This article has been accepted for publication in IEEE Control Systems Letters. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/LCSYS.2024.3523844

On the Stability of a Nonlinear MPC Scheme for Avoidance

Marcelo Alves dos Santos, Antonio Ferramosca, and Guilherme Vianna Raffo

***Abstract*—This work analyzes the stability properties of a nonlinear Model Predictive Control (MPC) scheme for avoidance. This control approach introduces an extra penalty for avoidance within the nonlinear tracking MPC framework. We demonstrate that, under a mild assumption on the avoidance penalty, the closed-loop system is Input-to-State Stable (ISS) with respect to this penalty. Furthermore, we discuss the conditions under which asymptotic stability can be achieved and present a simplified scheme with relaxed terminal constraints. To illustrate the effectiveness of the proposed strategy, we apply it to the control of a van der Pol oscillator subjected to non-convex constraints.**

***Index Terms*—Constrained control, Optimal control, Predictive control for nonlinear systems, Stability of nonlinear systems.**

I. Introduction

MPC has made significant progress in stability guarantees, multi-objective handling, and optimization efficiency [1]. These advancements, coupled with increasing computational power in embedded systems, have broadened its applicability to real-world scenarios. MPC's ability to integrate constraints and objectives makes it particularly well-suited for tasks like avoidance, where systems evolve without entering undesirable regions of the state or output space.

Although often associated with obstacle avoidance in navigation, avoidance is a broader concept encompassing scenarios where controlled systems avoid undesirable regions, whether physical obstacles or other constraints. This generalization highlights that avoidance capabilities extend beyond navigation problems.

Traditionally, avoidance in MPC has been addressed using multi-level approaches that separate reference generation from control [2], [3]. However, these methods often face challenges with feasibility and stability during set-point transitions. In this context, integrated MPC-based strategies embed avoidance directly into the optimization problem, addressing these issues and enabling the seamless incorporation of secondary tasks into primary control objectives, a concept increasingly explored by the robotics community [4].

Avoidance remains challenging due to its inherent non-convexity. Existing methods either directly impose avoidance constraints [5], [6] or relax them to enhance feasibility [7], [8], but recursive feasibility and stability guarantees are often overlooked, particularly in dynamic environments where non-feasible regions are previously unknown, meaning that their number and form can vary over time.

Building on these challenges, [9] proposed a linear MPC scheme with avoidance penalties, proving that the closed-loop system is Input-to-State Stable (ISS) and recursively feasible. Subsequently, [10] considered these ideas in the context of nonlinear systems, applying them to obstacle avoidance in robotics using a set-point tracking framework [11]. However, while effective, the controller in [10] lacked formal guarantees, as its mathematical properties, such as stability and recursive feasibility, were not rigorously analyzed. Instead, the focus was on practical implementation.

This work addresses the limitations of [10] by providing a rigorous mathematical analysis of the proposed nonlinear MPC scheme, extending it to a broader class of avoidance problems. Specifically, we prove the scheme's feasibility and stability for generic nonlinear systems, showing that the closed-loop system is ISS with respect to the avoidance penalty and recursively feasible for any initial feasible condition. While the analysis shares some similarities with the linear case in [9], the necessity of using Lipschitz continuity arguments, instead of optimality arguments, was essential to handle nonlinearities. In this context, this work can be seen as an extension of [9] to the nonlinear case.

Beyond the theoretical contributions, a numerical example using a van der Pol oscillator with non-convex constraints demonstrates the methodology's ability to address non-convexity, a central challenge in avoidance problems. This example shows that MPC-based avoidance strategies apply to a wide range of scenarios beyond obstacle avoidance in robotics, bridging the gap between theoretical insights and practical applications.

Notation and definitions

The set $\mathbb{I}_{0:N}$ denotes the set of integers $\{0, 1, \cdots, N\}$, and a matrix $0_{n,m} \in \mathbb{R}^{n\times m}$ denotes a zero matrix. The Euclidean

This work was partially supported by the project INCT (National Institute of Science and Technology) under the grants CNPq 465755/2014-3 and FAPESP 2014/50851-0, and by the Brazilian agencies CAPES through the Academic Excellence Program (PROEX), CNPq 317058/2023-1 and 422143/2023-5, and FAPEMIG APQ-03090-17. It was also supported by the Italian Ministry for University and Research grant PRIN 2022 "AGeNT", n. 20223F5SZB - CUP: F53D23000470006.

Marcelo Alves dos Santos is with the Graduate Program in Electrical Engineering, Federal University of Minas Gerais, Belo Horizonte, MG, Brazil (e-mail: cidomg32@ufmg.br).

Guilherme V. Raffo is with the Department of Electronics Engineering and the Graduate Program in Electrical Engineering, Federal University of Minas Gerais, Belo Horizonte, MG, Brazil (e-mail: raffo@ufmg.br).

Antonio Ferramosca is with the Department of Management, Information and Production Engineering, University of Bergamo, Bergamo, Italy (e-mail: antonio.ferramosca@unibg.it).

This article has been accepted for publication in IEEE Control Systems Letters. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/LCSYS.2024.3523844

norm of x is denoted as $\|x\| = \sqrt{x'x}$, with $'$ being the transpose operator. The absolute value of x is denoted as $|x|$. Given a set $X \subset \mathbb{R}^n$ and a variable $\lambda \in \mathbb{R}$, the set $\lambda X \subset \mathbb{R}^n$ is defined as $\lambda X = \{\lambda x : x \in X\}$. A bold lowercase variable $\boldsymbol{u}$ denotes a sequence of values of a signal $(u(0), u(1), \cdots, u(N-1))$, with $u(i)$ being the i-th element and N being the length of the sequence deduced by the context. A parameter-dependent signal is denoted by $\boldsymbol{u}(a)$, and its i-th element is $u(i;a)$.

II. Control Formulation

Consider a system described by a nonlinear discrete-time model of the form

$$\begin{aligned} x(k+1) &= f(x(k), u(k)), \\ y(k) &= h(x(k), u(k)), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ are, respectively, the system state, the current control input, and the controlled output at the sampling time k. The system solution for a given sequence of control inputs $\boldsymbol{u}$ and the initial state x is denoted as $x(j) = \phi(j; x, \boldsymbol{u}), j \in \mathbb{I}_{\geq 0}$, where $x = \phi(0; x, \boldsymbol{u})$.

The evolution of the system is subjected to the constraints

$$(x(k), u(k)) \in Z, \ \forall k \in \mathbb{I}_{\geq 0}, \tag{2}$$

with $Z \subset \mathbb{R}^{n+m}$ being a compact convex polyhedron containing the origin.

Any suitable target for the system (1) must be associated with an equilibrium point (x_s, u_s, y_s), which is described by

$$x_s = f(x_s, u_s), \ y_s = h(x_s, u_s), \tag{3}$$

with x_s, u_s, and y_s being, respectively, the steady state, steady input, and steady output.

The set of joint steady states and inputs, Z_s, and the set of reachable set-points, Y_r, can be defined as

$$Z_s = \{(x_s, u_s) : x_s = f(x_s, u_s), (x_s, u_s) \in \lambda Z\}, \tag{4}$$

$$Y_r = \{y_t : y_t = h(x_s, u_s), (x_s, u_s) \in Z_s\}, \tag{5}$$

with $\lambda \in (0, 1)$ being a constant defined to avoid loss of controllability related to having equilibrium points at active constraints [12]. It is worthwhile mentioning that since λ can be chosen arbitrarily close to 1, Z is arbitrarily close to λZ in the Hausdorff sense.

Assumption 1: The state vector is available at each sampling time, and the state-transition, $f(\cdot)$, and the state-output, $h(\cdot)$, maps are continuously differentiable at any equilibrium point. Moreover, based on continuity arguments, $f(\cdot)$ is bounded for bounded states with $f(0_{n,1}, 0_{m,1}) = 0_{n,1}$.

Assumption 2: The steady output y_s uniquely defines the equilibrium (x_s, u_s), and there exists a locally Lipschitz continuous function $g_x : \mathbb{R}^p \mapsto \mathbb{R}^n$ and a continuous function $g_u : \mathbb{R}^p \mapsto \mathbb{R}^m$ such that $x_s = g_x(y_s)$ and $u_s = g_u(y_s)$.

Therefore, we want to solve the set-point tracking control problem for the system (1) constrained by (2) considering that certain regions of the admissible space are non-feasible and, therefore, should be avoided. If the set-point is reachable, i.e. $y_t \in Y_r$, and can be achieved without getting inside the non-feasible regions, the closed-loop system should converge to y_t. Otherwise, the closed-loop system should converge to a steady output y_s that minimizes a given index.

For that, considering a previously unknown number N_o of non-feasible output regions O_i strictly contained in Y, the admissible output set can be defined as

$$\tilde{Y} = Y - \bigcup_{i=1}^{N_o} O_i, \tag{6}$$

with Y being the set of admissible outputs obtained from (2) considering the output-state map $h(\cdot)$. In this work, any output non-feasible set O_i is assumed known only at time instant k and time-invariant for prediction purposes, with no convexity assumptions made. In other words, the number and shapes of the non-feasible output regions may vary over time.

It is cumbersome to enforce the evolution of the system to lie inside the non-convex set $\tilde{Y}$ using hard constraints in an optimal control problem. For that reason, in [9], it was proposed a linear MPC scheme with avoidance features that adds to the tracking formulation of [13] an additional penalty for avoidance, $V_a(\cdot) \in \mathbb{R}$. This penalty together with the tracking MPC ingredients, namely, artificial variables (x_a, u_a, y_a) and an offset cost functional $V_o(\cdot) \in \mathbb{R}$, allows the controlled system to track a given set-point y_t while avoiding the non-feasible regions. In what follows, this idea will be further analyzed for nonlinear systems.

Let $y_a \in Y_r$ be an admissible artificial output such that $x_a = g_x(y_a)$ and $u_a = g_u(y_a)$. Furthermore, consider the offset cost functional $V_o(\cdot)$ to be a penalty on the deviation between the artificial steady output y_a and the target output y_t. Further, let the avoidance cost functional $V_a(\cdot)$ be a penalty designed to constrain the system output to the admissible space $\tilde{Y}$. Then, we can consider the following MPC cost functional

$$\begin{aligned} V_N(x, y_t, O_i; \boldsymbol{u}, y_a) = V_o(y_a - y_t) + V_a(y(j), y_a, O_i) + \\ \sum_{j=0}^{N-1} \ell(x(j) - x_a, u(j) - u_a) + V_f(x(N) - x_a) \end{aligned} \tag{7}$$

for all possible non-feasible regions O_i. Moreover, the cost functionals $\ell(\cdot)$ and $V_f(\cdot)$ denote, respectively, the stage cost and the terminal cost, which are assumed continuous. For more information on MPC fundamentals, refer to [14].

The MPC controller with avoidance features is derived from the solution of the optimization problem $P_N(x, y_t, O_i)$ having as parameters (x, y_t, O_i) and as decision variables $(\boldsymbol{u}, y_a)$, which is given by

$$\min_{\boldsymbol{u}, y_a} V_N(x, y_t, O_i; \boldsymbol{u}, y_a)$$

$$\text{s.t. } x(0) = x, \tag{8a}$$

$$x(j+1) = f(x(j), u(j)), j \in \mathbb{I}_{0:N-1}, \tag{8b}$$

$$y(j) = h(x(j), u(j)), j \in \mathbb{I}_{0:N-1}, \tag{8c}$$

$$(x(j), u(j)) \in Z, j \in \mathbb{I}_{0:N-1}, \tag{8d}$$

$$x_a = f(x_a, u_a), \tag{8e}$$

$$y_a = h(x_a, u_a), \tag{8f}$$

$$(x(N), y_a) \in \Omega_t^a, \tag{8g}$$

with the predicted trajectory subjected to the system dynamics and constraints (8a)-(8d), and with constraints (8e) and (8f)

This article has been accepted for publication in IEEE Control Systems Letters. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/LCSYS.2024.3523844

defining the artificial steady output related to an artificial equilibrium. Moreover, constraint (8g) enforces the terminal state to reach an invariant set for tracking Ω_t^a [11]. It is worthwhile mentioning that the set of constraints of $P_N(x, y_t, O_i)$ depends neither on y_t nor on O_i. Then, there exists a region $X_N \subseteq \mathbb{R}^n$ such that, for all $x \in X_N$, $y_t \in \mathbb{R}^n$, and $O_i \subset \mathbb{R}^p$, the optimization problem $P_N(x, y_t, O_i)$ is feasible [15, Proposition 2.4]. The set X_N is by definition the domain of attraction of the closed-loop system.

The optimal solution to the optimization problem $P_N(x, y_t, O_i)$ is denoted as $(\boldsymbol{u}^O, y_a^O)$. Furthermore, considering the receding horizon policy of MPC controllers, the optimal control law is implicitly given by the first element of the optimal control sequence, yielding

$$\kappa_N(x, y_t, O_i) = \boldsymbol{u}^O(0; x, y_t, O_i). \tag{9}$$

Assumption 3: The following conditions from the tracking MPC literature are sufficient to ensure asymptotic stability for a system in closed-loop with (9) without non-feasible regions O_i [11]:

3.1. Let α_ℓ be a $\mathcal{K}_\infty$-function such that $\ell(x - x_a, u - u_a) \geq \alpha_\ell(\|x - x_a\|)$ for all $(x, u) \in \mathbb{R}^{n+m}$ and $(x_a, u_a) \in Z_s$;

3.2. Let $\kappa(x, y_a)$ be a continuous control law defined over the set Ω_t^a, such that (x_a, u_a) is an asymptotically stable equilibrium point for the closed-loop system (1) controlled by $\kappa(x, y_a)$;

3.3. Let $\Omega_t^a \subseteq \mathbb{R}^{n+p}$ be an invariant set for tracking for the closed-loop system (1) controlled by $\kappa(x, y_a)$ such that, for all $(x, y_a) \in \Omega_t^a$, we have that $(x, \kappa(x, y_a)) \in Z$, $y_a \in Y_r$, and $(f(x, \kappa(x, y_a)), y_a) \in \Omega_t^a$;

3.4. Let $V_f(x - x_a)$ be a control Lyapunov function for the closed-loop system (1) controlled by $\kappa(x, y_a)$ such that, for all $(x, y_a) \in \Omega_t^a$, there exist constants $b > 0$ and $\sigma > 1$ which verify

$$V_f(x - x_a) \leq b\|x - x_a\|^\sigma \tag{10}$$

and

$$\begin{aligned} V_f(f(x, \kappa(x, y_a)) - x_a) - V_f(x - x_a) \leq \\ - \ell(x - x_a, \kappa_f(x, y_a) - u_a); \end{aligned} \tag{11}$$

3.5. Let $V_o\left(y_a - y_t\right) : \mathbb{R}^p \mapsto \mathbb{R}_{\geq 0}$ be a continuous, convex, and positive definite function with $V_o(0_{p,1}) = 0$, such that the minimizer

$$y_a^O = \arg \min_{y_a \in Y_r} V_o(y_a - y_t) \tag{12}$$

is unique for any y_t. Furthermore, there exists a $\mathcal{K}_\infty$-function α_o such that

$$V_o(y_a - y_t) - V_o(y_a^O - y_t) \geq \alpha_o(\|y_a - y_a^O\|). \tag{13}$$

To later derive the ISS property with respect to the avoidance cost, which is the main argument in the stability analysis, we consider the additional assumption.

Assumption 4: Let $V_a\left(y, y_a, O_i\right) : \mathbb{R}^{3p} \mapsto \mathbb{R}_{\geq 0}$ be a continuous function such that $S = \sup(V_a(y, y_a, O_i))$ exists.

Depending on the penalties used, only local convergence may be achieved due to local minima, and the non-convex constraint may not be exactly satisfied. As this work focuses on stability, where Assumption 4 is sufficient, we do not discuss possible avoidance penalties. For more on penalty methods, refer to the nonlinear programming literature [16].

III. Stability Analysis

When we consider the problem without any non-feasible output regions, the tracking MPC literature establishes that, for a given target y_t and for any feasible initial state x, the closed-loop system is stable and fulfills the constraints throughout the time (see Theorem 1 of [11]). Two conditions can be defined regarding the convergence of the system:

(i) If $y_t \in Y_r$, then the closed-loop system asymptotically converges to y_t;

(ii) If $y_t \notin Y_r$, then the closed-loop system asymptotically converges to a reachable steady output that minimizes

$$\arg \min_{y_a \in Y_r} V_o\left(y_a - y_t\right).$$

The stability analysis of the nonlinear MPC scheme for avoidance follows similar steps to those in [9] for the linear case. First, we prove recursive feasibility of the controlled system. Next, a shifted value function is defined to account for the avoidance cost, and upper, lower, and decrease bounds for the function are established. Finally, we show that the closed-loop system is ISS with respect to the bound S and, consequently, the avoidance cost functional.

Lemma 1: (Recursive feasibility) Consider that Assumptions 1 to 3 hold, then the system in closed-loop with $\kappa_N(x, y_t, O_i)$ is recursively feasible for any feasible state $x \in X_N$.

Proof: Recursive feasibility can be proven using standard arguments of the MPC literature. First, we consider an optimal control sequence $\boldsymbol{u}^O$ associated with the optimal predicted state sequence $\boldsymbol{x}^O$. Then, we choose a feasible input sequence $\tilde{\boldsymbol{u}}$ that is equal to the optimal sequence in all but the last element. Finally, leveraging the invariance of Ω_t^a, it can be shown that if $x \in X_N$, then $x(k+1) \in X_N$. For additional details, see Lemma 3 of [9]. ■

Before analyzing the existence of an ISS-Lyapunov function for system (1) constrained by (2) and in closed-loop with (9), we need to show that the optimal artificial triplet (x_a^O, u_a^O, y_a^O) converges to the steady triplet (x_s, u_s, y_s). This result was presented in [9] for linear systems; however, it cannot be applied directly to the nonlinear case. The next lemma addresses this issue and its proof is in the Appendix.

Lemma 2: (Steady condition convergence) Consider that Assumptions 1 to 4 hold for the system (1) constrained by (2). For any feasible initial state $x \in X_N$, target y_t, and bound S, let the optimal solution to $P_N(x, y_t, O_i)$ be such that $x = x_a^O$, $u = u_a^O$, and $y = y_a^O$. Furthermore, let (x_s, u_s, y_s) be the optimal triplet that satisfies (3), so that the function $V_o(y_a - y_t) + V_a(y, y_a, O_i)$ is minimized. Then, $x = x_s$, $u = u_s$, and $y = y_s$.

Consider the shifted value function defined by

$$V_s(x, y_t, O_i) = V_N(x, y_t, O_i) - S, \tag{14}$$

as a Lyapunov candidate for the problem $P_N(x, y_t, O_i)$.

Lemma 3: (ISS-Lyapunov function) Consider that Assumptions 1 to 4 hold, then it holds that

1) There exist $\mathcal{K}_\infty$-functions $\alpha_c(\cdot)$ and $\alpha_b(\cdot)$ such that

$$\alpha_b(\|x - x_s\|) - S \leq V_s(x, y_t, O_i) \leq \alpha_c(\|x - x_s\|); \quad (15)$$

2) There exists a $\mathcal{K}_\infty$-function $\alpha(\cdot)$ such that

$$\begin{aligned} V_s^O(x(k+1), y_t, O_i) - &V_s^O(x(k), y_t, O_i) \\ &\leq -\alpha(\|x - x_s\|) + S; \end{aligned} \quad (16)$$

3) There exist a $\mathcal{KL}$-function $\hat{\beta}(\cdot)$ and a $\mathcal{K}$-function $\hat{\gamma}(\cdot)$ such that

$$\begin{aligned} &V_s^O(x(k), y_t, O_i) \\ &\quad \leq \max\{\hat{\beta}(V_s^O(x(0), y_t, O_i), k), \hat{\gamma}(S)\}. \end{aligned} \quad (17)$$

Proof: This proof follows directly from the proofs of Lemmas 4 to 7 of [9], which is based on optimality, boundedness, and continuity arguments. Once Lemma 2 is considered, as proposed in this work, their extension to nonlinear systems is straightforward and will be omitted. ■

Theorem 1: (ISS-based avoidance) Consider that Assumptions 1 to 4 are satisfied. Then, the system (1) in closed-loop with the optimal control law $\kappa_N(x, y_t, O_i)$ is ISS with respect to the avoidance cost $V_a(y, y_a, O_i)$, i.e., there exist a $\mathcal{KL}$-function $\beta(\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that, for any feasible initial state $x \in X_N$, steady output $y_s \in Y_r$, and bound S, the solution $\phi(k; x, \boldsymbol{u})$ exists and satisfies

$$\|\phi(k; x, \boldsymbol{u}) - g_x(y_s)\| \leq \beta(\|x - g_x(y_s)\|, k) + \gamma(S), \quad (18)$$

for all $k \in \mathbb{I}_{>0}$.

Proof: Based on Lemma 3, $V_s(x, y_t, O_i)$ is an ISS-Lyapunov function for system (1) with bounds $\hat{\beta}(\cdot)$ and $\hat{\gamma}(\cdot)$. Then, from Lemma 3.5 of [17], we can state that the system is ISS since it assumes an ISS-Lyapunov function. Therefore, there exist a $\mathcal{KL}$-function $\beta(\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that $\|\phi(k; x, \boldsymbol{u}) - g_x(y_s)\| \leq \beta(\|x - g_x(y_s)\|, k) + \gamma(S)$ for all $k \in \mathbb{I}_{>0}$, concluding the proof. ■

Remark 1: The results of Theorem 1 should be interpreted as follows. In the presence of non-feasible output regions, the avoidance cost acts as a disturbance and only ISS can be ensured. Consequently, the closed-loop system converges to a bounded set around a steady state, which can be either desired or feasible. Therefore, when considering the avoidance problem, asymptotic stability, as defined in the literature, can only be recovered when the avoidance cost approaches zero. In other words, if required, the controlled system sacrifices asymptotic convergence for the sake of fulfilling the non-convex constraints imposed on the system.

Remark 2: The avoidance problem was defined with non-feasible regions in the output space, i.e., $O_i \subset \mathbb{R}^p$. However, it is easy to see that with simple manipulations, the results achieved will also hold if the problem is posed considering non-feasible regions at the state-level, i.e., $O_i \subset \mathbb{R}^n$.

IV. Simplified Control Scheme

A simple approach for the implementation of the considered nonlinear MPC with avoidance features leverages the feasibility enhancements present in the tracking MPC framework to avoid the computation of terminal invariant sets. Namely, the possibility of considering relaxed terminal equality constraints to enforce the terminal state to reach $x_a = g_x(y_a)$. Therefore, the invariant set for tracking comes down to $\Omega_t^{a,e} = \{(x, y_a) : x = g_x(y_a), y_a \in Y_r\}$.

In this formulation, the cost functional becomes

$$\begin{aligned} V_N^e(x, y_t, O_i; \boldsymbol{u}, y_a) = &V_o(y_a - y_t) + V_a(y(j), y_a, O_i) + \\ &\sum_{j=0}^{N-1} \ell(x(j) - x_a, u(j) - u_a), \end{aligned} \quad (19)$$

which does not require any terminal cost functional due to the constraint $x(N) = g_x(y_a)$.

Therefore, the simplified nonlinear MPC with avoidance features can be obtained by solving the optimization problem $P_N^e(x, y_t, O_i)$ having as parameters (x, y_t, O_i) and as decision variables $(\boldsymbol{u}, y_a)$, which is given by

$$\min_{\boldsymbol{u}, y_a} V_N(x, y_t, O_i; \boldsymbol{u}, y_a)$$

$$\begin{aligned}
\text{s.t. } & x(0) = x, && \text{(20a)} \\
& x(j+1) = f(x(j), u(j)), j \in \mathbb{I}_{0:N-1}, && \text{(20b)} \\
& y(j) = h(x(j), u(j)), j \in \mathbb{I}_{0:N-1}, && \text{(20c)} \\
& (x(j), u(j)) \in Z, j \in \mathbb{I}_{0:N-1}, && \text{(20d)} \\
& x_a = f(x_a, u_a), && \text{(20e)} \\
& y_a = h(x_a, u_a), && \text{(20f)} \\
& y_a \in Y_r, && \text{(20g)} \\
& x(N) = x_a. && \text{(20h)}
\end{aligned}$$

As usual in MPC with terminal equality constraint, a controllability assumption is required to the optimization problem $P_N^e(x, y_t, O_i)$ be feasible for a given prediction horizon N.

Assumption 5: The model function $f(x, u)$ is differentiable at any equilibrium point $(x_a, u_a) \in Z_s$ associated to a steady output $y_a \in Y_r$, and the linearized model given by the Jacobians $A(x_a, u_a)$ and $B(x_a, u_a)$ is controllable.

When we consider the simplified terminal equality tracking formulation without any non-feasible output regions, the asymptotic stability is maintained if, in addition to the previous assumptions, the controllability assumption holds and the prediction horizon satisfies $N \geq n$ [11]. Similarly, the closed-loop system remains ISS with respect to the avoidance cost.

Demonstrating this property is straightforward since Lemma 3 must hold for an invariant set for tracking Ω_t^a degenerated to a single steady condition (x_a, y_a). In fact, it always holds that $\Omega_t^{a,e} \subseteq \Omega_t^a$.

V. Numerical Example

To illustrate the proposed control strategy, consider the van der Pol oscillator, originally introduced by Balthazar Van der Pol to model triode oscillations in electrical circuits [18]. The system dynamics, with a forcing term, are given by

$$\dot{x}_1 = x_2,\ \dot{x}_2 = 0.2(1 - x_1^2)x_2 - x_1 + u,\ y = x_1,$$

This article has been accepted for publication in IEEE Control Systems Letters. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/LCSYS.2024.3523844

with $x = [x_1\ x_2]'$, u, and y being, respectively, the state vector, the input control, and the output. Furthermore, the system's states are constrained as $|x_1| \leq 6$ and $|x_2| \leq 2$, and the control input constraint is $|u| \leq 6$. An additional constraint limiting the rate of energy transfer in the system is considered, given by $|x_1 x_2| \leq 2$. It is noteworthy that the additional non-convex constraint makes the admissible state space a non-convex set. Therefore, the avoidance approach discussed in this work is a good candidate to solve this control problem.

The nonlinear MPC strategy is formulated considering a discrete-time nonlinear model for prediction, obtained using the first-order Euler approximation with a sampling time of $T_s = 0.5$s. To avoid the computation of invariant sets, we choose the simplified scheme with terminal equality constraints, considering a prediction horizon $N = 5$, a stage cost $\ell(\cdot) = \|x - x_a\|^2 + \|u - u_a\|^2$, and an offset cost $V_o(\cdot) = \kappa(y_a - y_t)^2$ with $\kappa = 10^4$.

Consider an avoidance cost function defined as

$$V_a(x, x_a, O_i) = \mu F\left(x_a, O_i\right) + \sum_{j=0}^{N} \mu F\left(x(j), O_i\right),$$

where $F\left(x, O_i\right)$ is a penalty given by

$$F\left(x, O_i\right) = \sum_{j=1}^{2} \left(\max\left\{0, g_j(x)\right\}\right)^2,$$

with $g_1(x) = x_1 x_2 - 2$ and $g_2 = -x_1 x_2 - 2$ defining the non-feasible region $O = \{x \in \mathbb{R}^2 : g_1(x) \leq 0, g_2(x) \leq 0\}$, and $\mu = 10^8$. For this example, note that it is more adequate to perform avoidance at the state-level. As highlighted in Remark 2, this can be done without loss of generality. In addition, while large penalty values can cause ill-conditioning in optimization problems, this issue is often mitigated using techniques like augmented Lagrangian methods [16]. In this example, however, no such problems arose, and the optimization was successfully performed with the CasADI Toolbox and IPOPT solver for MATLAB.

Fig. 1 shows the phase diagram of the van der Pol oscillator. The system tracks the set-point $y_t = 5$ from the initial condition $x(0) = [-5\ 0]'$, marked by a black square on the left side. The steady state associated to y_t is marked by an x on the right side. The non-feasible state region is highlighted in light red.

Notably, in the case without avoidance ($\mu = 0$), the closed-loop system maximizes the time the state x_2 remains active at its constraint, but this leads to violations of the energy transfer rate limit. This behavior is seen in Figs. 1 and 2, particularly in the solid blue line plots. In Fig. 2, the second plot clearly shows the system keeping $x_2 = 2$ for the maximum duration.

With avoidance features, the closed-loop system follows the boundary of the non-feasible region without entering it, ensuring the maximum allowed energy transfer rate is used. As expected, the system without avoidance converges faster. However, the system with avoidance achieves the best possible convergence time while respecting the constraints, as shown in Fig. 2.

VI. Conclusion

This work analyzed the stability of a nonlinear MPC scheme for avoidance. It was shown that with the avoidance cost bounded, the closed-loop system is ISS with respect to this cost and achieves asymptotic stability as the cost approaches zero. A simplified scheme with relaxed terminal constraints was also discussed and demonstrated with a van der Pol oscillator example, which used a non-convex constraint to limit energy transfer. This example illustrated that the control strategy can handle non-convex admissible spaces by solving an equivalent convex optimization problem. Future research will focus on efficient numerical methods for computing invariant sets for nonlinear systems.

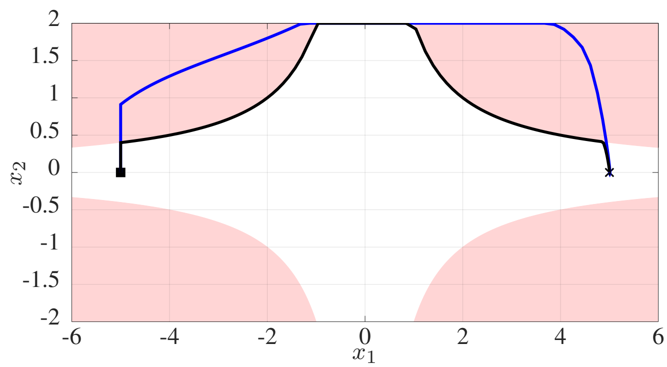

Fig. 1. The figure shows the phase diagram of the dynamical system. The black square marks the initial condition, while the black x marks the target's steady state. The solid black and blue lines represent the system's evolution with and without avoidance, respectively. The light red area indicates the region to be avoided.

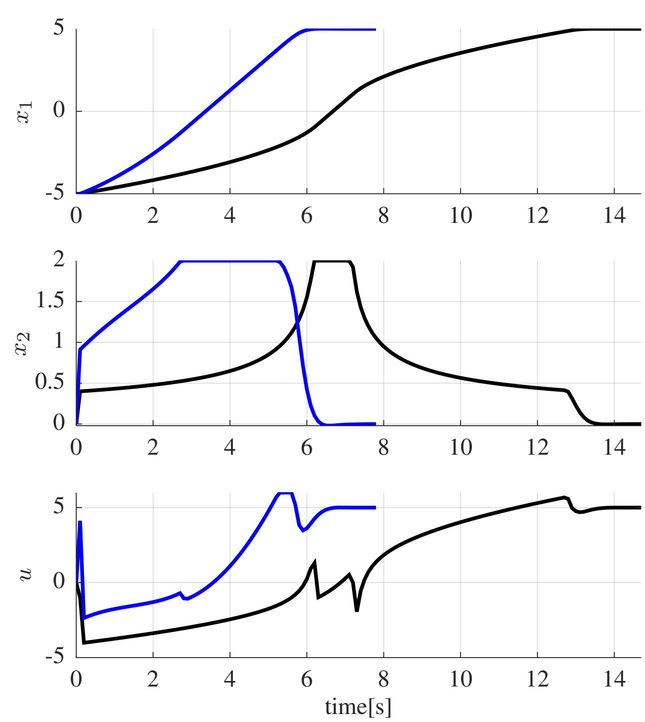

Fig. 2. The figure presents the time evolution of the states and the input. The solid black and solid blue lines represent the system controlled with avoidance and without avoidance, respectively.

Appendix I
Proof of Lemma 2

Consider that (x_a^O, u_a^O, y_a^O) is the optimal solution to $P_N(x, y_t, O_i)$. Then

$$V_N(x, y_t, O_i) = V_o(y_a^O - y_t) + V_a(y, y_a^O, O_i). \tag{21}$$

This lemma can be proved by contradiction using convexity and Lipschitz continuity arguments. For that, assume now that the stationary point is not optimal, i.e., $(x_a^O, u_a^O) \neq (x_s, u_s)$. Let us define

$$(\tilde{x}_a, \tilde{u}_a) = \gamma(x_a^O, u_a^O) + (1-\gamma)(x_s, u_s), \tag{22}$$

with $\gamma \in [0, 1]$. Since both (x_s, u_s) and (x_a^O, u_a^O) are in Z_s, and this set is convex, then a convex combination of these points, $(\tilde{x}_a, \tilde{u}_a)$, is also in Z_s.

Considering Assumptions 3.5. and 4, it is possible to obtain a convex cost functional that superiorly bounds the non-convex cost $V_o(y_a - y_t) + V_a(y, y_a, O_i)$. Then, we can define

$$V_B(y_a - y_t) = V_o(y_a - y_t) + S, \tag{23}$$

for any bound S, such that

$$V_B(\tilde{y}_a - y_t) \leq V_B(y_a^O - y_t) \tag{24}$$

for every γ with $\tilde{y}_a = h(\tilde{x}_a, \tilde{u_a})$. In other words, since the system is not at the optimal point (x_s, u_s), it is more convenient to move toward $(\tilde{x}_a, \tilde{u}_a)$ than to remain in (x_a^O, u_a^O).

Let $\tilde{\boldsymbol{u}}$ be a feasible control sequence that drives the system from (x_a^O, u_a^O) to $(\tilde{x}_a, \tilde{u}_a)$. This sequence is such that the j-th element is given by $\tilde{u}(j) = \kappa(\tilde{x}(j), \tilde{y}_a)$ and $\tilde{x}(j+1) = f(\tilde{x}(j), \tilde{u}(j))$, with $\tilde{x}(0) = x_a^O$. Additionally, from the Lipschitz continuity of the function $g_x(\cdot)$ (see Assumption 2), we have that $\|x_a^O - \tilde{x}_a\| \leq L_g \|y_a^O - \tilde{y}_a\|$, with $L_g > 0$ being the Lipschitz constant. Then, the cost to drive the system to $(\tilde{x}_a, \tilde{u}_a)$ in N steps is

$$\begin{aligned}
&V_N(x_a^O, y_t, O_i) = \\
&\quad \sum_{j=0}^{N-1} \ell(\tilde{x}(j) - \tilde{x}_a, \kappa(\tilde{x}(j), \tilde{y}_a) - \tilde{u}_a) + V_a(\tilde{y}(j), \tilde{y}_a, O_i) + \\
&\quad V_f(\tilde{x}(N) - \tilde{x}_a) + V_o(\tilde{y}_a - y_t) \\
&\leq \sum_{j=0}^{N-1} \ell(\tilde{x}(j) - \tilde{x}_a, \kappa(\tilde{x}(j), \tilde{y}_a) - \tilde{u}_a) + S + \\
&\quad V_f(\tilde{x}(N) - \tilde{x}_a) + V_o(\tilde{y}_a - y_t) \\
&\leq V_f(x_a^O - \tilde{x}_a) + V_o(\tilde{y}_a - y_t) + S \\
&\leq b\|x_a^O - \tilde{x}_a\|^\sigma + V_o(\tilde{y}_a - y_t) + S \\
&\leq b(L_g\|y_a^O - \tilde{y}_a\|)^\sigma + V_o(\tilde{y}_a - y_t) + S \\
&= L_g^\sigma b(1-\gamma)^\sigma \|x_a^O - x_s\|^\sigma + V_o(\tilde{y}_a - y_t) + S.
\end{aligned} \tag{25}$$

Define $W(\gamma) = L_g^\sigma b(1-\gamma)^\sigma \|x_a^O - x_s\|^\sigma + V_B(\tilde{y}_a - y_t)$ and notice that for $\gamma = 1$, $W(1) = V_B(y_a^O - y_t)$. Taking the partial derivative of this function with respect to γ and evaluating it for $\gamma = 1$, we obtain

$$\left.\frac{\partial W}{\partial \gamma}\right|_{\gamma=1} = g^{O'}(y_a^O - y_t), \tag{26}$$

with $g^{O'}(y_a^O - y_t) \in \partial V_B(y_a^O - y_t)$, where $\partial V_B(y_a^O - y_t)$ is defined as the subdifferential of $V_B(y_a^O - y_t)$.

From convexity and (24),

$$\left.\frac{\partial W}{\partial \gamma}\right|_{\gamma=1} = g^{O'}(y_a^O - y_t) \geq V_B(y_a^O - y_t) - V_B(\tilde{y}_a - y_t) > 0. \tag{27}$$

This means that there exists a value of $\gamma \in [0, 1)$ such that $V_B(\tilde{y}_a - y_t)$ is smaller than the value of the cost $V_B(\tilde{y}_a - y_t)$ for $\gamma = 1$, which is $V_B(y_a^O - y_t)$. This contradicts the optimality of the solution of $P_N(x, y_t, O_i)$. Then, it has to be $(x_a^O, u_a^O) = (x_s, u_s)$, with (x_s, u_s) being the minimizer of $V_o(y_a - y_t) + V_a(y, y_a, O_i)$, which concludes the proof.

References

[1] D. Q. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.

[2] P. Tatjewski, "Advanced control and on-line process optimization in multilayer structures," *Annual Reviews in Control*, vol. 32, no. 1, pp. 71–85, 2008.

[3] M. Ellis, H. Durand, and P. D. Christofides, "A tutorial review of economic model predictive control methods," *Journal of process control*, vol. 24, no. 8, pp. 1156–1178, 2014.

[4] M. Jacquet and A. Franchi, "Enforcing vision-based localization using perception constrained N-MPC for multi-rotor aerial vehicles," in *Proceedings of the IEEE/RSJ IROS*, 2022, pp. 1818–1824.

[5] S. V. Raković, S. Zhang, L. Dai, Y. Hao, and Y. Xia, "Convex model predictive control for collision avoidance," *IET Control Theory and Applications*, vol. 15, no. 9, pp. 1270–1285, 2021.

[6] X. Zhang, A. Liniger, and F. Borrelli, "Optimization-based collision avoidance," *IEEE Transactions on Control System Technology*, vol. 29, no. 3, pp. 972–983, 2021.

[7] B. Hermans, P. Patrinos, and G. Pipeleers, "A penalty method based approach for autonomous navigation using nonlinear model predictive control," *IFAC-PapersOnLine*, vol. 51, no. 20, pp. 234–240, 2018.

[8] J. C. Pereira, V. J. S. Leite, and G. V. Raffo, "Nonlinear model predictive control on SE(3) for quadrotor aggressive maneuvers," *Journal of Intelligent & Robotic Systems*, vol. 101, no. 3, pp. 1 – 15, 2021.

[9] M. A. Santos, A. Ferramosca, and G. V. Raffo, "Set-point tracking MPC with avoidance features," *Automatica*, vol. 159, p. 111390, 2024.

[10] ——, "Nonlinear model predictive control schemes for obstacle avoidance," *Journal of Control, Automation and Electrical Systems*, vol. 34, no. 5, pp. 891–906, 2023.

[11] D. Limon, A. Ferramosca, I. Alvarado, and T. Alamo, "Nonlinear MPC for tracking piece-wise constant reference signals," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3735–3750, 2018.

[12] C. V. Rao and J. B. Rawlings, "Steady states and constraints in model predictive control," *AIChE Journal*, vol. 45, no. 6, pp. 1266–1278, 1999.

[13] D. Limon, I. Alvarado, T. Alamo, and E. Camacho, "MPC for tracking piecewise constant references for constrained linear systems," *Automatica*, vol. 44, no. 9, pp. 2382–2387, 2008.

[14] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[15] J. B. Rawlings and D. Q. Mayne, *Model Predictive Control: Theory and Design*. Nob-Hill Publishing, 2009.

[16] D. G. Luenberger and Y. Ye, *Linear and Nonlinear Programming*. Springer, 2008.

[17] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.

[18] G. Israel, *Technological Innovation and New Mathematics: van der Pol and the Birth of Nonlinear Dynamics*. Birkhäuser Basel, 2004, pp. 52–77.