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Almost positive kernels on compact Riemannian manifolds

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Abstract

We show how to build a kernel $K_X(x, y) = \sum_{m=0}^{X} h(\lambda_m/\lambda_X)\varphi_m(x)\overline{\varphi_m(y)}$ on a compact Riemannian manifold $\mathcal{M}$, which is positive up to a negligible error and such that $K_X(x, x) \approx X$. Here $0 = \lambda_0^2 \leq \lambda_1^2 \leq \cdots$ are the eigenvalues of the Laplace–Beltrami operator on $\mathcal{M}$, listed with repetitions, and $\varphi_0, \varphi_1, \ldots$ an associated system of eigenfunctions, forming an orthonormal basis of $L^2(\mathcal{M})$. The function h is smooth up to a certain minimal degree, even, compactly supported in $[-1, 1]$ with $h(0) = 1$, and $K_X(x, y)$ turns out to be an approximation to the identity.

Keywords Approximation to the identity · Parametrix of the wave equation · Compact Riemannian manifold · Schwartz kernel

Mathemetics Subject Classification 58C40 · 42C15 · (11K38)

1 Introduction

Let $\mathcal{C} \subset \mathbb{R}^d$ be convex and symmetric and define the trigonometric polynomial

$$T_{\mathcal{C}}(x) = \frac{1}{\mathrm{card}((\frac{1}{2}\mathcal{C}) \cap \mathbb{Z}^d)} \sum_{\ell,k\in\frac{1}{2}\mathcal{C}} e^{2\pi i(\ell-k)\cdot x} = \frac{1}{\mathrm{card}((\frac{1}{2}\mathcal{C}) \cap \mathbb{Z}^d)} \left|\sum_{\ell\in\frac{1}{2}\mathcal{C}} e^{2\pi i\ell\cdot x}\right|^2$$
$$= \sum_{m\in\mathcal{C}} \frac{\mathrm{card}(\frac{1}{2}\mathcal{C} \cap (\frac{1}{2}\mathcal{C} + m) \cap \mathbb{Z}^d)}{\mathrm{card}((\frac{1}{2}\mathcal{C}) \cap \mathbb{Z}^d)} e^{2\pi i m\cdot x}.$$

The above identities immediately show that $T_{\mathcal{C}}(x) \geq 0$, that its Fourier coefficients vanish outside $\mathcal{C}$, that $\widehat{T}(0) = 1$, and that $T_{\mathcal{C}}(0) = \mathrm{card}(\frac{1}{2}\mathcal{C} \cap \mathbb{Z}^d)$.

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In particular, when $\mathcal{C}$ is the axis-parallel, symmetric box of sides $2Y_1, \ldots, 2Y_d$, then $T_{\mathcal{C}}$ is just the standard d-dimensional Fejér kernel

$$F_{2\lfloor Y_1/2\rfloor+1}(x_1)\ldots F_{2\lfloor Y_d/2\rfloor+1}(x_d),$$

where

$$F_n(t) = \sum_{-n\le m\le n}\left(1-\frac{|m|}{n}\right)e^{2\pi i m t}.$$

One could also let $\mathcal{C}$ be the ball centered at the origin and with radius Y, and in this case the non-vanishing Fourier coefficients of $T_{\mathcal{C}}$ are just those corresponding to the eigenvalues $4\pi^2|m|^2$ of the (positive) Laplace–Beltrami operator on the torus which are smaller than or equal to $4\pi^2 Y^2$. Notice that in this case, there are $\approx Y^d$ such eigenvalues, and that since $T_{\mathcal{C}}(0) = \operatorname{card}(\frac{1}{2}\mathcal{C}\cap\mathbb{Z}^d) \approx Y^d$, then $T_{\mathcal{C}}(0)$ is essentially the number of eigenvalues less than or equal to $4\pi Y^2$.

Let now $(\mathcal{M}, g)$ be a d-dimensional compact connected Riemannian manifold, where the Riemannian distance $d(x, y)$ and the Riemannian measure are normalized so that the total measure of $\mathcal{M}$ equals 1. Let $\left\{\lambda_m^2\right\}_{m=0}^{+\infty}$ be the sequence of eigenvalues of the (positive) Laplace–Beltrami operator Δ, listed in increasing order with repetitions, and let $\{\varphi_m\}_{m=0}^{+\infty}$ be an associated sequence of orthonormal eigenfunctions. In particular $\varphi_0 \equiv 1$ and $\lambda_0 = 0$. This allows to define the Fourier coefficients of $L^1(\mathcal{M})$ functions as

$$\widehat{f}(\lambda_m) = \int_{\mathcal{M}} f(x)\overline{\varphi_m(x)}dx,$$

where the integration is with respect to the Riemannian measure, and the associated Fourier series

$$\sum_{m=0}^{+\infty}\widehat{f}(\lambda_m)\varphi_m(x).$$

We would like to extend the construction of the above type of kernel to the case of Riemannian manifolds. In particular it would be very interesting to have a kernel of the form

$$K_X(x, y) = \sum_{m=0}^{X} a(\lambda_m, \lambda_X)\varphi_m(x)\overline{\varphi_m(y)} \tag{1}$$

which is nonnegative and such that $a(0, \lambda_X) = 1$, and $K_X(x, x) \gtrsim X$. If possible, it would be great to have $0 \le a(\lambda_m, \lambda_X) \le 1$.

Observe that by Weyl's estimates, X is essentially the number of eigenvalues λ_m^2 that are smaller than or equal to λ_X^2 (and this number is essentially λ_X^d). Thus, this type of kernel could be considered as a generalization to the case of manifolds of the kernel $T_{\mathcal{C}}$ defined above, when $\mathcal{C}$ is the ball of radius $Y \approx X^{1/d} \approx \lambda_X$.

We do not know if this type of kernels in a general manifold exist. Travaglini [19] proved that one can define certain Fejér kernels on compact Lie groups which are nonnegative. Furthermore, it is easy to see that, in a compact two-point homogeneous space, if a kernel has finite spectrum, then also its square (which is nonnegative) has finite spectrum and a suitable normalization has mean one. In particular, Askey [1] showed that the kernels corresponding to certain Cesàro means are positive in certain compact two-point homogeneous spaces, and conjectured their positivity in all such spaces (see also [2]).

The first natural choice that comes to mind when in need of one such kernel is the heat kernel

$$p_t(x, y) = \sum_{m=0}^{+\infty} e^{-\lambda_m^2 t} \varphi_m(x) \overline{\varphi_m(y)}, \quad t > 0.$$

It is well known that the above heat kernel is positive, all the coefficients are clearly between 0 and 1, the coefficient corresponding to λ_0^2 equals 1, and $p_t(x, x) \approx t^{-d/2}$ for small t. The only problem with it is therefore that the coefficients do not vanish for $m > t^{-d/2}$. It can be proved (see [4]) that

$$\left| \sum_{m=X+1}^{+\infty} e^{-\lambda_m^2 t} \varphi_m(x) \overline{\varphi_m(y)} \right| \lesssim t^{-d+1/2} (X^{2/d} t)^{d-3/2} e^{-X^{2/d} t}.$$

Thus, setting $t = cX^{-2/d} \log X$, the kernel

$$\widetilde{p}_t(x, y) = \sum_{m=0}^{X} e^{-\lambda_m^2 t} \varphi_m(x) \overline{\varphi_m(y)} = p_t(x, y) + O(X^{2-c-1/d} / \log X)$$

is positive up to the remainder $O(X^{2-c-1/d} / \log X)$, all its Fourier coefficients vary between 0 and 1, the coefficient corresponding to λ_0^2 equals 1, but $\widetilde{p}_t(x, x) \approx p_t(x, x) \approx X / \log^{d/2} X$. This strategy therefore gives a good estimate of the remainder, uniform in the variables x and y, but generates a logarithmic loss in the diagonal estimate of the kernel. Observe that the choice $t = cX^{-2/d}$ would give $p_t(x, x) \approx X$, as desired, but the remainder would be too big, precisely $O(X^{2-1/d})$.

Throughout the paper, we will denote $\mathcal{F}_d$ the d-dimensional Fourier transform

$$\mathcal{F}_d f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx,$$

and when f will be radial, with a slight abuse of notation, by $\mathcal{F}_d f(r)$ we will mean $\mathcal{F}_d f(z)$ for all those $z \in \mathbb{R}^d$ such that $|z| = r$. We will also denote $\mathcal{C}$ the cosine transform

$$\mathcal{C} f(s) = \int_{\mathbb{R}} f(t) \cos(st) dt$$

and its inverse (on even functions) $\mathcal{C}^{-1}$ by

$$\mathcal{C}^{-1} f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} f(s) \cos(st) ds = \frac{1}{2\pi} \mathcal{C} f(t).$$

Our main result is the following

Theorem 1 (i) *There exists a nonnegative function $\alpha_0 \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M})$, with $\alpha_0(x, x) = 1$ such that the following holds. Let h be an integrable radial function on $\mathbb{R}^d$, compactly supported in the ball centered at the origin and with radius 1 and such that for some $G > d+1$ and for some positive constant C,*

$$|\mathcal{F}_d h(t)| \leq C \frac{1}{(1+t)^{2G}}.$$

Then

$$K_X(x,y) := \sum_{m=0}^{X} h\left(\frac{\lambda_m}{\lambda_X}\right)\varphi_m(x)\overline{\varphi_m(y)}$$
$$= \frac{\alpha_0(x,y)}{(2\pi)^d}\lambda_X^d \mathcal{F}_d h\left(\frac{\lambda_X d(x,y)}{2\pi}\right) + O\left(\frac{\lambda_X^{d-2}}{(1+\lambda_X d(x,y))^{\lfloor 2G\rfloor - 2d-2}}\right).$$

(ii) *For any integer G there exists a (non vanishing) integrable radial function h defined in $\mathbb{R}^d$, compactly supported in the unit ball, such that $0 \le h(x) \le h(0) = 1$ for all x, and such that for all t*

$$0 \le \mathcal{F}_d h(t) \le C\frac{1}{(1+t)^{2G}}.$$

Point (ii) of Theorem 1 is in fact trivial (see the proof at the end of Sect. 3). The function h in point (ii) satisfies all the hypotheses of point (i). Furthermore, with this choice of h, K_X has all the properties we mentioned after Eq. (1), and non-negativity up to the remainder. In particular $h(\lambda_0/\lambda_X) = h(0) = 1$, $K_X(x,x) = \lambda_X^d \mathcal{F}_d h(0)/(2\pi)^d + O(\lambda_X^{d-2}) \approx X$ and $0 \le h(\lambda_m/\lambda_X) \le h(0) = 1$.

Here we can observe that a kernel K_X as in Theorem 1 is in fact an approximation to the identity, when G is sufficiently large.

Corollary 1 *Let h be an integrable radial function on $\mathbb{R}^d$, compactly supported in the ball centered at the origin and with radius 1, with $h(0) = 1$ and such that for $G > (3d+1)/2$, and for some positive constant C,*

$$|\mathcal{F}_d h(t)| \le C\frac{1}{(1+t)^{2G}}.$$

Then

$$K_X(x,y) := \sum_{m=0}^{X} h\left(\frac{\lambda_m}{\lambda_X}\right)\varphi_m(x)\overline{\varphi_m(y)}$$

is an approximation to the identity, in the sense that for all $x \in \mathcal{M}$ and for all $\delta > 0$,

$$\int_{\mathcal{M}} K_X(x,y)dy = 1,$$
$$\int_{\mathcal{M}} |K_X(x,y)|dy = \int_{\mathcal{M}} |K_X(y,x)|dy \le C,$$
$$\lim_{X\to+\infty}\int_{\{y: d(x,y)\ge\delta\}} |K_X(x,y)|dy = 0.$$

Proof It suffices to apply Theorem 1 (i), with a sufficiently large G to ensure the required integrability and decay. □

It follows by standard arguments that, for kernels as in Corollary 1, the means

$$K_X f(y) := \sum_{m=0}^{X} h\left(\frac{\lambda_m}{\lambda_X}\right)\widehat{f}(\lambda_m)\varphi_m(x) = \int_{\mathcal{M}} K_X(x,y) f(y)dy$$

converge uniformly to $f(x)$ as $X \to +\infty$ whenever f is continuous on $\mathcal{M}$, and in the L^p norm whenever f is in $L^p(\mathcal{M})$, for $1 \le p < +\infty$.

The basic idea in the proof of Theorem 1 is classic and consists in synthesizing the kernels $K_X(x, y)$ by means of the fundamental solution of the wave equation

$$K_X = \int_{\mathbb{R}} \mathcal{C}^{-1}h(t) \cos(\sqrt{\Delta}t)\, dt$$

(see [5, 6, 13, 14]). Then, the Hadamard construction of the parametrix of the wave equation (see [3, 12, 15]) naturally produces an expansion of the kernel K_X with an essentially Euclidean first term

$$\frac{\alpha_0(x, y)}{(2\pi)^d}\lambda_X^d \mathcal{F}_d h\left(\frac{\lambda_X d(x, y)}{2\pi}\right)$$

that can easily be made positive, and a smaller remainder. These combined techniques have been used already in several occasions by different authors [6, 7, 9]. In [7], though, a result as Theorem 1 is not clearly stated, and its proof appears somehow mixed up with the result that the authors were actually proving, and for which they needed one such kernel. In fact, essentially all the proofs of the theorems that we state here are already contained in [7]. In [9], a vague statement is given, and for the proof the reader is referred to [6, 18]. In [6] one can find a result, Theorem 2.3, which could be considered as one step of the proof of our result and that here corresponds somehow to our Theorem 4. Our intent here is to give this result in the simplest and most transparent possible form, with an explicit control of the remainder, so that other authors can use it even if they do not master all the technicalities involved in the proof, like the Hadamard construction of the parametrix of the wave equation, that we discuss in Sect. 2. In Sect. 3 we present all the steps needed to prove Theorem 1, and in the final Sect. 4 we show how one can apply Theorem 1 to give a direct proof of the main result of [7].

2 Hadamard construction of the parametrix of the wave equation

Let $\cos(\sqrt{\Delta}t)$ be the operator that associates to any function $f \in \mathcal{D}(\mathcal{M})$ (smooth functions on $\mathcal{M}$), the solution $u \in \mathcal{D}'(\mathcal{M})$ (distributions on $\mathcal{M}$) to the wave problem

$$\begin{cases} (\partial_t^2 + \Delta)u(t, x) = 0 & (t, x) \in \mathbb{R} \times \mathcal{M} \\ u(0, x) = f(x) & x \in \mathcal{M} \\ \partial_t u(0, x) = 0 & x \in \mathcal{M}\,. \end{cases}$$

It is easy to see that

$$(\cos(\sqrt{\Delta}t) f)(x) = \sum_{m=0}^{+\infty} \cos(t\lambda_m) \widehat{f}(\lambda_m)\varphi_m(x).$$

Notice in particular that since $\widehat{f}(\lambda_m)$ decays rapidly and $\|\varphi_m\|_\infty$ has polynomial growth, $(\cos(\sqrt{\Delta}t) f)(x)$ is in fact a smooth function, and as a distribution acts on smooth functions by integration

$$\begin{aligned}\langle\cos(\sqrt{\Delta}t) f, g\rangle_{\mathcal{D}'(\mathcal{M})} &= \int_{\mathcal{M}} \left(\sum_{m=0}^{+\infty} \cos(t\lambda_m) \widehat{f}(\lambda_m)\varphi_m(x)\right) g(x)dx \\ &= \sum_{m=0}^{+\infty} \cos(t\lambda_m) \widehat{f}(\lambda_m) \int_{\mathcal{M}} \varphi_m(x)g(x)dx.\end{aligned}$$

Observe now that for every $f, g \in \mathcal{D}(\mathcal{M})$, the function $t \mapsto \langle \cos(\sqrt{\Delta}t) f, g\rangle_{\mathcal{D}'(\mathcal{M})}$ is bounded and continuous, and this implies that it can be seen as a tempered distribution in $\mathcal{S}'(\mathbb{R})$. It obviously acts on smooth, rapidly decaying functions $h \in \mathcal{S}(\mathbb{R})$ by integration

$$\langle\langle \cos(\sqrt{\Delta}t) f, g\rangle_{\mathcal{D}'(\mathcal{M})}, h\rangle_{\mathcal{S}'(\mathbb{R})} = \int_{\mathbb{R}} \langle \cos(\sqrt{\Delta}t) f, g\rangle_{\mathcal{D}'(\mathcal{M})} h(t)dt.$$

In particular, notice that by the above formula, $\cos(\sqrt{\Delta}\cdot)$ can be seen as a (tempered) distribution on $\mathcal{M} \times \mathcal{M} \times \mathbb{R}$.

The following asymptotic expansion of the solution of the above mentioned wave problem is due to Hadamard, and its principal term is known as Hadamard parametrix.

Theorem 2 (See [15,Theorem 3.1.5]) *Given a d-dimensional Riemannian manifold $(\mathcal{M}, g)$, there exists $\varepsilon > 0$ and functions $\alpha_\nu \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M})$, so that if $Q > d + 3$ then for every $f \in \mathcal{D}(\mathcal{M})$*

$$\begin{aligned}\left(\cos(t\sqrt{\Delta}) f\right)(x) &= \int_{\mathcal{M}} \sum_{\nu=0}^{Q} \alpha_\nu(x, y)\partial_t(E_\nu - \check{E}_\nu)(t, d(x, y)) f(y)dy \\ &\quad + \int_{\mathcal{M}} R_Q(t, x, y) f(y)dy \qquad (2)\end{aligned}$$

where $R_Q \in \mathcal{C}^{Q-d-3}([-\varepsilon, \varepsilon] \times \mathcal{M} \times \mathcal{M})$ and

$$\left|\partial^\beta_{t,x,y} R_Q(t, x, y)\right| \le C\, |t|^{2Q+2-d-|\beta|}\,.$$

Furthermore $\alpha_0(x, y) \ge 0$ in $\mathcal{M} \times \mathcal{M}$, and $\alpha_0(x, x) = 1$.

Here we only want to recall that E_ν is a homogeneous distribution of degree $2\nu - d + 1$ supported on the forward light cone $\{(t, x) \in \mathbb{R}^{1+d} : t \ge 0, t^2 \ge |x|^2\}$, radial in x, and defined by

$$E_\nu(t, x) = \lim_{\varepsilon \to 0+} \nu!(2\pi)^{-d-1} \iint_{\mathbb{R}^{1+d}} e^{i(x\cdot\xi + tr)}(|\xi|^2 - (\tau - i\varepsilon)^2)^{-\nu-1} d\xi d\tau.$$

The distribution $\check{E}_\nu$ is the reflection of E_ν about the origin of $\mathbb{R}^{1+d}$. The distribution E_0 is the fundamental solution of the wave operator supported on the forward light cone, and for all $\nu = 1, 2, \dots,$ the distributions E_ν are defined in such a way that $(\partial_t^2 + \Delta)E_\nu = \nu E_{\nu-1}$. With a small abuse of notation, we shall sometimes write $\partial_t(E_\nu - \check{E}_\nu)(t, |z|)$ instead of $\partial_t(E_\nu - \check{E}_\nu)(t, z)$. Formula (2) has then to be interpreted in local coordinates (more precisely, normal coordinates in a coordinate patch centered at $x \in \mathcal{M}$), whenever the time t is smaller than the injectivity radius.

Finally, the distributions $\partial_t(E_\nu - \check{E}_\nu)(t, z)$ can be regarded as continuous radial functions of z with values in $\mathcal{S}'(\mathbb{R})$. Furthermore, when $0 \le \nu < d/2$, for every $z \in \mathbb{R}^d$ the inverse cosine transform $\mathcal{C}^{-1}\left(\partial_\cdot(E_\nu - \check{E}_\nu)(\cdot, z)\right)$ is a function and for all $t \in \mathbb{R}$

$$\mathcal{C}^{-1}\left(\partial_\cdot(E_\nu - \check{E}_\nu)(\cdot, z)\right)(t) = \pi^{-d/2} 2^{-\nu-d/2-1} |t|^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(t|z|)}{(t|z|)^{-\nu+d/2-1}}, \qquad (3)$$

whereas when $d/2 \leq \nu$, for every $z \in \mathbb{R}^d$ the distribution itself $\partial_t(E_\nu - \check{E}_\nu)(t, z)$ can be identified with the locally integrable function

$$t \mapsto C_\nu |t| (t^2 - |z|^2)_+^{\nu-1+(1-d)/2},$$

with $C_\nu = 2^{-2\nu}\pi^{(1-d)/2}\left(1 + \frac{1-d}{2}\right)$.

3 Analysis of the kernel

Let us now take an even continuous function $H \in L^1(\mathbb{R})$, and assume that its cosine transform $\mathcal{C}H(s)$ is compactly supported. Then for every $s \in \mathbb{R}$,

$$H(s) = \int_{\mathbb{R}} \mathcal{C}^{-1}H(t)\cos(st)dt.$$

Consider the operator $\mathcal{H}$ that maps any function $f \in \mathcal{D}(\mathcal{M})$ to the distribution $\mathcal{H}f = \sum_{m=0}^{+\infty} H(\lambda_m)\widehat{f}(\lambda_m)\varphi_m$. Since $\mathcal{H}f$ is in fact a smooth function, it acts on $\mathcal{D}(\mathcal{M})$ by integration,

$$\begin{aligned}
\langle \mathcal{H}f, g\rangle_{\mathcal{D}'(\mathcal{M})} &= \int_{\mathcal{M}} \left(\sum_{m=0}^{+\infty} H(\lambda_m)\widehat{f}(\lambda_m)\varphi_m(x)\right) g(x)dx \\
&= \sum_{m=0}^{+\infty} H(\lambda_m)\widehat{f}(\lambda_m) \int_{\mathcal{M}} g(x)\varphi_m(x)dx \\
&= \sum_{m=0}^{+\infty} \left(\int_{\mathbb{R}} \mathcal{C}^{-1}H(t)\cos(\lambda_m t)dt\right) \widehat{f}(\lambda_m) \int_{\mathcal{M}} g(x)\varphi_m(x)dx \\
&= \int_{\mathbb{R}} \mathcal{C}^{-1}H(t) \sum_{m=0}^{+\infty} \cos(\lambda_m t)\widehat{f}(\lambda_m) \left(\int_{\mathcal{M}} g(x)\varphi_m(x)dx\right) dt \\
&= \int_{\mathbb{R}} \mathcal{C}^{-1}H(t)\langle \cos(\sqrt{\Delta}t)f, g\rangle_{\mathcal{D}'(\mathcal{M})}dt. \qquad (4)
\end{aligned}$$

Theorem 3 *Let $H \in L^1(\mathbb{R})$ be even and continuous, and assume that its cosine transform $\mathcal{C}H(s) = \int_{\mathbb{R}} H(t)\cos(st)dt$ is supported in $[-\varepsilon, \varepsilon]$. Let $\mathcal{H}$ be the operator that maps any function $f \in \mathcal{D}(\mathcal{M})$ to the distribution $\mathcal{H}f = \sum_{m=0}^{+\infty} H(\lambda_m)\widehat{f}(\lambda_m)\varphi_m$. Then, for any $f, g \in \mathcal{D}(\mathcal{M})$,*

$$\begin{aligned}
&\langle \mathcal{H}f, g\rangle_{\mathcal{D}'(\mathcal{M})} \\
&= \sum_{0\leq\nu<d/2} \int_{\mathcal{M}}\int_{\mathcal{M}} \alpha_\nu(x, y) \\
&\quad \times \int_0^{+\infty} H(t)\pi^{-d/2}2^{-\nu-d/2}t^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(td(x, y))}{(td(x, y))^{-\nu+d/2-1}} dtg(x)dxf(y)dy \\
&\quad + \sum_{d/2\leq\nu\leq Q} C_\nu \int_{\mathcal{M}}\int_{\mathcal{M}} \alpha_\nu(x, y) \times \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t)|t| \left(t^2 - d(x, y)^2\right)_+^{\nu-1+(1-d)/2} \\
&\quad \times g(x)dxf(y)dy + \int_{\mathcal{M}}\int_{\mathcal{M}}\int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t)R_Q(t, x, y)dtg(x)f(y)dxdy.
\end{aligned}$$

Proof It follows from (4) and Theorem 2, that if $\mathcal{C}^{-1}H$ is supported in $[-\varepsilon, \varepsilon]$ then

$$\begin{aligned}
\langle \mathcal{H} f, g\rangle_{\mathcal{D}'(\mathcal{M})} &= \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t)\langle \cos(\sqrt{\Delta}t) f, g\rangle_{\mathcal{D}'(\mathcal{M})} dt \\
&= \sum_{\nu=0}^{Q} \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t) \left\langle \int_{\mathcal{M}} \alpha_\nu(\cdot, y)\partial_t(E_\nu - \check{E}_\nu)(t, d(\cdot, y)) f(y)dy, g\right\rangle_{\mathcal{D}'(\mathcal{M})} dt \\
&\quad + \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t) \left\langle \int_{\mathcal{M}} R_Q(t, \cdot, y) f(y)dy, g\right\rangle_{\mathcal{D}'(\mathcal{M})} dt \\
&= \sum_{\nu=0}^{Q} \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t) \int_{\mathcal{M}} \langle \alpha_\nu(\cdot, y)\partial_t(E_\nu - \check{E}_\nu)(t, d(\cdot, y)), g\rangle_{\mathcal{D}'(\mathcal{M})} f(y)dydt \\
&\quad + \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t) \int_{\mathcal{M}}\int_{\mathcal{M}} R_Q(t, x, y)g(x)dxf(y)dydt \\
&= \sum_{\nu=0}^{Q} \int_{\mathcal{M}} \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t) \langle \alpha_\nu(\cdot, y)\partial_t(E_\nu - \check{E}_\nu)(t, d(\cdot, y)), g\rangle_{\mathcal{D}'(\mathcal{M})} f(y)dtdy \\
&\quad + \int_{\mathcal{M}}\int_{\mathcal{M}}\int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t)R_Q(t, x, y)dtg(x)f(y)dxdy \\
&= \sum_{\nu=0}^{Q} \int_{\mathcal{M}} \langle \alpha_\nu(\cdot, y) \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t)\partial_t(E_\nu - \check{E}_\nu)(t, d(\cdot, y))dt, g\rangle_{\mathcal{D}'(\mathcal{M})} f(y)dy \\
&\quad + \int_{\mathcal{M}}\int_{\mathcal{M}}\int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t)R_Q(t, x, y)dtg(x)f(y)dxdy.
\end{aligned}$$

Let us now look closely to each of the terms of the above sum. If $0 \le \nu < d/2$, then

$$\begin{aligned}
&\int_{\mathcal{M}} \left\langle \alpha_\nu(\cdot, y) \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1}H(t)\partial_t(E_\nu - \check{E}_\nu)(t, d(\cdot, y))dt, g\right\rangle_{\mathcal{D}'(\mathcal{M})} f(y)dy \\
&= \int_{\mathcal{M}} \left\langle \alpha_\nu(\cdot, y) \int_{-\infty}^{+\infty} H(t)\mathcal{C}^{-1} \left(\partial_t(E_\nu - \check{E}_\nu)(t, d(\cdot, y))\right) dt, g\right\rangle_{\mathcal{D}'(\mathcal{M})} f(y)dy \\
&= \int_{\mathcal{M}} \left\langle \alpha_\nu(\cdot, y) \int_{-\infty}^{+\infty} H(t)\pi^{-d/2}2^{-\nu-d/2-1} |t|^{-2\nu-1+d} \right. \\
&\quad \left. \times \frac{J_{-\nu+d/2-1}(td(\cdot, y))}{(td(\cdot, y))^{-\nu+d/2-1}} dt,\ g\right\rangle_{\mathcal{D}'(\mathcal{M})} f(y)dy
\end{aligned}$$

and since now for any y, the distribution acting on g is a locally integrable function, it acts by integration, thus obtaining

$$\begin{aligned}
&\int_{\mathcal{M}}\int_{\mathcal{M}} \alpha_\nu(x, y) \int_{0}^{+\infty} H(t)\pi^{-d/2}2^{-\nu-d/2}t^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(td(x, y))}{(td(x, y))^{-\nu+d/2-1}} dt \\
&\times g(x)dxf(y)dy.
\end{aligned}$$

If instead $\nu \geq d/2$, then

$$\int_{\mathcal{M}} \left\langle \alpha_\nu(\cdot, y) \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1} H(t) \partial_t (E_\nu - \check{E}_\nu)(t, d(\cdot, y)) dt, g \right\rangle_{\mathcal{D}'(\mathcal{M})} f(y) dy$$

$$= C_\nu \int_{\mathcal{M}} \left\langle \alpha_\nu(\cdot, y) \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1} H(t) |t| \left(t^2 - d(\cdot, y)^2\right)_+^{\nu-1+(1-d)/2}, g \right\rangle_{\mathcal{D}'(\mathcal{M})} f(y) dy.$$

Again, for any y, the distribution acting on g is a locally integrable function, so that the above term equals

$$C_\nu \int_{\mathcal{M}} \int_{\mathcal{M}} \alpha_\nu(x, y) \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1} H(t) |t| \left(t^2 - d(x, y)^2\right)_+^{\nu-1+(1-d)/2} g(x) dx f(y) dy.$$

□

The formula in Theorem 3 also gives an explicit expression of the Schwartz kernel of $\mathcal{H}$. In particular, it is a function.

Theorem 4 *Let $H \in L^1(\mathbb{R})$ be even and continuous, and assume that its cosine transform $\mathcal{C}H(s) = \int_{\mathbb{R}} H(t) \cos(st) dt$ is supported in $[-\varepsilon, \varepsilon]$. Then*

$$\sum_{m=0}^{+\infty} H(\lambda_m) \varphi_m(x) \overline{\varphi_m(y)}$$

$$= \sum_{0 \leq \nu < d/2} \alpha_\nu(x, y) \int_0^{+\infty} H(t) \pi^{-d/2} 2^{-\nu-d/2} t^{-2\nu-1+d} \frac{J_{-\nu+d/2-1}(t d(x, y))}{(t d(x, y))^{-\nu+d/2-1}} dt$$

$$+ \sum_{d/2 \leq \nu \leq Q} C_\nu \alpha_\nu(x, y) \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1} H(t) |t| \left(t^2 - d(x, y)^2\right)_+^{\nu-1+(1-d)/2} dt$$

$$+ \int_{-\varepsilon}^{\varepsilon} \mathcal{C}^{-1} H(t) R_Q(t, x, y) dt.$$

Proof Since

$$\langle \mathcal{H} f, g \rangle_{\mathcal{D}'(\mathcal{M})} = \int_{\mathcal{M}} \int_{\mathcal{M}} \sum_{m=0}^{+\infty} H(\lambda_m) \varphi_m(x) \overline{\varphi_m(y)} g(x) f(y) dx dy,$$

it follows that the kernel can also be written as the function

$$\sum_{m=0}^{+\infty} H(\lambda_m) \varphi_m(x) \overline{\varphi_m(y)}.$$

By Theorem 3, one has the thesis. □

For smooth radial integrable functions on $\mathbb{R}^d$, $f(x) = f_0(|x|)$, the Fourier transform

$$\mathcal{F}_d f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$$

reduces essentially to the Hankel transform, given by (see [16, Chapter 4, Theorem 3.3])

$$\mathcal{F}_d f(\xi) = 2\pi |\xi|^{-\frac{d-2}{2}} \int_0^{\infty} f_0(s) J_{\frac{d-2}{2}}(2\pi |\xi| s) s^{\frac{d}{2}} ds. \tag{5}$$

As we mentioned before, with an abuse of notation, we will identify the function f with its radial profile f_0 and write $\mathcal{F}_d f(|\xi|)$ instead of $\mathcal{F}_d f(\xi)$. One can easily show that if f is an even smooth function on $\mathbb{R}$, then for any $t \in \mathbb{R}$,

$$\mathcal{C}^{-1} f(t) = \frac{1}{2\pi}\mathcal{C} f(t) = \frac{1}{2\pi}\mathcal{F}_1 f\left(\frac{|t|}{2\pi}\right).$$

With this notation, our kernel can be rewritten as

$$\begin{aligned}
&\sum_{m=0}^{+\infty} H(\lambda_m)\varphi_m(x)\overline{\varphi_m(y)} \\
&\quad = \sum_{0\le\nu<d/2} \alpha_\nu(x,y)\frac{\pi^\nu}{(2\pi)^d}\mathcal{F}_{d-2\nu} H\left(\frac{d(x,y)}{2\pi}\right) \\
&\qquad + \frac{1}{\pi}\sum_{d/2\le\nu\le Q} C_\nu\alpha_\nu(x,y)\int_{d(x,y)}^{+\infty}\mathcal{F}_1 H\left(\frac{t}{2\pi}\right) t\left(t^2-d(x,y)^2\right)^{\nu-1+(1-d)/2} dt \\
&\qquad + \frac{1}{\pi}\int_0^{+\infty}\mathcal{F}_1 H\left(\frac{t}{2\pi}\right) R_Q(t,x,y)dt. \qquad (6)
\end{aligned}$$

We have the following

Theorem 5 *Let $H \in L^1(\mathbb{R})$ be even and continuous, and assume that its Fourier transform $\mathcal{F}_d H$ is supported in $[0, \varepsilon/(2\pi))$. Then*

$$\begin{aligned}
&\left|\sum_{m=0}^{+\infty} H(\lambda_m)\varphi_m(x)\overline{\varphi_m(y)} - \alpha_0(x,y)\frac{1}{(2\pi)^d}\mathcal{F}_d H\left(\frac{d(x,y)}{2\pi}\right)\right| \\
&\quad \le C\sum_{1\le\nu\le Q}\int_{d(x,y)}^{+\infty} r^{2\nu-1}\left|\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\right| dr + C\int_0^{+\infty} r^{2Q+1}\left|\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\right| dr.
\end{aligned}$$

Proof We want to express the formula (6) in terms of $\mathcal{F}_d H$ rather then $\mathcal{F}_1 H$ or $\mathcal{F}_{d-2\nu} H$. This can be done by means of the following transplantation result (see [17, eq. (3.9)]): for $d > d' \ge 1$,

$$\mathcal{F}_{d'} H(s) = c_{d,d'}\int_s^{+\infty}(r^2-s^2)^{(d-d')/2-1} r\mathcal{F}_d H(r)dr. \qquad (7)$$

Thus, if $\mathcal{F}_d H$ is supported in $[0, \varepsilon/(2\pi)]$, then $\mathcal{F}_1 H$ is supported in $[0, \varepsilon/(2\pi)]$ too, and $\mathcal{C}H$ is supported in $[-\varepsilon, \varepsilon]$, as required. Also, if $\mathcal{F}_d H$ is nonnegative, so is $\mathcal{F}_1 H$.

Let us now assume $1 \le \nu < d/2$. Then

$$\begin{aligned}
\left|\mathcal{F}_{d-2\nu} H\left(\frac{d(x,y)}{2\pi}\right)\right| &= c_{d,d-2\nu}\left|\int_{d(x,y)/(2\pi)}^{+\infty}\left(r^2-\frac{d(x,y)^2}{(2\pi)^2}\right)^{\nu-1} r\mathcal{F}_d H(r)dr\right| \\
&= \frac{c_{d,d-2\nu}}{(2\pi)^{2\nu}}\left|\int_{d(x,y)}^{+\infty}\left(r^2-d(x,y)^2\right)^{\nu-1} r\mathcal{F}_d H\left(\frac{r}{2\pi}\right) dr\right| \\
&\le C\int_{d(x,y)}^{+\infty} r^{2\nu-1}\left|\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\right| dr.
\end{aligned}$$

Similarly, for $d \ge 2$

$$\begin{aligned}
&\int_{d(x,y)}^{+\infty} \mathcal{F}_1 H\left(\frac{t}{2\pi}\right) t(t^2-d(x,y)^2)^{\nu-1+(1-d)/2}dt \\
&= c_d \int_{d(x,y)}^{+\infty}\int_{t/(2\pi)}^{+\infty} \left(r^2-\frac{t^2}{(2\pi)^2}\right)^{(d-3)/2} r\mathcal{F}_d H(r)dr t(t^2-d(x,y)^2)^{\nu-1+(1-d)/2}dt \\
&= \frac{c_d}{(2\pi)^{d-1}}\int_{d(x,y)}^{+\infty} r\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\int_{d(x,y)}^{r}(r^2-t^2)^{(d-3)/2}t(t^2-d(x,y)^2)^{\nu-1+(1-d)/2}dtdr.
\end{aligned}$$

It can be proved easily that for some constant γ depending on $d \geq 2$ and on ν between $d/2$ and Q, for all $r \geq d(x,y)$

$$\int_{d(x,y)}^{r}(r^2-t^2)^{(d-3)/2}t(t^2-d(x,y)^2)^{\nu-1+(1-d)/2}dt \leq \gamma(r^2-d(x,y)^2)^{\nu-1}.$$

It follows that for all $d \geq 1$ and for all $d/2 \leq \nu \leq Q$,

$$\begin{aligned}
&\left|\frac{1}{\pi}\sum_{d/2\leq\nu\leq Q} C_\nu\alpha_\nu(x,y)\int_{d(x,y)}^{+\infty}\mathcal{F}_1 H\left(\frac{t}{2\pi}\right)t(t^2-d(x,y)^2)^{\nu-1+(1-d)/2}dt\right| \\
&\leq C\sum_{d/2\leq\nu\leq Q}\int_{d(x,y)}^{+\infty} r\left|\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\right|(r^2-d(x,y)^2)^{\nu-1}dr \\
&\leq C\sum_{d/2\leq\nu\leq Q}\int_{d(x,y)}^{+\infty} r^{2\nu-1}\left|\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\right|dr.
\end{aligned}$$

The same strategy can be used to estimate the last term of the kernel, the one involving the remainder R_Q. Indeed,

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{+\infty}\mathcal{F}_1 H\left(\frac{t}{2\pi}\right)R_Q(t,x,y)dt \\
&= \frac{c_d}{\pi}\int_0^{+\infty}\int_{t/2\pi}^{+\infty}(r^2-(t/2\pi)^2)^{(d-3)/2} r\mathcal{F}_d H(r)dr R_Q(t,x,y)dt \\
&= \frac{c_d}{\pi(2\pi)^{d-1}}\int_0^{+\infty} r\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\int_0^{r}(r^2-t^2)^{(d-3)/2}R_Q(t,x,y)dtdr.
\end{aligned}$$

It follows that

$$\left|\frac{1}{\pi}\int_0^{+\infty}\mathcal{F}_1 H\left(\frac{t}{2\pi}\right)R_Q(t,x,y)dt\right| \leq C\int_0^{+\infty}\left|\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\right| r^{2Q+1}dr.$$

□

Let us now fix a more specific choice for the function H. Let h be an integrable radial function on $\mathbb{R}^d$ and let η be a continuous integrable radial function on $\mathbb{R}^d$ with Fourier transform compactly supported in the ball centered at the origin and with radius $\varepsilon/(2\pi)$. Let us fix a nonnegative integer X, and define

$$H(|z|) = h\left(\frac{\cdot}{\lambda_X}\right) * \eta(z),$$

where the convolution is intended in $\mathbb{R}^d$. Observe that with the above choices, H is continuous, it belongs to $L^1(\mathbb{R})$ and

$$\mathcal{F}_d H(t) = \lambda_X^d \mathcal{F}_d h(\lambda_X t) \mathcal{F}_d \eta(t)$$

is supported in $[0, \varepsilon/(2\pi)]$, so that the previous theorem can be applied.

Theorem 6 *Let h be an integrable radial function on $\mathbb{R}^d$ such that for some $G > 0$, and for some positive constant C,*

$$|\mathcal{F}_d h(t)| \leq C \frac{1}{(1+t)^{2G}}.$$

Let η be a continuous integrable radial function on $\mathbb{R}^d$ with Fourier transform $\mathcal{F}_d \eta$ compactly supported in the ball centered at the origin and with radius $\varepsilon/(2\pi)$. Let us fix a nonnegative integer X, and define

$$H(|z|) = h\left(\frac{\cdot}{\lambda_X}\right) * \eta(z),$$

where the convolution is intended in $\mathbb{R}^d$. Then

$$\left| \sum_{m=0}^{+\infty} H(\lambda_m) \varphi_m(x) \overline{\varphi_m(y)} - \alpha_0(x, y) \frac{1}{(2\pi)^d} \mathcal{F}_d H\left(\frac{d(x, y)}{2\pi}\right) \right|$$

$$\leq C \begin{cases} \lambda_X^{d-2G} & \textit{if } 0 < G < 1, \\ \dfrac{\lambda_X^{d-2}}{(1+\lambda_X d(x, y))^{2G-2}} & \textit{if } G > 1,\ G \textit{ non integer,} \\ \dfrac{\lambda_X^{d-2}}{(1+\lambda_X d(x, y))^{2G-2}} + \lambda_X^{d-2G} \log \lambda_X & \textit{if } G \geq 1,\ G \textit{ integer.} \end{cases}$$

Proof Let Q be an integer greater than $d+1$ and than $G-1$. Then we may apply Theorem 5. For all $1 \leq \nu \leq Q$,

$$\begin{aligned} &\int_{d(x,y)}^{+\infty} r^{2\nu-1} \left| \mathcal{F}_d H\left(\frac{r}{2\pi}\right) \right| dr \\ &\leq C \lambda_X^d \|\mathcal{F}_d \eta\|_\infty \int_{d(x,y)}^{\varepsilon} \frac{1}{(1+\lambda_X r/(2\pi))^{2G}} r^{2\nu-1} dr \\ &= C (2\pi)^{2\nu} \lambda_X^{d-2\nu} \|\mathcal{F}_d \eta\|_\infty \int_{\lambda_X d(x,y)/(2\pi)}^{\lambda_X \varepsilon/2\pi} \frac{r^{2\nu-1}}{(1+r)^{2G}} dr \\ &\leq C \lambda_X^{d-2\nu} \int_{\lambda_X d(x,y)/(2\pi)}^{\lambda_X \varepsilon/2\pi} \frac{1}{(1+r)^{2G-2\nu+1}} dr \\ &\leq C \begin{cases} \dfrac{\lambda_X^{d-2\nu}}{(1+\lambda_X d(x, y))^{2G-2\nu}} & \text{if } \nu < G, \\ \lambda_X^{d-2G} & \text{if } \nu > G, \\ \lambda_X^{d-2G} \log(\lambda_X) & \text{if } \nu = G \text{ (with } G \text{ integer).} \end{cases} \end{aligned}$$

On the other hand,

$$\int_0^{+\infty} r^{2Q+1}\left|\mathcal{F}_d H\left(\frac{r}{2\pi}\right)\right| dr \le C\lambda_X^d \|\mathcal{F}_d\eta\|_\infty \int_0^{\varepsilon} \frac{1}{(1+\lambda_X r/(2\pi))^{2G}} r^{2Q+1} dr$$
$$= C(2\pi)^{2Q+2}\lambda_X^{d-2Q-2}\|\mathcal{F}_d\eta\|_\infty \int_0^{\lambda_X\varepsilon/2\pi} \frac{r^{2Q+1}}{(1+r)^{2G}} dr$$
$$\le C\lambda_X^{d-2G}.$$

It now follows from Theorem 5 that

$$\left|\sum_{m=0}^{+\infty} H(\lambda_m)\varphi_m(x)\overline{\varphi_m(y)} - \alpha_0(x,y)\frac{1}{(2\pi)^d}\mathcal{F}_d H\left(\frac{d(x,y)}{2\pi}\right)\right|$$
$$\le C\sum_{1\le\nu<G} \frac{\lambda_X^{d-2\nu}}{(1+\lambda_X d(x,y))^{2G-2\nu}} + C\sum_{\nu=G}\lambda_X^{d-2G}\log(\lambda_X) + C\sum_{G<\nu\le Q}\lambda_X^{d-2G},$$

and the thesis follows. □

We are now ready for the final step. The kernel we have found so far is not what we wanted, since H is not supported in $[0,\lambda_X]$. Therefore we need some further assumptions on h and η.

Theorem 7 *Let h be an integrable radial function on $\mathbb{R}^d$ such that for some $G>(d+2)/2$, and for some positive constant C,*

$$|\mathcal{F}_d h(t)| \le C\frac{1}{(1+t)^{2G}},$$

and assume that h is compactly supported in the ball centered at the origin and with radius 1. Let η be a continuous integrable radial function on $\mathbb{R}^d$ with Fourier transform $\mathcal{F}_d\eta$ compactly supported in the ball centered at the origin with radius $\varepsilon/(2\pi)$ and that equals 1 in the ball centered at the origin with radius $\varepsilon/(4\pi)$. Let $I(z)$ be defined by

$$I(z) = h(z/\lambda_X) - H(z) = \int_{\mathbb{R}^d}\left[h\left(\frac{z}{\lambda_X}\right) - h\left(\frac{z-y}{\lambda_X}\right)\right]\eta(y)dy.$$

Then

$$\left|\sum_{m=0}^{+\infty} I(\lambda_m)\varphi_m(x)\overline{\varphi_m(y)}\right| \le c\lambda_X^{-\lfloor 2G\rfloor+3d}.$$

Proof Let us first give an estimate on the function $I(z)$. Since $\eta(y)$ has rapid decay at infinity and $h(z)$ is supported in $\{|z|\le 1\}$, if $|z|\ge 2\lambda_X$ we have

$$|I(z)| \le \int_{\mathbb{R}^d}\left|h\left(\frac{z-y}{\lambda_X}\right)\eta(y)\right| dy \le \int_{\{|z-y|\le\lambda_X\}}\left|h\left(\frac{z-y}{\lambda_X}\right)\eta(y)\right| dy$$
$$\le c\int_{\{|y|\ge|z|-\lambda_X\}} |\eta(y)|dy \le C(1+|z|-\lambda_X)^{-M},$$

for some M as large as needed. Assume $|z|<2\lambda_X$. By [16, Theorem 1.7], the decay of the Fourier transform of h implies that $h\in\mathcal{C}^{\lfloor 2G\rfloor-d-1}(\mathbb{R}^d)$. By Taylor's theorem with integral reminder, setting $K=\lfloor 2G\rfloor-d-1\ge 1$, we can write

$$h\left(\frac{z}{\lambda_X}-\frac{y}{\lambda_X}\right)=h\left(\frac{z}{\lambda_X}\right)+\sum_{1\leq|\alpha|\leq K-1}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}h}{\partial x^\alpha}\left(\frac{z}{\lambda_X}\right)\left(-\frac{y}{\lambda_X}\right)^\alpha$$
$$+\sum_{|\alpha|=K}\frac{K}{\alpha!}\left(-\frac{y}{\lambda_X}\right)^\alpha\int_0^1(1-t)^{K-1}\frac{\partial^{|\alpha|}h}{\partial x^\alpha}\left(\frac{z}{\lambda_X}-t\frac{y}{\lambda_X}\right)dt$$

so that

$$h\left(\frac{z}{\lambda_X}-\frac{y}{\lambda_X}\right)=h\left(\frac{z}{\lambda_X}\right)+\sum_{1\leq|\alpha|\leq K-1}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}h}{\partial x^\alpha}\left(\frac{z}{\lambda_X}\right)\left(-\frac{y}{\lambda_X}\right)^\alpha+O\left(\left|-\frac{y}{\lambda_X}\right|^K\right).$$

It follows that

$$I(z)=\int_{\mathbb{R}^d}\left[h\left(\frac{z}{\lambda_X}\right)-h\left(\frac{z-y}{\lambda_X}\right)\right]\eta(y)\,dy$$
$$=-\sum_{1\leq|\alpha|\leq K-1}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}h}{\partial x^\alpha}\left(\frac{z}{\lambda_X}\right)\int_{\mathbb{R}^d}\left(-\frac{y}{\lambda_X}\right)^\alpha\eta(y)\,dy+\int_{\mathbb{R}^d}O\left(\frac{|y|^K}{\lambda_X^K}\right)\eta(y)\,dy$$

and since

$$\int_{\mathbb{R}^d}y^\alpha\eta(y)dy=\mathcal{F}_d(y^\alpha\eta(y))(0)=(-2\pi i)^{-|\alpha|}\frac{\partial^{|\alpha|}\mathcal{F}_d\eta}{\partial\xi^\alpha}(0)=0$$

we obtain

$$|I(z)|\leq c\lambda_X^{-K}.$$

The kernel

$$\sum_{m=0}^{+\infty}I(\lambda_m)\varphi_m(x)\overline{\varphi_m(y)}$$

can be estimated uniformly by means of Weyl's estimates on the eigenfunctions, $\|\varphi_m\|_\infty\leq c(1+\lambda_m)^{(d-1)/2}$. Indeed, if $M>2d-1$,

$$\left|\sum_{m=0}^{+\infty}I(\lambda_m)\varphi_m(x)\overline{\varphi_m(y)}\right|$$
$$\leq c\lambda_X^{-K}\sum_{\lambda_m\leq 2\lambda_X}(1+\lambda_m)^{d-1}+c\sum_{\lambda_m\geq 2\lambda_X}(1+\lambda_m-\lambda_X)^{-M}(1+\lambda_m)^{d-1}$$
$$\leq c\lambda_X^{-K+2d-1}+c\sum_{\lambda_m\geq 2\lambda_X}\lambda_m^{-M+d-1}$$
$$\leq c\lambda_X^{-K+2d-1}+c\sum_{k=1}^{+\infty}\sum_{2^k\lambda_X\leq\lambda_m\leq 2^{k+1}\lambda_X}\lambda_m^{-M+d-1}$$
$$\leq c\lambda_X^{-K+2d-1}+c\sum_{k=1}^{+\infty}\lambda_X^d2^{dk}(\lambda_X2^k)^{-M+d-1}$$
$$\leq c\lambda_X^{-K+2d-1}+c\lambda_X^{-M+2d-1}\sum_{k=1}^{+\infty}\left(2^{-M+2d-1}\right)^k\leq c\lambda_X^{-K+2d-1}+c\lambda_X^{-M+2d-1}.$$

Since we can take $M\geq K$, we have therefore proved the thesis. □

We are ready to state our final result.

Theorem 8 *Let h be an integrable radial function on $\mathbb{R}^d$ such that for some $G > (d+2)/2$, and for some positive constant C,*

$$|\mathcal{F}_d h(t)| \leq C \frac{1}{(1+t)^{2G}},$$

and assume that h is compactly supported in the ball centered at the origin and with radius 1. Then

$$\begin{aligned}&\left|\sum_{m=0}^{X} h\left(\frac{\lambda_m}{\lambda_X}\right) \varphi_m(x)\overline{\varphi_m(y)} - \alpha_0(x,y)\frac{\lambda_X^d}{(2\pi)^d}\mathcal{F}_d h\left(\frac{\lambda_X d(x,y)}{2\pi}\right)\right| \\ &\quad \leq C\frac{\lambda_X^{d-2}}{(1+\lambda_X d(x,y))^{2G-2}} + C\lambda_X^{3d-\lfloor 2G\rfloor}.\end{aligned}$$

Proof It suffices to observe that

$$\begin{aligned}&\sum_{m=0}^{+\infty} h\left(\frac{\lambda_m}{\lambda_X}\right) \varphi_m(x)\overline{\varphi_m(y)} - \alpha_0(x,y)\frac{\lambda_X^d}{(2\pi)^d}\mathcal{F}_d h\left(\frac{\lambda_X d(x,y)}{2\pi}\right) \\ &\quad = \sum_{m=0}^{+\infty} I\left(\lambda_m\right) \varphi_m(x)\overline{\varphi_m(y)} \\ &\qquad + \sum_{m=0}^{+\infty} H\left(\lambda_m\right) \varphi_m(x)\overline{\varphi_m(y)} - \alpha_0(x,y)\frac{1}{(2\pi)^d}\mathcal{F}_d H\left(\frac{d(x,y)}{2\pi}\right) \\ &\qquad + \alpha_0(x,y)\frac{1}{(2\pi)^d}\mathcal{F}_d H\left(\frac{d(x,y)}{2\pi}\right) - \alpha_0(x,y)\frac{\lambda_X^d}{(2\pi)^d}\mathcal{F}_d h\left(\frac{\lambda_X d(x,y)}{2\pi}\right).\end{aligned}$$

The estimates of the first two terms follow from the previous theorems. Concerning the last term, since

$$\mathcal{F}_d H\left(\frac{d(x,y)}{2\pi}\right) = \lambda_X^d \mathcal{F}_d h\left(\frac{\lambda_X d(x,y)}{2\pi}\right)\mathcal{F}_d \eta\left(\frac{d(x,y)}{2\pi}\right)$$

and since $\mathcal{F}_d\eta(t)$ equals 1 for $t \leq \varepsilon/4\pi$, it follows that

$$\alpha_0(x,y)\frac{1}{(2\pi)^d}\mathcal{F}_d H\left(\frac{d(x,y)}{2\pi}\right) - \alpha_0(x,y)\frac{\lambda_X^d}{(2\pi)^d}\mathcal{F}_d h\left(\frac{\lambda_X d(x,y)}{2\pi}\right)$$

equals zero when $d(x,y) \leq \varepsilon/2$, and when $d(x,y) \geq \varepsilon/2$ it is bounded in absolute value by

$$\|\alpha_0\|_\infty \frac{1}{(2\pi)^d}\lambda_X^d\left(\|\mathcal{F}_d\eta\|_\infty + 1\right)\frac{C}{(1+\lambda_X d(x,y))^{2G}} \leq C\lambda_X^{d-2G}.$$

□

Proof of Theorem 1. Since $G > d+1$, we can apply Theorem 8:

$$\begin{aligned}&\left|\sum_{m=0}^{X} h\left(\frac{\lambda_m}{\lambda_X}\right) \varphi_m(x)\overline{\varphi_m(y)} - \alpha_0(x,y)\frac{\lambda_X^d}{(2\pi)^d}\mathcal{F}_d h\left(\frac{\lambda_X d(x,y)}{2\pi}\right)\right| \\ &\quad \leq C\frac{\lambda_X^{d-2}}{(1+\lambda_X d(x,y))^{2G-2}} + C\lambda_X^{-\lfloor 2G\rfloor+3d} \leq C\frac{\lambda_X^{d-2}}{(1+\lambda_X d(x,y))^{\lfloor 2G\rfloor-2-2d}}.\end{aligned}$$

This proves point (i). As for point (ii), it suffices to set $h = \psi * \psi$ where

$$|\mathcal{F}_d \psi(\xi)| \leq C \frac{1}{(1+|\xi|)^G},$$

with ψ a nonnegative radial function, compactly supported in the ball centered at the origin and with radius $1/2$ and with $\|\psi\|_2 = 1$. □

4 An application

As we mentioned in the Introduction, an explicit expression of the kernel as a sum of a nonnegative term and a bounded remainder allows to simplify the original proof of the following theorem, a version of the Cassels–Montgomery inequality for compact manifolds recently proved in [7],

Theorem 9 *There exists a positive constant C such that for all integers N and X and for all finite sequences of N points in $\mathcal{M}$, $\{x(j)\}_{j=1}^N$, and positive weights $\{a_j\}_{j=1}^N$ we have*

$$\sum_{m=0}^{X} \left| \sum_{j=1}^{N} a_j \varphi_m (x(j)) \right|^2 \geq CX \sum_{j=1}^{N} a_j^2. \tag{8}$$

Proof It suffices to show the theorem for large X. Let $Y = \kappa X$, with κ a positive integer which will be chosen later. By [11, Theorem 2], the manifold $\mathcal{M}$ can be split into Y disjoint regions $\{\mathcal{R}_i\}_{i=1}^Y$ with measure $|\mathcal{R}_i| = 1/Y$ and such that each region contains a ball of radius $c_1 Y^{-1/d}$ and is contained in a ball of radius $c_2 Y^{-1/d}$, for appropriate values of c_1 and c_2 independent of Y. Call $\{\mathcal{B}_r\}_{r=1}^R$ the sequence of all the regions in $\{\mathcal{R}_i\}_{i=1}^Y$ which contain at least one of the points $x(j)$, K_r the cardinality of the set $\{j = 1, \ldots, N : x(j) \in \mathcal{B}_r\}$ and S_r the sum of the weights $\{a_j\}$ corresponding to points $x(j) \in \mathcal{B}_r$. Assume without loss of generality that

$$S_1 \geq S_2 \geq \cdots \geq S_R > 0.$$

Rename the sequence $\{x(j)\}_{j=1}^N$ as

$$\left\{x_{r,j}\right\}_{\substack{r=1,\ldots,R \\ j=1,\ldots,K_r}}$$

with $x_{r,j} \in \mathcal{B}_r$ for all $j = 1, \ldots, K_r$, and the sequence $\{a_j\}_{j=1}^N$ as

$$\left\{a_{r,j}\right\}_{\substack{r=1,\ldots,R \\ j=1,\ldots,K_r}}.$$

Observe that $S_r = \sum_{j=1}^{K_r} a_{r,j}$. Inequality (8) follows immediately from

$$\sum_{m=0}^{X} \left| \sum_{r=1}^{R} \sum_{j=1}^{K_r} a_{r,j} \varphi_m \left(x_{r,j}\right) \right|^2 \geq CX \sum_{r=1}^{R} \left(\sum_{j=1}^{K_r} a_{r,j} \right)^2. \tag{9}$$

Notice that, if h is as in the hypotheses of Theorem 1, then

$$\begin{aligned}
&\sum_{m=0}^{X}\left|\sum_{r=1}^{R}\sum_{j=1}^{K_r} a_{r,j}\varphi_m\left(x_{r,j}\right)\right|^2\\
&\geq \sum_{m=0}^{+\infty} h\left(\frac{\lambda_m}{\lambda_X}\right)\left|\sum_{r=1}^{R}\sum_{j=1}^{K_r} a_{r,j}\varphi_m\left(x_{r,j}\right)\right|^2\\
&= \sum_{r=1}^{R}\sum_{j=1}^{K_r}\sum_{s=1}^{R}\sum_{i=1}^{K_s} a_{r,j}a_{s,i}\left(\sum_{m=0}^{+\infty} h\left(\frac{\lambda_m}{\lambda_X}\right)\varphi_m\left(x_{r,j}\right)\overline{\varphi_m\left(x_{s,i}\right)}\right)\\
&\geq \sum_{r=1}^{R}\sum_{j=1}^{K_r}\sum_{s=1}^{R}\sum_{i=1}^{K_s} a_{r,j}a_{s,i}\frac{\alpha_0(x_{r,j},x_{s,i})}{2\pi}\lambda_X^d\mathcal{F}_d h\left(\frac{\lambda_X d(x_{r,j},x_{s,i})}{2\pi}\right)\\
&\quad - C\sum_{r=1}^{R}\sum_{j=1}^{K_r}\sum_{s=1}^{R}\sum_{i=1}^{K_s} a_{r,j}a_{s,i}\frac{\lambda_X^{d-2}}{\left(1+\lambda_X d(x_{r,j},x_{s,i})\right)^{\lfloor 2G\rfloor-2d-2}}.
\end{aligned}$$

Let κ large enough so that if $x, y \in \mathcal{B}_r$

$$\mathcal{F}_d h\left(\frac{\lambda_X d\left(x,y\right)}{2\pi}\right) \geq \frac{\mathcal{F}_d h\left(0\right)}{2} > 0.$$

Thus

$$\begin{aligned}
&\sum_{r=1}^{R}\sum_{j=1}^{K_r}\sum_{s=1}^{R}\sum_{i=1}^{K_s} a_{r,j}a_{s,i}\frac{\alpha_0(x_{r,j},x_{s,i})}{2\pi}\lambda_X^d\mathcal{F}_d h\left(\frac{\lambda_X d(x_{r,j},x_{s,i})}{2\pi}\right)\\
&\geq CX\sum_{r=1}^{R}\sum_{j=1}^{K_r}\sum_{i=1}^{K_r} a_{r,j}a_{r,i} = CX\sum_{r=1}^{R}\left(\sum_{j=1}^{K_r} a_{r,j}\right)^2.
\end{aligned}$$

In order to estimate the remainder, let us call z_r the center of the ball of radius $c_2Y^{-1/d}$ containing the region $\mathcal{B}_r$ and let $c_3 = 10c_2$. For every $r = 1,\ldots,R$ we will consider separately the contribution of those values of s for which $\mathcal{B}_s$ is near $\mathcal{B}_r$ (meaning that $\mathcal{B}_s$ is contained in the ball centered at z_r with radius $c_3Y^{-1/d}$) and the contribution of the remaining values of s, for which we will say that $\mathcal{B}_s$ is far from $\mathcal{B}_r$. Notice that there are at most

$$\frac{\left|B\left(z_r, c_3Y^{-1/d}\right)\right|}{Y^{-1}} \leq \frac{C\left(c_3Y^{-1/d}\right)^d}{Y^{-1}} \leq Cc_3^d$$

regions $\mathcal{B}_s$ near $\mathcal{B}_r$. Thus, since $\lambda_X \sim X^{1/d}$ and since $\sum_{j=1}^{K_r} a_{r,j} \geq \sum_{i=1}^{K_s} a_{s,i}$ for $r \leq s$, setting $M = \lfloor 2G\rfloor - 2d - 2$ we obtain,

$$\begin{aligned}
&\sum_{r=1}^{R}\sum_{j=1}^{K_r}\sum_{s=r}^{R}\sum_{i=1}^{K_s} a_{r,j}a_{s,i}\frac{\lambda_X^{d-2}}{\left(1+\lambda_X d\left(x_{r,j},x_{s,i}\right)\right)^M}\\
&\leq CX^{1-2/d}\sum_{r=1}^{R}\sum_{\substack{s=r\\ \mathcal{B}_s \text{ near } \mathcal{B}_r}}^{R}\sum_{j=1}^{K_r} a_{r,j}\sum_{i=1}^{K_s} a_{s,i}\\
&\quad + CX^{1-2/d}\sum_{r=1}^{R}\sum_{\substack{s=r\\ \mathcal{B}_s \text{ far from } \mathcal{B}_r}}^{R}\sum_{j=1}^{K_r} a_{r,j}\sum_{i=1}^{K_s} a_{s,i}\left(\lambda_X d\left(x_{r,j},x_{s,i}\right)\right)^{-M}
\end{aligned}$$

$$\leq CX^{1-2/d}\sum_{r=1}^{R}\left(\sum_{j=1}^{K_r}a_{r,j}\right)^2$$

$$+CX^{1-2/d}\sum_{r=1}^{R-1}\sum_{\substack{s=r+1\\ \mathcal{B}_s \text{ far from } \mathcal{B}_r}}^{R}\sum_{j=1}^{K_r}a_{r,j}\sum_{i=1}^{K_s}a_{s,i}\left(X^{1/d}d\left(x_{r,j},x_{s,i}\right)\right)^{-M}.$$

Using again that $\sum_{j=1}^{K_r}a_{r,j}\geq\sum_{i=1}^{K_s}a_{s,i}$ for $r\leq s$,

$$\sum_{r=1}^{R-1}\sum_{\substack{s=r+1\\ B_s \text{ far from } B_r}}^{R}\sum_{j=1}^{K_r}a_{r,j}\sum_{i=1}^{K_s}a_{s,i}\left(X^{1/d}d\left(x_{r,j},x_{s,i}\right)\right)^{-M}$$

$$=\sum_{r=1}^{R-1}\sum_{j=1}^{K_r}a_{r,j}\sum_{\ell=0}^{\infty}\sum_{\substack{s>r:\\ 2^{\ell-1}c_3Y^{-1/d}\leq d(z_r,z_s)\leq 2^{\ell}c_3Y^{-1/d}}}\sum_{i=1}^{K_s}a_{s,i}\left(X^{1/d}d\left(x_{r,j},x_{s,i}\right)\right)^{-M}$$

$$\leq C\sum_{r=1}^{R-1}\sum_{j=1}^{K_r}a_{r,j}\sum_{\ell=0}^{\infty}2^{-\ell M}\sum_{\substack{s>r:\\ d(z_r,z_s)\leq 2^{\ell}c_3Y^{-1/d}}}\sum_{i=1}^{K_s}a_{s,i}$$

$$\leq C\sum_{r=1}^{R-1}\sum_{j=1}^{K_r}a_{r,j}\sum_{\ell=0}^{\infty}2^{-\ell M}\frac{\left(2^{\ell}Y^{-1/d}\right)^d}{Y^{-1}}\sum_{j=1}^{K_r}a_{r,j}$$

$$\leq C\sum_{r=1}^{R-1}\left(\sum_{j=1}^{K_r}a_{r,j}\right)^2\sum_{\ell=0}^{\infty}2^{-\ell(M-d)}\leq C\sum_{r=1}^{R-1}\left(\sum_{j=1}^{K_r}a_{r,j}\right)^2.$$

□

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References

1. Askey, R.: Summability of Jacobi series. Trans. Am. Math. Soc. **179**, 71–84 (1973)
2. Askey, R., Gasper, G.: Positive Jacobi polynomial sums II. Am. J. Math. **98**, 709–737 (1976)
3. Bérard, P.H.: On the wave equation on a compact Riemannian manifold without conjugate points. Math. Z. **155**, 249–276 (1977)

4. Bilyk, D., Dai, F., Steinerberger, S.: General and refined Montgomery lemmata. Math. Ann. **373**, 1283–1297 (2019)
5. Brandolini, L., Colzani, L.: Localization and convergence of eigenfunction expansions. J. Fourier Anal. Appl. **5**, 431–447 (1999)
6. Brandolini, L., Colzani, L.: Decay of Fourier transforms and summability of eigenfunction expansions. Ann. Sc. Norm. Super. Pisa Cl. Sci. **4**(29), 611–638 (2000)
7. Brandolini, L., Gariboldi, B., Gigante, G.: On a sharp lemma of Cassels and Montgomery on manifolds. Math. Ann. **379**, 1807–1834 (2021)
8. Chavel, I., Randol, B., Dodziuk, J.: Eigenvalues in Riemannian Geometry. Elsevier Science, New York (1984)
9. Colzani, L., Gigante, G., Travaglini, G.: Trigonometric approximation and a general form of the Erdös Turán inequality. Trans. Am. Math. Soc. **363**(2), 1101–1123 (2011)
10. do Carmo, M.P.: Riemannian Geometry. Translated from the Second Portuguese Edition by Francis Flaherty. Mathematics: Theory and Applications. Birkhäuser Boston Inc, Boston (1992)
11. Gigante, G., Leopardi, P.: Diameter bounded equal measure partitions of Alfhors regular metric measure spaces. Discret. Comput. Geom. **57**, 419–430 (2017)
12. Hörmander, L.: The Analysis of Linear Partial Differential Operators, vol. III. Springer, Berlin (1994)
13. Pinsky, M.A.: Pointwise Fourier inversion and related eigenfunction expansions. Comm. Pure Appl. Math. **47**, 653–681 (1994)
14. Pinsky, M.A., Taylor, M.E.: Pointwise Fourier inversion: a wave equation approach. J. Fourier Anal. Appl. **3**, 647–703 (1997)
15. Sogge, C.D.: Hangzhou Lectures on Eigenfunctions of the Laplacian, Annals of Mathematics Studies, vol. 188. Princeton University Press, Princeton (2014)
16. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton (1971)
17. Stempak, K., Trebels, W.: Hankel multipliers and transplantation operators. Stud. Math. **126**, 51–66 (1997)
18. Taylor, M.: Pseudodifferential Operators. Princeton University Press, Princeton (1981)
19. Travaglini, G.: Fejér kernels for Fourier series on T^n and on compact Lie groups. Math. Z. **216**(2), 265–281 (1994)