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Old and new Weibull-Pareto distributions

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OLD AND NEW WEIBULL-PARETO DISTRIBUTIONS

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Abstract

Strangely enough, the same term *New Weibull-Pareto distribution* (NWPD) has been recently used for denoting different parametric models. We illustrate the difference among these models briefly, and suggest alternative definitions.

Keywords: T-X families, Weibull distribution; Weibull-G family; Weibull-Pareto distribution

1. Introduction – Some generalized families of distributions

The recent literature has suggested several ways of extending a given family of distributions. For example, the T-$X\{Y\}$ family proposed by Aljarrah et al (2014) is as a method of adding new parameters to an existing distribution (or family of distributions), in order to obtain more flexible new families of distributions. The method can be outlined as follows: given three random variables (r.v.), say X, Y and T, a new r.v. is defined by means of the cumulative distribution function (CDF)

$$F_Z(z) = F_T\left(F_Y^{-1}\left(F_X(z)\right)\right), \tag{1}$$

where F_Y^{-1} is the quantile function the r.v. Y. In this formula, F_T plays the role of a so-called "generator" distribution and F_X represents a so-called "baseline" distribution. Now, if X and/or Y and/or T, taken individually, are *parametric* families of distributions, say,

$$F_X(x) = F_X(x, \boldsymbol{\pi}),$$

$$F_T(t) = F_T(t, \boldsymbol{\lambda}),$$

$$F_Y(y) = F_Y(y, \boldsymbol{\rho}),$$

then the new distribution (family of distributions) defined by the above relationship (1) inherits the parameters (if any) of F_X, F_Y and F_T:

$$F_Z(z) = F_Z(z, \boldsymbol{\pi}, \boldsymbol{\lambda}, \boldsymbol{\rho}).$$

In particular, broad classes of distributions can be obtained extending the classical Weibull model, by the use of a Weibull distribution as generator. In Section 2 a brief review is provided of some of these families. Then, a singular case of degeneracy of a Weibull-Pareto (that is a model in which the baseline distribution is of Pareto-type) family of distributions is discussed in Section 3. Finally, in Section 4, alternative names for different classes of Weibull-Pareto distributions are suggested. But before going on further, let us remember the expressions of the CDF of some simple univariate continuous distributions, useful for the remainder of the paper.

Weibull distribution

A two parameter Weibull distribution with scale parameter δ $(\delta > 0)$ and shape parameter β $(\beta > 0)$, $V \sim W(\delta, \beta)$, is defined by the following CDF and probability density function (PDF):

$$F_V(v) = 1 - \exp\{-\delta v^\beta\},\ v > 0 \text{ (and 0 otherwise)},$$

$$f_V(v) = \delta\beta v^{\beta-1} \exp\{-\delta v^\beta\} = \delta\beta \exp\{-\delta v^\beta + (\beta - 1)\log v\},\ v > 0 \text{ (and 0 otherwise)}.$$

In particular, the Weibull distribution with scale parameter 1 and shape parameter β, $V \sim W(1, \beta)$, is defined by the CDF: $F_V(v) = 1 - \exp\{-v^\beta\}$, $v > 0$ (and 0 otherwise).

Pareto(I) distribution

A Pareto distribution of the first kind is denoted by $P(I)(\theta, \alpha)$, where $\alpha > 0$, $\theta > 0$ (Johnson et al. 1994, p.574; Arnold 1983, p.44), and is defined by the CDF:

$$F_V(v) = 1 - \left(\frac{\theta}{v}\right)^\alpha, v > \theta \text{ (and 0 otherwise).}$$

Pareto(II) distribution

A standard Pareto distribution of the second kind is denoted by $P(II)(\mu, \theta, \alpha)$, where $\alpha > 0$, $\theta > 0$, $-\infty < \mu < \infty$ (Johnson et al. 1994, p.575; Arnold 1983, p.45), and it is defined by the CDF:

$$F_V(v) = 1 - \theta^\alpha(\theta + v - \mu)^{-\alpha}, v > \mu \text{ (and 0 otherwise).}$$

Lomax distribution

When $\mu = 0$, the distribution of a r.v. $V \sim P(II)(0, \theta, \alpha)$, has a CDF:

$$F_V(v) = 1 - \theta^\alpha(\theta + v)^{-\alpha}, v > 0 \text{ (and 0 otherwise).}$$

This distribution it is also known as *Lomax distribution* (Lomax, 1954). Then, we will equivalently write $V \sim P(II)(0, \theta, \alpha)$ and $V \sim L(\theta, \alpha)$.

In particular, the Lomax distribution with shape parameter α equal to one, $L(\theta, 1)$ has the CDF:

$$F_V(v) = \frac{v}{\theta + v}, v > 0 \text{ (and 0 otherwise).}$$

Log-logistic distribution

A *log-logistic* distribution $LL(\theta, \alpha)$, with scale parameter θ and shape parameter α, where $\alpha > 0$, $\theta > 0$, is defined by the CDF:

$$F_V(v) = \frac{v^\alpha}{\theta^\alpha + v^\alpha}, v > 0 \text{ (and 0 otherwise).}$$

It is interesting to note that the following Pareto (II), Lomax and Log-logistic distributions are equivalent models, with parameterization as follows:

$$P(II)(0,1,1),\ L(1,1) \text{ and } LL(1,1).$$

In this case the quantile function is given by

$$F_V^{-1}(v) = \frac{v}{1 - v}, 1 < v < 1.$$

2. Families of Weibull-Pareto distributions

Interesting special cases of T-$X\{Y\}$ families of distributions are obtained by taking T of Weibull type. Indeed, if $T \sim W(\delta, \beta)$ and Y is a Lomax distribution $L(1,1)$ (i.e., equivalently, a standard *log-logistic* distribution $LL(1,1)$), i.e. a r.v. with CDF $F_Y(y) = \frac{y}{1+y}$, $y > 0$, then the relationship (1) becomes

$$F_Z(z) = F_T\left(\frac{G(z)}{1-G(z)}\right), \tag{2}$$

where the baseline distribution F_X is denoted by G. The transformation (2) is known as the *Weibull-G* class of distributions (Bourguignon et al, 2014).

Besides, the T-$X\{Y\}$ family of distributions may be alternatively specialized by taking $T \sim W(\delta, \beta)$ and a r.v. Y exponentially distributed with parameter one, $Y \sim \mathcal{E}(1)$. In this case, the CDF of Y is $F_Y(y) = 1 - e^{-y}$, $y > 0$, and the corresponding quantile function is given by $F_Y^{-1}(t) = -\log(1-t)$. Then, if the baseline distribution F_X is denoted by G, the CDF F_Z becomes

$$F_Z(z) = F_T\big(-\log\big(1 - G(z)\big)\big). \tag{3}$$

The transformation (3) is also known as the *Generalized Weibull-G* (GW-G) family of distributions (Cordeiro et al, 2015).

One of the most significant features the T-$X\{Y\}$ family of distributions is the great flexibility offered by its parameters. Nevertheless, it is important to note that sometimes the dimension of the parameter space collapses. Let p, l and r be, respectively, the dimension of parameter vectors $\boldsymbol{\pi}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$, and let $d = p + l + r$. As a general rule, the dimension of the parameter space of the resulting family of r.v. Z is $d^* \leq d$.

For example, if $T \sim W(\delta, \beta)$, $Y \sim \mathcal{E}(1)$ and $X \sim P(I)(\theta, \alpha)$, then F_Z becomes

$$F_Z(z) = F_T\big(-\log\big(1 - F_X(z)\big)\big) = 1 - \exp\left\{-\delta\left(-\log\left(1 - \left(1 - \left(\frac{\theta}{z}\right)^{\alpha}\right)\right)\right)^{\beta}\right\} =$$

$$= 1 - \exp\left\{-\delta\alpha^{\beta}\left(\log\left(\frac{z}{\theta}\right)\right)^{\beta}\right\}, z > \theta, \text{(and 0 otherwise)}.$$

Replacing $\delta\alpha^{\beta}$ by σ, the CDF can be simplified to a three-parameter model

$$F_Z(z) = 1 - \exp\left\{-\sigma\left(\log\left(\frac{z}{\theta}\right)\right)^{\beta}\right\}, z > \theta, \text{(and 0 otherwise)},$$

that corresponds to the PDF

$$f_Z(z) = \frac{\sigma\beta}{z}\left(\log\left(\frac{z}{\theta}\right)\right)^{\beta-1}\exp\left\{-\sigma\left(\log\left(\frac{z}{\theta}\right)\right)^{\beta}\right\}, z > \theta, \text{(and 0 otherwise)},$$

also known as the *Weibull-Pareto* distribution (Alzaatreh et al., 2013). In this case $p = r = 2$, $d = 4$, but $d^* = 3$.

Several families of "Weibull-Pareto" distributions can be viewed as instances of T-$X\{Y\}$ families of distributions, by taking a Weibull generator applied to a baseline "Pareto" distribution. Given the potential high number of different parameterizations, a large number of "Weibull-Pareto" classes of distributions can be defined. Indeed, recall that there exist several types of Pareto models, the so-called Pareto(I), Pareto(II), Pareto(III) and Pareto(IV), where the first three types of Pareto distributions can be identified as special cases of the Pareto(IV) family for an appropriate choice of the parameters. For a broad discussion of these models see Arnold (1983).

In particular the following known cases may be mentioned:

a) *Weibull-Pareto(I){Exponential}*. The above mentioned "WPD", as named by Alzaatreh et al (2013), can be obtained by taking

$$X \sim P(I)(\theta, \alpha); Y \sim \mathcal{E}(1); T \sim W(\delta, \beta).$$

As noted earlier, the WPD by Alzaatreh et al (2013) can also be viewed as a special case of the GW-G family of distributions (Cordeiro et al, 2015).

b) *Weibull-Pareto(I){Lomax}*. The "NWPD", as named by Aljarrah et al (2015), can be obtained by taking

$$X \sim P(I)(\theta, \alpha); Y \sim L(1,1); T \sim W(\delta, \beta).$$

One easily obtains the PDF:

$$f_Z(z) = \frac{\alpha\beta\delta}{\theta^\alpha} z^{\alpha-1} \left(\left(\frac{z}{\theta}\right)^\alpha - 1\right)^{\beta-1} \exp\left\{-\delta\left(\left(\frac{z}{\theta}\right)^\alpha - 1\right)^\beta\right\}, z > \theta \text{ (and 0 otherwise).}$$

c) *Weibull-Pareto(I){Lomax}*. The "NWPD", as named by Tahir et al (2016), can be obtained by taking

$$X \sim P(I)(\theta, \alpha); Y \sim L(1,1); T \sim W(1, \beta).$$

One easily obtains the PDF:

$$f_Z(z) = \frac{\alpha\beta}{\theta^\alpha} z^{\alpha-1} \left(\left(\frac{z}{\theta}\right)^\alpha - 1\right)^{\beta-1} \exp\left\{-\left(\left(\frac{z}{\theta}\right)^\alpha - 1\right)^\beta\right\}, z > \theta \text{ (and 0 otherwise).}$$

The NWPD by Tahir et al (2016) is clearly a special case of the above model (b).

d) *Weibull-Pareto(II){Lomax}*. The "NWPD" as named by Nasiru & Luguterah (2015), can be obtained by taking

$$X \sim P(II)(0, \theta, 1)\ ; Y \sim L(1,1); T \sim W(\delta, \beta).$$

3. A Weibull-Pareto(II){Lomax} distribution that degenerates to a Weibull model

In particular, in the above case (d), since $X \sim P(II)(0, \theta, 1)$ we find $F_X(z) = \frac{z}{\theta+z}$. Then

$$F_Y^{-1}\big(F_X(z)\big) = \frac{F_X(z)}{1-F_X(z)} = \frac{z}{\theta}.$$

Thus the CDF of the r.v. Z becomes

$$F_Z(z) = F_T\left(\frac{F_X(z)}{1-F_X(z)}\right) = 1 - \exp\left\{-\delta\left(\frac{z}{\theta}\right)^{\beta}\right\}, z > 0 \text{ (and 0 otherwise)},$$

and the corresponding PDF is

$$f_Z(z, \delta, \beta, \theta) = \frac{\beta\delta}{\theta}\left(\frac{z}{\theta}\right)^{\beta-1} \exp\left\{-\delta\left(\frac{z}{\theta}\right)^{\beta}\right\}, z > 0 \text{ (and 0 otherwise)}. \quad (4)$$

This distribution, that should be called *Weibull-Pareto(II){Lomax}* distribution, has been recently introduced—even if obtained in a different way—by Nasiru & Luguterah (2015, equation (5)) and named as *New Weibull-Pareto Distribution* (see also Mahmoud et al. 2016, El-Sayed et al. 2016, Hamedani, 2015). Unfortunately the parametric model (4) is not identifiable, because $d^* = 2 < d = 2 + 1$. Indeed, by setting $\frac{\delta}{\theta^{\beta}} = \omega$, $\omega > 0$, the model becomes identifiable, with a CDF of the form

$$F_Z(z, \omega, \beta) = 1 - \exp\{-\omega z^{\beta}\}, \omega > 0 \text{ (and 0 otherwise)}.$$

In conclusion, starting from a Weibull r.v. $T \sim W(\delta, \beta)$ and applying the transformation (1), one obtains another Weibull r.v., $Z \sim W\left(\frac{\delta}{\theta^{\beta}}, \beta\right)$. Then the r.v. Z reduces to a mere reparameterization of the r.v. X. In other words, the family (d) defined by Nasiru & Luguterah (2015) trivially coincides with the family of the two-parameter Weibull distributions, then, strictly speaking it does not exist.

Conclusion

Following the approach suggested by Aljarrah et al (2014), a plethora of variants of a model that could be called "Weibull-Pareto" are possible, depending on different choices of

1) the parameterization of a generator Weibull distribution,
2) the parameterization of a baseline Pareto distribution,
3) the distribution adopted for defining the quantile function in the transformation (1).

In this sense, (a), (b), (c) and (d) are all, literally, Weibull-Pareto distributions. Then, the term "new" in the NWPD should be treated with caution, simply because the "newness" is always relative (and vulnerable to obsolescence).

In conclusion, the known families of distributions (a), (b), (c) and (d) could be more precisely denoted and represented symbolically as:

(a) *Weibull-Pareto(I){Exponential}* $W(\delta,\beta)-P(I)(\theta,\alpha)\{\mathcal{E}(1)\}(\xi_1,\xi_2,\xi_3)$, a three-parameter distribution with PDF

$$f_Z(z)=\frac{\xi_1\xi_3}{z}\left(\log\left(\frac{z}{\xi_2}\right)\right)^{\xi_3-1}\exp\left\{-\xi_1\left(\log\left(\frac{z}{\xi_2}\right)\right)^{\xi_3}\right\},\ z>\xi_2,$$

where $\xi_1=\delta\alpha^{\beta}$, $\xi_2=\theta$ and $\xi_3=\beta$ $(\xi_1,\xi_2,\xi_3>0)$.

(b) *Weibull-Pareto(I){Lomax}* $W(\delta,\beta)-P(I)(\theta,\alpha)\{L(1,1)\}(\xi_1,\xi_2,\xi_3,\xi_4)$, a four-parameter distribution PDF

$$f_Z(z)=\frac{\xi_1\xi_3\xi_4}{\xi_2^{\xi_1}}z^{\xi_1-1}\left(\left(\frac{z}{\xi_2}\right)^{\xi_1}-1\right)^{\xi_3-1}\exp\left\{-\xi_4\left(\left(\frac{z}{\xi_2}\right)^{\xi_1}-1\right)^{\xi_3}\right\},\ z>\xi_2,$$

where $\xi_1=\alpha$, $\xi_2=\theta$, $\xi_3=\beta$ and $\xi_4=\delta$ $(\xi_1,\xi_2,\xi_3,\xi_4>0)$

(c) *Weibull-Pareto(I){Lomax}* $W(1,\beta)-P(I)(\theta,\alpha)\{L(1,1)\}(\xi_1,\xi_2,\xi_3)$, a three-parameter distribution with PDF

$$f_Z(z)=\frac{\xi_1\xi_3}{\xi_2^{\xi_1}}z^{\xi_1-1}\left(\left(\frac{z}{\xi_2}\right)^{\xi_1}-1\right)^{\xi_3-1}\exp\left\{-\left(\left(\frac{z}{\xi_2}\right)^{\xi_1}-1\right)^{\xi_3}\right\},\ z>\xi_2$$

where $\xi_1=\alpha$, $\xi_2=\theta$, $\xi_3=\beta$ $(\xi_1,\xi_2,\xi_3>0)$.

(d) *Weibull-Pareto(II){Lomax}* $W(\delta,\beta)-P(II)(\theta,1)\{L(1,1)\}(\xi_1,\xi_2)$, a two-parameter distribution with PDF

$$f_Z(z)=z^{\xi_2-1}\xi_1\xi_2\exp\{-\xi_1 z^{\xi_2}\},\ z>0$$

where $\xi_1=\frac{\delta}{\theta^{\beta}}$, $\xi_2=\beta$ $(\xi_1,\xi_2>0)$.

The last one model is nothing but the two-parameter Weibull family of distributions.

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