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One-Dimensional Fokker–Planck Equations and Functional Inequalities for Heavy Tailed Densities

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Abstract. We present and discuss connections between the problem of trend to equilibrium for one-dimensional Fokker–Planck equations modeling socio-economic problems, and one-dimensional functional inequalities of the type of Poincaré, Wirtinger and logarithmic Sobolev, with weight, for probability densities with polynomial tails. As main examples, we consider inequalities satisfied by inverse Gamma densities, taking values on $\mathbb{R}_+$, and Cauchy-type densities, taking values on $\mathbb{R}$.

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1. Introduction

The deep connections between the problem of trend to equilibrium for the classical Fokker–Planck equation of statistical physics and several inequalities from functional analysis, have been introduced and discussed by Markowich and Villani more than twenty years ago [22]. Few years later, mathematical modeling of socio-economic problems via well-established methods of kinetic theory led to consider a new class of Fokker–Planck equations, mainly one-dimensional in the space variable. In contrast to the classical one considered in [22], these equations are characterized by variable coefficient of diffusion, and by equilibria with polynomial tails [15]. The study of the relaxation problem related to this new type of Fokker–Planck equations put a new effort in deriving functional inequalities for their equilibria.

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In the following, we will deal with this challenging physical problem, in which the balance between the diffusion coefficient and the drift of the Fokker–Planck equation plays an essential role. In particular, we will show that the Fokker–Planck representation of the steady state induces in a natural way one-dimensional functional inequalities of the type of Poincaré, Wirtinger and logarithmic Sobolev, with weight, for probability densities with polynomial tails. While the classical Fokker–Planck equation treated in [22] is deeply connected with the Gaussian density, the main examples are furnished here by the inverse Gamma and Cauchy-type probability densities.

Let X be a random variable distributed with probability density $f(x)$, where $x \in \mathcal{I} \subseteq \mathbb{R}$, and let $w = w(x)$ be a fixed nonnegative, Borel measurable function on $\mathcal{I}$. The random variable X is said to satisfy a weighted Poincaré-type inequality with weight function $w(x)$, if for any bounded smooth function ϕ on $\mathcal{I}$

$$Var\,[\phi(X)] \le E\left\{w(X)[\phi'(X)]^2\right\}. \tag{1.1}$$

As usual, for a given random variable Y, $E(Y)$ denotes its expectation value, and

$$Var\,[\phi(X)] = \int_{\mathcal{I}} \phi^2(x)\, f(x)\, dx - \left(\int_{\mathcal{I}} \phi(x)\, f(x)\, dx\right)^2$$

is the variance of ϕ with respect to f. Likewise, X is said to satisfy a weighted logarithmic Sobolev inequality with weight function $w(x)$ if, for any bounded smooth function ϕ on $\mathcal{I}$

$$Ent\left[\phi^2(X)\right] \le E\left\{w(X)[\phi'(X)]^2\right\}. \tag{1.2}$$

Here

$$\begin{aligned} Ent\left[\phi^2(X)\right] = &\int_{\mathcal{I}} \phi^2(x) \log \phi^2(x)\, f(x)\, dx \\ &- \left(\int_{\mathcal{I}} \phi^2(x)\, f(x)\, dx\right) \log\left(\int_{\mathcal{I}} \phi^2(x)\, f(x)\, dx\right) \end{aligned}$$

denotes the entropy of ϕ^2 with respect to f. Last, the random variable X is said to satisfy a weighted Wirtinger-type inequality (of order p) with weight function $w_p(x)$, if for any bounded smooth function ϕ on $\mathcal{I}$, and $p \ge 1$

$$E\left\{|\phi(X) - E(\Phi(X))|^p\}\right] \le E\left\{w_p^p(X)|\phi'(X)|^p\right\}. \tag{1.3}$$

The inequalities are understood in the following sense: if the right-hand side is finite, then the inequalities hold true.

Abstract weighted Poincaré and logarithmic Sobolev inequalities are connected with the problem of large deviations of Lipschitz functions and measure concentration. In reason of that, the question whether a probability measure satisfies such functional inequalities has attracted a lot of attention in recent years [1,3–6,10,19].

In the probabilistic literature, inequality (1.1) is also known under the name of weighted Chernoff inequality, in reason of the analogous inequality with weight $w(x) = 1$ obtained by Chernoff [11] for the one-dimensional Gaussian density

$$g(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}, \qquad x \in \mathbb{R}. \tag{1.4}$$

Chernoff-type inequalities with weight were proven, few years later Chernoff's result, by Klaassen [21], who listed a number of probability densities for which the weight $w(x)$ was explicitly computable. A different proof of Chernoff-type inequalities with weight, valid for heavy tailed densities, has been recently obtained in [15] by resorting to the representation of these densities as equilibria of Fokker–Planck type equations with variable coefficient of diffusion and linear drift.

The analysis of [15] was deeply motivated by the study of Fokker–Planck type equations appearing in the modeling of social and economic phenomena, a challenging research activity in the communities of both physicists and applied mathematicians, who classified the fields of research with the names of socio-physics and, respectively, econophysics [15,23,26].

One of the typical features of these phenomena is related to the tails of the underlying steady distribution, which are often characterized by polynomial decay at infinity [24]. The classical example is furnished by the study of the distribution of wealth among trading agents, that leads to a Fokker–Planck type equation with variable coefficients of diffusion and linear drift [7,12]. This equation, which describes the time-evolution of the density $f(x,t)$ of a system of agents with personal wealth $x \geq 0$ at time $t \geq 0$ reads

$$\frac{\partial f(x,t)}{\partial t} = \frac{\sigma}{2}\frac{\partial^2}{\partial x^2}\left(x^2 f(x,t)\right) + \lambda \frac{\partial}{\partial x}\left((x-1)f(x,t)\right). \tag{1.5}$$

In (1.5), σ and λ denote positive constants related to essential properties of the trade rules of the agents. By fixing the mass density equal to unity, the unique steady state of Eq. (1.5) is the inverse Gamma density

$$f_\infty(x) = \frac{\mu^{1+\mu}}{\Gamma(1+\mu)}\frac{\exp\left(-\frac{\mu}{x}\right)}{x^{2+\mu}}, \tag{1.6}$$

characterized by the positive constant μ, given by

$$\mu = 2\frac{\lambda}{\sigma}. \tag{1.7}$$

This stationary distribution, in agreement with the analysis of the Italian economist Vilfredo Pareto [27], exhibits a power-law tail for large values of the wealth variable.

One of the physically relevant questions related to the Fokker–Planck equation (1.5) is the knowledge of the exact rate of relaxation to equilibrium of its solution, which, as it happens for the classical Fokker–Planck equation [33], is expected to be exponential in time. This rate of relaxation would in fact justify that in real economies the wealth distribution profile is always well fitted, for large values of the wealth variable, by the inverse Gamma [26].

The study of relaxation to equilibrium for the solution to equation (1.5) treasured the successful methodology used for the classical Fokker–Planck equation [22], corresponding to constant coefficient of diffusion and linear drift. Hence, the relaxation of the solution of (1.5) towards equilibrium has been usually investigated by looking at the time evolution of its Shannon entropy relative to the equilibrium density [15]. We recall that, given two probability densities $f(x)$ and $g(x)$, with $x \in \mathcal{I} \subseteq \mathbb{R}$, the Shannon entropy of f relative to g is defined by

$$H(f|g) = \int_{\mathcal{I}} f(x) \log \frac{f(x)}{g(x)}\, dx. \tag{1.8}$$

This argument has its roots in classical statistical physics, and, similarly to the classical kinetic theory of rarefied gases, identifies the steady state solution of the Fokker–Planck equation (1.5) as the target density to be reached monotonically in time in relative entropy [15]. The proof of the exponential convergence of the solution to the classical Fokker–Planck equation towards the Maxwellian (Gaussian) density (1.4) follows by applying the logarithmic Sobolev inequality [32,33], which establishes a sharp bound of the relative entropy in terms of the entropy production. This introduces a deep link between differential inequalities of Sobolev type and Fokker–Planck equations [22]. While the standard logarithmic Sobolev inequality allows us to prove exponential convergence of the solution towards the Maxwellian equilibrium density (1.4) in relative entropy, at the same time the evolution of the relative entropy of the solution density of the Fokker–Planck equation can be used to obtain a dynamical proof of the logarithmic Sobolev inequality [2,32,33]. This idea has been subsequently extended, in order to obtain sharp differential inequalities, to Fokker–Planck type equations with constant diffusion term and general drift by Otto and Villani [25].

As a matter of fact, however, even if various weaker results are available [31], a proof of exponential convergence in relative entropy of the solution to the Fokker–Planck equation (1.5) towards the inverse Gamma (1.6) is at present not available. In a recent paper [17] a possible motivation of this unpleasant difference in convergence between the classical and the wealth Fokker–Planck equations has been identified in the choice of a Maxwellian (constant) interaction kernel, made in [12], in the kinetic equation leading to (1.5). Without going into detail regarding this discussion about the modeling assumptions, that the interested reader can find in [17], the introduction of a variable collision kernel led to build a new Fokker–Planck equation with variable coefficient of diffusion and variable drift, still describing the time-evolution of the density $f(x,t)$ of a system of agents with personal wealth $x \geq 0$ at time $t \geq 0$. This new Fokker–Planck equations reads

$$\frac{\partial f(x,t)}{\partial t} = \frac{\sigma}{2} \frac{\partial^2}{\partial x^2} \left(x^{2+\delta} f(x,t) \right) + \lambda \frac{\partial}{\partial x} \left(x^{\delta}(x-1) f(x,t) \right). \tag{1.9}$$

In (1.9) δ is a positive constant, with $0 < \delta \leq 1$. Equation (1.9) has a unique equilibrium density of unit mass, still given by an inverse Gamma function

$$f_\infty^\delta(x) = \frac{\mu^{1+\delta+\mu}}{\Gamma(1+\delta+\mu)} \frac{\exp\left(-\frac{\mu}{x}\right)}{x^{2+\delta+\mu}}. \tag{1.10}$$

In (1.10) μ is the positive constant defined in (1.7). Hence, the presence of the constant δ is such that the Pareto index in the equilibrium density of the target Fokker–Planck equation is increased by the amount $\delta > 0$. Note that this class of Fokker–Planck type equations contains (1.5), which is obtained in the limit $\delta \to 0$.

As proven in [17], and in contrast to Eq. (1.5), the solution to the Fokker–Planck equation (1.9) has been shown to converge exponentially in relative entropy, with explicit rate, towards the equilibrium density (1.10). For this reason, Eq. (1.9) has been proposed as a better model for the description of the process of relaxation of the wealth distribution density in a multi-agent society [17].

A critical comparison of the two Fokker–Planck equations (1.5) and (1.9) allows us to come to some interesting conclusions, that will be at the basis of this paper. The inverse Gamma density is the steady state of different Fokker–Planck equations, which can share different properties in relation with convergence to equilibrium. Indeed, by looking at the computations in [17], the evolution of the Shannon entropy of the solution to the Fokker–Planck equation (1.9) relative to the equilibrium solution depends on the parameter δ, that appears in the entropy production term. This suggests that in order to get sharp differential inequalities for a certain probability density with heavy tails, one has to look for the most general class of Fokker–Planck type equations which have this probability density as steady state, aiming at finding the optimal one.

It is interesting to remark that this strategy is not restricted to differential inequalities of Sobolev type, but it can be fruitfully applied also to Chernoff (Poincaré) type inequalities, thus generalizing the result obtained in [15].

In addition to the class of inverse Gamma functions, this method can be applied to obtain weighted inequalities for the class of Cauchy-type densities of unit mass $f_\beta(x)$, $\beta > 1/2$, with $x \in \mathbb{R}$

$$f_\beta(x) = C_\beta \frac{1}{(1+x^2)^\beta}, \qquad C_\beta = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta)}{\Gamma(\beta - 1/2)}. \tag{1.11}$$

Weighted inequalities for Cauchy-type densities have been studied with geometric techniques in various papers [3,18,20,28,35]. In one dimension of the space variable, we will obtain similar results by resorting to this relationship with Fokker–Planck type equations. Also, we will show that the optimal results in weighted Poincaré inequality for Cauchy-type densities obtained in [6] can follow by resorting to this idea.

The content of the paper is as follows. In Sect. 2 we will prove an extension of the classical Chernoff inequality obtained in [15], and we will apply the result to Cauchy-type and inverse Gamma densities. Likewise, Sect. 3 will contain an improvement of an old result by Elcrat and MacLean [13], recently reconsidered in [29], about weighted Wirtinger inequalities on unbounded domains. In this case, the underlying probability density is characterized as the equilibrium density of a Fokker–Planck equation with a positive coefficient of diffusion with a suitable polynomial growth, and a *weak* coefficient of drift. We will then apply our result to Cauchy-type and inverse Gamma densities, and we will be able to obtain explicit weight functions in the inequalities for them. Last, Sect. 4 will contain the results about weighted logarithmic Sobolev inequalities for the same classes of probability densities. Both Sects. 2 and 4 will take advantage of the representation of Cauchy-type and inverse Gamma densities as steady solutions to Fokker–Planck type equations of type (2.1).

To end this Introduction, it is important to outline that the strategy used in this paper can be fruitfully used to obtain differential inequalities with weight for other densities.

2. Chernoff-Type Inequalities for Heavy Tailed Densities

The aim of this section is to prove that the class of densities (1.10) and (1.11) satisfies some sharp weighted inequalities of Chernoff type, resorting to their relationship with Fokker–Planck type equations. In the rest of this section, we refer to a class of Fokker–Planck type equations with variable coefficients of diffusion and drift in the form

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\left(P(x)f(x,t)\right) + \frac{\partial}{\partial x}\left(Q(x)f(x,t)\right), \tag{2.1}$$

where $x \in \mathcal{I} = (i_-, i_+) \subseteq \mathbb{R}$. To ensure mass conservation, we will further consider no-flux boundary conditions

$$\left.\frac{\partial}{\partial x}\left(P(x)f(x,t)\right) + Q(x)f(x,t)\right|_{x=\{i_+,i_-\}} = 0. \tag{2.2}$$

In Eq. (2.1) the coefficients of the diffusion $P(x)$ and the drift $Q(x)$ are smooth functions, and $P(x) \geq 0$. We suppose moreover that $P(x)$ and $Q(x)$ are such that, for any $a \in \mathcal{I}$, the steady state

$$f_\infty(x) = \frac{C_a}{P(x)} \exp\left\{-\int_a^x \frac{Q(y)}{P(y)}\, dy\right\}. \tag{2.3}$$

is a probability density supported in $\mathcal{I}$ for a given value of the constant C_a. Note that the steady state (2.3) satisfies the first order differential equation

$$\frac{\partial}{\partial x}\left(P(x)f_\infty(x)\right) + Q(x)f_\infty(x) = 0. \tag{2.4}$$

The case

$$Q(x) = x - M, \quad M \in (i_-, i_+) \tag{2.5}$$

has been considered and studied in [15], resorting to a clear proof that is closely related to the Fokker–Planck description of the equilibria, as given by (2.4). In [15] it was proven that, if X is a random variable distributed with density $f_\infty(x)$, $x \in \mathcal{I} \subseteq \mathbb{R}$, and f_∞ satisfies the differential equality

$$\frac{\partial}{\partial x}\left(P(x)f_\infty(x)\right) + (x - M)\, f_\infty(x) = 0, \quad x \in \mathcal{I}, \tag{2.6}$$

then for any smooth function ϕ defined on $\mathcal{I}$ such that $\phi(X)$ has finite variance, the following inequality holds true

$$Var[\phi(X)] \leq E\left\{P(X)[\phi'(X)]^2\right\} \tag{2.7}$$

with equality if and only if $\phi(X)$ is linear in X. Note that when $Q(x)$ is linear in x, the weight function w appearing in (2.7) coincides with the variable coefficient of diffusion $P(x)$. In what follows, we extend the result of [15] to cover a larger class of functions $Q(x)$.

2.1. An Extension of Chernoff-Type Inequality

Theorem 2.1. (Chernoff with weight) *Let X be a random variable distributed with density $f_\infty(x)$, $x \in \mathcal{I} = (i_-, i_+) \subseteq \mathbb{R}$. Let us suppose moreover that f_∞ satisfies (2.4), where $Q(x)$ is a C^1 function, satisfying $Q'(x) > 0$ on $\mathcal{I}$, such that*

$$\lim_{x\to i_-} Q(x) < 0, \qquad \lim_{x\to i_+} Q(x) > 0. \tag{2.8}$$

Let $w(x)$ be defined by

$$w(x) = \frac{P(x)}{Q'(x)}, \quad x \in \mathcal{I}. \tag{2.9}$$

Then, for any smooth function ϕ on $\mathcal{I}$, and $\phi(X)$ with finite variance, it holds

$$Var[\phi(X)] \le E\left\{w(X)[\phi'(X)]^2\right\},$$

that is

$$\int_{\mathcal{I}} \phi^2(x) f_\infty(x)\,dx - \left(\int_{\mathcal{I}} \phi(x) f_\infty(x)\,dx\right)^2 \le \int_{\mathcal{I}} w(x)\left(\phi'(x)\right)^2 f_\infty(x)\,dx. \tag{2.10}$$

Proof. Let ϕ be a smooth function on $\mathcal{I}$ such that $Var(\phi(X))$ is bounded. Since $\int_{\mathcal{I}} f_\infty(x)dx = 1$, for any given constants $A \in \mathbb{R}$ it holds

$$\begin{aligned} Var(\phi(X)) &= \int_{\mathcal{I}} \phi^2(x) f_\infty(x)\,dx - \left(\int_{\mathcal{I}} \phi(x) f_\infty(x)\,dx\right)^2 \\ &\le \int_{\mathcal{I}} (\phi(x) - A)^2 f_\infty(x)dx. \end{aligned}$$

Now, since $Q(x)$ is strictly increasing on $\mathcal{I}$, and satisfies conditions (2.8), there is a point $x_0 \in \mathcal{I}$ where $Q(x_0) = 0$. Let us consider the change of variable $x \to Q(x)$, which is invertible for $x \in \mathcal{I}$ due to the assumptions on Q and let us define

$$\phi(x) = \psi(Q(x)). \tag{2.11}$$

If we set $A = \phi(x_0) = \psi(0)$, we get

$$\begin{aligned} \int_{\mathcal{I}} (\phi(x) - \psi(0))^2 f_\infty(x))dx &= \int_{\mathcal{I}} (\psi(Q(x)) - \psi(0))^2 f_\infty(x)dx \\ &= \int_{\mathcal{I}} \left(\int_0^{Q(x)} \psi'(s)ds\right)^2 f_\infty(x)dx \\ &= \int_{\mathcal{I}} \left(\int_0^1 \psi'(t\,Q(x))Q(x)dt\right)^2 f_\infty(x)dx. \end{aligned}$$

Now by Jensen's inequality

$$\begin{aligned} &\int_{\mathcal{I}} f_\infty(x) \left(\int_0^1 \psi'(t\,Q(x))Q(x)dt\right)^2 dx \\ &\quad\le \int_{\mathcal{I}} f_\infty(x) \left(\int_0^1 \left(\psi'(t\,Q(x))Q(x)\right)^2 dt\right) dx \\ &\quad= \int_{\mathcal{I}} f_\infty(x) Q^2(x) \left(\int_0^1 \left(\psi'(t\,Q(x))\right)^2 dt\right) dx. \end{aligned} \tag{2.12}$$

Using that f_∞ satisfies (2.4) we get

$$\begin{aligned}
&\int_{\mathcal{I}} f_\infty(x)Q^2(x)\left(\int_0^1 (\psi'(t\,Q(x)))^2\,dt\right)dx\\
&\quad= -\int_{\mathcal{I}} \partial_x\big(P(x)f_\infty(x)\big)Q(x)\left(\int_0^1 (\psi'(t\,Q(x)))^2\,dt\right)dx\\
&\quad= \left[-P(x)f_\infty(x)Q(x)\left(\int_0^1 (\psi'(t\,Q(x)))^2\,dt\right)\right]_{i_-}^{i_+}\\
&\qquad+\int_{\mathcal{I}} P(x)f_\infty(x)\partial_x\left(Q(x)\int_0^1 (\psi'(t\,Q(x)))^2\,dt\right)dx. \qquad (2.13)
\end{aligned}$$

The boundary term in (2.13), due to assumption (2.8) is non positive. In view of the identity

$$\partial_y(y\beta(ty)) = \partial_t(t\beta(ty)),$$

valid for any function $\beta(\cdot)$, and variables y and t we get

$$\begin{aligned}
&\int_{\mathcal{I}} P(x)f_\infty(x)\partial_x\left(Q(x)\int_0^1 (\psi'(t\,Q(x)))^2\,dt\right)dx\\
&\quad= \int_{\mathcal{I}} P(x)f_\infty(x)Q'(x)\;\partial_y\left(y\int_0^1 (\psi'(t\,y))^2\,dt\right)\Big|_{y=Q(x)}dx\\
&\quad= \int_{\mathcal{I}} P(x)f_\infty(x)Q'(x)\int_0^1 \partial_y\big(y\psi'(t\,y)^2\big)\Big|_{y=Q(x)}dtdx\\
&\quad= \int_{\mathcal{I}} P(x)f_\infty(x)Q'(x)\int_0^1 \partial_t\left(t\psi'(t\,Q(x))^2\right)dtdx\\
&\quad= \int_{\mathcal{I}} P(x)f_\infty(x)Q'(x)\psi'(Q(x))^2dx. \qquad (2.14)
\end{aligned}$$

Now, differentiation of (2.11) gives

$$\phi'(x) = \psi'(Q(x))Q'(x),$$

and

$$\psi'(Q(x)) = \frac{\phi'(x)}{Q'(x)}. \qquad (2.15)$$

Replacing equality (2.15) into the last integral in (2.14), and using the relation (2.9) it follows that

$$\int_{\mathcal{I}} P(x)f_\infty(x)Q'(x)\psi'(Q(x))^2dx = \int_{\mathcal{I}} w(x)\left(\phi'(x)\right)^2 f_\infty(x)dx.$$

Finally we have

$$Var(\phi(X)) \le \int_{\mathcal{I}} w(x)\left(\phi'(x)\right)^2 f_\infty(x)dx,$$

and the proof is completed. □

Remark 2.2. Even if the main applications of Theorem 2.1 refer to probability densities with heavy tails, it is interesting to remark that the case $P(x) = 1$ leads to a functional inequality related to weighted Poincaré inequalities, known as the Brascamp–Lieb inequality [8]. If $P(x) = 1$, the steady state f_∞ takes the form

$$f_\infty(x) = C_a \exp\left\{-\int_a^x Q(y)\,dy\right\}, \tag{2.16}$$

and the condition $Q'(x) > 0$ of Theorem 2.1 corresponds to the assumption that the potential

$$V(x) = \int_a^x Q(y)\,dy$$

is strictly convex. In this case inequality (2.10) can be written in terms of the strictly convex potential $V(x)$ to give

$$Var[\phi(X)] \le \int_{\mathcal{I}} \frac{1}{V''(x)} \left(\phi'(x)\right)^2 f_\infty(x)\,dx. \tag{2.17}$$

Inequality (2.17) is exactly the Brascamp–Lieb inequality in dimension one [8].

Theorem 2.1 allows us to prove Chernoff-type inequalities with weight for various families of probability densities on the line with heavy tails. We present below two examples, which refer to the family of the Cauchy-type densities (1.11) in the range $\beta > 1/2$, and to the family of inverse Gamma densities (1.6). To maintain the analogy with the Cauchy-type densities, we will write the inverse Gamma densities in the form

$$h_{\beta,m}(x) = \frac{C_{\beta,m}}{x^{2\beta}} e^{-\frac{m}{x}}, \quad x \in \mathbb{R}_+, \tag{2.18}$$

where $\beta > 1/2$ and $m > 0$. The constant $C_{\beta,m}$ is explicit, and it is such that the functions $h_{\beta,m}$ have unitary mass.

It is important to remark that, as far as the Cauchy-type densities are concerned, the same inequalities with weight have been recently obtained in [6], by resorting to the spectral gap of a convenient Markovian diffusion operator, and then by using a recent result [4], which allows the authors to estimate precisely this spectral gap. While for $\beta > 3/2$ the present proof is similar to that of [6], the method of proof in the case $1/2 < \beta \le 3/2$ is new, and makes a substantial use of Theorem 2.1.

2.2. Chernoff with Weight for Cauchy-Type Densities

Theorem 2.3. (Chernoff for Cauchy-type densities) *Let X be a random variable distributed with the Cauchy-type density (1.11), with $\beta > 1/2$. For any smooth function $\phi(x)$, with $x \in \mathbb{R}$ such that $\phi(X)$ has finite variance, one has the bounds*

$$Var[\phi(X)] \le \frac{1}{\rho(\beta)} E\left\{(1 + X^2)[\phi'(X)]^2\right\}, \tag{2.19}$$

where

$$\rho(\beta) = \begin{cases} \left(\beta - \frac{1}{2}\right)^2 & \frac{1}{2} < \beta \le \frac{3}{2} \\ 2(\beta - 1) & \beta > \frac{3}{2}. \end{cases} \tag{2.20}$$

Proof. We start by remarking that the Cauchy-type density f_β defined in (1.11) can be characterized as the stationary state of a whole family of Fokker–Planck type equations, which depend on two positive parameters α and λ related to satisfy the constraint

$$\alpha(1+\lambda) = \beta. \tag{2.21}$$

For our purposes, we will assume that $\alpha \in (1/2, 1]$ and let $\lambda > 0$. This choice guarantees that we can obtain from relation (2.21) all values of $\beta > 1/2$. It can be easily checked that this family of Fokker–Planck type equations is given by

$$\partial_t f = \partial_x^2 \left((1+x^2)^\alpha f\right) + \lambda \partial_x \left(\frac{2\alpha x}{(1+x^2)^{1-\alpha}} f \right), \quad x \in \mathbb{R}, t > 0.$$

Indeed f_β satisfies, for all $x \in \mathbb{R}$, the differential equation

$$\partial_x \left((1+x^2)^\alpha f_\beta\right) = -\frac{2\alpha\lambda x}{(1+x^2)^{1-\alpha}} f_\beta. \tag{2.22}$$

Equation (2.22) is of the type (2.4), with

$$P(x) = (1+x^2)^\alpha, \quad Q(x) = \frac{2\alpha\lambda x}{(1+x^2)^{1-\alpha}}, \quad x \in \mathbb{R}. \tag{2.23}$$

In the allowed range of the constant α, the function Q satisfies all the assumptions of Theorem 2.1. Indeed, the function Q is differentiable on $\mathbb{R}$ and for all $\alpha > 1/2$

$$Q'(x) = \frac{2\alpha\lambda(1+x^2(2\alpha-1))}{(1+x^2)^{2-\alpha}} > 0, \quad x \in \mathbb{R}.$$

Moreover

$$\lim_{x \to -\infty} Q(x) = -\infty, \qquad \lim_{x \to +\infty} Q(x) = +\infty.$$

So $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly monotone, smooth transformation. Moreover, since $\alpha \le 1$ we have $2\alpha - 1 \le 1$, so that

$$\frac{P(x)}{Q'(x)} = \frac{(1+x^2)^2}{2\alpha\lambda(1+x^2(2\alpha-1))} \le \frac{1}{2\alpha\lambda(2\alpha-1)}(1+x^2), \quad x \in \mathbb{R}. \tag{2.24}$$

Therefore, from inequality (2.10) of Theorem 2.1 we obtain

$$\int_{\mathbb{R}} \phi^2(x) f_\beta(x)\, dx - \left(\int_{\mathbb{R}} \phi(x) f_\beta(x)\, dx \right)^2 \\ \le \frac{1}{2\alpha\lambda(2\alpha-1)} \int_{\mathbb{R}} (1+x^2)\left(\phi'(x)\right)^2 f_\beta(x)\, dx.$$

We can now look for the optimal value of the constant $2\alpha\lambda(2\alpha-1))$ under the constraints $\alpha \in (1/2, 1]$, $\lambda > 0$ and (2.21). Since (2.21) implies $\lambda = \beta/\alpha - 1$, the optimal value is obtained by maximizing the function

$$\rho_\beta(\alpha) = 2(\beta - \alpha)(2\alpha - 1).$$

To this end, since

$$\rho'_\beta(\alpha) = 2(2\beta - 4\alpha + 1)$$

we obtain

$$\rho'_\beta(\alpha) \geq 0 \iff \alpha \leq \frac{\beta}{2} + \frac{1}{4}.$$

If $\frac{1}{2} < \frac{\beta}{2} + \frac{1}{4} \leq 1$, then $\alpha_{\max} = \frac{\beta}{2} + \frac{1}{4}$, while if $\frac{\beta}{2} + \frac{1}{4} > 1$, then $\alpha_{\max} = 1$. Denoting by $\rho(\beta) = \max\{\rho_\beta(\alpha), \frac{1}{2} < \alpha \leq 1\}$, we then find

$$\rho(\beta) = \begin{cases} \left(\beta - \frac{1}{2}\right)^2 & \frac{1}{2} < \beta \leq \frac{3}{2} \\ 2(\beta - 1) & \beta > \frac{3}{2}. \end{cases} \tag{2.25}$$

This completes the proof. □

Remark 2.4. In the case $\beta > 3/2$ the optimal constant $\rho(\beta)$ is obtained by choosing $\alpha_{\max} = 1$. In this case $Q(x) = 2\lambda x$ is therefore linear, and the proof of Chernoff inequality was already obtained in [15]. Moreover, in this range of the parameter β, the Cauchy-type density has finite variance and this implies that the function ϕ in Theorem 2.3 can be chosen to be linear in x. Since we proved in [15] that Chernoff inequality with weight further guarantees that there is equality in (2.19) if and only if $\phi(x)$ is linear in x, we can conclude that for $\beta > 3/2$ the constant $\rho(\beta)$ is sharp.

2.3. Chernoff with Weight for Inverse Gamma Densities

Theorem 2.5. (Chernoff for inverse Gamma-type densities) *Let X be a random variable distributed with density (2.18) for $x \in \mathbb{R}_+$, $\beta > 1/2$, $m > 0$. For any smooth function ϕ on $\mathbb{R}_+$ such that the variance of $\phi(X)$ is finite, it holds*

$$Var[\phi(X)] \leq \frac{1}{\rho(\beta)} E\left\{X^2[\phi'(X)]^2\right\}, \tag{2.26}$$

where

$$\rho(\beta) = \begin{cases} \left(\beta - \frac{1}{2}\right)^2 & \frac{1}{2} < \beta \leq \frac{3}{2} \\ 2(\beta - 1) & \beta > \frac{3}{2}. \end{cases} \tag{2.27}$$

Proof. The proof follows along the same lines of Theorem 2.3. Indeed, $h_{\beta,m}$ is the stationary state of a whole family of Fokker–Planck type equations, which depend on two positive parameters α and λ, where

$$2\alpha + \lambda = 2\beta. \tag{2.28}$$

For our purposes, we will take $\alpha \in (1/2, 1]$ and $\lambda > 0$. Consequently, the exponent 2β of x in the inverse Gamma density is greater than one. The family of Fokker–Planck type equations having $h_{\beta,m}$ as stationary state is defined by

$$\partial_t h = \partial_x^2\left(x^{2\alpha} h\right) + \lambda \partial_x\left(\left(x - \frac{m}{\lambda}\right) x^{2\alpha-2} h\right), \quad x \in \mathbb{R}_+, t > 0. \tag{2.29}$$

Thus, $h_{\beta,m}$ satisfies, for all $x \in \mathbb{R}_+$

$$\partial_x\left(x^{2\alpha} h_{\beta,m}\right) = -\lambda\left(x - \frac{m}{\lambda}\right) x^{2\alpha-2} h_{\beta,m}. \tag{2.30}$$

Equation (2.30) is of the type (2.4), with

$$P(x) = x^{2\alpha}; \quad Q(x) = \lambda\left(x - \frac{m}{\lambda}\right) x^{2\alpha-2}, \quad x \in \mathbb{R}_+ \tag{2.31}$$

In the allowed range of the constant α, the function Q satisfies all the assumptions of Theorem 2.1. The function Q is differentiable on $\mathbb{R}_+$ and for $1/2 < \alpha \leq 1$

$$Q'(x) = \lambda(2\alpha - 1)x^{2\alpha-2} - m(2\alpha - 2)x^{2\alpha-3} \geq \lambda(2\alpha - 1)x^{2\alpha-2} > 0, \quad x \in \mathbb{R}_+.$$

Moreover, for $\alpha < 1$

$$\lim_{x \to 0^+} Q(x) = -\infty, \quad \lim_{x \to +\infty} Q(x) = +\infty.$$

When $\alpha = 1$ the function $Q(x)$ is defined also for $x = 0$ and $Q(0) = -m < 0$. So $Q : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a strictly monotone, smooth transformation. Moreover

$$\frac{P(x)}{Q'(x)} \leq \frac{x^{2\alpha}}{\lambda(2\alpha - 1)x^{2\alpha-2}} = \frac{x^2}{\lambda(2\alpha - 1)}, \quad x \in \mathbb{R}_+. \tag{2.32}$$

We apply Theorem 2.1 and for all $\alpha \in (1/2, 1]$ and $\lambda > 0$ satisfying (2.28) we obtain

$$\int_{\mathbb{R}_+} \phi^2(x)h_{\beta,m}(x)\,dx - \left(\int_{\mathbb{R}_+} \phi(x)h_{\beta,m}(x)\,dx\right)^2$$
$$\leq \frac{1}{\lambda(2\alpha - 1)} \int_{\mathbb{R}^*} x^2 \left(\phi'(x)\right)^2 h_{\beta,m}(x)\,dx.$$

Thanks to (2.28), we can substitute the value $\lambda = 2\beta - 2\alpha$ in the constant $\lambda(2\alpha - 1)$. This leads to maximize the constant

$$\rho_\beta(\alpha) = (2\beta - 2\alpha)(2\alpha - 1),$$

with respect to α. To this end, since

$$\rho'_\beta(\alpha) = 2\left(2\beta - 4\alpha + 1\right),$$

we obtain

$$\rho'_\beta(\alpha) \geq 0 \iff \alpha \leq \frac{2\beta + 1}{4}.$$

So, if $\frac{1}{2} < \frac{2\beta+1}{4} \leq 1$, then $\alpha_{\max} = \frac{2\beta+1}{4}$, and if $\frac{2\beta+1}{4} > 1$, then $\alpha_{\max} = 1$. Denoting by $\rho(\beta) = \max\{\rho_\beta(\alpha), \frac{1}{2} < \alpha \leq 1\}$, we obtain

$$\rho(\beta) = \begin{cases} \left(\beta - \frac{1}{2}\right)^2 & \frac{1}{2} < \beta \leq \frac{3}{2} \\ 2(\beta - 1) & \beta > \frac{3}{2} \end{cases}$$

and this completes the proof. □

Remark 2.6. The conclusions of Remark (2.4) remain true.

We can now rewrite the Chernoff inequality (2.26) in terms of the standard notation of the inverse Gamma functions with parameters $\kappa > 0$ and $m > 0$, that is

$$f_{\kappa,m} = \frac{m^\kappa}{\Gamma(\kappa)} \frac{1}{x^{1+\kappa}} e^{-m/x}, \quad x \in \mathbb{R}_+. \tag{2.33}$$

We then obtain

$$\int_{\mathbb{R}_+} \phi^2(x) f_{\kappa,m}(x)\,dx - \left(\int_{\mathbb{R}_+} \phi(x) f_{\kappa,m}(x)\,dx\right)^2 \\ \le \frac{1}{\gamma(\kappa)} \int_{\mathbb{R}_+} x^2 \left(\phi'(x)\right)^2 f_{\kappa,m}(x)\,dx. \tag{2.34}$$

In (2.34) the optimal constants $\gamma(\kappa)$ are given by

$$\gamma(\kappa) = \begin{cases} \kappa^2/4 & 0 < \kappa \le 2 \\ \kappa - 1 & \kappa > 2. \end{cases}$$

It is immediate to check that inequality (2.34), for $\kappa > 2$, reduces to equality when $\phi(x)$ is linear in x.

3. Wirtinger-Type Inequalities for Heavy Tailed Densities

Let X be a random variable with an absolutely continuous density $f(x)$, $x \in \mathcal{I} = (i_-, i_+) \subseteq \mathbb{R}$ such that $f(x) > 0$ in $\mathcal{I}$, and let $F(x)$, $x \in \mathcal{I}$, denote its distribution function, defined as usual by the formula

$$F(x) = \int_{i_-}^{x} f(y)\,dy \le 1. \tag{3.1}$$

Let $\bar{x}$ denote the median of the random variable X, that is the value where the increasing function $F(x)$ satisfies $F(\bar{x}) = 1/2$. Last, let $K(x)$ be defined as the nonnegative function

$$K(x) = \frac{F(x)}{f(x)} \quad \text{if } x \le \bar{x}; \qquad K(x) = \frac{1 - F(x)}{f(x)} \quad \text{if } x \ge \bar{x}. \tag{3.2}$$

Then, $K(x)$ is a continuous function on $\mathcal{I}$, and we for $x \in \mathcal{I}$, $x \ne \bar{x}$ have the identity

$$f(x) = -\frac{x - \bar{x}}{|x - \bar{x}|} \frac{d}{dx} \left[K(x) f(x)\right]. \tag{3.3}$$

Note that (3.3) is a clean way to characterize the density $f(x)$ as the steady state of a Fokker–Planck equation of type (2.1) where the diffusion coefficient is the continuous nonnegative function

$$P(x) = K(x),$$

and the drift term is

$$\tilde{Q}(x) = \frac{x - \bar{x}}{|x - \bar{x}|}. \tag{3.4}$$

The drift term (3.4) satisfies conditions (2.8) at the boundaries of $\mathcal{I}$. Note that, unlike the drift terms considered in Sect. 2, the smoothness of the drift term is lost.

The action of the drift induced by the function (3.4) can be easily understood by observing that the drift equation

$$\frac{\partial S(x,t)}{\partial t} = \tilde{Q}(x) \frac{\partial S(x,t)}{\partial x} \tag{3.5}$$

with initial data the distribution function $S_0(x)$, $x \in \mathcal{I}$, is explicitly solvable, and its solution is given by

$$S(x,t) = \begin{cases} S_0(x - \bar{x} - t) & x - \bar{x} < 0, \\ S_0(x - \bar{x} + t) & x - \bar{x} > 0, \end{cases}$$

which implies linear in time convergence of the solution towards a Dirac delta localized in $x = \bar{x}$. Hence, while a linear drift implies exponential in time convergence towards a delta function, the action of the drift term (3.5), and consequently the convergence results of the solution to the Fokker–Planck equation towards the steady state are expected to be weaker.

Using expression (3.3) we prove the following

Theorem 3.1. (Wirtinger with weight) *Let X be a random variable distributed with density $f(x)$, $x \in \mathcal{I} = (i_-, i_+) \subseteq \mathbb{R}$, and let $K(x)$ be defined by (3.2). Then, for any smooth function ϕ on $\mathcal{I}$ such that $E\left[|\phi(X)|^p\right]$ is bounded, $1 \le p < +\infty$, it holds*

$$E\left[|\phi(X) - E(\phi(X))|^p\right] \le E\left[w_p^p(X)\,|\phi'(X)|^p\right], \tag{3.6}$$

where the weight function $w_p(x)$ is given by

$$w_p(x) = 2pK(x). \tag{3.7}$$

Proof. Let us first suppose that the function ϕ satisfies the condition $\phi(\bar{x}) = 0$. In this case, we can directly make use of the argument of proof in [13]. Thanks to (3.3), we have

$$\begin{aligned}\int_{i_-}^{\bar{x}} |\phi(x)|^p f(x)\,dx &= \int_{i_-}^{\bar{x}} |\phi(x)|^p \frac{d}{dx}\left[K(x)f(x)\right]\,dx \\ &= |\phi(x)|^p K(x) f(x)\Big|_{i_-}^{\bar{x}} - \int_{i_-}^{\bar{x}} K(x)f(x)\frac{d}{dx}|\phi(x)|^p\,dx. \end{aligned}\tag{3.8}$$

Now, since $\phi(\bar{x}) = 0$,

$$\begin{aligned}&|\phi(x)|^p K(x) f(x)\Big|_{i_-}^{\bar{x}} \\ &\quad = |\phi(\bar{x})|^p F(\bar{x}) - \lim_{x\to i_-} |\phi(x)|^p F(x) = -\lim_{x\to i_-} |\phi(x)|^p F(x) \le 0,\end{aligned}$$

and the contribution of the boundary term is nonpositive on the interval $(i_-, \bar{x})$. Therefore (3.8) implies the inequality

$$\int_{i_-}^{\bar{x}} |\phi(x)|^p f(x)\,dx \le p\int_{i_-}^{\bar{x}} K(x)f(x)|\phi(x)|^{p-1}|\phi'(x)|\,dx.$$

The same argument can be used on the interval $(\bar{x}, i_+)$, to obtain

$$\int_{\bar{x}}^{i_+} |\phi(x)|^p f(x)\,dx \le p\int_{\bar{x}}^{i_+} K(x)f(x)|\phi(x)|^{p-1}|\phi'(x)|\,dx.$$

Consequently, if $\phi(\bar{x}) = 0$, we have the inequality

$$\int_{\mathcal{I}} |\phi(x)|^p f(x)\,dx \le p\int_{\mathcal{I}} K(x)f(x)|\phi(x)|^{p-1}|\phi'(x)|\,dx. \tag{3.9}$$

If $p = 1$, (3.9) reduces to

$$E\left[|\phi(X)|\right] \le E\left[K(X)\,|\phi'(X)|\right]. \tag{3.10}$$

If $1 < p < +\infty$, Hölder's inequality implies

$$\begin{aligned}&\int_{\mathcal{I}} K(x) f(x) |\phi(x)|^{p-1} |\phi'(x)|\, dx \\ &\quad \le \left[\int_{\mathcal{I}} \left(K(x)|\phi'(x)|\right)^p f(x)\, dx\right]^{1/p} \left[\int_{\mathcal{I}} |\phi(x)^p f(x)\, dx\right]^{1-1/p}, \end{aligned} \tag{3.11}$$

which, combined with (3.9), shows that, for any function ϕ satisfying $\phi(\bar{x}) = 0$, it holds

$$E\left[|\phi(X)|^p\right] \le p^p E\left[K(X)^p\,|\phi'(X)|^p\right]. \tag{3.12}$$

The general case is an easy consequence of the previous argument. Indeed, since $f(\cdot)$ is a probability density on $\mathcal{I}$, for $1 \le p < +\infty$ we have

$$\begin{aligned}&\int_{\mathcal{I}} \left|\phi(x) - \int_{\mathcal{I}} \phi(y)\, f(y)\, dy\right|^p f(x)\, dx \\ &\quad = \int_{\mathcal{I}} \left|\int_{\mathcal{I}} \left(\phi(x) - \phi(y)\right)\, f(y)\, dy\right|^p f(x)\, dx \\ &\quad \le \int_{\mathcal{I}\times\mathcal{I}} |\phi(x) - \phi(y)|^p\, f(x) f(y)\, dx\, dy \\ &\quad = \int_{\mathcal{I}\times\mathcal{I}} |\phi(x) - \phi(\bar{x}) - (\phi(y) - \phi(\bar{x}))|^p\, f(x) f(y)\, dx\, dy \\ &\quad \le 2^{p-1} \int_{\mathcal{I}\times\mathcal{I}} \left(|\phi(x) - \phi(\bar{x})|^p + |\phi(y) - \phi(\bar{x})|^p\right) f(y) f(x) dx\, dy \\ &\quad = 2^p \int_{\mathcal{I}} |\phi(x) - \phi(\bar{x})|^p\, f(x)\, dx = 2^p \int_{\mathcal{I}} |\psi(x)|^p\, f(x)\, dx, \end{aligned} \tag{3.13}$$

where the function $\psi(x)$ in (3.13) is such that $\psi(\bar{x}) = 0$. At this point, we can apply (3.12) to the function ψ to get the general inequality (3.6). □

Unlike the result of [13], the function ϕ is not required to satisfy particular boundary conditions at the point i_-. For example, it is not necessary, in the case $\mathcal{I} = \mathbb{R}_+$, that the function ϕ satisfies $\phi(0) = 0$, as required by Corollary to Theorem 1 of [13]. We further remark that Theorem 3.1 improves analogous result in [29].

3.1. Wirtinger Inequalities with Weight for Cauchy-Type Densities

In this short section, we apply Theorem 3.1 to recover inequalities for the class of Cauchy-type densities, with an explicit expression of the weight function $K(x)$. We prove the following result

Theorem 3.2. *Let X be a random variable distributed with the Cauchy-type density (1.11), with $\beta > 1/2$. For any smooth function $\phi(x)$, with $x \in \mathbb{R}$, such that $E\left[|\phi(X)|^p\right]$ is bounded, $1 \le p < +\infty$, one has the inequality*

$$E\left[|\phi(X) - E(\phi(X))|^p\right] \le (2p\gamma_\beta)^p\, E\left[(1 + X^2)^{p/2}\,|\phi'(X)|^p\right], \tag{3.14}$$

where

$$\gamma_\beta = \sqrt{\pi}\,\frac{\Gamma\left(\beta - \frac{1}{2}\right)}{2\,\Gamma(\beta)}. \tag{3.15}$$

Proof. For any given positive constant $\beta > 1/2$, let us consider a generalized Gaussian density $f_\beta(x)$, as given by (1.11). Then, if $x > 0$ formula (3.2) gives

$$K(x) = \frac{\int_x^{+\infty} f_\beta(y)\,dy}{f_\beta(x)} = \int_x^{+\infty} \left(\frac{1+x^2}{1+y^2}\right)^\beta dy.$$

The integral on the right-hand side can be evaluated by substitution, setting $1+y^2 = (1+x^2)(1+z^2)$, and we obtain

$$K(x) = \int_0^{+\infty} \frac{z(1+x^2)}{\sqrt{z^2(1+x^2)+x^2}}\frac{1}{(1+z^2)^\beta}\,dz. \tag{3.16}$$

Since

$$\frac{z(1+x^2)}{\sqrt{z^2(1+x^2)+x^2}} \le \sqrt{1+x^2},$$

we obtain

$$K(x) \le \sqrt{1+x^2}\int_0^{+\infty} \frac{1}{(1+z^2)^\beta}\,dz = \gamma_\beta\sqrt{1+x^2}, \tag{3.17}$$

where γ_β is given by (3.15). Since $f_\beta(x)$ is an even function, the same result holds when $x < 0$. □

We remark that in the case $p = 1$, in any dimension $n \ge 1$ the same weight function $K(x)$ has been obtained in [3], but , with a different constant.

3.2. Wirtinger Inequalities with Weight for Inverse Gamma Densities

Last, we apply Theorem 3.1 to recover inequalities for the class of inverse Gamma densities. In this case, the expression of the weight function $K(x)$ depends on the value of the median of the distribution, which is not explicitly available. We prove

Theorem 3.3. *Let X be a random variable distributed with density $h_{\beta,m}$ defined as in (2.18), for $x \in \mathbb{R}_+$, $\beta > 1/2$, $m > 0$. For any smooth function ϕ on $\mathbb{R}_+$ such that $E\left[|\phi(X)|^p\right]$ is finite, it holds*

$$E\left[|\phi(X) - E(\phi(X))|^p\right] \le (pD(\beta,m))^p E\left\{X^p[\phi'(X)]^p\right\}, \tag{3.18}$$

where

$$D(\beta,m) = \frac{1}{\bar{x}_{\beta,m} h_{\beta,m}\left(\bar{x}_{\beta,m}\right)}, \tag{3.19}$$

and $\bar{x}_{\beta,m}$ is the median of the random variable X.

Proof. For any pair of positive constants β, m, let $\bar{x}_{\beta,m}$ denote the median of the random variable X with density $h_{\beta,m}$ given by (2.18), and distribution function $H_{\beta,m}(x)$. Then, if $x \le \bar{x}_{\beta,m}$

$$
\begin{aligned}
\frac{H_{\beta,m}(x)}{h_{\beta,m}(x)} &= \int_0^x \left(\frac{x}{y}\right)^{2\beta} \exp\left\{-\frac{m}{x}\left(\frac{x}{y}-1\right)\right\} dy \\
&= x\int_0^1 z^{-2\beta} \exp\left\{-\frac{m}{x}\left(\frac{1}{z}-1\right)\right\} dz \\
&\le x\int_0^1 z^{-2\beta} \exp\left\{-\frac{m}{\bar{x}_{\beta,m}}\left(\frac{1}{z}-1\right)\right\} dz.
\end{aligned} \tag{3.20}
$$

Indeed, the value of the integral on the second line of (3.20) is non decreasing with respect to x, as it can be easily verified by direct inspection. Likewise, if $x \ge \bar{x}_{\beta,m}$ one shows that

$$
\begin{aligned}
\frac{1-H_{\beta,m}(x)}{h_{\beta,m}(x)} &= \int_x^{+\infty} \left(\frac{x}{y}\right)^{2\beta} \exp\left\{-\frac{m}{x}\left(\frac{x}{y}-1\right)\right\} dy \\
&\le x\int_1^{+\infty} z^{-2\beta} \exp\left\{-\frac{m}{\bar{x}_{\beta,m}}\left(\frac{1}{z}-1\right)\right\} dz.
\end{aligned} \tag{3.21}
$$

On the other hand we have

$$
\begin{aligned}
&\int_0^1 z^{-2\beta} \exp\left\{-\frac{m}{\bar{x}_{\beta,m}}\left(\frac{1}{z}-1\right)\right\} dy \\
&\quad = e^{m/\bar{x}_{\beta,m}} \int_0^1 z^{-2\beta} \exp\left\{-\frac{m}{\bar{x}_{\beta,m} z}\right\} dz \\
&\quad = \frac{1}{C_{\beta,m}} e^{m/\bar{x}_{\beta,m}} (\bar{x}_{\beta,m})^{2\beta-1} \int_0^{\bar{x}_{\beta,m}} C_{\beta,m} u^{-2\beta} \exp\left\{-\frac{m}{u}\right\} du \\
&\quad = \frac{1}{C_{\beta,m}} e^{m/\bar{x}_{\beta,m}} (\bar{x}_{\beta,m})^{2\beta-1} \int_0^{\bar{x}_{\beta,m}} h_{\beta,m}(u)\, du = \frac{1}{2}\left[\bar{x}_{\beta,m} h_{\beta,m}(\bar{x}_{\beta,m})\right]^{-1}.
\end{aligned} \tag{3.22}
$$

In fact, by definition of median, the last integral into (3.22) is equal to 1/2. Clearly, the same result holds for the last integral into (3.21). This concludes the proof. □

4. Logarithmic Sobolev Inequalities for Heavy Tailed Densities

As mentioned in the Introduction, the relationships between the classical Fokker–Planck equation and logarithmic Sobolev inequalities are well-known [2,22,25,32,33]. These connections mainly refer to Fokker–Planck type equations with a constant coefficient of diffusion.

As a matter of fact, in his pioneering paper on diffusion equations [14], Feller remarked that for one-dimensional Fokker–Planck equations one can always reduce to the case of constant coefficient of diffusion by a suitable change of variables. However, a different balance between the coefficient of diffusion and the drift in Fokker–Planck equations which share the same steady state, results in different

Fokker–Planck equations with constant coefficient of diffusion. Hence, as already shown in Sect. 2, different pairs of coefficients of diffusion and drift give rise in general to inequalities with different weights.

In this last section we will directly apply Feller's idea to both Cauchy-type and Gamma densities to obtain weighted logarithmic Sobolev inequalities in the form (1.2). Similarly to the analysis of Sect. 2, we will refer to suitable classes of Fokker–Planck type equations (2.1), well adapted to the derivation of the result. Let

$$f_\beta(x) = \frac{C_\beta}{(1+|x|^2)^\beta}, \quad x \in \mathbb{R}^n$$

denote a Cauchy-type probability density in $\mathbb{R}^n$, $n \geq 1$, where $\beta > n/2$.

For any probability density $f \in L_1(\mathbb{R}^n)$, absolutely continuous with respect to f_β, we can rewrite inequality (1.2) by assuming

$$\phi(x) = \sqrt{\frac{f(x)}{f_\beta(x)}}, \quad x \in \mathbb{R}^n,$$

that implies

$$\int_{\mathbb{R}^n} \phi(x)^2 f_\beta(x)\, dx = 1.$$

Then, inequality (1.2) can be written in the physically relevant form

$$\int_{\mathbb{R}^n} f \log \frac{f}{f_\beta} dx \leq \frac{1}{\beta - 1} \int_{\mathbb{R}^n} w(x) \left| \nabla \sqrt{\frac{f}{f_\beta}} \right|^2 f_\beta \, dx. \tag{4.1}$$

Moreover, since

$$4 \left| \nabla \sqrt{\frac{f}{f_\beta}} \right|^2 f_\beta = \left| \nabla \log \frac{f}{f_\beta} \right|^2 f$$

inequality (4.1) is equivalently written as

$$\int_{\mathbb{R}^n} f \log \frac{f}{f_\beta} dx \leq \frac{1}{4(\beta - 1)} \int_{\mathbb{R}^n} w(x) \left| \nabla \log \frac{f}{f_\beta} \right|^2 f \, dx. \tag{4.2}$$

It is known, after Bobkov and Ledoux [3], that these densities satisfy a weighted Log-Sobolev inequality in the range $\beta \geq \frac{n+1}{2}$ if $n > 1$, and $\beta > 1$ if $n = 1$, where the weight function is expressed by

$$w(x) = (1 + |x|^2)^2, \tag{4.3}$$

and so the weight function in inequality (4.1) does not depend on the value of the parameter β characterizing the Cauchy-type density. The weight obtained by Bobkov and Ledoux in [3] for the Cauchy distributions is near optimal, and has been recently improved. In dimension $n \geq 1$, the optimal weight in the Log-Sobolev inequality has been shown by Hebisch and Zegarlinski in Theorem 5.7 of [20] to be

$$\tilde{w}(x) = c_\beta (1 + x^2) \log(e + x^2).$$

The constant value c_β in front of the weight is not explicit. In dimension one, similar weights have been obtained by Saumard [28], still in absence of an explicit constant.

By means of the relationship between Fokker–Planck type equations and Log-Sobolev inequalities, we will first show that in dimension one the result by Bobkov and Ledoux can be improved. Then, by resorting to the same analogy, we will show that inequality (4.2) still holds with a weight of the same order at infinity as the one in [20], with in addition an explicit value of the constant.

4.1. Logarithmic Sobolev Inequalities for Cauchy-Type Densities

The main result of this section is the following.

Theorem 4.1. (Log–Sobolev for Cauchy-type densities) *Let X be a random variable distributed with the Cauchy-type probability density (1.11), with $\beta > 1/2$. For any bounded smooth function $\phi(x)$, with $x \in \mathbb{R}$, such that $\phi(X)$ has finite entropy, and for all $1 < \alpha < 2\beta$ one has the bound*

$$Ent\left[\phi^2(X)\right] \le \frac{2}{\rho_{\beta,\alpha}} E\left\{(1+X^2)^\alpha [\phi'(X)]^2\right\}. \tag{4.4}$$

In (4.4) the constant $\rho_{\beta,\alpha}$ is given by

$$\rho_{\beta,\alpha} = \begin{cases} (2\beta-\alpha)\left(\frac{\alpha-1}{2-\alpha}\right)^{3-2\alpha} & 1 < \alpha < \frac{3}{2} \\ 2\beta - 3/2 & \frac{3}{2} \le \alpha < 2\beta, \quad 2\beta > \frac{3}{2}. \end{cases} \tag{4.5}$$

Remark 4.2. Since the weight function is monotonically increasing in terms of the parameter α, for any given $\beta > 1/2$ the weighted inequality (4.4) holds true for all values of $\alpha > 1$. Before entering into the technical details of the proof, let us compare inequality (4.4) with the analogous one proven by Bobkov and Ledoux, as given by (4.1). First of all, since the exponent $\alpha > 1$ of the weight function is only subject to the constraint to be less than $2\beta > 1$, for any value of β we can always choose $\alpha < 2$ to satisfy the inequality. Hence we have a smaller weight, which, however, for values of α close to one has a worse constant $\rho_{\beta,\alpha}$. In any case, the weight $w(x) = 1 + x^2$ can not be reached, since $\rho_{\beta,\alpha} \to 0$ as $\alpha \to 1$. The best result is obtained in the interval $3/2 \le \alpha \le 2$, since the constant $\rho_{\beta,\alpha}$ satisfies $\rho_{\beta,\alpha} > 2(\beta - 1)$ and at the same time the weight function in (4.4) is smaller than the one in (4.1). Last, when $\beta > 1$, by setting $\alpha = 2$ we recover exactly the result by Bobkov and Ledoux with a smaller constant.

Proof. We proceed by proving an equivalent inequality of type (4.2), for a smooth probability density f, absolutely continuous with respect to f_β. Then, for any bounded, smooth function ϕ, we define

$$f(x) = \frac{f_\beta(x)\phi^2(x)}{\int_{\mathbb{R}} f_\beta(x)\phi^2(x)\,dx}$$

and we will recover inequality (4.4) in the general form. As in the proof of Chernoff inequality in Theorem 2.3, we observe that f_β is a stationary state of the family of Fokker–Planck type equations

$$\partial_t f = \partial_x^2\left((1+x^2)^\alpha f\right) + \lambda \partial_x\left(2\alpha x(1+x^2)^{\alpha-1} f\right), \quad x \in \mathbb{R}, t > 0. \tag{4.6}$$

Unlike the proof of Theorem 2.3, we assume now the conditions $\alpha > 1$ and $\lambda > -1/2$, still subject to the constraint $\alpha(1+\lambda) = \beta$. This choice is coherent with the lower bound $\beta > 1/2$ in the statement of the theorem. It is remarkable that for negative values of the parameter λ, the Fokker–Planck equation (4.6), while having the correct steady state, is the balance between a diffusion operator and an *anti-drift* term. This is a situation that is typical of mathematical models which describe granular materials (cf. [9] and the references therein).

In order to proceed, we make use of an equivalent formulation of the Fokker Planck equation in terms of the function $F = f/f_\beta$. Skipping details, that can be found in [15], one shows that F satisfies the evolution equation

$$\frac{\partial F}{\partial t} = (1+x^2)^\alpha \frac{\partial^2 F}{\partial x^2} - 2\alpha\lambda x(1+x^2)^{\alpha-1}\frac{\partial F}{\partial x}. \tag{4.7}$$

Following the original argument of Feller [14], we introduce a change of variables to make the diffusion coefficient equal to unity. To this end, let us define

$$G(y,t) = F(x,t), \tag{4.8}$$

with

$$\frac{dy}{dx} = \frac{1}{(1+x^2)^{\alpha/2}}. \tag{4.9}$$

Owing to (4.9) we obtain

$$\frac{\partial F}{\partial x} = \frac{1}{(1+x^2)^{\alpha/2}}\frac{\partial G}{\partial y}$$

and

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{(1+x^2)^{\alpha}}\frac{\partial^2 G}{\partial y^2} - \frac{\alpha x}{(1+x^2)^{\alpha/2+1}}\frac{\partial G}{\partial y}.$$

Therefore the right hand side of (4.7) becomes

$$\frac{\partial^2 G}{\partial y^2} - \alpha x(1+x^2)^{\frac{\alpha}{2}-1}\frac{\partial G}{\partial y} - 2\alpha\lambda x(1+x^2)^{\frac{\alpha}{2}-1}\frac{\partial G}{\partial y}.$$

We denote by $x = x(y)$ the inverse of the increasing function $y(x)$, defined by (4.9). Hence equation (4.7) turns into a Fokker Planck equation with coefficient of diffusion equal to one

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial y^2} - W'(y)\frac{\partial G}{\partial y}, \tag{4.10}$$

where $W'(y)$ is the drift term

$$W'(y) = \alpha(1+2\lambda)x(y)(1+x^2(y))^{\frac{\alpha}{2}-1}. \tag{4.11}$$

Equation (4.10) is the adjoint of the Fokker–Planck equation

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2} + \frac{\partial}{\partial y}(W'(y)g) \tag{4.12}$$

still with diffusion coefficient equal to one, and steady state

$$g_\infty(y) = Ce^{-W(y)}. \tag{4.13}$$

As shown in [34], it is useful to introduce a further version of the Fokker–Planck equation (4.7), that highlights an interesting feature of the change of variables (4.9). For given $t > 0$, let $X(t)$ denote the random process with probability density $f(x,t)$, solution of the Fokker–Planck equation (4.6), and let

$$\mathcal{F}(x,t) = P(X(t) \le x) = \int_{-\infty}^{x} f(y,t)\,dy \tag{4.14}$$

denote its probability distribution. Integrating both sides of Eq. (4.6) on $(-\infty, x)$, it follows by simple computations that $\mathcal{F}(x,t)$ satisfies the equation

$$\frac{\partial \mathcal{F}}{\partial t} = (1+x^2)^{\alpha}\frac{\partial^2 \mathcal{F}}{\partial x^2} + 2\alpha(1+\lambda)x(1+x^2)^{\alpha-1}\frac{\partial \mathcal{F}}{\partial x}. \tag{4.15}$$

As before, let us define

$$\mathcal{G}(y,t) = \mathcal{F}(x,t), \tag{4.16}$$

where $y = y(x)$ is defined through (4.9). Then, using the same computations leading from (4.7) to (4.10) it is immediate to show that $\mathcal{G}$ satisfies

$$\frac{\partial \mathcal{G}}{\partial t} = \frac{\partial^2 \mathcal{G}}{\partial y^2} + W'(y)\frac{\partial \mathcal{G}}{\partial y}. \tag{4.17}$$

Hence, if for given $t > 0$, $Y(t)$ denotes the random process with probability density $g(x,t)$, solution of the Fokker–Planck equation (4.12), $\mathcal{G}(y,t)$ is the distribution function of the process $Y(t)$. This relation implies an explicit connection between the solutions to the equations (4.6) and (4.12). Indeed, differentiating the identity (4.16), one obtains for all $t \ge 0$

$$g(y(x),t) = f(x,t)(1+x^2)^{\frac{\alpha}{2}}, \tag{4.18}$$

and

$$g_\infty(y(x)) = f_\beta(x)(1+x^2)^{\frac{\alpha}{2}}. \tag{4.19}$$

The properties of the steady state $g_\infty(y)$ can be easily deduced from (4.19). Recalling that $\alpha > 1$, the change of variable (4.9) implies

$$y(x) = \int_0^x \frac{1}{(1+t^2)^{\frac{\alpha}{2}}}\,dt,$$

and since the integral function belongs to $L_1(\mathbb{R})$, then $\lim_{x\to\pm\infty} y(x) = \pm a(\alpha)$. Thus, $y(x)$ is contained in the strip $\mathbb{M} = [-a, a]$. We are now ready to prove inequality (4.4). Actually, Fokker–Planck equations of type (4.12) have been introduced as a useful working tool to get logarithmic Sobolev inequalities for probability densities different from the standard Gaussian [25]. The argument follows from Bakry and Emery theorem [2], which can be immediately applied thanks to the particular form of (4.12). More precisely, given the equilibrium density $g_\infty = Ce^{-W(y)}$ defined on a complete manifold $\mathbb{M} = [-a, a] \subset \mathbb{R}$, Bakry and Emery criterion guarantees that for all smooth probability densities g on $\mathbb{M}$ absolutely continuous with respect g_∞, it holds

$$\int_{\mathbb{M}} g(y)\log\frac{g(y)}{g_\infty(y)}dy \le \frac{1}{2\rho}\int_{\mathbb{M}}\left(\frac{d}{dy}\log\frac{g(y)}{g_\infty(y)}\right)^2 g(y)dy, \tag{4.20}$$

provided that the function W is strongly convex, with

$$W''(y) \geq \rho > 0. \tag{4.21}$$

In our case

$$W(y) = \int_0^y \alpha(1+2\lambda)x(s)(1+x^2(s))^{\frac{\alpha}{2}-1}\, ds.$$

Resorting to condition (4.9) we easily obtain

$$W''(y) = \alpha(1+2\lambda)\frac{1+(\alpha-1)x^2(y)}{(1+x^2(y))^{2-\alpha}}$$

The even function

$$z(x) = \frac{1+(\alpha-1)x^2(y)}{(1+x^2(y))^{2-\alpha}}.$$

attains its minimum in the point $\bar{x} = 0$ if $\alpha \geq \frac{3}{2}$, and in the point $\bar{x} = \frac{(3-2\alpha)^{\frac{1}{2}}}{\alpha-1}$, if $1 < \alpha < \frac{3}{2}$. Consequently

$$W''(y) \geq \alpha(1+2\lambda), \quad \alpha > \frac{3}{2}$$

whereas

$$W''(y) \geq \alpha(1+2\lambda)z(\bar{x}) = \alpha(1+2\lambda)\left(\frac{\alpha-1}{2-\alpha}\right)^{3-2\alpha}, \quad 1 < \alpha \leq \frac{3}{2}.$$

Let us notice that, as $\alpha \to 1$, the convexity condition is lost.

Finally, for $\alpha > 1$, and for any smooth probability density function g absolutely continuous with respect to g_∞, we get the logarithmic Sobolev inequality

$$\int_{\mathbb{M}} g(y)\log\frac{g(y)}{g_\infty(y)}dy \leq \frac{1}{2\rho}\int_{\mathbb{M}}\left(\frac{d}{dy}\log\frac{g(y)}{g_\infty(y)}\right)^2 g(y)dy \tag{4.22}$$

with $\rho = \alpha(1+2\lambda)z(\bar{x}) = \rho_{\alpha,\lambda}$.

The last step relies in rewriting inequality (4.22) in terms of the original Cauchy density f_β. This can be obtained easily by resorting again to the change of variables (4.9). In view of (4.8) and (4.18) the integral on the left-hand side of (4.22) becomes

$$\int_{-\infty}^{+\infty} g(y(x))\left(\log\frac{g(y(x))}{g_\infty(y(x))}\right)\frac{1}{(1+x^2)^{\frac{\alpha}{2}}}\, dx = \int_{\mathbb{R}} f(x)\log\frac{f(x)}{f_\beta(x)}\, dx.$$

Likewise, the integral on the right-hand side of (4.22) becomes

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(\frac{d}{dx}\log\frac{g(y(x))}{g_\infty(y(x))}\right)^2(1+x^2)^\alpha g(y(x))\frac{1}{(1+x^2)^{\frac{\alpha}{2}}}dx\\ &= \int_{\mathbb{R}}(1+x^2)^\alpha\left(\frac{d}{dx}\log\frac{f(x)}{f_\beta(x)}\right)^2 f(x)\, dx.\end{aligned}$$

Finally, inequality (4.22), written in terms of f and f_β reads

$$\int_{\mathbb{R}} f(x)\log\frac{f(x)}{f_\beta(x)}\, dx \leq \frac{1}{2\rho_{\alpha,\lambda}}\int_{\mathbb{R}}(1+x^2)^\alpha\left(\frac{d}{dx}\log\frac{f(x)}{f_\beta(x)}\right)^2 f(x)\, dx.$$

Resorting to the relation

$$\beta = \alpha(1+\lambda)$$

we replace $\lambda = \frac{\beta}{\alpha} - 1$ and we get $\rho_{\alpha,\lambda} = \rho_{\beta,\alpha}$ with

$$\rho_{\beta,\alpha} = \begin{cases} (2\beta-\alpha)\left(\frac{\alpha-1}{2-\alpha}\right)^{3-2\alpha} & 1<\alpha<\frac{3}{2} \\ 2\beta-\alpha & \frac{3}{2}\le\alpha<2\beta. \end{cases}$$

If $\alpha \ge 3/2$, it is obvious that $2\beta-\alpha < 2\beta - 3/2$. Considering that the weight function is increasing in α, it is convenient to replace this value to get a sharper constant. This concludes the proof. □

4.2. Weighted Logarithmic Sobolev Inequalities for Inverse Gamma Densities

As discussed in the Introduction, sharp logarithmic Sobolev inequalities for inverse–Gamma type densities are directly connected to the study of convergence to equilibrium for Fokker–Planck type equations like (1.9), of interest in the study of wealth distribution in a western society. The result that follows is essentially contained in the paper [17] and it is here reported with few details to make it possible to compare it with the result for the Cauchy-type densities obtained in the previous section. Like in Sect. 2, we use expression (2.18) that allows for a direct comparison with the result of Theorem 4.1.

Theorem 4.3. (Log–Sobolev for inverse Gamma-type densities) *Let X be a random variable distributed with the inverse Gamma probability density $h_{\beta,m}(x)$ defined by (2.18), with $\beta > 1/2$ and $m > 0$. For any bounded smooth function $\phi(x)$, with $x \in \mathbb{R}_+$ such that $\phi(X)$ has finite entropy, and for all $1 < \alpha \le \frac{3}{2}$ and $\alpha < \beta + \frac{1}{2}$ one has the bound*

$$Ent\left[\phi^2(X)\right] \le \frac{2}{\rho_{\beta,\alpha,m}} E\left\{X^{2\alpha}[\phi'(X)]^2\right\}. \tag{4.23}$$

In inequality (4.23) $\rho_{\beta,\alpha,m}$ is given by

$$\rho_{\beta,\alpha,m} = \begin{cases} \frac{1}{2}\left(\frac{2\beta-\alpha}{\frac{3}{2}-\alpha}\right)^{3-2\alpha}(m(2-\alpha))^{2\alpha-2}(\alpha-1)^{5-4\alpha} & 1<\alpha<\frac{3}{2} \\ \frac{m}{2} & \alpha=\frac{3}{2}, \quad \beta>1. \end{cases} \tag{4.24}$$

Proof. We proceed as in Theorem 4.1, by proving the equivalent inequality

$$\int_{\mathbb{R}_+} h(x)\log\frac{h(x)}{h_{\beta,m}(x)}\,dx \le \frac{1}{2\rho_{\beta,\alpha,m}}\int_{\mathbb{R}_+} x^{2\alpha}\left(\frac{d}{dx}\log\frac{h(x)}{h_{\beta,m}(x)}\right)^2 h(x)\,dx \tag{4.25}$$

for any probability density h, smooth and absolutely continuous with respect to $h_{\beta,m}$. As shown in Theorem 2.5, $h_{\beta,m}$ is the stationary state of the family of Fokker–Planck type equations (2.29), depending on the two parameters α and λ satisfying the constraint (2.28). Unlike the proof of Theorem 2.5, we assume now the conditions $1 < \alpha \le 3/2$ and $\lambda > -1$, still subject to the constraint $2\alpha + \lambda = 2\beta$. This choice is coherent with the lower bound $\beta > 1/2$ and $\alpha < \beta + 1/2$ in the statement of the theorem.

In terms of the function $H = h/h_{\beta,m}$, (2.29) reads

$$\partial_t H = x^{2\alpha}\partial_x^2 H - \lambda\left(x - \frac{m}{\lambda}\right)x^{2\alpha-2}\partial_x H, \quad x\in\mathbb{R}_+, t>0. \tag{4.26}$$

We proceed as in Theorem 4.1, and we change variable to transform the Fokker–Planck type equation (4.26) into a new one with coefficient of diffusion equal to one. This is done by setting

$$L(y,t) = H(x,t),$$

with

$$\frac{dy}{dx} = -\frac{1}{x^\alpha}, \quad x \in \mathbb{R}_+. \tag{4.27}$$

In terms of L, the right-hand side of (4.26) becomes

$$\frac{\partial^2 L}{\partial y^2} - \left(mx^{\alpha-2} - (\alpha+\lambda)x^{\alpha-1}\right)\frac{\partial L}{\partial y}$$

where $x = x(y)$ is the inverse of the decreasing function $y(x)$, defined by (4.27). In this case the function $y(x)$ can be computed explicitly to give

$$y(x) = \frac{1}{\alpha-1}\frac{1}{x^{\alpha-1}} \tag{4.28}$$

so that $y \in \mathbb{R}_+$. Equation (4.26) turns into

$$\frac{\partial L}{\partial t} = \frac{\partial^2 L}{\partial y^2} - U'(y)\frac{\partial L}{\partial y}, \tag{4.29}$$

where the drift term equals

$$U'(y) = m(\alpha-1)^{\frac{2-\alpha}{\alpha-1}} y^{\frac{2-\alpha}{\alpha-1}} - \frac{\alpha+\lambda}{\alpha-1}\frac{1}{y}. \tag{4.30}$$

Equation (4.29) is the adjoint of the Fokker–Planck equation

$$\frac{\partial l}{\partial t} = \frac{\partial^2 l}{\partial y^2} + \frac{\partial}{\partial y}(U'(y)l) \tag{4.31}$$

with diffusion coefficient still equal to one and steady state $l_\infty(y) = Ce^{-U(y)}$. In this case, we recognize that l_∞ is a generalized Gamma density [30]

$$l_{\beta,\alpha,m}(y) = \frac{C_{\beta,\alpha,m}}{y^{\frac{2\beta-\alpha}{\alpha-1}}} e^{-m(\alpha-1)^{\frac{1}{\alpha-1}} y^{\frac{1}{\alpha-1}}}, \quad y \in \mathbb{R}_+.$$

Proceeding as in the proof of Theorem 4.1, we conclude that the relation between the inverse–Gamma density $h_{\beta,m}$ and the generalized Gamma density $l_{\beta,\alpha,m}$ is given by

$$h_{\beta,m}(x) = l_{\beta,\alpha,m}(y(x))\left|\frac{dy}{dx}\right|. \tag{4.32}$$

To apply Bakry and Emery criterion to $l_{\beta,\alpha,m}$, we find a positive lower bound on U''. Since

$$U''(y) = \frac{1}{y^2(\alpha-1)}\left(m(2-\alpha)(\alpha-1)^{\frac{2-\alpha}{\alpha-1}} y^{\frac{1}{\alpha-1}} + \alpha + \lambda\right), \quad y > 0, \tag{4.33}$$

for $\alpha = 3/2$ we obtain

$$U''(y) \geq \frac{m}{2} := \rho\left(\beta, \frac{3}{2}, m\right), \quad y > 0. \tag{4.34}$$

If now $1 < \alpha < \frac{3}{2}$, U'' achieves its minimum in

$$\bar{y} = \left(\frac{\alpha + \lambda}{m(2-\alpha)\left(\frac{3}{2} - \alpha\right)} \right)^{\alpha - 1} \frac{1}{(\alpha-1)^{3-2\alpha}}.$$

Owing to (2.28) we write

$$\lambda = 2\beta - 2\alpha.$$

Then the minimum of the function U'' is given by

$$U''(\bar{y}) := \rho_{\beta,\alpha,m} > 0$$

with

$$\rho_{\beta,\alpha,m} = \frac{1}{2} \left(\frac{2\beta - \alpha}{\frac{3}{2} - \alpha} \right)^{3-2\alpha} (m(2-\alpha))^{2\alpha-2} (\alpha - 1)^{5-4\alpha}. \tag{4.35}$$

It is easy to verify that

$$\lim_{\alpha \to \frac{3}{2}} \rho_{\alpha,m,\beta} = \frac{m}{2}.$$

We observe that we can not choose $\alpha > 3/2$, since in this case

$$\lim_{y \to +\infty} U''(y) = 0,$$

and the strict convexity of $U(y)$ is lost.

If $\alpha \le 3/2$ we apply Bakry–Emery criterion as in [16] and we get the logarithmic Sobolev inequality for the generalized Gamma density $l_{\beta,\alpha,m}$

$$\int_{\mathbb{R}_+} l(y) \log \frac{l(y)}{l_{\beta,\alpha,m}(y)} dy \le \frac{1}{2\rho} \int_{\mathbb{R}_+} \left(\frac{d}{dy} \log \frac{l(y)}{l_{\beta,\alpha,m}(y)} \right)^2 l(y) dy \tag{4.36}$$

where $\rho = \rho_{\beta,\alpha,m}$ as in (4.34) and (4.35). Turning back to the original variables gives the result. □

As in Sect. 2, we can rewrite inequality (4.23) in terms of the standard notation of the inverse Gamma functions (2.33). For any given $\kappa > 0$ and $1 < \alpha \le 3/2$ with $\alpha < \kappa/2 + 1$ we obtain

$$\int_{\mathbb{R}_+} h(x) \log \frac{h(x)}{h_{\kappa,m}(x)} \, dx \le \frac{1}{2\rho_{\kappa,\alpha,m}} \int_{\mathbb{R}_+} x^{2\alpha} \left(\frac{d}{dx} \log \frac{h(x)}{h_{\kappa,m}(x)} \right)^2 h(x) \, dx, \tag{4.37}$$

or, equivalently, if X is a random variable distributed with probability density function (2.33)

$$Ent\left[\phi^2(X)\right] \le \frac{2}{\rho_{\kappa,\alpha,m}} E\left\{ X^{2\alpha} [\phi'(X)]^2 \right\}. \tag{4.38}$$

In inequalities (4.37) and (4.38) the constant $\rho_{\kappa,\alpha,m}$ is given by

$$\rho_{\kappa,\alpha,m} = \begin{cases} \frac{1}{2} \left(\frac{\kappa+1-\alpha}{\frac{3}{2}-\alpha} \right)^{3-2\alpha} (m(2-\alpha))^{2\alpha-2} (\alpha-1)^{5-4\alpha} & 1 < \alpha < \frac{3}{2} \\ \frac{m}{2} & \alpha = \frac{3}{2}, \quad \kappa > 1. \end{cases} \tag{4.39}$$

4.3. The Optimal Weight Growth for Cauchy-Type Densities

We will now be concerned with further results about weighted logarithmic Sobolev inequalities for both Cauchy-type and inverse Gamma densities, which include the results in [20, 28] about the optimal growth at infinity of the weight functions for Cauchy-type densities. Compared with the results of Sect. 4, the interval of values allowed for the exponent β is now smaller. However, the method of Sect. 4 is suitable to quantify explicitly the values of the constants, in contrast with the result in [28].

Theorem 4.4. (Optimal weight for Cauchy-type densities) *Let X be a random variable distributed with the Cauchy-type probability density (1.11), with $\beta > 1$. For any bounded smooth function $\phi(x)$, with $x \in \mathbb{R}$, such that $\phi(X)$ has finite entropy one has the bound*

$$Ent\left[\phi^2(X)\right] \le \frac{1}{\beta-1} E\left\{(1+X^2)\left[1+\log(1+X^2)\right][\phi'(X)]^2\right\}. \tag{4.40}$$

Proof. We proceed as in the proof of Theorem 4.1 of Sect. 4.1. Let us restrict the values of the parameter β to the interval $\beta > 1$. It is immediate to verify that in this range of β the Cauchy-type density f_β, as defined by (1.11), is a stationary state of the Fokker–Planck type equation

$$\begin{aligned}\partial_t f = \partial_x^2 &\left[(1+x^2)(1+\log(1+x^2))f\right] \\ &+\partial_x\left\{2x\left[\beta-2+(\beta-1)\log(1+x^2)\right]f\right\}, \quad x\in\mathbb{R},\, t>0.\end{aligned} \tag{4.41}$$

The Fokker–Planck equation (4.41) has been built to ensure that the target weight function coincides with the coefficient of diffusion. Then, the drift term is consequently derived in such a way that the Cauchy-type density f_β is the unique steady state of unitary mass. As described in Sect. 4.1, we make use of the equivalent formulation of the Fokker–Planck equation in terms of the function $F = f/f_\beta$. F then satisfies the evolution equation

$$\begin{aligned}\frac{\partial F}{\partial t} &= \left[(1+x^2)(1+\log(1+x^2))\right]\frac{\partial^2 F}{\partial x^2} \\ &\quad -\left\{2x\left[\beta-2+(\beta-1)\log(1+x^2)\right]f\right\}\frac{\partial F}{\partial x}.\end{aligned} \tag{4.42}$$

Let us now introduce the change of variables to make the diffusion coefficient equal to unity. To this end, let us define as before

$$G(y,t) = F(x,t), \tag{4.43}$$

with

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1+x^2)(1+\log(1+x^2))}}. \tag{4.44}$$

Clearly, the solution $y = y(x)$ to the differential equation (4.44) is increasing and it can be inverted. We denote by $x = x(y)$ the inverse function.

Owing to (4.44) we easily obtain that $G(y,t)$ satisfies a Fokker–Planck equation with coefficient of diffusion equal to one

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial y^2} - W'(y)\frac{\partial G}{\partial y}, \tag{4.45}$$

where the drift term $W'(y) = \tilde{W}'(x(y))$, and

$$\tilde{W}'(x) = x\frac{2(\beta-1) + (2\beta-1)\log(1+x^2)}{\sqrt{(1+x^2)(1+\log(1+x^2))}}. \tag{4.46}$$

Equation (4.45) is the adjoint of the Fokker–Planck equation

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2} + \frac{\partial}{\partial y}(W'(y)g) \tag{4.47}$$

still with diffusion coefficient equal to one, and steady state

$$g_\infty(y) = Ce^{-W(y)}. \tag{4.48}$$

It remains to show that the drift term $W'(y)$ is strongly convex. Resorting to the relationship (4.44), we easily obtain

$$\frac{dW'(y)}{dy} = \frac{d\tilde{W}'(x)}{dx}\frac{dx}{dy}\bigg|_{x=x(y)} = \frac{d\tilde{W}'(x)}{dx}\sqrt{(1+x^2)(1+\log(1+x^2))}\bigg|_{x=x(y)}.$$

Simple computations show that

$$\frac{dW'(y)}{dy} \geq 2(\beta-1). \tag{4.49}$$

Indeed

$$\frac{dW'(y)}{dy} = 2(\beta-1) + A(u,v),$$

where, if $u = x^2$ and $v = \log(1+x^2)$, $A(u,v)$ denotes the nonnegative function

$$A(u,v) = \frac{(2\beta-1)(1+u)v(1+v) + 2(2\beta-1)u(1+v)}{(1+u)(1+v)} - \frac{4(\beta-1)u + 2(3\beta-2)uv + (2\beta-1)uv^2}{(1+u)(1+v)} \geq \frac{2u+uv}{(1+u)(1+v)}. \tag{4.50}$$

We can now end up by the same computations we did in theorem 4.1. This allows us to recover the result in [20, 28] with the same growth of the weight at infinity, and with explicit constant. □

Remark 4.5. The result of Theorem 4.4 is stronger than the result of Theorem 4.1, since the constant in front of the integral 4.4 in Theorem 4.1 converges to infinity as $\alpha \to 0$. However, as far as the exponential convergence to infinity in relative entropy of the solution to the Fokker–Planck type equation (4.6) is concerned, inequality (4.40) does not lead to a better result. Indeed, for $0 < \alpha < 1$, the inequality

$$1 + \log(1+x^2) \leq \frac{1}{\alpha^p}(1+x^2)^\alpha,$$

is satisifed with

$$p = p(\alpha) = 1 - \frac{1-\alpha}{\log 1/\alpha},$$

and this implies that $1/\alpha^{p(\alpha)}$ tends to zero as α tends to zero.

4.4. The Optimal Weight Growth for Inverse Gamma Densities

Last, we recover the optimal weight for the class of inverse Gamma densities. We prove

Theorem 4.6. (Optimal weight growth for inverse Gamma densities) *Let X be a random variable distributed with the inverse Gamma probability density (2.18), with $\beta > 1$. For any bounded smooth function $\phi(x)$, with $x \in \mathbb{R}_+$, such that $\phi(X)$ has finite entropy one has the bound*

$$Ent\left[\phi^2(X)\right] \le \frac{2}{\rho_{\beta,m}} E\left\{X^2 \log(1+X)[\phi'(X)]^2\right\}, \tag{4.51}$$

where

$$\rho_{\beta,m} = \min\left\{\frac{m}{2}, (\beta-1)\right\}. \tag{4.52}$$

Proof. We proceed as in Sect. 4.3. Let us fix $\beta > 1$. The inverse Gamma density $h_{\beta,m}$, as defined by (2.18), is a stationary state of the Fokker–Planck type equation

$$\begin{aligned}\partial_t h &= \partial_x^2\left[x^2\log(1+x)h\right]\\ &\quad+\partial_x\left\{\left[(2(\beta-1)x-m)\log(1+x)-\frac{x^2}{1+x}\right]h\right\}, \quad x\in\mathbb{R}_+,\ t>0.\end{aligned} \tag{4.53}$$

Also in this case, the Fokker–Planck equation (4.53) has been built to ensure that the target weight function coincides with the coefficient of diffusion. Then, the drift term is consequently evaluated to obtain the inverse Gamma density (2.18) as steady state.

We make use of the equivalent formulation of the Fokker Planck equation in terms of the function $H = h/h_{\beta,m}$. H then satisfies the evolution equation

$$\begin{aligned}\frac{\partial H}{\partial t} &= x^2\log(1+x)\frac{\partial^2 H}{\partial x^2}\\ &\quad-\left[(2(\beta-1)x-m)\log(1+x)-\frac{x^2}{1+x}\right]\frac{\partial H}{\partial x}.\end{aligned} \tag{4.54}$$

Let us now change variables in order to render the diffusion coefficient equal to unity. We set

$$L(y,t) = H(x,t), \tag{4.55}$$

with

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2\log(1+x)}}. \tag{4.56}$$

Let us denote by $x = x(y)$ the inverse function of the increasing function $y = y(x)$, solution to the differential equation (4.56).

Owing to (4.56) we obtain the Fokker–Planck equation satisfied by $L(y,t)$

$$\frac{\partial L}{\partial t} = \frac{\partial^2 L}{\partial y^2} - U'(y)\frac{\partial L}{\partial y}, \tag{4.57}$$

where the drift term $U'(y) = \tilde{U}'(x(y))$, and $\tilde{U}'(x)$ is easily computed

$$\tilde{U}'(x) = \left[(2\beta - 1) - \frac{m}{x}\right]\sqrt{\log(1+x)} - \frac{x}{2(1+x)}\frac{1}{\sqrt{\log(1+x)}}. \tag{4.58}$$

The drift term $U'(y)$ is strongly convex. Thanks to (4.56), we have

$$\frac{dU'(y)}{dy} = \frac{d\tilde{U}'(x)}{dx}\frac{dx}{dy}\bigg|_{x=x(y)} = \frac{d\tilde{U}'(x)}{dx}\sqrt{x^2\log(1+x)}\bigg|_{x=x(y)}.$$

Let us set

$$D(x) = \frac{d\tilde{U}'(x)}{dx}\sqrt{x^2\log(1+x)}.$$

Through simple computation we obtain

$$\begin{aligned} D(x) = m\frac{2(1+x)\log(1+x) - x}{2x(1+x)} + \frac{x}{2(1+x)}\left(2\beta - 1 - \frac{1}{1+x}\right) \\ + \frac{x^2}{4(1+x)^2}\frac{1}{\log(1+x)}. \end{aligned} \tag{4.59}$$

Let us now observe that the function

$$z(x) = \frac{2(1+x)\log(1+x) - x}{x}$$

is increasing from the value $z(0) = 1$, while

$$2\beta - 1 - \frac{1}{1+x} \geq 2(\beta - 1).$$

Hence, discarding the nonnegative last term in (4.59) we have

$$D(x) \geq \frac{1}{1+x}\left(\frac{m}{2} + (\beta - 1)x\right),$$

which implies

$$D(x) \geq \min\left\{\frac{m}{2}, (\beta - 1)\right\}.$$

□

5. Conclusions

The recent developments of mathematical modeling of social and economic phenomena led to the study of new types of Fokker–Planck equations characterized by steady state solutions with fat tails. For a precise study of the convergence to equilibrium of the solution to these equations, functional inequalities with weight are the main mathematical tool. In this paper we showed how to make use of these Fokker–Planck type equations to obtain one-dimensional functional inequalities, sometimes expressed in sharp form. This method is closely connected to kinetic theory, and more in general to statistical physics, and gives a new light to the meaning of these inequalities, in agreement with the results obtained in the case of the classical Fokker–Planck equation [22, 32, 33].

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References

[1] Bakry, D., Cattiaux, P., Guillin, A.: Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. J. Funct. Anal. **254**, 727–759 (2008)

[2] Bakry, D., Émery, M.: Diffusions hypercontractives. In: Séminaire de probabilités, XIX, 1983/84, vol. 1123 of Lecture Notes in Math., pp. 177–206. Springer, Berlin (1985)

[3] Bobkov, S.G., Ledoux, M.: Weighted Poincaré-type inequalities for Cauchy and other convex measures. Ann. Probab. **3**(7), 403–427 (2009)

[4] Bonnefont, M., Joulin, A.: Intertwining relations for one-dimensional diffusions and application to functional inequalities. Pot. Anal. **41**, 1005–1031 (2014)

[5] Bonnefont, M., Joulin, A., Ma, Y.: Spectral gap for spherically symmetric log-concave probability measures, and beyond. J. Funct. Anal **270**, 2456–2482 (2016)

[6] Bonnefont, M., Joulin, A., Ma, Y.: A note on spectral gap and weighted Poincaré inequalities for some one-dimensional diffusions. ESAIM, PS **20**, 18–29 (2016)

[7] Bouchaud, J.F., Mézard, M.: Wealth condensation in a simple model of economy. Physica A **282**, 536–545 (2000)

[8] Brascamp, H.J., Lieb, E.H.: On extensions of the Brunn–Minkovski and Prékopa–Leindler theorems, including inequalities for log-concave functions, and with an application to the diffusion equation. J. Funct. Anal. **22**, 366–389 (1976)

[9] Caglioti, E., Villani, C.: Homogeneous cooling states are not always good approximation to granular flows. Arch. Ration. Mech. Anal. **163**, 32–343 (2002)

[10] Cattiaux, P., Gozlan, N., Guillin, A., Roberto, C.: Functional inequalities for heavy tailed distributions and application to isoperimetry. Electron. J. Prob. **15**, 346–385 (2010)

[11] Chernoff, H.: A note on an inequality involving the normal distribution. Ann. Probab. **9**(3), 533–535 (1981)

[12] Cordier, S., Pareschi, L., Toscani, G.: On a kinetic model for a simple market economy. J. Stat. Phys. **120**, 253–277 (2005)

[13] Elcrat, A.R., MacLean, A.: Weighted Wirtinger and Poincaré inequalities on unbounded domains. Indiana Univ. Math. J. **29**(3), 321–332 (1980)

[14] Feller, W.: The parabolic differential equations and the associated semi-groups of transformations. Ann. Math. **55**(3), 468–519 (1952)

[15] Furioli, G., Pulvirenti, A., Terraneo, E., Toscani, G.: Fokker–Planck equations in the modelling of socio-economic phenomena. Math. Mod. Methods Appl. Sci. **27**(1), 115–158 (2017)

[16] Furioli, G., Pulvirenti, A., Terraneo, E., Toscani, G.: Wright–Fisher-type equations for opinion formation, large time behavior and weighted logarithmic-Sobolev inequalities. Ann. IHP Analyse Non Linéaire **36**, 2065–2082 (2019)

[17] Furioli, G., Pulvirenti, A., Terraneo, E., Toscani, G.: Non-Maxwellian kinetic equations modeling the evolution of wealth distribution. Math. Mod. Methods Appl. Sci. **30**(4), 685–725 (2020)

[18] Gentil, I., Zugmeyer, S.: A family of Beckner inequalities under various curvature-dimension conditions. Bernoulli **27**(2), 751–771 (2021)

[19] Gozlan, N.: Poincaré inequalities and dimension free concentration of measure. Ann. Inst. H. Poincaré Probab. Stat. **46**, 708–739 (2010)

[20] Hebisch, W., Zegarliński, B.: Coercive inequalities on metric measure spaces. J. Funct. Anal. **258**(3), 814–851 (2010)

[21] Klaassen, C.A.: On an inequality of Chernoff. Ann. Probab. **13**(3), 966–974 (1985)

[22] Markowich, P.A., Villani, C.: On the trend to equilibrium for the Fokker–Planck equation: an interplay between physics and functional analysis. Mat. Contemp. **19**, 1–29 (2000)

[23] Naldi, G., Pareschi, L., Toscani, G. (eds.): Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences. Birkhauser, Boston (2010)

[24] Newman, M.E.: Power laws, Pareto distributions and Zipf's law. Contemp. Phys. **46**(5), 323–351 (2005)

[25] Otto, F., Villani, C.: Generalization of an inequality by Talagrand and Links with the logarithmic Sobolev inequality. J. Funct. Anal. **173**, 361–400 (2000)

[26] Pareschi, L., Toscani, G.: Interacting Multiagent Systems: Kinetic Equations and Monte Carlo Methods. Oxford University Press, Oxford (2014)

[27] Pareto, V.: Cours d'Économie Politique. Lausanne and Paris (1897)

[28] Saumard, A.: Weighted Poincaré inequalities, concentration inequalities and tail bounds related to Stein kernels in dimension one. Bernoulli **25**, 3978–4006 (2019)

[29] Saumard, A., Wellner, J.A.: On the isoperimetric constant, covariance inequalities and L_p-Poincaré inequalities in dimension one. Bernoulli **25**, 1794–1815 (2019)

[30] Stacy, E.W.: A generalization of the gamma distribution. Ann. Math. Stat. **33**, 1187–1192 (1962)

[31] Torregrossa, M., Toscani, G.: On a Fokker–Planck equation for wealth distribution. Kinet. Relat. Models **11**(2), 337–355 (2018)

[32] Toscani, G.: Sur l'inégalité logarithmique de Sobolev. C. R. Acad. Sci. Paris Sér. I Math **324**, 689–694 (1997)

[33] Toscani, G.: Entropy production and the rate of convergence to equilibrium for the Fokker–Planck equation. Q. Appl. Math. Vol. LVI **I**, 521–541 (1999)

[34] Toscani, G.: Entropy-type inequalities for generalized Gamma densities. Ricerche di Matematica **70**, 35–50 (2021)

[35] Zhang, Z., Qian, B., Liu, W.: Optimal weighted Poincaré and log-Sobolev inequalities for Cauchy measures. arXiv:hal-00528966 (2010)

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