Eur. Phys. J. C (2025) 85:336
https://doi.org/10.1140/epjc/s10052-025-14010-6

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Rotating Casimir wormholes

Remo Garattini[1,2,a], **Athanasios G. Tzikas**[1,b]

[1] Dipartimento di Ingegneria e Scienze Applicate, Università degli Studi di Bergamo, Viale Marconi 5, 24044 Dalmine, Bergamo, Italy
[2] I.N.F.N.-sezione di Milano, Milan, Italy

Received: 26 November 2024 / Accepted: 27 February 2025

Abstract A Casimir wormhole is a traversable wormhole powered by a Casimir energy source within a static reference frame. A natural extension of this system is the inclusion of rotation. We will explore two basic configurations: one with radially varying Casimir plates and another with parametrically fixed plates. In both cases, we will show that rotations do not alter the structure of a Casimir wormhole, and the behavior observed in a static frame is reaffirmed. Since the case with radially varying plates predicts a constant angular velocity as a solution, we must introduce an exponential cutoff and an additional scale to prevent rotations at infinity. This adjustment is not necessary when the plates are kept parametrically fixed. Moreover, the consistency of the Einstein field equations is ensured with the help of an additional source without an accompanying energy density, which we interpret as a thermal stress tensor.

1 Introduction

A Casimir wormhole (CW) [1] is a solution of the Einstein field equations (EFE) representing a traversable wormhole (TW) with a source described by a stress-energy tensor (SET) of the form

$$T_{\nu}^{\mu}|_{\mathrm{C}} = -\frac{\hbar c \pi^2}{720 d^4} \mathrm{diag}\left(-1, 3, -1, -1\right). \tag{1}$$

The distance d represents the separation between two parallel, closely spaced, uncharged metallic plates in vacuum at almost zero temperature. To obtain a CW, one has to replace the parametrically fixed distance d with a radially varying variable r. In other words, the quadruple $(\rho(r), p_r(r), p_t(r), p_t(r))$, which includes the energy density $\rho(r)$, the radial pressure $p_r(r)$, and the tangential pressure $p_t(r)$, represents the physical source as follows:

$$T_{\nu}^{\mu}|_{\mathrm{C}} = -\frac{r_1^2}{\kappa r^4} \mathrm{diag}\left(-1, 3, -1, -1\right) \quad \text{with}$$
$$r_1^2 = \frac{\pi^3}{90}\left(\frac{\hbar G}{c^3}\right) = \frac{\pi^3}{90}\ell_{\mathrm{P}}^2. \tag{2}$$

By using the conventional line element of a TW [2–4]

$$\mathrm{d}s^2 = -e^{2\Phi(r)}\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - b(r)/r} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right), \tag{3}$$

the EFE can be written as[1,2]

$$\frac{b'(r)}{r^2} = \kappa\rho(r) \tag{4}$$

$$\frac{2}{r}\left(1 - \frac{b(r)}{r}\right)\Phi'(r) - \frac{b(r)}{r^3} = \kappa p_r(r) \tag{5}$$

$$\left(1 - \frac{b(r)}{r}\right)\left[\Phi''(r) + \Phi'(r)\left(\Phi'(r) + \frac{1}{r}\right)\right] - \frac{b'(r)\,r - b(r)}{2r^2}\left(\Phi'(r) + \frac{1}{r}\right) = \kappa p_t(r) \tag{6}$$

in which $b(r)$ and $\Phi(r)$ denote the shape function and the redshift function, respectively. The EFE can be supplemented with the expression for the conservation of the SET, which

[1] Throughout the paper, the field equations are not examined in an orthonormal frame. Instead, Schwarzschild-like coordinates (t, r, θ, φ) are used for convenience in solving equations.

[2] We must observe that by assuming the form of the SET (1), a consistent calculation is obtained if we adopt the configuration considered in [3], where the plates have a spherical form. As noted in [3], this approximation introduces an error that remains small when we are very close to the throat, which is precisely the condition required in this paper. In this way, even with the inclusion of rotation, the physical setup remains essentially the same.

[a] e-mail: remo.garattini@unibg.it (corresponding author)
[b] e-mail: athanasios.tzikas@unibg.it

is written in the same reference frame as

$$p_r'(r) = \frac{2}{r}(p_t(r) - p_r(r)) - (\rho(r) + p_r(r))\,\Phi'(r). \quad (7)$$

The solution to the previous field equations with the Casimir source (2) is given by

$$\Phi(r) = \ln\left(\frac{3r}{3r + r_0}\right) \quad (8)$$

$$b(r) = \frac{2}{3}r_0 + \frac{r_0^2}{3r}. \quad (9)$$

Such a solution satisfies the following relationship between the energy density and the radial pressure

$$p_r(r) = \omega\rho(r) \quad (\omega = 3) \quad (10)$$

with

$$r_0 = \sqrt{3}\,r_1. \quad (11)$$

It is interesting to observe that a solution can also be derived from the original form of the SET represented by (1). For this purpose, we consider the following setting:

$$\Phi(r) = 0 \quad (12)$$

$$b(r) = r_0 - \frac{r_1^2}{3d^4}\left(r^3 - r_0^3\right). \quad (13)$$

As also discussed in Refs. [1,5], the shape function (13) does not represent a TW in the strict sense. However, it can be observed that there exists a value $\bar{r}$ such that

$$b(\bar{r}) = 0 \iff \bar{r} = r_0\sqrt[3]{1 + \frac{3d^4}{r_0^2 r_1^2}}. \quad (14)$$

Therefore, a TW can be defined for $\Phi(r) = 0$ in the interval $[r_0, +\infty)$ and for

$$b(r) = \begin{cases} r_0 - \frac{r_1^2}{3d^4}\left(r^3 - r_0^3\right) & r_0 \le r < \bar{r} \\ 0 & r \ge \bar{r}. \end{cases}$$

The previous profile can be justified by imposing an equation of state (EoS) of the form

$$\omega(r) = \begin{cases} \frac{d^4}{r_1^2 r^3}\left(r_0 - \frac{r_1^2}{3d^4}\left(r^3 - r_0^3\right)\right) & r_0 \le r < \bar{r} \\ 0 & r \ge \bar{r} \end{cases}$$

with $\omega(r_0) = \frac{d^4}{r_1^2 r_0^2}$. If we compare the above expression for $\omega(r)$ with

$$\frac{p_r(r)}{\rho(r)} = 3 = \omega(r), \quad (15)$$

we observe that this condition is satisfied only at the throat, where its value is

$$r_0 = \frac{\sqrt{3}d^2}{3r_1}. \quad (16)$$

Note that, this time, Eq. (16) predicts an enormously large throat size. It is interesting to observe that the solution in (16) allows for the evaluation of the location of $\bar{r}$. Indeed, substituting (16) into (14), one obtains

$$\bar{r} = \sqrt[3]{10}\,r_0. \quad (17)$$

This implies that the boundary $\bar{r}$ is located very close to the throat. Note also that the plates are inside this boundary as shown in Eq. (14). One can observe that the structure of the TW defined by (I) resembles an absurdly benign traversable wormhole (ABTW) [5], with the distinction that the energy density is non-vanishing outside the boundary $\bar{r}$. We remind the reader that an ABTW is defined by

$$b(r) = \begin{cases} r_0\left(1 - \lambda(r - r_0)\right)^2 & r_0 \le r < \bar{r} \\ 0 & r \ge \bar{r} \end{cases} \quad (18)$$

with $\Phi(r) = 0$ everywhere and $\bar{r} = r_0 + 1/\lambda$. This means that we have to assume that the energy density must be zero for $r > \bar{r}$. In other words, outside $\bar{r}$, we recover Minkowski spacetime. Such an assumption is not unusual, as the negativity of the energy density appears only between the plates and not outside of them. A comment on this point is in order: In his book, Visser [4] proposed a realistic model for the total stress-energy tensor (SET), represented by

$$T_\sigma^{\mu\nu} = \sigma\hat{t}^\mu\hat{t}^\nu\left[\delta(z) + \delta(z - a)\right] + \Theta(z)\,\Theta(a - z)\,\frac{\hbar c\pi^2}{720a^4}\left[\eta^{\mu\nu} - 4\hat{z}^\mu\hat{z}^\nu\right] \quad (19)$$

where $\hat{t}^\mu$ is a unit time-like vector, $\hat{z}^\mu$ is a normal vector to the plates, and σ is the mass density of the plates. He concluded that the mass of the plates compensates for the negative energy density to such an extent that it prevents the creation of a TW. However, this does not imply that the general structure of a TW cannot be investigated. Rather, one could explore the conditions necessary to minimize the positive effects of the mass of the plates. A more suitable approach for addressing this issue could be a thin-shell model, which is beyond the scope of this paper.

What we have introduced is valid for a static TW, but what happens when we introduce rotations? Historically, the first proposal of a rotating TW was made by Teo [6], while an interesting analysis of the properties of the SET describing such a wormhole was performed by Perez Bergliaffa and Hibberd [7]. Kuhfittig found a solution that exhibits a less severe violation of the weak energy condition (WEC) compared to the static case [8]. Kim considered scalar perturbations around a rotating TW, while an investigation of slow rotation was conducted by Kashargin and Sushkov [9,10]. In this paper, we aim to answer the following question: *Is it possible to introduce rotations in such a way that a Casimir wormhole is obtained in the non-rotating limit, even in the case of parametrically fixed plates?* To answer this question, we need to construct a spacetime that describes such a rotating wormhole. The rest of the paper is organized as follows: In Sect. 2, we examine the structure of both the rotating metric and the SET. In Sect. 3, we examine the EFE for a rotating TW with the Casimir plates positioned at a distance either parametrically fixed or radially varying. We summarize and conclude in Sect. 4. Units in which $\hbar = c = k = 1$ are used throughout the paper and will be reintroduced whenever it is necessary.

2 Setting up the rotating spacetime and stress-energy tensor

Let us introduce the following spacetime metric

$$\mathrm{d}s^2 = -e^{2\Phi(r,\theta)}\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-b(r,\theta)/r} + r^2K^2(r,\theta)\left[\mathrm{d}\theta^2 + \sin^2\theta\,\left(\mathrm{d}\varphi - \Omega(r,\theta)\mathrm{d}t\right)^2\right] \tag{20}$$

representing a stationary, axisymmetric traversable wormhole. The functions $\Phi(r,\theta)$, $b(r,\theta)$, $K(r,\theta)$, and $\Omega(r,\theta)$ are arbitrary functions of r and θ. The range of the radial coordinate is $[r_0, +\infty)$, while the angular variable satisfies $\theta \in [0,\pi]$. The function $\Omega(r,\theta)$ represents the angular velocity, and $K(r,\theta)$ is associated with the proper radial distance. In the limit of $\Omega(r,\theta) \to 0$, the line element (20) must reproduce the Morris–Thorne metric (3). We can rearrange the above line element in the following manner

$$\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + 2g_{t\varphi}\mathrm{d}t\mathrm{d}\varphi + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\varphi\varphi}\,\mathrm{d}\varphi^2 \tag{21}$$

where

$$g_{tt} = -\left(e^{2\Phi(r,\theta)} - r^2K^2(r,\theta)\sin^2\theta\,\Omega^2(r,\theta)\right) \tag{22}$$

$$g_{rr} = \frac{1}{1-b(r,\theta)/r} \tag{23}$$

$$g_{t\varphi} = -r^2K^2(r,\theta)\sin^2\theta\,\Omega(r,\theta) \tag{24}$$

$$g_{\theta\theta} = r^2K^2(r,\theta) \tag{25}$$

$$g_{\varphi\varphi} = r^2K^2(r,\theta)\sin^2\theta. \tag{26}$$

The ergoregion appears when g_{tt} vanishes, namely

$$\Phi(r,\theta) = \ln\left(rK(r,\theta)\Omega(r,\theta)\sin\theta\right). \tag{27}$$

Note also that the discriminant of the metric (20) is given by

$$-g_{tt}g_{\varphi\varphi} + g_{t\varphi}^2 = e^{2\Phi(r,\theta)}r^2K^2(r,\theta)\sin^2\theta \tag{28}$$

which implies that an event horizon appears when $e^{2\Phi(r,\theta)} = 0$. However, such a possibility can be excluded if we aim to reproduce a TW in the limit of vanishing rotation. In order to reproduce the correct behavior when the rotation stops, we assume that the redshift function and the shape function are given by (8) and (9) on one hand, and by (12) and (13) on the other.

Now that the spacetime has been introduced, we turn our attention to the anisotropic source tensor. This source includes a radial pressure p_r, a transverse pressure p_t, an energy density ρ, and a thermal stress tensor τ, all described by the following SET

$$T_{\mu\nu} = \left(\rho + \tau_\rho\right)u_\mu u_\nu + \left(p_r + \tau_r\right)n_\mu n_\nu + \left(p_t + \tau_t\right)\sigma_{\mu\nu}. \tag{29}$$

The unit time-like vector u_μ is the fluid four-velocity, and n_μ is a unit space-like vector orthogonal to u_μ, implying the relations $n^\mu n_\mu = 1$, $u^\mu u_\mu = -1$, and $n^\mu u_\mu = 0$. The thermal stress tensor arises from relativistic thermodynamic considerations, implying that the matter source possesses both mass and heat as local forms of energy [11]. Moreover, it has been decomposed into an energy component τ_ρ, a radial component τ_r, and a transverse component τ_t. Here,

$$\sigma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu \tag{30}$$

is a projection operator onto a two-surface orthogonal to u_μ and n_μ, i.e.,

$$u_\mu\sigma^{\mu\nu}\mathrm{v}_\nu = n_\mu\sigma^{\mu\nu}\mathrm{v}_\nu = 0 \quad \forall \mathrm{v}_\nu. \tag{31}$$

The vector n_μ can be written as

$$n_\mu = \sqrt{\frac{1}{1-b(r,\theta)/r}}\,(0,1,0,0). \tag{32}$$

Keeping in mind the anisotropic structure of the SET, we would like to introduce rotations. To do this, we define two Killing vectors

$$k_t^\alpha = \delta_t^\alpha \quad \text{and} \quad k_\varphi^\alpha = \delta_\varphi^\alpha \tag{33}$$

whose linear combination allows us to build the four-velocity of the fluid source

$$u^\mu = \left(u^t, 0, 0, u^\varphi\right) = u^t\,(1, 0, 0, \Omega_0) \tag{34}$$

with Ω_0 being the angular velocity measured by a distant observer, defined by

$$\Omega_0 = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{u^\varphi}{u^t}. \tag{35}$$

Since u^μ is a time-like vector, it must satisfy

$$u^\mu u_\mu = \left(u^t\right)^2\left[g_{tt} + 2g_{t\varphi}\Omega_0 + \Omega_0^2 g_{\varphi\varphi}\right] = -1 \tag{36}$$

with

$$g_{tt} + 2g_{t\varphi}\Omega_0 + \Omega_0^2 g_{\varphi\varphi} < 0. \tag{37}$$

In terms of the metric (20), one finds that (36) becomes

$$\begin{aligned}\left(u^t\right)^2 &= \frac{-1}{\left[g_{tt} + 2g_{t\varphi}\Omega_0 + \Omega_0^2 g_{\varphi\varphi}\right]}\\ &= \frac{1}{e^{2\Phi(r,\theta)} - r^2K^2(r,\theta)\sin^2\theta\left(\Omega(r,\theta) - \Omega_0^2\right)^2}\end{aligned} \tag{38}$$

while the inequality (37) becomes

$$\frac{e^{2\Phi(r,\theta)}}{r^2K^2(r,\theta)\sin^2\theta} > \left(\Omega(r,\theta) - \Omega_0^2\right)^2. \tag{39}$$

This means that

$$\Omega_- < \Omega(r,\theta) < \Omega_+ \tag{40}$$

with

$$\Omega^\pm = \Omega_0 \pm \frac{e^{\Phi(r,\theta)}}{rK(r,\theta)\sin\theta}. \tag{41}$$

It is useful to rearrange u^μ in the following manner

$$u^\mu = \frac{e^{-\Phi(r,\theta)}}{\left(1 - v^2\right)^{\frac{1}{2}}}\,(1, 0, 0, \Omega_0) \tag{42}$$

where

$$v = rK(r,\theta)\sin\theta\,(\Omega(r,\theta) - \Omega_0)\,e^{-\Phi(r,\theta)} \tag{43}$$

is the proper velocity of the matter with respect to a zero angular momentum observer. With the help of the line element (20), we can compute the covariant form of u_μ, which is given by

$$\begin{aligned}u_t &= \frac{e^{-\Phi(r,\theta)}}{\left(1 - v^2\right)^{\frac{1}{2}}}\left(g_{tt} + \Omega_0 g_{t\varphi}\right)\\ u_\varphi &= \frac{e^{-\Phi(r,\theta)}}{\left(1 - v^2\right)^{\frac{1}{2}}}\left(g_{t\varphi} + \Omega_0 g_{\varphi\varphi}\right).\end{aligned}$$

With this information, we can write the components of the SET (29) as follows:

$$\begin{aligned}T_{tt} &= \left(\rho + \tau_\rho\right)u_tu_t + (p_t + \tau_t)\,(g_{tt} + u_tu_t)\\ &= \left(\rho + \tau_\rho + p_t + \tau_t\right)u_tu_t + (p_t + \tau_t)\,g_{tt} &(44)\\ T_{rr} &= (p_r + \tau_r)\,n_rn_r = (p_r + \tau_r)\,g_{rr} &(45)\\ T_{\theta\theta} &= (p_t + \tau_t)\,\sigma_{\theta\theta} = (p_t + \tau_t)\,g_{\theta\theta} &(46)\\ T_{\varphi\varphi} &= \left(\rho + \tau_\rho\right)u_\varphi u_\varphi + (p_t + \tau_t)\,g_{\varphi\varphi} + (p_t + \tau_t)\,u_\varphi u_\varphi\\ &= \left(\rho + \tau_\rho + p_t + \tau_t\right)u_\varphi u_\varphi + (p_t + \tau_t)\,g_{\varphi\varphi} &(47)\\ T_{t\varphi} &= T_{\varphi t} = \left(\rho + \tau_\rho\right)u_tu_\varphi + (p_t + \tau_t)\,\sigma_{t\varphi}\\ &= \left(\rho + \tau_\rho + p_t + \tau_t\right)u_tu_\varphi + (p_t + \tau_t)\,g_{t\varphi}. &(48)\end{aligned}$$

For the specific case under investigation, the property $\rho + p_t = 0$ holds. Therefore, the previous components (44)–(48) can be rearranged as

$$\begin{aligned}T_{tt} &= \left(\tau_\rho + \tau_t\right)u_tu_t + (p_t + \tau_t)\,g_{tt} &(49)\\ T_{rr} &= (p_r + \tau_r)\,g_{rr} &(50)\\ T_{\theta\theta} &= (p_t + \tau_t)\,g_{\theta\theta} &(51)\\ T_{\varphi\varphi} &= \left(\tau_\rho + \tau_t\right)u_\varphi u_\varphi + (p_t + \tau_t)\,g_{\varphi\varphi} &(52)\\ T_{t\varphi} &= T_{\varphi t} = \left(\tau_\rho + \tau_t\right)u_tu_\varphi + (p_t + \tau_t)\,g_{t\varphi\varphi} &(53)\end{aligned}$$

where τ_ρ, τ_t, and τ_r need to be determined. It is straightforward to see that T_{rr} and $T_{\theta\theta}$ are not affected by the rotation. It is also convenient to introduce a particular reference frame, called ZAMO (zero-angular-momentum observer), in which $u_\varphi = u^tg_{t\varphi} + u^\varphi g_{\varphi\varphi} = 0$. This means that

$$\Omega_0 = \frac{u^\varphi}{u^t} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \Omega(r,\theta) \quad\Longrightarrow\quad v = 0. \tag{54}$$

Moreover, we find that

$$u_t = \frac{e^{-\Phi(r,\theta)}}{\left(1 - v^2\right)^{\frac{1}{2}}}\left(g_{tt} + \Omega_0 g_{t\varphi}\right) = -e^{\Phi(r,\theta)}, \tag{55}$$

as expected for a ZAMO. With these ingredients, the previous SET can be written as

$$T_{tt} = p_t g_{tt} + \tau_t \left(g_{tt} + e^{2\Phi(r,\theta)}\right) = p_t g_{tt} + \tau_t g_{\varphi\varphi}\Omega^2(r,\theta) \tag{56}$$

$$T_{rr} = p_r n_r n_r = p_r g_{rr} \tag{57}$$

$$T_{\theta\theta} = p_t \sigma_{\theta\theta} = p_t g_{\theta\theta} \tag{58}$$

$$T_{\varphi\varphi} = p_t g_{\varphi\varphi} \tag{59}$$

$$T_{t\varphi} = T_{\varphi t} = p_t g_{t\varphi} = -p_t \Omega(r,\theta) g_{\varphi\varphi}. \tag{60}$$

3 Einstein field equations for a rotating traversable wormhole

One crucial aspect of rotations is the emergence of a constraint related to the field equation $G_{r\theta}$, which must vanish [7,8]. This implies that some restrictions are necessary. For this purpose, we assume that

$$b(r,\theta) \to b(r) \tag{61}$$

$$K(r,\theta) \to 1 \tag{62}$$

$$\Phi(r,\theta) \to \Phi(r). \tag{63}$$

With this choice, the aforementioned constraint equation reduces to

$$G_{r\theta} = \frac{r^2}{2}\sin^2(\theta)\, e^{-2\Phi(r)} \frac{\partial\Omega(r,\theta)}{\partial\theta}\frac{\partial\Omega(r,\theta)}{\partial r} = 0, \tag{64}$$

and it can be satisfied if

$$\Omega(r,\theta) \to \Omega(r) \tag{65}$$

or

$$\Omega(r,\theta) \to \Omega(\theta). \tag{66}$$

In this paper, we will adopt the choice (65). The restriction of $b(r,\theta)$ to $b(r)$ is also dictated by the presence of a singularity in the scalar curvature R, which is proportional to

$$R \sim \frac{\partial_\theta b(r,\theta)}{(r-r_0)^2}. \tag{67}$$

This singularity can be eliminated by imposing the condition that $\partial_\theta b(r,\theta) = 0$ [6]. We can observe that the SET components (50)–(52) are not affected by the rotation. In the next subsection, we will examine the rotating structure of a Casimir wormhole. The components of the SET (49)–(53) suggest considering firstly those equations that are independent of the rotation. We will consider two models regarding the Casimir plates:

(a) Rotating Casimir wormhole with radially varying plates
(b) Rotating Casimir wormhole with fixed plates.

We begin by considering the case (a).

3.1 Rotating Casimir wormhole with radially varying plates

We begin by considering the EFE that does not include angular velocity in the SET. To this end, the first equation we will examine is $G_{rr} = \kappa T_{rr}$, which is described by

$$\frac{r-b(r)}{4r^2}\left(r^3\left(\Omega'(r)\right)^2 \sin^2\theta e^{-2\Phi(r)} + 8\Phi'(r)\right) - \frac{b(r)}{r^3} + \frac{3r_1^2}{r^4} - \kappa\tau_r(r) = 0. \tag{68}$$

At the throat, the previous equation reduces to

$$-r_0^2 + 3r_1^2 = r_0^4 \kappa \tau_r(r_0) \tag{69}$$

upon using the throat condition $b(r_0) = r_0$. It can be immediately seen that a solution is

$$r_0 = \sqrt{3} r_1 \quad \text{and} \quad \tau_r(r_0) = 0, \tag{70}$$

which is characteristic of a Casimir wormhole. To extend the solution outside the throat, we substitute (8) and (9) into the EFE (68) to obtain

$$\frac{\sin^2(\theta)\, r^2 (r-r_0)(3r+r_0)^3 \left(\Omega'(r)\right)^2}{108\kappa} = \tau_r(r). \tag{71}$$

Of course, in the vicinity of the throat, the left-hand side vanishes, and a solution emerges if we assume that $\tau_r(r)$ also vanishes. However, in this approximation, $\Omega(r)$ remains undetermined. Another solution with $\tau_r(r) = 0$ arises when $\Omega(r) = \Omega$, with Ω being constant. This time, the solution is valid for all $r \in [r_0, +\infty)$.

The next equation we will consider is the EFE $G_{\theta\theta} = \kappa T_{\theta\theta}$, which can be reduced to

$$\frac{(9r+r_0)\,r_0^2}{3r^4\,(3r+r_0)} = \kappa\tau_t\,(r) + \frac{r_1^2}{r^4}. \tag{72}$$

To obtain (72), we have plugged (8), (9), and $\Omega\,(r) = \Omega$ into (A3), and we have also used the relationship $r_0 = \sqrt{3}r_1$. Isolating $\tau_t\,(r)$, one finds

$$\tau_t\,(r) = \frac{2r_0^2}{\kappa r^3\,(3r+r_0)}. \tag{73}$$

The next equation we are about to examine is the EFE $G_{\phi\phi} = \kappa T_{\phi\phi}$, which can be reduced to

$$r_0^2\,(9r+r_0) = -\frac{\left((\Omega-\Omega_0)^2\left(3\tau_\rho\,(r)\,\kappa\,r^4 - r_0^2\right)(3r+r_0)^3\sin^2(\theta) + 9r_0^2\,(9r+r_0)\right)}{\left((\Omega-\Omega_0)\,(3r+r_0)\sin(\theta)\right)^2 - 9}. \tag{74}$$

To obtain (74), we have plugged (8), (9), (73), $\Omega\,(r) = \Omega$, and $r_0 = \sqrt{3}r_1$ into (A4). A solution of this equation is

$$\tau_\rho\,(r) = -\frac{2r_0^2}{\kappa r^3\,(3r+r_0)}. \tag{75}$$

Alternatively, one can observe that if we assume

$$\tau_\rho\,(r) = \frac{r_0^2}{3\kappa r^4}, \tag{76}$$

Eq. (74) becomes

$$(\Omega-\Omega_0)\,(3r+r_0)\sin(\theta)^2 = 0 \tag{77}$$

and is satisfied only if $\Omega = \Omega_0$, that is, in a ZAMO frame. The last two EFE that need to be examined are $G_{tt} = \kappa T_{tt}$ and $G_{t\phi} = \kappa T_{t\phi}$. We begin with $G_{t\phi} = \kappa T_{t\phi}$. From (A5) and with the help of (8), (9), (73), $\Omega\,(r) = \Omega$, and $r_0 = \sqrt{3}r_1$, one can write

$$-A\,(\Omega,r)\,\tau_\rho\,(r) - B\,(\Omega,r) = 0, \tag{78}$$

where

$$A\,(\Omega,r) = \frac{\left(-1+(\Omega-\Omega_0)\,\Omega\left(r+\frac{r_0}{3}\right)^2\left(\sin^2(\theta)\right)\right)r^2\,(\Omega-\Omega_0)\left(\sin^2(\theta)\right)}{\left((\Omega-\Omega_0)\left(r+\frac{r_0}{3}\right)\sin(\theta)\right)^2 - 1} \tag{79}$$

$$B\,(\Omega,r) = \frac{2\left(\sin^2(\theta)\right)\left(-1+(\Omega-\Omega_0)\,\Omega\left(r+\frac{r_0}{3}\right)^2\left(\sin^2(\theta)\right)\right)(\Omega-\Omega_0)\,r_0^2}{3r\left(\left((\Omega-\Omega_0)\left(r+\frac{r_0}{3}\right)\sin(\theta)\right)^2 - 1\right)\left(r+\frac{r_0}{3}\right)\kappa}. \tag{80}$$

Solving for $\tau_\rho\,(r)$, one finds (75). This implies that the solution (76) can be discarded. Finally, we must examine the EFE $G_{tt} = \kappa T_{tt}$, as described by (A1). Even in this last EFE, with the help of (8), (9), (73), $\Omega\,(r) = \Omega$, and $r_0 = \sqrt{3}r_1$, one can write, for $\Omega \neq \Omega_0$,

$$\frac{C\,(\Omega,r)}{F\,(\Omega,r)}\tau_\rho\,(r) + \frac{r_0^2\,(D\,(\Omega,r) - E\,(\Omega,r))}{3r^2\kappa F\,(\Omega,r)} = \frac{9r_0^2 G\,(\Omega,r)}{\kappa r^2\,(3r+r_0)^2}, \tag{81}$$

where we have defined

$$C\,(\Omega,r) = r^2\left((3r+r_0)^2\,(\Omega-\Omega_0)\,\Omega\left(\sin^2(\theta)\right) - 9\right)^2 \tag{82}$$

$$D\,(\Omega,r) = 81 + (3r+r_0)^4\,(\Omega-\Omega_0)^2\,\Omega^2\sin^4(\theta) \tag{83}$$

$$E\,(\Omega,r) = 9\,(3r+r_0)\Big[(3r+r_0)\Big(2\,(\Omega-\Omega_0)^2 + \Omega_0\,(2\Omega+\Omega_0)\Big) - 2r_0\Omega_0^2\Big]\sin^2(\theta) \tag{84}$$

$$F\,(\Omega,r) = (3r+r_0)^2\left((3r+r_0)^2\,(\Omega-\Omega_0)^2\left(\sin^2(\theta)\right) - 9\right) \tag{85}$$

$$G\,(\Omega,r) = \frac{1}{27}\left(\Omega^2\,(9r+r_0)\,(3r+r_0)\sin^2(\theta) - 9\right). \tag{86}$$

Equation (81) admits no solution for $\Omega \neq \Omega_0$. Nevertheless, for the special case $\Omega = \Omega_0$ (ZAMO), Eq. (74) reduces to an identity, and $\tau_\rho\,(r)$ can no longer be determined. The same occurs for Eq. (78). Therefore, in a ZAMO frame, Eq. (81) becomes

$$\frac{r^2\tau_\rho\,(r)}{(3r+r_0)^2} + \frac{r_0^2\left(9 - \left((3r+r_0)\,(9r+r_0)\,\Omega^2\sin^2(\theta)\right)\right)}{27r^2\kappa\,(3r+r_0)^2} = \frac{r_0^2\left(9 - \Omega^2\,(9r+r_0)\,(3r+r_0)\sin^2(\theta)\right)}{27\kappa r^2\,(3r+r_0)^2}, \tag{87}$$

and the only solution is

$$\tau_\rho\,(r) = 0. \tag{88}$$

Apparently, a contradiction arises between (75) and (88). However, this is not the case, as we are compelled to set $\Omega = \Omega_0$, which means that the only consistent solution is the vanishing of $\tau_\rho\,(r)$. Even though the constant Ω is a solution of the EFE, it has the unpleasant feature of not vanishing at large distances. This implies that a dragging effect will be present even at infinity. To address this, we observe that, by considering the definition of an ergosurface, and to avoid a change in the signature, we can assume to remain outside the ergoregion defined by (27), namely

$$\frac{3}{(3r+r_0)\sin\theta} > \Omega. \tag{89}$$

Due to the inequality (89), one can argue that the farther the distance from the throat, the smaller the value of Ω becomes, even though the decrease in the dragging velocity is not very rapid. Of course, this argument can only be applied if we are far from the values $\theta = 0$ and $\theta = \pi$. Note that the inequality also represents an upper bound when we are at the throat. Indeed, we have

$$\frac{3}{4r_0 \sin\theta} = \frac{\sqrt{3}}{4r_1 \sin\theta} > \Omega \tag{90}$$

by using (70). Due to the inequalities (89) and (90), a better strategy is needed to describe the vanishing of the rotation when $r > r_0$. To this end, we propose the following profile:

$$\Omega(r) = \Omega \exp(-\mu(r - r_0)). \tag{91}$$

Such a profile can be introduced at the cost of considering a non-vanishing $\tau_r(r)$. Indeed, Eq. (68) is satisfied provided that one considers

$$\tau_r(r) = \frac{(3r + r_0)^3 \left(\sin^2(\theta)\right)(r - r_0)\,\Omega^2\mu^2 e^{-2\mu(r-r_0)}}{108\kappa\, r^2}. \tag{92}$$

As we can see, $\tau_r(r)$ vanishes at the throat, as in the pure constant Ω case, and is highly suppressed for $r > r_0$. When $\Omega(r)$ is plugged into (72), we obtain the following modification for $\tau_t(r)$:

$$\tau_t(r) = -\frac{3\Omega^2 (3r + r_0)^3 \mu^2 (r - r_0)\left(\sin^2(\theta)\right) e^{-2\mu(r-r_0)}}{108\kappa r^2} + \frac{2r_0^2}{\kappa r^3 (3r + r_0)}. \tag{93}$$

Close to the throat, the first term of (93) can be neglected compared to the second term, and the form of (73) is recovered. For $r > r_0$, the exponential suppresses the first term, and the behavior of (73) persists. By plugging $\tau_t(r)$ as described by (93) and (91) into the EFE (81), one finds that (88) remains valid. Moreover, Eq. (88) is satisfied both for large values of μ and r, and for $\mu = 0$ and close to the throat. Regarding the EFE $G_{\phi\phi} = \kappa T_{\phi\phi}$ and the EFE $G_{t\phi} = \kappa T_{t\phi}$, we note that for $\mu = 0$ and close to the throat, both are satisfied, while for large values of μ and r, they vanish, but with different numerical coefficients.

It remains to check the violation of the null energy condition (NEC). We recall that the NEC is violated if

$$\rho(r) + p_r(r) \leq 0. \tag{94}$$

Since the violation of the NEC is relevant close to the throat, it is not necessary to introduce $\tau_r(r)$ and $\Omega(r) = \Omega$. Therefore, in this context, we can write

$$G_{\mu\nu}u^\mu u^\nu + G_{\mu\nu}n^\mu n^\nu = \kappa\left(T_{\mu\nu}u^\mu u^\nu + T_{\mu\nu}n^\mu n^\nu\right) = \kappa(\rho(r) + p_r(r)) \tag{95}$$

or, in other words,

$$-\frac{2r_0^2\left((3r + r_0)(9r + 2r_0)(\Omega - \Omega_0)^2 \sin^2\theta - 18\right)}{r^4\left((3r + r_0)^2 (\Omega - \Omega_0)^2 \sin^2\theta - 9\right)} = \kappa(\rho(r) + p_r(r)). \tag{96}$$

When $\Omega = \Omega_0$, we obtain

$$-\frac{4r_0^2}{3r^4} = \kappa(\rho(r) + p_r(r)), \tag{97}$$

as expected. Nevertheless, if we approach the throat before considering a ZAMO in (96), we obtain

$$\frac{4\left(9 - 22r_0^2(\Omega - \Omega_0)^2 \sin^2\theta\right)}{r_0^2\left(16r_0^2(\Omega - \Omega_0)^2 \sin^2\theta - 9\right)} = \kappa(\rho(r_0) + p_r(r_0)). \tag{98}$$

It is straightforward to see that there are two values of Ω such that $\rho(r_0) + p_r(r_0) = 0$. These are

$$\Omega_{1,2} = \Omega_0 \pm \frac{3\sqrt{22}}{22r_0 \sin\theta}. \tag{99}$$

On the other hand, the denominator of (98) vanishes for

$$\Omega_{3,4} = \Omega_0 \pm \frac{3}{4r_0 \sin\theta}. \tag{100}$$

This means that for values of Ω such that

$$\Omega_3 > \Omega > \Omega_1 \tag{101}$$

$$\Omega_2 > \Omega > \Omega_4, \tag{102}$$

the NEC is not violated. Moreover, from (98), one finds in a ZAMO frame that

$$-\frac{4}{r_0^2} = \kappa(\rho(r_0) + p_r(r_0)). \tag{103}$$

In other words, it appears that

$$\lim_{r\to r_0}\lim_{\Omega\to\Omega_0} \kappa(\rho(r) + p_r(r)) \neq \lim_{\Omega\to\Omega_0}\lim_{r\to r_0} \kappa(\rho(r) + p_r(r)). \tag{104}$$

3.2 Rotating Casimir wormhole with fixed plates

In this subsection, we consider the profile given by (13) and aim to apply the same method used for the SET in (2), but now with the SET described by (1). The EFE are formally the same as in the previous subsection, with the only modification being the change in the plates' separation from r to d on the SET components. The first EFE we wish to examine, when rotations come into play, is the equation $G_{rr} = \kappa T_{rr}$, which reads

$$\frac{r-b(r)}{4r^2}\left(r^3\left(\Omega'(r)\right)^2\sin^2\theta e^{-2\Phi(r)}+8\Phi'(r)\right) -\frac{b(r)}{r^3}+\frac{3r_1^2}{d^4}-\kappa\tau_r(r)=0. \tag{105}$$

Since the condition $b(r_0) = r_0$ must be satisfied, we observe that, at the throat, the previous equation simplifies to

$$-\frac{1}{r_0^2}+\frac{3r_1^2}{d^4}-\kappa\tau_r(r_0)=0. \tag{106}$$

Its solution is given by (16), along with the assumption that $\tau_r(r_0) = 0$. Isolating $\tau_r(r)$ from (105), we obtain

$$\tau_r(r) = \frac{\left(r^7+9r^5r_0^2-10r^4r_0^3\right)\sin^2(\theta)\left(\Omega'(r)\right)^2+40r^3-40r_0^3}{36r_0^2\kappa r^3} \tag{107}$$

where we have used (12), (13), and (16). Because of the boundary (17), we are forced to remain in the vicinity of the throat. Thus, we can write

$$\tau_r(r) \simeq \frac{10\left(r^3-r_0^3\right)}{9r_0^2\kappa r^3}. \tag{108}$$

Note that $\Omega(r)$ still remains undetermined. Therefore, we next consider the EFE $G_{\theta\theta} = \kappa T_{\theta\theta}$. Isolating $\tau_t(r)$ from (A3), one gets

$$\tau_t(r) = \frac{\sin^2(\theta)\left(\Omega'(r)\right)^2\left(10r^4r_0^3-r^7-9r^5r_0^2\right)-8r^3+20r_0^3}{36r_0^2\kappa r^3}. \tag{109}$$

Once again, in proximity of the throat, the equation is independent of $\Omega(r)$ and reduces to

$$\tau_t(r) \simeq \frac{5r_0^3-2r^3}{9\kappa r_0^2 r^3}. \tag{110}$$

As a result, we will evaluate $G_{\phi\phi} = \kappa T_{\phi\phi}$. From (A4), it is easy to show that, in the vicinity of the throat, $G_{\phi\phi}$ reduces to

$$G_{\phi\phi} \simeq \frac{\left(r\left(\Phi'(r)\right)+1\right)}{2\kappa r}\left(b(r)-\left(b'(r)\right)r\right)=\frac{r^3+5r_0^3}{9r_0^2\kappa r}, \tag{111}$$

while $T_{\phi\phi}$ becomes

$$T_{\phi\phi} = \frac{r^3+5r_0^3}{9r_0^2\kappa r}\sin^2(\theta) +\frac{\left(20r_0^3-8r^3\right)\left(\Omega(r)-\Omega_0\right)^2 r\sin^4(\theta)}{36r_0^2\kappa\left(1-r^2\left(\sin^2(\theta)\right)\left(\Omega(r)-\Omega_0\right)^2\right)} \tag{112}$$

Still, $\Omega(r)$ remains undetermined. However, if we adopt the ZAMO frame with the additional assumption that $\Omega(r) = \Omega$ with Ω constant, then $G_{\phi\phi} = \kappa T_{\phi\phi}$ is satisfied.

It remains to evaluate $G_{tt} = \kappa T_{tt}$ and $G_{t\phi} = \kappa T_{t\phi}$. We begin with $G_{tt} = \kappa T_{tt}$. By examining G_{tt} from (A1), one finds in vicinity of the throat that

$$G_{tt} \simeq \frac{2b'(r)e^{2\Phi(r)}-r\Omega(r)^2\sin^2(\theta)\left(b'(r)r-b(r)\right)}{2r^2} -\frac{\sin^2(\theta)\Omega(r)^2\left(b'(r)r+b(r)-2r\right)\Phi'(r)}{2} +\Omega'(r)\sin^2(\theta)\Omega(r)\left(\frac{b'(r)r}{2}-4r+\frac{7b(r)}{2}\right). \tag{113}$$

By plugging (12), (13), and (16) into (113), we obtain

$$G_{tt} \simeq \frac{2b'(r)-r\Omega^2\sin^2(\theta)\left(b'(r)r-b(r)\right)}{2r^2} = \frac{\sin^2(\theta)\Omega^2\left(r^3+5r_0^3\right)-3r}{9r_0^2\kappa r} \tag{114}$$

where we have also used that $\Omega(r) = \Omega$. On the other hand, T_{tt} becomes

$$T_{tt} \simeq \left(\frac{r_1^2}{\kappa d^4}+\tau_t(r)\right)\left(r^2\Omega^2\sin^2(\theta)-1\right)+\frac{\tau_t(r)\left(r^2\sin^2(\theta)\left(\Omega^2-\Omega\Omega_0\right)-1\right)^2}{1-r^2\sin^2(\theta)\left(\Omega-\Omega_0\right)^2} = \frac{3r^5\Omega^2\left(\Omega-\Omega_0\right)^2\sin^4(\theta)+\left(\left(-6\Omega^2+6\Omega\Omega_0-\Omega_0^2\right)r^3-5\Omega_0^2r_0^3\right)\sin^2(\theta)+3r}{9r_0^2 r\left(r^2\sin^2(\theta)\left(\Omega-\Omega_0\right)^2-1\right)\kappa} \tag{115}$$

where we have set $\tau_\rho(r) = 0$ and used (109). Even in this case, the ZAMO frame significantly simplifies the expression for T_{tt}. Indeed, we find

$$T_{tt} = \frac{\Omega^2\left(r^3 + 5r_0^3\right)\sin^2(\theta) - 3r}{9\kappa r_0^2 r}, \tag{116}$$

and as a result, the EFE $G_{tt} = \kappa T_{tt}$ is satisfied.

The last equation to examine is $G_{t\phi} = \kappa T_{t\phi}$. As with the other equations, we write $G_{t\phi}$ in proximity of the throat as

$$G_{t\phi} \simeq -\frac{\Omega}{2\kappa r}\left(\left(b'(r)\right)r - b(r)\right) = \frac{\sin^2(\theta)\,\Omega\left(r^3 + 5r_0^3\right)\sin^2(\theta)}{9\kappa r_0^2 r}, \tag{117}$$

where we have used the condition that $\Omega(r) = \Omega$ together with (12), (13), and (16). The component $T_{t\phi}$, in the same approximation, becomes

$$T_{t\phi} \simeq -\frac{r^3 + 5r_0^3}{9r_0^2\kappa r}\Omega\sin^2(\theta), \tag{118}$$

where we have also used the ZAMO frame and Eq. (109). Therefore, even the last EFE is satisfied, at least near the throat. Since the boundary (17) is very close to the throat, the approximation we used is consistent. However, for this model, we still need to check the NEC violation. In this context, we can write

$$G_{\mu\nu}u^\mu u^\nu + G_{\mu\nu}n^\mu n^\nu = \kappa\left(T_{\mu\nu}u^\mu u^\nu + T_{\mu\nu}n^\mu n^\nu\right) = \kappa\left(\rho(r) + p_r(r) + \tau_r(r)\right) \tag{119}$$

or, in other words,

$$\frac{2r^3 + 10r_0^3 - 15r^2r_0^3(\Omega - \Omega_0)^2\sin^2(\theta)}{9r_0^2\left((\Omega - \Omega_0)^2 r^2\sin^2(\theta) - 1\right)r^3} = \kappa\left(\rho(r) + p_r(r) + \tau_r(r)\right). \tag{120}$$

At the throat, we find

$$\frac{4 - 5r_0^2(\Omega - \Omega_0)^2\sin^2(\theta)}{3r_0^2\left((\Omega - \Omega_0)^2 r_0^2\sin^2(\theta) - 1\right)} = \kappa\left(\rho(r_0) + p_r(r_0)\right). \tag{121}$$

For the ZAMO frame, we obtain

$$-\frac{4}{3r_0^2} = \kappa\left(\rho(r_0) + p_r(r_0)\right), \tag{122}$$

confirming the violation of the NEC. This time,

$$\lim_{r\to r_0}\lim_{\Omega\to\Omega_0}\kappa\left(\rho(r) + p_r(r)\right) = \lim_{\Omega\to\Omega_0}\lim_{r\to r_0}\kappa\left(\rho(r) + p_r(r)\right). \tag{123}$$

For this TW as well, it can be shown that $\rho(r_0) + p_r(r_0) = 0$ outside the ZAMO frame. Indeed, from (121), we obtain

$$\frac{4 - 5r_0^2(\Omega - \Omega_0)^2\sin^2(\theta)}{3r_0^2\left((\Omega - \Omega_0)^2 r_0^2\sin^2(\theta) - 1\right)} = 0. \tag{124}$$

The numerator vanishes for

$$\Omega_{5,6} = \Omega_0 \pm \frac{2\sqrt{5}}{5r_0\sin(\theta)} \tag{125}$$

and can be compared with (99). On the other hand, the denominator vanishes for

$$\Omega_{7,8} = \Omega_0 \pm \frac{1}{r_0\sin(\theta)}. \tag{126}$$

This means that for the following values of Ω

$$\Omega_7 > \Omega > \Omega_5 \tag{127}$$

$$\Omega_6 > \Omega > \Omega_8, \tag{128}$$

the NEC is not violated. For the fixed-plates case, it is possible to extract additional information on the angular velocity from the definition of the ergosurface. Indeed, this occurs when

$$1 = rK(r,\theta)\Omega(r,\theta)\sin\theta = r\Omega\sin\theta, \tag{129}$$

where we have used (12) and $\Omega(r) = \Omega$. To stay outside the ergoregion, we must impose the following inequality:

$$1 > r\Omega\sin\theta. \tag{130}$$

At the throat, this becomes

$$\frac{1}{r_0\sin\theta} > \Omega \implies \frac{3r_1}{\sqrt{3d^2}\sin\theta} > \Omega, \tag{131}$$

implying that the rotation of the TW is very small.

4 Conclusions

In this paper, we have extended the study of Casimir wormholes by including rotations. Given the complexity of deriving entirely new solutions, we adopted the following strategy: The powering source is assumed to be a Casimir device generating a SET represented by (1). To ensure consistency, we imposed the condition that when rotation ceases, the static solution is recovered. Consequently, we had to distinguish between two scenarios:

(a) the ordinary Casimir wormhole with radially varying plates, and
(b) the TW powered by the SET (1), where the plates are parametrically fixed.

For the ordinary Casimir wormhole, we found that a solution is represented by a constant angular velocity Ω. The SET is given by (2), with the inclusion of the thermal tensor where only the transverse component survives, ensuring the consistency of the EFE. Nevertheless, the constant angular velocity Ω has an unpleasant feature: it extends throughout the entire space, implying that dragging effects remain detectable even at infinite distances. To address this issue, we modified the angular velocity by including a damping factor of exponential form. This adjustment suppresses the rotational effects, thereby confining them to the vicinity of the throat. To avoid a change in the signature, we restricted the analysis to the region outside the ergoregion. This assumption also enabled us to establish an upper bound for the angular velocity. Interestingly, on the ergosurface, the TW exhibits ultra-spinning behavior. This property is a consequence of having a Planckian radius for the throat, as described by (11). Regarding the NEC, it is found to be violated at the throat, even if we adopt the ZAMO frame. However, a form of non-commutative behavior arises between the throat region and the ZAMO frame, as expressed by (104). Concerning case (b), it is found that the static TW profile predicts a large throat radius, as given by (16), which is consistent with the result found in Ref. [5]. Additionally, a natural external boundary appears close to the throat. This implies that, in the rotating case (b), the constant Ω solution does not require any modification. Regarding the NEC in this second case, we have observed a commutative behavior, as described by (123). On the other hand, in the region outside the ergoregion, the inequality (131) leads to a very slow rotation, which contrasts with the prediction made in case (a). Finally, concerning the double limit described by (104) and (123), at this stage of the investigation, the origin of such behavior remains unknown.

Data Availability Statement This manuscript has no associated data. [Authors' comment: This is a theoretical study and no experimental data has been listed.]

Code Availability Statement This manuscript has no associated code/software. [Authors' comment: This is a theoretical study without a specific code.]

Funded by SCOAP[3].

Appendix A: Einstein field equations

$G_{tt} = \kappa T_{tt}$:

$$\begin{aligned}
&\left(\frac{r_1^2}{\kappa r^4} + \tau_t(r)\right)\left(-e^{2\Phi(r)} + r^2\Omega(r)^2\sin^2(\theta)\right) + \frac{\left(\tau_\rho(r)+\tau_t(r)\right)\left(e^{-\Phi(r)}\right)^2\left(-e^{2\Phi(r)} + r^2\sin^2(\theta)\left(\Omega(r)^2-\Omega(r)\Omega_0\right)\right)^2}{1-r^2\sin^2(\theta)\left(\Omega(r)-\Omega_0\right)^2\left(e^{-\Phi(r)}\right)^2}\\
&= \frac{\sin^2(\theta)\, r^2}{\kappa}\left(1-\frac{b(r)}{r}\right)\Omega(r)\left(\Phi''(r)\,\Omega(r) - \Omega''(r) + \left(\Phi'(r)\right)^2\Omega(r)\right)\\
&\quad - \frac{3\sin^2(\theta)\,(r-b(r))\left(r^2\Omega(r)^2\left(\sin^2(\theta)\right)e^{-2\Phi(r)} + \frac{1}{3}\right)\left(\Omega'(r)\right)^2 r}{4}\\
&\quad + \frac{2b'(r)\,e^{2\Phi(r)} - r\Omega(r)^2\sin^2(\theta)\left(b'(r)\,r - b(r)\right)}{2r^2} - \frac{\sin^2(\theta)\,\Omega(r)^2\left(b'(r)\,r + b(r) - 2r\right)\Phi'(r)}{2}\\
&\quad + \Omega'(r)\sin^2(\theta)\,\Omega(r)\left(r\,(r-b(r))\left(\Phi'(r)\right) + \frac{b'(r)\,r}{2} - 4r + \frac{7b(r)}{2}\right)
\end{aligned} \tag{A1}$$

$G_{t\varphi} = \kappa T_{t\varphi}$:

$$-\left(\frac{r_1^2}{\kappa\, r^4} + \tau_t\,(r)\right) r^2 \Omega\,(r) \sin^2(\theta)$$
$$+\frac{\left(\tau_\rho\,(r) + \tau_t\,(r)\right)\left(e^{-\Phi(r)}\right)^2 \left(-e^{2\Phi(r)} + r^2 \sin^2(\theta)\left(\Omega\,(r)^2 - \Omega\,(r)\,\Omega_0\right)\right)\left(\Omega\,(r) - \Omega_0\right) r^2 \sin^2(\theta)}{1 - r^2 \sin^2(\theta)\left(\Omega\,(r) - \ Omega_0\right)^2 \left(e^{-\Phi(r)}\right)^2}$$
$$= \frac{r^2 \sin^2(\theta)}{\kappa}\left(1 - \frac{b\,(r)}{r}\right)\left(-\Omega\,(r)\,\Phi''\,(r) + \frac{\Omega''\,(r)}{2} - \Omega\,(r)\left(\Phi'\,(r)\right)^2\right) - \frac{3 \sin^2(\theta)\, e^{-2\Phi(r)}\Omega\,(r)\, r^2 \left(\Omega'\,(r)\right)^2}{4}$$
$$-\frac{\sin^2(\theta)\,\Omega'\,(r)}{4\kappa}\left(2r^2\left(1 - \frac{b\,(r)}{r}\right)\Phi'\,(r) + \left(b'\,(r)\right) r - 8r + 7b\,(r)\right)$$
$$-\frac{\sin^2(\theta)\,\Omega\,(r)}{2\kappa r}\left(\left(r\left(\left(b'\,(r)\right) r - 2r + b\,(r)\right)\Phi'\,(r) + \left(b'\,(r)\right) r - b\,(r)\right)\right) \tag{A5}$$

$G_{rr} = \kappa T_{rr}$:

$$\frac{r^2}{32\pi}\left(1 - \frac{b(r)}{r}\right)\left(r^3 \sin^2\theta\ (\Omega'(r))^2 e^{-2\Phi(r)} + 8\Phi'(r)\right)$$
$$-\frac{b(r)}{8\pi} = \tau_r(r) r^3 - \frac{3r_1^2}{\kappa r} \tag{A2}$$

$G_{\theta\theta} = \kappa T_{\theta\theta}$:

$$\left(1 - \frac{b\,(r)}{r}\right)\Phi''\,(r) + \left(1 - \frac{b\,(r)}{r}\right)\left(\Phi'\,(r)\right)^2$$
$$+\frac{\Phi'\,(r)}{2r}\left(\left(1 - \frac{b\,(r)}{r}\right) + \left(1 - b'\,(r)\right)\right)$$
$$-\frac{r^4 \sin^2(\theta)\, e^{-2\Phi(r)}}{4r^2}\left(\Omega'\,(r)\right)^2\left(1 - \frac{b\,(r)}{r}\right)$$
$$+\frac{b\,(r) - b'\,(r)\, r}{2r^3} = \left(\frac{r_1^2}{r^4} + \kappa\tau_t\,(r)\right) \tag{A3}$$

$G_{\varphi\varphi} = \kappa T_{\varphi\varphi}$:

$$\frac{r^2 \sin^2(\theta)}{\kappa}\left(1 - \frac{b\,(r)}{r}\right)$$
$$\left(\Phi''\,(r) - \frac{3\left(\sin^2(\theta)\right) e^{-2\Phi(r)} r^2 \left(\Omega'\,(r)\right)^2}{4}\right.$$
$$\left. + \left(\Phi'\,(r)\right)^2 + \frac{1}{r}\Phi'\,(r)\right)$$
$$+\frac{\left(r\left(\Phi'\,(r)\right) + 1\right)}{2\kappa r}\left(b\,(r) - \left(b'\,(r)\right) r\right)$$
$$= \left(\frac{r_1^2}{\kappa r^4} + \tau_t\,(r)\right) r^2 \left(\sin^2(\theta)\right)$$
$$+\frac{\left(\tau_\rho\,(r) + \tau_t\,(r)\right)\left(e^{-\Phi(r)}\right)^2 \left(\Omega\,(r) - \Omega_0\right)^2 r^4 \left(\sin^4(\theta)\right)}{1 - r^2\left(\sin^2(\theta)\right)\left(\Omega\,(r) - \Omega_0\right)^2 \left(e^{-\Phi(r)}\right)^2} \tag{A4}$$

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