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Irregularity of the Bergman Projection on Smooth Unbounded Worm Domains

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Abstract. In this work, we consider smooth unbounded worm domains $\mathcal{Z}_\lambda$ in $\mathbb{C}^2$ and show that the Bergman projection, densely defined on the Sobolev spaces $H^{s,p}(\mathcal{Z}_\lambda)$, $p \in (1,\infty)$, $s \geq 0$, does not extend to a bounded operator $P_\lambda : H^{s,p}(\mathcal{Z}_\lambda) \to H^{s,p}(\mathcal{Z}_\lambda)$ when $s > 0$ or $p \neq 2$. The same irregularity was known in the case of the non-smooth unbounded worm. This improved result shows that the irregularity of the projection is not a consequence of the irregularity of the boundary but instead of the infinite windings of the worm domain.

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1. Introduction

Let ϕ be a non-negative smooth function on $\mathbb{R}$, such that

- ϕ is convex
- $\phi^{-1}(0) = (-\infty, 0]$.

Notice that $\phi'(t) > 0$ for $t > 0$ and that there exists $a > 0$, such that $\phi(a) = 1$. For $\lambda > 0$, we set

$$\mathcal{Z}_\lambda = \Big\{ (z_1, z_2) : \big| z_1 - \mathrm{e}^{i \log |z_2|^2} \big|^2 < 1 - \phi(\log(\lambda |z_2|)^{-2}) \Big\}. \qquad (1)$$

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Then, $\mathcal{Z}_\lambda$ is smooth, unbounded, and pseudoconvex (see Theorem 1.1). Moreover, $\{\mathcal{Z}_\lambda\}_{\lambda>0}$ is a nested family of domains whose union is the unbounded non-smooth worm

$$\mathcal{W} = \Big\{(z_1, z_2) : \big|z_1 - \mathrm{e}^{i \log|z_2|^2}\big|^2 < 1,\ z_2 \neq 0\Big\}. \tag{2}$$

The domain $\mathcal{W}$ was studied in [13], where three main facts were proved (see the enumerated list below). For $p \in [1, \infty]$ and $s \geq 0$, given any domain Ω, denote by $H^{s,p} = H^{s,p}(\Omega)$ the standard Sobolev space on Ω. When $s = k$ is an integer, $H^{s,p}$ consists of functions with k-derivatives in $L^p(\Omega)$, and for non-integer s, $H^{s,p}$ can be defined by interpolation; see Sect. 2. For $p \in [1, \infty]$, let $A^p(\Omega) := L^p(\Omega)\cap\mathrm{Hol}(\Omega)$ denote the Bergman space. In [13], it was proved that:

(i) the space $A^2(\mathcal{W}) \neq \{0\}$, so that the Bergman projection $P : L^2(\mathcal{W}) \to A^2(\mathcal{W})$ is a non-trivial orthogonal projector;
(ii) the operator P, initially defined on a dense subspace of $L^p(\mathcal{W})$, extends to a bounded operator $P : L^p(\mathcal{W}) \to L^p(\mathcal{W})$ (if and) only if $p = 2$;
(iii) the operator P, initially defined on a dense subspace of $H^{s,2}(\mathcal{W})$, extends to a bounded operator $P : H^{s,2}(\mathcal{W}) \to H^{s,2}(\mathcal{W})$ (if and) only if $s = 0$.

The goal of this paper is to show that also in the case of the unbounded smooth worms $\mathcal{Z}_\lambda$, $\lambda > 0$, the Bergman projection P_λ on $\mathcal{Z}_\lambda$ cannot be extended to a bounded operator $P_\lambda : H^{s,p}(\mathcal{Z}_\lambda) \to H^{s,p}(\mathcal{Z}_\lambda)$ when $s > 0$ or $p \neq 2$. Observe that, since $\mathcal{Z}_\lambda \subseteq \mathcal{W}$ for all $\lambda > 0$, (i) above implies that $A^2(\mathcal{Z}_\lambda)$, and hence, P_λ are non-trivial. We now state our main results.

Theorem 1.1. *Let $\lambda > 0$ and $\mathcal{Z}_\lambda$ be defined as in (1). Then, $\mathcal{Z}_\lambda$ is smooth, unbounded, and pseudoconvex, and its boundary is strongly pseudoconvex except at the points $\mathcal{A} := \{(z_1, z_2) : z_1 = 0, |z_2| \geq 1/\lambda)\}$. Moreover, the Bergman space $A^p(\mathcal{Z}_\lambda)$ is infinite dimensional for all $p \in (0, \infty)$.*

Theorem 1.2. *Let $\lambda > 0$, $\mathcal{Z}_\lambda$ be defined as above, and let P_λ denote the Bergman projection on $\mathcal{Z}_\lambda$. If P_λ, initially defined on the dense subspace $(L^2 \cap H^{s,p})(\mathcal{Z}_\lambda)$, $p \in (1, \infty)$ and $s \geq 0$, extends to a bounded operator*

$$P_\lambda : H^{s,p}(\mathcal{Z}_\lambda) \to H^{s,p}(\mathcal{Z}_\lambda),$$

then necessarily $s = 0$ and $p = 2$.

The problem of the regularity of the Bergman projection on worm domains has been an object of active and intense research. In the seminal paper [1], D. Barrett considered the smoothly *bounded* worm domain

$$\mathcal{W}_\mu = \Big\{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - \mathrm{e}^{i \log|z_2|^2}|^2 < 1 - \eta(\log|z_2|^2)\Big\}, \tag{3}$$

where η is smooth, non-negative, convex, $\eta^{-1}(0) = [-\mu, \mu]$, and such that $\mathcal{W}_\mu$ is smooth, bounded, and pseudoconvex; see, e.g., [10, Proposition 2.1]. Barrett showed that the Bergman projection on $\mathcal{W}_\mu$ does not preserve the Sobolev space $H^{s,2}(\mathcal{W}_\mu)$ if $s \geq \pi/\mu$, whereas in [12], it was then shown that the Bergman projection on $\mathcal{W}_\mu$ does not preserve L^p if $\left|\frac{1}{2} - \frac{1}{p}\right| \geq \frac{\pi}{\mu}$. We further mention in particular [2,3,6,8,12,13]. We also refer the reader to [11] for an expository account of the subject, and to [14,28,29] for some interesting connections between Bergman spaces on worm domains and the Müntz–Szász problem for the Bergman space in one complex dimension.

In the next section, we prove Theorem 1.1, whereas in Sect. 3, we introduce the tools that we need to deal with Sobolev spaces on smoothly bounded domains. In Sect. 4, we prove Theorem 1.2, and in Sect. 5, we discuss some open problems and future work.

2. The Unbounded Smooth Worm

Consider the domains $\mathcal{Z}_\lambda$. It is clear that they are unbounded, that $\mathcal{Z}_\lambda \subseteq \mathcal{Z}_{\lambda'}$ if $0 < \lambda < \lambda'$, and that $\bigcup_{\lambda>0} \mathcal{Z}_\lambda = \mathcal{W}$. It is also immediate to see that

$$\mathcal{Z}_\lambda \subseteq \{z_1 : 0 < |z_1| < 2\} \times \{z_2 : |z_2| > 1/(\mathrm{e}^{a/2}\lambda)\}.$$

Since $\mathcal{Z}_\lambda \subseteq \mathcal{W}$, where $\mathcal{W}$ is as in (2), [13, Proposition 2.3] gives that $A^2(\mathcal{Z}_\lambda)$ is infinite dimensional. Similar calculations also show that also the spaces $A^p(\mathcal{Z}_\lambda)$ are infinite dimensional, $p \in (0, \infty]$. Explicitly, for $\alpha \in \mathbb{C}$, for $z = (z_z, z_2) \in \mathcal{W}$, let

$$L(z) = \log\left(z_1 \mathrm{e}^{-i \log |z_2|^2}\right) + i \log |z_2|^2, \quad E_\alpha(z_1, z_2) := \mathrm{e}^{\alpha L(z)},$$

where $\log(z)$ denotes the principal branch of the logarithm on $\mathbb{C} \setminus (-\infty, 0]$. Then, $L, E_\alpha \in \mathrm{Hol}(\mathcal{W})$ by [13, Lemma 2.2]. Moreover, for $j \in \mathbb{Z}$, $m \in \mathbb{N}$, $c \in \mathbb{R}, c > \log 2$, $\alpha = \mathrm{Re}(\alpha) + i(\frac{j}{2} + \frac{1}{p})$, setting

$$F_{\alpha,c,j,m}(z) := \frac{E_\alpha(z) z_2^j}{(L(z) - c)^m}$$

and arguing as in [13, Proposition 2.3], it is simple to see that $F_{\alpha,c,j,m} \in A^p(\mathcal{W})$ if $\mathrm{Re}(\alpha) > -2/p$ and $m > 1/p$, where $p \in (0, \infty]$. Hence, $A^p(\mathcal{W})$ is infinite dimensional.

The argument to show that $\mathcal{Z}_\lambda$ is smooth and pseudoconvex is standard, but we repeat it for the sake of completeness. Letting ρ denote the defining function of $\mathcal{Z}_\lambda$, we observe that

$$\begin{aligned} \rho(z_1, z_2) &= \left|z_1 - \mathrm{e}^{i \log |z_2|^2}\right|^2 - 1 + \phi(\log(\lambda |z_2|)^{-2}) \\ &= |z_1|^2 - 2\mathrm{Re}\big(z_1 \mathrm{e}^{-i \log |z_2|^2}\big) + \phi(\log(\lambda |z_2|)^{-2}), \end{aligned}$$

so that

$$\partial \rho(z_1, z_2) = \left(\overline{z}_1 - \mathrm{e}^{-i \log |z_2|^2}, -\frac{2}{z_2} \mathrm{Im}\big(z_1 \mathrm{e}^{-i \log |z_2|^2}\big) - \frac{1}{z_2} \phi'(\log(\lambda |z_2|)^{-2})\right).$$

Let $(z_1, z_2) \in b\mathcal{Z}_\lambda$ be such that $\partial_{z_1} \rho(z_1, z_2) = 0$. Then, $z_1 = \mathrm{e}^{i \log |z_2|^2}$, so that $\phi(\log(\lambda |z_2|)^{-2}) = 1$. The assumptions on ϕ imply that $\partial_{z_2} \rho \neq 0$ at such points. Thus, $\mathcal{Z}_\lambda$ is smooth, and it is clearly unbounded, since it contains points (z_1, z_2) with $|z_2|$ arbitrarily large.

To show that $\mathcal{Z}_\lambda$ is pseudoconvex, arguing as in [10], we observe that locally a branch of z_2^2 is defined and that the local defining function $\mathrm{e}^{\arg z_2^2} \rho$ equals

$$|z_1|^2 \mathrm{e}^{\arg z_2^2} - 2\mathrm{Re}\big(z_1 \mathrm{e}^{-i \log z_2^2}\big) + \phi(\log(\lambda |z_2|)^{-2}) \mathrm{e}^{\arg z_2^2}.$$

The first two terms are plurisubharmonic, while the third one satisfies the differential inequality

$$\Delta\big(\phi(\log(\lambda|z_2|)^{-2})\mathrm{e}^{\arg z_2^2}\big) = \Delta\big(\phi(\log(\lambda|z_2|)^{-2})\big)\mathrm{e}^{\arg z_2^2}$$
$$+\phi(\log(\lambda|z_2|)^{-2})\Delta\big(\mathrm{e}^{\arg z_2^2}\big) \geq 0,$$

since ϕ is smooth and convex. Hence, $\mathcal{Z}_\lambda$ is pseudoconvex. Moreover, the defining function is strictly plurisubharmonic at every boundary point where $z_1 \neq 0$.

Next, at $(z_1, z_2) \in b\mathcal{Z}_\lambda$, the complex tangent space is spanned by the vector

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} 2\mathrm{Im}\big(z_1\mathrm{e}^{-i\log|z_2|^2}\big) + \phi'(\log(\lambda|z_2|)^{-2}) \\ z_2(\overline{z}_1 - \mathrm{e}^{-i\log|z_2|^2}) \end{pmatrix}.$$

Since

$$\partial^2_{z_1,\overline{z}_1}\rho = 1,\ \partial^2_{z_1,\overline{z}_2}\rho = \frac{i}{\overline{z}_2}\mathrm{e}^{-i\log|z_2|^2},$$
$$\partial^2_{z_2,\overline{z}_2}\rho = \frac{1}{|z_2|^2}\Big(2\mathrm{Re}\big(z_1\mathrm{e}^{-i\log|z_2|^2}\big) + \phi''(\log(\lambda|z_2|)^{-2})\Big),$$

and $2\mathrm{Re}(z_1\mathrm{e}^{-i\log|z_2|^2}) = |z_1|^2 + \phi(\log(\lambda|z_2|)^{-2})$ on the boundary, the Levi form is given by

$$\begin{aligned}
&\mathcal{L}_\rho(z; v)\\
&\quad = (v_1, v_2)\begin{pmatrix} 1 & \frac{i}{\overline{z}_2}\mathrm{e}^{-i\log|z_2|^2} \\ -\frac{i}{z_2}\mathrm{e}^{i\log|z_2|^2} & \frac{1}{|z_2|^2}\big(|z_1|^2 + \phi(\log(\lambda|z_2|)^{-2}) + \phi''(\log(\lambda|z_2|)^{-2})\big) \end{pmatrix}\\
&\qquad \times \begin{pmatrix} \overline{v}_1 \\ \overline{v}_2 \end{pmatrix}\\
&\quad = v_1^2 + 2v_1\mathrm{Re}\left(i\frac{\mathrm{e}^{-i\log|z_2|^2}}{\overline{z}_2}\overline{v}_2\right) + |v_2|^2\frac{1}{|z_2|^2}\Big(|z_1|^2\\
&\qquad + \phi(\log(\lambda|z_2|)^{-2}) + \phi''(\log(\lambda|z_2|)^{-2})\Big)\\
&\quad = v_1^2 - 2v_1\mathrm{Im}\big(z_1\mathrm{e}^{-i\log|z_2|^2}\big) + \big(1 - \phi(\log(\lambda|z_2|)^{-2})\big)\Big(|z_1|^2\\
&\qquad + \phi(\log(\lambda|z_2|)^{-2}) + \phi''(\log(\lambda|z_2|)^{-2})\Big)\\
&\quad = 2\mathrm{Im}\big(z_1\mathrm{e}^{-i\log|z_2|^2}\big)\phi'(\log(\lambda|z_2|)^{-2}) + (\phi')^2(\log(\lambda|z_2|)^{-2})\\
&\qquad + \big(1 - \phi(\log(\lambda|z_2|)^{-2})\big)\Big(|z_1|^2 + \phi(\log(\lambda|z_2|)^{-2}) + \phi''(\log(\lambda|z_2|)^{-2})\Big).
\end{aligned}$$

It follows that the boundary points $\{(0, z_2) : |z_2| \geq 1/\lambda\}$ are of weak pseudoconvexity. This proves Theorem 1.1. $\square$

3. Sobolev Spaces on Smoothly Bounded Domains and on $\mathcal{Z}_\lambda$

In this section, we collect the results on Sobolev spaces on smoothly bounded domains and prove a few properties that we shall need later. We begin by recalling the definition and a few standard results from the theory of function spaces on smoothly *bounded* domains; see, e.g., [31, Chapter 3] and [17]. In what follows the space $H^{s,p}(\mathbb{R}^d)$ is defined by means of the Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ and $\mathcal{D}'(\Omega)$ is the dual of the space $C_c^\infty(\Omega)$ of smooth functions with compact support in Ω. Namely

$$H^{s,p}(\mathbb{R}^d) = \Big\{ f \in \mathcal{D}'(\mathbb{R}^d) : \mathcal{F}^{-1}\Big((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\Big) \in L^p(\mathbb{R}^d)\Big\}.$$

Definition 3.1. Let Ω be a smoothly bounded domain in $\mathbb{R}^d$, $s \geq 0$ and $p \in (1,\infty)$. We define

$$\begin{aligned} H^{s,p}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) : \exists F \in H^{s,p}(\mathbb{R}^d) \,\big|\, F_{|\Omega} = f,\\ \|f\|_{H^{s,p}(\Omega)} \quad := \inf\{\|F\|_{H^{s,p}(\mathbb{R}^d)} : F_{|\Omega} = f\}\Big\}. \end{aligned}$$

We also denote by $H_0^{s,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in the $H^{s,p}(\Omega)$-norm. Then, for $s<0$ and $p \in (1,\infty)$, we define $H^{s,p}(\Omega)$ as the dual of $H_0^{-s,p'}(\Omega)$, where $p' = p/(p-1)$ is the exponent conjugate to p.[1]

When $s=k$ is a non-negative integer, the space $H^{k,p}(\Omega)$ has a natural characterization. On the space $C^\infty(\overline{\Omega})$, consider the norm

$$\|\psi\|_{W^{k,p}(\Omega)} := \sum_{|\alpha|\leq k} \|\partial^\alpha \psi\|_{L^p(\Omega)} < \infty,$$

and define $W^{k,p}(\Omega)$ as the closure of $C^\infty(\overline{\Omega})$ with respect to this norm. Then, $W^{k,p}(\Omega)$ is isomorphic to $H^{k,p}(\Omega)$, with equivalence of norms; see, e.g., [31].

Using the complex interpolation method, we have that when $s>0$

$$H^{s,p}(\Omega) = [H^{k,p}(\Omega), H^{k+1,p}(\Omega)]_\theta, \tag{4}$$

where $\theta \in (0,1)$ and $s = k+\theta$, cf. [31] or [17], so that $[W^{k,p}(\Omega), W^{k+1,p}(\Omega)]_\theta$ is isomorphic as Banach space to $H^{s,p}(\Omega)$, $s=k+\theta$. For the complex interpolation method, we refer to [4].

Since Ω is a bounded, smooth domain, the multiplier operator $f \mapsto \chi_\Omega f$ is bounded on $H^{s,p}(\mathbb{R}^d)$ when $0 \leq s < \frac{1}{p}$, $p \in (1,\infty)$. This fact in turn implies the key property that $C_c^\infty(\Omega)$ is dense in $H^{s,p}(\Omega)$ when $0 \leq s < \frac{1}{p}$—see [31, Theorem 3.4.3].

We now prove a result that is probably well known, but for which we do not know a precise reference.

Lemma 3.2. *For $-1/p' < s < 1/p$, the spaces $H^{s,p}(\Omega)$ and $H^{-s,p'}(\Omega)$ are mutually dual with respect to the $L^2(\Omega)$ pairing of duality.*

Proof. Observe that, by duality, we may assume that $0 \leq s < 1/p$. Since $H^{s,p}(\Omega) = H_0^{s,p}(\Omega)$ in the given range, $H^{-s,p'}(\Omega) = \big(H^{s,p}(\Omega)\big)^*$ with the L^2-pairing of duality.

Conversely, let$\ell \in \big(H^{-s,p'}(\Omega)\big)^*$. Since the multiplication $f \mapsto \chi_\Omega f$ is bounded on $H^{s,p}(\mathbb{R}^d)$, $H^{s,p}(\Omega)$ can be identified with the subspace of

[1]We remark that the definition of $H^{s,p}(\Omega)$ with $s<0$ is the same as in [17] but different from the one in [31].

$H^{s,p}(\mathbb{R}^d)$ of functions vanishing on Ω^c. Therefore, also $H^{-s,p'}(\Omega)$ can be identified with the elements of $\big(H^{s,p}(\mathbb{R}^d)\big)^* = H^{-s,p'}(\mathbb{R}^d)$ that annihilate functions of $H^{s,p}(\mathbb{R}^d)$ vanishing on Ω^c. Therefore, by the Hahn–Banach theorem, there exists $L \in \big(H^{-s,p'}(\mathbb{R}^d)\big)^* = H^{s,p}(\mathbb{R}^d)$ with the same norm, that agrees with ℓ on $H^{-s,p'}(\Omega)$. Hence, there exists $F \in H^{s,p}(\mathbb{R}^d)$, such that $\ell(u) = \int_\Omega Fu = \int_\Omega (\chi_\Omega F)u$, where $\chi_\Omega F \in H^{s,p}(\Omega)$; that is, $\big(H^{-s,p'}(\Omega)\big)^* = H^{s,p}(\Omega)$. □

Next, we need an extension of a result by E. Ligocka, namely [16, Theorem 2]. We denote by $H^{s,p}_{\mathrm{har}}(\Omega)$ the subspace of $H^{s,p}(\Omega)$ consisting of harmonic functions. Let $\varrho : \mathbb{R}^d \to \mathbb{R}$ be a smooth defining function (see [15]) for Ω and let $L^p_{\mathrm{har}}(\Omega, |\varrho|^q)$ be the subspace of $L^p(\Omega, |\varrho|^q\, dm)$ consisting of harmonic functions on Ω, $p \in (1,\infty)$. In [16, Theorem 2], Ligocka proved that for $s \geq 0$ $p \in (1,\infty)$

(i) $H^{s,p}_{\mathrm{har}}(\Omega)$ and $H^{-s,p'}_{\mathrm{har}}(\Omega)$ are mutually dual with respect to the $L^2(\Omega)$-inner product;

(ii) $H^{-s,p'}_{\mathrm{har}}(\Omega)$ is isomorphically equivalent (as a Banach space) to $L^{p'}_{\mathrm{har}}(\Omega, |\varrho|^{sp'})$.

We shall need the following extension of (ii).

Lemma 3.3. *Let $s \in \mathbb{R}$, $s < 1/p$, $p \in (1,\infty)$. Then, $H^{s,p}_{\mathrm{har}}(\Omega)$ is isomorphically equivalent (as a Banach space) to $L^p_{\mathrm{har}}(\Omega, |\varrho|^{-sp})$.*

Proof. As mentioned, the case $s \leq 0$ is proved in [16, Theorem 2]. Next, let $0 < s < 1/p$. If $f \in L^p_{\mathrm{har}}(\Omega, |\varrho|^{-sp})$ and $g \in L^{p'}_{\mathrm{har}}(\Omega, |\varrho|^{sp'})$, we have that

$$\begin{aligned}\left|\int_\Omega fg\,\mathrm{d}V\right| &= \left|\int_\Omega (|\varrho|^{-s}f)(|\varrho|^{s}g)\,\mathrm{d}V\right| \leq \|f\|_{L^p_{\mathrm{har}}(\Omega,|\varrho|^{-sp})}\|g\|_{L^{p'}_{\mathrm{har}}(\Omega,|\varrho|^{sp'})}\\ &\leq C\|f\|_{L^p_{\mathrm{har}}(\Omega,|\varrho|^{-sp})}\|g\|_{H^{-s,p'}_{\mathrm{har}}(\Omega)},\end{aligned}$$

so that by (i) above

$$\begin{aligned}\|f\|_{H^{s,p}_{\mathrm{har}}(\Omega)} &\leq C\|f\|_{(H^{-s,p'}_{\mathrm{har}}(\Omega))^*} = \sup\left\{\left|\int_\Omega fg\,\mathrm{d}V\right| : \|g\|_{H^{-s,p'}_{\mathrm{har}}(\Omega)} \leq 1\right\}\\ &\leq C\|f\|_{L^p_{\mathrm{har}}(\Omega,|\varrho|^{-sp})}.\end{aligned}$$

Conversely, let $f \in H^{s,p}_{\mathrm{har}}(\Omega)$. It is well known that the mapping $H^{s,p}(\Omega) \ni f \mapsto |\varrho|^{-s}f \in L^p(\Omega)$ is bounded when $0 \leq s < 1/p$; see, e.g., [21, Theorem 2, 1.3.1] or [16, p. 256]. Then, we have

$$\begin{aligned}\|f\|_{L^p_{\mathrm{har}}(\Omega,|\varrho|^{-sp})} &= \||\varrho|^{-s}f\|_{L^p(\Omega)}\\ &\leq \|f\|_{H^{s,p}_{\mathrm{har}}(\Omega)}.\end{aligned}$$

This proves the lemma. □

We now define Sobolev spaces on the smooth unbounded domains $\mathcal{Z}_\lambda$.

Definition 3.4. For k a non-negative integer and $p \in (1,\infty)$, define the space (of test functions)

$$\mathcal{T}(\overline{\mathcal{Z}_\lambda}) := \left\{ \psi \in C^\infty(\overline{\mathcal{Z}_\lambda}) : \|\psi\|_{H^{k,p}(\mathcal{Z}_\lambda)} := \sum_{|\alpha|\le k} \|D_z^\alpha \psi\|_{L^p(\mathcal{Z}_\lambda)} < \infty \right\},$$

where $D_z := (\partial_{z_1}, \partial_{\bar z_1}; \partial_{z_2}, \partial_{\bar z_2})$. We define $H^{k,p}(\mathcal{Z}_\lambda)$ as the closure of $\mathcal{T}(\overline{\mathcal{Z}_\lambda})$ with respect to the norm $\|\cdot\|_{H^{k,p}(\mathcal{Z}_\lambda)}$. For $s = k+\theta$ with $0<\theta<1$, we define $H^{s,p}(\mathcal{Z}_\lambda)$, $p \in (1,\infty)$, by complex interpolation, as

$$H^{s,p}(\mathcal{Z}_\lambda) := [H^{k,p}(\mathcal{Z}_\lambda), H^{k+1,p}(\mathcal{Z}_\lambda)]_\theta.$$

See, e.g., [4].

Finally, we point out the following fact that we will need later.

Remark 3.5. Let $\mu(\lambda) = \log \lambda^2$, and consider the domain $\mathcal{W}_{\mu(\lambda)}$ as defined in (3), where η is given by

$$\eta(t) = \phi(t - \log \lambda^2) + \phi(-t - \log \lambda^2),$$

so that $\mathcal{W}_{\mu(\lambda)} \subseteq \mathcal{Z}_\lambda$. Observe then that the restriction operator $H^{s,p}(\mathcal{Z}_\lambda) \ni f \mapsto f_{|W_{\mu(\lambda)}} \in H^{s,p}(\mathcal{W}_{\mu(\lambda)})$ is well defined and norm decreasing when $s = k$ is a non-negative integer and $p \in (1,\infty)$, and then, by interpolation, also when $s \ge 0$ and $p \in (1,\infty)$. Analogously, for all $\lambda' > \lambda$

$$\|f\|_{H^{s,p}(\mathcal{W}_{\mu(\lambda)})} \le \|f\|_{H^{s,p}(\mathcal{Z}_{\lambda'})}.$$

4. Irregularity of the Bergman Projection

The proof of Theorem 1.2 will combine some new ideas with Barrett's arguments [1] and results from [13]. We first extend [13, Corollary 5.5] to the case of the Sobolev spaces $H^{s,p}(\mathcal{W}_\mu)$.

Proposition 4.1. *Let $\mathcal{W}$ be the unbounded non-smooth worm, K_w be its Bergman kernel at $w \in \mathcal{W}$, and $\mathcal{W}_\mu$ be the smoothly bounded worm as in* (3). *Suppose $p \in (1,\infty)$. Then, the following properties hold*:

- (i) *if $s \in \left(\frac{2}{p}-1, \infty\right)$ (the region $R \cup T_1 \cup T_2$ union the open segments of end points $(0,0)$ and $(1,1)$ and $(0,0)$ and $(\frac{1}{2},0)$, resp., in Fig. 1), then $K_w \notin H^{s,p}(\mathcal{W}_\mu)$;*
- (ii) *if $s = \frac{2}{p}-1$ and $p \in (1,2)$ (the open segment of end points $(\frac{1}{2},0)$, $(1,1)$ in Fig. 1), then $\|K_w\|_{H^{s,p}(\mathcal{W}_\mu)} \to \infty$ as $\mu \to \infty$.*

Proof. We first observe that the cases $p = 2$, $s > 0$, and $p > 2$, $s = 0$, that appear in (i), are proved in [13, Corollary 5.5].

We now recall some notation from [13, Corollary 5.5]. We let $S(\mathrm{e}^{i\log|z_2|^2}, \varepsilon)$ denote the angular sector in the z_1-plane

$$S(\mathrm{e}^{i\log|z_2|^2}, \varepsilon) = \left\{ z_1 = r\mathrm{e}^{i(t+\log|z_2|^2)} : |t| < \delta,\, 0 < r < \varepsilon \right\},$$

with $0 < \delta < \pi/2$. For $\varepsilon > 0$ sufficiently small, the set

$$G_\mu = \left\{ z = (z_1, z_2) \in \mathbb{C}^2 : |\log|z_2|^2| < \mu,\, z_1 \in S(\mathrm{e}^{i\log|z_2|^2}, \varepsilon) \right\} \tag{5}$$

is contained in $\mathcal{W}_\mu$. Then, from [13, (5.8) and p. 1180], for $w \in \mathcal{W}$ and $z \in G_\mu$, we have the estimate

$$\begin{aligned} |K_w(z)| &\ge \frac{C}{|z_1||z_2|} \frac{1}{\big(\log(|z_1|/2) + \log(|w_1|/2)\big)^2 + (\pi + 2\mu)^2} \\ &\ge C_w \frac{1}{|z_1||z_2|(\log^2|z_1| + \mu^2)}, \end{aligned} \tag{6}$$

where C_w does not depend on μ. Therefore, arguing as in [13, Corollary 5.5], for $s < 1/p$, we have

$$\begin{aligned} &\|K_w\|^p_{L^p(\mathcal{W}_\mu, |\rho|^{-sp})} \\ &\ge C_w \int_{|\log|z_2|^2| \le \mu} \frac{1}{|z_2|^p} \int_{S(1,\varepsilon)} \frac{1}{\big||\zeta|^2 - 2\mathrm{Re}\zeta\big|^{sp} \big[|\zeta|(\log^2|\zeta| + \mu^2)\big]^p} \, \mathrm{d}V(\zeta)\, \mathrm{d}V(z_2) \\ &= C_w 2\pi \frac{\sinh\big(|1-p/2|\mu\big)}{|1-p/2|} \int_{|t|\le\delta} \int_0^\varepsilon \frac{1}{|r - 2\cos t|^{sp} r^{p(s+1)-1} (\log^2 r + \mu^2)^p} \, \mathrm{d}r\mathrm{d}t \\ &\ge C'_w 2\pi \frac{\sinh\big(|1-p/2|\mu\big)}{|1-p/2|} \int_0^\varepsilon \frac{1}{r^{p(s+1)-1}(\log^2 r + \mu^2)^p} \, \mathrm{d}r. \end{aligned} \tag{7}$$

(i) Suppose then that $s \in \big(\frac{2}{p} - 1, \frac{1}{p}\big)$. From Lemma 3.3, $K_w \in H^{s,p}(\mathcal{W}_\mu)$ if and only if $K_w \in L^p(\mathcal{W}_\mu |\varrho|^{-sp})$. From (7), it then follows that $K_w \notin H^{s,p}(\mathcal{W}_\mu)$ when $s \in (\frac{2}{p} - 1, \frac{1}{p})$. We now use the natural embedding $H^{s,p}(\mathcal{W}_\mu) \subseteq H^{s',p}(\mathcal{W}_\mu)$ when $0 \le s' \le s$ (see [31, Theorem 3.3.1]). It follows that $K_w \notin H^{s,p}(\mathcal{W}_\mu)$ for all p, s, such that $p \in (1, \infty)$ and $s > \frac{2}{p} - 1$. This proves (i).

(ii) We look at the estimate in (7) when $s = \frac{2}{p} - 1$ (notice that $s < 1/p$ in this case) and observe that

$$\begin{aligned} \|K_w\|^p_{L^p(\mathcal{W}_\mu, |\rho|^{-sp})} &\ge C'_w 2\pi \frac{\sinh\big(|1-p/2|\mu\big)}{|1-p/2|} \int_0^\varepsilon \frac{1}{r(\log^2 r + \mu^2)^p} \, \mathrm{d}r \\ &= C'_w 2\pi \frac{\sinh\big(|1-p/2|\mu\big)}{|1-p/2|} \frac{1}{\mu^{2p-1}} \int_{\frac{1}{\mu}\log\frac{1}{\varepsilon}}^\infty \frac{1}{(1+t^2)^p} \, \mathrm{d}t. \end{aligned} \tag{8}$$

Clearly, if $p \ne 2$, the right-hand side above tends to ∞ if $\mu \to \infty$. The rest of the proof will show that the same is true for $\|K_w\|_{H^{s,p}(\mathcal{W}_\mu)}$. We observe in passing that, on the other hand, if $p = 2$, the right-hand side above remains bounded (actually, it tends to 0) when $\mu \to \infty$, in accordance to the fact that $\|K_w\|_{L^2(\mathcal{W}_\mu)} \le \|K_w\|_{L^2(\mathcal{W})} < \infty$.

To conclude the proof of (ii), we will bound $\|K_w\|_{H^{s,p}(\mathcal{W}_\mu)}$ from below. This will require several steps. We begin by setting for $j = 1, 2$ and $\mu > 2$

$$G^j_\mu = \big\{z = (z_1, z_2) \in \mathbb{C}^2 : j < |z_2| < \mathrm{e}^{\mu/2}/j,\ z_1 \in S(\mathrm{e}^{i\log|z_2|^2}, \varepsilon/j)\big\}.$$

Keeping in mind (5), we see that $G^2_\mu \subseteq G^1_\mu \subseteq G_\mu$. We define a cut-off function $\psi : \mathbb{C}^2 \to [0, +\infty)$ by setting $\psi(z_1, z_2) = \psi_1(z_1)\psi_2(z_2)$, where

- $\psi_1 \in C^\infty_c(\mathbb{C})$, $\psi_1(z_1) = 1$ for $|z_1| < \varepsilon/2$ and $\psi_1(z_1) = 0$ for $|z_1| \ge \varepsilon$;

- $\psi_2 \in C_c^\infty(\mathbb{C})$ is identically equal to 1 on the annulus $\{z_2 : 2 < |z_2| < \mathrm{e}^{\mu/2}/2\}$, is supported in a compact subset of $\{z_2 : 1 < |z_2| < \mathrm{e}^{\mu/2}\}$, and has uniformly bounded derivatives.

Following the lines of the computations in (7) and (8), we can find for each $p \in (1,2)$ and each $s = \frac{2}{p} - 1 > 0$ a constant $C' > 0$, independent of μ, such that $\|K_w\|_{L^p(G^2_\mu, |\rho|^{-ps})} \geq C'\mathrm{e}^{\mu/p}$, whence

$$\|\psi K_w\|_{L^p(G^2_\mu, |\rho|^{-ps})} \geq C'\mathrm{e}^{\mu/p} \tag{9}$$

for $\mu > 2$. Now, let us consider the map $(z_1, z_2) \mapsto (z_1\mathrm{e}^{-i\log|z_2|^2}, z_2)$. It is a C^∞-diffeomorphism from (a neighborhood of) G^1_μ onto (a neighborhood of) its image

$$\widetilde{G}^1_\mu := \left\{\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2 : \zeta_1 \in S(1,\varepsilon),\, 1 < |\zeta_2| < \mathrm{e}^{\mu/2}\right\},$$

included in the (Lipschitz) domain

$$\Lambda = \left\{w = (t + iu, w_2) \in \mathbb{C}^2 : t > (\tan\delta)^{-1}|u|,\, 1 < |w_2| < \mathrm{e}^{\mu/2}\right\}.$$

We denote by $\Psi : \Lambda \to \mathbb{C}^2$ the inverse mapping $(w_1, w_2) \mapsto (w_1\mathrm{e}^{i\log|w_2|^2}, w_2)$. We will later precompose Ψ with the map $(w_1', w_2) \mapsto (\tau(w_1'), w_2)$ with $\tau(t' + iu) = t' + (\tan\delta)^{-1}|u| + iu$. The preimage of Λ through this map (as well as Λ itself) is contained in the half-space

$$\mathcal{H} := \{(\zeta, w_2) \in \mathbb{C}^2 : \mathrm{Re}\zeta > 0\}.$$

We compute

$$\begin{aligned}
\|\psi K_w\|^p_{L^p(G^2_\mu, |\rho|^{-ps})} &\leq \|\psi K_w\|^p_{L^p(G^1_\mu, |\rho|^{-ps})} \\
&= \int_{\widetilde{G}^1_\mu} |(\psi K_w)\circ\Psi(w)|^p |\rho\circ\Psi(w)|^{-ps} |\det(J\Psi)(w)|\,\mathrm{d}V(w) \\
&= \int_{\widetilde{G}^1_\mu} |(\psi K_w)\circ\Psi(w)|^p (1 - |w_1 - 1|^2)^{-ps} |\det(J\Psi)(w)|\,\mathrm{d}V(w) \\
&\leq \int_{\widetilde{G}^1_\mu} |(\psi K_w)\circ\Psi(w)|^p (1 - |w_1 - 1|^2)^{-ps}\,\mathrm{d}V(w) \\
&= \int_{\Lambda} |(\psi K_w)\circ\Psi(t + iu, w_2)|^p (2t - t^2 - u^2)^{-ps}\,\mathrm{d}t\mathrm{d}u\,\mathrm{d}V(w_2),
\end{aligned}$$

where we took into account the fact that $w \in \widetilde{G}^1_\mu$ implies $|w_2| > 1$, whence $|\det(J\Psi)(w)| < 1$. On the support of $\psi\circ\Psi$, we have $t^2 + u^2 = |w_1|^2 < \varepsilon^2$. Up to shrinking ε to have $\varepsilon < \left(1 + (\tan\delta)^{-2}\right)^{-1}$, we get that $2t - t^2 - u^2 \geq t$ in the support of $\psi\circ\Psi$. We obtain

$$\begin{aligned}
&\|\psi K_w\|^p_{L^p(G^2_\mu, |\rho|^{-ps})} \\
&\quad\leq \int_{\Lambda} |(\psi K_w)\circ\Psi(t + iu, w_2)|^p t^{-ps}\,\mathrm{d}t\mathrm{d}u\,\mathrm{d}V(w_2) \\
&\quad= \int_{\mathcal{H}} |(\psi K_w)\circ\Psi(\tau(t' + iu), w_2)|^p (t' + (\tan\delta)^{-1}|u|)^{-ps}\,\mathrm{d}t'\mathrm{d}u\,\mathrm{d}V(w_2)
\end{aligned}$$

$$\begin{aligned}&\le \int_{\mathcal{H}} |(\psi K_w)\circ \Psi(\tau(t'+iu),w_2)|^p (t')^{-ps}\,\mathrm{d}t'\mathrm{d}u\,\mathrm{d}V(w_2)\\ &= \|f\|^p_{L^p(\mathcal{H},(\mathrm{Re}\zeta)^{-ps})},\end{aligned}$$

where $f(t'+iu,w_2) = (\psi K_w)\circ\Psi(\tau(t'+iu),w_2)$. For the half-space $\mathcal{H}$, when $p\in(1,\infty)$, and $0<s<1/p$, we have the well-known estimate

$$\|h\|_{L^p(\mathcal{H},(\mathrm{Re}\zeta)^{-ps})} \le C''\|h\|_{H^{s,p}(\mathcal{H})}$$

for all $h\in H^{s,p}(\mathcal{H})$; see, e.g., [31, Proposition 2.8.6/1, Proposition 3.3.2]. We conclude that

$$\|\psi K_w\|_{L^p(G^2_\mu,|\rho|^{-ps})} \le C''\|f\|_{H^{s,p}(\mathcal{H})}. \tag{10}$$

Our next aim is going back from $f(t'+iu,w_2) = (\psi K_w)\circ\Psi(\tau(t'+iu),w_2)$ to $(\psi K_w)\circ\Psi$. Taking into account that $\tau(t'+iu) = t' + (\tan\delta)^{-1}|u| + iu$, we compute

$$\begin{aligned}\partial_u f(t'+iu,w_2) &= \partial_u\Big((\psi K_w)\circ\Psi(\tau(t'+iu),w_2)\Big)\\ &= (\tan\delta)^{-1}\,\mathrm{sgn}(u)\Big(\partial_1((\psi K_w)\circ\Psi)\Big)(\tau(t'+iu),w_2)\\ &\quad + \Big(\partial_2((\psi K_w)\circ\Psi)\Big)(\tau(t'+iu),w_2),\end{aligned}$$

where we identified $\mathcal{H}$ with $\mathbb{R}^4_+ = \{(x_1,x_2,x_3,x_4) : x_1>0\}$ and we let ∂_1 and ∂_2 denote the partial derivatives w.r.t. x_1 and x_2, resp. Hence

$$\begin{aligned}&\big\|\partial_u f(t'+iu,w_2)\big\|_{L^p(\mathcal{H})}\\ &\quad\le C_\delta\Big(\Big\|\Big(\partial_1((\psi K_w)\circ\Psi)\Big)(\tau(t'+iu),w_2)\Big\|_{L^p(\mathcal{H})}\\ &\qquad + \Big\|\Big(\partial_2((\psi K_w)\circ\Psi)\Big)(\tau(t'+iu),w_2)\Big\|_{L^p(\mathcal{H})}\Big).\end{aligned}$$

Thus, there exists $C'_\delta>0$, such that, for all $g\in H^{1,p}(\mathcal{H})$ and for $\widetilde{g}(t'+iu,w_2) = g(\tau(t'+iu),w_2)$, the inequality

$$\|\widetilde{g}\|_{H^{r,p}(\mathcal{H})} \le C'_\delta\|g\|_{H^{r,p}(\mathcal{H})}$$

holds for $r=0,1$. By interpolation, the same inequality holds for all $r\in[0,1]$. Using this bound in the estimate (10) and (later), the fact that all the derivates of the components of Ψ are uniformly bounded on the support of ψ, and we obtain

$$\|\psi K_w\|_{L^p(G^2_\mu,|\rho|^{-ps})} \le C\|(\psi K_w)\circ\Psi\|_{H^{s,p}(\mathcal{H})} \le C'\|\psi K_w\|_{H^{s,p}(\mathcal{W}_\mu)},$$

where all constants are independent of μ. Now, the assumptions on ψ_2 guarantee that the function ψ has $|D^\alpha_z\psi(z)|\le 1$ for all multiindices α. Hence, multiplication by ψ is a bounded operator, whose norm is independent of μ, on $H^{k,p}(\mathcal{W}_\mu)$ for all $k\in\mathbb{N}_0$ (whence on $H^{s,p}(\mathcal{W}_\mu)$ for all $s\ge 0$). Thus, there exists a constant C'', independent of μ, such that

$$\|\psi K_w\|_{L^p(G^2_\mu,|\rho|^{-ps})} \le C''\|K_w\|_{H^{s,p}(\mathcal{W}_\mu)}.$$

This bound and (9) complete the proof of (ii). $\square$

To prove Theorem 1.2, we need two preliminary lemmas. We denote by $\|T\|_{(X,X)}$ the operator norm of $T : X \to X$.

Lemma 4.2. *For $\lambda, \lambda' > 0$, the domain $\mathcal{Z}_\lambda$ is biholomorphic to $\mathcal{Z}_{\lambda'}$. Moreover, the Bergman projection P_λ induces a bounded operator on $L^p(\mathcal{Z}_\lambda)$ for some $\lambda > 0$ if and only if P_λ induces a bounded operator on $L^p(\mathcal{Z}_\lambda)$ for every $\lambda > 0$, and in this case, $\|P_\lambda\|_{(L^p(\mathcal{Z}_\lambda),L^p(\mathcal{Z}_\lambda))}$ is independent of λ.*

Proof. To show that the domains $\mathcal{Z}_\lambda$ are all biholomorphic to each other, it suffices to observe that for all $r, \lambda > 0$

$$\Phi_\lambda : \mathcal{Z}_r \ni (w_1, w_2) \mapsto (w_1 \mathrm{e}^{-i \log \lambda^2}, w_2/\lambda) \in \mathcal{Z}_{\lambda r} \tag{11}$$

is a biholomorphic map, since $\Phi_\lambda \in GL(2, \mathbb{C})$ and $\Phi_\lambda(\mathcal{Z}_r) = \mathcal{Z}_{\lambda r}$. Moreover, $\det \Phi'_\lambda = \mathrm{e}^{-i \log \lambda^2}/\lambda$ and $T_{\lambda,p} f := (\det \Phi'_\lambda)^{2/p} f \circ \Phi_\lambda$ is an isometric isomorphism $T_{\lambda,p} : L^p(\mathcal{Z}_{\lambda r}) \to L^p(\mathcal{Z}_r)$

$$\|T_{\lambda,p} f\|_{L^p(\mathcal{Z}_r)} = \|f\|_{L^p(\mathcal{Z}_{\lambda r})}, \tag{12}$$

that also gives an isometric isomorphism $T_{\lambda,p} : A^p(\mathcal{Z}_{\lambda r}) \to A^p(\mathcal{Z}_r)$ when restricted to $A^p(\mathcal{Z}_{\lambda r})$, $p \in [1, \infty]$. Recalling the transformation rule for the Bergman projections

$$P_r(\det \Phi'_\lambda f \circ \Phi_\lambda) = \det \Phi'_\lambda (P_{\lambda r} f) \circ \Phi_\lambda$$

for every $f \in L^2(\mathcal{Z}_\lambda)$, since $\det \Phi'_\lambda$ is constant, it follows that $P_r(f \circ \Phi_\lambda) = (P_{\lambda r} f) \circ \Phi_\lambda$, for all $f \in L^2(\mathcal{Z}_\lambda)$ and $\lambda > 0$. This implies that (also when $p \neq 2$)

$$P_r(T_{\lambda,p} f) = T_{\lambda,p}(P_{\lambda r} f)$$

for all $f \in (L^2 \cap L^p)(\mathcal{Z}_{\lambda r})$. Since $(L^2 \cap L^p)(\mathcal{Z}_{\lambda r})$ is dense in $L^p(\mathcal{Z}_{\lambda r})$ and $T_{\lambda,p}(L^2 \cap L^p)(\mathcal{Z}_{\lambda r})$ is dense in $L^p(\mathcal{Z}_r)$, for $f \in L^p(\mathcal{Z}_{\lambda r})$, we have

$$\|P_{\lambda r} f\|_{L^p(\mathcal{Z}_{\lambda r})} = \|T_{\lambda,p} P_{\lambda r} f\|_{L^p(\mathcal{Z}_r)} = \|P_r(T_{\lambda,p} f)\|_{L^p(\mathcal{Z}_r)}.$$

Since $T_{\lambda,p} : L^p(\mathcal{Z}_{\lambda r}) \to L^p(\mathcal{Z}_r)$ is an isometric isomorphism, the equality of the operator norms of P_λ easily follows. $\square$

Lemma 4.3. *Let $s > 0$, $p \in (1, \infty)$ and suppose P_λ induces a bounded operator on $H^{s,p}(\mathcal{Z}_\lambda)$ for some $\lambda > 0$. Then, $P_{\lambda'}$ induces a bounded operator on $H^{s,p}(\mathcal{Z}_{\lambda'})$ for all $\lambda' > \lambda$ and, setting $N_{s,p}(\lambda) = \|P_\lambda\|_{(H^{s,p}(\mathcal{Z}_\lambda),H^{s,p}(\mathcal{Z}_\lambda))}$, we have*

$$N_{s,p}(\lambda') \leq N_{s,p}(\lambda), \tag{13}$$

for all $\lambda' > \lambda$.

Proof. For $r > 0$, let $T_{r,p}$ be as in the proof of Lemma 4.2. We argue as in [1]. Recalling that $D_z = (\partial_{z_1}, \partial_{\bar{z}_1}; \partial_{z_2}, \partial_{\bar{z}_2})$, if $\alpha = (a_1, b_1; a_2, b_2)$ is a given multi-index, we have that

$$D_z^\alpha (f \circ \Phi_r)(z) = \mathrm{e}^{i(b_1 - a_1) \log r^2} r^{-(a_2 + b_2)} (D_z^\alpha f)(\Phi_r(z)).$$

Therefore, for $\lambda > 0, r > 1$, and k a positive integer, using (12), we have

$$\begin{aligned}\|T_{r,p}f\|_{H^{k,p}(\mathcal{Z}_\lambda)} &= \sum_{|\alpha|\le k} \|D_z^\alpha T_{r,p} f\|_{L^p(\mathcal{Z}_\lambda)}\\ &\le \sum_{|\alpha|\le k} \|T_{r,p} D_z^\alpha f\|_{L^p(\mathcal{Z}_\lambda)} = \|f\|_{H^{k,p}(\mathcal{Z}_{r\lambda})}. \end{aligned} \tag{14}$$

Next, observe that, using the transformation rule and a change of variables, for $z \in \mathcal{Z}_r$

$$\begin{aligned}(D_z^\alpha P_{r\lambda} f)(\Phi_r(z)) &= D_z^\alpha \int_{\mathcal{Z}_{r\lambda}} K_{r\lambda}(\Phi_r(z), w) f(w)\,\mathrm{d}V(w)\\ &= D_z^\alpha \int_{\mathcal{Z}_{r\lambda}} |\det \Phi_r'|^{-2} K_\lambda(z, \Phi_r^{-1}(w)) f(w)\,\mathrm{d}V(w)\\ &= D_z^\alpha \int_{\mathcal{Z}_\lambda} K_\lambda(z, w') f(\Phi_r(w'))\,\mathrm{d}V(w')\\ &= D_z^\alpha (P_\lambda(f\circ\Phi_r))(z),\end{aligned}$$

so that $T_{r,p}(D_z^\alpha P_{r\lambda} f) = D_z^\alpha(P_\lambda T_{r,p} f)$.

Therefore, assuming that P_λ is bounded on $H^{s.p}(\mathcal{Z}_\lambda)$, for $r > 1$, using both the fact that $T_{r,p} : L^p(\mathcal{Z}_{r\lambda}) \to L^p(\mathcal{Z}_\lambda)$ is an isometry and (14), we have

$$\begin{aligned}\|P_{r\lambda} f\|_{H^{k,p}(\mathcal{Z}_{r\lambda})} &= \sum_{|\alpha|\le k} \|D_z^\alpha P_{r\lambda} f\|_{L^p(\mathcal{Z}_{r\lambda})}\\ &= \sum_{|\alpha|\le k} \|T_{r,p} D_z^\alpha P_{r\lambda} f\|_{L^p(\mathcal{Z}_\lambda)}\\ &= \sum_{|\alpha|\le k} \|D_z^\alpha (P_\lambda T_{r,p} f)\|_{L^p(\mathcal{Z}_\lambda)}\\ &= \|(P_\lambda T_{r,p} f)\|_{H^{k,p}(\mathcal{Z}_\lambda)}\\ &\le N_{k,p}(\lambda) \|T_{r,p} f\|_{H^{k,p}(\mathcal{Z}_r)}\\ &\le N_{k,p}(\lambda) \|f\|_{H^{k,p}(\mathcal{Z}_{r\lambda})}. \end{aligned} \tag{15}$$

Therefore, $N_{k,p}(r\lambda) \le N_{k,p}(\lambda)$ for all integers k and $r > 1$. Thus, by interpolation, for all $s > 0$ and $r > 1$, (13) follows. □

Proof of Theorem 1.2. Step 1. Suppose that P_{λ_0} is bounded on $L^p(\mathcal{Z}_{\lambda_0})$ for some $\lambda_0 > 0$ and some $p \in (1,\infty)$. Hence, P_λ is bounded on $L^p(\mathcal{Z}_\lambda)$ for all $\lambda > \lambda_0$ by Lemma 4.2. Fix $f \in C_c^\infty(\mathcal{W})$, where $\mathcal{W}$ is the non-smooth unbounded worm and suppose that $\operatorname{supp} f \subseteq \mathcal{Z}_\lambda$ for all $\lambda \ge \lambda_0$. For all such λ's, denoting by χ_λ the characteristic function of $\mathcal{Z}_\lambda$

$$\|\chi_\lambda P_\lambda f\|_{L^p(\mathcal{W})} = \|P_\lambda f\|_{L^p(\mathcal{Z}_\lambda)} \le C\|f\|_{L^p(\mathcal{Z}_\lambda)} = C'$$

for some constant C' independent of λ. In the second-before-last inequality, we have used Lemma 4.2. Then, there exist a sequence $\{\lambda_n\}$, $\lambda_n \to \infty$ as $n \to \infty$, and $h \in (L^2 \cap L^p)(\mathcal{W})$, such that $\chi_{\lambda_n} P_{\lambda_n} f \to h$ in the weak-$*$ topology, as $n \to \infty$. It is easy to see that $h \in \mathrm{Hol}(\mathcal{W})$ arguing as follows.

Let ψ be smooth, and compactly supported in $\mathcal{W}$. Then, denoting by $\mathrm{d}V$ the Lebesgue volume form, for $j=1,2$, we have

$$\begin{aligned}\langle(\overline{\partial}_{z_j}h),\psi\rangle &= \int_{\mathcal{W}}(\overline{\partial}_{z_j}h)\overline{\psi}\,\mathrm{d}V = -\int_{\mathcal{W}}h\overline{(\partial_{z_j}\psi)}\,\mathrm{d}V = -\lim_{n\to\infty}\int_{\mathcal{Z}_{\lambda_n}}P_{\lambda_n}f\overline{(\partial_{z_j}\psi)}\,\mathrm{d}V\\ &= \lim_{n\to\infty}\int_{\mathcal{Z}_{\lambda_n}}(\overline{\partial}_{z_j}P_{\lambda_n}f)\overline{\psi}\,\mathrm{d}V = 0.\end{aligned}$$

Hence, $\overline{\partial}_{z_j}h=0$, $j=1,2$, and therefore, h is holomorphic. We claim that $h=Pf$, where P denotes the Bergman projection on $\mathcal{W}$. It suffices to show that $f-h\perp A^2(\mathcal{W})$. To this end, let $g\in A^2(\mathcal{W})$. Then

$$\int_{\mathcal{W}}(f-h)\overline{g}\,\mathrm{d}V = \lim_{n\to\infty}\int_{\mathcal{W}}(f-\chi_{\lambda_n}P_{\lambda_n}f)\overline{g}\,\mathrm{d}V = \lim_{n\to\infty}\int_{\mathcal{Z}_{\lambda_n}}(f-P_{\lambda_n}f)\overline{g}\,\mathrm{d}V = 0,$$

since the restriction of g to $\mathcal{Z}_\lambda$ belongs to $A^2(\mathcal{Z}_\lambda)$ for all $\lambda>0$, as well.

Seeking a contradiction, we suppose $p\neq 2$ and remark that

$$\begin{aligned}\|Pf\|_{L^p(\mathcal{W})} &= \sup\left\{\left|\int_{\mathcal{W}}Pf\overline{\psi}\,\mathrm{d}V\right| :\ \psi\in C_c^\infty(\mathcal{W}), \|\psi\|_{L^{p'}(\mathcal{W})}\le 1\right\}\\ &= \sup\left\{\lim_{n\to\infty}\left|\int_{\mathcal{Z}_{\lambda_n}}P_{\lambda_n}f\overline{\psi}\,\mathrm{d}V\right| :\ \psi\in C_c^\infty(\mathcal{W}), \|\psi\|_{L^{p'}(\mathcal{W})}\le 1\right\}\\ &\le \lim_{n\to\infty}\|P_{\lambda_n}f\|_{L^p(\mathcal{Z}_{\lambda_n})}\\ &\le C\|f\|_{L^p(\mathcal{W})}.\end{aligned}$$

This implies that $P: L^p(\mathcal{W})\to L^p(\mathcal{W})$ is bounded, contradicting [13, Theorem 1.1]. Therefore, P_{λ_0} cannot be bounded on $L^p(\mathcal{Z}_{\lambda_0})$ for $p\in(1,\infty)$ and $p\neq 2$. We also observe that, by interpolation with the case $p=2$, P_{λ_0} cannot be bounded on L^1 and L^∞ either.

Step 2. To prove the irregularity of P_λ in the Sobolev scale, we first show that P_λ is densely defined by showing that $(L^2\cap H^{s,p})(\mathcal{Z}_\lambda)$ is dense in $H^{s,p}(\mathcal{Z}_\lambda)$. Let $\varphi\in C_c^\infty(\mathbb{C}^2)$, $\varphi=1$ on the ball $B(0,1)$ and set $\varphi^\varepsilon(\,\cdot\,)=\varphi(\varepsilon\cdot)$. Given $f\in H^{s,p}(\mathcal{Z}_\lambda)$, let $f_{(\varepsilon)}:=f\varphi^\varepsilon$. It is easy to check that $f_{(\varepsilon)}\in(L^2\cap H^{s,p})(\mathcal{Z}_\lambda)$ and that $f_{(\varepsilon)}\to f$ as $\varepsilon\to 0^+$ in $H^{s,p}(\mathcal{Z}_\lambda)$.

Step 3. Let us show that it suffices to consider the case $s\in(0,1/p)$ (the region $T_1\cup T_3$ in Fig. 1). Suppose we have a bounded extension $P_\lambda: H^{s,p}(\mathcal{Z}_\lambda)\to H^{s,p}(\mathcal{Z}_\lambda)$ for some $s\ge 1/p$ and $p\in(1,\infty)$ (the region R in Fig. 1). Interpolating with $L^2(\mathcal{Z}_\lambda)$, we obtain a bounded extension $P_\lambda: H^{s_\theta,p_\theta}(\mathcal{Z}_\lambda)\to H^{s_\theta,p_\theta}(\mathcal{Z}_\lambda)$, where $\theta\in(0,1)$, $s_\theta=\theta s$, $\frac{1}{p_\theta}=\frac{\theta}{p}+\frac{1-\theta}{2}$. By taking θ small enough, we obtain that $0<s_\theta<1/p_\theta$.

Step 4. We show that, if $p\in(1,\infty)$, $s\in(0,1/p)$, and $P_\lambda: H^{s,p}(\mathcal{Z}_\lambda)\to H^{s,p}(\mathcal{Z}_\lambda)$ is bounded, then $K_w\in H^{s,p}(\mathcal{W}_{\mu(\lambda)})$ and $K_w\in H^{-s,p'}(\mathcal{W}_{\mu(\lambda)})$.

Lemma 4.3 gives bounded extensions $P_{\lambda'}: H^{s,p}(\mathcal{Z}_{\lambda'})\to H^{s,p}(\mathcal{Z}_{\lambda'})$ for all $\lambda'>\lambda$ as well as

$$\|P_{\lambda'}\|_{(H^{s,p}(\mathcal{Z}_{\lambda'}),H^{s,p}(\mathcal{Z}_{\lambda'}))}\le N_{s,p}(\lambda)$$

for all $\lambda'>\lambda$.

Fix $w \in \mathcal{W}$ and let $K_w = K(\cdot, w)$ denote the Bergman kernel of $\mathcal{W}$ at w. If we choose $\varphi_w \in C_c^\infty$ supported in a ball centered at w within $\mathcal{W}$, with radial symmetry and with $\int \varphi_w = 1$, then $P\varphi_w = K_w$.

Then, for all $\lambda' > \lambda$ large enough for $\operatorname{supp} \varphi_w \subseteq \mathcal{Z}_\lambda$, using Remark 3.5, we have that

$$\begin{aligned}\|P_{\lambda'}\varphi_w\|_{H^{s,p}(\mathcal{W}_{\mu(\lambda)})} &\leq \|P_{\lambda'}\varphi_w\|_{H^{s,p}(\mathcal{Z}_{\lambda'})} \leq N_{s,p}(\lambda)\|\varphi_w\|_{H^{s,p}(\mathcal{Z}_{\lambda'})}\\ &= N_{s,p}(\lambda)\|\varphi_w\|_{H^{s,p}(\mathbb{C}^2)}.\end{aligned}$$

Therefore, $\{P_{\lambda'}\varphi_w\}_{\lambda'}$ is a family of functions contained in the ball of radius $N_{s,p}(\lambda)\|\varphi_w\|_{H^{s,p}(\mathbb{C}^2)}$ centered at the origin in $H^{s,p}(\mathcal{W}_{\mu(\lambda)})$. Since we are assuming $0 < s < 1/p$, using Lemma 3.2 and the Hahn–Banach theorem, we have that $\{P_{\lambda'}\varphi_w\}_{\lambda' > \lambda}$ admits a subsequence weak-$*$ converging to a function h in $H^{s,p}(\mathcal{W}_{\mu(\lambda)})$. Recalling that $H^{s,p}(\mathcal{W}_{\mu(\lambda)})$ is the dual of $H^{-s,p'}(\mathcal{W}_{\mu(\lambda)})$ with respect to the $L^2(\mathcal{W}_{\mu(\lambda)})$ inner product, this implies that for all $g \in C_c^\infty(\mathcal{W}_{\mu(\lambda)})$, we have

$$\int_{\mathcal{W}} (\chi_{\lambda'_n} P_{\lambda'_n}\varphi_w) g \,\mathrm{d}V = \int_{\mathcal{W}_{\mu(\lambda)}} (P_{\lambda'_n}\varphi_w) g \to \int_{\mathcal{W}_{\mu(\lambda)}} hg \,\mathrm{d}V$$

as $n \to \infty$.

Arguing as in step 1, we have that (up to refinements) $\chi_{\lambda'_n} P_{\lambda'_n}\varphi_w$ converges to $P\varphi_w = K_w$ in the weak-$*$ topology of $L^2(\mathcal{W}_{\mu((\lambda)})$. Thus

$$\int_{\mathcal{W}_{\mu(\lambda)}} K_w g \,\mathrm{d}V = \int_{\mathcal{W}_{\mu(\lambda)}} hg \,\mathrm{d}V$$

for all $g \in C_c^\infty(\mathcal{W}_{\mu(\lambda)})$. This implies that $h = K_w$ on $\mathcal{W}_{\mu(\lambda)}$, whence $K_w \in H^{s,p}(\mathcal{W}_{\mu(\lambda)})$.

To prove that $K_w \in H^{-s,p'}(\mathcal{W}_{\mu(\lambda)})$, we use Lemma 3.2. For all $\lambda' > \lambda$, we have

$$\begin{aligned}&\|P_{\lambda'}\varphi_w\|_{H^{-s,p'}(\mathcal{W}_{\mu(\lambda)})}\\ &\quad= \sup\left\{\left|\int_{\mathcal{W}_{\mu(\lambda)}} P_{\lambda'}\varphi_w\overline{\psi}\,\mathrm{d}V\right| : \ \psi \in C_c^\infty(\mathcal{W}_{\mu(\lambda)}),\ \|\psi\|_{H^{s,p}(\mathcal{W}_{\mu(\lambda)})} \leq 1\right\}\\ &\quad= \sup\left\{\left|\int_{\mathcal{Z}_{\lambda'}} P_{\lambda'}\varphi_w\overline{\psi}\,\mathrm{d}V\right| : \ \psi \in C_c^\infty(\mathcal{W}_{\mu(\lambda)}),\ \|\psi\|_{H^{s,p}(\mathcal{W}_{\mu(\lambda)})} \leq 1\right\}\\ &\quad= \sup\left\{\left|\int_{\mathcal{Z}_{\lambda'}} \varphi_w\overline{P_{\lambda'}\psi}\,\mathrm{d}V\right| : \ \psi \in C_c^\infty(\mathcal{W}_{\mu(\lambda)}),\ \|\psi\|_{H^{s,p}(\mathcal{W}_{\mu(\lambda)})} \leq 1\right\}\\ &\quad\leq \|\varphi_w\|_{H^{-s,p'}(\mathbb{C}^2)}\|P_{\lambda'}\psi\|_{H^{s,p}(\mathcal{Z}_{\lambda'})}\\ &\quad\leq N_{s,p}(\lambda)\|\varphi_w\|_{H^{-s,p'}(\mathbb{C}^2)}.\end{aligned}$$

We now argue as before and conclude that $K_w \in H^{-s,p'}(\mathcal{W}_{\mu(\lambda)})$.

We split the remaining part of the argument into three steps: one concerning the region T_1, one concerning the region T_3, and one concerning the line segment separating them; see Fig. 1.

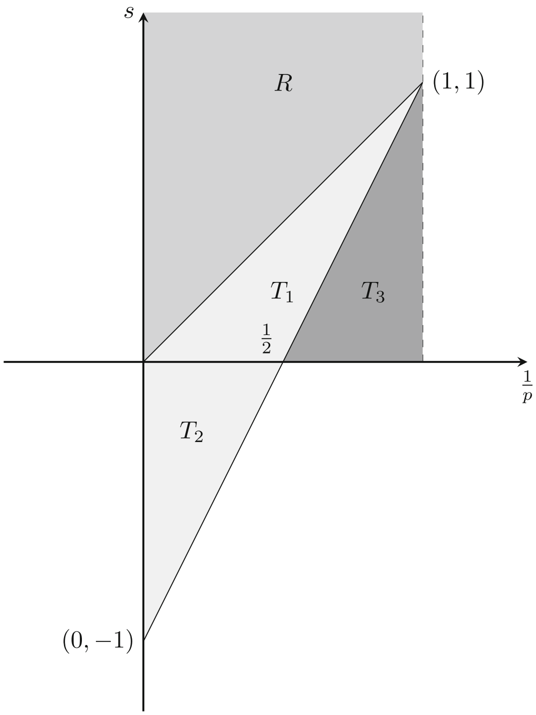

Figure 1. Diagram for the proofs of Proposition 4.1 and Theorem 1.2

Step 5. We assume $p \in (1,\infty)$ and $s > \max(\frac{2}{p}-1, 0)$ and $P_\lambda : H^{s,p}(\mathcal{Z}_\lambda) \to H^{s,p}(\mathcal{Z}_\lambda)$ is bounded. By step 4, we have that $K_w \in H^{s,p}(\mathcal{W}_{\mu(\lambda)})$. Then, Proposition 4.1 immediately gives a contradiction.

Step 6. We assume that $p \in (1,2)$, $0 < s < \frac{2}{p} - 1$ and $P_\lambda : H^{s,p}(\mathcal{Z}_\lambda) \to H^{s,p}(\mathcal{Z}_\lambda)$ is bounded. Notice that $\frac{2}{p} - 1 < \frac{1}{p}$, so that, by step 4, we obtain that $K_w \in H^{-s,p'}(\mathcal{W}_{\mu(\lambda)})$, where $-s > 1 - \frac{2}{p} = \frac{2}{p'} - 1$. However, again, this is false by Proposition 4.1(i) and we have reached a contradiction. Hence, the projector P_λ does not extend to a bounded operator $P_\lambda : H^{s,p}(\mathcal{Z}_\lambda) \to H^{s,p}(\mathcal{Z}_\lambda)$.

Step 7. Finally, let $p \in (1,2)$ and $s = \frac{2}{p} - 1$ and suppose that $P_\lambda : H^{s,p}(\mathcal{Z}_\lambda) \to H^{s,p}(\mathcal{Z}_\lambda)$ is bounded. Again, Lemma 4.3 gives that $P_{\lambda'} : H^{s,p}(\mathcal{Z}_{\lambda'}) \to H^{s,p}(\mathcal{Z}_{\lambda'})$ is bounded and

$$\|P_{\lambda'}\|_{(H^{s,p}(\mathcal{Z}_{\lambda'}),H^{s,p}(\mathcal{Z}_{\lambda'}))} \leq N_{s,p}(\lambda)$$

for all $\lambda' > \lambda$. Take any $\mu > \pi$ and let λ' sufficiently large, so that $\mathcal{W}_\mu \subseteq \mathcal{Z}_{\lambda'}$. Let $\varphi_w \in C_c^\infty(\mathcal{Z}_{\lambda'})$ be as in step 4. We have shown that there exists a sequence $\{P_{\lambda_n}\varphi_w\}$, such that $P_{\lambda_n}\varphi_w \to K_w$ weak-$*$ in $H^{s,p}(\mathcal{Z}_{\lambda'})$ as $\lambda_n \to \infty$, so that

$$\begin{aligned}\|K_w\|_{H^{s,p}(\mathcal{W}_\mu)} &\leq \|K_w\|_{H^{s,p}(\mathcal{Z}_{\lambda'})} \leq \lim_{n\to\infty} \|P_{\lambda_n}\|_{(H^{s,p}(\mathcal{Z}_{\lambda_n}),H^{s,p}(\mathcal{Z}_{\lambda_n}))}\|\varphi_w\|_{H^{s,p}(\mathbb{C}^2)}\\ &\leq CN_{s,p}(\lambda)\end{aligned}$$

independent of μ. This contradicts Proposition 4.1(ii) and the proof is complete. $\square$

5. Final Remarks and Open Questions

We wish to conclude by indicating a number of open problems. First of all, we recall that the exact range of regularity on the Lebesgue Sobolev spaces $H^{s,p}$ of the Bergman projection on the smoothly bounded domain $\mathcal{W}_\mu$ is not known. Clearly, to prove a positive result, one needs to have precise information on the Bergman kernel itself. In fact, also the precise behaviour of the kernel near the critical annulus $\mathcal{A} = \{(0, z_2) : \mathrm{e}^{-\mu/2} < |z_2| < \mathrm{e}^{\mu/2}\}$ on the boundary of $\mathcal{W}_\mu$ remains to be understood.

The equivalence of the regularity of the Bergman projections on $(0, q)$-forms and the Neumann operator $\mathcal{N}$, proved in [5], was later exploited by M. Christ [7] to show that P_μ does not preserve $C^\infty(\overline{\mathcal{W}_\mu})$. These results heavily relied on the boundedness of the domain $\mathcal{W}_\mu$. We believe that the Neumann operator $\mathcal{N}$ on $\mathcal{Z}_\lambda$ is as irregular as the Bergman projection P_λ, but this problem has not been addressed and (to the best of our knowledge) is open.

Finally, we mention the boundary analogue of this problem, namely the study of the behaviour of the Szegő projection on $\mathcal{Z}_\lambda$. Given a smooth domain $\Omega = \{z : \rho(z) < 0\} \subseteq \mathbb{C}^n$, the Hardy space $H^2(\Omega, \mathrm{d}\sigma)$ is defined as

$$H^2(\Omega, \mathrm{d}\sigma) = \left\{ f \in \mathrm{Hol}(\Omega) : \sup_{\varepsilon > 0} \int_{\partial\Omega_\varepsilon} |f|^2 \mathrm{d}\sigma_\varepsilon < \infty \right\},$$

where $\Omega_\varepsilon = \{z : \rho(z) < -\varepsilon\}$ and $\mathrm{d}\sigma_\varepsilon$ is the induced surface measure on $\partial\Omega_\varepsilon$. Then, $H^2(\Omega, \mathrm{d}\sigma)$ can be identified with a closed subspace of $L^2(\partial\Omega, \mathrm{d}\sigma)$, that we denote by $H^2(\partial\Omega, \mathrm{d}\sigma)$, where σ is the induced surface measure on $\partial\Omega$. The Szegő projection is the orthogonal projection

$$S_\Omega : L^2(\partial\Omega, \mathrm{d}\sigma) \to H^2(\partial\Omega, \mathrm{d}\sigma);$$

see [30] for the case of *bounded* domains. The regularity of S_Ω when Ω is a (model) worm domain was studied in a series of papers [20, 22–24, 26, 27]. In particular, in [20], it was announced that $S_{\mathcal{W}_\mu}$ does not preserve $L^p(\partial\mathcal{W}_\mu)$ when $\left|\frac{1}{2} - \frac{1}{p}\right| \geq \frac{\pi}{\mu}$, in analogy to the case of the Bergman projection. L. Lanzani and E. Stein also studied the L^p-regularity of the Szegő and other projections on the boundary on bounded domains under minimal smoothness conditions [18, 19], whereas a definition of Hardy spaces and associated Szegő projection for singular domains was studied, for instance, in [9, 25]. It is certainly of interest to consider the case of the Szegő projection also in the case of the domains $\mathcal{Z}_\lambda$.

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