where each node is a joint outcome of all the variables at the corresponding decision stage and each path through the tree represents a specific scenario.

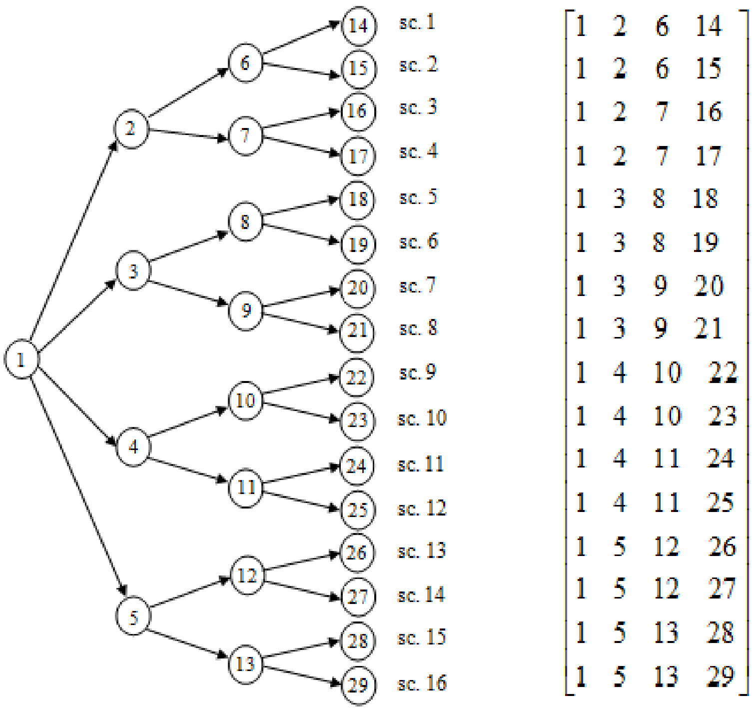


Figure 3.1: Example of scenario tree structure and nodal partition matrix

The conditional structure of the tree can be conveniently represented by the trees' nodal partition matrix, in which each row corresponds to a scenario and each column to a time-stage. In Figure 4.1, the tree on the left is associated with the partition matrix on the right.

Chapter 4

Optimal DB Pension Fund Management

The purpose of this thesis is precisely to identify the optimal DSP strategy with dynamic allocation rebalancing rule and multi-critical objective over a certain time span as a function of the portfolio manager's targets in a DB pension plan. In particular, we consider a portfolio optimization problem which presents financial returns and market risks with a long-term investor plan under stochastic correlation. In this chapter, we describe the mathematical formulation of DSP for DB pension fund which will be implement for a case study in the next chapter. The case study has been designed for a large institutional investors in order to identify the optimal asset allocation over certain planning horizon with inclusion of capital and liability constraints generated by a DB pension fund.

4.1 Problem Set-up

We consider a 10-year multi-stage dynamic stochastic optimization problem with scenario generation under stochastic correlation based on European market standards for a DB pension fund manager facing stochastic liabilities. The optimal problem is formulated as an expected shortfall minimization with respect to a pension wealth goal. The investment universe includes cash equivalents, fixed income, equity and commodity as indirect real asset[1]. For each such investment opportunity a dedicated statistical model has been implemented to generate future price and return scenarios for describing the uncertainty that investment manager is facing over time. However this task is not straightforward, as it requires long-term forecasting for all investment classes and dealing with a stochastic liabilities and at the same time ability to assess short, medium and long-term implications coming from allocation decision. Dynamic stochastic programming is the technique of choice to solve this kind of the problem.

Numerical results are presented in chapter 5 for specifications of the dynamic optimization problem over a long-term horizon with several decision stages.

4.2 Under-Funding Risk

There is a very important requirement to monitor and manage the probability of under-funding risk for the fund, that is the confidence level with which the pension fund will be able to meet its targets. In considering the risks faced by

[1]Indirect real assets are real assets with historically preserved the real rate of return which are strongly correlated with inflation, but are not directly linked to inflation (i.e. commodity and real state).

a pension fund, we must look at the fund in its entirety. On the liability side, funds are exposed to interest rates, inflation and mortality [6]. On the asset side, they face exposure to risk in different markets [24]. Hence they operate in a multi-factor environment, exposed to a wide range of asset classes and liabilities which also have varying degrees of correlation amongst themselves. Because of this, the worst situation for pension funds is falling asset prices and increasing of funds' future liabilities [42].

In fact, analyzing the current and projected future status of the fund, including review of the funding ratio of the plan is an important issue that can shed more light on the ALM framework. In particular, in the modeling framework we consider expected shortfall with respect to the fund's target levels in order to monitor the fund's status and probability of liquidity shortfall to be able to determine the optimal pension fund investment strategy and fully fund the pension obligation over the long-term.

4.3 Mathematical Instance of Pension Fund ALM

The mathematical formulation is based on dynamic stochastic optimization technique with objective function and relative constraints to solve the DB pension fund dynamic ALM problem. From a mathematical viewpoint, we focus on multi-stage stochastic optimization problem over a long horizon [12, 145, 27, 52, 143, 144, 71, 47, 44, 42, 45, 9, 29, 35]. And more recently, exploring ALM alternative modeling and optimization approaches [1, 31, 116].

4.3.1 Asset Returns Model

An investor's strategy depends on the random coefficients which must be derived from the data process simulations along the specified scenario tree. Furthermore, for the optimization modeling and scenario generation, we need to know about the behavior of our portfolio over time horizon. Since our portfolio is changing over time and is exposed to some stochastic factors, we introduce asset price and returns models in discrete-time with the modelling framework consisting of a Two-Layers Asset Simulation Model (TLASM). The model structure is close to the framework originally employed by [29] with introducing relevant risk factors for DB pension fund and stochastic correlation among asset indexes.

Alternative asset price return simulation has been introduced by [105] as cascade simulators and some implementation of CASM (Cascade Asset Simulation Model) is given in [47, 42, 5]. For short, long term interest rate and inflation, Cox-Ingersoll-Ross (CIR) dynamics models has been introduced and implemented by [37, 28, 29]. Also, gaussian economic factor model (EFM) introduced to capture the dynamics of the whole term structure of bond price [17, 112, 45].

4.3.1.1 Correlation Model

In a multistage scenario setting when introducing a stochastic correlation assumption, we need to first employ a dynamic correlation model driven by a set of random factors and then interface such model with a scenario representation of the stochastic programming problem. Stochastic correlation was introduced as Dynamic Conditional Correlation model by [55] with analyzing the performance of the model for large covariance matrices. We consider a

DCC model and then present an approach for scenario tree generation consistent with such assumption. For more details on DCC model and application, see [55, 56, 118, 69, 128, 58, 41].

Let $r_{i,t}$ be the return of asset $i \in \mathcal{I}$ at time $t \in \mathcal{T}$. We indicate with r_t the return vector with components $r_{i,t}$. Given r_0 and $\omega \in (\Omega, \mathbb{F}, \mathbb{P})$ as a generic random source of risk, for $t = 1, 2, ..., T$ we define the stochastic differential equations for return process as:

$$r_t = r_{t-dt} + \mu(t,\omega)dt + \Gamma(t,\omega)\sqrt{dt}e(\omega) \tag{4.1}$$

where $\mu(t,\omega)dt$ is an instantaneous drift. We assume in section 4.3.1.2 a generic model for the stochastic mean function and in section 4.3.1.3 dedicated for different asset indexes. Let's focus on $\Gamma(t,\omega)$ and $e(\omega)$ where $e \sim N(0,1)$. $\Gamma(t,\omega)$ is a random covariance matrix that admits decomposition $\Gamma_t = D_t C_t D_t$. Where D_t is a diagonal matrix with elements $\sigma_{i,t}$ with $i = 1, 2, ..., \mathcal{I}$, includes the $\mathcal{I}$ returns' conditional standard deviations, or *volatilities* in financial terminology which can be defined by any type of a univariate GARCH process, while $C_t := \{c_{i,j,t}\}$ includes the time-varying correlation coefficients between asset i and asset j at time t.

The DCC is a natural extension of the GARCH models. In the DCC model [55] the relationship between conditional correlations and conditional variance is obtained expressing the returns, $\rho_{i,t}$ as

$$\rho_{i,t} = \varepsilon_{i,t}\sigma_{i,t} \qquad where \qquad \varepsilon_{i,t} \sim N(0,1). \tag{4.2}$$

At the first stage we assume a *Threshold Autoregressive Conditional Heteroskedasticity* (TARCH) process of the first order for $\sigma^2_{i,t}$. [66] and [140] introduced independently the TARCH models which allows for asymmetric shocks to volatility. In particular, referring to [57], we define the conditional variance $\sigma^2_{i,t}$ for the TARCH(Q,K,J) model as

$$\sigma^2_{i,t} = \varpi + \sum_{q=1}^{\mathcal{Q}} \varphi_q \rho^2_{i,t-q} + \sum_{k=1}^{\mathcal{K}} \psi_k \sigma^2_{i,t-k} + \sum_{j=1}^{\mathcal{J}} \nu_j \mathbb{I}_{t-j}(\rho_{i,t-j})^2 \qquad i = 1, 2, ..., \mathcal{I} \tag{4.3}$$

where ϖ, ψ, φ and ν are deterministic coefficients. The parameters are estimated through *Maximum Likelihood Estimation* (MLE) method. The indicator function $\mathbb{I}_{t-j}(\cdot)$, takes value 1 if the residual at time $t-j$ is negative, and zero otherwise. Positive innovation at time t has an impact on the volatility at time t+1 equal to φ times the ρ^2_i, while a negative innovation has impact equal to $(\varphi+\nu)$ times the ρ^2_i. The presence of the leverage effect would imply that the coefficient ν is positive, that is, that a negative innovation has a greater impact than a positive innovation. For more details about TARCH process see [140, 66, 57] and references therein.

According to [57], TARCH volatility process is a way of parametrizing the sign of the innovation that may influence the volatility in addition to its magnitude. They indicated that the sign of the innovation has a significant influence on the volatility of returns and TARCH model implies that a positive and negative innovation at time t have different impact on the volatility at time $t + 1$.

In DCC model [55], the conditional correlation matrix is modeled as

$$C_t = \tilde{Q}_t^{-0.5} Q_t \tilde{Q}_t^{-0.5} \tag{4.4}$$

where $Q_t = \{q_{ij,t}\}$ is the conditional covariance matrix and $\tilde{Q}_t$ is the diagonal matrix with the square root of the i th diagonal element of the Q_t. Dynamics of Q_t can be consider with following the DCC model [55] assumption as

$$Q_t = (1 - \alpha - \beta)\bar{Q} + \alpha \varepsilon_{t-1} \varepsilon'_{t-1} + \beta Q_{t-1} \tag{4.5}$$

where $\varepsilon_t \equiv [\varepsilon_{1,t}, \varepsilon_{2,t}, ..., \varepsilon_{I,t}]'$, $\varepsilon_{i,t} = \rho_{i,t}/\sigma_{i,t}$ and $\bar{Q}$ is the unconditional covariance matrix of ε. α and β parameters estimated through MLE function such that $\alpha, \beta \geqslant 0$ and $\alpha + \beta < 1$ to ensure positive definiteness and stationarity, respectively. Q_t should be positive definite in order to guarantee that correlation matrix C_t has ones on the diagonal and all other elements are in the interval $[-1, 1]$. Accordingly the variance-covariance process dynamics can be expressed as

$$\Gamma_t = D_t(\tilde{Q}_t^{-0.5} Q_t \tilde{Q}_t^{-0.5}) D_t. \tag{4.6}$$

For two assets, the elements of matrix $Q_t \in \mathbb{R}^{2,2}$ are shown as follow:

$$\begin{bmatrix} q_{11,t} & q_{12,t} \\ q_{21,t} & q_{22,t} \end{bmatrix} = (1-\alpha-\beta) \begin{bmatrix} \bar{q}_{11} & \bar{q}_{12} \\ \bar{q}_{21} & \bar{q}_{22} \end{bmatrix} + \alpha \begin{bmatrix} \varepsilon_{1,t-1} \\ \varepsilon_{2,t-1} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t-1} & \varepsilon_{2,t-1} \end{bmatrix} + \beta \begin{bmatrix} q_{11,t-1} & q_{12,t-1} \\ q_{21,t-1} & q_{22,t-1} \end{bmatrix} =$$

$$= \begin{bmatrix} (1-\alpha-\beta)\bar{q}_{11} + \alpha\varepsilon_{1,t-1}^2 + \beta q_{11,t-1} & (1-\alpha-\beta)\bar{q}_{12} + \alpha\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \beta q_{12,t-1} \\ (1-\alpha-\beta)\bar{q}_{21} + \alpha\varepsilon_{2,t-1}\varepsilon_{1,t-1} + \beta q_{21,t-1} & (1-\alpha-\beta)\bar{q}_{22} + \alpha\varepsilon_{2,t-1}^2 + \beta q_{22,t-1} \end{bmatrix}.$$

Then we have dynamic correlation matrix $C_t = \tilde{Q}_t^{-0.5} Q_t \tilde{Q}_t^{-0.5}$ with component of $c_{12,t}$ for two assets:

$$c_{12,t} = \begin{bmatrix} \frac{1}{\sqrt{q_{11,t}}} & 0 \\ 0 & \frac{1}{\sqrt{q_{22,t}}} \end{bmatrix} \begin{bmatrix} q_{11,t} & q_{12,t} \\ q_{21,t} & q_{22,t} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{q_{11,t}}} & 0 \\ 0 & \frac{1}{\sqrt{q_{22,t}}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{q_{12,t}}{\sqrt{q_{11,t}}\sqrt{q_{22,t}}} \\ \frac{q_{21,t}}{\sqrt{q_{22,t}}\sqrt{q_{11,t}}} & 1 \end{bmatrix}.$$

In each time period asset returns $r_{i,t}$ will depend on prior and current realizations of the risk factors, current correlations, dynamic volatilities and random events. We define $\xi_{i,t} = [r_{i,t}, \mu_{i,t}, \sigma_{i,t}, c_{ij,t}]$ as a set of random variables of asset i at time t and accordingly $\xi_t \in \mathbb{R}^I$ as a vector of realization and will determine the portfolio revisions according to the following scheme:

$$\xi_1, x_1 \rightsquigarrow \xi_2, x_2 \rightsquigarrow ... \rightsquigarrow \xi_T.$$

We want to evaluate the optimal strategy allowing time varying volatility and correlation. Therefore, in the tree process (session 4.3.1.2), we specify the mean function of $\sigma_{i,t}$ and $c_{ij,t}$ to generate the scenarios.

Remarks:

- DCC allows us to model the correlation dynamics between asset classes such that we can evaluate whether different markets co-moved to a greater extent during the crisis.
- DCC model is mean reverting as long as $\alpha + \beta < 1$.
- There are some conditions on the parameters α and β to guarantee Γ_t to be positive definite. The scalars α and β must satisfy: $\alpha \geqslant 0$ and $\beta \geqslant 0$ and $\alpha + \beta < 1$. In addition C_1, the starting value of C_t, has to be positive definite to guarantee Γ_t to be positive definite.

4.3.1.2 Tree Generation

The tree generation of the return process scenarios is based on an interface between a Monte Carlo simulator and the methodology adopted to derive the ALM model coefficients: we distinguish accordingly between a discrete, model-based, *coefficient tree procces* and a core *risk process* which is first defined relying on consistent financial assumptions and then approximated. In figure 4.1, we have example of four scenarios leads to seven nodes $(n_1, n_2, ..., n_7)$ to describe the random variables. In what follows, we denote the vector process including the risk factors with ζ_t and elements ζ_t^j, leading to a tree process denoted by ζ_n^j, $n \in \mathcal{N}_t$, while the coefficient process for the ALM problem is $r_{i,n}$ which, as shown below, will depend on ζ_n^j and may include an autoregressive component. Figure 4.1 illustrates the scenario tree structure with decision stages where each node is a joint outcome of all the variables at the corresponding decision stage and each path through the tree represents a specific scenario. In tree process we consider the correlation matrix C_t, dynamic volatilities σ_t and asset return r_t, leading to a scenario generation form denoted by c_{i,j,t_n} which is correlation between assets i and j at time t corresponding to node n, σ_{i,t_n} dynamic volatility of each asset i at time t corresponding to node n and $r_{i,n}$ as return of asset i at each node n, $n \in \mathcal{N}_t$.

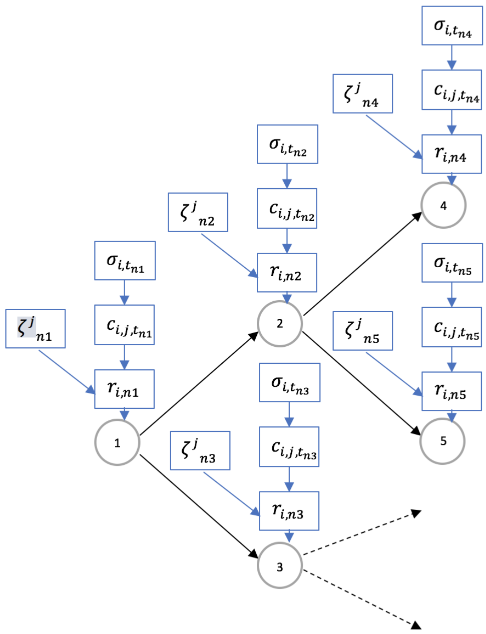


Figure 4.1: Example of return scenario tree process with stochastic variables

Through the tree approximation, in abstract, we generate a discrete probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with associated random events ω whose dynamics will determine the risk factors and then the coefficient process: in a dynamic setting every decision is required to be measurable with respect to the current σ-algebra: $\{x_t|\mathbb{F}_t\}$. In the case study we will consider a ALM problem with asset returns determined by a two-layer economic and financial model, popular in literature for long-term financial planning [105, 47, 42, 5, 28, 29].

In the case study we consider a stage partition based on $t \in \mathcal{T} = \{0,0.5,1,2,3,5,7,10\}$ and generate the scenario tree accordingly. The current time will be denoted by $t = 0$ which is the beginning of the decision horizon, while T is the end of the decision horizon. More generally in the tree notation $n \in \mathcal{N}_t$ denote the set of nodes at stage t. For $t > 0$ every $n \in \mathcal{N}_t$ has a unique ancestor $n-$ and for $t < T$ a non-empty set of children nodes $n+$. We denote with t_n the time period associated with node $n : t_n - t_{n-}$ will then denote the time length between node $n-$ and node n. The set of ancestors of node $n : n-, n--, ..., n_0$ is $a(n)$. The probability distribution $\mathbb{P}$ is considered on the leaf nodes of the scenario tree so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. A scenario is a path from the root to a leaf node and represents a joint realization of the random variables along the path to the planning horizon. The price returns $r_{i,n}$ of all assets $i \in \mathcal{I}$ together with relevant risk factors ζ_n^j and an equity risk premium λ_n in node n are defined in terms of current market values. Following equations (4.1), (4.3), (4.5) and (4.6) , for $t \in \mathcal{T}, n \in \mathcal{N}_t$ we consider the following nodal transitions along the scenario tree, for each asset i:

$$r_{i,n} = \beta^i r_{i,n-} + \mu_{i,n}(t_n - t_{n-}) + \sigma_{i,t_n} \sqrt{t_n - t_{n-}} \sum_j c_{i,j,t_n} \omega_{j,n} \tag{4.7}$$

$$\mu_{i,n} = \sum_{k \neq i} \beta^k r_{k,n \vee a(n)} + \sum_j \beta^j \zeta_{j,n \vee a(n)} \tag{4.8}$$

$$\sigma_{i,t_n} = \varpi + \sum_{q=1}^{\mathcal{Q}} \varphi_q \rho_{i,t_n-q}^2 + \sum_{k=1}^{\mathcal{K}} \psi_k \sigma_{i,t_n-k} + \sum_{j=1}^{\mathcal{J}} \nu_j \mathbb{I}_{t-j} (\rho_{i,t_n-j})^2 \tag{4.9}$$

$$c_{i,j,t_n} = \frac{(1-\alpha-\beta)\bar{q}_{i,j} + \alpha\varepsilon_{i,n-1}\varepsilon_{j,n-1} + \beta q_{i,j,n-1}}{\sqrt{((1-\alpha-\beta)\bar{q}_{i,i} + \alpha\varepsilon_{i,n-1}^2 + \beta q_{i,i,n-1})((1-\alpha-\beta)\bar{q}_{j,j} + \alpha\varepsilon_{j,n-1}^2 + \beta q_{j,j,n-1})}} \tag{4.10}$$

Where σ_{i,t_n} at each time t corresponding to node n is derived from TARCH(1,1,1). Model (4.7) implies a sequence of dependencies induced by correlated random events in the financial market: we assume $\omega_n \in N(0, t_n - t_{n-})$. The terms σ_{i,t_n} and c_{i,j,t_n} capture the market overall risky conditions: as mentioned during turbulent times we assist to increasing volatilities but also, with a severe impact on portfolio diversification to correlation clustering. We apply the above stochastic model to a recent crisis period in European financial markets, from 2009 to 2011.

4.3.1.3 Assets Simulation and Scenario Approximation

The pension fund's optimal decision depends on the set of returns $r_{i,n}$ at each nodes n in scenario tree approximation for every asset classes. We have considered four asset classes include: cash equivalents, fixed income, equity and indirect real asset such as commodities, where each of them carry a different returns and cash flow structure. The risk factors ζ^j $(j = 1, 2, 3, 4)$ for the asset classes are considered: the 12-month interest rate for the Euro area for short-term deposits in the money market, the 10-year Euro benchmark interest rate for long-term deposits, the consumer price index inflation rate in the Euro area, GDP out-put gap and the MSCI Europe equity benchmark. In equations (4.11 & 4.12), for $j = 1, 2$ dynamic model of the GDP out-put gap ζ_n^1 and inflation rate ζ_n^2 for all $n \in N_t$ are assumed.

$$\zeta_n^1 = \beta_0^1 + \beta_1^1 \zeta_{n^-}^1 + \beta_2^1 \zeta_{n^{--}}^1 + \sigma^1 \sqrt{\Delta t} \sum_{r \geq 1} c_{1,r} e_n^r \tag{4.11}$$

$$\zeta_n^2 = \beta_0^2 + \beta_1^2 \zeta_{n^-}^2 + \beta_2^2 \zeta_n^1 + \beta_3^2 \zeta_{n^-}^1 + \beta_4^2 \zeta_{n^{--}}^1 + \sigma^2 \sqrt{\Delta t} \sum_{r \geq 2} c_{2,r} e_n^r \tag{4.12}$$

The short-term interest rate ζ_n^3 for the Euro area in the money market and the 10-year Euro benchmark interest rate ζ_n^4 for the long-term deposits have been derived by equations (4.13) and (4.14), respectively.

$$\zeta_n^3 = \beta_0^3 + \beta_1^3 \zeta_n^2 + \beta_2^3 \zeta_n^1 + \sigma^3 \sqrt{\Delta t} \sum_{r \geq 3} c_{3,r} e_n^r \tag{4.13}$$

$$\zeta_n^4 = \beta_0^4 + \beta_1^4 \zeta_{n^-}^4 + \beta_2^4 \zeta_n^1 + \beta_3^4 \zeta_{n^-}^2 + \sigma^4 \sqrt{\Delta t} \sum_{r \geq 4} c_{4,r} e_n^r \tag{4.14}$$

The evolution of MSCI Europe equity risk premium λ_n is determined via equation (4.15). This is assumed to depend on the long interest rate ζ_n^4, the inflation ζ_n^2, constant volatility σ^j and random variables e_n with normal distribution $N(0,1)$.

$$\lambda_n = \beta_0^j + \beta_1^j \zeta_{n^-}^4 + \beta_2^j \zeta_{n^-}^2 + \sigma^\lambda \sqrt{\Delta t} e_n \tag{4.15}$$

The set of estimated coefficients and risk factors provides an input to generate the asset returns scenarios and determines the returns' evolution over the decision horizon. The asset returns of each benchmark in each node $n \in N_t$ for scenario generation can now be determined as following formulas for the different asset classes $i \in \mathcal{I} = \{i_0, i_1, i_2, i_3, i_4, i_5\}$:

Cash equivalents:

- Money market

The evolution of the return of money market r_0 is defined by the performance of the inflation and GDP out-put gap (see equation 4.13) . The correlation between these variables is positive, hence the yield of money market increases (decreases) by the higher (lower) rate of inflation and GDP out-put gap (economic cycle) in the Euro zone.

Fixed income:

- JPM Global Government Bond EMU

$$r_{1,n} = \beta_0^1 + \beta_1^1 r_{1,n^-} + \beta_2^1 r_{4,n} + \beta_3^1 \zeta_n^1 + \beta_4^1 \zeta_n^4 + \sigma_{t_n}^1 \sqrt{\Delta t} \sum_r c_{1,r,t_n} e_n^r \tag{4.16}$$

- JPM Global Government Bond ex-EMU

$$r_{2,n} = \beta_0^2 + \beta_1^2 r_{2,n^-} + \beta_2^2 r_{4,n} + \beta_3^2 \zeta_n^2 + \sigma_{t_n}^2 \sqrt{\Delta t} \sum_r c_{2,r,t_n} e_n^r \tag{4.17}$$

- Barclays Euro Government Inflation Linked bond

$$r_{3,n} = \beta_0^3 + \beta_1^3 \zeta_n^2 + \beta_2^3 r_{4,n^-} + \beta_3^3 \zeta_n^3 + \sigma_{t_n}^3 \sqrt{\Delta t} \sum_r c_{3,r,t_n} e_n^r \tag{4.18}$$

The risk factors that affect on the performance of the fixed incomes' benchmarks are the inflation rate, economic cycle, interest rate and risk premium on the equity market. In the statistical models, the estimated coefficients show the dependence of the government bonds return with the stock risk premium is negative. Consequently, the returns of the government bonds increasing (decreasing) by the decrease (increase) in the stock risk premium.

Equity:

- MSCI Europe equity

$$r_{4,n} = \left(r_{0,n^-} + \lambda_n\right) + \sigma^4 \sqrt{\Delta t} e_n \tag{4.19}$$

The equity return is modeled relying on the performance of the equity risk premium (see equation 4.15) and short-term interest rate in Euro zone. The risk premium is dependent on the long-term interest rate, inflation rate and economic cycle.

Indirect real asset:

- Commodities

$$r_{5,n} = \beta_0^5 + \beta_1^5 r_{4,n} + \beta_2^5 r_{3,n^-} + \beta_3^5 \zeta_{n^-}^1 + \beta_4^5 \zeta_n^2 + \sigma_{t_n}^5 \sqrt{\Delta t} \sum_r c_{5,r,t_n} e_n^r \quad (4.20)$$

The commodity return is evaluated by a stable relationship with the GDP out-put gap, inflation rate, equity market and Euro government inflation linked bond yields.

The dynamic ALM returns $r_{i,n}$ along the tree are derived for a given input tree structure through MC simulations. The following algorithm 1 describes the key steps of the scenario tree generation process:

Algorithm 1 Stylized return scenario generation using stochastic correlation

for $j = 1, 2, 3, 4$ **do**
  for $t \in \mathcal{T}$ **do**
    Inputζ_0^j and β_0^j, β_1^j, β_2^j, β_3^j & σ^j, $c_{j,r}$
    for $n \in \mathcal{N}_t$ **do**
      Generate risk factors ζ_n^j through eq.s (4.11) to (4.14)
      Derive equity risk premium λ_n via eq. (4.15)
      for $i = 1, 2, 3, 4, 5$ **do**
        Estimate dynamic volatility $\sigma_{t_n}^i$ by TARCH process
        Derive DCC matrix c_{i,r,t_n}
        Input β_0^i, β_1^i, β_2^i, β_3^i, β_4^i & $\sigma_{t_n}^i$, c_{i,r,t_n}
        Compute asset returns $r_{i,n}$ for $i = (0, 1, 2, 3, 4, 5)$ via eq.s (4.13) and (4.16) to (4.20)
      end for i
    end for n
  end for t
end for j

4.3.1.4 Statistical Assumption and Estimation

The economic and financial risk factors are modelled with parameters and correlations fitted to quarterly data in a two level asset simulation fashion:

- Level 1: Economic and financial risk factors variables
- Level 2: Equities, commodity and fixed income

The statistical coefficients are estimated through the method of Ordinary Least Squares (OLS) by Gnu Regression, Econometrics and Time-series Library (gretl) as a statistical package with given quarterly past data.

The estimated coefficients of risk factors $\beta_0^j, \beta_1^j, \beta_2^j, \beta_3^j$ together with the standard deviations σ_j of equations (4.11), (4.12), (4.13), (4.14) and (4.15) are portrayed in Table 4.1. Furthermore, Δt defines the time increment between the nodes n^- and n. Correlation is introduced directly on the realizations e_n^r of four standard normal variables through the Choleski elements $c_{j,r}$ of the correlation matrix with normal distribution $N(0,1)$ and illustrated in Table 4.2.

Table 4.1: Risk factors coefficients and volatilities

	β_0	β_1	β_2	β_3	Volatility (%)
Out-put gap	0.0006	1.7767	-0.9089	—	1.14
Inflation	0.0030	1.2160	-0.3846	0.01815	0.92
Short rate	0.0092	0.8589	0.3058	—	1.53
Long rate	0.0024	0.8759	0.1621	-0.1507	1.41
Risk premium	0.1717	-3.0922	-4.0528	12.9808	20.07

Table 4.2: Risk factors correlation matrix

	Long rate	Inflation	Out-put gap	Short rate
Long rate	1.0000	0.6775	0.4203	0.8338
Inflation	0.6775	1.0000	0.2336	0.6075
Out-put gap	0.4203	0.2336	1.0000	0.6379
Short rate	0.8338	0.6075	0.6379	1.0000

Furthermore, the estimated returns coefficients $\beta_0^i, \beta_1^i, \beta_2^i, \beta_3^i, \beta_4^i$ by the OLS method together with the standard deviations σ_i of equations (4.16), (4.17), (4.18), (4.19) and (4.20) are illustrated in table (4.3). Table 4.4 shows estimated benchmark constant correlation matrix specification for the case study. We used the constant volatility, σ_i in case of testing the model under constant correlation assumption.

Table 4.3: Estimated coefficients for asset return models

	β_0	β_1	β_2	β_3	β_4	Vol. (%)
Bond EMU	0.0437	0.5678	-0.0799	-0.5783	-0.5069	4.18
Bond ex-EMU	0.0344	0.2574	-0.1003	0.2532	—	3.21
Infl. Link. bond	0.0303	-0.2051	-0.0666	0.8330	—	4.65
MSCI Europe	—	—	—	—	—	19.74
Commodity	-0.0362	13.2071	0.0055	0.0665	–	20.98

Table 4.4: Constant correlation matrix of the benchmarks

	EMU	ex-EMU	Infl. Link.	MSCI	Commodity
Bond EMU	1.0000	0.4901	0.6649	-0.5252	-0.0262
Bond ex-EMU	0.4901	1.0000	0.2426	-0.5448	-0.3139
Infl. Link. bond	0.6649	0.2426	1.0000	0.1637	0.2170
MSCI	-0.5252	-0.5448	0.1637	1.0000	0.5517
Commodity	-0.0262	-0.3139	0.2170	0.5517	1.0000

4.3.1.5 Correlation Analysis

An analysis of assets correlations helps us to understand the changing undercurrents between different asset classes during the financial crisis and post crisis periods. In the layer one of our asset simulation, factors are connected through multi-variate set-up but they are assumed to have constant correlation. But in the second layer we used DCC assumption for the modeling of market benchmarks returns. Since correlation clustering changes dynamically during and after the crisis with introducing stochastic correlation among market benchmarks into scenario generation we can consider correlation shocks that may come from the financial market. Therefore, considering stochastic correlation assumption is needed in order to approximate return distribution and drive the optimal ALM solution with more effective strategy.

We have implemented stochastic correlation to analyze relationships for the entire period from Q1 2009 to find out the impact of correlation breakdown (during the Euro sovereign crisis 2009-2011) on the long-term planning. Table 4.5 reports the result of time-varying stochastic correlation forecasting at every year. All other intermediate quarterly periods are not reported to avoid complicity but have been considered in scenario approximation. Fig-

Table 4.5: Long-term simulated stochastic correlation among benchmarks

	Q1/2009	Q1/2010	Q1/2011	Q1/2012	Q1/2013	Q1/2014	Q1/2015	Q1/2016	Q1/2017	Q1/2018	Q1/2019
Emu-Ex Emu	0.8764	0.9329	-0.0043	0.8568	0.9070	0.9577	0.8886	0.9303	0.9297	0.6715	0.8872
Emu-Inf. bond	0.8751	0.9340	-0.0403	0.8646	0.9173	0.9569	0.8862	0.9553	0.9341	0.7424	0.8767
Emu-Stock	0.2521	0.2871	-0.4857	0.0470	-0.5368	-0.7396	0.2028	0.5137	0.4359	-0.2094	0.2279
Emu-Comm.	0.7507	0.6989	-0.3793	-0.1111	-0.1438	-0.1158	0.3627	-0.2231	0.5293	0.1940	0.3114
Ex Emu-Inf. bond	0.9628	0.9494	0.9492	0.9492	0.9246	0.9619	0.9162	0.9081	0.9187	0.2189	0.8181
Ex Emu-Stock	0.5108	0.3952	0.6513	0.2808	-0.4554	-0.6772	0.3155	0.5052	0.5127	0.3806	0.4359
Ex Emu-Comm.	0.7449	0.7136	0.6918	0.1251	-0.0398	-0.0432	0.4696	-0.0592	0.6102	-0.2525	0.4097
Inf. bond-Stock	0.4887	0.3477	0.6398	0.2365	-0.4993	-0.7285	0.3585	0.5302	0.4563	-0.5303	0.1292
Inf. bond-Comm.	0.7718	0.7499	0.7273	0.1401	-0.0286	-0.0252	0.5387	-0.1798	0.5794	0.6386	0.3321
Stock-Comm.	0.5584	0.6366	0.7833	0.6771	0.6371	0.4369	0.7223	0.2275	0.8359	-0.2979	0.5876

Emu: JPM Blobal Emu - Ex Emu: JPM Blobal Ex-Emu - Inf. bond: Barclays inflation linked bond- Stock: MSCI Europe index - Comm.: Dow Jones commodity ure 4.2 shows correlation clustering of three asset indexes from 2009 to 2011 which has been derived through DCC model. The correlation is fickle among asset classes during different period. Therefore investors' strategy seems to be weaken or simply disappearing during a crisis owing to correlation changing behavior. Further, the asset return scenario generation ability to fit the actual market price without considering dynamic stochastic correlation would be to wither in the risk opacity of the crisis situation.

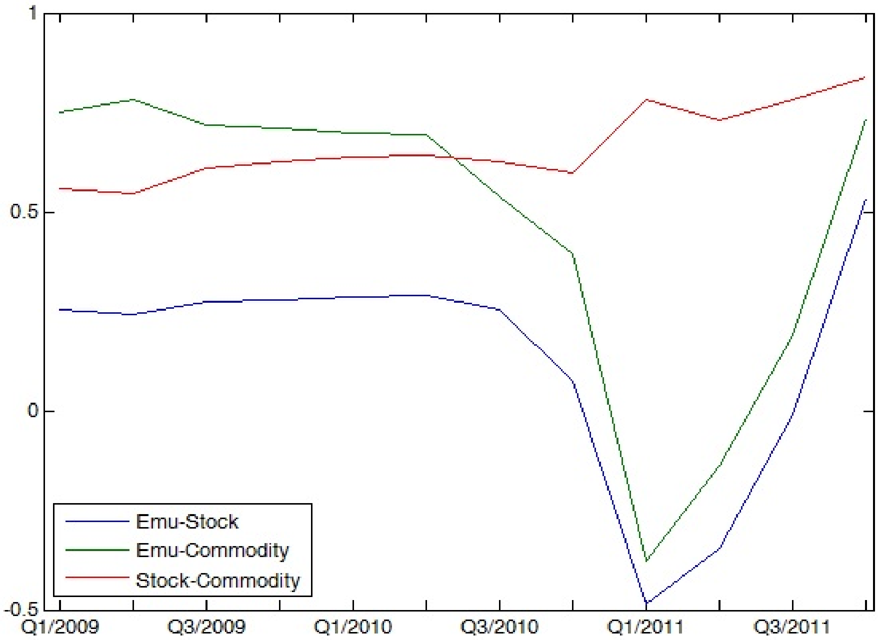


Figure 4.2: Correlation clustering

4.3.1.6 Results of Scenario Approximation and Model Validation

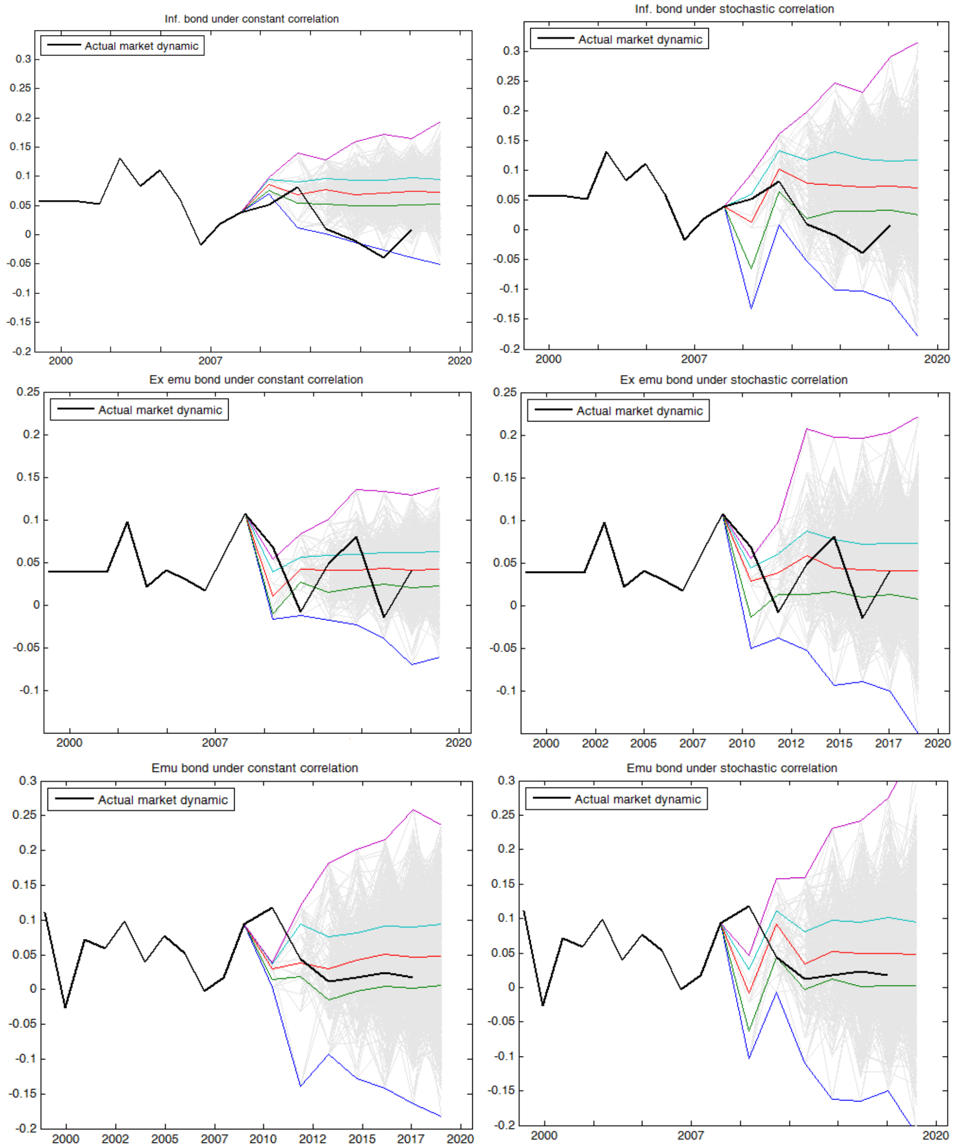
We consider the historical data from the time series of past market data and we look at historical back-testing in which asset returns statistical models (see section 4.3.1) are fitted to the data up to a trading time t and simulate to some chosen horizon $t+T$. Accordingly, we can move to the optimization part by knowing that our statistical models are qualified in back-testing analyses.

All implementation results have been performed through a set of software modules combining MATLAB R2014a for simulation and scenario generation as the development tool (see main MATLAB codes in appendices).

In particular for model validation and forecasting we consider a scenario tree generated over a 10-year planning horizon, split into non-homogeneous decision stages $t \in \mathcal{T} = \{0, 0.5, 1, 2, 3, 5, 7, 10\}$ specified as a discrete time and stages setting.

We illustrate in the Figure 4.3 a set of output trees generated for representative asset index classes based on the introduced scenario generation with stochastic -right side- versus constant correlation -left side- models. The scenario tree is generated across time, with the some assessment of the plausibility of our asset scenarios with respect to the observed market dynamic at each stages up to Q4 2015. The different techniques for scenario generation are ex-post analyzed on actual market dynamic with the same estimated coefficients and assumption. We consider the periods of 2009 until 2016 during which the crisis, post crisis and recent market situation took place. In almost all cases, the sample space distribution of the generated scenarios with stochastic correlation includes actual market returns over the simulation period seen to that date which includes also recent market instability. The results show in the period of crisis in most of the cases scenario generation with stochastic correlation performed better to capture the actual value through the generated scenarios.

All in all, the evidence shows that during and after the crisis periods, generating scenarios under stochastic correlation approach would have dominated the other model with constant correlation and the ex-post evidence proves consistent results.



Inf. bond under constant correlation
Actual market dynamic
Inf. bond under stochastic correlation
Actual market dynamic
Ex emu bond under constant correlation
Actual market dynamic
Ex emu bond under stochastic correlation
Actual market dynamic
Emu bond under constant correlation
Actual market dynamic
Emu bond under stochastic correlation
Actual market dynamic

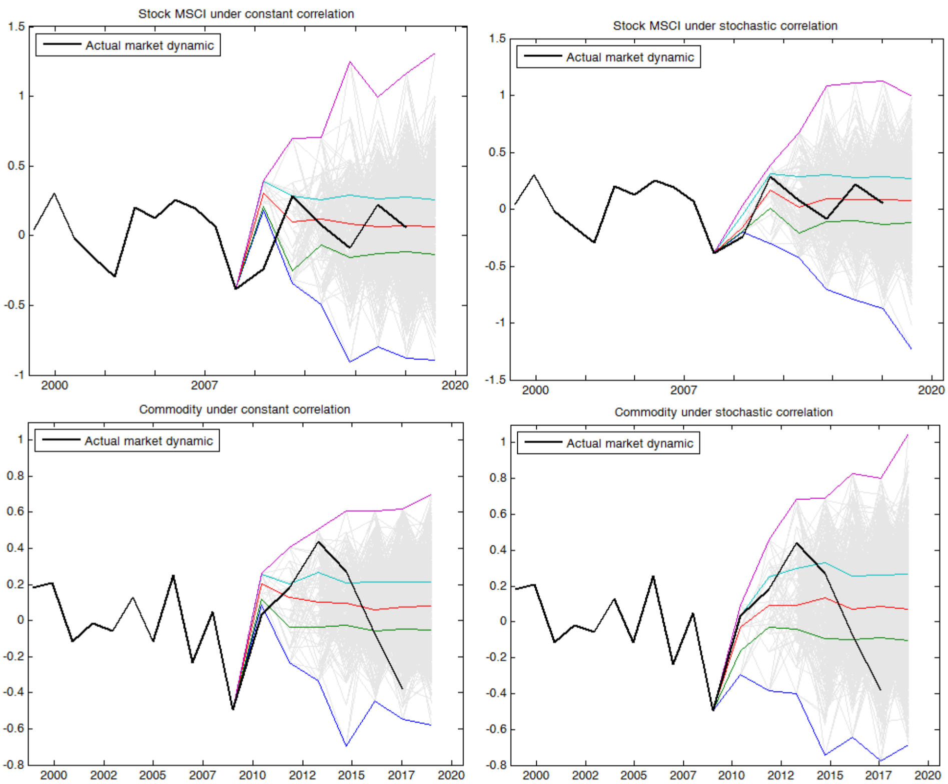


Figure 4.3: Asset returns scenario generation of investment universe under constant correlation -left side- and stochastic correlation -right side- (red teregectory is the mean scenario, green and turquoise tragectories 25% and 75% quantile, blue and megenta tragectories lower and upper boundary of the distribution)

4.3.2 Liability Model and Funding Condition

The defined benefit obligation (DBO) should be paid as annual payment to the pensioners at each year. For ease of implementation pension members carry constant survival and mortality intensities and we use a simplified model of the employed liability pricing for the DBO estimation. We assume pension liabilities determined by actuarial assumptions and scenario dependent wage inflation linked. The funds' pension payments at node n is:

$$L_n = p^s_{t_n} \sum_{k=1}^{K} b^a_k (1 + \zeta^2_n) \qquad n \in \mathcal{N}_t \tag{4.21}$$

where K is a number of pensioners at the fund, t_n the reference time of node n, the annual pension benefit payment to each pensioner denoted by b^a and ζ^2_n is the inflation rate at node n which has been evaluated by equation (4.12). Also, for each year has been considered survival probability for pensioners, say p^s. We consider a net pension payments L^{NET}_n at node n as the difference between pension benefits payments and (employer and employee) contributions C_n:

$$L^{NET}_n = C_n - L_n \tag{4.22}$$

The funds' discounted DBO at each year is measured by the following formula:

$$\Lambda_n = \sum_{m \in \mathcal{C}_n} p_m (L_m (1 + r_l)^{-(t_m - t_n)}) \tag{4.23}$$

where $m \in \mathcal{C}_n$ indicates the nodes in the sub-tree originating from node n, p_m define the conditional probabilities associated with the subtree originating from node n and r_l is the discounted value. The DBO is directly related to the Funding Ratio and we focus on the FR dynamic stability as long-term management target. This is defined as:

$$\Phi_n = \frac{W_n}{\Lambda_n} \tag{4.24}$$

where W_n is the value of funds' wealth (total assets) at node n. FR below 1 express a PF underfunding condition and above 1 shows an overfunding condition.

4.3.3 Optimization Framework

In this section, we describe the framework of optimization with objective function and relative financial and regulatory constraints. The assets return must be derived from the stochastic formula of previous section and input to the following optimization problem.

4.3.3.1 Model Decision Variables

Decisions on trading and holding assets of the fund's portfolio and borrowing strategies define the control variables of the problem. In the optimization problem, the following decision variables are considered: $x_{i,0}$ is the value of the position in asset i at $t = 0$, $x_{i,h,n}$ denotes the value in node n of holdings in asset i purchased in node h, $x^+_{i,n}$ the value of asset i bought in node n and $x^-_{i,h,n}$ is the value of asset i sold in node n which was purchased in node h. The value of investment in asset i in node n given by $x_{i,n}$ and wealth process in node n is denoted by W_n.

4.3.3.2 Objective Function

The following objective function (4.25) is based on the trade-off dynamic tracking minimum guarantee return with consider the short and medium-term shortfalls h^j_n $(j = 1, 2)$ with respect to targets $\widetilde{W}^j$ and long-term portfolio wealth. The objective function is subject to several financial and institutional constraints over a certain time horizon to make sure a dynamic ALM approach is capable of combining long-term allocation with possibility of the intermediate term shortfalls control.

$$max_{x \in X}\{\alpha_3 \cdot \mathbb{E}(W_n)_{n \in \mathcal{N}_T} - (\alpha_1 \cdot \mathbb{E}(h_n^1)_{n \in \mathcal{N}_t^1} + \alpha_2 \cdot \mathbb{E}(h_n^2)_{n \in \mathcal{N}_t^2})\} \tag{4.25}$$

In model (4.25), for all $n \in \mathcal{N}_t$ and $t \in \mathcal{T}$ time interval considered given by $\mathcal{T}$ ={0,0.5,1,2,3,5,7,10}. We define the shortfalls for $\mathcal{N}_t^j$ by

$$h_n^j = \phi(W_n^j, \widetilde{W^j}) = max(0, \widetilde{W^j} - W_n^j) \quad j = 1,2 \tag{4.26}$$

where $\widetilde{W^j}$ and W_n^j are portfolio target and tree values, respectively. In (4.25) we have $\sum_j \alpha_j = 1, \alpha_j \in (0,1)$ and $\mathbb{E}(W_n)$ represents the (unconditional) expectation of terminal portfolio values at $t = T, n \in \mathcal{N}_T$ while we denote with $\mathcal{N}_t^j$, $j = 1,2$ the j-th target associated stages and accordingly $\mathbb{E}(h_n^j)$, $n \in \mathcal{N}_t^j$ will denote the expectations adapted to $\mathcal{N}_t^j$.

The α_j represent the relative emphasis that fund manager would like to put on different targets and the value of them can be chosen freely and set the level of risk aversion. The higher value of α_j for $j = 1,2$, the higher importance given to the shortfalls and the less to the expected of the final wealth, and hence the more risk-averse the optimal portfolio allocation will be and vice versa.

All in all, with considering multi-critical objective function, we are able to track the targets return and funding ratio of the pension fund over time.

4.3.3.3 Model Constraints

The dynamic ALM problem is maximization of the objective function in (4.25) under uncertainty subject to an extended set of linear constraints. Hence, the relative following constraints need to be considered in the DB pension fund ALM modeling.

Inventory balance constraints

This set of constraints involves buying, selling, and holding variables for each asset. They give the quantity invested in each asset at each time over the horizon. The first-stage decision, or root-node decision also referred to as the implementable decision of the multi-stage stochastic problem is the only one under complete uncertainty regarding the markets' future evolution. We consider an initial portfolio assets $\overset{0}{x}_i$ prior to rebalancing and $x_{i,0}$ portfolio holdings in asset i after rebalancing. For $n = 0$ and $i \in \mathcal{I}$ where $\mathcal{I} = \{i_0, i_1, i_2, i_3, i_4, i_5\}$ includes six asset classes and i_0 representing an investment in the money market, thus yielding a maybe very low but positive income. We distinguish a cash position $z_n = z_n^+ - z_n^-$ accounting for the evolution of liquidity in ALM problem.

$$x_{i,0} = \overset{0}{x}_i + x_{i,0}^+ - x_{i,0}^- \quad i \in \mathcal{I} \tag{4.27}$$

Moreover, up to the horizon T, we have for $t \in \mathcal{T}$ and $n \in \mathcal{N}_t$ where $\mathcal{N}_t$ set of nodes at stage t:

$$x_{i,n,n} = x_{i,n}^+ \qquad i \in \mathcal{I} \tag{4.28}$$

$$x_{i,n}^- = \sum_{h \in a(n)} x_{i,h,n}^- \qquad i \in \mathcal{I} \tag{4.29}$$

$$x_{i,h,n} = \sum_{h \in a(n)} x_{i,h,n}(1 + r_{i,n}) - x_{i,n}^- \qquad i \in \mathcal{I} \tag{4.30}$$

$$x_{i,n} = \sum_{h \in a(n)} x_{i,h,n} + x_{i,n,n} \qquad i \in \mathcal{I} \tag{4.31}$$

Cash balance constraints

These are the set of constraints of the model which refer respectively to the first period and the remaining periods before the horizon. The cash balance model explains the value movement in the fund and introduces a proper accounting for cash inflows, outflows and within the fund at each stage. For $n = 0$ at time $t = 0$ we have cash surplus z_0 which is affected by investment, trading strategy and pension payment from the input portfolio at inception stage. Given an initial cash $\bar{z}_0$ we have

$$z_0 = \bar{z}_0 + \sum_{i \in I} (x^-_{i,0} - x^+_{i,0}) - L^{NET}_0 = 0 \tag{4.32}$$

On subsequent stages for $n \in \mathcal{N}_t$ and $t > 0$, we consider the cash flows generated in each node as follow:

$$z_n = z_{n^-}(1 + r_n) + \sum_{i \in I} (x^-_{i,n} - x^+_{i,n}) - z^+_n + z^-_n - L^{NET}_n = 0 \tag{4.33}$$

Where positive z^+_n and negative z^-_n cash positions with current return on previous cash account plus the rebalancing decisions $x^+_{i,h,n}$ and $x^-_{i,h,n}$ together with the net pension payment L^{NET}_n which has to be paid to the pensioners generate cash flows until end of the horizon.

Policy constraints

These constraints limit the amount invested in an asset to be less than some proportion of the fund wealth and typically lower l and upper u bounds of the assets are problem dependent.

$$W_n l_i \leq x_{i,n} \leq u_i W_n \qquad i \in \mathcal{I}, n \in \mathcal{N}_t \;\;\&\;\; 0 \leq l_i, u_i \leq 1 \tag{4.34}$$

Cash account bond:

$$W_n l_z \leq z_n \leq u_z W_n \qquad n \in \mathcal{N}_t \;\;\&\;\; 0 \leq l_z, u_z \leq 1 \tag{4.35}$$

Turnover constraint

The turnover constraint fixed in optimization problem to limit the approximate change in the fraction of total portfolio wealth invested in some assets

i at prior stage and current stage be less or equal than some proportion of the fund wealth and take the form:

$$\sum_{i\in\mathcal{I}}\sum_{h\in a(n)} x_{i,h,n} + \sum_{i\in\mathcal{I}} x^{+}_{i,n} \leq \theta \sum_{i\in\mathcal{I}} x_{i,n^-}(1 + r_{i,n})$$
$$n \in \mathcal{N}_t \;\; \& \;\; 0 \leq \theta \leq 1 \tag{4.36}$$

Short sale constraints

In this model, short sale constraints restrict the fund strategy to long-only positions, meaning that all assets holding, buying and selling must be non-negative

$$x_{i,n} \geq 0 \qquad i \in \mathcal{I}, n \in \mathcal{N}_t \tag{4.37}$$

$$x^{+}_{i,n} \geq 0 \qquad i \in \mathcal{I}, n \in \mathcal{N}_t \tag{4.38}$$

$$x^{-}_{i,n} \geq 0 \qquad i \in \mathcal{I}, n \in \mathcal{N}_t \tag{4.39}$$

Horizon decision

This constraint ensure that is not possible to have new investments at the horizon. So the portfolio value is determined only by the assets value of the investments realized in the previous stage and only in case of liquidity shortfall in order to pay the pension payment at the horizon there is possibility to sell assets up to covering the pension payment.

$$x^{+}_{i,n} = 0 \qquad i \in \mathcal{I}, n \in \mathcal{N}_T \tag{4.40}$$

$$x^{-}_{i,n} \leq L^{NET}_{n} \qquad i \in \mathcal{I}, n \in \mathcal{N}_T \tag{4.41}$$

Wealth constraints

These constraints determines the portfolio wealth at each node.

Total Wealth. The total wealth of the fund at each node n is defined as the sum of all asset holdings:

$$W_n = \sum_i x_{i,n} + z_n \qquad i \in \mathcal{I}, n \in \mathcal{N}_t \tag{4.42}$$

Expected Final Wealth. This constraint evaluates expected wealth of the fund at end of the horizon.

$$E(W_n) = \sum_n W_n p_n \qquad n \in \mathcal{N}_T \tag{4.43}$$

Where the probability of each node denoted by p_n.

Chapter 5

Case Study: A Pension Fund ALM Problem

In this chapter, we implement the theoretical model of ALM from previous chapter under some assumptions to real data for a DB pension fund in order to test their applicability and interpret solutions. The case study is designed for a DB pension fund's management in order to identify the optimal asset allocation over certain planning horizon with inclusion of capital and liability constraints. Liabilities are determined under the assumptions of constant pension fund future pension payments and their current market value (current fund obligation) under assumption of constant pension fund population by discounting all future pension payments. The case study implementation framework is based on:

1. Financial market analysis modeling using real historical data;
2. Scenario generation and risk assessment;
3. Using optimization problem described in Chapter 4 with the market scenarios based on the market model under points 1 and 2;

4. ALM solution analysis and recommended decision.

5.1 Main Assumptions, Methods and Tools

This case study is designed on 10 year linear multistage stochastic programming for a portfolio manager facing stochastic asset returns and liabilities from the DB pension fund. The investment portfolio under management includes cash, money market, bonds, equity and commodities. The behavior of each asset is simulated trough Monte Carlo method over time and generated quarterly returns using a tree structure. Table 5.1 described the assumption of time and space specification for scenario tree approximations.

Table 5.1: Scenario tree approximation setting

Decision stages	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Scenario No.
Stage time structure	H&N	6m	1y	2y	3y	5y	7y	10y
Branching structure	4	4	4	4	2	2	2	2048
	4	4	4	3	2	2	2	1536
	4	4	4	2	2	2	2	1024
	4	4	3	2	2	2	2	768
	4	4	2	2	2	2	2	512

Moreover, we consider structure of the objective function as described in Figure (5.1) the trade-off between short, medium and long-term decision criteria for optimal DB pension fund investment manager.

The formulation of the objective function has been formed as multi-critical dynamic tracking targets (see equation 4.25) with minimizing the probability to have shortfalls against the targets' level and maximizing portfolio wealth at the horizon with different combination of α_j $(j = 1, 2, 3)$.

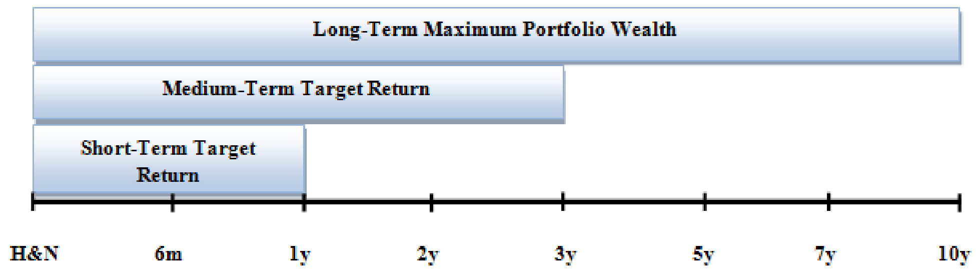


Figure 5.1: Explicit trade-off between short, medium and long-term targets

We solve the multi-stage stochastic problem with defined ambition level for targets as reported in Table 5.2.

Table 5.2: Targets ambition level

Target	Time horizon	Ambition level
Short	1^{st} year	4% annual return
Medium	3^{st} year	5% annual return
Long	10^{st} year	Maximize return

Furthermore, with regards to the discounted liabilities, the initial portfolio total reserves is considered amount to €8,250,000 which leading to 100% initial pension fund's funding ratio. Figures 5.2 and 5.3 report respectively the annual pension payment over 10-year horizon and mean scenario case of inflation linked pension payment at decision stages which has been considered in the problem. Pension benefit payments have been paid at decision stages with considering interest rate between decision stage and the payment year stage.

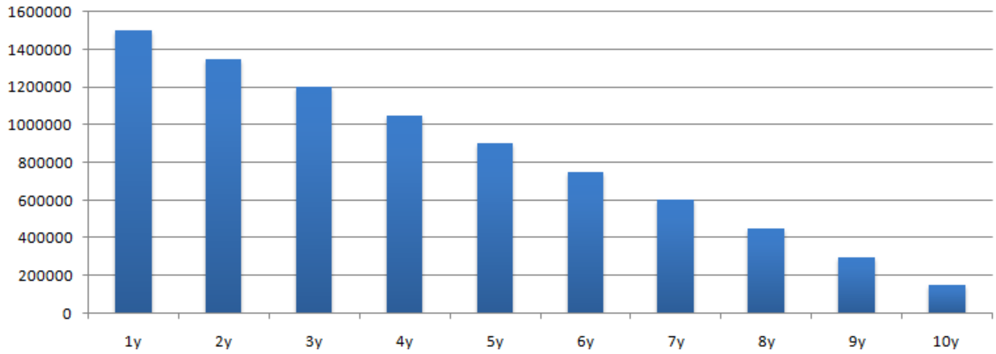


Figure 5.2: Annual pension payment

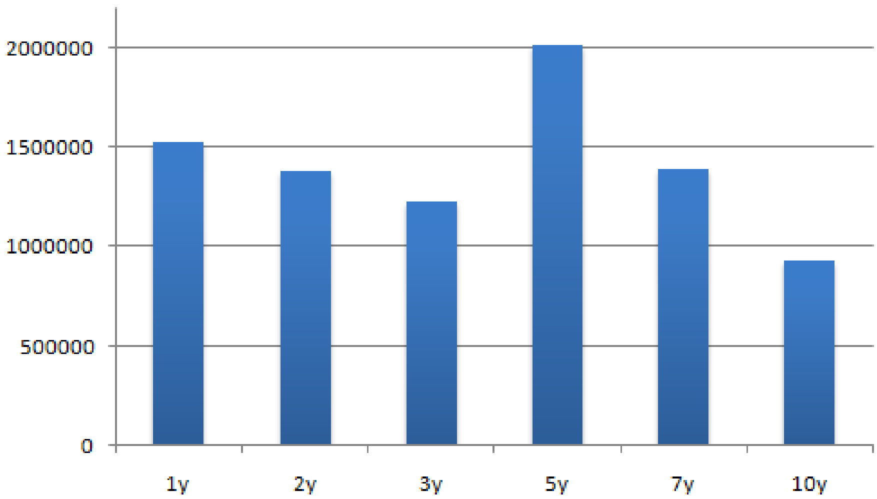


Figure 5.3: Pension payment at decision stages (mean case scenario)

Lower and upper asset bonds information relates to the various asset allocation constraints and should be considered in the optimization structure. Taking into account asset bounds is necessary not only for policy (regulatory) restriction as minimum level of diversification or operational limits but also to avoid too extreme and hardly implementation portfolio changes thus improving the robustness of the optimization results. The lower and upper asset bounds are illustrated in Table 5.3. Furthermore, Table 5.3 reports also maximum portfolio turnover limit which is considered 50% per every decision stages thus the overall portfolio changes can not exceed the bound from each stage to the next one. The portfolio turnover limit allows fairly high but not excessive asset allocation flexibility over time.

Table 5.3: Bounds constraints

Assets	Lower bound	Upper bound
Cash	5%	100%
Gov. bonds + MKT	0%	100%
Inf. bond	0%	30%
Equity	0%	20%
Commodity	0%	20%
Portfolio Turnover		50%

The results of maximizing the objective value in equation (4.25) under the several linear constraints are the output of the solution algorithm. Hence, the implementation results are generated through a set of software modules combining MATLAB (R2014a) as the development tool (see main Matlab codes in appendices), GAMS (24.1.3) as the model generator (see main Gams codes in appendices) using commercial solver (e.g. MOSEK LP and MLP) and Excel as the input and output data collection while the operating system is Windows 8 with an Intel processor (Core i5 - 2.66GHz) and 4GB of RAM.

5.2 ALM Results

We consider an ALM model over the 10-year planning horizon for a DB pension fund from Q1 2009 and include investment universe according to the Table 2.1 with financial and regulatory constraints. The input to the optimization problem has been generated following the procedure presented by concentrating on the scenario generation with a number of 2048 scenarios at the horizon. We analyze the results of the optimal first stage here and now decision under different issues for both scenario generation with stochastic versus constant correlation. For this analysis, we run the problem with and without considering turnover limit constrain and focusing on $\alpha_1 = 0.5$, $\alpha_2 = 0.3$ and $\alpha_3 = 0.2$ target calibration.

5.2.1 Optimal H&N Decision Analysis

We plot in Figures 5.4 and 5.5 the H&N decisions after solving the optimization problems under scenario generation with asset return simulation model under the structure of stochastic mean with both stochastic versus constant correlation. In Figures 5.4 and 5.5, the H&N decision under stochastic correlation on the left and under constant correlation on the right for both cases with and without activated turnover constraint are portrayed respectively. The dynamic ALM model generates an optimal portfolio at time zero consistent with the introduced stochastic correlation with an increasing portion of investment from risky assets to risk free and riskless assets in the portfolio compere to the ALM result under constant correlation for both activated and deactivated turnover constraint.

As it's happened during a severe Euro sovereign crisis such as the one recorded in 2009, default events during the market instability will lead portfolio managers to abandon the risky side of the asset classes. Therefore, we show in the here and now strategy, the dynamic policy under stochastic correlation technic would be able to heavily penalize speculative assets except risk free money market and riskless assets such as bonds in order to prevent default risk in the systematic risk as 2009 European crisis period.

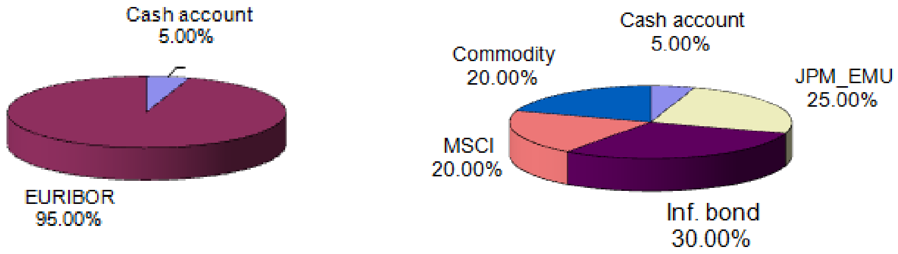


Figure 5.4: Here and now solution under stochastic versus constant correlation without turn over constraint

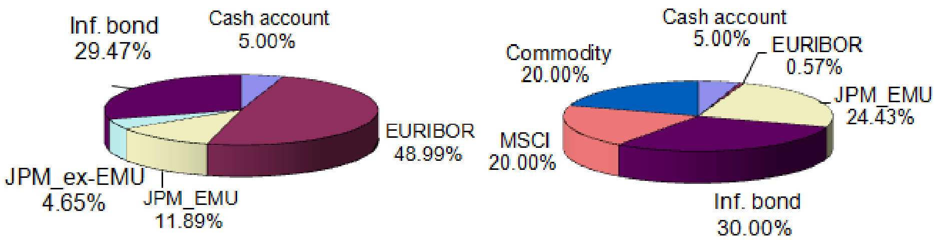


Figure 5.5: Here and now solution under stochastic versus constant correlation with turnover constrant

5.2.2 Historical Backtest

We look at historical backtest in which optimal dynamic ALM asset strategies under the models with stochastic versus constant correlation are fitted to actual market returns in the period of European sovereign crisis (2009-2011) and post crisis (2012). We considered European MSCI equity as a benchmark. We analyzed and portrayed the performance of historical backtesting with both deactivated and activated turnover constraint in Figures 5.6 and 5.7, respectively. As shown in the Figures 5.6 and 5.7, the dynamic strategy with stochastic correlation yields superior performance over the backtesting period comparing to other cases (multi-stage under constant correlation, Euro MSCI equity benchmark and static optimization) in both cases with and without turnover limit.

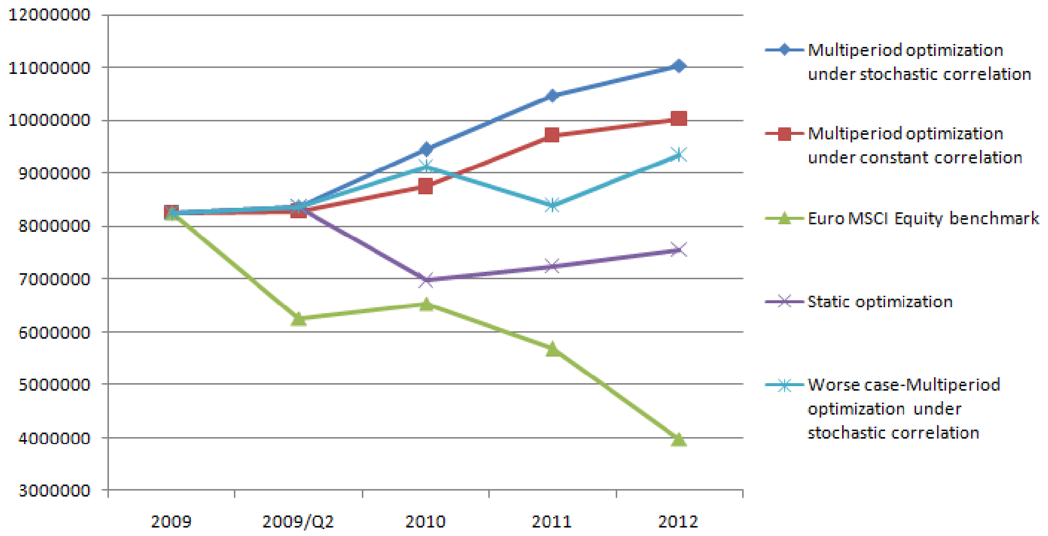


Figure 5.6: Strategies historical backtesting without turnover constraint

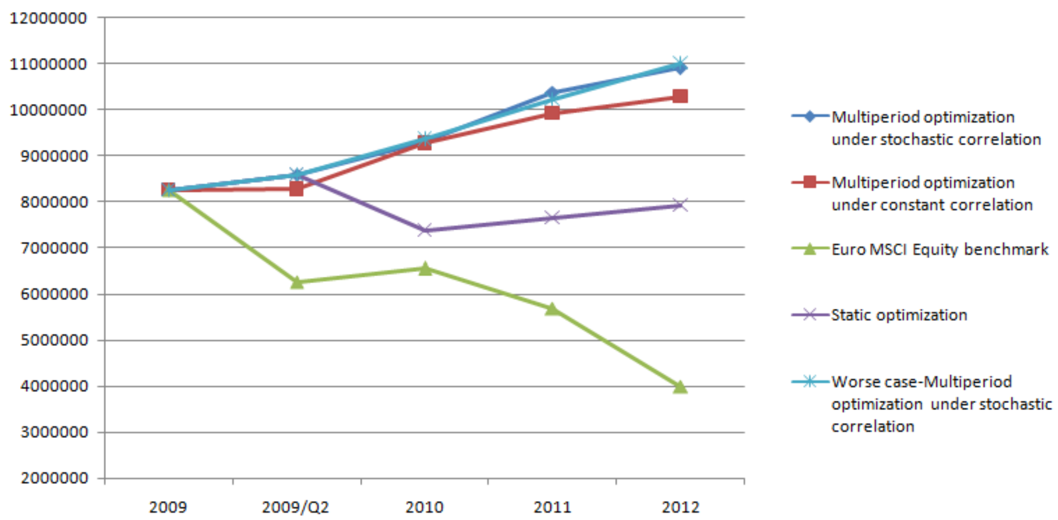


Figure 5.7: Strategies historical backtesting with activated turnover constraint

5.2.3 Strategy Comparison

To complete the comparative analysis between the solution of the optimal decision planning under stochastic and constant correlation, we evaluated (from an initial portfolio value of 8250000) the performance of a multi-priod optimization strategy with stochastic correlation against constant correlation and static optimization. As it is evident in Figures 5.8 and 5.9, the dynamic strategy with stochastic correlation yields superior performance in both cases without and with activated turnover constraint, respectively. The worst cases scenario of the dynamic optimization with stochastic correlation solutions are also generating a non-negative wealth at the end of the test period. The worst case lines display the behavior of the worst possible strategy based on the generated scenarios.

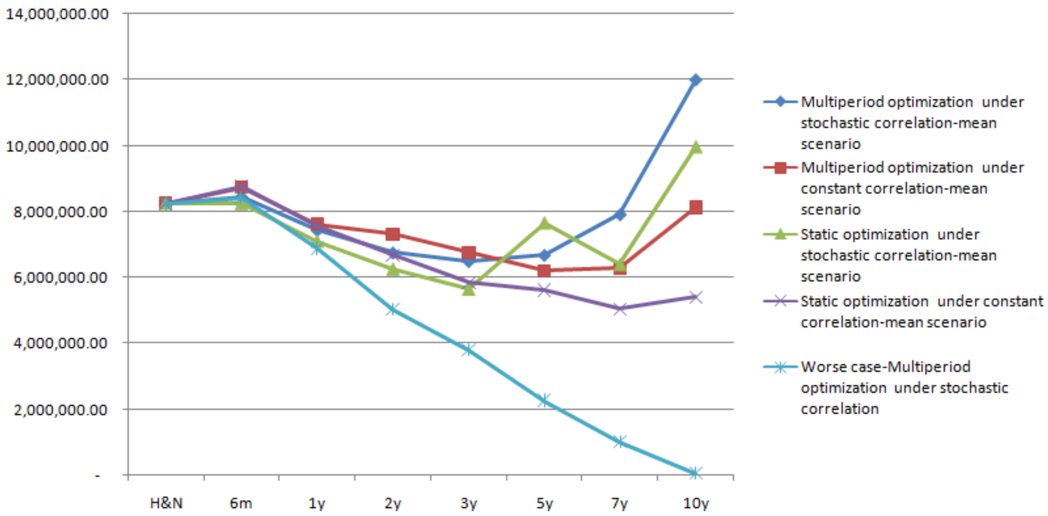


Figure 5.8: Strategies comparison without turnover constraint

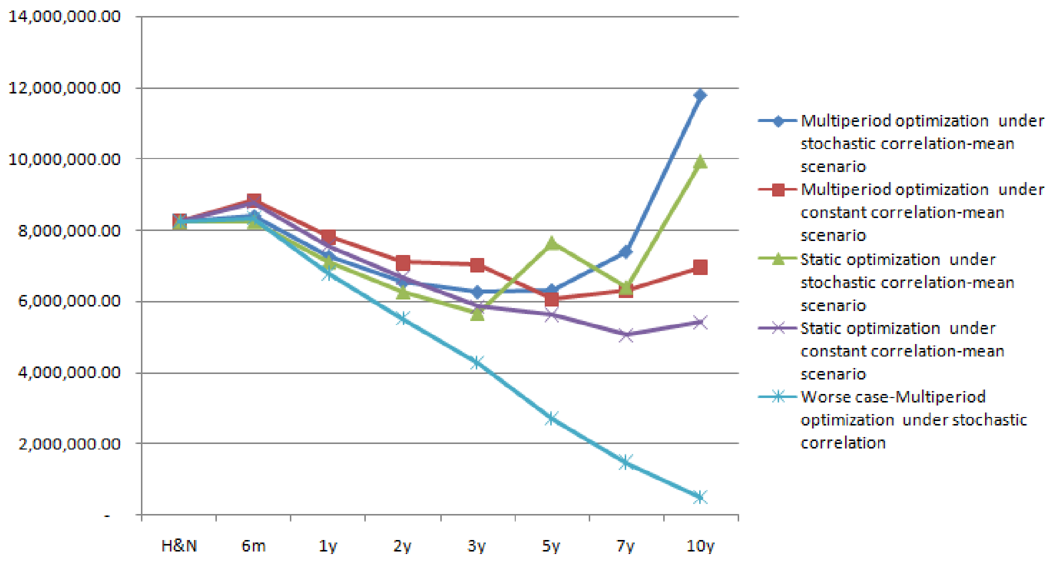
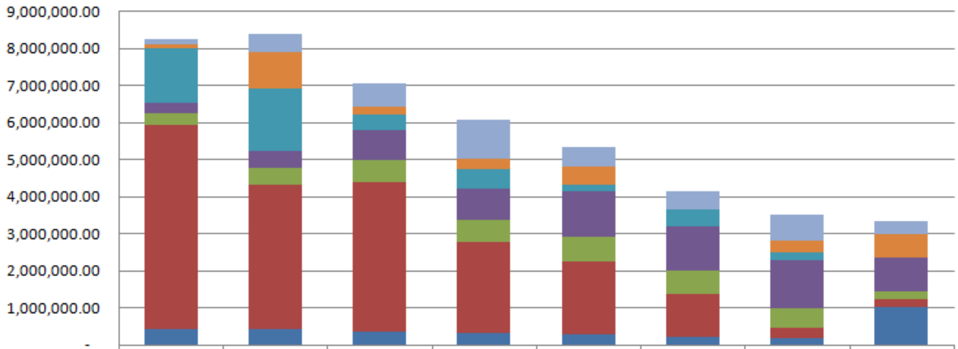


Figure 5.9: Strategies comparison with activated turnover constraint

5.2.4 Target Function's Calibration

We presented an extended set of results with different combinations on the targets by the multi-stage stochastic programming. Different experiments of optimization results have been carried out by changing the α values in objective function (4.25). We evaluated the DB pension fund's strategies with respect to changes in the relevant short, medium and long term decision criteria in different periods. In the first case of the target trade-off, the emphasis is assumed to be put only on short term target ($\alpha_1 = 100\%$, $\alpha_2 = 0\%$, $\alpha_3 = 0\%$) with activated turnover constraint. In this case, the only goal is to minimize the probability of shortfall of missing the target ambition level after the first year.

Figure 5.10 illustrates the dynamic asset allocation strategy for the BD pension fund investor along the average wealth scenario under emphasis on only short-term target. At time $t = 0$, an optimal here & now decision rep-



	H&N	6m	1y	2y	3y	5y	7y	10y
Commodity	117,794.74	501,855.24	654,745.72	1,051,067.	519,485.13	500,635.42	701,300.10	348,078.46
MSCI	113,413.88	973,817.65	198,580.05	282,069.52	521,282.91	-	300,028.94	624,690.26
Inf. Bond	1,485,371.	1,677,259.	433,755.61	516,415.18	177,805.33	462,182.05	220,133.91	-
JPM_ex-EMU	292,981.94	473,520.62	803,377.19	842,027.61	1,219,449.	1,191,523.	1,313,226.	913,572.01
JPM_EMU	285,002.60	453,424.19	571,516.37	599,133.39	658,255.28	631,479.13	527,596.00	222,256.68
EURIBOR	5,542,935.	3,904,033.	4,057,490.	2,473,165.	1,987,437.	1,159,204.	268,889.92	205,575.01
Cash account	412,500.00	420,205.82	353,656.07	303,362.06	267,563.97	207,632.90	175,325.03	1,029,337.

Figure 5.10: Dynamic asset allocation under stochastic correlation- mean case scenarios ($\alpha_1 = 100\%$, $\alpha_2 = 0\%$, $\alpha_3 = 0\%$)

resents the best asset allocation to be immediately implemented under full uncertainty. On the following stages, scenario dependent strategies contribute to minimize the expected shortfall with respect to the target ambition level.

The probability of the target achievement over the short and medium term and also long term portfolio return over the 10-year horizon under such assumption are reported in Figure 5.11. The short target ambition level is achieved 100% while the likelihood to meet the medium target at the third year is 76% and portfolio return at the 10 year horizon is 41% of the initial wealth.

The relevant results of the MSP under only short-term target assumption are reported in Figures 5.10 and 5.11. What follows is an attempt to a deep analysis of the trade-off among short and medium-term target achievement and long-term increasing portfolio wealth to shed more light on the mechanism of making intermediate decision with long-time investment horizons

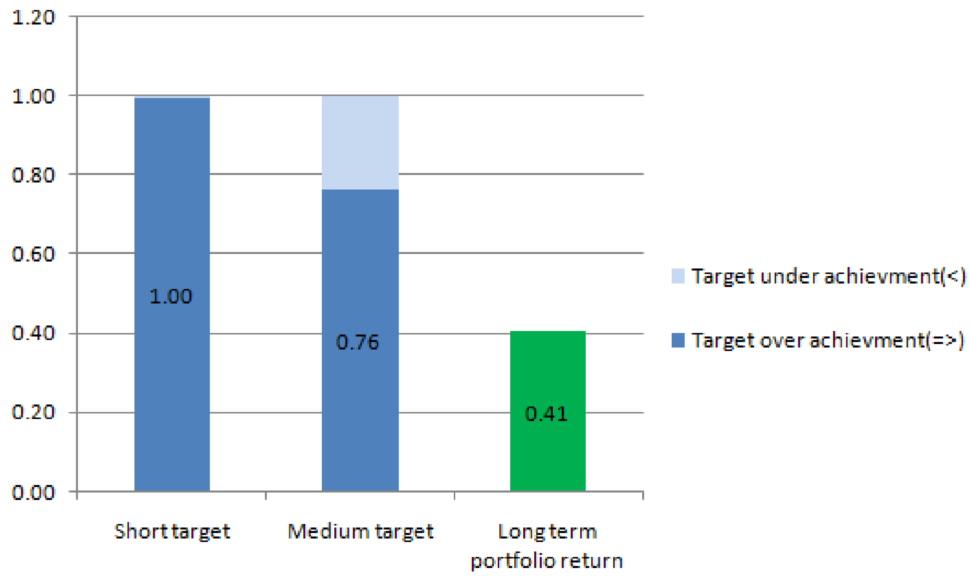


Figure 5.11: Target achievement probability and long-term portfolio return - mean case scenario under stochastic correlation (α_1 = 100%, α_2 = 0%, α_3 = 0%)

and the way their risks of shortfall are controlled.

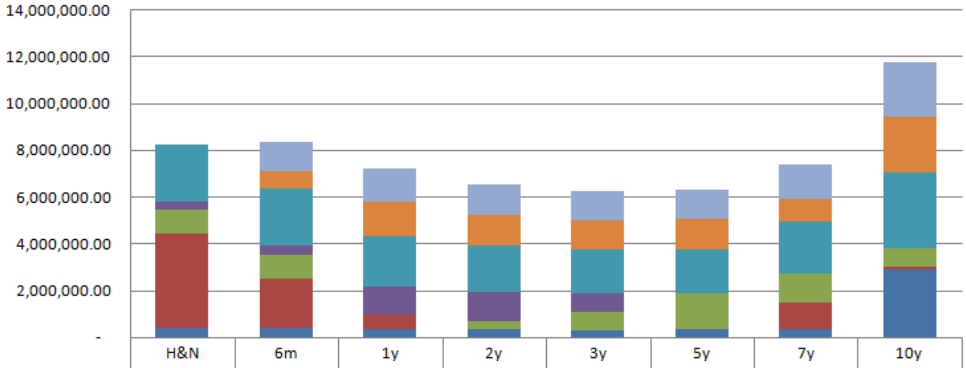
The ALM results are compared to identify the best investment strategy in terms of probability of meeting the targets ambition levels and consequence maximizing portfolio wealth with combination of α_1, α_2 and α_3. The optimal combination of α_j is evaluated based on the results of different calibration of the targets balance. Three particular cases where the target balance is fully focused on short, medium and long-term respectively together with the optimal emphasis among the target trade-off with 2048 number of scenario and under assumption of activated turnover constraint are reporting in Table 5.4.

Table 5.4 summarizes the results of different calibrations, the first row reports the short-term emphasis with α_1 = 100%, α_2 = 0% and α_3 = 0% while the second row is based on the medium-term emphasis with α_1 = 0%, α_2 = 100%, α_3 = 0% and the third row for the long-term target focused case

Table 5.4: Short, medium and long-term targets balance with stochastic correlation - mean case scenario

Targets Balance			H&N Portfolio Weighs (Mean Scenario)						Target Achiev. Prob.		Port. Value
Short	Medium	Long	Cash	Money MKT	Gov. Bonds	Inf. Bond	Equity	Comm.	Short	Medium	Long
100%	0%	0%	5%	67.19%	7%	18%	1.37%	1.43%	100%	76%	41%
0%	100%	0%	5%	48.69%	16.84%	29.47%	0%	0%	99%	84%	41%
0%	0%	100%	5%	50.09%	15.44%	29.47%	0%	0%	100%	85%	140%
50%	30%	20%	5%	48.99%	16.54%	29.47%	0%	0%	100%	89%	143%

with $\alpha_1 = 0\%$, $\alpha_2 = 0\%$, $\alpha_3 = 100\%$. The optimal calibration ($\alpha_1 = 50\%$, $\alpha_2 = 30\%$, $\alpha_3 = 20\%$) is illustrated on the last row of the Table 5.4 since the target achievement probability is 100% for the short and 89% medium-term while the expected portfolio return at horizon is 143% of the initial wealth. It is remarkable that the optimal target balance is well calibrated compared to other cases. The problem has been run with different combination of α_j to figure out discrimination of the optimal target trade-off but are not reported to avoid complexity. Moreover, dynamic asset allocation and dynamic wealth



	H&N	6m	1y	2y	3y	5y	7y	10y
Commodity	-	1,297,923	1,450,476	1,305,304	1,250,542	1,260,441	1,477,716	2,357,879
MSCI	-	727,921.8	1,450,476	1,305,304	1,250,542	1,260,441	960,082.5	2,357,879
Inf. bond	2,431,137	2,459,661	2,175,715	1,957,956	1,875,813	1,890,661	2,216,574	3,247,680
JPM_ex-EMU	383,563.0	389,952.4	1,205,872	1,276,380	766,247.1	-	-	-
JPM_EMU	981,095.4	983,666.9	-	355,249.6	796,930.4	1,575,551	1,279,294	801,081.6
EURIBOR	4,041,704	2,111,313	607,223.7	-	-	-	1,085,484	103,731.5
Cash account	412,500.0	419,496.8	362,619.2	326,326.0	312,635.5	315,110.3	369,429.0	2,921,144

Figure 5.12: Dynamic asset allocation under stochastic correlation- mean case scenarios ($\alpha_1 = 50\%$, $\alpha_2 = 30\%$, $\alpha_3 = 20\%$)

of mean scenario for the optimal target balance case have been portrayed in Figure 5.12. In particular, the portfolio wealth starting from 8,250,000 leads to 100% funding ratio with respect to the discounted liabilities over the horizon and the value of wealth has been evaluated in mean cases scenario after paying all the fund's liabilities over the 10-year horizon.

Figure 5.13 reports the target achievement likelihood and the expected portfolio return of the optimal target balance case for the mean case scenario.

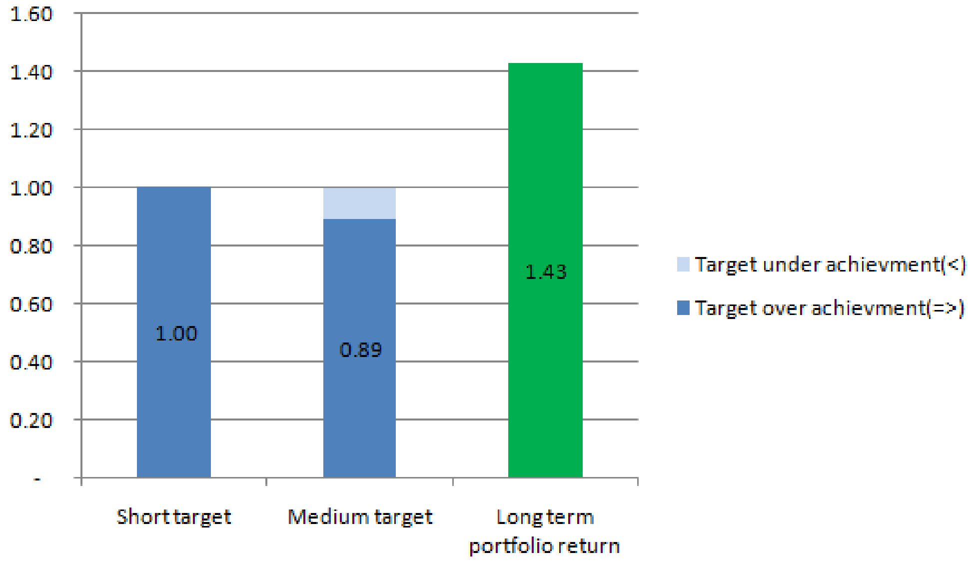


Figure 5.13: Target achievement probability and long-term portfolio return - mean case scenario under stochastic correlation($\alpha_1 = 50\%$, $\alpha_2 = 30\%$, $\alpha_3 = 20\%$)

All in all, the results show that the trade-off between short, medium and long-term can favor to meet the higher target achievement and portfolio return with more sustainable investment strategy. Accordingly, managerial decision process should be fed by inter-temporal targets' trade-off analysis and management incentive plans should be improved in the ALM framework.

5.2.5 H&N Solution as Function of Tree

We present the results for the H&N decision under stochastic correlation including the impact of a change in the number of scenarios. We analyzed the optimal H&N solution as a function of the tree structure with different numbers of scenarios to have an efficient sample space. More specifically, for each scenario tree (512, 768, 1024, 1536, 2048), the optimization problem has been solved from randomly generated scenario samples to test the H&N

solution stability with respect to each problem. As it is evident from Figure 5.14, increasing the number of scenarios leads to a smooth change in the first stage input strategy, while by increasing the number of scenarios from 1536 to 2048, the decision remains fairly similar.

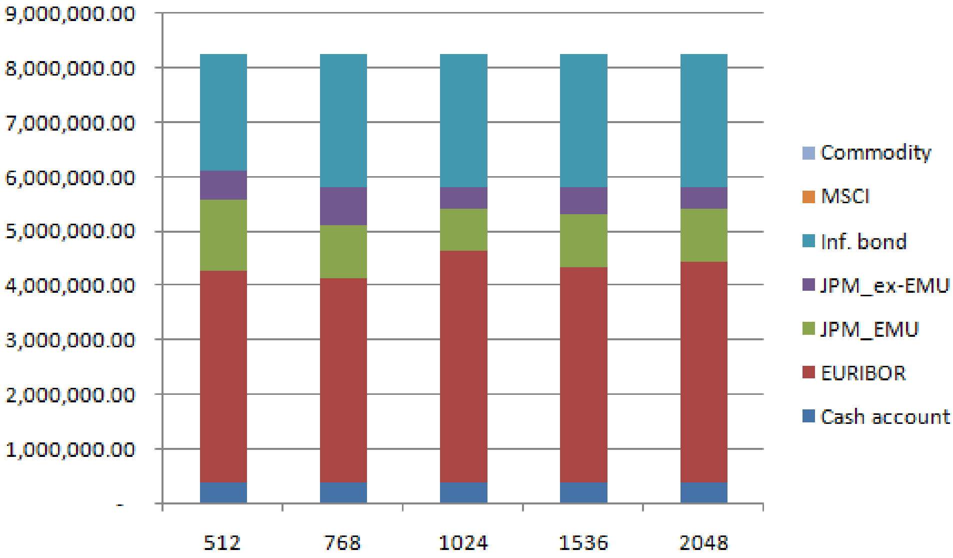


Figure 5.14: H&N Solution as function of different scenario numbers

5.2.6 Funding Ratio Evaluation

We show in Figure 5.15 the evaluation of the FR stage-by-stage across time and scenarios under optimal target calibration and stochastic correlation assumptions. At the end of the first stage (6 month), the FR values in each of 4 nodes is increasing value from left to right. At the end of the second stage (1 year), increasing 16 nodes, so forth until the sixth stage (7 year) we have 1024 nodes and 2048 nodes at the horizon. The constant red line in each plot indicates an equilibrium condition of FR equal to 1.

The result reported in Figure 5.15 shows the achievement of a funding surplus across all scenarios. The ALM solution under stochastic correlation refers to ex-ante information with the problem solution leads to full recovery of a funding surplus consistently with the PF managerial goals. Such surplus is achieved satisfying risk capital and policy constraints.

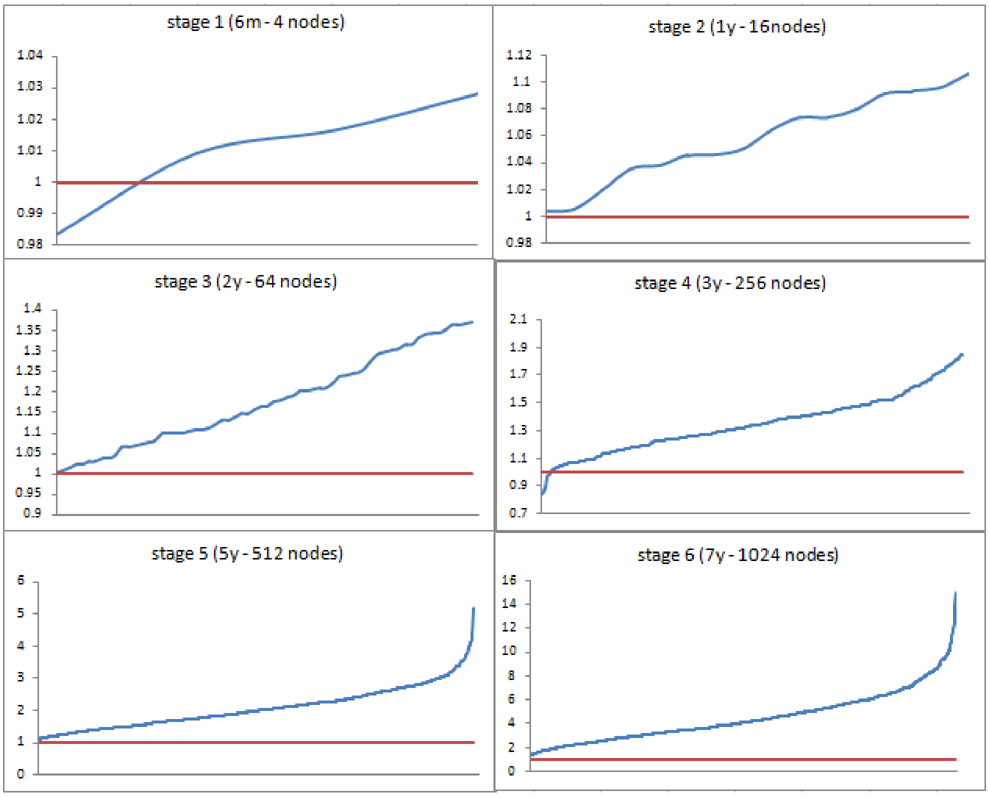


Figure 5.15: Funding Ratio (Y axis) at stage nodes (X axis)

5.2.7 Worst Case Scenario Analysis

According to the results reported in the previous section, a well combination of target balance on objective function raises the portfolio return and minimizes the possible shortfall with respect to the target levels. In this section, we are analyzing the investment strategy of the DB pension fund management under the worst case scenario to show that the dynamic approach leads to hedging the strategies of the planning horizon. One way to make the optimization model more challenging is to taste the model under the worst case scenario situation to see how the dynamic ALM model can cope with such condition. Figures 5.16 and 5.17 illustrate the target achievement likelihood with expected portfolio wealth of the optimal target balance case and dynamic asset allocation under the worst case scenario, respectively.

A different picture arises if the situation of the worst case scenario is assumed without changing any other assumption. Under these extreme conditions, we analyze the impact on the target achievement likelihood, portfolio wealth at horizon and the portfolio strategy. According to the Figure 5.16, the investment policy would keep the portfolio wealth after paying all liabilities still positive at horizon with 96% short and 61% medium-term target achievement under such a worst situation with big enough scenario tree to capture the stochastics by a good collection of fat-tail scenarios.

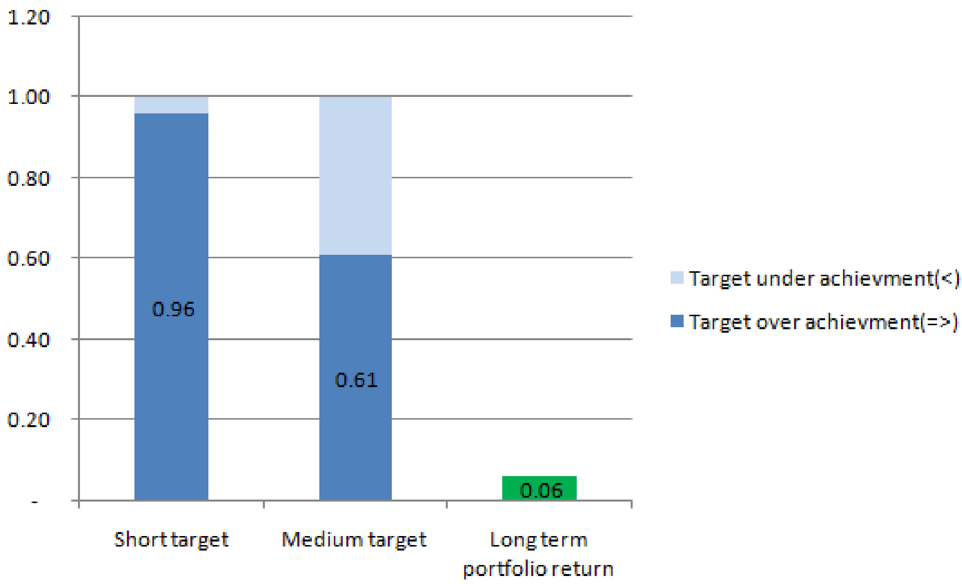
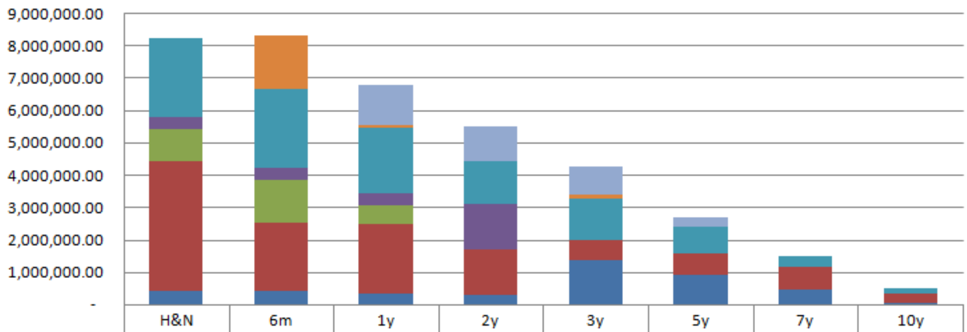


Figure 5.16: Target achievement probability and long-term portfolio return - worst case scenario under stochastic correlation (α_1 = 50%, α_2 = 30%, α_3 = 20%)

Considering the Figure 5.17 which reports the portfolio investment asset allocation with a dramatic decrease of the wealth can affect the pension fund's liquidity over a 10-year horizon. The optimal portfolio strategy will keep the overall fund's wealth positive until the end of the horizon with a relatively sustainable target achievement likelihood over the short and medium-term objectives.

The investment dynamic asset allocation in Figure 5.17 shows that under such a scenario the portfolio will modify the strategy to maintain a sufficient liquidity level. Under worst case situation, the portfolio is kept mainly in low-risk fixed income assets and that equity rebalancing decisions help preserve sufficient liquidity at first year. Clearly, the worst case situation shifts in asset allocation patterns. With Such moving into more conservative asset allocation, the investment risk can be locked in portfolio losses. In such a case, the primarily goal of an investment manager is to limit any liquidity shortfall while maximizing the expected portfolio wealth.



	H&N	6m	1y	2y	3y	5y	7y	10y
Commodity	-	-	1,247,566	1,105,104	854,408.3	299,203.2	-	-
MSCI	-	1,665,075	65,688.25	-	138,542.0	-	-	-
Inf. Bond	2,431,137	2,434,880	2,036,742	1,328,986	1,281,612	811,075.4	310,823.9	149,793.6
JPM_ex-EMU	383,563.0	387,859.1	380,331.0	1,380,049	-	-	-	-
JPM_EMU	981,095.4	1,300,592	575,271.4	-	-	-	-	-
EURIBOR	4,041,704	2,120,699	2,144,085	1,435,105	618,157.4	675,775.9	694,909.8	324,552.8
Cash account	412,500.0	416,268.7	339,457.1	276,276.0	1,379,321	917,530.1	466,957.2	24,965.60

Figure 5.17: Dynamic asset allocation - worst case scenarios under stochastic correlation (α_1 = 50%, α_2 = 30%, α_3 = 20%)

All in all, compared with the investment strategy with the mean case scenario in Figure 5.12 the portfolio allocation under the worst case scenario represents more risk averse decision with and appropriate relevant portfolio rebalancing over the time. In this case, we can keep the positive portfolio wealth at the horizon with relatively good intermediate target achievement probability.

Chapter 6

Conclusion

As discussed earlier, over the past several decades financial institutions such as DB pension funds were faced with downside potential of the financial markets. The recent financial crisis has strongly influenced the performance of DB pension funds in many countries. Pension funds struggle to cover their liabilities, having serious trouble maintaining a respectable funding ratio. Accordingly, we have presented the key elements of a dynamic ALM model, so as to effectively incorporate how several relevant short and medium-run risks of the DB PF portfolio over 10-year horizon can be controlled in a bid to ensure the long-term stability of funding ratio. To this end, we introduced a strategic level multi-stage stochastic optimization model with considering the trade-off between the risk of shortfall with respect to the target ambition levels and the funding ratio in the form of the final expected wealth and discounted liabilities over the horizon. Inter-temporal trade-off is checked by weighted parameters for each target to help the decision makers attach on each one a different importance. This approach can be operationalized through the multi-stage stochastic programming which allows integrating an uncertainty scenario model of market and investment risk for pension fund with a realistic representation of the dynamic ALM method.

We have presented also a long-term scenario generation approach which is built on concepts from existing methods, integrating a set of economic and financial risk factors requirements for a DB pension fund within a two layer simulation framework. The method has been developed based on the assumption of scenario approximation using stochastic correlations for ALM problems, which may lead to a sufficient representation of the randomness underlying the decision-making process.

Measuring correlations among assets represents a key task for risk management in financial markets. The introduction of stochastic correlation in a structure statistical model is a challenging methodological task and this technique has been tested on actual market dynamic and we report evidence of its effectiveness compared to the method with constant correlation. Since correlation clustering changes dynamically during and after the crisis, introducing stochastic correlation among assets into scenario generation is needed in order to approximate return distribution and drive the optimal ALM solution with more effective strategy. We presented effectiveness result of optimal first stage decision and dynamic portfolio diversification under stochastic correlation versus constant correlation. The numerical evidences support that relative to constant correlation assumption and one period optimization, the multi-period approach under stochastic correlation leads to superior hedging results and the recovery of fully funded PF conditions. The results illustrated over a 10-year horizon the optimal market risk control depends on correlation dynamics: positive correlation clustering leads to increasing investment in short-term liquid and riskless instrument. Moreover, a well-balanced emphasis on short, medium and long term target function can increase portfolio wealth and minimize the probability of shortfall with respect to the target levels. This well-balance emphasis hold true even if the market situation is under the worst case scenario. Nevertheless, there is not any standard receipt to assess the optimal weight for targets since it can depend on the business cycle as well as on the specific target levels. This motivates the use of a dynamic ALM tool to assess the optimal investment strategy and the related well target balance among different targets if and when necessary. This approach allows a pension fund's managers who wish to have sufficient liquidity and control interest and inflation rate risks to update their investment strategies or portfolios selections.

Bibliography

[1] Aro, H., & Pennanen, T. (2016). Liability-driven investment in longevity risk management. In Consigli, G., Kuhn, D., and Brandimarte, P., editors, International Series in Operations Research and Management Science. Springer, in print.

[2] Amzallag, A., Kapp, D., & Kok Sorensen, C. (2014). The impact of regulating occupational pensions in Europe on investment and financial stability. ECB Occasional Paper, (154).

[3] Andonov, A., Bauer, R., & Cremers, M. (2013). Pension Fund Asset Allocation and Liability Discount Rates: Camouflage and Reckless Risk Taking by U.S. Public Plans?.

[4] Antolin, P., & Stewart, F. (2009). Private pensions and policy responses to the financial and economic crisis.

[5] Arbeleche, S., & Dempster, M. A. H. (2003). Econometric modelling for global asset liability management. Working Paper, Centre for Financial Research, University of Cambridge, in preparation.

[6] Babel, B., Bomsdorf, E., & Kahlenberg, J. (1955). Future mortality improvements in the G7 countries. age, 1965, 1975.

[7] Bassanini, F., & Reviglio, E. (2011). Financial Stability, Fiscal Consolidation and Long-Term Investment after the Crisis. OECD Journal: Financial Market Trends, 2011(1), 31-75.

[8] Bernaschi, M., Briani, M., Papi, M., & Vergni, D. (2007). Scenario-generation methods for an optimal public debt strategy. Quantitative Finance, 7(2), 217-229.

[9] Bertocchi, M., Consigli, G., & Dempster, M. A. (2011). Stochastic Optimization Methods in Finance and Energy. Springer.

[10] Bessis, J. (2015). Risk management in banking. John Wiley & Sons.

[11] Birge, J.R. & Louveaux, F. (Eds) (2011). Introduction to Stochastic Programming. 2nd ed., (Springer: New York).

[12] Birge, Z., & Louveaux, S. (1997). Principles on stochastic programming.

[13] Boender, C., & van Hoogdalem, S. (1998). Defined benefit/defined contribution: een risico-rendement perspectief. Research Memorandum, 1998, 61.

[14] Bogentoft, E., Edwin Romeijn, H., & Uryasev, S. (2001). Asset/liability management for pension funds using CVaR constraints. The Journal of Risk Finance, 3(1), 57-71.

[15] Bolla, L., Wittig, H., & Kohler, A. (2016). The liability market value as benchmark in pension fund performance measurement. Journal of Pension Economics and Finance, 15(01), 90-111.

[16] Bradley, S. P., & Crane, D. B. (1973). Management of commercial bank government security portfolios: An optimization approach under uncertainty. Journal of bank research, 4(1), 18-30.

[17] Campbell, J. Y. (2000). Asset pricing at the millennium. The Journal of Finance, 55(4), 1515-1567.

[18] Cariño, D. R., Kent, T., Myers, D. H., Stacy, C., Sylvanus, M., Turner, A. L., Watanabe, K. and Ziemba, W. T. (1994). The Russel-Yasuda-Kasai model: An asset-liability model for a Japanese insurance company using multistage stochastic programming. Interfaces 24(1), 29-49.

[19] Cariño, D. R., & Turner, A. L. (1998). Multi period asset allocation with derivative assets. Worldwide asset and liability modeling, 10, 182.

[20] Chellathurai, T. & Dravian, T., (2002). Generalized Markowitz mean-variance principles for multi-period portfolio selection problems. Journal of the Royal Society.

[21] Chernoff, J. (2003a). Asset liability management back with a twist. Pensions and Investments, 21.

[22] Chernoff, J. (2003b). Revolution in pension investing has begun. Pensions and Investments.

[23] Chiu, M. C., & Wong, H. Y. (2012). Mean-variance asset-liability management: Cointegrated assets and insurance liability. European Journal of Operational Research, 223(3), 785-793.

[24] Clark, G. L., Munnell, A. H., & Orszag, J. M. (2006). The Oxford handbook of pensions and retirement income (Vol. 13). Oxford University Press on Demand.

[25] Consigli, G., (2007). Asset-Liability Management for Individual Investors. In Zenios, S. A., & Ziemba, W. T. (Eds.). (2007b). Handbook

of Asset and Liability Management: Applications and case studies (Vol. 2). Elsevier.

[26] Consigli, G., Brandimarte, P., & Kuhn, D. (2015). Financial Optimization: optimization paradigms and financial planning under uncertainty. Or Spectrum, 37(3), 553-557.

[27] Consigli, G., & Dempster, M. A. H. (1998). Dynamic stochastic programming for asset-liability management. Annals of Operations Research, 81, 131-162.

[28] Consigli, G., di Tria, M., Gaffo, M., Iaquinta, G., Moriggia, V., & Uristani, A. (2011). Dynamic portfolio management for property and casualty insurance. In Stochastic Optimization Methods in Finance and Energy (pp. 99-124). Springer New York.

[29] Consigli, G., Iaquinta, G., Moriggia, V., di Tria, M., & Musitelli, D. (2012). Retirement planning in individual asset-liability management. IMA Journal of Management Mathematics, 23(4), 365-396.

[30] Consigli, G., Iaquinta, G., & Moriggia, V. (2012). Path-dependent scenario trees for multistage stochastic programmes in finance. Quantitative Finance, 12(8), 1265-1281.

[31] Consigli, G., Kuhn, D., and Brandimarte, P. (2016). Optimal financial decision making in finance. In Consigli, G., Kuhn, D., and Brandimarte, P., editors, Optimization paradigms and decision making under uncertainty for financial applications. Springer U.S., in print.

[32] Consigli, G., & Moriggia, V. (2014). Applying stochastic programming to insurance portfolios stress-testing. Quantitative Finance Letters, 2(1), 7-13.

[33] Consiglio, A., Cocco, F., & Zenios, S. A. (2001). Asset and liability modeling for participating policies with guarantees. Journal of Risk and Insurance.

[34] Consiglio, A., & De Giovanni, D. (2008). Evaluation of insurance products with guarantee in incomplete markets. Insurance: Mathematics and Economics, 42(1), 332-342.

[35] Consiglio, A., Tumminello, M., & Zenios, S. A. (2015). Designing and pricing guarantee options in defined contribution pension plans. Insurance: Mathematics and Economics, 65, 267-279.

[36] Consiglio, A., & Zenios, S. A. (2003). Model Error in Enterprise-wide Risk Management: Insurance Policies with Guarantees. Advances in Operational Risk: Firm-wide Issues for Financial Institutions, 179-196.

[37] Cox, J. C., Ingersoll Jr, J. E., & Ross, S. A. (1985). A theory of the term structure of interest rates. Econometrica: Journal of the Econometric Society, 385-407.

[38] Crouhy, M., Galai, D., & Mark, R. (2001). Prototype risk rating system. Journal of banking & finance, 25(1), 47-95.

[39] Davis, M. H., & Lleo, S. (2015). Jump-diffusion asset-liability management via risk-sensitive control. OR Spectrum, 37(3), 655-675.

[40] de Lange P. E., Fleten S-E. & Gaivoronski A. (2003). Modeling financial reinsurance in the casualty insurance business via stochastic programming. PhD thesis, Department of Industrial Economics and Technology Management, N.T.N.U., Trondheim (NO).

[41] D'Ecclesia, R. L. (2017). Time Varying Correlation: a Key Indicator in Finance. In Consigli G., Stefani S. & Zambruno G., editors, Recent advances in Commodity and Financial Modeling: Quantitative methods in Banking, Finance, Insurance, Energy and Commodity markets. International Series in Operations Research and Management Science. Springer, forthcoming.

[42] Dempster, M. A. H., Germano, M., Medova, E. A., Murphy, J. K., Ryan, D., & Sandrini, F. (2009). Risk profiling defined benefit pension schemes. To appear in: Journal of Portfolio Management.

[43] Dempster, M. A., Germano, M., Medova, E. A., Rietbergen, M. I., Sandrini, F., & Scrowston, M. (2006). Managing guarantees. The Journal of Portfolio Management, 32(2), 51-61.

[44] Dempster, M. A. H., Germano, M., Medova, E. A., Rietbergen, M. I., Sandrini, F., & Scrowston, M. (2007). Designing minimum guaranteed return funds. Quantitative Finance, 7(2), 245-256.

[45] Dempster, M. A., Germano, M., Medova, E. A., Rietbergen, M. I., Sandrini, F., & Scrowston, M. (2011). Designing minimum guaranteed return funds. In Stochastic Optimization Methods in Finance and Energy (pp. 21-42). Springer New York.

[46] Dempster, M. A. H., & Thompson, R. T. (1999). EVPIbased importance sampling solution proceduresfor multistage stochastic linear programmeson parallel MIMD architectures. Annals of Operations Research, 90, 161-184.

[47] Dempster, M. A., Germano, M., Medova, E. A., & Villaverde, M. (2003). Global asset liability management. British Actuarial Journal, 9(01), 137-195.

[48] Dert, C. (1995). Asset liability management for pension funds: a multistage chance constrained programming approach (Doctoral dissertation, Erasmus School of Economics (ESE)).

[49] Di Domenica, N., Mitra, G., Valente, P., & Birbilis, G. (2007). Stochastic programming and scenario generation within a simulation framework: an information systems perspective. Decision Support Systems, 42(4), 2197-2218.

[50] Di Tria, M., (2014). Relevant Decision Criteria for Optimal Property and Casualty Portfolio Selection. ESMT Master's Thesis.

[51] Drijver, S. J., Haneveld, W. K. K., & Vlerk, M. H. (2000). Asset Liability Management modeling using multi-stage mixed-integer Stochastic Programming. University of Groningen.

[52] Dupačová, J. (2001). Output analysis for approximated stochastic programs. In Stochastic Optimization: Algorithms and Applications (pp. 1-29). Springer US.

[53] Dupačová, J., & Kozmík, V. (2015). Structure of risk-averse multistage stochastic programs. OR Spectrum, 37(3), 559-582

[54] Dupačová, J., & Polívka, J. (2009). Asset-liability management for Czech pension funds using stochastic programming. Annals of Operations Research, 165(1), 5-28.

[55] Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. Journal of Business & Economic Statistics, 20(3), 339-350.

[56] Engle, R. (2009). Anticipating correlations: a new paradigm for risk management. Princeton University Press.

[57] Engle, R. F., & Patton, A. J. (2001). What good is a volatility model. Quantitative finance, 1(2), 237-245.

[58] Engle, R., & Figlewski, S. (2015). Modeling the dynamics of correlations among implied volatilities. Review of Finance, 19(3), 991-1018.

[59] Ermoliev, Y. M., & Wets, R. B. (1988). Numerical techniques for stochastic optimization. Springer-Verlag.

[60] Fabozzi, F. J., Focardi, S. M., & Jonas, C. L. (2005). Market experience with modeling for defined-benefit pension funds: evidence from four countries. Journal of Pension Economics and Finance, 4(03), 313-327.

[61] Fleten, S. E., Høyland, K., & Wallace, S. W. (2002). The performance of stochastic dynamic and fixed mix portfolio models. European Journal of Operational Research, 140(1), 37-49.

[62] Franzen, D. (2010). Managing investment risk in defined benefit pension funds.

[63] Frühwirth-Schnatter, S., & Geyer, A. L. (1998). Bayesian estimation of econometric multifactor Cox Ingersoll Ross models of the term structure of interest rates via MCMC methods. Working Paper, Department of Statistics, Vienna University of Economics and Business Administration.

[64] Gaivoronski, A. A., Høyland, K., & de Lange, P. E. (2001). Statutory regulation of casualty insurance companies: an example from Norway with stochastic programming analysis. In Stochastic Optimization: Algorithms and Applications (pp. 55-85). Springer US.

[65] Glasserman, P. (2003). Monte Carlo methods in financial engineering (Vol. 53). Springer Science & Business Media.

[66] Glosten, L. R., Jagannathan, R., & Runkle, D. E. (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. The journal of finance, 48(5), 1779-1801.

[67] Guigues, V., & Römisch, W. (2012a). Sampling-based decomposition methods for multistage stochastic programs based on extended polyhedral risk measures. SIAM Journal on Optimization, 22(2), 286-312.

[68] Guigues, V., & Römisch, W. (2012b). SDDP for multistage stochastic linear programs based on spectral risk measures. Operations Research Letters, 40(5), 313-318.

[69] Hafner, C. M., & Franses, P. H. (2009). A generalized dynamic conditional correlation model: simulation and application to many assets. Econometric Reviews, 28(6), 612-631.

[70] Hainaut, D., & Deelstra, G. (2011). Optimal funding of defined benefit pension plans. Journal of Pension Economics and Finance, 10(01), 31-52.

[71] Herzog, F., Dondi, G., Keel, S., Schumani, L. M., & Geering, H. P. (2007). Solving ALM problems via sequential stochastic programming. Quantitative Finance, 7(2), 231-244.

[72] Hilli, P., Koivu, M., Pennanen, T., & Ranne, A. (2003). A stochastic programming model for asset liability management of a finnish pension company. Stochastic Programming E-Print Series. Annals of Operations Research, to appear.

[73] Hilli, P., Koivu, M., Pennanen, T., & Ranne, A. (2007). A stochastic programming model for asset liability management of a Finnish pension company. Annals of Operations Research, 152(1), 115-139.

[74] Hochreiter, R., Pflug, G., & Paulsen, V. (2007). Design and management of unit-linked life insurance contracts with guarantees. Handbook of asset and liability management, 2, 627-662.

[75] Horniman, M. D., Jobst, N. J., Lucas, C. A. & Mitra, G. (2000). Constructing efficient portfolios: Alternative models and discrete constraints-a computational study. Technical Report, Department of Mathematical Sciences, Brunel University.

[76] Høyland, K., & Wallace, S. W. (2001). Generating scenario trees for multistage decision problems. Management Science, 47(2), 295-307. Chicago

[77] IMF. (2007). Global Financial Stability Report: Market Developments and Issues. International Monetary Fund.

[78] Ito, C. (1995). Managing the risk of pension assets. Financial Risk and the Corporate Treasury.

[79] Kaut, M., Vladimirou, H., Wallace, S. W., & Zenios, S. A. (2007). Stability analysis of portfolio management with conditional value-at-risk. Quantitative Finance, 7(4), 397-409.

[80] Kaut, M., & Wallace, S. W. (2003). Evaluation of scenario-generation methods for stochastic programming.

[81] Kall, P., & Mayer, J. (2005). Stochastic linear programming: models, theory and computation. International series in operations research and management science.

[82] King, A. J., (2002). Duality and martingales: a stochastic programming perspective on contingent claims. Mathematical Programming, 91(3), 543-562.

[83] Konicz, A. K., & Mulvey, J. M. (2015). Optimal savings management for individuals with defined contribution pension plans. European Journal of Operational Research, 243(1), 233-247.

[84] Konicz, A. K., Pisinger, D., Rasmussen, K. M., & Steffensen, M. (2015). A combined stochastic programming and optimal control approach to personal finance and pensions. OR Spectrum, 37(3), 583-616.

[85] Kouwenberg, R., (2001). Scenario generation and stochastic programming models for asset liability management. European Journal of Operational Research, 134(2), 279-292.

[86] Kreuser, J. L., (2012). Correcting the Financial Crisis Failures of Asset-Liability Management (ALM) Risk Management. In Swarup, B., (Eds.). Asset Liability Management for Financial Institutions: Balancing Financial Stability with Strategic Objectives. Bloomsbury Academic.

[87] Kusy, M. I., & Ziemba, W. T. (1986). A bank asset and liability management model. Operations Research, 34(3), 356-376.

[88] Leippold, M., Trojani, F., & Vanini, P. (2011). Multiperiod mean-variance efficient portfolios with endogenous liabilities. Quantitative Finance, 11(10), 1535-1546.

[89] Li, C., & Li, Z. (2012). Multi-period portfolio optimization for asset-liability management with bankrupt control. Applied Mathematics and Computation, 218(22), 11196-11208.

[90] Li, Z.F., & Yao, H.X. (2014). Dynamic Portfolio Selection of Risky Assets under Uncertain Exit Time and Stochastic Market Environment. Systems Engineering: Theory and Practice 34, 2737-2747.

[91] Liang, X., Bai, L., & Guo, J. (2014). Optimal Time-Consistent Portfolio and Contribution Selection for Defined Benefit Pension Schemes Under Mean-Variance Criterion. ANZIAM Journal, 56(1).

[92] Markowitz, H. (1952). Portfolio selection*. The journal of finance, 7(1), 77-91.

[93] Markowitz, H. (1959). Portfolio selection: efficient diversification of investments. Cowies Foundation Monograph, (16).

[94] Marti, K., & Pflug, G. C. (2004). Dynamic stochastic optimization (Vol. 532). Springer Science & Business Media.

[95] Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. The review of Economics and Statistics, 247-257.

[96] Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. Journal of economic theory, 3(4), 373-413.

[97] Maurer, R. (2015). Integrated risk management for defined benefit pensions: models and metrics. Journal of Pension Economics and Finance, 14(02), 151-160.

[98] Mitra, G., & Schwaiger, K. (Eds.). (2011). Asset and liability management handbook. Palgrave Macmillan.

[99] Mulvey, J. M., Bauerfeind, T., Simsek, K. D., & Vural, M. T. (2011). Performance Enhancements for Defined Benefit Pension Plans. In Stochastic Optimization Methods in Finance and Energy (pp. 43-71). Springer New York.

[100] Mulvey, J. M., & Erkan, H. G. (2003). Risk management of a P/C insurance company scenario generation, simulation and optimization. In Simulation Conference, 2003. Proceedings of the 2003 Winter (Vol. 1, pp. 364-371). IEEE.

[101] Mulvey, J. M., & Erkan, H. G. (2005). Decentralized Risk Management for Global Property and Casualty. Applications of Stochastic programming, 5, 503.

[102] Mulvey, J. M., Fabozzi, F. J., Pauling, W. R., Simsek, K. D., & Zhang, Z. (2005). Modernizing the defined-benefit pension system. The Journal of Portfolio Management, 31(2), 73-82.

[103] Mulvey, J. M., Gould, G., & Morgan, C. (2000). An asset and liability management system for Towers Perrin-Tillinghast. Interfaces, 30(1), 96-114.

[104] Mulvey, J. M., & Kim, W. C. (2010). Multistage financial planning models: Integrating stochastic programs and policy simulators. In Stochastic Programming (pp. 257-275). Springer New York.

[105] Mulvey, J. M. (1996). Generating scenarios for the Towers Perrin investment system. Interfaces, 26(2), 1-15.

[106] Mulvey, J. M., Pauling, B., Britt, S., & Morin, F. (2007). Dynamic financial analysis for multinational insurance companies. Handbook of Asset and Liability Management: Applications and Case Studies, 2, 543-589.

[107] Mulvey, J. M., Simsek, K. D., & Zhang, Z. (2006). Improving investment performance for pension plans. Journal of Asset Management, 7(2), 93-108.

[108] Mulvey, J. M., Simsek, K. D., Zhang, Z., Fabozzi, F. J., & Pauling, W. R. (2008). OR PRACTICE-Assisting Defined-Benefit Pension Plans. Operations research, 56(5), 1066-1078.

[109] Mulvey, J. M., & Vladimirou, H. (1989). Stochastic network optimization models for investment planning. Annals of Operations Research, 20(1), 187-217.

[110] Mulvey, J. M., & Vladimirou, H. (1992). Stochastic network programming for financial planning problems. Management Science, 38(11), 1642-1664.

[111] Natali, D. (2011). Pensions after the financial and economic crisis: a comparative analysis of recent reforms in Europe.

[112] Nelson, C. R., & Siegel, A. F. (1987). Parsimonious modeling of yield curves. Journal of business, 473-489.

[113] OECD, (2014). Institutional Investors and Long-term Investment. Project report.

[114] OECD, (2008). Private Pension Outlook. URL: http://www.oecd.org/pensions/private-pensions/42169565.pdf.

[115] Olson, R. L. (2005). The School of Hard Knocks: The Evolution of Pension Investing at Eastman Kodak. RIT Cary Graphic Arts Press.

[116] Pachamanova, D., Gulpinar, N. & Canakoglu E. Robust data-driven approaches to pension fund asset-liability management under uncertainty. In Consigli, G., Kuhn, D. & Brandimarte, P. editors, International Series in Operations Research and Management Science. Springer, 2016, in print.

[117] Peng, J. (2008). State and local pension fund management. CRC Press.

[118] Pesaran, B., & Pesaran, M. H. (2007). Modelling volatilities and conditional correlations in futures markets with a multivariate t distribution.

[119] Pflug, G. C., & Romisch, W. (2007). Modeling, measuring and managing risk. World Scientific.

[120] Philpott, A. B., & de Matos, V. L. (2012). Dynamic sampling algorithms for multi-stage stochastic programs with risk aversion. European Journal of Operational Research, 218(2), 470-483.

[121] Pirbhai, M., Mitra, G., & Kyriakis, T. (2003). Asset liability management using stochastic programming.

[122] Reus, L., & Mulvey, J. M. (2015). Multistage stochastic optimization for private equity investments. Journal of Asset Management, 16(5), 342-362.

[123] Rudolf, M., & Ziemba, W. T. (2004). Intertemporal surplus management. Journal of Economic Dynamics and Control, 28(5), 975-990.

[124] Rudloff, B., Street, A., & Valladão, D. M. (2014). Time consistency and risk averse dynamic decision models: Definition, interpretation and practical consequences. European Journal of Operational Research, 234(3), 743-750.

[125] Ruszczynski, A., & Shapiro, A. (2003). Stochastic Programming, volume 10 of Handbooks in Operations Research and Management Science.

[126] Ryan, R. J., & Fabozzi, F. J. (2002). Rethinking pension liabilities and asset allocation. The Journal of Portfolio Management, 28(4), 7-15.

[127] Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. The review of economics and statistics, 239-246.

[128] Silvennoinen, A., & Thorp, S. (2013). Financialization, crisis and commodity correlation dynamics. Journal of International Financial Markets, Institutions and Money, 24, 42-65.

[129] Sundaresan, S., & Zapatero, F. (1997). Valuation, optimal asset allocation and retirement incentives of pension plans. Review of Financial Studies, 10(3), 631-660.

[130] Shapiro, A. (2009). On a time consistency concept in risk averse multistage stochastic programming. Operations Research Letters, 37(3), 143-147.

[131] Shapiro, A. (2011). Analysis of stochastic dual dynamic programming method. European Journal of Operational Research, 209(1), 63-72.

[132] Shapiro, A., & Dentcheva, D. (2014). Lectures on stochastic programming: modeling and theory (Vol. 16). SIAM.

[133] Steinbach, M. C. (1999). Markowitz revisited: Single period and multi-period mean-variance models. Technical Report SC 99-30, Konrad-Zuse-Zentrum für Informationstechnik Berlin.

[134] Swarup, B., (Eds.). (2012). Asset Liability Management for Financial Institutions: Balancing Financial Stability with Strategic Objectives. Bloomsbury Academic.

[135] Szegö, G. P. (Ed.). (2004). Risk measures for the 21st century. Wiley.

[136] Tower, I., & Impavido, G. (2009). How the financial crisis affects pensions and insurance and why the impacts matter (No. 9-151). International Monetary Fund.

[137] Wallace, S. W., & Ziemba, W. T. (Eds.). (2005). Applications of stochastic programming (Vol. 5). Siam.

[138] Yao, H., Lai, Y., Ma, Q., & Jian, M. (2014). Asset allocation for a DC pension fund with stochastic income and mortality risk: A multi-period mean-variance framework. Insurance: Mathematics and Economics, 54, 84-92.

[139] Yao, H., Zeng, Y., & Chen, S. (2013). Multi-period mean-variance asset-liability management with uncontrolled cash flow and uncertain time-horizon. Economic Modelling, 30, 492-500.

[140] Zakoian, J. M. (1994). Threshold heteroskedastic models. Journal of Economic Dynamics and control, 18(5), 931-955.

[141] Zeng, Y., & Li, Z. (2011). Asset-liability management under benchmark and mean-variance criteria in a jump diffusion market. Journal of Systems Science and Complexity, 24(2), 317-327.

[142] Zenios, S. A. (2008). Practical financial optimization.

[143] Zenios, S. A., & Ziemba, W. T. (Eds.). (2007a). Handbook of Asset and Liability Management: Theory and Methodology (Vol. 1). (North-Holland: Amsterdam).

[144] Zenios, S. A., & Ziemba, W. T. (Eds.). (2007b). Handbook of Asset and Liability Management: Applications and case studies (Vol. 2). Elsevier.

[145] Ziemba, W. T., & Mulvey, J. M. (1998). Worldwide asset and liability modeling (Vol. 10). Cambridge University Press.

[146] Ziemba, W. T. (2003). The stochastic programming approach to asset, liability, and wealth management.

Appendix

ALM-General Algebraic Modeling System (GAMS) main code

```
* PARAMETERS
$SET OnDebug N
$SETGLOBAL WDir ../../tmp/
$ONEMPTY

* OPTIONS
$EOLCOM //
$INLINECOM /* */

$INCLUDE "%Wdir%AllSets.inc";

alias (Node,n,m,h);
alias (Asset, i,j,i2);
alias (Stage,s,s1,s2);
alias (S,Sh,Sm,Sn,Sleaf);

$INCLUDE "%Wdir%constants.inc";
* Load initial asset position
$INCLUDE "%Wdir%x0.inc";          //Initial wealth
* Load Asset features
$INCLUDE "%Wdir%r.inc";            // price return of asset i at node n

* Load Tree structure

$INCLUDE "%Wdir%StageOfNode.inc";
$INCLUDE "%Wdir%NodeAncestor.inc";
$INCLUDE "%Wdir%Ancs.inc";  // Path of Node Ancestors
Set AncsWith(Node,Node);    // Path of Node Ancestors including Node
AncsWith(h,n)$(Ancs(h,n) Or Ord(h)=Ord(n)) = Yes;

$INCLUDE "%Wdir%Probability.inc";
$INCLUDE "%Wdir%StageTimeWeight.inc";
$INCLUDE "%Wdir%LiabStream.inc";
$INCLUDE "%Wdir%ttm.inc";
$INCLUDE "%Wdir%StageTime.inc";

* Save last stage
Scalar LastStage;
       LastStage = card(s);

* Save Time horizon
Scalar T;
       T=sum(s,StageTimeWeight(s));

* Define some sets
Set root(Node), leaf(Node);
root(Node)$(Ord(Node)=1) = Yes;
leaf(Node)$(StageOfNode(node) eq LastStage) = Yes;
```

```
*******************************
$SETGLOBAL MKT /i2/
set MKT(Asset) %MKT%;

$SETGLOBAL Bond_EMU /i3/
set Bond_EMU(Asset) %Bond_EMU%;

$SETGLOBAL Bond_exEMU /i4/
set Bond_exEMU(Asset) %Bond_exEMU%;

$SETGLOBAL Bond_inf /i5/
set Bond_inf(Asset) %Bond_inf%;

$SETGLOBAL Gov_bond /i3,i4/
set Gov_bond(Asset) %Gov_bond%;

$SETGLOBAL FX_In /i2,i3,i4/
set FX_In(Asset) %FX_In%;
*******************************
Set AlternInv(Asset);
AlternInv(i)=No;

Options

         LP  = MOSEK
         MIP = MOSEK
         NLP = CONOPT
         limrow = 0 // to avoid rows truncation in lst file
         limcol = 0 // to avoid cols truncation in lst file
         iterlim = 200000000 // Iteration limits
         reslim  = 1000000000
;

Variables
CapX(n)            Asset portfolio value in node n (without bank-account)
CapXtot(n)         Asset portfolio value in node n
Wealth(n)          Wealth in node n
Wealth_declag(n)          Wealth in node n (decision lag)
LiabReserve(n)     Liability (year term)
Final_Wealth(n)    Final_Wealth in node n
Expected_Wealth    Expected Wealth at final stage
x(n,i)            amount held of asset i in node n
RiskCapital(n)    Tot Asset - Liability Reserve
objval           Multi-criteria object function value type

wealth_stage1
Expected_wealth_stage2
Expected_wealth_stage3
Expected_wealth_stage4
first_target_wealth_1
Expected_First_obj_2

wealth_obj1
```

```
second_target_wealth_1
Expected_second_obj_2
Expected_second_obj_3
wealth_obj2
wealth_obj3

third_target_wealth_1
Expected_third_obj_2
Expected_Wealth1   Expected Wealth at first stage

Liab_1y
Liab_2y
Liab_3y
Liab_5y
Liab_7y
Liab_10y

;

Positive Variables
Expected_First_shortfall
Expected_second_shortfall


xp(n,i)  amount bought of asset i in node n
xm(n,i)  amount sold of asset i in node n
xh(m,n,i)  amount held of i-th asset in node n bought in node m
xhm(m,n,i) amount sold of i-th asset in node n bought in node m

liabi(n)
;
************************ Equation implementation sections    ***********
Set no_Slack(i); no_Slack(i) = not(Slack(i));
Set NodeAndStage(n,s); NodeAndStage(n,s)$(Ord(s) = StageOfNode(n)) = Yes;
Set NAS(n,s), NAS2(n,s);
NAS(n,s) = NodeAndStage(n,s);
NAS2(n,s) = NodeAndStage(n,s);

/* ======================================================================
*                   FIRST STAGE WEALTH AND ALLOCATION DECISION EQUATION
   =====================================================================*/
* Asset Balance (first Stage)
Equation Inventory_Asset_Balance_Eq_0_0;
Inventory_Asset_Balance_Eq_0_0(Slack,root(N))..x(n,Slack)=e=xp(n,Slack)-xm(n,Slack);
Equation Inventory_Asset_Balance_Eq_0 Inventory Asset Balance Equation at first stage;
Inventory_Asset_Balance_Eq_0(no_Slack(i),root(N))..xh(n,n,i)=e=x0(i)+xp(n,i)-xhm(n,n,i);
Equation Inventory_Asset_Balance_Eq_01;
Inventory_Asset_Balance_Eq_01(no_Slack(i),root(N))..x(n,i)=e=xh(n,n,i);
Equation Inventory_Asset_Balance_Eq_02;
Inventory_Asset_Balance_Eq_02(no_Slack(i),root(N))..xm(n,i)=e=xhm(n,n,i);
Equation Inventory_Asset_Balance_Eq_03;
Inventory_Asset_Balance_Eq_03(no_Slack(i),root(N))..x0(i)=g=xhm(n,n,i);
xm.UP(root,i)$(Not(slack(i)))=x0(i);
```

```
* cash flow balance

Equation Cash_Flow_Balance_Eq_0;
        Cash_Flow_Balance_Eq_0(Slack,NodeAndStage(root(N),s))          ..
                                                    0 =e= x0(Slack)
                                                    + sum(i, xm(n,i)-xp(n,i) )
                                                     - LiabStream(n)
                                                     ;

/* ================================================================================
*                  NEXT STAGES WEALTH AND ALLOCATION DECISION EQUATION
   ==============================================================================*/
* Asset Balance (Next stages)

Equation Inventory_Asset_Balance_Eq_t Inventory Asset Balance Equation at node n;
Inventory_Asset_Balance_Eq_t(n,Slack(i))$(Not(root(n))).. x(n,i)=e=xp(n,i) - xm(n,i);

Equation Inventory_Asset_Balance_Eq_t1;
Inventory_Asset_Balance_Eq_t1(n,no_Slack(i))$(Not(root(n)))..
xm(n,i)=e=sum(h$(Ancs(h,n) ), xhm(h,n,i));

Equation Inventory_Asset_Balance_Eq_t2;
Inventory_Asset_Balance_Eq_t2(h,m,no_Slack(i),nas(n,s))
$(Not(root(n)) and (Ancs(h,n)) And ord(m) eq anc(n)) ..
xh(h,n,i) =e=   xh(h,m,i) * (1+r(n,i)*StageTimeWeight(s))  - xhm(h,n,i);

Equation Inventory_Asset_Balance_Eq_t3;
Inventory_Asset_Balance_Eq_t3(n,no_Slack(i))$(Not(root(n)))   ..
xh(n,n,i) =e=   xp(n,i);//*(1-u(i));

Equation Inventory_Asset_Balance_Eq_t4;
Inventory_Asset_Balance_Eq_t4(n,no_Slack(i),s)$(Not(root(n)))  ..
x (n,i)   =e=   sum(h$AncsWith(h,n), xh (h,n,i));


       xhm.FX(n,n,i)$(not Root(n) and not Slack(i))=0;
       xhm.FX(h,n,i)$(AncsWith(h,n) and Slack(i))=0;


Parameter zetap(n); zetap(n) = sum(i$Slack(i), r(n,i) );
Parameter zetam(n); zetam(n) = sum(i$FxIn_MoneyMkt(i),r(n,i)+Spread_E3m);
*=========================================================
* cash flow balance
Equation Cash_Flow_Balance_Eq          Cash Flow Balance at node n;
Cash_Flow_Balance_Eq(M,NodeAndStage(N,S),slack(i))$(Not(root(n)) And Ord(m)eqanc(n))..
                0 =e= ( xp(m,i)  * (1+(zetap(n)*(1-theta1))*  StageTimeWeight(s))
                       - xm(m,i)  * (1+zetam(n)*StageTimeWeight(s))  )
                         - xp(n,i) +  xm(n,i)
                         + sum(j$(Not(slack(j))), xm(n,j)-xp(n,j) )
                         - (LiabStream(n))
                        ;
```

```
/* ===========================================================================
*                            Policy contraints
   ===========================================================================*/
*   Maximum exposition
Equation Equity_maxpos(n)    Maximum exposition to equity position;
Equity_maxpos(n)  ..   sum( i$(Equity(i)),x(n,i) )  =l= (0.2)*CapXtot(n);
Equation Commodity_maxpos(n)   Maximum exposition to Commodity position;
Commodity_maxpos(n) ..  sum( i$(Commodity(i)),x(n,i) )=l= (0.2)*CapXtot(n);

Equation bond_inf_maxpos(n)      Maximum exposition to Bond position;
bond_inf_maxpos(n) .. sum( i$(bond_inf(i)),x(n,i) )=l= (0.3)*CapXtot(n);

Equation Gov_bond_maxpos(n)      Maximum exposition to Gov. Bond position;
Gov_bond_maxpos(n).. sum( i$(Gov_bond(i)),x(n,i) ) =l= (1)*CapXtot(n);

Equation MKT_maxpos(n)      Maximum exposition to Gov. Bond position;
MKT_maxpos(n).. sum( i$(MKT(i)),x(n,i) )        =l= (1)*CapXtot(n);

Equation Slack_maxpos;
Slack_maxpos(n)..   sum( i$(Slack(i)),x(n,i) ) =l= (1)*CapXtot(n);


*   Minimum exposition
Equation bond_inf_minpos(n)  Minimum exposition to inf Bond position;
bond_inf_minpos(n)   ..   sum( i$(bond_inf(i)),x(n,i) ) =g= (0)*CapXtot(n);

Equation Gov_bond_minpos(n)      Maximum exposition to Gov. Bond position;
Gov_bond_minpos(n) .. sum( i$(Gov_bond(i)),x(n,i) )   =g= (0)*CapXtot(n);
Equation MKT_minpos(n)      Maximum exposition to MKT position;
MKT_minpos(n)    ..   sum( i$(MKT(i)),x(n,i) )       =g= (0)*CapXtot(n);

Equation Commodity_minpos(n);
Commodity_minpos(n)  ..  sum( i$(Commodity(i)),x(n,i) ) =g= (0)*CapXtot(n);
Equation Slack_minpos(n);
Slack_minpos(n)  ..   sum( i$(Slack(i)),x(n,i) )   =g= (0.05)*CapXtot(n);
Equation Equity_minpos(n);
Equity_minpos(n)  ..   sum( i$(Equity(i)),x(n,i) )   =g= (0)*CapXtot(n);

// -------- Turnover Constraint ---------------------------------------------

Equation MaxTurnover      Max Turnover constraint definition;
       MaxTurnover (NodeAndStage(h,sh),nas(n,sn),nas2(m,sm))
                $( not root(n)
                   And Ancs(h,n)
                   And Ord(m) = Anc(n))      ..
  Sum(i$(Not Slack(i)), xp(n,i) +xm(n,i))

=L=
  sum( i$(Not Slack(i)),(0.5) *  x(m,i) * (1+r(n,i)) )
;
```

```
/* ===========================================================================
*                                  HORIZON DECISION
   ===========================================================================*/
xp.fx(leaf,i)$(Not(slack(i)))=0;
xm.up(leaf,i)$(Not(slack(i)))=LiabStream(leaf);

/* ===========================================================================
*                              RISK REWARD DEFINITION EQUATION
   ===========================================================================*/

* Total Asset Portfolio value (X)
Equation Def_Cap_X(n)            Total portfolio value (excluding bank account);
         Def_Cap_X(n)    .. CapX(n)      =e= sum(i$(not slack(i)),x(n,i));
Equation Def_Cap_Xtot(n)         Total portfolio value;
         Def_Cap_Xtot(n).. CapXtot(n)   =e= sum(i,x(n,i));


* Wealth definiton(W)
Equation Def_Wealth(n)           Wealth definition equation;
Def_Wealth(n)   .. Wealth(n) =e= CapXtot(n) ;
* Wealth definiton(W) decision lagged
Equation Def_Wealth_declag(n,m)  Wealth definition equation (decision lag);
Def_Wealth_declag(n,m)$(ord(m) eq anc(n))
.. Wealth_declag(n) =e= CapXtot(n) - sum(i,xp(n,i)) + sum(i,xm(n,i)) ;

* Final Wealth
Equation Def_Final_Wealth(n)     Final Wealth definition equation (Wealth);
Def_Final_Wealth(leaf(n))  ..   Final_Wealth(n) =e= Wealth(n);
* Expected Final Wealth definition(E(W(T))
Equation Def_Expected_Wealth     Expected wealth equation;
Def_Expected_Wealth  ..
Expected_Wealth =e= sum(leaf(n), Final_Wealth(n)*Probability (n));


//---------------------------------------------------------------------------------

Equation Def_wealth_obj1       wealth at first target stage1;
Def_wealth_obj1(n)$(StageOfNode(n) eq 3)  .. wealth_obj1(n) =e= (Wealth(n));

Equation Def_first_target_wealth_1    Target wealth in Target Stage1;
Def_first_target_wealth_1 ..first_target_wealth_1=e=sum((slack), (x0(slack)*(1.04)));

*part two
Equation Def_Expected_first_obj_2      wealth in Target Stage1;
Def_Expected_first_obj_2 ..
Expected_first_obj_2 =e= sum((n)$(StageOfNode(n) eq 3),
((Wealth(n)+Liabstream(n))*Probability (n)));

Equation Def_Expected_First_shortfall;
Def_Expected_First_shortfall ..
Expected_First_shortfall =g= (first_target_wealth_1 - Expected_First_obj_2);
```

```
Equation Def_Liab_1y      liability in Target Stage1;
Def_Liab_1y ..
Liab_1y  =e= sum((n)$(StageOfNode(n) eq 3),((Liabstream(n))*Probability (n)));

Equation Def_wealth_obj2   wealth at second target stage2;
Def_wealth_obj2(n)$(StageOfNode(n) eq 5) .. wealth_obj2(n) =e= ( Wealth(n));
*part one

Equation Def_second_target_wealth_1     Target wealth in Target Stage2;
Def_second_target_wealth_1 ..
second_target_wealth_1 =e= (first_target_wealth_1 - Liab_1y)*((1.05)**2);
*part two
Equation Def_Expected_second_obj_2      wealth in Target Stage2;
Def_Expected_second_obj_2 .. Expected_second_obj_2 =e=
sum((n)$(StageOfNode(n) eq 5),(((Wealth(n)+Liabstream(n))*Probability (n))));

Equation Def_Expected_second_shortfall     Expected Shortfall in Target Stage2;
Def_Expected_second_shortfall .. Expected_second_shortfall =g=
(second_target_wealth_1 - Expected_second_obj_2);

Equation Def_Liab_2y      liability in Target Stage2;
Def_Liab_2y ..
Liab_2y  =e= sum((n)$(StageOfNode(n) eq 4),((Liabstream(n))*Probability (n)));

Equation Def_Liab_3y      liability in Target Stage2;
Def_Liab_3y ..
Liab_3y  =e= sum((n)$(StageOfNode(n) eq 5),((Liabstream(n))*Probability (n)));
Equation Def_Liab_5y      liability in 5 year;
Def_Liab_5y ..
Liab_5y  =e= sum((n)$(StageOfNode(n) eq 6),((Liabstream(n))*Probability (n)));

//---------------------------------------------------------------------------------
Equation Def_LiabReserve(n,s)        Liability Definition (year term);
Def_LiabReserve(n,s)$(nas(n,s) and not root(n) and StageTimeweight(s)>0) ..
LiabReserve(n) =e= LiabStream(n);

Equation OBF1 objective function;
OBF1 .. objval =e= -((0.5)*Expected_first_shortfall+
                   (0.3)*Expected_second_shortfall)+((0.2)*(Expected_Wealth));

Model Model_1 personal AIM model obj.func= E(W(T))
        /all /
;

/* ======================================================================
*                           STANDARD OPTIMIZATION
   ======================================================================*/
$set Mode StandardOpt
*SOLVE MAXIMIZING objval USING LP;
SOLVE Model_1 MAXIMIZING objval USING LP;
```

```
parameter dummy(n);
dummy(n) = (BondsUp)*CapXtot.l(n);
display bondsup, dummy;
display LiabStream;
$INCLUDE "OutOpt.gms";
Scalar OBF_1;
OBF_1= (objval.l);
display 'OBF1='
display OBF_1
Scalar EWT;
EWT = (Expected_wealth.l);
display 'EWT='
display EWT
Scalar W_X;
W_X = sum(leaf(n), CapXtot.l(n)*Probability (n));
display 'W_X='
display W_X

$exit
* =========================================================================
put Expected_Wealth.l
loop(n,put Wealth.l(n));
loop(n,put CapX.l(n));
loop(n,put CapXtot.l(n));
loop(i,put x.l("n1",i));
put /;
);
putclose;
```

Main Matlab codes of the scenario tree generation

```
function ScenGen_run
% Start time to compute computational time
tic;
run DirSet;
Parameters_read; % Read all parameter models (and store in Param structure)
% Scenario generation structure
SG.tnodi=[1 4 4 4 4 2 2 2];
SG.stage=[0 1 2 3 4 5 6 7];
set_init_date=2009;
dete1='30/6/2009'; dete1= datenum(dete1,'dd/mm/yyyy');  %6 month
dete2='31/12/2009'; dete2= datenum(dete2,'dd/mm/yyyy'); %1 year
dete3='31/12/2010'; dete3= datenum(dete3,'dd/mm/yyyy'); %2 year
dete4='31/12/2011'; dete4= datenum(dete4,'dd/mm/yyyy'); %3 year
dete5='31/12/2013'; dete5= datenum(dete5,'dd/mm/yyyy'); %5 year
dete6='31/12/2015'; dete6= datenum(dete6,'dd/mm/yyyy'); %7 year
dete7='31/12/2018'; dete7= datenum(dete7,'dd/mm/yyyy'); %10 year
 SG.timepoints = [dete1, dete2, dete3, dete4, dete5, dete6, dete7];
SG.Init_date='31/12/2008'; SG.Init_date= datenum(SG.Init_date,'dd/mm/yyyy');
SG.TimePoints = [0 0.5000 1 2 3 5 7 10];
SG.DatePoints = {'31/12/2008','30/6/2009','31/12/2009', '31/12/2010','31/12/2011',...
'31/12/2013','31/12/2015','31/12/2018'};
end
SG.timelenght=[0 6/12 6/12 1 1 2 2 3];
SG.StageTimeWeight=[0 6/12 6/12 1 1 2 2 3];

IntSet=InternationalSetting;
% nodes number computation
SG.nnodes = sum(cumprod(SG.tnodi));

SG.nstages=numel(SG.TimePoints);
if ( SG.nstages~= numel(SG.tnodi))
    error('The number of stages doesn''t match the nodes structure');
end

SG=Create_tree(Param,SG);
SG.tempo=toc;
save([WorkDir '\SGtree'],'SG');
save ('SG');

msgbox(['Scenario Tree generated (' num2str(SG.tempo) ' secs) .'],'Generation complete','modal' );

end
```

```
% Create tree script
function [SG]=Create_tree(Parameters,SG)

Hwb=waitbar(0,'Create Tree: execution started','NAME','Tree building');

%% tree structure via Nodal Part Matrix
[SG.npm,SG.NodeAncestor,SG.stageOFnode]=NodalPartMat(SG.nstages,SG.tnodi);

waitbar(0,Hwb,'Read values at initial date...');

IntSet=InternationalSetting;
% History read to SG.Init_date
[History,BenchList]=HistoryRead_to_date_gen(SG.Init_date,IntSet.DateFormat);

% Read risk factors value at date_init
HistoryRead_date;

set(Hwb,'NAME','Fill nodes');
waitbar(0,Hwb,'Fill nodes...');

% Tree initialize
Riskfactor_init=cell2mat(struct2cell(SG.init));

SG.Bench_product=[0.00; Riskfactor_init];
SG.Bench_product2=[0.00; Riskfactor_init(5:end-1)];
SG.Riskfactor(:,1) =Riskfactor_init(1:5);
SG.Bench_returns(:,1) = [0.00; Riskfactor_init(5:end-1)];
SG.Liab_Stream(:,1)=[0];

% Tree filling
SG = fill_nodes(Parameters,SG,2,SG.nnodes, History,Hwb);

%IPS parameter load
 DirSet;
 IPS=GetValueFromXls([WorkDir '\IPSset.xls'],'IPS_details','auto');

StageTimeWeight_node=SG.StageTimeWeight(SG.stageOFnode(1));

set(Hwb,'NAME','Fill nodes: Other Factors');
waitbar(0,Hwb,'Fill nodes...');
```

```
fromNodo=2;toNodo=SG.nnodes;
for node=fromNodo:toNodo
    if exist('Hwb','var')
        waitbar((node-fromNodo)/(toNodo-fromNodo),Hwb,['node #' num2str(node)]);
    end
    StageTimeWeight_node=SG.StageTimeWeight(SG.stageOFnode(node));
    if SG.NodeAncestor(node)==1
    StageTimeWeight_Anc=1;
    else
    StageTimeWeight_Anc=SG.StageTimeWeight(SG.stageOFnode(SG.NodeAncestor(node)));
    end

end
SG.OtherFactor(:,1) = 0;
close(Hwb);
dummy=0;
```

Fill nodes

```
function [SG, varargout] =fill_nodes(P,SG, fromNodo, toNodo, varargin)
% fill_nodes function
% **input**
%       param = all parameter models
%       SG = tree structure and current value
%   fromNodo    = initial node
%       toNodo      = end node
%   err         = random shock generated (RiskFactors x  SG.numnodes(toNodo-fromNodo+1))
% **output**
%   SG        = tree structure and value updated

fill_nnodes = toNodo - fromNodo+1;
DirSet;
IntSet=InternationalSetting;

%  Pre-Allocate space
SG.Riskfactor = [SG.Riskfactor zeros(size(SG.Riskfactor,1),fill_nnodes)];
SG.Bench_returns=[SG.Bench_returns zeros(size(SG.Bench_returns,1),fill_nnodes)];
SG.Liab_Stream=[SG.Liab_Stream zeros(size(SG.Liab_Stream,1),fill_nnodes)];
if nargin>1
    Hwb=varargin{2}; %Handle to the waitbar
end

clear varargin;
gen_ndx=0;

SG.qBench{SG.NodeAncestor(fromNodo)}=[SG.Bench_product(1: end,SG.NodeAncestor(fromNodo))];
SG.qBench2{SG.NodeAncestor(fromNodo)}=[SG.Bench_product2(1: end,SG.NodeAncestor(fromNodo))];

[History]=HistoryRead_to_date_gen(SG.Init_date,IntSet.DateFormat);
```

```
Layer_1_value=[History.Output_gap.value; History.Inflation_rate.value; History.EMU10y.value;...
    History.EURIBOR_12_month.value]';

        %% first layer correlation matrix
        corr_Layer_1=corrcoef(Layer_1_value);
        ch=chol(corr_Layer_1);
        %% second layer correlation matrix
        Layer_2_value=[History.JPM_EMU_GOV.value; History.JPMexEMU.value;...
      History.Gov_inf.value; History.Equity_MSCI.value; History.Commodity.value]';
        % correlation matrix (CCC)
        corr_Layer_2=corrcoef(Layer_2_value);
        ch2=chol(corr_Layer_2);

        % DCC correlation matrix
[m,n] = size(Layer_2_value);
[p, lik, Ht,~,~,~, hsigma] = dcc_mehdi(Layer_2_value,[],1,0,20);
T= length(Layer_2_value);
HT=Ht(:,:,:);
DCCrho = [];
index=[];
k=1;
for i=1:n
    for j=i+1:n
        DCCrho(k,:)= vec(HT(i,j,:)) ./ sqrt(vec(HT(i,i,:)) .* vec(HT(j,j,:))); %correlation matrix
        index(k,:)=[i;j];
        k=k+1;
    end
end
corr1=DCCrho(1:4,:);
corr2=[DCCrho(1,:); DCCrho(5:7,:)];
corr3=[DCCrho(2,:); DCCrho(5,:); DCCrho(8:9,:)];
corr4=[DCCrho(3,:); DCCrho(6,:); DCCrho(8,:); DCCrho(10,:)];
corr5=[DCCrho(4,:); DCCrho(7,:); DCCrho(9:10,:)];

sigma_t=hsigma;

i2=1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Choose statistical MODEL
 %model=2; %CCC benchmark and risk factors correlation
 model=3; %DCC benchmark correlation +CCC risk factor

 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% coefficients
  sigma_p= std (History.Inflation_rate.value);

  bita0_p=0.0122819; bita1_p=0.431745; bita2_p=-0.623634; bita3_p=1.41837; bita4_p=-0.801184;

  %(output gap) Model%
  sigma_y= std (History.Output_gap.value);
  bita0_y=0.000183972; bita1_y=1.80948; bita2_y=-0.895028;

  %Money market (Euribor 12 month) Model %
   sigma_r=std(History.EURIBOR_12_month.value);
   bita0_r=0.0391083; bita1_r=-0.248289; bita2_r=0.588875;

   %Interest rate Model %
   sigma_l=std(History.EMU10y.value);
   bita0_l=0.0191300; bita1_l=0.774867; bita2_l=0.0872683; bita3_l=-0.457248;
```

```
%%%%%%% Equity risk permium
sigma_lamda=0.200474735;
bita0_lamda=1.05636; bita1_lamda=-13.2855; bita2_lamda=-21.2597;
% MSCI Europe TR Index %
 sigma_S=std(History.Equity_MSCI.value);

% JPM Global Gov. Bond EMU %
 sigma_g_1=std(History.JPM_EMU_GOV.value);
 bita0_g_1=0.112593; bita1_g_1=0.347006; bita2_g_1=-0.129258; bita3_g_1=-0.748494; bita4_g_1=-1.68531;

 % JPM Global Gov. Bond ex EMU %
  sigma_g_2=std(History.JPMexEMU.value);
  bita0_g_2=0.0482851; bita1_g_2=0.421198; bita2_g_2=-0.0658721; bita3_g_2=-0.892567;

 % Barclays Euro Gov. Inflation Linked %
     sigma_g_3=std(History.Gov_inf.value);
     bita0_g_3=0.163523; bita1_g_3=-0.286524; bita2_g_3=-0.134399; bita3_g_3=-1.91733;

     %Growth rate GDP
     sigma_gG=std(History.g_GDP.value);
     bita0_gG=0.00158146; bita1_gG=0.492478; bita2_gG=0.0312635;

    % Commodity model %
     sigma_C=std(History.Commodity.value);
     bita0_C=-0.00183939; bita1_C=0.586250; bita2_C=1.34962; bita3_C=4.71747; bita4_C=-3.03868;

    delta_t=1/4;

       p=[];
[t]=find_date_near_ndx(History.Inflation_rate.date,SG.Init_date,'mm/dd/yyyy');
 p_a=History.Inflation_rate.value(t);   %one leg behind (just in historical data)
 p_b=History.Inflation_rate.value(t-1); %two legs behind(just in historical data)
       y=[];
 y_a=History.Output_gap.value(t);   %one leg behind (just in historical data)
 y_b=History.Output_gap.value(t-1); %two legs behind(just in historical data)
 y_c=History.Output_gap.value(t-2);
       l=[];
 l_a=History.EMU10y.value(t);   %one leg behind (just in historical data)
 l_b=History.EMU10y.value(t-1); %two legs behind(just in historical data)
       r=[];
 r_a=History.EURIBOR_12_month.value(t);   %one leg behind (just in historical data)
 r_b=History.EURIBOR_12_month.value(t-1); %two legs behind(just in historical data)
       lamda=[];
       gG=[];
gG_a=History.g_GDP.value(t);
gG_b=History.g_GDP.value(t-1);
       S=[];
S_a=History.Equity_MSCI.value(t);   %one leg behind (just in historical data)
S_b=History.Equity_MSCI.value(t-1); %two legs behind(just in historical data)
          g_1=[];
g_1_a=History.JPM_EMU_GOV.value(t);   %one leg behind (just in historical data)
g_1_b=History.JPM_EMU_GOV.value(t-1); %two legs behind(just in historical data)
         g_2=[];
g_2_a=History.JPMexEMU.value(t);   %one leg behind (just in historical data)
g_2_b=History.JPMexEMU.value(t-1); %two legs behind(just in historical data)
```

```
g_2_b=History.JPMexEMU.value(t-1); %two legs behind(just in historical data)

           g_3=[];
g_3_a=History.Gov_inf.value(t);   %one leg behind (just in historical data)
g_3_b=History.Gov_inf.value(t-1); %two legs behind(just in historical data)

        C=[];
C_a=History.Commodity.value(t);
C_b=History.Commodity.value(t-1);

        cash=[];

        Liability=[];

  for nodo=fromNodo:toNodo      %% go to the next node

  if exist('Hwb','var')
      waitbar((nodo-fromNodo)/(toNodo-fromNodo),Hwb,['node #' num2str(nodo)]);
  end

  old_t=datenum(SG.DatePoints{SG.stageOFnode(SG.NodeAncestor(nodo))},'dd/mm/yyyy'); %IntSet.DateFormat);
  new_t=datenum(SG.DatePoints{SG.stageOFnode(nodo)},'dd/mm/yyyy'); %IntSet.DateFormat);

  t_cursor=old_t;

  %%%%%%%%%%%%%%%%%%% Fit DCC index
  if t_cursor==new_t
      i11=i11+1;
      i2=i2+1;
  end
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

  %monthly simulation
  SG.qBench{nodo}=[];  % cells to save quartely values
  SG.qBench2{nodo}=[];
  last_q=t_cursor;
  addtodate_last_q=addtodate(last_q,3,'month');

  month_num=0;

      Bench_proc2=[];
while t_cursor<new_t
       gen_ndx=gen_ndx+1;
       t_ndx=t_cursor-old_t+1;
       % Save the drift vector at time t
      r_t =   Bench_proc2;

      %%normal distribution (mean zero sigma one)
      if  model==2 || model==3
      dis1=randn(4,1);
      end
```

```
if model==2
 dis2=randn(5,1);
 elseif model==3
 dis2=randn(4,1);
 end
 dis3=randn;

 %output gap
 if model==2 || model==3
 y(nodo)= bita0_y+ bita1_y*y_a+ bita2_y*y_b+ sigma_y* sqrt(delta_t)* ch(3,:)* dis1;
 end

 if  model==2 || model==3
 p(nodo)= bita0_p+ bita1_p*p_a+ bita2_p*y(nodo)+ bita3_p*y_a+ bita4_p*y_b+  sigma_p*sqrt(delta_t)* ch(2,:)* dis1;
 end

 %Euribor 12 month
 if model==2 || model==3
 r(nodo)= bita0_r+ bita1_r*p(nodo)+ bita2_r*y(nodo)+ sigma_r* sqrt(delta_t)* ch(4,:)* dis1;
 end

%Interest rate

if model==2 || model==3
 l(nodo)= bita0_l+ bita1_l*l_a+ bita2_l*y(nodo)+ bita3_l*p_a + sigma_l* sqrt(delta_t)* ch(1,:)* dis1;
end
  %Equity risk permium
 lamda(nodo)= bita0_lamda+ bita1_lamda*l_a+ bita2_lamda*p_a+ sigma_lamda* sqrt(delta_t)* dis3;

 %%%%%%%%%%%%% Benchmarks %%%%%%%%%%%%%

 % MSCI Europe
 if model==2
 S(nodo)= (r_a+ lamda(nodo))+sigma_S* sqrt(delta_t)* ch2(4,:)* dis2; %% Model_2
  elseif model==3
 S(nodo)= (r_a+ lamda(nodo))+sigma_t(i2,4)* sqrt(delta_t)* corr4(:,i2)' * dis2; %% Model_2
 end

 % JPM Global Gov. Bond EMU
if model==2
 g_1(nodo)= bita0_g_1+ bita1_g_1*g_1_a+ bita2_g_1*S(nodo)+bita3_g_1*y(nodo)+ bita4_g_1*l(nodo)+...
sigma_g_1* sqrt(delta_t)* ch2(1,:)*dis2; %% Model_2
 elseif model==3 || model==4
 g_1(nodo)= bita0_g_1+ bita1_g_1*g_1_a+ bita2_g_1*S(nodo)+bita3_g_1*y(nodo)+ bita4_g_1*l(nodo)+...
 sigma_t(i2,1)* sqrt(delta_t)* corr1(:,i2)' *dis2; %% Model_3
 end


 % JPM Global Gov. Bond ex EMU
 if model==2
 g_2(nodo)= bita0_g_2+ bita1_g_2*g_2_a+ bita2_g_2*S(nodo)+bita3_g_2*p(nodo)+...
 sigma_g_2* sqrt(delta_t)* ch2(2,:)*dis2; %% Model_2
  elseif model==3
 g_2(nodo)= bita0_g_2+ bita1_g_2*g_2_a+ bita2_g_2*S(nodo)+bita3_g_2*p(nodo)+...
 sigma_t(i2,2)* sqrt(delta_t)* corr2(:,i2)' *dis2; %% Model_3
 end
```

```
% Barclays Euro Gov. Inflation Linked
if model==2
g_3(nodo)= bita0_g_3 + bita1_g_3 *p(nodo)+ bita2_g_3 *r_a + bita3_g_3*l(nodo) +...
sigma_g_3* sqrt(delta_t)*ch2(3,:)* dis2; %% Model_2
elseif model==3
g_3(nodo)= bita0_g_3 + bita1_g_3 *p(nodo)+ bita2_g_3 *r_a + bita3_g_3*l(nodo) +...
sigma_t(i2,3)* sqrt(delta_t)* corr3(:,i2)' * dis2; %% Model_3
end

% Commodity model
if model==2
C(nodo)= bita0_C+bita1_C*S(nodo) + bita2_C*g_3_a+bita3_C*y_a+ bita4_C*p(nodo)+...
sigma_C* sqrt(delta_t)* ch2(5,:)* dis2; %model_2 corr. benchmark
elseif model==3
C(nodo)= bita0_C + bita1_C*S(nodo) + bita2_C*g_3_a+bita3_C*y_a+ bita4_C*p(nodo)+...
sigma_t(i2,5)* sqrt(delta_t)* corr5(:,i2)' * dis2; %model_3 corr. benchmark
end
cash(nodo)=0.00;
 if (addtodate_last_q < SG.Init_date+364)
     Liability(nodo)=0;
 else
if (addtodate_last_q == (SG.Init_date+364))
 Liability(nodo)=(1500000*(1+p(nodo)));
else
if (addtodate_last_q == (SG.Init_date+729))
 Liability(nodo)=(1350000*(1+p(nodo)));
else
if (addtodate_last_q  == SG.Init_date+1094)
   Liability(nodo)=(1200000*(1+p(nodo)));
else
if (addtodate_last_q  == SG.Init_date+1825)
   Liability(nodo)=(1050000*(1+ ((p_a+p(nodo))/2) ))+ (900000*(1+p(nodo)));
else
if (addtodate_last_q  == SG.Init_date+2555)
   Liability(nodo)=(750000*(1+ ((p_a+p(nodo))/2) ))+ (600000*(1+p(nodo)));
else
  if(addtodate_last_q == SG.Init_date+ 3651)
      li=((p(nodo)-p_a)/3);
   Liability(nodo)=(450000*(1+(p_a+(li)))*(1+ii)^2)+(300000*(1+(p_a+(2*li))))+ 150000*(1+p(nodo));
  end
end
  end
end
  end
end
 end
```

```
        Bench_proc=[cash(nodo); p(nodo); y(nodo); r(nodo); l(nodo); gG(nodo); g_1(nodo);...
        g_2(nodo);g_3(nodo) ; S(nodo); C(nodo)];
        Bench_proc2=[cash(nodo); r(nodo); g_1(nodo); g_2(nodo); g_3(nodo); S(nodo); C(nodo)];
        Riskfactor=[y(nodo); p(nodo); l(nodo); gG(nodo); r(nodo)];
        Bench_returns=[cash(nodo); r(nodo); g_1(nodo); g_2(nodo); g_3(nodo); S(nodo); C(nodo)];
        p_b = p_a; p_a=p(nodo);   y_c=y_b; y_b = y_a; y_a=y(nodo);
        r_b = r_a; r_a=r(nodo);   l_b = l_a; l_a=l(nodo);
        S_b = S_a; S_a=S(nodo);   g_1_b = g_1_a; g_1_a=g_1(nodo);
        g_2_b = g_2_a; g_2_a=g_2(nodo); g_3_b = g_3_a; g_3_a=g_3(nodo);
        gG_b = gG_a; gG_a=gG(nodo); C_b = C_a; C_a=C(nodo);
        month_num=month_num+1;

        r_t_dt =Bench_proc2;
        t_cursor=datemnth(t_cursor,3,0,0,1);

         if t_cursor>=addtodate_last_q
          SG.qBench{nodo}=[SG.qBench{nodo} Bench_proc];
          SG.qBench2{nodo}=[SG.qBench2{nodo} Bench_proc2];

          last_q=t_cursor;
          addtodate_last_q=addtodate(last_q,3,'month');
         end
end
        SG.Riskfactor(1: end,nodo)=Riskfactor;
        SG.Bench_returns(1: end,nodo)=Bench_returns;
       SG.Liab_Stream(1: end,nodo)= (Liability(nodo));
 end
end
```