Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 56-1 (2023) 210–215

Nonlinear MPC for Tracking Piecewise-Constant Reference Signals: the Positive Semidefinite Stage Cost Case

Giacomo Galuppini [*,⋆] **Lalo Magni** [**] **Antonio Ferramosca** [***]

[*] *Dipartimento di Ingegneria Civile e Architettura, University of Pavia, Pavia 27100, Italy (e-mail: giacomo.galuppini01@ateneopv.it)*
[**] *Dipartimento di Ingegneria Civile e Architettura, University of Pavia, Pavia 27100, Italy (e-mail: lalo.magni@unipv.it)*
[***] *Department of Management, Information and Production Engineering, University of Bergamo, Italy (e-mail: antonio.ferramosca@unibg.it)*

Abstract: Model Predictive Control (MPC) is a successful control strategy, with solid theoretical and practical backgrounds. Currently, several stabilizing MPC formulations are available to deal with tracking of piecewise constant references. In particular, it is well understood that, in many cases, the use of artificial reference variables in the optimisation problem allows to sensibly extend the region of attraction of the controller. This work proposes a modified MPC for tracking formulation which is able to guarantee nominal stability also in presence of positive semidefinite stage cost. This can be particularly useful when dealing with high order and/or black-box models, as it allows penalizing the outputs or a subset of states of the system without compromising stability. The algorithm design is based on terminal ingredients and a cost detectability assumption which is explicitly accounted for in the algorithm formulation. Such assumption can be verified by means of input-output-to-state stability arguments, as well as dissipativity ones, thus exploiting techniques already available in the literature.

Keywords: Nonlinear Model Predictive Control, Tracking, Nonlinear Detectability.

1. INTRODUCTION

Model Predictive Control (MPC) is a successful control strategy, widely employed in the industrial frameworks and thoroughly investigated in the academic one. Specifically, stability, robustness, constraint satisfaction results, as well as computationally tractable implementations are nowadays available for both linear and nonlinear systems (Rawlings et al. (2017)).

While MPC was originally developed to face regulation problems, several stabilizing and robust algorithm formulations have been proposed to deal with tracking of piecewise or asymptotically constant references (Magni and Scattolini (2005); Limón et al. (2008); Rawlings et al. (2017); Limon et al. (2018)), periodic references (Limon et al. (2015); Koehler et al. (2021)), zone regions (Ferramosca et al. (2010); Liu et al. (2019)) or general trajectories generated by exosystems (Magni et al. (2001); Köhler et al. (2018)). Moreover, the use of artificial reference variables was introduced in Limón et al. (2008) and further discussed in Ferramosca et al. (2019, 2014); D'jorge et al. (2018); Limon et al. (2018); Köhler et al. (2020), to extend the region of attraction of the tracking control scheme. As a matter of fact, while standard tracking algorithm may loose feasibility due to setpoint changes, the use of artificial reference variables ensures recursive feasibility and convergence to an admissible setpoint.

In both regulation and tracking frameworks, the computation of each MPC control action requires the minimization of a suitable cost function, which penalizes the distance of the system state from the desired setpoint, as well as the necessary control effort, at each instant along the prediction horizon(Rawlings et al. (2017)). However, several practical applications favour or require the use of cost functions weighting the output of the system, rather than its state. As an example, this occurs with MPC schemes based on high-order, black-box models, for which a standard approach would result in a cumbersome tuning process. While the MPC stage cost is typically required to be positive definite (PD) to ensure stability properties of the closed-loop scheme, the aforementioned situation may result in a positive semidefinite (PSD) cost function. In this scenario, a suitable detectability assumption must be introduced to recover closed-loop stability (Grimm et al. (2005); Rawlings et al. (2017)). In particular, nonlinear detectability or input-output-to-state stability (Cai and Teel (2008); Allan et al. (2021)) assumptions are exploited in the recent literature to prove closed-loop stability, for both MPC schemes with and without terminal ingredients (Grimm et al. (2005); Rawlings et al. (2017); Koehler et al. (2021)). Additionally, Höger and Grüne (2019) suggests that dissipativity results can be exploited to prove nonlinear detectability, thus allowing to leverage further results

⋆ Corresponding Author

10.1016/j.ifacol.2023.02.036

already available in the literature.
However, stability properties of tracking MPC schemes involving both a PSD stage cost and the use of artificial reference variables has not been explored in the literature yet. This work faces this issue by proposing a novel Nonlinear MPC (NMPC) algorithm for tracking, with guaranteed stability properties. Specifically, the proposed NMPC extends the original approach from Limon et al. (2018), and copes with the PSD stage cost by explicitly accounting for nonlinear detectability in the overall cost function which is minimized at each iteration. Stability of the closed-loop scheme is achieved by rather standard assumptions on terminal ingredients. The proposed NMPC retains the advantages of the original algorithm, and eases the tuning process in presence of high-order and/or black-box models of the system under control. Finally, the proposed NMPC for tracking is demonstrated on a simulated example, to highlight the benefits of the approach.

Notation

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function, if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}_\infty$-function, if it is a $\mathcal{K}$-function and it is not bounded above. The inverse of $\mathcal{K}$-function α is denoted as α^{-1}. Let a bold variable $\mathbf{u}$ denote a sequence $\{u(0), u(1), ..., u(N-1)\}$, where $u(i)$ is the i-th component and N is the length of the sequence. For a given $z \in \mathbb{R}^n$, let $|z|$ denote its L^2-norm and $|z|_M$ its M-norm, with M a matrix of suitable dimensions.

2. PROBLEM STATEMENT

Consider a nonlinear time invariant, discrete time system:

$$x^+ = f(x,u), \quad y = h(x,u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the system output, and x^+ is the successor state. Assume $f(x,u)$ and $h(x,u)$ are continuous at any equilibrium point. State, input and output at sampling time k are denoted as $x(k)$, $u(k)$ and $y(k)$, respectively. The solution of the system for an input sequence $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$ where $x = \phi(0; x, \mathbf{u})$.

The system is subject to hard constraints on state and control:

$$(x(k), u(k)) \in \mathcal{Z} \;\; \forall k \geq 0 \tag{2}$$

with $\mathcal{Z} \subset \mathbb{R}^{n+m}$ a closed set with not empty interior.

Let the steady state, input, and output of the plant be denoted as (x_s, u_s, y_s). To avoid equilibrium points lying on active constraints, define the restricted constraint set as:

$$\hat{\mathcal{Z}} = \{z : z + e \in \mathcal{Z}, \forall\, |e| < \epsilon_z\}$$

where $\epsilon_z > 0$ is an arbitrarily small constant. Then, the set of admissible equilibrium states such that the constraints are not active is defined as follows:

$$\mathcal{Z}_s = \{(x,u) \in \hat{\mathcal{Z}} : x = f(x,u)\}$$
$$\mathcal{Y}_s = \{y = h(x,u) : (x,u) \in \mathcal{Z}_s\}$$

Note that, for the MPC for tracking proposed in this work to be meaningful, $\mathcal{Z}_s$ must be non empty. Finally, let $\mathcal{Y}_t$ be the largest convex subset of $\mathcal{Y}_s$.

Assumption 1. It is assumed that the steady output y_s univocally defines the equilibrium point (x_s, u_s), hence, for any given y_s, there exists a unique steady state and input (x_s, u_s) such that $x_s = f(x_s, u_s)$ and $y_s = h(x_s, u_s)$.

It is also assumed that there exists a locally Lipschitz continuous function $g_x : \mathcal{Y}_t \to \mathbb{R}^n$ and a continuous function $g_u : \mathcal{Y}_t \to \mathbb{R}^m$ such that $x_s = g_x(y_s)$ and $u_s = g_u(y_s)$.

Remark: Assumption 1 is satisfied if $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are continuously differentiable and the Jacobian

$$\begin{bmatrix} \left(\dfrac{\partial f(x,u)}{\partial x}(x_s,u_s) - \mathbf{I}_n\right) & \dfrac{\partial f(x,u)}{\partial u}(x_s,u_s) \\ \dfrac{\partial h(x,u)}{\partial x}(x_s,u_s) & \dfrac{\partial h(x,u)}{\partial u}(x_s,u_s) \end{bmatrix}$$

is nonsingular for all $(x_s, u_s) \in \mathcal{Z}_s$.

3. MPC FORMULATION

In this section, a new MPC for tracking with PSD stage cost will be presented. This formulation, which, extends the algorithm presented in Limon et al. (2018), is of particular relevance in case of output tracking, or tracking of a small subset of states of a high order system. In order to prove asymptotic stabilty of the closed loop, a nonlinear detectability assumption is needed.

For a given state x and setpoint y_t, the cost function of the proposed MPC is given by:

$$\begin{aligned} V_{N_c,N_p}(x, y_t, \mathbf{u}, y_s) = &\sum_{j=0}^{N_c-1} \ell(x(j) - x_s, u(j) - u_s) \\ &+ \sum_{j=N_c}^{N_p-1} \ell(x(j) - x_s, \kappa(x(j), y_s) - u_s) \\ &+ V_f(x(N_p) - x_s, y_s) + V_O(y_s - y_t) \\ &+ \psi(x - x_s) \end{aligned}$$

where $\mathbf{u}$ is the sequence of control inputs, y_s is the artificial reference, $x(j) = \phi(j; x, \mathbf{u})$, $x_s = g_x(y_s)$, and $u_s = g_u(y_s)$; N_c and N_p are the control and prediction horizon, respectively. The function $l : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is the stage cost function, the function $\kappa : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ is a local control law, the function $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is the terminal cost function, the function $V_O : \mathbb{R}^p \to \mathbb{R}_{\geq 0}$ is the offset cost function, and the function $\Psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is the cost detectability function.

The MPC for tracking control law is derived from the solution of the optimisation problem $P_{N_c,N_p}(x, y_t)$ given by:

$$\min_{\mathbf{u}, y_s} V_{N_c,N_p}(x, y_t, \mathbf{u}, y_s) \tag{3}$$

s. t.

$$x(0) = x \tag{4}$$
$$x(j+1) = f(x(j), u(j)) \;\forall j = 0, ..., N_c - 1 \tag{5}$$
$$(x(j), u(j)) \in \mathcal{Z} \;\forall j = 0, ..., N_c - 1 \tag{6}$$
$$x(j+1) = f(x(j), \kappa(x(j), y_s)) \;\forall j = N_c, ..., N_p - 1 \tag{7}$$
$$(x(j), \kappa(x(j), y_s)) \in \mathcal{Z} \;\forall j = N_c, ..., N_p - 1 \tag{8}$$
$$x_s = f(x_s, u_s), \;\; y_s = h(x_s, u_s) \tag{9}$$
$$(x(N_p), y_s) \in \Gamma \tag{10}$$

where $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^p$ is the terminal set.

Denote the optimal solution to this optimisation problem and the corresponding optimal cost function as $(\mathbf{u}^0(x, y_t), y_s^0(x, y_t))$ and $V^0_{N_c,N_p}(x, y_t)$, respectively. Moreover, let $x_s^0 = g_x(y_s^0)$ and $u_s^0 = g_u(y_s^0)$. According to the receding horizon policy, the closed-loop control law $\kappa_{N_c,N_p}(x, y_t)$ is then given by:

$$\kappa_{N_c,N_p}(x, y_t) = \mathbf{u}^0(0; x, y_t)$$

The stage cost function, the offset cost function and the cost detectability function must fulfill the following assumptions:

Assumption 2. (1) Given a PSD stage cost function $\ell(\cdot)$ and a cost detectability function $\Psi(\cdot)$, there exists constants γ_0, ϵ_0 and a continuous PD function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that:

$$\Psi(x - x_s) \leq \gamma_0 \mu(|x - x_s|)$$
$$\Psi(x^+ - x_s) - \Psi(x - x_s) \leq -\epsilon_0 \mu(|x - x_s|) + \ell(x - x_s, u - u_s)$$

for any $(x, u) \in \hat{\mathcal{Z}}$, and any $(x_s, u_s) \in \mathcal{Z}_s$.

(2) The offset cost function $V_O : \mathbb{R}^p \to \mathbb{R}$ is a convex PD function such that the minimizer:

$$y_s^* = \arg \min_{y_s \in \mathcal{Y}_t} V_O(y_s - y_t)$$

is unique. Moreover, there exists a $\mathcal{K}_\infty$ function α_O such that:

$$V_O(y_s - y_t) - V_O(y_s^* - y_t) \geq \alpha_O(|y_s - y_s^*|)$$

□

Remark: note that, for system (1), Assumption 2.1 requires $\mu(\cdot)$ to be nonlinearly detectable from the stage cost $\ell(\cdot)$. Given $\ell(\cdot)$, a suitable cost detectability function can be computed by relying on input-output-to-state stability (Cai and Teel (2008); Allan et al. (2021)) or strict dissipativity methodologies (Grune and Guglielmi (2018); Höger and Grüne (2019)).

The terminal ingredients must fulfil the following conditions:

Assumption 3. (1) Let Γ be a *positive invariant set for tracking* for the system $x^+ = f(x, \kappa(x, y_s))$, as defined in Limon et al. (2018).

(2) Let $\kappa(x, y_s)$ be a control law such that for all $(x, y_s) \in \Gamma$ the equilibrium point $x_s = g_x(y_s)$ and $u_s = g_u(y_s)$ is an asymptotically stable equilibrium point for the system $x^+ = f(x, \kappa(x, y_s))$. Besides, $\kappa(x, y_s)$ is continuous at (x, y_s) for all $y_s \in \mathcal{Y}_t$.

(3) Let $V_f(x - x_s, y_s)$ be a PD function such that for all $(x, y_s) \in \Gamma$ there exist constants $b > 0$ and $\sigma > 1$ which verify:

$$V_f(x - x_s, y_s) \leq b|x - x_s|^\sigma$$

and:

$$V_f(f(x, \kappa(x, y_s)) - x_s, y_s) - V_f(x - x_s, y_s) \leq -\ell(x - x_s, \kappa(x, y_s) - u_s)$$

where $x_s = g_x(y_s)$, $u_s = g_u(y_s)$.

□

Remark: note that an invariant set for tracking Γ can be conveniently computed with the methodology discussed in Limon et al. (2018).

The following theorem presents the main result of this work. Specifically, it ensures closed-loop stability and convergence of the system controlled by the proposed MPC for tracking.

Theorem 1. Suppose that Assumptions 1-3 hold, and consider a given constant setpoint y_t. Then, for any feasible initial state x, the system controlled by the MPC controller $\kappa_{Nc,Np}(x, y_t)$ derived from the solution of (3-10) is stable, fulfills the constraints, and converges to an equilibrium point such that

(1) If $y_t \in \mathcal{Y}_t$, then $\lim_{k\to\infty} |y(k) - y_t| = 0$.
(2) If $y_t \notin \mathcal{Y}_t$, then $\lim_{k\to\infty} |y(k) - y_s^*| = 0$.

where

$$y_s^* = \arg \min_{y_s \in \mathcal{Y}_t} V_O(y_s - y_t)$$

□

The proof of Theorem 1 requires two steps. In the first one, recursive feasibility of the optimisation problem must be proved for any setpoint y_t. In the second one, asymptotic stability of the equilibrium (x_s^*, u_s^*) must be proved. Moreover, the proof is based on some technical lemmas, stated and proved in Appendix.

Recursive Feasibility

Recursive feasibility directly follows from the proof in Limon et al. (2018), Appendix A.

Stability

This part of the proof is conducted by showing that the function

$$W(x, y_t) = V^0_{Nc,Np}(x, y_t) - V_O(y_s^* - y_t)$$

is a Lyapunov function for the closed-loop system in a neighborhood of the equilibrium point.

Assuming that ϵ is sufficiently small to guarantee that the terminal control law $u = \kappa(x, y_s^*)$ is admissible for all $|x - x_s^*| \leq \epsilon$, it is proved that there exists three of suitable $\mathcal{K}_\infty$ functions, α_W, β_W and $\beta_{\Delta W}$, such that:

$$\alpha_W(|x - x_s^*|) \leq W(x, y_t) \leq \beta_W(|x - x_s^*|)$$
$$W(f(x, \kappa_{N_c,N_p}(x, y_t)), y_t) - W(x, y_t) \leq -\beta_{\Delta W}(|x - x_s^*|)$$

Lower bound.

Consider the following lower bound for the optimal cost $V^0_{Nc,Np}(x, y_t)$:

$$V^0_{Nc,Np}(x, y_t) \geq \ell(x - x_s^0, u - u_s^0) + V_O(y_s^0 - y_t) + \Psi(x - x_s^0)$$

and note that:

$$V^0_{Nc,Np}(x, y_t) = W(x, y_t) + V_O(y_s^* - y_t)$$

then:

$$W(x, y_t) \geq \ell(x - x_s^0, u - u_s^0) + V_O(y_s^0 - y_t) - V_O(y_s^* - y_t) + \Psi(x - x_s^0)$$

From Assumption 2.2 and Lipschitz continuity of $g_x(\cdot)$, it holds:

$$\begin{aligned} V_O(y_s^0 - y_t) - V_O(y_s^* - y_t) &\geq \alpha_O(|y_s^0 - y_s^*|) \\ &\geq \alpha_O(L_g^{-1}|x_s^0 - x_s^*|) \\ &= \hat{\alpha}_O(|x_s^0 - x_s^*|) \end{aligned}$$

Then:

$$W(x, y_t) \geq \ell(x - x_s^0, u - u_s^0) + \hat{\alpha}_O(|x_s^0 - x_s^*|) + \Psi(x - x_s^0)$$

Note that Assumption 2.1 can be rearranged as:

$$\begin{aligned}\ell(x - x_s^0, u - u_s^0) + \Psi(x - x_s^0) &\geq \epsilon_0 \mu(|x - x_s^0|) \\ &\quad + \Psi(x^+ - x_s^0) \\ &\geq \epsilon_0 \mu(|x - x_s^0|)\end{aligned}$$

Finally, combining the previous two:

$$\begin{aligned}W(x, y_t) &\geq \hat{\alpha}_O(|x_s^0 - x_s^*|) + \epsilon_0 \mu(|x - x_s^0|) \\ &\geq \alpha_W(|x_s^0 - x_s^*| + |x - x_s^0|) \\ &\geq \alpha_W(|x - x_s^*|)\end{aligned}$$

Upper bound.

Let $\mathbf{u}_k$ be the sequence of future inputs derived from the local control law taking x as initial state and y_s^* as reference. This sequence is feasible, and the corresponding cost function upper bounded by the terminal cost, as follows:

$$\begin{aligned}V^0_{Nc,Np}(x, y_t) &\leq V_{Nc,Np}(x, y_t; \mathbf{u}_k, y_s^*) \\ &\leq V_f(x - x_s^*, y_s^*) + V_O(y_s^* - y_t) \\ &\quad + \Psi(x - x_s^*)\end{aligned}$$

Then:

$$V^0_{Nc,Np}(x, y_t) - V_O(y_s^* - y_t) \leq V_f(x - x_s^*, y_s^*) + \Psi(x - x_s^*)$$

Therefore:

$$W(x, y_t) \leq V_f(x - x_s^*, y_s^*) + \Psi(x - x_s^*)$$

Finally, using the upper bounds from Assumptions 2.1 and 3.3:

$$\begin{aligned}W(x, y_t) &\leq b|x - x_s^*|^\sigma + \gamma_0 \mu(|x - x_s^*|) \\ &\leq \beta_W(|x - x_s^*|)\end{aligned}$$

Cost decrease.

Define the successor state, $x^+ = f(x, \kappa_{N_c,N_p}(x, y_t))$ and the following feasible sequence:

$$\begin{aligned}\tilde{\mathbf{u}}^+ = \{&u^0(1), .., u^0(N_c - 1), \\ &\kappa_{N_c,N_p}(x^0(N_c), y_s^0), ..., \kappa_{N_c,N_p}(x^0(N_p), y_s^0)\}, \\ \tilde{y_s}^+ = y_s^0 &\end{aligned}$$

Moreover, assume $x \neq x_s^0(x, y_t)$, and define $\tilde{V}_{Nc,Np}(x^+, y_t; \tilde{\mathbf{u}}^+, y_s^0)$ as the cost function evaluated at the feasible solution $(\tilde{\mathbf{u}}^+, y_s^0)$.

Then:

$$\begin{aligned}\Delta W(x, y_t) &= W(x^+, y_t) - W(x, y_t) \\ &= V^0_{Nc,Np}(x^+, y_t) - V_O(y_s^* - y_t) \\ &\quad - V^0_{Nc,Np}(x, y_t) + V_O(y_s^* - y_t) \\ &= V^0_{Nc,Np}(x^+, y_t) - V^0_{Nc,Np}(x, y_t)\end{aligned}$$

Note that, by optimality, it holds:

$$V^0_{Nc,Np}(x^+, y_t) \leq \tilde{V}_{Nc,Np}(x^+, y_t; \tilde{\mathbf{u}}^+, y_s^0)$$

By means of standard procedures in MPC stability proofs (Rawlings et al. (2017)) and Assumption 3.3, it is possible to obtain:

$$\begin{aligned}&\tilde{V}_{Nc,Np}(x^+, y_t; \tilde{\mathbf{u}}^+, y_s^0) - V^0_{Nc,Np}(x, y_t) \leq \\ &\quad - \ell(x - x_s^0, u(0)^0 - u_s) + \Psi(x^+ - x_s^0) - \Psi(x - x_s^0)\end{aligned}$$

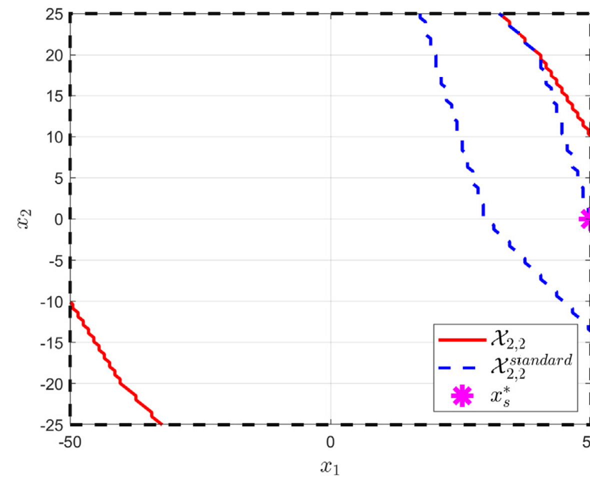

Fig. 1. Comparison of feasible regions of the proposed MPC for tracking ($\mathcal{X}_{2,2}$) and for a standard MPC for tracking ($\mathcal{X}_{2,2}^{standard}$) in case of $y_t = 49.99$.

Therefore:

$$\begin{aligned}\Delta W(x, y_t) &\leq -\ell(x - x_s^0, u(0)^0 - u_s) \\ &\quad + \Psi(x^+ - x_s^0) - \Psi(x - x_s^0)\end{aligned}$$

In view of Assumption 2.1, it holds:

$$\Delta W(x, y_t) \leq -\epsilon_0 \mu(|x - x_s^0|)$$

Finally, using Lemma 2:

$$\begin{aligned}\Delta W(x, y_t) &\leq -\epsilon_0 \mu(\alpha_d(|x - x_s^*|)) \\ &\leq -\beta_{\Delta_W}(|x - x_s^*|)\end{aligned}$$

4. NUMERICAL EXAMPLE

This Section demonstrates the proposed MPC scheme with a numerical example, inspired by Höger and Grüne (2019), and compares the results to a standard MPC for tracking, which does not exploit auxiliary reference variables (i.e. $y_s = y_t$ is set whenever admissible).

Consider the following system:

$$x^+ = Ax + Bu, \quad y = Cx$$

with:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \quad C = [1 \;\; 0]$$

subject to state and input constraints:

$$x_1 \in [-50; 50] \quad x_2 \in [-25; 25] \quad u \in [-20; 20]$$

Moreover, consider the following PSD stage cost:

$$\ell(x - x_s, u - u_s) = |x - x_s|^2_Q + |u - u_s|^2_R$$

with:

$$Q = C'C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1$$

Note that Assumption 1 is satisfied, and that, according to the analysis carried out in Höger and Grüne (2019), nonlinear detectability holds for the stage cost, with:

$$\Psi(x - x_s) = |x - x_s|^2_{P_\Psi}$$

$$P_\Psi = \frac{1}{10}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \quad \gamma_0 = \lambda_{max}(P_\Psi), \quad \epsilon_0 = 3/40$$

$$\mu(|x - x_s|) = |x - x_s|^2$$

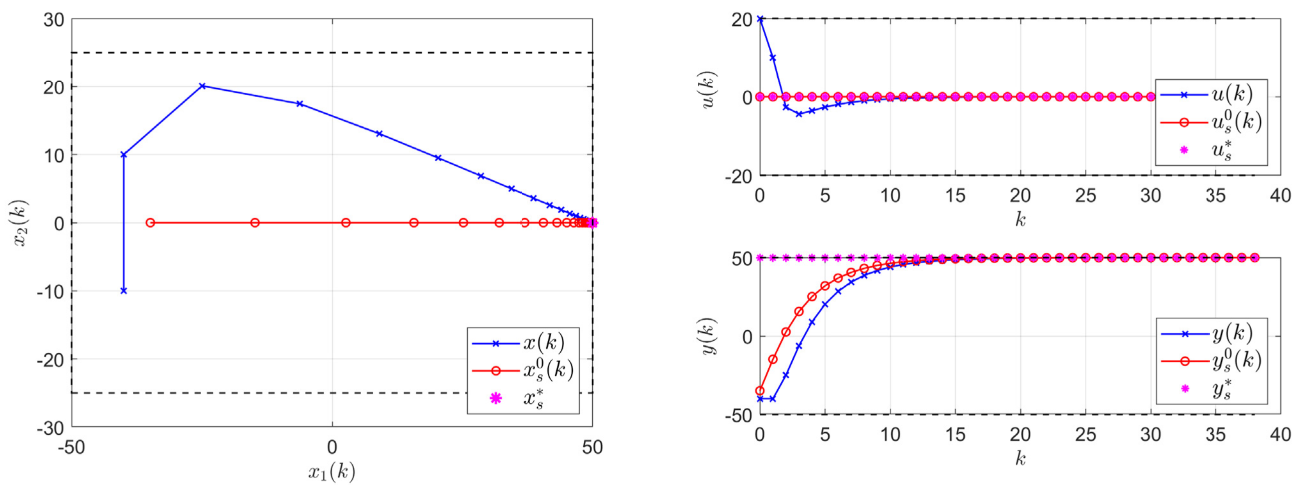

Fig. 2. Result of a simulation with the proposed MPC for tracking, in case of $y_t = 49.99$.

where $\lambda_{max}(P_\Psi)$ denotes the maximum eigenvalue of P_Ψ. Set then:

$$V_O(y_s - y_t) = |y_s - y_t|^2$$

and synthesize an infinite horizon LQR with weights Q and R as auxiliary control law (note that the couple (A, C) is observable). Let P_{lq} denote the solution of the associated algebraic Riccati equation, and set:

$$V_f(x - x_s, y_s) = |x - x_s|^2_{P_{lq}} \qquad P_{lq} = \begin{bmatrix} 2 & 1 \\ 1 & 1.5 \end{bmatrix}$$

$$\Gamma = \{(x, y_s) : |x - g_x(y_s)|^2_{P_{lq}} \leq \gamma_{lq}\}$$

with γ_{lq} chosen so that the auxiliary control law locally fulfils state and input constraints. Finally, pick:

$$N_c = N_p = 2$$

For sake of comparison with a standard MPC for tracking, consider now an output target $y_t \in \mathcal{Y}_t$. e.g. $y_t = 49.99$. Fig. 1 depicts the feasible region for the proposed MPC for tracking, $\mathcal{X}_{2,2}$, and that of a standard MPC for tracking with the same tuning, $\mathcal{X}_{2,2}^{standard}$. As expected from Limon et al. (2018), the use of artificial reference variables allows obtaining a sensitive enlargement of the domain of attraction of the controller. Moreover, it should be also stressed that the domain of attraction of the proposed MPC for tracking does not depend on the output target y_t. With the novel results proved in this work, the tracking approach based on artificial reference variables can be applied also in case of PSD stage cost, as in this example.

Fig. 2 depicts the results of a closed-loop simulation with initial state $x(0) = [-40\ -10]'$ and further stresses the role of artificial reference variables. As a matter of fact, despite the use of a short horizon and a small terminal set (note that Γ could have been set in a less conservative way by following the procedure discussed in Limon et al. (2018)), the proposed MPC successfully manages to regulate the state of the system by suitably adjusting the auxiliary reference variables at each time instant.

5. CONCLUSION

This work proposes a nonlinear MPC for tracking, which guarantees stability even in presence of positive semidefinite stage cost. Stabiltiy results are based on a cost detectability assumption, which is explicitly accounted for in the algorithm design, and can be verified by means of techniques already available in the literature on input-output-to-state stability and dissipativity. Moreover, the MPC for tracking exploits artificial variables to extend the region of attraction of the controller, as demonstrated with a numerical example.

REFERENCES

Allan, D.A., Rawlings, J., and Teel, A.R. (2021). Nonlinear detectability and incremental input/output-to-state stability. *SIAM Journal on Control and Optimization*, 59(4), 3017–3039.

Cai, C. and Teel, A.R. (2008). Input–output-to-state stability for discrete-time systems. *Automatica*, 44(2), 326–336.

D'jorge, A., Anderson, A., González, A.H., and Ferramosca, A. (2018). A robust economic mpc for changing economic criterion. *International Journal of Robust and Nonlinear Control*, 28(15), 4404–4423.

Ferramosca, A., Limón, D., Alvarado, I., Alamo, T., and Camacho, E.F. (2019). Mpc for tracking with optimal closed-loop performance. *Automatica*, 8(45), 1975–1978.

Ferramosca, A., Limon, D., and Camacho, E.F. (2014). Economic mpc for a changing economic criterion for linear systems. *IEEE Transactions on Automatic Control*, 59(10), 2657–2667.

Ferramosca, A., Limon, D., González, A.H., Odloak, D., and Camacho, E.F. (2010). Mpc for tracking zone regions. *Journal of Process Control*, 20(4), 506–516.

Grimm, G., Messina, M.J., Tuna, S.E., and Teel, A.R. (2005). Model predictive control: for want of a local control lyapunov function, all is not lost. *IEEE Transactions on Automatic Control*, 50(5), 546–558.

Grune, L. and Guglielmi, R. (2018). Turnpike properties and strict dissipativity for discrete time linear quadratic optimal control problems. *SIAM Journal on Control and Optimization*, 56(2), 1282–1302.

Höger, M. and Grüne, L. (2019). On the relation between detectability and strict dissipativity for nonlinear discrete time systems. *IEEE Control Systems Letters*, 3(2), 458–462.

Khalil, H.K. (2002). Nonlinear systems third edition. *Patience Hall*, 115.

Koehler, J., Muller, M.A., and Allgower, F. (2021). Constrained nonlinear output regulation using model predictive control. *IEEE Transactions on Automatic Control*.

Köhler, J., Müller, M.A., and Allgöwer, F. (2018). Nonlinear reference tracking: An economic model predictive control perspective. *IEEE Transactions on Automatic Control*, 64(1), 254–269.

Köhler, J., Müller, M.A., and Allgöwer, F. (2020). A nonlinear tracking model predictive control scheme for dynamic target signals. *Automatica*, 118, 109030.

Limón, D., Alvarado, I., Alamo, T., and Camacho, E.F. (2008). Mpc for tracking piecewise constant references for constrained linear systems. *Automatica*, 44(9), 2382–2387.

Limon, D., Ferramosca, A., Alvarado, I., and Alamo, T. (2018). Nonlinear mpc for tracking piece-wise constant reference signals. *IEEE Transactions on Automatic Control*, 63(11), 3735–3750.

Limon, D., Pereira, M., de la Pena, D.M., Alamo, T., Jones, C.N., and Zeilinger, M.N. (2015). Mpc for tracking periodic references. *IEEE Transactions on Automatic Control*, 61(4), 1123–1128.

Liu, S., Mao, Y., and Liu, J. (2019). Model-predictive control with generalized zone tracking. *IEEE Transactions on Automatic Control*, 64(11), 4698–4704.

Magni, L., De Nicolao, G., and Scattolini, R. (2001). Output feedback and tracking of nonlinear systems with model predictive control. *Automatica*, 37(10), 1601–1607.

Magni, L. and Scattolini, R. (2005). On the solution of the tracking problem for non-linear systems with mpc. *International journal of systems science*, 36(8), 477–484.

Rawlings, J.B., Mayne, D.Q., and Diehl, M. (2017). *Model predictive control: theory, computation, and design*, volume 2. Nob Hill Publishing Madison, WI.

LEMMATA

The following lemmas are required for the proof of Theorem 1.

Lemma 1. Consider system (1) subject to constraints (2). Suppose that Assumptions 1-3 hold. Consider a given setpoint y_t, and assume that for a given state x, the optimal solution to $P_{N_c,N_p}(x, y_t)$ is such that $x = x_s^0(x, y_t) = g_x(y_s^0(x, y_t))$. Then, $V^0_{N_c,N_p}(x, y_t) = V_O(y_s^* - y_t)$.

Proof. Considering that the optimal solution to problem $P_{N_c,N_p}(x, y_t)$ is (x_s^0, u_s^0, y_s^0), since $x = x_s^0$, the optimal value cost function is $V^0_{N_c,N_p}(x, y_t) = V_O(y_s^0 - y_t)$.

The lemma is proved by contradiction. Assume that $V_O(y_s^0 - y_t) > V_O(y_s^* - y_t)$. Then, since y_s^* is the unique minimizer of $V_O(\cdot)$, $y_s^0 \neq y_s^*$. Define then $\hat{y}_s$ as:

$$\hat{y}_s = \beta y_s^0 + (1 - \beta) y_s^*$$

From the definition of set $\mathcal{Y}_t$, it follows that $(x_s^0, u_s^0, y_s^0) \in \hat{\mathcal{Z}}$ and it is in the interior of $\mathcal{Z}$. Therefore, there exists a $\hat{\beta} \in [0, 1)$ such that for $\hat{y}_s$ given by a $\beta \in [\hat{\beta}, 1]$, the sequence of inputs $\hat{\mathbf{u}}$ generated by the local control law is such that $(\hat{\mathbf{u}}, \hat{y}_s)$ is a feasible solution of $P_{N_c,N_p}(x, y_t)$.

From the Lipschitz continuity of the function $g_x(\cdot)$, it holds that $|x_s^0 - \hat{x}_s| \leq L_g|y_s^0 - \hat{y}_s|$, where $L_g > 0$ is the Lipschitz constant of $g_x(\cdot)$. Taking into account that $y_s^0 - \hat{y}_s = (1 - \beta)(y_s^0 - y_s^*)$ and the optimality of the solution, the following holds:

$$\begin{aligned}
V_O(y_s^0 - y_t) &= V^0_{N_c,N_p}(x, y_t) \\
&\leq V_{N_c,N_p}(x, y_t, \hat{\mathbf{u}}, \hat{y}_s) \\
&= \sum_{j=0}^{N_p-1} \ell(x(j) - \hat{x}_s, (\kappa(x(j), \hat{y}_s) - \hat{u}_s) \\
&\quad + V_f(x(N_p) - \hat{x}_s, \hat{y}_s) + V_O(\hat{y}_s - y_t) \\
&\quad + \psi(x - \hat{x}_s) \\
&\leq V_f(x_s^0 - \hat{x}_s, \hat{y}_s) + V_O(\hat{y}_s - y_t) + \psi(x_s^0 - \hat{x}_s) \\
&\leq b|x_s^0 - \hat{x}_s|^\sigma + V_O(\hat{y}_s - y_t) + \gamma_0\mu(|x_s^0 - \hat{x}_s|) \\
&\leq b(L_g|y_s^0 - \hat{y}_s|)^\sigma + V_O(\hat{y}_s - y_t) \\
&\quad + \gamma_0\mu(L_g|y_s^0 - \hat{y}_s|) \\
&= L_g^\sigma(1-\beta)^\sigma|y_s^0 - y_s^*|^\sigma + V_O(\hat{y}_s - y_t) \\
&\quad + \gamma_0\mu(L_g(1-\beta)|y_s^0 - y_s^*|)
\end{aligned}$$

From the convexity of $V_O(\cdot)$, it follows:

$$V_O(\hat{y}_s - y_t) \leq \beta V_O(y_s^0 - y_t) + (1 - \beta)V_O(y_s^* - y_t)$$

Therefore:

$$\begin{aligned}
V_O(y_s^0 - y_t) \leq\ & L_g^\sigma(1-\beta)^\sigma|y_s^0 - y_s^*|^\sigma \\
& + \beta V_O(y_s^0 - y_t) + (1-\beta)V_O(y_s^* - y_t) \\
& + \gamma_0\mu(L_g(1-\beta)|y_s^0 - y_s^*|)
\end{aligned}$$

which leads to the following inequality:

$$\begin{aligned}
V_O(y_s^0 - y_t) - V_O(y_s^* - y_t) \leq\ & L_g^\sigma(1-\beta)^{\sigma-1}|y_s^0 - y_s^*|^\sigma \\
& + \gamma_0\mu(L_g(1-\beta)|y_s^0 - y_s^*|)
\end{aligned}$$

Since $\sigma > 1$, taking the limit of both sides of the inequality as β approaches 1 from the right,leads to:

$$V_O(y_s^0 - y_t) - V_O(y_s^* - y_t) \leq 0$$

which in turn leads to a contradiction and concludes the proof. □

Lemma 2. (Adapted from Ferramosca et al. (2014), Lemma 6). Suppose that Assumptions 1-3 hold. Then, there exists a $\mathcal{K}_\infty$ function α_d verifying:

$$|x - x_s^0| \geq \alpha_d(|x - x_s^*|)$$

for all $y_t \in \mathcal{Y}_t$ and all feasible x, with $x_s^* = g_x(y_s^*)$.

Proof. Because of the convexity of the sets $\mathcal{Z}$ and $\mathcal{Z}_s$, $|x - x_s^0|$ is a continuous function. Moreover, consider these two cases:

(1) $|x - x_s^0| = 0 \iff |x - x_s^*| = 0$.
In fact, $|x - x_s^0| = 0 \implies |x - x_s^*| = 0$ follows from Lemma 1; moreover, $|x - x_s^*| = 0 \implies |x - x_s^0| = 0$ follows by optimality.

(2) $|x - x_s^*| > 0 \implies |x - x_s^0| > 0$.
Assume by contradiction that $|x - x_s^0| = 0$. Then, from the previous point, it would be $|x - x_s^*| = 0$, which contradicts the hypothesis that $|x - x_s^*| > 0$.

Then, since $\mathcal{Z}$ is compact, there exists a $\mathcal{K}_\infty$ function α_d verifying (Khalil (2002)):

$$|x - x_s^0| \geq \alpha_d(|x - x_s^*|)$$