Quasi-Lie Bialgebroids, Dirac Structures, and Deformations of Poisson Quasi-Nijenhuis Manifolds

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Abstract

We show how to deform a Poisson quasi-Nijenhuis manifold by means of a closed 2-form. Then we interpret this procedure in the context of quasi-Lie bialgebroids, as a particular case of the so called twisting of a quasi-Lie bialgebroid. Finally, we frame our result in the setting of Courant algebroids and Dirac structures.

Keywords Poisson quasi-Nijenhuis manifolds · Quasi-Lie bialgebroids · Courant algebroids · Dirac structures

1 Introduction

A tensor N of type $(1, 1)$ on a manifold $\mathcal{M}$ is called a Nijenhuis operator if its Nijenhuis torsion vanishes—see (11). Such a geometrical object was introduced in Nijenhuis (1951) and is still the subject of interesting investigations (see Bolsinov and Konyaev 2022 and references therein). Moreover, it is very useful in the theory of integrable systems thanks to the notion of Poisson–Nijenhuis (PN) manifold (Kosmann-Schwarzbach and Magri 1990; Magri et al. 1985), i.e., a manifold endowed with a Poisson tensor π and a Nijenhuis operator N fulfilling suitable compatibility conditions—see (6). This approach to integrability was put in a more general context in Dorfman (1987), Dorfman (1993), using suitable pairs of Dirac structures.

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Poisson quasi-Nijenhuis (PqN) manifolds, see Stiénon and Xu (2007), are a generalization of PN manifolds, where π and N are still compatible but the Nijenhuis torsion of N is just required to be controlled by a suitable 3-form (so it is not necessarily zero). Some preliminary results about the relevance of PqN manifolds in the study of integrable systems have recently been obtained in Falqui et al. (2020), Falqui et al. (2023). Much remains to be done as far as the construction of a family of functions in involution is concerned.

As shown in Kosmann-Schwarzbach (1996), one can associate a Lie bialgebroid, see Mackenzie and Xu (1994), to a PN manifold. In a similar way, a quasi-Lie bialgebroid can be associated to a PqN manifold, see Stiénon and Xu (2007) and Roytenberg (2002). A general setting to describe Lie bialgebroids and quasi-Lie bialgebroids is supplied by the notions of Courant algebroid and Dirac structure (Courant 1990; Liu et al. 1997)—for the definition of Courant algebroid, see Appendix A and references therein. Indeed, Lie bialgebroids are in one-to-one correspondence with pairs of transversal Dirac structures in a Courant algebroid. If one of the transversal structure is not Dirac but Lagrangian, one obtains a quasi-Lie bialgebroid (see, e.g., Stiénon and Xu (2007) and references therein). Moreover, in Iglesias-Ponte et al. (2012) (following Roytenberg (2002)) it is shown how to deform a quasi-Lie bialgebroid (A, A^*) thanks to the so called twisting by a section of $\bigwedge^2 A$.

The aim of this paper is to present a result about the deformation of a PqN manifold into another PqN manifold, by means of a closed 2-form. This is a first step to generalize the ideas in Dorfman (1993), with the aim of constructing (under suitable assumptions) an involutive family of functions on a PqN manifold. We will give a direct proof of our result in Sect. 2.1, a proof in the setting of twists of quasi-Lie bialgebroids in Sect. 2.2, and a proof in the context of Courant algebroids and Dirac structures in Sect. 3.

Conventions and Notations. Hereafter all manifolds will be smooth of class C^∞ and defined over the real numbers $\mathbb{R}$. In the same vein, all vector bundles considered will be real and smooth of class C^∞ and their sections will be always considered smooth of the same class. As far as the notation is concerned, a vector bundle whose total space, base and canonical projection are E, $\mathcal{M}$ and, respectively, p will be denoted by the triple $(E, p, \mathcal{M})$ or, more simply, by E, if this will not be cause of confusion. The space of (global) sections of E will be denoted by $\Gamma(E)$. For the general notions about Lie algebroids used hereafter we refer the reader to the monograph Mackenzie (2005).

2 Poisson Quasi-Nijenhuis Manifolds and Quasi-Lie Bialgebroids

In the first part of this section we recall the definition of Poisson quasi-Nijenhuis manifold and we present a result, generalizing that in Falqui et al. (2023), concerning the deformations of PqN manifolds. The second part is devoted to a recollection of definitions and results on quasi-Lie bialgebroids, and to an alternative proof of our result.

2.1 Poisson Quasi-Nijenhuis Manifolds

First of all, recall (see, e.g., Kosmann-Schwarzbach and Magri 1990) that any bivector π on a manifold $\mathcal{M}$ induces a bracket on the space $\Gamma(T^*\mathcal{M})$ of 1-forms, given by

$$[\alpha, \beta]_\pi = L_{\pi^\sharp \alpha} \beta - L_{\pi^\sharp \beta} \alpha - \mathrm{d}\langle \beta, \pi^\sharp \alpha \rangle, \tag{1}$$

where $\pi^\sharp : T^*\mathcal{M} \to T\mathcal{M}$ is defined as $\langle \beta, \pi^\sharp \alpha \rangle = \pi(\alpha, \beta)$, and that (1) is a Lie bracket if and only if π is Poisson. Suppose that it is so, and extend (1) to a degree -1, $\mathbb{R}$-bilinear bracket on $\Gamma(\bigwedge^\bullet T^*\mathcal{M})$, still denoted by $[\cdot, \cdot]_\pi$, such that for all $\eta \in \Gamma(\bigwedge^q T^*\mathcal{M})$ and $\eta' \in \Gamma(\bigwedge^{q'} T^*\mathcal{M})$

(K1) $[\eta, \eta']_\pi = -(-1)^{(q-1)(q'-1)}[\eta', \eta]_\pi$;

(K2) $[\alpha, f]_\pi = i_{\pi^\sharp \alpha} \mathrm{d}f = \langle \mathrm{d}f, \pi^\sharp \alpha \rangle$ for all $f \in C^\infty(\mathcal{M})$ and for all 1-forms α;

(K3) $[\eta, \cdot]_\pi$ is a graded derivation of $(\Gamma(\bigwedge^\bullet T^*\mathcal{M}), \wedge)$, that is, for any differential form η'',

$$[\eta, \eta' \wedge \eta'']_\pi = [\eta, \eta']_\pi \wedge \eta'' + (-1)^{(q-1)q'} \eta' \wedge [\eta, \eta'']_\pi. \tag{2}$$

It follows that the graded Jacobi identity,

$$(-1)^{(q_1-1)(q_3-1)}[\eta_1, [\eta_2, \eta_3]_\pi]_\pi + (-1)^{(q_2-1)(q_1-1)}[\eta_2, [\eta_3, \eta_1]_\pi]_\pi$$
$$+ (-1)^{(q_3-1)(q_2-1)}[\eta_3, [\eta_1, \eta_2]_\pi]_\pi = 0, \tag{3}$$

holds (see, e.g., Marle (2008), Proposition 5.4.9), where q_i is the degree of η_i, and that, for any differential form η and for any $f \in C^\infty(\mathcal{M})$,

$$[f, \eta]_\pi = i_{\pi^\sharp \mathrm{d}f}\, \eta. \tag{4}$$

Remark 1 The bracket on $\Gamma(\bigwedge^\bullet T^*\mathcal{M})$ introduced above is an instance of the so called *generalized Schouten bracket*, which can be defined on the exterior algebra of the vector space $\Gamma(A)$ of any Lie algebroid $(A, [\cdot, \cdot]_A, \rho_A)$, see pages 418-419 in Mackenzie and Xu (1994). The generalized Schouten bracket associated to the Lie algebroid $(T\mathcal{M}, [\cdot, \cdot], \mathrm{id})$ is the usual Schouten bracket defined on $\Gamma(\bigwedge^\bullet T\mathcal{M})$, while the one stemming from (1) is associated to the Lie algebroid defined by the Poisson bivector field π. Note that the sign conventions in the above formulas are the same as in Mackenzie and Xu (1994) and in agreement with the ones adopted in Dorfman (1993), see Proposition 2.13. Indeed, if read at the level of the generalized Schouten bracket $[\cdot, \cdot]_\pi$, it becomes, for 1-forms α_i and β_j,

$$[\alpha_1 \wedge \cdots \wedge \alpha_n, \beta_1 \wedge \cdots \wedge \beta_m]_\pi$$
$$= \sum_{i=1}^n \sum_{j=1}^m (-1)^{i+j}[\alpha_i, \beta_j]_\pi \wedge \alpha_1 \wedge \cdots \wedge \hat{\alpha}_i \wedge \cdots \wedge \alpha_n \wedge \beta_1 \wedge \cdots \wedge \hat{\beta}_j \wedge \cdots \wedge \beta_m,$$

that can be obtained applying iteratively (K1) and (K3) to its left-hand side.

The following example points to a useful identity.

Example 2 If $\Omega \in \Gamma(\bigwedge^2 T^*\mathcal{M})$ and $\Omega^\flat : T\mathcal{M} \to T^*\mathcal{M}$ is defined as usual by $\Omega^\flat(X) = i_X\Omega$, then for all $X, Y, Z \in \Gamma(T\mathcal{M})$

$$[\Omega, \Omega]_\pi(X, Y, Z) = 2 \sum_{\circlearrowleft(X,Y,Z)} \left(\langle [\Omega^\flat X, \Omega^\flat Y]_\pi, Z \rangle) - \mathcal{L}_{\pi^\sharp \Omega^\flat X}(\Omega(Y, Z)) \right), \quad (5)$$

which follows by applying to our setting Formula (2.31) and Proposition 2.15 in Dorfman (1993).

Recall now that a Poisson tensor π and a $(1, 1)$ tensor field $N : T\mathcal{M} \to T\mathcal{M}$ on $\mathcal{M}$ are said to be *compatible* (Kosmann-Schwarzbach and Magri 1990; Magri et al. 1985) if

$$N\pi^\sharp = \pi^\sharp N^* \quad \text{and} \quad [\alpha, \beta]_{\pi_N} = [N^*\alpha, \beta]_\pi + [\alpha, N^*\beta]_\pi - N^*[\alpha, \beta]_\pi \text{ for all 1-forms } \alpha, \beta, \tag{6}$$

where $N^* : T^*\mathcal{M} \to T^*\mathcal{M}$ is the transpose of N and π_N is the bivector field defined by $\pi_N^\sharp = N\pi^\sharp$ (notice that it is a bivector thanks to the first condition).

Finally, given a p-form α with $p \geq 1$, the p-form $i_N\alpha$ is defined as

$$i_N\alpha(X_1, \ldots, X_p) = \sum_{i=1}^p \alpha(X_1, \ldots, NX_i, \ldots, X_p), \tag{7}$$

while $i_N f = 0$ for all functions f. After introducing

$$\mathrm{d}_N = i_N \circ \mathrm{d} - \mathrm{d} \circ i_N, \tag{8}$$

it can be proved, see Kosmann-Schwarzbach (1996), that the compatibility between a Poisson tensor π and a $(1, 1)$ tensor field N is equivalent to requiring that d_N is a derivation of $[\cdot, \cdot]_\pi$, a fact that we will often use in the rest of the paper. Moreover, for any k-form η one has that

$$(\mathrm{d}_N\eta)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \mathcal{L}_{NX_i} \left(\eta(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \right)$$
$$+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j]_N, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}), \tag{9}$$

where

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]. \tag{10}$$

In Stiénon and Xu (2007) a *Poisson quasi-Nijenhuis (PqN) manifold* was defined as a quadruple $(\mathcal{M}, \pi, N, \phi)$ such that:

- the Poisson bivector π and the $(1, 1)$ tensor field N are compatible;

- the 3-forms ϕ and $i_N\phi$ are closed;
- $T_N(X, Y) = \pi^\sharp(i_{X \wedge Y}\phi)$ for all vector fields X and Y, where $i_{X \wedge Y}\phi$ is the 1-form defined as $\langle i_{X \wedge Y}\phi, Z \rangle = \phi(X, Y, Z)$, and

$$T_N(X, Y) = [NX, NY] - N[X, Y]_N \tag{11}$$

is the *Nijenhuis torsion* of N.

A slightly more general definition of PqN manifold was recently proposed in Bursztyn and Drummond (2019)—see also Bursztyn et al. (2022), where the PqN structures are recast in the more general framework of the Dirac-Nijenhuis ones. Another interesting generalization, given by the so called PqN manifolds with background, was considered in Antunes (2008), see also Cordeiro and Nunes da Costa (2010).

If $\phi = 0$, then the torsion of N vanishes and $\mathcal{M}$ becomes a *Poisson–Nijenhuis (PN) manifold* (see Kosmann-Schwarzbach and Magri (1990) and references therein). In this case, π_N is a Poisson tensor compatible with π, so that $(\mathcal{M}, \pi, \pi_N)$ is a bi-Hamiltonian manifold.

The following theorem generalizes a result in Falqui et al. (2023), where the starting point was a PN manifold.

Theorem 3 *Let $(\mathcal{M}, \pi, N, \phi)$ be a PqN manifold and let Ω be a closed 2-form. If $\widehat{N} = N + \pi^\sharp \Omega^\flat$ and*

$$\widehat{\phi} = \phi + \mathrm{d}_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi, \tag{12}$$

then $(\mathcal{M}, \pi, \widehat{N}, \widehat{\phi})$ is a PqN manifold. In particular, if Ω is a solution of the non homogeneous Maurer-Cartan equation

$$\mathrm{d}_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi = -\phi, \tag{13}$$

then $(\mathcal{M}, \pi, \widehat{N})$ is a PN manifold.

Proof It is similar to the one given in Falqui et al. (2023), corresponding to the case $\phi = 0$. We present here only the main points, focussing on the differences.

To prove the compatibility between π and $\widehat{N}$, we notice that $\mathrm{d}\Omega = 0$ implies

$$\mathrm{d}_{\pi^\sharp \Omega^\flat} = [\Omega, \cdot]_\pi. \tag{14}$$

In fact, both $\mathrm{d}_{\pi^\sharp \Omega^\flat}$ and $[\Omega, \cdot]_\pi$ are graded derivation of $(\Omega^\bullet(\mathcal{M}), \wedge)$, anticommuting with d and coinciding on $C^\infty(\mathcal{M})$—see Falqui et al. (2023) for details. The previous identity entails that

$$\mathrm{d}_{\widehat{N}} = \mathrm{d}_{N + \pi^\sharp \Omega^\flat} = \mathrm{d}_N + [\Omega, \cdot]_\pi.$$

Hence $\mathrm{d}_{\widehat{N}}$ is a derivation of $[\cdot, \cdot]_\pi$, yielding the compatibility between $\widehat{N}$ and π.

The closedness of $\widehat{\phi}$ easily follows from that of ϕ and Ω, recalling that $d \circ d_N = -d_N \circ d$ and that d is a derivation of $[\cdot, \cdot]_\pi$. Moreover,

$$
\begin{aligned}
d_{\widehat{N}}\widehat{\phi} &= (d_N + [\Omega, \cdot]_\pi)(\phi + d_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi) \\
&= d_N\phi + d_N^2\Omega + \frac{1}{2}[d_N\Omega, \Omega]_\pi - \frac{1}{2}[\Omega, d_N\Omega]_\pi + [\Omega, \phi]_\pi + [\Omega, d_N\Omega]_\pi \\
&\quad + \frac{1}{2}[\Omega, [\Omega, \Omega]_\pi]_\pi \\
&= [\phi, \Omega]_\pi + [d_N\Omega, \Omega]_\pi + [\Omega, \phi]_\pi + [\Omega, d_N\Omega]_\pi + \frac{1}{2}[\Omega, [\Omega, \Omega]_\pi]_\pi = 0,
\end{aligned}
\tag{15}
$$

thanks to $d_N^2 = [\phi, \cdot]_\pi$, the fact that d_N is a derivation of $[\cdot, \cdot]_\pi$, the commutation rule (K1), and the graded Jacobi identity (3). Furthermore, observe that $d_N\phi = 0$ as a consequence of (8). Then $i_{\widehat{N}}\widehat{\phi}$ is closed because of (8) and $d\widehat{\phi} = 0$.

Finally, if α is any differential form,

$$
\begin{aligned}
d_{\widehat{N}}^2\alpha &= (d_N + [\Omega, \cdot]_\pi)(d_N\alpha + [\Omega, \alpha]_\pi) \\
&= d_N^2\alpha + [d_N\Omega, \alpha]_\pi - [\Omega, d_N\alpha]_\pi + [\Omega, d_N\alpha]_\pi + [\Omega, [\Omega, \alpha]_\pi]_\pi \\
&= [\phi, \alpha]_\pi + [d_N\Omega, \alpha]_\pi + \frac{1}{2}[[\Omega, \Omega]_\pi, \alpha]_\pi \\
&= [\widehat{\phi}, \alpha]_\pi.
\end{aligned}
\tag{16}
$$

As shown in Lemma 3.7 of Stiénon and Xu (2007), this implies that $T_{\widehat{N}}(X, Y) = \pi^\sharp(i_{X \wedge Y}\widehat{\phi})$ for all vector fields X, Y. □

Remark 4 If $\mathfrak{G}_{\Omega_c^2}$ is the (additive) group of closed 2-forms on $\mathcal{M}$, the theorem above implies that the set of the PqN-structures on $\mathcal{M}$ carries the following $\mathfrak{G}_{\Omega_c^2}$-action:

$$
\Omega.(\mathcal{M}, \pi, N, \phi) = (\mathcal{M}, \pi, N + \pi^\sharp\Omega^\flat, \phi + d_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi).
\tag{17}
$$

In fact, if $\Omega_1, \Omega_2 \in \mathfrak{G}_{\Omega_c^2}$, then

$$
\begin{aligned}
&\Omega_1.(\Omega_2.(\mathcal{M}, \pi, N, \phi)) \\
&= \Omega_1.(\mathcal{M}, \pi, N + \pi^\sharp\Omega_2^\flat, \phi + d_N\Omega_2 + \frac{1}{2}[\Omega_2, \Omega_2]_\pi)) \\
&= (\mathcal{M}, \pi, N + \pi^\sharp(\Omega_1 + \Omega_2)^\flat, \phi + d_N\Omega_2 \\
&\quad + \frac{1}{2}[\Omega_2, \Omega_2]_\pi + d_{N+\pi^\sharp\Omega_2^\flat}\Omega_1 + \frac{1}{2}[\Omega_1, \Omega_1]_\pi) \\
&= (\mathcal{M}, \pi, N + \pi^\sharp(\Omega_1 + \Omega_2)^\flat, \phi + d_N(\Omega_1 + \Omega_2) + \frac{1}{2}[\Omega_1 + \Omega_2, \Omega_1 + \Omega_2]_\pi),
\end{aligned}
$$

where the last equality was obtained observing that $d_{N+\pi^\sharp\Omega_2} = d_N + d_{\pi^\sharp\Omega_2} = d_N + [\Omega_2, \cdot]_\pi$ since $d\Omega_2 = 0$, and $[\Omega_1, \Omega_2]_\pi = [\Omega_2, \Omega_1]_\pi$, see Property (K1).

The aim of the rest of the paper is two give two alternative proofs of Theorem 3, in the frameworks of quasi-Lie bialgebroids (see next subsection) and Dirac structures (see Sect. 3).

2.2 Quasi-Lie Bialgebroids

Suppose that $p : A \to \mathcal{M}$ is a Lie algebroid (see, e.g., Mackenzie (2005)) with anchor $\rho_A : A \to T\mathcal{M}$ and Lie bracket $[\cdot, \cdot]_A$, defined on the space of sections $\Gamma(A)$ and then extended to $\Gamma(\bigwedge^\bullet A)$, see Remark 1. Recall that, given $P \in \Gamma(\bigwedge^k A^*)$, one can define $d_A P \in \Gamma(\bigwedge^{k+1} A^*)$ as

$$(d_A P)(\alpha_1, \ldots, \alpha_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\rho_A(\alpha_i)\left(P(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1})\right)$$
$$+ \sum_{i<j}(-1)^{i+j}P([\alpha_i, \alpha_j]_A, \alpha_1 \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}),$$
(18)

for all $\alpha_1, \ldots, \alpha_{k+1} \in \Gamma(A)$, and that d_A is a degree-1 derivation of $\Gamma(\bigwedge^\bullet A^*)$ such that $d_A^2 = 0$.

Definition 5 (Stiénon and Xu (2007)) A quasi-Lie bialgebroid is a triple (A, d_{A^*}, ϕ), where

- A is a Lie algebroid
- d_{A^*} is a degree-1 derivation of $\Gamma(\bigwedge^\bullet A)$, both with respect to the wedge product and $[\cdot, \cdot]_A$
- $\phi \in \Gamma(\bigwedge^3 A)$ satisfies $d_{A^*}\phi = 0$ and $d_{A^*}^2 = [\phi, \cdot]_A$.

Under these assumptions, one can define a morphism $\rho_{A^*} : A^* \to T\mathcal{M}$ by

$$\rho_{A^*}(X)(f) = X(d_{A^*}f), \qquad \forall X \in \Gamma(A^*), \ f \in C^\infty(\mathcal{M}), \tag{19}$$

a bracket in $\Gamma(A^*)$ by

$$[X, Y]_{A^*}(\alpha) = \rho_{A^*}(X)(\alpha(Y)) - \rho_{A^*}(Y)(\alpha(X)) - (d_{A^*}\alpha)(X, Y), \quad \forall\alpha \in \Gamma(A), \tag{20}$$

and show that d_{A^*} is explicitly given by a formula which is analog to (18). If $\phi = 0$, then $(A^*, \rho_{A^*}, [\cdot, \cdot]_{A^*})$ is also a Lie algebroid, and (A, A^*) turns out to be a Lie bialgebroid (see, e.g., Stiénon and Xu (2007)).

The first alternative proof of Theorem 3 hinges on the following two results.

Proposition 6 (Stiénon and Xu 2007, Proposition 3.5) *The quadruple (M, π, N, ϕ) is a Poisson quasi-Nijenhuis manifold if and only if $((T^*M)_\pi, d_N, \phi)$ is a quasi-Lie bialgebroid and ϕ is closed.*

Proposition 7 *If (A, d_{A^*}, ϕ) is a quasi-Lie bialgebroid, $\Omega \in \bigwedge^2 A$, and*

$$\widehat{d}_{A^*} = d_{A^*} + [\Omega, \cdot]_A, \qquad \widehat{\phi} = \phi + d_A\Omega + \frac{1}{2}[\Omega, \Omega]_A,$$

then $(A, \widehat{d}_{A^*}, \widehat{\phi})$ is a quasi-Lie bialgebroid too, called the twist *of (A, d_{A^*}, ϕ) by Ω.*

This statement can be found, for example, at the beginning of Section 4.4 of Iglesias-Ponte et al. (2012). Its proof consists in showing that the new defined triple satisfies what is needed to form a quasi-Lie bialgebroid, i.e., $\widehat{d}_{A^*}\widehat{\phi} = 0$ and $\widehat{d}_{A^*}^2 = [\widehat{\phi}, \cdot]_A$, which can be both checked by a direct computation completely analogous to (15) and, respectively, (16).

Remark 8 Note that if $\phi = 0$ Proposition 6 reduces to the correspondence between Lie bialgebroids and PN manifolds described in Kosmann-Schwarzbach (1996).

Going back to the proof Theorem 3, let $((T^*\mathcal{M})_\pi, d_N, \phi)$ be the quasi-Lie bialgebroid associated to the PqN manifold $(\mathcal{M}, \pi, N, \phi)$, and let Ω be any 2-form on $\mathcal{M}$. Then we obtain the twist $((T^*\mathcal{M})_\pi, d_N + [\Omega, \cdot]_\pi, \widehat{\phi})$, where

$$\widehat{\phi} = \phi + d_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi.$$

If $d\Omega = 0$, we know from (14) that $[\Omega, \cdot]_\pi = d_{\pi^\sharp \Omega^\flat}$. Therefore

$$d_N + [\Omega, \cdot]_\pi = d_{N+\pi^\sharp \Omega^\flat} = d_{\widehat{N}}, \qquad \text{where } \widehat{N} = N + \pi^\sharp \Omega^\flat.$$

The last step is to realize that the quasi-Lie bialgebroid $\big((T^*\mathcal{M})_\pi, d_{\widehat{N}}, \widehat{\phi}\big)$ comes from a PqN manifold, i.e., that $\widehat{\phi}$ is closed. This follows, as already observed just above Formula (15), from the fact that both ϕ and Ω are closed.

Remark 9 Proposition 7 and a computation very similar to the one in Remark 4 show that the (additive) group $\mathfrak{G}_{\Omega^2}$ of all 2-forms on $\mathcal{M}$ acts on the set of all quasi-Lie bialgebroids of the form $((T^*\mathcal{M})_\pi, d_N, \phi)$ by *twisting* them, i.e.,

$$\Omega.((T^*\mathcal{M})_\pi, d_N, \phi) = ((T^*\mathcal{M})_\pi, d_N + [\Omega, \cdot]_\pi, \phi + d_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi). \quad (21)$$

The latter restricts to an action of $\mathfrak{G}_{\Omega_c^2}$, see Remark 4, on the set of all quasi-Lie bialgebroids of the form $((T^*\mathcal{M})_\pi, d_N, \phi)$ with $d\phi = 0$. Finally, the bijection between the set of PqN structures $(\mathcal{M}, \pi, N, \phi)$ and the quasi-Lie bialgebroids of the form $((T^*\mathcal{M})_\pi, d_N, \phi)$ invoked in Proposition 6 intertwines (21) with (17).

3 Courant Algebroids and Dirac Structures

In this section we give a second alternative proof of Theorem 3 in the framework of Dirac structures. To this end, we start by recalling how quasi-Lie bialgebroids are related to Courant algebroids, see (Roytenberg 2002; Stiénon and Xu 2007)—more precisely, how quasi-Lie bialgebroids correspond to the pairs formed by a Dirac structure and a complementary Lagrangian subbundle in a given Courant algebroid, see the proof of part (ii) of Theorem 2.6 in Stiénon and Xu (2007). For the reader convenience, we recall the definition of a Courant algebroid in Appendix A.

Given a quasi-Lie bialgebroid (A, d_{A^*}, ϕ), and using the notations in and after Definition 5, we consider $E = A^* \oplus A$ together with the non degenerate symmetric pairing

$$\langle X_1 + \alpha_1, X_2 + \alpha_2 \rangle_E = \frac{1}{2} \left(X_1 (\alpha_2) + X_2 (\alpha_1) \right),$$

the bundle map $\rho \colon E \to TM$ given by

$$\rho(X + \alpha) = \rho_{A^*}(X) + \rho_A(\alpha),$$

and the bracket in $\Gamma(E)$ defined by

$$
\begin{aligned}
[\![\alpha, \beta]\!] &= [\alpha, \beta]_A \\
[\![X, Y]\!] &= [X, Y]_{A^*} + i_{X \wedge Y} \phi \\
[\![\alpha, Y]\!] &= \left(i_\alpha (d_A Y) + \frac{1}{2} d_A (Y(\alpha)) \right) - \left(i_Y (d_{A^*} \alpha) + \frac{1}{2} d_{A^*} (Y(\alpha)) \right),
\end{aligned}
\tag{22}
$$

for all $X, Y \in \Gamma(A^*)$ and $\alpha, \beta \in \Gamma(A)$, where the bracket $[\cdot, \cdot]_{A^*}$ was defined in (20) while d_A is the differential defined on $\Gamma(\bigwedge^\bullet A^*)$ by the algebroid A. Then $(E, \langle \cdot, \cdot \rangle_E, [\![\cdot, \cdot]\!], \rho)$ is a Courant algebroid, A is a Dirac structure (i.e., it is maximal isotropic and its space of sections is closed under $[\![\cdot, \cdot]\!]$), and A^* is a Lagrangian (i.e., maximal and isotropic) subbundle of E. Note that, if $\phi = 0$, i.e., in the case of a Lie bialgebroid, A^* is a Dirac structure too.

On the other hand, let $(E, \langle \cdot, \cdot \rangle_E, [\![\cdot, \cdot]\!], \rho)$ be a Courant algebroid and let $A \subset E$ be a Dirac structure (it follows that A has an induced Lie algebroid structure). Suppose that there exists a Lagrangian subbundle L which is transversal to A. Then we can identify A^* with L through

$$
\begin{aligned}
A^* &\to L \\
X &\mapsto \tilde{X}
\end{aligned}
\tag{23}
$$

where $X(\alpha) = 2 \langle \tilde{X}, \alpha \rangle_E$ for all $\alpha \in A$. Since we have the identification $\Gamma(\bigwedge^\bullet A) \simeq \Gamma(\bigwedge^\bullet L^*)$, we can define $d_L : \Gamma(\bigwedge^\bullet A) \to \Gamma(\bigwedge^\bullet A)$ as in (18), that is,

$$
\begin{aligned}
(d_L \psi)(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \rho(\tilde{X}_i) \left(\tilde{\psi}(\tilde{X}_1, \ldots, \hat{\tilde{X}}_i, \ldots, \tilde{X}_{k+1}) \right) \\
&\quad + \sum_{i<j} (-1)^{i+j} \tilde{\psi}([\tilde{X}_i, \tilde{X}_j]_L, \tilde{X}_1 \ldots, \hat{\tilde{X}}_i, \ldots, \hat{\tilde{X}}_j, \ldots, \tilde{X}_{k+1}),
\end{aligned}
\tag{24}
$$

for all $X_1, \ldots, X_{k+1} \in \Gamma(A^*)$, where $\psi \in \Gamma(\bigwedge^k A)$ corresponds to $\tilde{\psi} \in \Gamma(\bigwedge^k L^*)$, and $[\cdot, \cdot]_L$ is the L-component of $[\![\cdot, \cdot]\!]$ with respect to the splitting $E = L \oplus A$. Notice that d_L is a derivation, but $d_L^2 \neq 0$ since L in general is not a Dirac structure. Finally, let $\varphi \in \Gamma(\bigwedge^3 A)$ be defined as

$$\varphi(X_1, X_2, X_3) = 2 \langle [\![\tilde{X}_1, \tilde{X}_2]\!], \tilde{X}_3 \rangle_E \quad \text{for all } X_1, X_2, X_3 \in \Gamma(A^*). \tag{25}$$

Then $(A, \mathrm{d}_L, \varphi)$ turns out to be a quasi-Lie bialgebroid.

Example 10 Consider the quasi-Lie bialgebroid $(T^*\mathcal{M}, \mathrm{d}, \phi)$, where $A = T^*\mathcal{M}$ with its trivial Lie algebroid structure, i.e., with zero bracket and zero anchor, d is the Cartan differential and ϕ is any closed 3-form on $\mathcal{M}$. Applying (22) to the this setting, one finds

$$\llbracket \alpha, \beta \rrbracket = 0$$
$$\llbracket X, Y \rrbracket = [X, Y] + i_{X \wedge Y}\phi$$
$$\llbracket \alpha, Y \rrbracket = -\left(i_Y(\mathrm{d}\alpha) + \frac{1}{2}\mathrm{d}(Y(\alpha))\right),$$

for all $X, Y \in \Gamma(T\mathcal{M})$ and $\alpha, \beta \in \Gamma(T^*\mathcal{M})$, since, in that case, $\mathrm{d}_A = 0$, $[\cdot, \cdot]_A = 0$, $\mathrm{d}_{A^*} = \mathrm{d}$ and $[\cdot, \cdot]_{A^*} = [\cdot, \cdot]$ is the usual Lie bracket defined on $\Gamma(T\mathcal{M})$. Computing the bracket so obtained on a pair of general sections of E, one has

$$\llbracket X + \alpha, Y + \beta \rrbracket = [X, Y] + i_X(\mathrm{d}\beta) - i_Y(\mathrm{d}\alpha) + \frac{1}{2}(\mathrm{d}(X(\beta) - \mathrm{d}(Y(\alpha))) + i_{X \wedge Y}\phi,$$

(26)

i.e., E is the Courant algebroid $\mathbb{T}\mathcal{M}^\phi$, obtained twisting the standard one by the closed 3-form ϕ, see Example 19. Note that $T\mathcal{M}$ is a Lagrangian subbundle of E transversal to the Dirac structure $T^*\mathcal{M} \subset E$. In particular $(T\mathcal{M}, T^*\mathcal{M})$ is the pair corresponding to $(T^*\mathcal{M}, \mathrm{d}, \phi)$. Note now that if Ω is any closed 2-form in M, its graph

$$\mathrm{Gr}(\Omega) = \{X + \Omega^\flat X \in T\mathcal{M} \oplus T^*\mathcal{M} \mid X \in \Gamma(T\mathcal{M})\}$$

(27)

is a Lagrangian subbundle of $\mathbb{T}\mathcal{M}^\phi$, transversal to $T^*\mathcal{M}$. A simple computation, using (24) and (25), shows that the quasi-Lie bialgebroid corresponding to $(\mathrm{Gr}(\Omega), T^*\mathcal{M})$ is $(T^*\mathcal{M}, \mathrm{d}, \phi)$.

Remark 11 It is worth recalling that every closed 2-form Ω defines an automorphism of $\mathbb{T}\mathcal{M}^\phi$ via the formula

$$\Omega.(X + \alpha) = X + i_X\Omega + \alpha, \quad \forall X + \alpha \in \Gamma(\mathbb{T}\mathcal{M}^\phi),$$

(28)

see for example Gualtieri (2011), just above Proposition 2.2, the proof of Lemma 3.1 in Bressler (2007) and, in a more general setting, Remark 4.2 in Roytenberg (2002). In particular, the image of a Dirac structure under (28) is still a Dirac structure. More in general, if Ω is any 2-form on $\mathcal{M}$, (28) sends a Lagrangian subbundle to a Lagrangian subbundle and a pair of transversal Lagrangian subbundles to a pair of transversal Lagrangian subbundles. Note that, in the previous example, the pair $(\mathrm{Gr}(\Omega), T^*\mathcal{M})$ is obtained applying (28) to the pair of transversal Lagrangian subbundles $(T\mathcal{M}, T^*\mathcal{M})$. On the other hand, since $\mathrm{d}\Omega = 0$ and the Poisson structure π is trivial, i.e., identically zero, (21) yields

$$\Omega.(T^*\mathcal{M}, \mathrm{d}, \phi) = (T^*\mathcal{M}, \mathrm{d}, \phi).$$

(29)

As opposed to what is stated in part (ii) of Theorem 2.6 in Stiénon and Xu (2007), this example seems to suggest that the correspondence between quasi-Lie bialgebroids and pairs of a Lagrangian subbundle and of a transversal Dirac structure of a given Courant algebroid, described in the first part of this section is, in general, not one-to-one. On the other hand, the above example and its generalization, contained in Theorem 12, propound the existence of a one-to-one correspondence between the $\Omega_c^2(\mathcal{M})$-orbits of quasi-Lie bialgebroids of the type $((T^*\mathcal{M})_\pi, d_N, \phi)$ and the $\Omega_c^2(\mathcal{M})$-orbits of pairs of a Lagrangian subbundle and of a transversal Dirac structure of the type $(\mathrm{Gr}(\Omega), T^*\mathcal{M})$ in a Courant algebroid of the type $E = (T\mathcal{M})_N \oplus (T^*\mathcal{M})_\pi$, where $\Omega_c^2(\mathcal{M})$ is the (additive) group of the closed 2-forms on $\mathcal{M}$ acting, on these sets, via (21) and, respectively, (28).

We are now ready to present the third proof of Theorem 3. We start from a PqN manifold $(\mathcal{M}, \pi, N, \phi)$ to construct the quasi-Lie bialgebroid $((T^*\mathcal{M})_\pi, d_N, \phi)$ as recalled in Sect. 2.2, and therefore the Courant algebroid $E = (T\mathcal{M})_N \oplus (T^*\mathcal{M})_\pi$ with its Lagrangian subbundle $A^* = T\mathcal{M}$ and its Dirac structure $A = T^*\mathcal{M}$. Given a closed 2-form Ω, we apply Formula (28) to each member of the pair $(T\mathcal{M}, T^*\mathcal{M})$, see Remark 11, to get $(L = \mathrm{Gr}(\Omega), T^*\mathcal{M})$, where L is a *non-integrable* Lagrangian subbundle of E. Note that the action of Ω on the original pair leaves $T^*\mathcal{M}$ fixed. Applying the construction presented in the first part of this section to the new pair $(L, T^*\mathcal{M})$, we obtain the quasi-Lie bialgebroid $((T^*\mathcal{M})_\pi, d_L, \varphi)$ which, thanks to Theorem 12, we are able to identify with $((T^*\mathcal{M})_\pi, d_{\widehat{N}}, \widehat{\phi})$. Finally, applying Proposition 6 to this quasi-Lie bialgebroid we conclude that the quadruplet $(\mathcal{M}, \pi, \widehat{N}, \widehat{\phi})$ is a Poisson quasi-Nijenhuis manifold, yielding our (third) proof of Theorem 3. In this way we are left with proving the following

Theorem 12 *If the 2-form Ω is closed, the quasi-Lie bialgebroid $((T^*\mathcal{M})_\pi, d_L, \varphi)$ coincides with $((T^*\mathcal{M})_\pi, d_{\widehat{N}}, \widehat{\phi})$, where $\widehat{N} = N + \pi^\sharp \Omega^\flat$ and $\widehat{\phi}$ is given by (12).*

The first step to prove this theorem is the computation of the bracket (22) between two sections of L. To this aim, we need two preliminary results.

Lemma 13 *For any closed 2-form Ω, the following relation holds:*

$$[\Omega^\flat X, \Omega^\flat Y]_\pi = \Omega^\flat[X, Y]_{\pi^\sharp \Omega^\flat} + \frac{1}{2}i_{X\wedge Y}[\Omega, \Omega]_\pi. \tag{30}$$

A direct proof of this lemma is given in Appendix B. Another proof can be spelled out along the lines of the following

Remark 14 For any 2-form Ω, not necessarily closed, a tedious but straightforward computation which uses (5) shows that

$$[\Omega^\flat X, \Omega^\flat Y]_\pi = \Omega^\flat[X, Y]_\Omega^\pi + \frac{1}{2}i_{X\wedge Y}[\Omega, \Omega]_\pi, \tag{31}$$

where

$$[X, Y]_\Omega^\pi = \mathcal{L}_{\Omega^\flat X}^\pi Y - \mathcal{L}_{\Omega^\flat Y}^\pi X - d_\pi(\Omega(X, Y))$$

and

$$\mathcal{L}_\alpha^\pi Y = \mathcal{L}_{\pi^\sharp \alpha} Y + \pi^\sharp (i_Y(\mathrm{d}\alpha)),$$

for all $X, Y \in \Gamma(T\mathcal{M})$ and $\alpha \in \Gamma(T^*\mathcal{M})$. Then one can obtain (30) from the fact that $[X, Y]_\Omega^\pi = [X, Y]_{\pi^\sharp \Omega^\flat}$ for all vector field X, Y if Ω is closed.

Lemma 15 *For any 2-form Ω, one has that*

$$\mathrm{d}_N(i_{X \wedge Y}\Omega) = i_X \mathrm{d}_N i_Y \Omega - i_Y \mathrm{d}_N i_X \Omega - i_{X \wedge Y} \mathrm{d}_N \Omega - i_{[X,Y]_N}\Omega. \tag{32}$$

Proof Using (9), we obtain

$$\begin{aligned}
\mathrm{d}_N \Omega(X, Y, Z) &= \mathcal{L}_{NX}\langle i_Y \Omega, Z\rangle - \mathcal{L}_{NY}\langle i_X \Omega, Z\rangle + \mathcal{L}_{NZ}\langle i_X \Omega, Y\rangle \\
&\quad - \Omega([X, Y]_N, Z) + \Omega([X, Z]_N, Y) - \Omega([Y, Z]_N, X) \\
&= \mathcal{L}_{NX}\langle i_Y \Omega, Z\rangle - \mathcal{L}_{NY}\langle i_X \Omega, Z\rangle + \langle \mathrm{d}(\Omega(X, Y)), NZ\rangle \\
&\quad - \langle i_{[X,Y]_N}\Omega, Z\rangle - \langle i_Y \Omega, [X, Z]_N\rangle + \langle i_X \Omega, [Y, Z]_N\rangle
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\langle i_X \mathrm{d}_N i_Y \Omega, Z\rangle = \mathrm{d}_N(i_Y \Omega)(X, Z) &= \mathcal{L}_{NX}\langle i_Y \Omega, Z\rangle - \mathcal{L}_{NZ}\langle i_Y \Omega, X\rangle - \langle i_Y \Omega, [X, Z]_N\rangle \\
\langle i_Y \mathrm{d}_N i_X \Omega, Z\rangle = \mathrm{d}_N(i_X \Omega)(Y, Z) &= \mathcal{L}_{NY}\langle i_X \Omega, Z\rangle - \mathcal{L}_{NZ}\langle i_X \Omega, Y\rangle - \langle i_X \Omega, [Y, Z]_N\rangle.
\end{aligned}$$

Substituting these relations into (33), we find (32) evaluated on an arbitrary vector field Z. □

Lemma 16 *If $\mathrm{d}\Omega = 0$, then*

$$[\![X + \Omega^\flat X, Y + \Omega^\flat Y]\!] = [X, Y]_{\widehat{N}} + \Omega^\flat [X, Y]_{\widehat{N}} + i_{X \wedge Y}\left(\phi + \mathrm{d}_N \Omega + \frac{1}{2}[\Omega, \Omega]_\pi\right). \tag{34}$$

Proof First we compute the vector field component of the left-hand side of (34). By definition (22), it is given by

$$\begin{aligned}
&[X, Y]_N + i_{\Omega^\flat X}(\mathrm{d}_\pi Y) - i_{\Omega^\flat Y}(\mathrm{d}_\pi X) + \mathrm{d}_\pi(\Omega(X, Y)) \\
&= [X, Y]_N - i_{\Omega^\flat X}(\mathcal{L}_Y \pi) + i_{\Omega^\flat Y}(\mathcal{L}_X \pi) - \pi^\sharp \mathrm{d}(\Omega(X, Y)) \\
&= [X, Y]_N - \mathcal{L}_Y(i_{\Omega^\flat X}\pi) + i_{\mathcal{L}_Y(\Omega^\flat X)}\pi + \mathcal{L}_X(i_{\Omega^\flat Y}\pi) - i_{\mathcal{L}_X(\Omega^\flat Y)}\pi - \pi^\sharp \mathrm{d}(\Omega(X, Y)) \\
&= [X, Y]_N + [\pi^\sharp \Omega^\flat X, Y] + [X, \pi^\sharp \Omega^\flat Y] \\
&\quad + \pi^\sharp \left(\mathcal{L}_Y(\Omega^\flat X) - \mathcal{L}_X(\Omega^\flat Y) - \mathrm{d}(\Omega(X, Y))\right).
\end{aligned}$$

Now, the last three terms are simply given by

$$\begin{aligned}
\mathcal{L}_Y(i_X \Omega) - \mathcal{L}_X(i_Y \Omega) - (\mathrm{d} \circ i_Y)i_X \Omega &= (i_Y \circ \mathrm{d})i_X \Omega - \mathcal{L}_X(i_Y \Omega) \\
&= i_Y((\mathrm{d} \circ i_X)\Omega) - i_{[X,Y]}\Omega - i_Y(\mathcal{L}_X \Omega) \\
&= -i_Y i_X \mathrm{d}\Omega - i_{[X,Y]}\Omega = -\Omega^\flat[X, Y]
\end{aligned}$$

since Ω is closed. Hence we have shown that the vector field component of the left-hand side of (34) is

$$[X, Y]_N + [\pi^\sharp \Omega^\flat X, Y] + [X, \pi^\sharp \Omega^\flat Y] - \pi^\sharp \Omega^\flat [X, Y] = [X, Y]_{N+\pi^\sharp \Omega^\flat} = [X, Y]_{\widehat{N}}.$$

The 1-form component of the left-hand side of (34) is

$$[\Omega^\flat X, \Omega^\flat Y]_\pi + i_{X \wedge Y} \phi - i_Y d_N i_X \Omega + i_X d_N i_Y \Omega - d_N(\Omega(X, Y)). \quad (35)$$

Using Lemma 13 for the first term and Lemma 15 for the last three, the sum (35) turns out to be

$$\Omega^\flat [X, Y]_{\pi^\sharp \Omega^\flat} + \frac{1}{2} i_{X \wedge Y} [\Omega, \Omega]_\pi + i_{X \wedge Y} \phi + i_{X \wedge Y} d_N \Omega + i_{[X, Y]_N} \Omega$$
$$= \Omega^\flat [X, Y]_{\widehat{N}} + i_{X \wedge Y} \left(\phi + d_N \Omega + \frac{1}{2} [\Omega, \Omega]_\pi \right).$$

$\square$

We can now prove Theorem 12, i.e., that the quasi-Lie bialgebroid $((T^*\mathcal{M})_\pi, d_L, \varphi)$ coincides with $((T^*\mathcal{M})_\pi, d_{\widehat{N}}, \widehat{\phi})$, where $\widehat{N} = N + \pi^\sharp \Omega^\flat$ and $\widehat{\phi} = \phi + d_N \Omega + \frac{1}{2} [\Omega, \Omega]_\pi$. First of all, we notice that the identification (23) in this case is simply $\tilde{X} = X + \Omega^\flat X$, where $X \in \Gamma(T\mathcal{M})$. By definition (25) of φ and using Lemma 16, we have that

$$\begin{aligned}
\varphi(X, Y, Z) &= 2\langle [\![\tilde{X}, \tilde{Y}]\!], \tilde{Z} \rangle_E \\
&= 2\langle [\![X + \Omega^\flat X, Y + \Omega^\flat Y]\!], Z + \Omega^\flat Z \rangle_E \\
&= 2\left\langle [X, Y]_{\widehat{N}} + \Omega^\flat [X, Y]_{\widehat{N}} + i_{X \wedge Y} \left(\phi + d_N \Omega + \frac{1}{2} [\Omega, \Omega]_\pi \right), Z + \Omega^\flat Z \right\rangle_E \\
&= \langle \Omega^\flat Z, [X, Y]_{\widehat{N}} \rangle + \left\langle \Omega^\flat [X, Y]_{\widehat{N}} + i_{X \wedge Y} \left(\phi + d_N \Omega + \frac{1}{2} [\Omega, \Omega]_\pi \right), Z \right\rangle \\
&= \left\langle i_{X \wedge Y} \left(\phi + d_N \Omega + \frac{1}{2} [\Omega, \Omega]_\pi \right), Z \right\rangle \\
&= \widehat{\phi}(X, Y, Z)
\end{aligned}$$

for all $X, Y, Z \in \Gamma(T\mathcal{M})$. So we are left with showing that d_L acts as $d_{\widehat{N}}$. But this immediately follows from

$$\rho(X + \Omega^\flat X) = NX + \pi^\sharp \Omega^\flat X = \widehat{N} X$$
$$\text{and} \quad [X + \Omega^\flat X, Y + \Omega^\flat Y]_L = [X, Y]_{\widehat{N}} + \Omega^\flat [X, Y]_{\widehat{N}}.$$

Indeed, if $\psi \in \Gamma(\bigwedge^k A)$ corresponds to

$\tilde{\psi} \in \Gamma(\bigwedge^k L^*)$, then from (24) we obtain

$$
\begin{aligned}
(\mathrm{d}_L \psi)(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \rho(\tilde{X}_i) \left(\tilde{\psi}(\tilde{X}_1, \ldots, \hat{\tilde{X}}_i, \ldots, \tilde{X}_{k+1}) \right) \\
&\quad + \sum_{i<j} (-1)^{i+j} \tilde{\psi}([\tilde{X}_i, \tilde{X}_j]_L, \tilde{X}_1 \ldots, \hat{\tilde{X}}_i, \ldots, \hat{\tilde{X}}_j, \ldots, \tilde{X}_{k+1}) \\
&= \sum_{i=1}^{k+1} (-1)^{i+1} \left(\widehat{N} X_i \right) \left(\psi(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \right) \\
&\quad + \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j]_{\widehat{N}}, X_1 \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \\
&= (\mathrm{d}_{\widehat{N}} \psi)(X_1, \ldots, X_{k+1}).
\end{aligned}
$$

Appendix A: Definition of Courant Algebroid

In this appendix we will recall the definition of Courant algebroid following Liu et al. (1997)—see also Courant (1990), where this definition appeared for the first time. To this end, let $\mathcal{M}$ be a manifold. A Courant algebroid (over $\mathcal{M}$) is a quadruplet $(E, \langle \cdot, \cdot \rangle_E, [\![\cdot, \cdot]\!], \rho)$, where

(1) E is a vector bundle over $\mathcal{M}$;
(2) $\langle \cdot, \cdot \rangle_E : \Gamma(E) \times \Gamma(E) \to \mathbb{R}$ is a symmetric, non-degenerate, and $C^\infty(\mathcal{M})$-bilinear form;
(3) $[\![\cdot, \cdot]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is an $\mathbb{R}$-bilinear, skew-symmetric bracket and
(4) $\rho : E \to TM$ is a bundle-map, called the *anchor* of the Courant algebroid, inducing the $\mathbb{R}$-linear operator $\mathcal{D} : C^\infty(M) \to \Gamma(E)$, via the formula

$$
\langle \mathcal{D} f, A \rangle_E := \frac{1}{2} \rho(A)(f), \qquad \forall A \in \Gamma(E), \ f \in C^\infty(\mathcal{M}), \tag{A1}
$$

satisfying the following compatibility conditions. For all $A, B, C \in \Gamma(E)$ and $f, g \in C^\infty(\mathcal{M})$,

 (i) ρ is *bracket-compatible*, i.e., $\rho([\![A, B]\!]) = [\rho(A), \rho(B)]$;
 (ii) $[\![[\![A, B]\!], C]\!] + [\![[\![B, C]\!], A]\!] + [\![[\![C, A]\!], B]\!] = \frac{1}{3} \mathcal{D} \big(\langle [\![A, B]\!], C \rangle_E + \langle [\![B, C]\!], A \rangle_E + \langle [\![C, A]\!], B \rangle_E \big)$;
(iii) $[\![A, f B]\!] = f [\![A, B]\!] + \rho(A)(f) B - \langle A, B \rangle_E \mathcal{D}(f)$;
(iv) $\langle \mathcal{D}(f), \mathcal{D}(g) \rangle_E = 0$;
 (v) $\rho(A) \langle B, C \rangle_E = \langle [\![A, B]\!] + \mathcal{D} \langle A, B \rangle_E, C \rangle_E + \langle B, [\![A, C]\!] + \mathcal{D} \langle A, C \rangle_E \rangle_E$.

A few comments are in order.

Remark 17 Note that:

- as shown in Uchino (2002), conditions (iii) and (iv) follow from the other conditions;

- $\mathcal{D}$ defined in (A1) is a differential operator in the sense that it satisfies the Leibniz identity;
- $[\![\cdot, \cdot]\!]$ is not a Lie bracket since it does not satisfies the Jacobi's identity, see item (ii) of the previous list.

Example 18 (Quadratic Lie algebras) A Courant algebroid over a point, i.e., if $\mathcal{M} = \{pt\}$, is the same as a quadratic Lie algebra, i.e., a Lie algebra endowed with a symmetric, non-degerate and ad-invariant bilinear form. In fact, if $\mathcal{M} = \{pt\}$, then $\rho = 0$, which forces the condition $\mathcal{D} = 0$. In this way $[\![\cdot, \cdot]\!]$ becomes a Lie bracket and $\langle \cdot, \cdot \rangle_E$ becomes a non-degenerate, symmetric and ad-invariant bilinear form, see items (ii) and (v) above.

Example 19 (Standard Courant algebroid) In this case $E = \mathbb{T}\mathcal{M} = T\mathcal{M} \oplus T^*\mathcal{M}$, $\rho : E \to T\mathcal{M}$ is the projection on the first summand, and $\langle X + \alpha, Y + \beta \rangle_E = \frac{1}{2}(\langle \alpha, Y \rangle + \langle \beta, X \rangle)$. In particular, $\langle \mathcal{D}f, X + \alpha \rangle_E = \frac{1}{2}X(f)$. Moreover,

$$
\begin{aligned}
[\![X + \alpha, Y + \beta]\!] &= [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2}\mathrm{d}(i_Y \alpha - i_X \beta) \\
&= [X, Y] + i_X \mathrm{d}\beta - i_Y \mathrm{d}\alpha + \frac{1}{2}\mathrm{d}(i_X \beta - i_Y \alpha) \quad (A2)
\end{aligned}
$$

for all $f \in C^\infty(\mathcal{M})$ and $X + \alpha, Y + \beta \in \Gamma(E)$. Note that the pair $(T\mathcal{M}, T^*\mathcal{M})$ forms a Lie bialgebroid, i.e., a Lie quasi-bialgebroid such that $\phi = 0$, see Definition 5. It turns out that $T\mathcal{M}$ carries the structure of Lie algebroid defined by the standard Lie brackets on vector fields and by the identity as anchor map, while $T^*\mathcal{M}$ carries the trivial Lie algebroid structure, i.e., with zero Lie bracket and zero anchor map.

The bracket (A2) can be modified by twisting it with the term $i_{X \wedge Y}\phi$, where ϕ is any closed 3-form on $\mathcal{M}$. The resulting structure is called *twisted* Courant algebroid and it is denoted with $\mathbb{T}\mathcal{M}^\phi$. Twisted Courant algebroids were introduced by Ševera, see Ševera (2017), who proved that a Courant algebroid E fits into the exact sequence

$$
0 \longrightarrow T^*\mathcal{M} \xrightarrow{\rho^*} E \xrightarrow{\rho} T\mathcal{M} \longrightarrow 0,
$$

if and only if E is isomorphic to $\mathbb{T}\mathcal{M}^\phi$ for some closed 3-form. In the exact sequence ρ denotes the anchor of E.

Appendix B: Proof of Lemma 13

We will now prove that, for all vector fields X, Y, the identity

$$
[\Omega^\flat X, \Omega^\flat Y]_\pi = \Omega^\flat [X, Y]_{\pi^\sharp \Omega^\flat} + \frac{1}{2}i_{X \wedge Y}[\Omega, \Omega]_\pi
$$

holds true. Since $d\Omega = 0$, we can use (14) to compute $[\Omega, \Omega]_\pi = d_{\pi^\sharp \Omega^\flat} \Omega$. Then (9) entails that

$$[\Omega, \Omega]_\pi (X, Y, Z) = \sum_{\circlearrowleft (X,Y,Z)} \left(\mathcal{L}_{\pi^\sharp \Omega^\flat Z} (\Omega(X, Y)) - \Omega \left([X, Y]_{\pi^\sharp \Omega^\flat}, Z \right) \right) \quad \text{(B1)}$$

for any vector fields X, Y, and Z. To show that (30) holds, we define

$$A(X, Y, Z) = \langle [\Omega^\flat X, \Omega^\flat Y]_\pi - \Omega^\flat [X, Y]_{\pi^\sharp \Omega^\flat}, Z \rangle$$

and we first show that $A(X, Y, Y) = 0$, so that $A(X, Y, Z) = -A(X, Z, Y)$ for all $X, Y, Z \in \Gamma(T\mathcal{M})$. Indeed,

$$
\begin{aligned}
A(X, Y, Y) &= \Big\langle \mathcal{L}_{\pi^\sharp \Omega^\flat X}(\Omega^\flat Y) - \mathcal{L}_{\pi^\sharp \Omega^\flat Y}(\Omega^\flat X) - d\langle \Omega^\flat Y, \pi^\sharp \Omega^\flat X \rangle \\
&\quad - \Omega^\flat \left([\pi^\sharp \Omega^\flat X, Y] + [X, \pi^\sharp \Omega^\flat Y] - \pi^\sharp \Omega^\flat [X, Y] \right), Y \Big\rangle \\
&= \mathcal{L}_{\pi^\sharp \Omega^\flat X} \langle \Omega^\flat Y, Y \rangle - \underline{\langle \Omega^\flat Y, [\pi^\sharp \Omega^\flat X, Y] \rangle} - \mathcal{L}_{\pi^\sharp \Omega^\flat Y} \langle \Omega^\flat X, Y \rangle \\
&\quad + \langle \Omega^\flat X, [\pi^\sharp \Omega^\flat Y, Y] \rangle - \mathcal{L}_Y \langle \Omega^\flat Y, \pi^\sharp \Omega^\flat X \rangle \\
&\quad + \underline{\langle \Omega^\flat Y, [\pi^\sharp \Omega^\flat X, Y] \rangle} + \langle \Omega^\flat Y, [X, \pi^\sharp \Omega^\flat Y] \rangle - \langle \Omega^\flat \pi^\sharp \Omega^\flat Y, [X, Y] \rangle \\
&= -\mathcal{L}_{\pi^\sharp \Omega^\flat Y} \langle \Omega^\flat X, Y \rangle - \langle \Omega^\flat [\pi^\sharp \Omega^\flat Y, Y], X \rangle \\
&\quad - \mathcal{L}_Y \langle \Omega^\flat Y, \pi^\sharp \Omega^\flat X \rangle - \langle \Omega^\flat Y, \mathcal{L}_{\pi^\sharp \Omega^\flat Y} X \rangle \\
&\quad + \langle \Omega^\flat \pi^\sharp \Omega^\flat Y, \mathcal{L}_Y X \rangle \\
&= -\underline{\mathcal{L}_{\pi^\sharp \Omega^\flat Y} \langle \Omega^\flat X, Y \rangle} - \langle \Omega^\flat [\pi^\sharp \Omega^\flat Y, Y], X \rangle \\
&\quad - \overline{\mathcal{L}_Y \langle \Omega^\flat Y, \pi^\sharp \Omega^\flat X \rangle} - \underline{\mathcal{L}_{\pi^\sharp \Omega^\flat Y} \langle \Omega^\flat Y, X \rangle} \\
&\quad + \langle \mathcal{L}_{\pi^\sharp \Omega^\flat Y}(\Omega^\flat Y), X \rangle + \overline{\mathcal{L}_Y \langle \Omega^\flat \pi^\sharp \Omega^\flat Y, X \rangle} - \langle \mathcal{L}_Y (\Omega^\flat \pi^\sharp \Omega^\flat Y), X \rangle \\
&= \langle -\Omega^\flat [\pi^\sharp \Omega^\flat Y, Y] + \mathcal{L}_{\pi^\sharp \Omega^\flat Y}(\Omega^\flat Y) - \mathcal{L}_Y (\Omega^\flat \pi^\sharp \Omega^\flat Y), X \rangle,
\end{aligned}
$$

and we have that

$$
\begin{aligned}
&- \Omega^\flat [\pi^\sharp \Omega^\flat Y, Y] + \mathcal{L}_{\pi^\sharp \Omega^\flat Y}(\Omega^\flat Y) - \mathcal{L}_Y (\Omega^\flat \pi^\sharp \Omega^\flat Y) \\
&= -i_{\mathcal{L}_{\pi^\sharp \Omega^\flat Y} Y}(\Omega) + \mathcal{L}_{\pi^\sharp \Omega^\flat Y}(i_Y \Omega) - \mathcal{L}_Y (i_{\pi^\sharp \Omega^\flat Y} \Omega) \\
&= i_Y \left(\mathcal{L}_{\pi^\sharp \Omega^\flat Y} \Omega \right) - (d \circ i_Y + i_Y \circ d) \left(i_{\pi^\sharp \Omega^\flat Y} \Omega \right) \\
&= i_Y \left(\mathcal{L}_{\pi^\sharp \Omega^\flat Y} \Omega \right) - \left(i_Y \circ d \circ i_{\pi^\sharp \Omega^\flat Y} \right) \Omega \\
&= i_Y \left(\mathcal{L}_{\pi^\sharp \Omega^\flat Y} \Omega \right) - i_Y \left(\mathcal{L}_{\pi^\sharp \Omega^\flat Y} \Omega - i_{\pi^\sharp \Omega^\flat Y} d\Omega \right),
\end{aligned}
$$

which vanishes if Ω is closed. Now,

$$2A(X, Y, Z) = A(X, Y, Z) - A(X, Z, Y)$$

$$= \left\langle \mathcal{L}_{\pi^\sharp \Omega^\flat X}(\Omega^\flat Y) - \mathcal{L}_{\pi^\sharp \Omega^\flat Y}(\Omega^\flat X) - \mathrm{d}\langle \Omega^\flat Y, \pi^\sharp \Omega^\flat X\rangle - \Omega^\flat [X, Y]_{\pi^\sharp \Omega^\flat}, Z \right\rangle$$

$$- \left\langle \mathcal{L}_{\pi^\sharp \Omega^\flat X}(\Omega^\flat Z) - \mathcal{L}_{\pi^\sharp \Omega^\flat Z}(\Omega^\flat X) - \mathrm{d}\langle \Omega^\flat Z, \pi^\sharp \Omega^\flat X\rangle - \Omega^\flat [X, Z]_{\pi^\sharp \Omega^\flat}, Y \right\rangle$$

$$= \mathcal{L}_{\pi^\sharp \Omega^\flat X}\langle \Omega^\flat Y, Z\rangle - \langle \Omega^\flat Y, [\pi^\sharp \Omega^\flat X, Z]\rangle - \mathcal{L}_{\pi^\sharp \Omega^\flat Y}\langle \Omega^\flat X, Z\rangle$$

$$+ \langle \Omega^\flat X, [\pi^\sharp \Omega^\flat Y, Z]\rangle$$

$$- \left\langle \mathrm{d}\langle \Omega^\flat Y, \pi^\sharp \Omega^\flat X\rangle, Z\right\rangle - \Omega\left([X, Y]_{\pi^\sharp \Omega^\flat}, Z\right)$$

$$- \langle \mathcal{L}_{\pi^\sharp \Omega^\flat X}(\Omega^\flat Z), Y\rangle + \mathcal{L}_{\pi^\sharp \Omega^\flat Z}\langle \Omega^\flat X, Y\rangle - \langle \Omega^\flat X, [\pi^\sharp \Omega^\flat Z, Y]\rangle$$

$$+ \left\langle \mathrm{d}\langle \Omega^\flat Z, \pi^\sharp \Omega^\flat X\rangle, Y\right\rangle + \Omega\left([X, Z]_{\pi^\sharp \Omega^\flat}, Y\right)$$

$$= \mathcal{L}_{\pi^\sharp \Omega^\flat X}\left(\Omega(Y, Z)\right) + \mathcal{L}_{\pi^\sharp \Omega^\flat Y}\left(\Omega(Z, X)\right) + \mathcal{L}_{\pi^\sharp \Omega^\flat Z}\left(\Omega(X, Y)\right)$$

$$- \Omega\left([X, Y]_{\pi^\sharp \Omega^\flat}, Z\right) - \Omega\left([Z, X]_{\pi^\sharp \Omega^\flat}, Y\right)$$

$$- \langle \Omega^\flat Y, [\pi^\sharp \Omega^\flat X, Z]\rangle + \langle \Omega^\flat X, [\pi^\sharp \Omega^\flat Y, Z]\rangle - \left\langle \mathrm{d}\langle \Omega^\flat Y, \pi^\sharp \Omega^\flat X\rangle, Z\right\rangle$$

$$- \langle \mathcal{L}_{\pi^\sharp \Omega^\flat X}(\Omega^\flat Z), Y\rangle - \langle \Omega^\flat X, [\pi^\sharp \Omega^\flat Z, Y]\rangle + \left\langle \mathrm{d}\langle \Omega^\flat Z, \pi^\sharp \Omega^\flat X\rangle, Y\right\rangle.$$

The last six terms are equal to

$$\langle \Omega^\flat Y, \mathcal{L}_Z(\pi^\sharp \Omega^\flat X)\rangle - \Omega\left([\pi^\sharp \Omega^\flat Y, Z] + [Y, \pi^\sharp \Omega^\flat Z], X\right) - \mathcal{L}_Z\langle \Omega^\flat Y, \pi^\sharp \Omega^\flat X\rangle$$

$$- \langle\!\!\!\!\!\diagdown(\mathrm{d} \circ i_{\pi^\sharp \Omega^\flat X})(i_Z \Omega), Y\rangle - \langle (i_{\pi^\sharp \Omega^\flat X} \circ \mathrm{d})(i_Z \Omega), Y\rangle$$

$$+ \langle\!\!\!\!\!\diagdown\mathrm{d}\langle i_Z \Omega, \pi^\sharp \Omega^\flat X\rangle, Y\rangle = -\langle \mathcal{L}_Z(\Omega^\flat Y), \pi^\sharp \Omega^\flat X\rangle$$

$$- \Omega\left([\pi^\sharp \Omega^\flat Y, Z] + [Y, \pi^\sharp \Omega^\flat Z], X\right) + \langle (i_Y \circ \mathrm{d})(i_Z \Omega), \pi^\sharp \Omega^\flat X\rangle$$

$$= -\langle \mathcal{L}_Z(i_Y \Omega), \pi^\sharp \Omega^\flat X\rangle - \Omega\left([\pi^\sharp \Omega^\flat Y, Z] + [Y, \pi^\sharp \Omega^\flat Z], X\right)$$

$$+ \langle i_Y(\mathcal{L}_Z \Omega - i_Z(\mathrm{d}\Omega)), \pi^\sharp \Omega^\flat X\rangle$$

$$= -\langle i_{[Z,Y]}\Omega, \pi^\sharp \Omega^\flat X\rangle - \Omega\left([\pi^\sharp \Omega^\flat Y, Z] + [Y, \pi^\sharp \Omega^\flat Z], X\right)$$

$$= -\Omega\left([Y, Z]_{\pi^\sharp \Omega^\flat}, X\right),$$

where we have used again the fact that $\mathrm{d}\Omega = 0$. So we conclude that

$$2A(X, Y, Z) = \sum_{\circlearrowleft(X,Y,Z)} \left(\mathcal{L}_{\pi^\sharp \Omega^\flat X}\left(\Omega(Y, Z)\right) - \Omega\left([X, Y]_{\pi^\sharp \Omega^\flat}, Z\right)\right).$$

The comparison between this formula and (B1) ends the proof.

References

Antunes, P.: Poisson quasi-Nijenhuis structures with background. Lett. Math. Phys. **86**, 33–45 (2008)

Bolsinov, A.V., Konyaev, A.Y., Matveev, V.S.: Nijenhuis geometry. Adv. Math. **394**, 52 (2022)

Bressler, P.: The first Pontryagin class. Compos. Math. **143**, 1127–1163 (2007)

Bursztyn, H., Drummond, T.: Lie theory of multiplicative tensors. Math. Ann. **375**, 1489–1554 (2019)

Bursztyn, H., Drummond, T., Netto, C.: Dirac structures and Nijenhuis operators. Math. Z. **302**, 875–915 (2022)

Cordeiro, F., Nunes da Costa, J.M.: Reduction and construction of Poisson quasi-Nijenhuis manifolds with background. Int. J. Geom. Methods Mod. Phys. **7**, 539–564 (2010)

Courant, T.: Dirac manifolds. Trans. Am. Math. Soc. **319**, 631–661 (1990)

Dorfman, I.Y.: Dirac structures of integrable evolution equations. Phys. Lett. A **125**, 240–246 (1987)

Dorfman, I.: Dirac Structures and Integrability of Nonlinear Evolution Equations. Wileys, Chichester (1993)

Falqui, G., Mencattini, I., Ortenzi, G., Pedroni, M.: Poisson quasi-Nijenhuis manifolds and the Toda system. Math. Phys. Anal. Geom. **23**, 17 (2020)

Falqui, G., Mencattini, I., Pedroni, M.: Poisson quasi-Nijenhuis deformations of the canonical PN structure. J. Geom. Phys. **186**, 10 (2023)

Gualtieri, M.: Generalized complex geometry. Ann. Math. **174**, 75–123 (2011)

Iglesias-Ponte, D., Laurent-Gengoux, C., Xu, P.: Universal lifting theorem and quasi-Poisson groupoids. J. Eur. Math. Soc. **14**, 681–731 (2012)

Kosmann-Schwarzbach, Y.: The Lie bialgebroid of a Poisson–Nijenhuis manifold. Lett. Math. Phys. **38**, 421–428 (1996)

Kosmann-Schwarzbach, Y., Magri, F.: Poisson–Nijenhuis structures. Ann. Inst. Henri Poincaré **53**, 35–81 (1990)

Liu, Z.-J., Weinstein, A., Xu, P.: Manin triples for Lie bialgebroids. J. Differ. Geom. **45**, 547–574 (1997)

Mackenzie, K.C.H.: General Theory of Lie Groupoids and Lie Algebroids. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (2005)

Mackenzie, K.C.H., Xu, P.: Lie bialgebroids and Poisson groupoids. Duke Math. J. **73**, 415–452 (1994)

Magri, F., Morosi, C., Ragnisco, O.: Reduction techniques for infinite-dimensional Hamiltonian systems: some ideas and applications. Commun. Math. Phys. **99**, 115–140 (1985)

Marle, C.-M.: Calculus on Lie algebroids, Lie groupoids and Poisson manifolds. Dissertationes Math. **457**, 57 (2008)

Nijenhuis, A.: X_{n-1}-forming sets of eigenvectors. Proc. Kon. Ned. Akad. Amsterdam **54**, 200–212 (1951)

Roytenberg, D.: Quasi-Lie bialgebroids and twisted Poisson manifolds. Lett. Math. Phys. **61**, 123–137 (2002)

Ševera, P.: Letters to Alan Weinstein about Courant algebroids. (2017). arXiv:1707.00265
Stiénon, M., Xu, P.: Poisson quasi-Nijenhuis manifolds. Commun. Math. Phys. **270**, 709–725 (2007)
Uchino, K.: Remarks on the definition of a Courant algebroid. Lett. Math. Phys. **60**, 171–175 (2002)

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