## Purely nonlocal Hamiltonian formalism for systems of hydrodynamic type

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#### Abstract

We study purely nonlocal Hamiltonian structures for systems of hydrodynamic type. In the case of a semi-Hamiltonian system, we show that such structures are related to quadratic expansions of the diagonal metrics naturally associated with the system.

## Introduction

In the last three decades many papers have been devoted to Hamiltonian structures for systems of hydrodynamic type:

$$u_t^i = v_i^i(u)u_x^j, \qquad i = 1, \dots, n.$$
 (0.1)

The starting point of this research was the paper [7] (see also [8]) where Dubrovin and Novikov introduced an important class of local Hamiltonian structures, called *Hamiltonian structures of hydrodynamic type*. Such operators have the form

$$P^{ij} = g^{ij}(u)\frac{d}{dx} - g^{is}\Gamma^j_{sk}(u)u^k_x, \qquad (0.2)$$

where  $g^{ij}$  are the contravariant components of a flat pseudo-Riemannian metric and  $\Gamma_{sk}^{j}$  are the Christoffel symbols of the associated Levi-Civita connection. Nonlocal extensions of the

bracket (0.2), related to metrics of constant curvature, were considered by Ferapontov and Mokhov in [11]; further generalizations, of the form

$$P^{ij} = g^{ij} \frac{d}{dx} - g^{is} \Gamma^j_{sk} u^k_x + \sum_{\alpha} \varepsilon_{\alpha} \left( w_{\alpha} \right)^i_k u^k_x \left( \frac{d}{dx} \right)^{-1} \left( w_{\alpha} \right)^j_h u^h_x, \qquad \varepsilon_{\alpha} = \pm 1, \qquad (0.3)$$

were considered by Ferapontov in [10]. Here we have defined

$$\left(\frac{d}{dx}\right)^{-1} = \frac{1}{2} \int_{-\infty}^{x} dx - \frac{1}{2} \int_{x}^{+\infty} dx,$$

and the index  $\alpha$  can take values over a finite, infinite or even continuous set. In the case det  $g^{ij} \neq 0$ , the operator (0.3) defines a Poisson structure if and only if the tensor  $g^{ij}$  defines a pseudo-Riemannian metric, the coefficients  $\Gamma_{sk}^{j}$  are the Christoffel symbols of its Levi-Civita connection  $\nabla$ , and the affinors  $w_{\alpha}$  satisfy the conditions

$$[w_{\alpha}, w_{\alpha'}] = 0, \qquad (0.4a)$$

$$g_{ik}(w_{\alpha})_j^k = g_{jk}(w_{\alpha})_i^k, \tag{0.4b}$$

$$\nabla_k (w_\alpha)^i_j = \nabla_j (w_\alpha)^i_k, \tag{0.4c}$$

$$R_{kh}^{ij} = \sum_{\alpha} \varepsilon_{\alpha} \left\{ (w_{\alpha})_k^i (w_{\alpha})_h^j - (w_{\alpha})_k^j (w_{\alpha})_h^i \right\},$$
(0.4d)

where  $R_{kh}^{ij} = g^{is} R_{skh}^{j}$  are the components of the Riemann curvature tensor of the metric g. As observed by Ferapontov, if the sum over  $\alpha$  goes from 1 to m, then these equations are the Gauss-Mainardi-Codazzi equations for an n-dimensional submanifold N with flat normal connection embedded in a (n + m)-dimensional pseudo-euclidean space.

It is important to point out that the metric g, in general, does not uniquely fix the Hamiltonian structure (0.3); this arbitrariness is related to the fact that the affinors satisfying the Gauss-Peterson-Mainardi-Codazzi equations, and hence the corresponding embedding, may not be unique.

More precisely, let the set of affinors  $\mathbf{w} = \{w_{\alpha}\}$  satisfy equations (0.4) for a given metric g, with associated Levi-Civita connection  $\nabla$  and curvature tensor R. Let another set of affinors  $\mathbf{W} = \{W_{\beta}\}$  satisfy the conditions

$$[W_{\beta}, w_{\alpha}] = 0, \quad [W_{\beta}, W_{\beta'}] = 0,$$
  

$$g_{ik}(W_{\beta})_{j}^{k} = g_{jk}(W_{\beta})_{i}^{k},$$
  

$$\nabla_{k}(W_{\beta})_{j}^{i} = \nabla_{j}(W_{\beta})_{k}^{i},$$
  

$$\sum_{\beta} \epsilon_{\beta} \left\{ (W_{\beta})_{k}^{i} (W_{\beta})_{h}^{j} - (W_{\beta})_{k}^{j} (W_{\beta})_{h}^{i} \right\} = 0, \quad \epsilon_{\beta} = \pm 1.$$

It then follows trivially that the union  $w \cup W$  also satisfies equations (0.4) with the same

metric g. This means that the expression

$$P^{ij} = g^{ij} \frac{d}{dx} - g^{is} \Gamma^j_{sk} u^k_x + \sum_{\alpha} \varepsilon_{\alpha} (w_{\alpha})^i_k u^k_x \left(\frac{d}{dx}\right)^{-1} (w_{\alpha})^j_h u^h_x$$
$$+ \sum_{\beta} \epsilon_{\beta} (W_{\beta})^i_k u^k_x \left(\frac{d}{dx}\right)^{-1} (W_{\beta})^j_h u^h_x$$

is a Poisson bivector; further, it is compatible with (0.3) as one can easily check by rescaling  $W \rightarrow \lambda W$ . In this way one obtains a pencil  $P_2 - \lambda P_1$  of Hamiltonian structures, where  $P_2$  is the Hamiltonian structure of Ferapontov type (0.3) and  $P_1$  is a *purely nonlocal* Hamiltonian structure of the form

$$P_1^{ij} = \sum_{\beta} \epsilon_{\beta} \left( W_{\beta} \right)_k^i u_x^k \left( \frac{d}{dx} \right)^{-1} (W_{\beta})_h^j u_x^h.$$
(0.5)

Summarizing, the study of the arbitrariness of the nonlocal tail in Hamiltonian operators of Ferapontov type (0.3) leads us to consider nonlocal operators of the form (0.5). Apart from purely nonlocal structures associated with *flat* metrics considered by Mokhov in [18] and a few isolated examples [3, 22], such operators have not been considered much in the literature. A more systematic study of some such structures, a subclass of Mokhov's, was given recently in [19], where it was shown that some such purely nonlocal Poisson operators can appear as inverses of local symplectic operators. The aim of this paper is to study Poisson operators of the form (0.5) in greater generality, to find the conditions they must satisfy, and to construct some classes of examples. In the case of a semihamiltonian hierarchy, there is a remarkable relationship between the symmetries  $W_{\alpha}$  appearing in the operator, and the metrics naturally associated with the hierarchy - these are expanded as quadratic forms in the  $W_{\alpha}$ . In particular we find such operators associated with reductions of the Benney equations, and with semisimple Frobenius manifolds admitting a superpotential.

# 1 Purely nonlocal Hamiltonian formalism of hydrodynamic type

Let us consider purely nonlocal operators of the form (0.5); the aim of this section is to determine necessary and sufficient conditions for (0.5) to be a Poisson operator, namely to satisfy the skew symmetry condition and the Jacobi identity. For this purpose it is more convenient to consider, instead of the differential operator (0.5), its associated bracket

$$\{F,G\} = \int \frac{\delta F}{\delta u^{i}} P^{ij} \frac{\delta G}{\delta u^{j}} dx$$
$$= \iint \frac{\delta F}{\delta u^{i}(x)} \Pi^{ij}(x,y) \frac{\delta G}{\delta u^{j}(y)} dy dx,$$
(1.1)

where we have introduced

$$\Pi^{ij}(x,y) = \sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(x)^{i}_{s} u^{s}_{x} \nu(x-y) W_{\alpha}(y)^{j}_{l} u^{l}_{y}, \qquad (1.2)$$

and

$$\nu(x-y) = \frac{1}{2} \text{sgn}(x-y).$$
(1.3)

The functionals F and G appearing in the bracket are of local type, not depending on the x-derivatives of u. The skew symmetry of this bracket is trivially satisfied, so we need to find the conditions on the  $W_{\alpha}$  such that the Jacobi identity

$$\{\{G, H\}, F\} + \{\{F, G\}, H\} + \{\{H, F\}, G\} = 0$$

holds for every F, G, H.

**Proposition 1.1** Suppose that the affinors  $W_{\alpha}$  are not degenerate and that they have a simple spectrum, then the bracket (1.1) with (1.2) satisfies the Jacobi identity if and only if the following conditions are satisfied:

$$(W_{\beta})_{q}^{m}\partial_{m}(W_{\alpha})_{l}^{k} + (W_{\beta})_{l}^{m}\partial_{m}(W_{\alpha})_{q}^{k} + (W_{\alpha})_{m}^{k}\partial_{l}(W_{\beta})_{q}^{m} + (W_{\alpha})_{m}^{k}\partial_{q}(W_{\beta})_{l}^{m} =$$
$$= (W_{\alpha})_{q}^{m}\partial_{m}(W_{\beta})_{l}^{k} + (W_{\alpha})_{l}^{m}\partial_{m}(W_{\beta})_{q}^{k} + (W_{\beta})_{m}^{k}\partial_{l}(W_{\alpha})_{q}^{m} + (W_{\beta})_{m}^{k}\partial_{q}(W_{\alpha})_{l}^{m}$$
(1.4)

$$[W_{\alpha}, W_{\beta}] = 0, \qquad \forall \alpha, \beta, \tag{1.5}$$

$$\sum_{\alpha} \epsilon_{\alpha} \left( (W_{\alpha})_k^i (W_{\alpha})_h^j - (W_{\alpha})_h^i (W_{\alpha})_k^j \right) = 0.$$
(1.6)

**Remark 1** The first two conditions say that the flows  $u_{t_{\alpha}}^{i} = (W_{\alpha})_{j}^{i} u_{x}^{j}$  commute.

*Proof* As noticed in [8], in order to prove the Jacobi identity we can restrict our attention to *linear* functionals of the type

$$F = \int f_i(x) u^i dx, \quad G = \int g_i(x) u^i dx \quad H = \int h_i(x) u^i dx.$$

We have

$$\{\{G, H\}, F\} = \iint \frac{\delta\{G, H\}}{\delta u^m(t)} \Pi^{mi}(t, x) \frac{\delta F}{\delta u^i(x)} dx dt$$
$$= \iint \frac{\delta\{G, H\}}{\delta u^m(t)} \Pi^{mi}(t, x) f_i(x) dx dt, \qquad (1.7)$$

where

$$\frac{\delta\{G,H\}}{\delta u^m(t)} = \frac{\delta}{\delta u^m(t)} \iint \frac{\delta G}{\delta u^j(y)} \Pi^{jk}(y,z) \frac{\delta H}{\delta u^k(z)} \, dy \, dz$$
$$= \iint g_j(y) \frac{\delta \Pi^{jk}(y,z)}{\delta u^m(t)} \, h_k(z) \, dy \, dz.$$

Hence, (1.7) can be reduced to

$$\{\{G,H\},F\} = \iiint f_i(x) g_j(y) h_k(z) S^{ijk}(x,y,z) dx dy dz,$$

where we have introduced the quantity

$$S^{jki}(y,z,x) = \int \frac{\delta \Pi^{jk}(y,z)}{\delta u^m(t)} \Pi^{mi}(t,x) dt.$$

In this way, the Jacobi identity reads

$$\iiint f_i(x) g_j(y) h_k(z) \left( S^{jki}(y, z, x) + S^{kij}(z, x, x) + S^{ijk}(x, y, z) \right) dx \, dy \, dz = 0,$$

This has to be satisfied for every function  $f_i$ ,  $g_j$ ,  $h_k$ , so that we must require

$$S^{jki}(y, z, x) + S^{kij}(z, x, x) + S^{ijk}(x, y, z) = 0.$$
 (1.8)

Let us consider the quantities  $S^{jki}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{x})$  more explicitly. We have

$$\frac{\delta \Pi^{jk}(y,z)}{\delta u^m(t)} = \sum_{\alpha} \epsilon_{\alpha} \left\{ \left[ \frac{\partial W_{\alpha}(y)_p^j}{\partial u^m(t)} \delta(y-t) \, u_y^p + W_{\alpha}(y)_m^j \, \delta'(y-t) \right] W_{\alpha}(z)_q^k \, u_z^q \right. \\ \left. + W_{\alpha}(y)_p^j \, u_y^p \left[ \frac{\partial W_{\alpha}(z)_q^k}{\partial u^m(t)} \, \delta(z-t) \, u_z^q + W_{\alpha}(z)_m^k \, \delta'(z-t) \right] \right\} \, \nu(y-z),$$

and so

$$S^{jki}(y, z, x) = \left(\sum_{\alpha} \epsilon_{\alpha} \frac{\partial W_{\alpha}(y)_{p}^{j}}{\partial u^{m}(y)} W_{\alpha}(z)_{q}^{k}\right) \Pi^{mi}(y, x) u_{y}^{p} u_{z}^{q} \nu(y - z)$$

$$+ \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(y)_{p}^{j} \frac{\partial W_{\alpha}(z)_{q}^{k}}{\partial u^{m}(z)}\right) \frac{\partial \Pi^{mi}(y, x)}{\partial y} u_{z}^{q} \nu(y - z)$$

$$+ \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(y)_{p}^{j} \frac{\partial W_{\alpha}(z)_{q}^{k}}{\partial u^{m}(z)}\right) \Pi^{mi}(z, x) u_{y}^{p} u_{z}^{q} \nu(y - z)$$

$$+ \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(y)_{p}^{j} W_{\alpha}(z)_{m}^{k}\right) \frac{\partial \Pi^{mi}(z, x)}{\partial z} u_{y}^{p} \nu(y - z).$$

Now we evaluate:

$$\frac{\partial \Pi^{mi}(y,x)}{\partial y} = \left(\sum_{\alpha} \epsilon_{\alpha} \frac{\partial W_{\alpha}(y)_{p}^{m}}{\partial u^{l}(y)} W_{\alpha}(x)_{s}^{i}\right) u_{x}^{s} u_{y}^{p} u_{y}^{l} \nu(y-x)$$
$$+ \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(y)_{p}^{m} W_{\alpha}(x)_{s}^{i}\right) u_{x}^{s} u_{yy}^{p} \nu(y-x)$$
$$+ \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(y)_{p}^{m} W_{\alpha}(x)_{s}^{i}\right) u_{x}^{s} u_{y}^{p} \delta(y-x),$$

so that we can write

$$\begin{split} S^{jki}(y,z,x) &= A^{kji}_{qpls}(z,y,x) \, u^s_x \, u^p_y \, u^l_y \, u^q_z \, \nu(y-x) \, \nu(y-z) \\ &+ A^{jki}_{pqls}(y,z,x) \, u^s_x \, u^p_y \, u^q_z \, u^l_z \, \nu(z-x) \, \nu(y-z) \\ &+ B^{kji}_{qps}(z,y,x) \, u^s_x \, u^p_{yy} \, u^q_z \, \nu(y-x) \, \nu(y-z) \\ &+ B^{jki}_{pqs}(y,z,x) \, u^s_x \, u^p_y \, u^q_{zz} \, \nu(z-x) \, \nu(y-z) \\ &+ B^{kji}_{qps}(z,y,x) \, u^s_x \, u^p_y \, u^q_z \, \delta(y-x) \, \nu(y-z) \\ &+ B^{jki}_{pqs}(y,z,x) \, u^s_x \, u^p_y \, u^q_z \, \delta(z-x) \, \nu(y-z), \end{split}$$

Here we have introduced the notation:

$$A_{qpls}^{kji}(z, y, x) = \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(z)_{q}^{k} \partial_{m} W_{\alpha}(y)_{p}^{j}\right) \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(y)_{l}^{m} W_{\alpha}(x)_{s}^{i}\right) + \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(z)_{q}^{k} W_{\alpha}(y)_{m}^{j}\right) \left(\sum_{\alpha} \epsilon_{\alpha} \partial_{l} W_{\alpha}(y)_{p}^{m} W_{\alpha}(x)_{s}^{i}\right), \quad (1.9)$$

$$B_{qps}^{kji}(z,y,x) = \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(z)_{q}^{k} W_{\alpha}(y)_{m}^{j}\right) \left(\sum_{\alpha} \epsilon_{\alpha} W_{\alpha}(y)_{p}^{m} W_{\alpha}(x)_{s}^{i}\right).$$
(1.10)

Cyclically permuting with respect to i, j, k and x, y, z, and then rearranging the terms, it is possible to rewrite condition (1.8) in the form

$$\begin{split} & \left[ A_{splq}^{ijk}(x,y,z) - A_{qpls}^{kji}(z,y,x) \right] \, u_x^s \, u_y^p \, u_y^l \, u_z^q \, \nu(x-y) \, \nu(y-z) \\ & + \left[ B_{spq}^{ijk}(x,y,z) - B_{qps}^{kji}(z,y,x) \right] \, u_x^s \, u_{yy}^p \, u_z^q \, \nu(x-y) \, \nu(y-z) \\ & + \left[ B_{spq}^{ijk}(x,y,z) - B_{qps}^{kji}(z,y,x) \right] \, u_x^s \, u_y^p \, u_z^q \, \delta(x-y) \, \nu(y-z) \end{split}$$

+ (cyclic permutations) = 0,

and this reduces to the following conditions (for if these are satisfied, then all the others vanish identically):

$$A_{splq}^{ijk}(x, y, z) + A_{slpq}^{ijk}(x, y, z) = A_{qpls}^{kji}(z, y, x) + A_{lpqs}^{kji}(z, y, x),$$
(1.11)

$$B_{spq}^{ijk}(x, y, z) = B_{qps}^{kji}(z, y, x)$$
(1.12)

$$B_{spq}^{ijk}(x,x,z) + B_{psq}^{ijk}(x,x,z) = B_{qps}^{kji}(z,x,x) + B_{qsp}^{kji}(z,x,x)$$
(1.13)

that follow immediately from (1.4), (1.5), (1.6). The converse is also true if the affinors  $W_{\alpha}$  are not degenerate, and have a simple spectrum.

## 2 Semi-Hamiltonian systems

Let us consider now the important class of diagonalizable, semi-Hamiltonian systems of hydrodynamic type. These systems were introduced by Tsarev in [23], and correspond to the

class of systems which are integrable by the *generalized hodograph method*. More precisely, a diagonal system of hydrodynamic type

$$u_t^i = v^i(u)u_x^i, \tag{2.1}$$

is called *semi-Hamiltonian* [23] if the coefficients  $v^i(u)$  satisfy the system of equations

$$\partial_j \left( \frac{\partial_k v^i}{v^i - v^k} \right) = \partial_k \left( \frac{\partial_j v^i}{v^i - v^j} \right) \qquad \forall i \neq j \neq k \neq i,$$
(2.2)

where  $\partial_i = \frac{\partial}{\partial \lambda^i}$ . The  $v^i$  are usually called *characteristic velocities*. Equations (2.2) are the integrability conditions for three different systems: the first, given by

$$\frac{\partial_j w^i}{w^i - w^j} = \frac{\partial_j v^i}{v^i - v^j},\tag{2.3}$$

which provides the characteristic velocities  $w^i(u)$  of the symmetries of (0.1):

$$u_{\tau}^{i} = w^{i}(u)u_{x}^{i}$$
  $i = 1, ..., n;$ 

the second is a system whose solutions are the conserved densities H(u) of (0.1):

$$\partial_i \partial_j H - \Gamma^i_{ij} \partial_i H - \Gamma^j_{ji} \partial_j H = 0, \qquad \Gamma^i_{ij} = \frac{\partial_j v^i}{v^j - v^i}, \tag{2.4}$$

and the third is

$$\partial_j \ln \sqrt{g_{ii}} = \frac{\partial_j v^i}{v^j - v^i}, \qquad i \neq j,$$
(2.5)

which relates the characteristic velocities of the system to a class of diagonal metrics  $g_{ii}(u)$ . In [10] Ferapontov noticed that these metrics represent all possible candidates for the construction of Hamiltonian operators for the system, whether of local type (0.2), or nonlocal (0.3).

Now let us consider a purely nonlocal Hamiltonian formalism. Let

$$u_t^i = v^i u_x^i, \tag{2.6}$$

be a semi-Hamiltonian system and let

$$u_{t_{\alpha}}^{i} = W_{\alpha}^{i} u_{x}^{i},$$

be a set of symmetries satisfying condition (1.6); if the affinors are diagonal, this takes the form:

$$\sum_{\alpha} \epsilon_{\alpha} W^{i}_{\alpha} W^{j}_{\alpha} = 0 \qquad i \neq j.$$
(2.7)

Then, according to the results of Section 1, the operator

$$P = \sum_{\alpha} \epsilon_{\alpha} W^{i}_{\alpha} u^{i}_{x} \left(\frac{d}{dx}\right)^{-1} W^{j}_{\alpha} u^{j}_{x}$$
(2.8)

defines a Hamiltonian structure. Moreover, the flows generated by the Hamiltonian densities which solve the linear system (2.4) are symmetries of (2.6).

More precisely, we consider

$$u_{\tau}^{i} = w^{i}u_{x}^{i}, \qquad w^{i} := P^{ij}\partial_{j}H = \sum_{\alpha}\epsilon_{\alpha}W_{\alpha}^{i}K^{\alpha},$$

where the functions  $K^{\alpha}$  are the fluxes of conservation laws given by

$$\partial_{t_{\alpha}}H = \partial_x K^{\alpha}. \tag{2.9}$$

For  $i \neq j$  we get:

$$\partial_j w^i = \partial_j \left( \sum_{\alpha} \epsilon_{\alpha} W^i_{\alpha} K^{\alpha} \right) = \sum_{\alpha} \epsilon_{\alpha} \partial_j W^i_{\alpha} K^{\alpha} + \sum_{\alpha} \epsilon_{\alpha} W^i_{\alpha} \partial_j K^{\alpha},$$

so by using equations (2.3) and (2.9) we obtain

$$\partial_j w^i = \frac{\partial_j v^i}{v^j - v^i} \sum_{\alpha} \epsilon_{\alpha} \left( W^j_{\alpha} - W^i_{\alpha} \right) K^{\alpha} + \sum_{\alpha} \epsilon_{\alpha} W^i_{\alpha} W^j_{\alpha} \partial_i H$$
$$= \frac{\partial_j v^i}{v^j - v^i} \left( w^j - w^i \right).$$

Hence the Hamiltonian flows generated by conserved densities indeed belong to the semi-Hamiltonian hierarchy containing (2.6). The converse problem, namely whether an arbitrary flow

$$u^i_\tau = X^i = w^i u^i_x \tag{2.10}$$

commuting with (2.6) is Hamiltonian with respect to the purely nonlocal Poisson structure (2.8), turns out to be much more difficult to solve. However, we can say that the Hamiltonian structure (2.8) is conserved along any flow (2.10) of the hierarchy:

#### Proposition 2.2 Let

$$\Pi^{ij}(x,y) = \sum_{\alpha} W^{i}_{\alpha}(x) \, u^{i}_{x} \, \nu(x-y) \, W^{j}_{\alpha}(y) \, u^{j}_{y} \tag{2.11}$$

be a purely nonlocal Poisson bivector, and suppose that the commuting flows

 $u_{t_{\alpha}}^{i} = W_{\alpha}^{i}(u^{1}(x), \dots, u^{n}(x))u_{x}^{i},$ 

belong to a semi-Hamiltonian hierarchy. If

$$u_t^i = X^i(x) = w^i(x)u_x^i,$$
 (2.12)

is an arbitrary flow of this hierarchy, then

 $\operatorname{Lie}_X \Pi = 0.$ 

*Proof.* For the bivector (2.11) and the vector field (2.12) it is not difficult to show [9] that the expression  $\text{Lie}_X \Pi$  takes the form:

$$\begin{aligned} [\operatorname{Lie}_{X}\Pi]^{ij} &= X^{k}(x) \, \frac{\partial \Pi^{ij}(x,y)}{\partial u^{k}(x)} + \partial_{x}X^{k}(x) \, \frac{\partial \Pi^{ij}(x,y)}{\partial u^{k}_{x}} \\ & X^{k}(y) \, \frac{\partial \Pi^{ij}(x,y)}{\partial u^{k}(y)} + \partial_{y}X^{k}(y) \, \frac{\partial \Pi^{ij}(x,y)}{\partial u^{k}_{y}} \\ & - \frac{\partial X^{i}(x)}{\partial u^{k}(x)} \, \Pi^{kj}(x,y) - \frac{\partial X^{i}(x)}{\partial u^{k}_{x}} \, \partial_{x}\Pi^{kj}(x,y) \\ & - \frac{\partial X^{j}(y)}{\partial u^{k}(y)} \, \Pi^{ik}(x,y) - \frac{\partial X^{j}(y)}{\partial u^{k}_{y}} \, \partial_{y}\Pi^{ik}(x,y). \end{aligned}$$

Rearranging and simplifying, we obtain

$$\begin{split} \left[\operatorname{Lie}_{X}\Pi\right]^{ij} &= \sum_{\alpha} w^{k}(x) \, u_{x}^{k} \, \frac{\partial W_{\alpha}^{i}(x)}{\partial u^{k}(x)} \, u_{x}^{i} \, \nu(x-y) \, W_{\alpha}^{j}(y) \, u_{y}^{j} \\ &+ \sum_{\alpha} \left[ w^{k}(x) \, u_{xx}^{k} + \frac{\partial w^{k}(x)}{\partial u^{l}(x)} \, u_{x}^{k} \, u_{x}^{l} \right] \, \delta_{k}^{i} \, W_{\alpha}^{i}(x) \, \nu(x-y) \, W_{\alpha}^{j}(y) \, u_{y}^{j} \\ &+ \sum_{\alpha} \, w^{k}(y) \, u_{y}^{k} \, W_{\alpha}^{i}(x) \, u_{x}^{i} \, \nu(x-y) \, \frac{\partial W_{\alpha}^{j}(y)}{\partial u^{k}(y)} \, u_{y}^{j} \\ &+ \sum_{\alpha} \left[ w^{k}(y) \, u_{yy}^{k} + \frac{\partial w^{k}(y)}{\partial u^{l}(y)} \, u_{y}^{k} \, u_{y}^{l} \right] \, \delta_{k}^{j} \, W_{\alpha}^{i}(x) \, u_{x}^{i} \, \nu(x-y) \, W_{\alpha}^{j}(y) \\ &- \sum_{\alpha} \, \frac{\partial w^{i}(x)}{\partial u^{k}(x)} \, u_{x}^{i} \, W_{\alpha}^{k}(x) \, u_{x}^{k} \, \nu(x-y) \, W_{\alpha}^{j}(y) \, u_{y}^{j} \\ &- \sum_{\alpha} \, \delta_{k}^{i} \, w^{i}(x) \left[ W_{\alpha}^{k}(x) \, u_{x}^{k} \, \nu(x-y) \, W_{\alpha}^{j}(y) \, u_{y}^{j} \\ &- \sum_{\alpha} \, \delta_{k}^{i} \, w^{i}(x) \, W_{\alpha}^{k}(x) \, u_{x}^{k} \, \delta(x-y) \, W_{\alpha}^{j}(y) \, u_{y}^{j} \\ &- \sum_{\alpha} \, \delta_{k}^{j} \, w^{j}(y) \, W_{\alpha}^{i}(x) \, u_{x}^{i} \, \nu(x-y) \left[ W_{\alpha}^{k}(y) \, u_{yy}^{i} + \frac{\partial W_{\alpha}^{k}(y)}{\partial u^{l}(y)} \, u_{y}^{k} \, u_{y}^{l} \right] \\ &+ \sum_{\alpha} \, \delta_{k}^{j} \, w^{j}(y) \, W_{\alpha}^{i}(x) \, u_{x}^{i} \, \delta(x-y) \, W_{\alpha}^{k}(y) \, u_{y}^{k}. \end{split}$$

The terms containing the second derivatives vanish identically. Collecting the remaining

terms and using the properties of the delta function we obtain

$$\begin{aligned} \left[\operatorname{Lie}_{X}\Pi\right]^{ij} &= \left[w^{j}(x) - w^{i}(x)\right] \left(\sum_{\alpha} W_{\alpha}^{i}(x) W_{\alpha}^{j}(x)\right) u_{x}^{i} u_{x}^{j} \,\delta(x-y) \\ &+ \sum_{\alpha} \left[\frac{\partial W_{\alpha}^{i}(x)}{\partial u^{k}(x)} \left(w^{k}(x) - w^{i}(x)\right) - \frac{\partial w^{i}(x)}{\partial u^{k}(x)} \left(W_{\alpha}^{k}(x) - W_{\alpha}^{i}(x)\right)\right] u_{x}^{k} u_{x}^{i} \nu(x-y) W_{\alpha}^{j}(y) u_{y}^{j} \\ &+ \sum_{\alpha} \left[\frac{\partial W_{\alpha}^{j}(y)}{\partial u^{k}(y)} \left(w^{k}(y) - w^{j}(y)\right) - \frac{\partial w^{j}(y)}{\partial u^{k}(y)} \left(W_{\alpha}^{k}(y) - W_{\alpha}^{j}(y)\right)\right] W_{\alpha}^{i}(x) u_{x}^{i} \nu(x-y) u_{y}^{j} u_{y}^{k}.\end{aligned}$$

which is identically zero, because:

$$\sum_{\alpha} W^i_{\alpha}(x) W^j_{\alpha}(x) = 0, \qquad i \neq j,$$

and

$$\frac{\partial_k w^i}{w^k - w^i} = \frac{\partial_k W^i_{\alpha}}{W^k_{\alpha} - W^i_{\alpha}}, \qquad k \neq i.$$

Thus, indeed,

$$\operatorname{Lie}_X \Pi = 0.$$

## **3** Quadratic expansion of the metric

Remarkably, in the case of semi-Hamiltonian systems, the existence of purely nonlocal Poisson structures is related to a quadratic expansion

$$g^{ii}\delta^{ij} = \sum_{\alpha} \epsilon_{\alpha} W^{i}_{\alpha} W^{j}_{\alpha}, \qquad (3.1)$$

of the *contravariant* components of a metric g, whose covariant components satisfy (2.5), namely:

$$\partial_j \ln \sqrt{g^{ii}} = -\frac{\partial_j v^i}{v^j - v^i}, \qquad i \neq j.$$

For  $i \neq j$ , the former identity follows from (2.7), while for the diagonal components we have the following

**Proposition 3.3** Let a diagonal system (2.1) be semi-Hamiltonian, and suppose we have a set of symmetries  $W_{\alpha}$  satisfying condition (2.7) for certain  $\epsilon_{\alpha} = \pm 1$ . Then, the set of functions

$$Q^{i} := \sum_{\alpha} \epsilon_{\alpha} \left( W_{\alpha}^{i} \right)^{2},$$

satisfies the system

$$\partial_j \ln \sqrt{Q^i} = -\frac{\partial_j v^i}{v^j - v^i}, \qquad i \neq j.$$

*Proof.* For  $i \neq j$ , we have

$$\partial_j Q^i = \partial_j \left( \sum_{\alpha} \epsilon_{\alpha} (W^i_{\alpha})^2 \right) = 2 \sum_{\alpha} \epsilon_{\alpha} W^i_{\alpha} \partial_j W^i_{\alpha}$$
$$= 2 \sum_{\alpha} \epsilon_{\alpha} W^i_{\alpha} \left( W^j_{\alpha} - W^i_{\alpha} \right) \frac{\partial_j v^i}{v^j - v^i} = -2 Q^i \frac{\partial_j v^i}{v^j - v^i}, \qquad i \neq j.$$

**Remark 2** If  $g_{ii}$  is a metric of Egorov type, that is

 $g_{ii} = \partial_i H$ 

for a suitable function H, then it is known (see e.g. [20]) that the characteristic velocities  $W^i_{\alpha}$  can be written as

$$W^{i}_{\alpha} = -\frac{\partial_{i}K_{\alpha}}{\partial_{i}H} = -\frac{\partial_{i}K_{\alpha}}{q_{ii}},$$
(3.2)

where the  $K_{\alpha}$  are densities of conservation laws. In this case, equation (3.1) can be written as

$$g^{ii}\delta^{ij} = \frac{1}{g_{ii}\,g_{jj}}\sum_{\alpha}\epsilon_{\alpha}\partial_i K_{\alpha}\partial_j K_{\alpha},$$

that is

$$g_{ii}\,\delta_{ij} = \sum_{\alpha} \epsilon_{\alpha}\partial_i K_{\alpha}\partial_j K_{\alpha}.$$

This suggests that, in the case of Egorov metrics, the existence of a quadratic expansion for a solution of the linear system (2.5) is related to the existence of an embedding of our n dimensional manifold N in a pseudo-euclidean space with coordinates  $K_{\alpha}$ , in which the metric g plays the role of the first fundamental form.

We have proved that any purely nonlocal Hamiltonian structure constructed for a semi-Hamiltonian system is necessarily related with one of the metrics which solves (2.5). This relation has been obtained with a direct calculation using the diagonal coordinates frame, in which both the symmetries and the metric are diagonal.

In order to give a coordinate-free formulation of this, it is convenient to interpret the characteristic velocities of the symmetries entering in the quadratic expansion of the metric as a vector fields on our *n*-dimensional manifold N. Such a change of point of view leads us naturally to introduce an algebraic structure on the tangent bundle TN of our *n*-dimensional manifold N - each fibre  $T_uN$  has the structure of an associative semisimple multiplicative algebra, and the bundle admits a holonomic basis of idempotents. This means that first, there exists a basis  $(Z_1, \ldots, Z_n)$  of idempotents:

$$Z_i(u) \cdot Z_j(u) = \delta_{ij} Z_j(u).$$

Second, if this basis commutes, (is holonomic), then there exists a set of coordinates, called *canonical coordinates*,  $(u^1, \ldots, u^n)$  such that

$$Z_i = \frac{\partial}{\partial u^i}.$$

An invariant form of the condition (3.1) can now be easily obtained by noting the following.

Lemma 3.1 Any diagonalizable system of hydrodynamic type

$$u_t^i = v_j^i(u)u_x^j, \tag{3.3}$$

can be written in the form

$$v_j^i(u) = c_{jk}^i(u) X^k(u),$$
 (3.4)

where the  $X^k$  are now the components of a vector field and the  $c^i_{jk}$  are the (u-dependent) structure "constants" of a associative semisimple algebra admitting a holonomic basis of idempotents.

*Proof.* Indeed system (3.4) becomes diagonal in canonical coordinates - and in such coordinates, the structure constants are simply

$$c^i_{jk} = \delta^i_j \delta^i_k. \tag{3.5}$$

These  $c_{jk}^i$  are evidently the structure constants of an associative algebra. Conversely, given a diagonal system of hydrodynamic type, we can define the structure constants by identifying the canonical coordinates with the given Riemann invariants by means of (3.5). This identification will clearly depend on the choice of the Riemann invariants.

#### Theorem 1 Let

 $u_{t_{\alpha}}^{i} = (W_{\alpha})_{j}^{i}(u)u_{x}^{j} = c_{jk}^{i}(u)X_{\alpha}^{k}(u)u_{x}^{j}$ (3.6)

be *n* commuting diagonal systems of hydrodynamic type defined by the structure constants of an associative semisimple algebra, admitting a holonomic basis of idempotents and by *n* vector fields  $X_{\alpha}$  ( $\alpha = 1, ..., m$ ).

Suppose that the metric

$$g^{ij} = \left(\sum_{\alpha=1}^{n} \epsilon_{\alpha} X_{\alpha} \otimes X_{\alpha}\right)^{ij}.$$
(3.7)

is nondegenerate and satisfies the following condition

$$g^{kl}c^{i}_{lm} = g^{il}c^{k}_{lm}.$$
 (3.8)

Then:

1. Denoting by  $\nabla$  the Levi-Civita connection associated with g, we have

$$g_{ik}(W_{\alpha})_{j}^{k} = g_{jk}(W_{\alpha})_{i}^{k}, \qquad \nabla_{k}(W_{\alpha})_{j}^{i} = \nabla_{j}(W_{\alpha})_{k}^{i}$$
(3.9)

2. The affinors  $(W_{\alpha})_{j}^{i}$  satisfy the conditions (1.4,1.5,1.6) and therefore the operator

$$\sum_{\alpha=1}^{M} \epsilon_{\alpha} \left(W_{\alpha}\right)_{k}^{i} u_{x}^{k} \left(\frac{d}{dx}\right)^{-1} \left(W_{\alpha}\right)_{h}^{j} u_{x}^{h}$$

is a purely nonlocal Hamiltonian operator.

#### Proof.

1. Condition (3.8) implies that the metric (3.7) is diagonal in canonical coordinates. Moreover it implies the first of conditions (3.9). In order to prove the second of conditions (3.9) we observe that, in canonical coordinates, it reads

$$\partial_j \ln \sqrt{g_{ii}} = \frac{\partial_j W^i_\alpha}{W^j_\alpha - W^i_\alpha}, \qquad i \neq j.$$

Taking into account that, in canonical coordinates, the vector fields  $W_{\alpha}$  ( $\alpha = 1, ..., m$ ) coincide with the characteristic velocities of the systems (3.6) and satisfy the condition (2.7), we obtain the result by computations of the Proposition 3.3.

2. Conditions (1.4,1.5) follow from the commutativity of the flows (3.6). Condition (1.6) follows immediately from (3.7,3.8). Indeed

$$\sum_{\alpha} \epsilon_{\alpha} \left( (W_{\alpha})_k^i (W_{\alpha})_h^j - (W_{\alpha})_h^i (W_{\alpha})_k^j \right) = \sum_{\alpha} \epsilon_{\alpha} (c_{kl}^i c_{hm}^j - c_{hl}^i c_{km}^j) (X_{\alpha})^l (X_{\alpha})^m =$$
$$= (c_{kl}^i c_{hm}^j - c_{hl}^i c_{km}^j) g^{lm}.$$

Using (3.8) we get

$$(c_{kl}^{i}c_{hm}^{j} - c_{hl}^{i}c_{km}^{j})g^{lm} = g^{js}(c_{kl}^{i}c_{hs}^{l} - c_{hl}^{i}c_{ks}^{l})$$

which vanishes due to associativity.

**Remark 3** We should point out that the second part of the theorem only uses the associativity property; the assumption of semisimplicity is only needed for the first part, which uses canonical coordinates.

## 4 Reductions of the Benney system

We recall the basic facts about the Benney chain and its reductions (for details see for example [12] and references therein). The Benney chain is the following infinite system of quasilinear PDEs:

$$A_t^k = A_x^{k+1} + kA^{k-1}A_x^0, \qquad k = 0, 1, 2, \dots,$$
(4.1)

in the infinitely many variables  $A^k(x, t)$ , which are usually called moments. Introducing the formal series

$$\lambda = p + \sum_{k=0}^{\infty} \frac{A^k}{p^{k+1}},$$

we can encode the whole system in the single equation

$$\lambda_t = p\lambda_x - A_x^0 \lambda_p, \tag{4.2}$$

which is the second flow of the dispersionless KP hierarchy; by considering the inverse of  $\lambda$  with respect to p, we obtain the series

$$p = \lambda - \sum_{k=0}^{\infty} \frac{H_k}{\lambda^{k+1}},\tag{4.3}$$

whose coefficients are conserved densities of the Benney chain, each of them polynomial in the moments.

**Remark 4** In many important examples, and in particular for all the reductions defined below, the series  $\lambda$  can be thought as the asymptotic expansion at infinity of an analytic function  $\lambda(p, x, t)$ . In this case, the generating function (4.3) is obtained by inverting the function  $\lambda$  with respect to p, and then expanding asymptotically around infinity.

A *reduction* of the Benney chain is a suitable restriction of the system (4.1) to the case when all the moments  $A^k$  can be expressed in terms of finitely many variables; as proved in [13], all reductions of the Benney chain are diagonalizable, that is they can be written in the form

$$\lambda_t^i = v^i(\lambda)\lambda_x^i,\tag{4.4}$$

and they satisfy the semi-Hamiltonian condition (2.2). As can easily be understood, in the case of a reduction the corresponding function  $\lambda$  depends on the variables x and t only through  $\lambda^1, \ldots, \lambda^n$ , that is

$$\lambda(p, x, t) = \lambda(p, \lambda^1(x, t), \dots, \lambda^n(x, t)).$$

In this case, and assuming the linear independence of the  $\lambda_x^i$ , (4.2) is equivalent to the system

$$\frac{\partial \lambda}{\partial \lambda^j} = \frac{\frac{\partial A^0}{\partial \lambda^j}}{p - v^j} \frac{\partial \lambda}{\partial p}, \qquad j = 1, \dots, n,$$
(4.5)

which is a system of *n* Loewner equations, which describe – for instance – families of conformal maps from the upper complex half plane to the upper complex half plane with *n* arbitrary slits [14]. The analytic properties of the conformal map solutions of (4.5) are intimately related to the properties of the corresponding reduction. For example, the critical points of  $\lambda$  are the characteristic velocities  $v^i$  and its critical values are Riemann invariants:

$$\frac{\partial \lambda}{\partial p}(v^i) = 0, \qquad \lambda(v^i) = \lambda^i.$$

Moreover, the coefficients of the expansion at  $\lambda = \infty$  of the functions

$$W_i(\lambda, \lambda^1, \dots, \lambda^n) = \frac{1}{p(\lambda) - v^i} = \sum_{n=1}^{\infty} \frac{w_{(n)}^i}{\lambda^n}$$
(4.6)

are characteristic velocities of symmetries. Finally, as proved by the present authors in [12], reductions of the Benney system associated with the function  $\lambda(p, \lambda^1, \dots, \lambda^n)$  are Hamiltonian with respect to the Hamiltonian structures

$$P^{ij} = \varphi_i(\lambda^i) \,\lambda^{\prime\prime}(v^i) \delta^{ij} \frac{d}{dx} + \Gamma_k^{ij} \,\lambda_x^k - \frac{1}{2\pi i} \sum_{l=1}^n \int_{C_l} w^i(\lambda) \lambda_x^i \left(\frac{d}{dx}\right)^{-1} w^j(\lambda) \lambda_x^j \,\varphi_l(\lambda) \,d\lambda \tag{4.7}$$

where the contour  $C_i$  is the image of a sufficiently small closed contour around the point  $p = v^i$  in the *p*-plane with respect to the analytic continuation of the conformal map  $\lambda(p)$ , the functions  $\varphi_i$  are arbitrary functions of  $\lambda$ , and

$$w^i(\lambda) := -\frac{\frac{\partial p}{\partial \lambda}}{(p(\lambda) - v^i)^2} = \frac{\partial W_i}{\partial \lambda}$$

As all reductions of the Benney chain are semi-Hamiltonian systems, in addition to the Hamiltonian structures (4.7) we can obtain a family of purely nonlocal Hamiltonian structures if we can expand the contravariant components of the metrics

$$g_{\varphi}^{ii} = \varphi_i(\lambda^i)\lambda^{''}(v^i), \tag{4.8}$$

in terms of symmetries of the system.

**Theorem 2** The components of a metric associated with a reduction of the Benney chain admit the following quadratic expansion

$$g_{\varphi}^{ii}\delta^{ij} = \varphi_i(\lambda^i)\lambda''(v^i)\delta^{ij} = \frac{1}{2\pi i}\sum_{k=1}^n \int_{C_k} W_i(\lambda)W_j(\lambda)\varphi_k(\lambda)\,d\lambda,\tag{4.9}$$

where the  $W_i(\lambda)$  are the generating functions of the symmetries (4.6) and the contours  $C_k$  are the same as in (4.7).

*Proof* The proof is a straightforward computation of the integral:

$$\frac{1}{2\pi i} \sum_{k=1}^{n} \int_{C_{k}} W_{i}(\lambda) W_{j}(\lambda) \varphi_{k}(\lambda) d\lambda = \sum_{k=1}^{n} \operatorname{Res}_{\lambda = \lambda^{k}} \left[ \frac{\varphi_{k}(\lambda) d\lambda}{(p(\lambda) - v^{i})(p(\lambda) - v^{j})} \right]$$
$$= \sum_{k=1}^{n} \operatorname{Res}_{p = v^{k}} \left[ \frac{\frac{\partial \lambda}{\partial p}}{(p - v^{i})(p - v^{j})} \varphi_{k}(\lambda(p)) dp \right] = \varphi_{i}(\lambda^{i}) \lambda''(v^{i}) \delta^{ij},$$

the last step being due to the fact that  $p = v^k$  are critical points of  $\lambda$ , so that the differential turns out to be regular at all these points for  $i \neq j$ , and also for i = j and  $k \neq i$ .

**Remark 5** In the Benney case, it is known [12] that the metric associated with a reduction are of Egorov type, and more precisely of the form

$$(g_{\varphi})_{ii} = \frac{1}{\varphi_i(\lambda^i)}.$$

Moreover, the function  $p(\lambda)$  satisfies a Loewner system of the form

$$\partial_i p = -\frac{\partial_i A^0}{p(\lambda) - v^i} = -W^i(\lambda) \,\partial_i A^0,$$

obtained by (4.5) by using the implicit function theorem. Using Remark 2 about the Egorov metrics, we conclude that the covariant metrics associated with a reduction of the Benney chain can be written as

$$(g_{\varphi})_{ii} \,\delta_{ij} = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{C_k} \frac{\partial_i p \partial_j p}{\varphi_k(\lambda)} \,d\lambda. \tag{4.10}$$

## 5 Semi-Hamiltonian systems related to semisimple Frobenius manifolds

We have seen that these purely nonlocal Hamiltonian structures are connected with a geometrical structure where the tangent space of a manifold has the structure of an associative multiplicative algebra. The most important examples of these are *Frobenius manifolds*. A Frobenius manifold [4, 5] is a manifold M endowed with a commutative, associative multiplicative structure  $\cdot$  on the tangent spaces together with a flat metric  $\eta$ , invariant with respect to the product  $\cdot$ . This means that the third order tensor c defined by

$$c(u, v, w) = (u \cdot v, w)$$

(where u, v, w are arbitrary vector fields and (, ) is the scalar product defined by  $\eta$ ) is symmetric.

It is easy to check that this condition, combined with requiring the symmetry of the fourth order tensor

$$\nabla_z c(u, v, w)$$

implies that, in flat coordinates  $v^1, \ldots, v^n$ , the structure constants of  $\cdot$  can be written (locally) as third derivatives of a function F, called the *Frobenius potential*:

$$c_{\alpha\beta\gamma} = \eta_{\alpha\delta} c^{\delta}_{\beta\gamma} = \frac{\partial^3 F}{\partial v^{\alpha} \partial v^{\beta} \partial v^{\gamma}}.$$

The definition of a Frobenius manifold also involves two special vector fields: the first, usually denoted by e, is the unit of the product  $\cdot$  and can be identified with the vector field  $\frac{\partial}{\partial v^1}$ ; the second, called the *Euler vector field*, encodes the quasi-homegeneity properties of the Frobenius potential F:

$$Lie_E(F) = (3-d)F,$$
 (5.1)

where d is a constant. In flat coordinates E is a linear vector field and the condition (5.1) becomes

$$\sum_{\alpha} \left( d_{\alpha} t^{\alpha} + r^{\alpha} \right) \frac{\partial F}{\partial v^{\alpha}} = (3 - d)F.$$

where  $r^{\alpha}$  is a constant, non-vanishing only if  $d_{\alpha} = 0$ .

Any Frobenius manifold possesses a second flat metric defined, in flat coordinates for the first metric, by the formula

$$g^{\alpha\beta} = E^{\epsilon}c^{\alpha\beta}_{\epsilon}.$$

Using the Dubrovin-Novikov results, starting from the two flat metrics  $\eta$  and g one can define the following pair of Hamiltonian structures of hydrodynamic type:

$$P_1^{\alpha\beta} = \eta^{\alpha\beta}\partial_x \tag{5.2}$$

$$P_2^{\alpha\beta} = g^{\alpha\beta}\partial_x + \Gamma_{\gamma}^{\alpha\beta}u_x^{\gamma} = E^{\epsilon}c_{\epsilon}^{\alpha\beta}\partial_x + \left(\frac{d-1}{2} + d_{\beta}\right)c_{\gamma}^{\alpha\beta}.$$
 (5.3)

It turns out [4] that  $P_1$  and  $P_2$  are compatible and therefore define a bi-Hamiltonian hierarchy of hydrodynamic type. According to well-known results (see for instance [2]), the Hamiltonian densities of such a hierarchy can be taken as the coefficients of the expansion at  $\lambda = \infty$ 

$$c^{\alpha}(x,\lambda) = c_1^{\alpha}(x) + \frac{c_0^{\alpha}(x)}{\lambda} + \frac{c_1^{\alpha}(x)}{\lambda^2} + \dots \qquad \lambda \to \infty,$$
(5.4)

of the Casimirs of the pencil

 $P_2 - \lambda P_1.$ 

Since the Casimirs of a Hamiltonian structure of hydrodynamic type coincide with the flat coordinates of the corresponding metric, it follows that the Casimirs (5.4) are given by the flat coordinates of the pencil of metrics

$$g - \lambda \eta,$$
 (5.5)

and thus satisfy the Gauss-Manin system:

$$(\nabla^* - \lambda \nabla) dc^{\alpha} = 0. \tag{5.6}$$

Here  $\nabla^*$  is the Levi-Civita connection for the metric g, and  $\nabla$  is the Levi-Civita connection for the metric  $\eta$ .

In this way, given a Frobenius manifold, it is possible to define a bi-Hamiltonian hierarchy of hydrodynamic type. In flat coordinates for the metric  $\eta$  the equations of such a hierarchy read

$$v_{t_{\alpha,k}}^{\beta} = \eta^{\beta\gamma} \partial_x \frac{\delta H_{\alpha,k}}{\delta v^{\gamma}}, \qquad \alpha, \beta = 1, \dots, n, \quad k = -1, 0, 1, \dots$$
(5.7)

where

$$H_{\alpha,k} = \int c_{\alpha,k} \, dx.$$

We now focus our attention on a special class of Frobenius manifolds.

A Frobenius manifold M is called *semisimple* [4] if at a generic point  $v \in M$  the Frobenius algebra  $T_v M$  is semisimple. The canonical coordinates  $(u^1, \ldots, u^n)$ , whose existence is not an additional assumption but follows from the general properties of these manifolds, can be obtained as solution of the equation

$$\det(g - \lambda \eta) = 0.$$

It turns out that, in canonical coordinates, the metrics  $\eta$  and g are diagonal and that the metric  $\eta$  is of Egorov type. Moreover such canonical coordinates are Riemann invariants of the hierarchy (5.7).

Given a semisimple Frobenius manifold with d < 1 it is possible to define a function  $\lambda(p, u^1, \ldots, u^n)$  called its *superpotential*, having the following properties (for details see [4, 5, 9]):

- it is defined as the inverse of a special solution of the Gauss-Manin system (5.6).
- its critical values are the canonical coordinates.
- the covariant components of the metric  $\eta$  in canonical coordinates can be written as

$$\eta_{ij} = -\sum_{i=1}^{n} \operatorname{res}_{p=p_i} \frac{\partial_i \lambda \partial_j \lambda}{\lambda_p} \, dp = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\partial_i \lambda \partial_j \lambda}{\lambda_p} \, dp \tag{5.8}$$

where  $\Gamma$  are "small" closed contours around the critical points  $p_1, \ldots, p_n$  of  $\lambda$ .

Using these results it is easy to prove the following theorem

**Theorem 3** Let M be a semisimple Frobenius manifold with d < 1 and let  $\lambda(p, u^1, ..., u^n)$  be its superpotential, then the covariant and contravariant components of the metric  $\eta$  in

canonical coordinates admit the following quadratic expansions

$$\eta_{ij} = \frac{1}{2\pi i} \int_C \partial_i p(\lambda, u^1, \dots, u^n) \partial_j p(\lambda, u^1, \dots, u^n) d\lambda$$
$$\eta^{ij} = \frac{1}{2\pi i} \int_C W_i(\lambda, u^1, \dots, u^n) W_j(\lambda, u^1, \dots, u^n) d\lambda$$

where C are the images of the contour  $\Gamma$  in the  $\lambda$  plane and the functions  $W^i(\lambda, u^1, \ldots, u^n)$ , defined by:

$$W^{i}(\lambda, u^{1}, \dots, u^{n}) = \frac{\partial_{i} p(\lambda, u^{1}, \dots, u^{n})}{\eta_{ii}}$$

are generating functions of the symmetries of the semi-Hamiltonian hierarchy associated with M.

*Proof.* The first quadratic expansion can be obtained just by changing the variable  $p \rightarrow \lambda$  in the integral (5.8). Raising the indices we get the second quadratic expansion. In order to prove that the functions  $W^i$  are generating functions of symmetries it is sufficient to observe that the metric  $\eta_{ii}$  is of Egorov type and that the inverse of the superpotential is a generating function of Hamiltonian densities.

**Remark 6** Starting from a Frobenius manifold one can define a hierarchy of integrable PDEs also in the following way. Let  $\nabla$  be the Levi Civita connection associated with the metric  $\eta$  and  $(X_{(\alpha,0)}, \alpha = 1, ..., n)$  be a basis of covariantly constant vector fields. One can define the primary flows of the hierarchy as

$$u_{t_{(\alpha,0)}}^i = c_{jk}^i X_{(\alpha,0)}^k u_x^j, \ i = 1, \dots, n.$$

and the "higher flows"

$$u_{t_{(\alpha,n)}}^{i} = c_{jk}^{i} X_{(\alpha,n)}^{k} u_{x}^{j}, \quad i = 1, \dots, n,$$
(5.9)

recursively, by means of the relations

$$\nabla_i X^k_{(\alpha,n)} = c^i_{kl} X^k_{(\alpha,n-1)}.$$

The hierarchy defined in this way is usually called the principal hierachy. It is equivalent to the hierachy defined above in terms of coefficients of the Casimirs of the pencil (5.5) since the flows (5.9) are related to the flows (5.7) just by triangular linear transformations (see [4, 9] for details).

This shows that in the case of hierarchies of quasilinear PDEs associated with a Frobenius manifold the "factorization" (3.4) has a natural interpretation: the structure constants coincide with the structure constants defining the Frobenius structure, and the vector fields X have a precise geometrical meaning.

### 6 The classical shallow water equations

Let us consider the classical shallow water system, given by

$$h_t = (hu)_x,$$

$$u_t = uu_x + h_x.$$
(6.1)

A related problem was solved by Riemann, [21], using the hodograph transformation.

This system can be seen as the elementary 2–component reduction of the Benney chain associated with the rational map [1, 24]:

$$\lambda = p + \frac{h}{p-u},\tag{6.2}$$

Moreover, (6.1) is an element of a bi-Hamiltonian hierarchy associated with a 2 dimensional Frobenius manifold, with Frobenius potential

$$F(h, u) = \frac{1}{2}hu^2 + h\log h,$$

in this setting, the function (6.2) is the superpotential. Let us recall that under the change of coordinates

$$r^1 = u - 2\sqrt{h},$$
  
$$r^2 = u + 2\sqrt{h},$$

the system takes the diagonal form

$$r_t^1 = \frac{1}{4} (3r^1 + r^2) r_x^1$$
  
$$r_t^2 = \frac{1}{4} (r^1 + 3r^2) r_x^2.$$

The general solution of the linear system (2.5), in this case, is

$$g_{ii} = \varphi_i(r^i)\partial_i A^0, \quad i = 1, 2, \tag{6.3}$$

where  $\varphi_i(r^i)$  are arbitrary functions of a single variable and

$$A_0 = \frac{(r^1 - r^2)^2}{16}$$

We will show now that the quadratic expansion of the contravariant components of the metrics (6.3) can be reduced to a finite sum, so that we can construct families of purely nonlocal Poisson brackets involving only a finite number of symmetries. We proceed as follows: first, we extend  $\lambda(p)$  from the upper half plane to a rational function on the whole Riemann sphere. Then, we note that although the extended  $\lambda$  is univalent, its inverse with respect to p is not, and we have to consider two functions  $p_+(\lambda)$  and  $p_-(\lambda)$  each of them defined on one sheet of a double covering of the Riemann sphere, with branch points at  $r^1$  and  $r^2$ . The two functions are easily found to be

$$p_{+}(\lambda) = \frac{1}{2}\lambda + \frac{1}{4}(r_{1} + r_{2}) + \frac{1}{2}\sqrt{(r_{2} - \lambda)(r_{1} - \lambda)},$$
$$p_{-}(\lambda) = \frac{1}{2}\lambda + \frac{1}{4}(r_{1} + r_{2}) - \frac{1}{2}\sqrt{(r_{2} - \lambda)(r_{1} - \lambda)},$$

their main difference being in the behaviour at infinity, for:

$$p_+(\lambda) = \lambda + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty_+,$$

while

$$p_{-}(\lambda) = \frac{u}{2} + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty_{-}$$

Thus, we can construct two generating functions of the symmetries, the first is given by

$$w^i(\lambda) = \frac{1}{p_+(\lambda) - v^i},$$

whose expansion at infinity is

$$w^{i}(\lambda) = \sum_{n=1}^{\infty} \frac{w_{n}^{i}}{\lambda^{n}},$$
(6.4)

and where the first few coefficients are given by

$$w_1^1 = 1, \qquad w_1^2 = 1,$$
  

$$w_2^1 = \frac{3}{4}r^1 + \frac{1}{4}r^2, \qquad w_2^2 = \frac{1}{4}r^1 + \frac{3}{4}r^2,$$
  

$$w_3^1 = \frac{5}{8}(r^1)^2 + \frac{1}{4}r^1r^2 + \frac{1}{8}(r^2)^2, \qquad w_3^2 = \frac{1}{8}(r^1)^2 + \frac{1}{4}r^1r^2 + \frac{5}{8}(r^2)^2$$

For the second generating function, an easy calculation shows that the analogous generating function constructed from  $p_{-}(\lambda)$  is related with the first by

$$\frac{1}{p_{-}(\lambda) - v^{i}} = w_{0}^{i} - \frac{1}{p_{+}(\lambda) - v^{i}}, \qquad i = 1, 2,$$

where

$$w_0^1 = -\frac{4}{r^1 - r^2}, \qquad w_0^2 = \frac{4}{r^1 - r^2},$$

is an extra symmetry, not appearing in the expansion (6.4).

Reducing the integral (4.10) to the sum of residues around the two points at infinity,  $\infty_+, \infty_-$ , we obtain a finite quadratic expansion of the components of the metric tensor in terms of symmetries:

$$g_{(k)}^{ii} \delta^{ij} = \frac{(r^i)^k \delta_{ij}}{\partial_i A^0} = -\operatorname{Res}_{\lambda = \infty_+} \frac{\lambda^k d\lambda}{(v^i - p_+(\lambda))(v^j - p_+(\lambda))} - \operatorname{Res}_{\lambda = \infty_-} \frac{\lambda^k d\lambda}{(v^i - p_-(\lambda))(v^j - p_-(\lambda))}$$
$$= \operatorname{Res}_{\lambda = \infty_+} \frac{w_0^i \lambda^k d\lambda}{p_+(\lambda) - v^j} + \operatorname{Res}_{\lambda = \infty_+} \frac{w_0^j \lambda^k d\lambda}{p_+(\lambda) - v^i} - 2 \operatorname{Res}_{\lambda = \infty_+} \frac{\lambda^k d\lambda}{(v^i - p_+(\lambda))(v^j - p_+(\lambda))}$$
$$= w_0^i w_{k+1}^j + w_{k+1}^i w_0^j - 2 \sum_{s=1}^k w_s^i w_{k-s+1}^j.$$

Therefore, for k = 0, 1, ..., the corresponding purely nonlocal Poisson operators have the form

$$P_{(k)}^{ij} = w_0^i r_x^i \left(\frac{d}{dx}\right)^{-1} w_{k+1}^j r_x^j + w_{k+1}^i r_x^i \left(\frac{d}{dx}\right)^{-1} w_0^j r_x^j - 2 \sum_{s=1}^k \left(w_s^i r_x^i \left(\frac{d}{dx}\right)^{-1} w_{k-s+1}^j r_x^j\right).$$

We consider now the flows generated by the simplest of these structures, namely  $P_{(0)}$ . As Hamiltonian density we consider the generating function  $p_+(\lambda)$ ; the quantities we want to evaluate are thus the flows

$$\mu^{i}(\lambda) := \sum_{j=1}^{2} P^{ij}_{(0)} \frac{\partial p_{+}(\lambda)}{\partial \lambda^{j}}.$$

Explicitly, and assuming vanishing boundary conditions

$$\lim_{|x|\to\infty}r^i(x,t)=0$$

these are found to be

$$\mu^{1}(\lambda) = -4 \frac{p_{+}(\lambda) - \lambda}{r^{1} - r^{2}} - 2 \ln\left(\sqrt{\lambda - r_{1}} + \sqrt{\lambda - r_{2}}\right) + \ln 4\lambda,$$
  
$$\mu^{2}(\lambda) = 4 \frac{p_{+}(\lambda) - \lambda}{r^{1} - r^{2}} - 2 \ln\left(\sqrt{\lambda - r_{1}} + \sqrt{\lambda - r_{2}}\right) + \ln 4\lambda.$$

By comparing the coefficients of the asymptotic expansions at infinity of  $\mu^i(\lambda)$  and  $p_+(\lambda)$ we obtain, for instance, that the characteristic velocities  $w_2$  of the systems are generated by the Hamiltonian density  $H^0$ , while the symmetry  $w_3$  is obtained from  $H^1$ . For the shallow water hierarchy, there exist another Poisson structure which sends the Hamiltonian density  $H^0$  to the system with characteristic velocities  $w_2$ . This is the third *local* Poisson structure of the system, generated by the flat metric  $g_{ii} = \frac{2}{(\lambda^i)^2} \partial_i A^0$ . We denote this structure as  $P_{loc}$ , and we call  $z^i(\lambda)$  the corresponding flows, so that

$$z^{i}(\lambda) = \sum_{j=1}^{2} P_{loc}^{ij} \frac{\partial p_{+}(\lambda)}{\partial \lambda^{j}}.$$

With little difficulty, it can be shown that the two Hamiltonian hierarchies  $z^i(\lambda)$  and  $\mu^i(\lambda)$  are related by

$$z^{i}(\lambda) = \frac{\lambda^{2}}{2} \frac{d^{2}\mu^{i}(\lambda)}{d \lambda^{2}},$$

so that the coefficients of the expansion at infinity are related by

$$z_k^i = \frac{k(k+1)}{2}\,\mu_k^i.$$

Moreover, the original generating function of the symmetries  $w^i(\lambda)$  can written in terms of  $\mu^i(\lambda)$  as

$$w^i(\lambda) = \frac{1}{\lambda} - \frac{d\,\mu^i(\lambda)}{d\,\lambda},$$

so that the coefficients satisfy the relation

$$w_{k+1}^i = -(k+1)\,\mu_k^i.$$

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