# Stabilization of Switched Affine Systems With Dwell-Time Constraint

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***Abstract*—This article addresses the problem of stabilization of switched affine systems under dwell-time constraint, giving guarantees on the bound of the quadratic cost associated with the proposed state switching control law. Specifically, two switching rules are presented relying on the solution of differential Lyapunov equalities and Lyapunov–Metzler inequalities, from which the stability conditions are expressed. The first one allows to regulate the state of linear switched systems to zero, whereas the second one is designed for switched affine systems, proving practical stability of the origin. In both cases, the determination of a guaranteed cost associated with each control strategy is shown. In the cases of linear and affine systems, the existence of the solution for the Lyapunov–Metzler condition is discussed, and guidelines for the selection of a solution ensuring suitable performance of the system evolution are provided. The theoretical results are finally assessed by means of three examples.**

***Index Terms*—Dwell time, Lyapunov–Metzler inequalities, Lyapunov stability, switched affine systems.**

## I. Introduction

SWITCHED systems are a pivotal class characterized by the interplay of continuous and discontinuous dynamics. These systems consist of multiple subsystems, each governed by distinct dynamical equations, and a switching signal that manages the transitions between them [1], [2]. In the literature, it is well established that switched systems do not inherently possess the stability properties of their subsystems. Therefore, the primary challenges in studying switched systems are investigating their stability under various classes of switching signals [3] and defining an effective switching strategy to achieve stabilization [4].

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In the context of system stabilization, it is important to distinguish between stabilization by a time-driven switching signal and by a state-feedback switching signal. It is not surprising to understand that there are systems that cannot be stabilized using a time-driven switching signal but are stabilizable through a well-designed state-feedback strategy [4]. A notable exception is given by the class of positive, more in general monotone, switched systems [5]. A further characterization of the set of possible stabilizing switching signals is given by the concept of (hard or average) dwell-time switching [6], [7], which limits the frequency of the commutation signal among the system's modes. Without this limit, the commutation can be arbitrarily fast, and the associated state-dependent feedback law can be designed through the so-called Lyapunov–Metzler inequalities, as in [4]. A review of results on stability and stabilization of switched linear systems can be found in [8] and [9].

### A. State of the Art for Switched Affine Systems

As an extension of switched linear systems, switched affine ones are used to model many processes characterized by variable structures, such as power converters [10], traffic congestion models [11], biochemical networks [12], and robotic locomotion [13]. In such practical applications, the affine term can model the presence of an energy source acting on the system (e.g., the presence of voltage generators in the microgrid, or the incoming traffic flow in an intersection). Unlike switched linear systems, switched affine systems can have different equilibrium points for each subsystem, and the control objective is to design a switching law that regulates the state to a point outside the set of subsystems' equilibria.

Designing stabilizing strategies for switched affine systems remains an open and challenging problem. For example, the pioneering work [14] demonstrates that the existence of a convex combination of subsystems forming a linear Hurwitz average system is sufficient to establish a switching strategy for regulating the state to the desired operating point. An optimal control strategy for switched piecewise affine autonomous systems, aimed at minimizing a performance index over an infinite time horizon, is proposed in [15]. A local stability result is proposed in [16], while a robust sampled-data approach was pursued in [17]. Furthermore, in [18], a state-dependent switching law is developed to stabilize switched affine systems, even when state measurements are affected by perturbations and noise. When the subsystems' equilibria are unknown, Deaecto et al. [19] provided an approach based on a two-step procedure expressed in terms of linear matrix inequalities (LMIs). In addition, Egidio et al. [20] addressed the global stabilization of continuous-time switched affine systems with rank-deficient convex combinations of their dynamic matrices. Finally, the recent paper [21] proposes a systematic method to translate a model-based condition, expressed using the framework of LMIs, into a data-driven condition. However, these control strategies permit potentially arbitrarily fast switching.

While arbitrarily fast switching can generate average solutions in the Filippov sense [22], dwell-time switching techniques are necessary for systems that require a finite commutation frequency, due to physical limitations, or to prevent actuator wear and tear. Conversely, while arbitrarily fast switching allows global asymptotic stabilization to the desired operating point, the dwell-time constraint prevents the asymptotic stabilization to this desired point (unless it coincides with the equilibrium point of one of the subsystems). Various strategies have been proposed to practically stabilize switched affine systems under a dwell-time constraint. For instance, Albea et al. [23] proposed a state-dependent switching control strategy that ensures a dwell-time constraint by using time-regularization within a hybrid system framework. In [24], a dwell-time constraint is inherently enforced by guaranteeing global asymptotic stability of a limit cycle for affine switched systems. Similarly, Xu et al. [25] analyzed the stability of both continuous-time and discrete-time switched affine systems via dwell-time switching. This approach constructs a discretized Lyapunov function to provide sufficient conditions to ensure practical stability of the state trajectories. Finally, Albea and Seuret [26] designed periodic time- and event-triggered control laws for switched affine systems using a hybrid dynamical system approach. This work, along with an appropriate optimization problem, formulates a stabilization result to ensure uniform global asymptotic stability of an attractor near the desired operating point for both types of controllers.

### B. Contributions With Respect to the State-of-the-Art

The present work, inspired by Allerhand and Shaked [27], introduces a novel dwell-time switching strategy aimed at regulating the state of a linear switched system to the origin. This proposed switching law utilizes a time-varying Lyapunov function derived from the solution of a differential Lyapunov equality and Lyapunov–Metzler inequalities. A key innovation of our approach compared to existing literature on switched linear systems with dwell time is its ability to provide a guaranteed bound on the $\mathcal{H}_2$ cost function. This bound aligns with that found in the literature regarding the control of switched linear systems with arbitrarily fast switching [4]. Such a cost can indeed be viewed as a quantitative measure of the quality of a fixed switching policy associated with the system evolution for a given initial condition.

Then, strength of this work is in providing a well-structured and rigorous method to generalize the proposed switching strategy to switched affine systems. Specifically, we prove practical stability of the origin and, differently from our previous findings [28], a guaranteed bound on the average $\mathcal{H}_2$ cost function, under dwell-time constraint. It is demonstrated that this bound on the average $\mathcal{H}_2$ cost function can be determined, in a simplified setting, as the solution to an optimization problem based on LMI and a line search procedure. The interpretation of this cost arises from the objective of formulating a stabilizing control policy that minimizes the average energy of the performance output. Such a measure can be of a paramount importance in many engineering applications in the presence of energy sources, such as the power converter control or the traffic congestion problem, discussed in this article.

Unlike other works in the literature, such as [14], [23], and [26], by virtue of the existence of a solution of Lyapunov–Metzler inequalities, our proposed switching strategies do not depend on the existence of a convex combination of subsystems that generates a Hurwitz average linear system. Therefore, the proposed switching law has the merit of being able to be successfully applied to switched affine systems that do not admit such linear Hurwitz convex combination, e.g., the traffic congestion control problem presented in [11]. In this article, we also provide some guidelines, even in the limit case of dwell-time tending to zero, to tune the parameters of the proposed switching law, for practitioners who wish to implement it in practice.

### C. Structure of This Article

This article is organized as follows. In Section II, the problem of stabilization of switched affine systems is formalized. First, the design of a dwell-time switching law for switched linear systems is discussed in depth in Section III. Then, the extension to the case of switched affine systems is introduced and rigorously analyzed in Section IV. Guidelines on the design of the switching law tuning parameters are provided in Section V. Finally, the effectiveness of the proposed algorithms is demonstrated through three examples in Section VI, while conclusions are gathered in Section VII.

*Notations:* The transpose of a matrix $A$ is denoted by $A'$. The set of reals is notated as $\mathbb{R}$, while the sets of nonnegative real and natural numbers are $\mathbb{R}_{\geq 0}$ and $\mathbb{N}_{\geq 0}$, respectively. Signals in the time domains are denoted by lowercase letters, such as $x(t)$, or just $x$. We indicated with $\mathcal{M}$ the class of irreducible Metzler matrices, which consist of all matrices $\Pi = [\pi_{i,j}] \in \mathbb{R}^{M\times M}$, with $\pi_{i,j}$ being the $(i, j)$th entry of $\Pi$, such that $\pi_{i,j} \geq 0$ for all $i \neq j$, and $\sum_{j=1}^{M} \pi_{i,j} = 0$, for all $i \in \{1, \dots, M\}$. For Hermitian matrices, $X > 0$ (respectively, $X \geq 0$) indicates that $X$ is positive (respectively, semipositive) definite. For given matrices $A$ and $B$ of compatible dimensions, $A \otimes B$ indicates the Kronecker product of the two matrices, whereas the Kronecker sum is defined as $A \oplus B = A \otimes I + I \otimes B$, with $I$ being the identity matrix [29, Ch. 7]. The symbols $0_n$ and $1_n$ indicate the $n$-dimensional column vectors with all zero entries and all one entries, respectively. Similarly, $0_{n\times n}$ indicates the $n$-by-$n$ null matrix. Finally, $\mathcal{B}_r\{x\}$ denotes the closed ball $\{y \in \mathbb{R}^n | \|y - x\| \leq r\}$.

## II. Problem Setup

Consider the switched affine system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}, \quad x(t_0) = x_0 \tag{1a}$$

$$z(t) = Cx(t) \tag{1b}$$

where the state $x \in \mathbb{R}^n$ is available for feedback for all $t \geq 0$, $z \in \mathbb{R}^p$ is a performance output, and $x_0$ is the initial condition. Let $\Omega := \{1, \ldots, M\}$, and denote with $\Omega_i$ the set of integers $\{1, \ldots, i-1, i+1, \ldots, M\}$, with $M \in \mathbb{N}_{>0}$. Then, considering a set of matrices $A_i \in \mathbb{R}^{n\times n}$ and vectors $b_i \in \mathbb{R}^n$, $i \in \Omega$, be given, the switching law $\sigma(t)$, for each $t \geq 0$, is such that

$$A_{\sigma(t)} \in \{A_1, \ldots, A_M\}, \quad b_{\sigma(t)} \in \{b_1, \ldots, b_M\}$$

with $\{t_k\}$, $k = 0, 1, \ldots, \infty$, being the monotonically increasing sequence of time instants such that $A_{\sigma(t_k^-)} \neq A_{\sigma(t_k)}$ or $b_{\sigma(t_k^-)} \neq b_{\sigma(t_k)}$. In what follows, we will say that the switching signal $\sigma(t)$ has dwell time $T$ if $t_{k+1} - t_k \geq T > 0$ for all $k = 0, 1, \ldots, \infty$, with $t_0$ being the initial time instant. The control objective is that of designing a switching law capable of stabilizing an operating point $x^{\mathrm{e}} \in \mathbb{R}^n$, that is, defining $\breve{x} = x - x^{\mathrm{e}}$ and $\breve{z} = z - Cx^{\mathrm{e}}$, we would like to stabilize the system

$$\dot{\breve{x}}(t) = \breve{A}_{\sigma(t)}\breve{x}(t) + \breve{b}_{\sigma(t)}, \quad \breve{x}(t_0) = \breve{x}_0$$
$$\breve{z}(t) = C\breve{x}(t)$$

where $\breve{A}_{\sigma(t)} = A_{\sigma(t)}$, $\breve{b}_{\sigma(t)} = A_{\sigma(t)}x^{\mathrm{e}} + b_{\sigma(t)}$, and $\breve{x}_0 = x(t_0) - x^{\mathrm{e}}$. Nevertheless, without loss of generality, in the following, we consider the case where $x^{\mathrm{e}} = 0$, so that, we will solve the problem of stabilizing the origin of system (1).

The design of the switching law presented in this work takes inspiration from the stripped-down stabilization problem with dwell time of the linear switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(t_0) = x_0 \tag{2a}$$
$$z(t) = Cx(t) \tag{2b}$$

presented in [27]. The idea of [27] is to design a switching-dependent quadratic Lyapunov function taking into account the solution of the system dynamics within a time horizon equal to the dwell time $T$. Moreover, in [27] and in the present work, both the switching instants and the index of the subsystem to be activated are obtained by means of continuous-time monitoring of the state. Making reference to system (2), the approach in [27, §III, Thm. 1] may be presented as follows. Assuming that there exists a collection $\{P_1, \ldots, P_M\}$ of positive-definite matrices, and $\Pi = [\pi_{i,j}] \in \mathcal{M}$ such that the following Lyapunov–Metlzer inequalities hold:

$$P_iA_i + A_i'P_i + \sum_{j\in\Omega_i} \pi_{i,j}\left(\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT} - P_i\right) < 0 \quad \forall\, i \in \Omega \tag{3}$$

assuming $\sigma(t_k) = i$, then the switching law is defined as

$$\begin{aligned}
&\sigma(t) = i \quad \forall t \in [t_k, t_k + T]\\
&\sigma(t) = i \quad \forall t > t_k + T\\
&\qquad \text{if} \quad x(t)'P_ix(t) \leq x(t)'\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT}x(t) \quad \forall j \in \Omega_i\\
&\sigma(t_{k+1}) = \operatorname*{argmin}_{j\in\Omega_i} x(t_{k+1})'\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT}x(t_{k+1})\\
&\qquad \text{otherwise.}
\end{aligned} \tag{4}$$

Then, it is proved in [27, §III, Thm. 1] that the switching law (4) globally asymptotically stabilizes the system $\dot{x}(t) = A_{\sigma(t)}x(t)$ with dwell-time constraint $T$. Conceptually, the law in (4) compares the current value of the Lyapunov function $V(x,t) = x'(t)P_{\sigma(t)}x(t)$ of the active subsystem with the Lyapunov function corresponding to the other subsystems evaluated at $T$ time units ahead of the current time instant. It is worth to notice that the law (4), although it does not limit the application to affine linear switched systems, is designed without taking into account the affine term $b_{\sigma(t)}$ and its effect on the performance index, commonly defined as

$$J(x_0, t) = \int_{t_0}^{t} z'(\tau)z(\tau)\mathrm{d}\tau. \tag{5}$$

The latter may indeed be considered as a measure of the efficiency of the algorithm and is of great importance when selecting the best approach in applications, providing a robustness indicator of the process behavior.

The cost (5) is not easily manageable in [27], and the switching law (4) is not readily extendable to include the affine term. Motivated by the above two challenges, we now present an alternative approach to provide some results on the cost, also including in the switching law the term $b_\sigma$.

More specifically, we address the following problems:

i) *design a state-feedback switching law* $\sigma(x(t), t) : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \Omega$ *with dwell time* $T$ *that globally asymptotically stabilizes the origin of the switched linear system (2)*, and provide a compatible bound to $J(x_0, t)$ for each $t \geq t_0$;

ii) *design a state-feedback switching law* $\sigma(x(t), t) : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \Omega$ *with dwell time* $T$ *that globally practically stabilizes the origin of the switched affine system (1)*, and provide a compatible bound to $J(x_0, t)$ for each $t \geq t_0$.

We here recall the definition of practical stability (see, e.g., [30, Def. 2]) for the switched affine system (1).

*Definition 1:* Assume that a time interval $\mathcal{T}$ ($\mathcal{T} = [t_0, \infty)$ or $\mathcal{T} = [t_0, t_f)$) and a switching law $\sigma(x(t), t) : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \Omega$ are given. The switched affine system (1) is said to be *practically stable over* $\mathcal{T}$ *under* $\sigma$ if, for some $R > 0$, there exists $r > 0$ such that $x(t) \in \mathcal{B}_R\{0_n\}$, for all $t \in \mathcal{T}$, whenever $x(t_0) \in \mathcal{B}_r\{0_n\}$.

In the following, we consider the interval $\mathcal{T} = [t_0, \infty)$, and the dependence on the switching law $\sigma$ will be clear from the context. Accordingly, we shall refer to the aforementioned property simply as *practical stability*.

For the sake of simplicity, in the following, $\sigma(x(t), t)$ is indicated as $\sigma(t)$.

## III. Dwell-Time State-Feedback Switching Law For Linear Switched Systems

In this section, to solve the first problem previously formulated, we propose an enhanced state-feedback switching control strategy, providing, differently from [27], an estimation on the performance cost (5).

Consider system (2) and let us define, for each $j \in \Omega$, the following quantities:

$$Y_{1,j} = e^{A_j'T}X_je^{A_jT}, \quad Y_{2,j} = \int_0^T e^{A_j'\tau}C'Ce^{A_j\tau}\mathrm{d}\tau$$

where $X_j$ is the solution to a Lyapunov–Metzler inequality, introduced in the next theorem.

*Theorem 1:* Consider the switched linear system (2) and assume that there exist constant symmetric positive-definite matrices $X_i \in \mathbb{R}^{n\times n}$ and matrix $\Pi = [\pi_{i,j}] \in \mathcal{M}$, solutions of the Lyapunov–Metzler inequalities

$$A_i'X_i + X_iA_i + \sum_{j\in\Omega_i} \pi_{i,j}(Y_{1,j} + Y_{2,j} - X_i) + C'C < 0 \quad (7)$$

for any $i \in \Omega$. Consider positive-definite symmetric time-varying matrices $P_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ such that

$$-\dot{P}_i = A_i'P_i + P_iA_i + C'C \qquad \forall t \in [t_k, t_k + T) \quad \forall i \in \Omega \quad (8a)$$

$$P_i(t) = X_i \qquad \forall t \in [t_k + T, t_{k+1}) \quad \forall i \in \Omega. \quad (8b)$$

Then, the dwell-time switching strategy

$$\sigma(t) = i \quad \forall t \in [t_k, t_k + T] \quad (9a)$$

$$\sigma(t) = i \quad \forall t > t_k + T \quad (9b)$$

$$\text{if} \quad x(t)'(Y_{1,j} + Y_{2,j})x(t) \geq x(t)'X_ix(t) \quad \forall j \in \Omega_i$$

$$\sigma(t_{k+1}) = \operatorname*{argmin}_{j\in\Omega_i} x(t_{k+1})'(Y_{1,j} + Y_{2,j})x(t_{k+1}) \quad (9c)$$

where

$$t_{k+1} := \inf_{t>t_k+T} \{t | \exists j : x(t)' [Y_{1,j} + Y_{2,j} - X_i] x(t) < 0\} \quad (10)$$

guarantees that the origin of system (2) is globally exponentially stable. Furthermore, the performance index (5) is bounded as

$$J(x_0, t) \leq x(t_0)'P_{\sigma(t_0)}(t_0)x(t_0). \quad (11)$$

*Proof:* Let us consider the following Lyapunov function:

$$V(x,t) = \begin{cases} x(t)'P_{\sigma(t)}(t)x(t), & t \in [t_k, t_k + T) \\ x(t)'X_{\sigma(t)}x(t), & t \in [t_k + T, t_{k+1}) \end{cases} \quad (12)$$

where $t_{k+1}$ is defined as in (10). Note that this function is continuous, by construction, at time instant $t_k + T$. Initially, consider the interval $[t_k, t_k + T)$, with the subsystem $i$ being active, i.e., $\sigma(t) = i$, for all $t \in [t_k, t_k + T)$. Then, from (12), $V(x,t) = x(t)'P_i(t)x(t)$ and

$$\begin{aligned}\dot{V}(x,t) &= \dot{x}(t)'P_i(t)x(t) + x(t)'\dot{P}_i(t)x(t) + x(t)'P_i(t)\dot{x}(t) \\ &= x(t)'(A_i'P_i(t) + P_i(t)A_i + \dot{P}_i)x(t) \\ &= -x(t)'C'Cx(t) \quad (13)\end{aligned}$$

where the last equality follows from (8a).

Consider now the interval $[t_k + T, t_{k+1})$. Since switching has not been triggered yet, then $\sigma(t) = i$ for all $t \in [t_k + T, t_{k+1})$. Hence, from (12), $V(x,t) = x(t)'X_ix(t)$ and

$$\begin{aligned}\dot{V}(x,t) &= \dot{x}(t)'X_ix(t) + x(t)'X_i\dot{x}(t) \\ &= x(t)'(A_i'X_i + X_iA_i)x(t) \\ &\leq -x(t)'\left[\sum_{j\in\Omega_i} \pi_{i,j}(Y_{1,j} + Y_{2,j} - X_i) + C'C\right]x(t) \\ &\leq -x(t)'C'Cx(t) \quad (14)\end{aligned}$$

where the first inequality comes from (7), while the last inequality comes from (9b) in the switching rule.

Let us now consider the jumps of the Lyapunov function at the switching instants, i.e.,

$$\begin{aligned}\Delta V(x(t_{k+1}), t_{k+1}) &= V(x(t_{k+1}), t_{k+1}) - V(x(t_{k+1}^-), t_{k+1}^-) \\ &= x(t_{k+1})'P_j(t_{k+1})x(t_{k+1}) - x(t_{k+1})'P_i(t_{k+1}^-)x(t_{k+1}).\end{aligned}$$

Note that from (8)

$$\begin{aligned}P_j(t_{k+1}) &= e^{A_j'T}X_je^{A_jT} + \int_0^T e^{A_j'\tau}C'Ce^{A_j\tau}\mathrm{d}\tau \\ &= Y_{1,j} + Y_{2,j} \\ P_i(t_{k+1}^-) &= X_i\end{aligned}$$

where $P_j(t_{k+1})$ is obtained through backward integration of (8a) in the interval $[t_{k+1}, t_{k+1} + T)$ with final condition $P_j(t_{k+1} + T) = X_j$, while $P_i(t_{k+1}^-)$ is obtained from (8b). Hence, by replacing the above terms in the definition of $\Delta V(x(t_{k+1}), t_{k+1})$, we obtain

$$\Delta V(x(t_{k+1}), t_{k+1}) = x(t_{k+1})'(Y_{1,j} + Y_{2,j} - X_i)x(t_{k+1}) < 0 \quad (15)$$

where the inequality comes from considering the switching law condition (9b) and the definition of the switching instant (10).

Thus, from (13)–(15), one has that $\dot{V}(x,t) \leq -x(t)'C'Cx(t)$ for almost all $t \in [t_k, t_{k+1}]$ and $\Delta V(x(t_{k+1}), t_{k+1}) < 0$ at the switching instant. Hence, iterating the above steps results in

$$\dot{V}(x,t) \leq -x(t)'C'Cx(t) \quad (16)$$

for all $t \geq t_0$, thus implying global exponential stability of the origin. Finally, integrating both sides of (16), it holds that

$$\begin{aligned}J(x_0, t) &= \int_{t_0}^t z(\tau)'z(\tau)\mathrm{d}\tau \leq V(x(t_0), t_0) \\ &= x(t_0)'P_{\sigma(t_0)}(t_0)x(t_0)\end{aligned}$$

thus proving the theorem statement. ■

The previous theorem shows that the switching law (9) globally asymptotically stabilizes the origin of the switched linear system (2) while providing a guaranteed upper-bound for the $\mathcal{H}_2$ cost.

*Remark 3.1 (Applicability of the switching control law):* It is worth emphasizing that requiring the existence of matrices $P_i(t)$ such that conditions (8) are satisfied does not require matrices $A_i$ to be Hurwitz, since (8) is assumed to hold only within a finite time interval. ▽

*Remark 3.2 (Initialization of the switching signal):* Different approaches are possible to initialize the switching signal $\sigma$. For instance, a convenient approach is to initialize the switching signal as $\sigma(t_0) = \operatorname{argmin}_\sigma V(x(t_0), t_0)$ to minimize the cost (11). ▽

Note that the matrix functions $P_i(t)$ evolve periodically, each satisfying the differential equation (8a) over the interval $t \in [t_k, t_k + T)$, with the terminal condition $P_i(t_k + T) = X_i$ holding for all $k \in \{1, 2, \ldots\}$.

Furthermore, solving only the Lyapunov–Metzler inequalities (7) suffices to implement the switching strategy (9). In fact, only the terms $X_j$, achieved as a solution to (7), and the terms $Y_{1,j}$ and $Y_{2,j}$, which can be computed from $X_j$ and the subsystems' matrices, are required for the implementation of the switching strategy (9). Therefore, the need to compute the analytic solution to the differential Lyapunov equation (8) is avoided. Moreover, it is worth noticing that the proposed solution provides a bound on the performance index depending on the system's initial condition, in analogy with existing literature results on arbitrarily fast switching [4].

An interesting result can be obtained if the switching strategy (9) is applied to the switched affine system (1), as it is shown in the next corollary.

*Corollary 2:* Consider the switched affine system (1) and assume that there exist constant symmetric positive-definite matrices $X_i \in \mathbb{R}^{n\times n}$ and matrix $\Pi = [\pi_{i,j}] \in \mathcal{M}$, solutions of the Lyapunov–Metzler inequalities

$$\hat{A}_i' X_i + X_i \hat{A}_i + \sum_{j\in\Omega_i} \pi_{i,j}(Y_{1,j} + Y_{2,j} - X_i) + C'C < 0 \tag{17}$$

for any $i \in \Omega$, with $\hat{A}_i = A_i + \frac{\epsilon}{2} I$ and the scalar $\epsilon > 0$. Consider positive-definite symmetric time-varying matrices $P_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ such that

$$-\dot{P}_i = \hat{A}_i' P_i + P_i \hat{A}_i + C'C \quad \forall t \in [t_k, t_k + T) \quad \forall i \in \Omega \tag{18a}$$

$$P_i(t) = X_i \quad \forall t \in [t_k + T, t_{k+1}) \quad \forall i \in \Omega. \tag{18b}$$

Then, the dwell-time switching strategy (9) guarantees that the origin of system (1) is practically stable. Furthermore, the cost is bounded as

$$\begin{aligned} J(x_0, t) &= \int_{t_0}^{t} z(\tau)' z(\tau) \mathrm{d}\tau \\ &\leq x(t_0)' P_{\sigma(t_0)}(t_0) x(t_0) + \delta(t - t_0) \end{aligned} \tag{19}$$

where

$$\delta = \frac{1}{\epsilon} \max_{i\in\Omega} \left\{ \sup_{t\in[t_0, t_0+T)} b_i' P_i(t) b_i \right\} \tag{20}$$

*Proof:* The proof follows similar reasoning as in Theorem 1, by selecting the Lyapunov function as in (12). Initially, consider the interval $[t_k, t_k + T)$, with the subsystem $i$ being active, i.e., $\sigma(t) = i$, for all $t \in [t_k, t_k + T)$. Then, computing the Lyapunov function time derivative within the interval $[t_k, t_k + T)$, the equality (13) becomes

$$\dot{V}(x, t) = -x(t)' C' C x(t) - \epsilon x(t)' P_i(t) x(t) + 2 b_i' P_i(t) x(t). \tag{21}$$

Considering the time interval $[t_k + T, t_{k+1})$, instead, the derivative of the Lyapunov function is upper bounded as

$$\dot{V}(x, t) \leq -x(t)' C' C x(t) - \epsilon x(t)' X_i x(t) + 2 b_i' X_i x(t). \tag{22}$$

Finally, the Lyapunov function difference at the switching instants still satisfies condition (15) due to the switching law definition.

Hence, from (21), one obtains

$$\dot{V}(x, t) \leq -x(t)' C' C x(t) + \delta_i \quad \forall t \in [t_k, t_k + T) \tag{23}$$

with

$$\begin{aligned} \delta_i &:= \frac{1}{\epsilon} \sup_{t\in[t_k, t_k+T)} b_i' P_i(t) b_i \\ &= \frac{1}{\epsilon} \sup_{t\in[t_0, t_0+T)} b_i' P_i(t) b_i \end{aligned}$$

where the equality derives from the periodic evolution of functions $P_i(t)$ with respect to the intervals $[t_k, t_k + T)$. Similarly, from (22), one obtains

$$\begin{aligned} \dot{V}(x, t) &\leq -x(t)' C' C x(t) + \frac{1}{\epsilon} b_i' X_i b_i \\ &\leq -x(t)' C' C x(t) + \delta_i \quad \forall t \in [t_k + T, t_{k+1}) \end{aligned} \tag{24}$$

where the last inequality derives from $\delta_i \geq \frac{1}{\epsilon} b_i' X_i b_i$ due to continuity of $P_i(t)$ at $t = t_k + T$. At the switching instants condition, (15) holds. Therefore, for all $t \in \mathbb{R}_{\geq 0}$, it holds that

$$\dot{V}(x, t) \leq -x(t)' C' C x(t) + \delta \tag{25}$$

with $\delta := \max_{i\in\Omega}\{\delta_i\}$ and $\Delta V(x(t_k), t_k) < 0$ for all $k$. Thus, practical stability of the origin, as in Definition 1, is implied by standard Lyapunov-based arguments. Integrating both sides of (25), it holds that

$$\int_{t_0}^{t} z(\tau)' z(\tau) \mathrm{d}\tau \leq V(x(t_0), t_0) + \delta(t - t_0) \tag{26}$$

thus obtaining (19). ■

*Remark 3.3 (Relation with Theorem 1):* The conditions specified in Theorem 1 are precisely the ones required for Corollary 2 achieved by setting $\epsilon = 0$. In other words, the prerequisites for both results are identical, except for the specific value of $\epsilon$. In Corollary 2, it would be desirable to find the largest positive $\epsilon$ that satisfies conditions (17) and (18) in order to minimize the slope of the cost upper bound $\delta$. ▽

*Remark 3.4 (Average cost ultimate bound):* From (19), it is clear that, in the long run

$$\lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} z(\tau)' z(\tau) \mathrm{d}\tau \leq \delta\,.$$

▽

Finally, it must be noted that the switching law (9) does not explicitly take into account the affine terms $b_i$. In the following, an alternative switching strategy accounting for the affine terms is presented.

## IV. Practical Stabilization for Switched Affine Systems

Differently from the control law proposed in the previous section, it is now desirable to include an affine term in the evaluation of the switching condition. Therefore, an easy-to-implement switching strategy is hereafter presented, as a natural extension of the solution reported in Theorem 1. It is worth highlighting that, differently from Corollary 2, the proposed switching strategy explicitly takes into account the affine term in the definition of the switching law.

Let us introduce the following extended system:

$$\dot{\tilde{x}}(t) = \tilde{A}_{\sigma(t)}\tilde{x}(t), \quad \tilde{x}(t_0) = \tilde{x}_0 \tag{27a}$$

$$z(t) = \tilde{C}\tilde{x}(t) \tag{27b}$$

with $\tilde{x} = [x' \quad \bar{x}]'$, $\tilde{x}_0 = [x_0' \quad 1]'$, and

$$\tilde{A}_{\sigma(t)} = \begin{bmatrix} A_{\sigma(t)} & b_{\sigma(t)} \\ 0_n' & 0 \end{bmatrix}, \quad \tilde{C} = [C \quad 0_p]. \tag{28}$$

It is easy to see that system (27) is equivalent to system (1). Moreover, the following auxiliary variables are used hereafter:

$$\tilde{Y}_{1,j} = e^{\tilde{A}_j' T}\tilde{X}_j e^{\tilde{A}_j T}, \quad \tilde{Y}_{2,j} = \int_0^T e^{\tilde{A}_j'\tau}\tilde{C}'\tilde{C}e^{\tilde{A}_j\tau}\mathrm{d}\tau$$

with $\tilde{X}_j$ being the solution to a Lyapunov–Metzler inequality, introduced in Theorem 3, structured as

$$\tilde{X}_j = \begin{bmatrix} X_j & 0_n \\ 0_n' & 1 \end{bmatrix} \tag{29}$$

$X_j$ being symmetric and positive definite. Finally, let us define

$$\tilde{I} = \begin{bmatrix} 0_{n\times n} & 0_n \\ 0_n' & 1 \end{bmatrix}.$$

In the following, the proposed dwell-time switching law for practical stabilization of the switched affine system (1) is formulated.

*Theorem 3:* Consider the switched system (27) and assume that there exist constant symmetric positive-definite matrices $\tilde{X}_i \in \mathbb{R}^{(n+1)\times(n+1)}$, $i \in \Omega$, structured as in (29), matrix $\Pi = [\pi_{i,j}] \in \mathcal{M}$, and scalar $\varepsilon > 0$, solutions of the Lyapunov–Metzler inequalities

$$\tilde{A}_i'\tilde{X}_i + \tilde{X}_i\tilde{A}_i + \sum_{j\in\Omega_i}\pi_{i,j}(\tilde{Y}_{1,j} + \tilde{Y}_{2,j} - \tilde{X}_i) + \tilde{C}'\tilde{C} < \varepsilon\tilde{I}. \tag{30}$$

Consider positive-definite symmetric time-varying matrices $\tilde{P}_i(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$-\dot{\tilde{P}}_i = \tilde{A}_i'\tilde{P}_i + \tilde{P}_i\tilde{A}_i + \tilde{C}'\tilde{C} \qquad \forall t\in[t_k, t_k+T) \quad \forall i \in \Omega \tag{31a}$$

$$\tilde{P}_i(t) = \tilde{X}_i \qquad \forall t \in [t_k + T, t_{k+1}) \quad \forall i \in \Omega. \tag{31b}$$

Then, the dwell-time switching strategy

$$\sigma(t) = i \quad \forall t \in [t_k, t_k + T] \tag{32a}$$

$$\sigma(t) = i \quad \forall t > t_k + T \tag{32b}$$

$$\text{if} \quad \tilde{x}(t)'(\tilde{Y}_{1,j} + \tilde{Y}_{2,j})\tilde{x}(t) \geq \tilde{x}(t)'\tilde{X}_i\tilde{x}(t) \quad \forall j \in \Omega_i$$

$$\sigma(t_{k+1}) = \operatorname*{argmin}_{j\in\Omega_i} \tilde{x}(t_{k+1})'(\tilde{Y}_{1,j} + \tilde{Y}_{2,j})\tilde{x}(t_{k+1}) \tag{32c}$$

where

$$t_{k+1} := \inf_{t>t_k+T}\left\{t \middle| \exists j : \tilde{x}(t)'\left[\tilde{Y}_{1,j} + \tilde{Y}_{2,j} - \tilde{X}_i\right]\tilde{x}(t) < 0\right\} \tag{33}$$

guarantees that the origin of system (1) is practically stable. Furthermore, the performance index (5) is bounded as

$$J(x_0, t) = \int_{t_0}^{t} z(\tau)'z(\tau)\mathrm{d}\tau \leq x(t_0)'P_{\sigma(t_0)}(t_0)x(t_0) + \varepsilon(t - t_0). \tag{34}$$

*Proof:* Let us consider the following Lyapunov function:

$$V(\tilde{x}, t) = \begin{cases} \tilde{x}(t)'\tilde{P}_{\sigma(t)}(t)\tilde{x}(t), & t \in [t_k, t_k + T) \\ \tilde{x}(t)'\tilde{X}_{\sigma(t)}\tilde{x}(t), & t \in [t_k + T, t_{k+1}) \end{cases} \tag{35}$$

where $t_{k+1}$ is defined as in (33). Note that this function is continuous, by construction, at time instant $t_k + T$. Initially, consider the interval $[t_k, t_k + T)$, with the subsystem $i$ being active, i.e., $\sigma(t) = i$, for all $t \in [t_k, t_k + T)$. Then, from (35), $V(\tilde{x}, t) = \tilde{x}(t)'\tilde{P}_i(t)\tilde{x}(t)$ and

$$\begin{aligned}\dot{V}(\tilde{x}, t) &= \dot{\tilde{x}}(t)'\tilde{P}_i(t)\tilde{x}(t) + \tilde{x}(t)'\dot{\tilde{P}}_i(t)\tilde{x}(t) + \tilde{x}(t)'\tilde{P}_i(t)\dot{\tilde{x}}(t) \\ &= \tilde{x}(t)'(\tilde{A}_i'\tilde{P}_i(t) + \tilde{P}_i(t)\tilde{A}_i + \dot{\tilde{P}}_i)\tilde{x}(t) \\ &= -\tilde{x}(t)'\tilde{C}'\tilde{C}\tilde{x}(t) = -x(t)'C'Cx(t)\end{aligned} \tag{36}$$

where the third equality follows from (31a).

Consider now the interval $[t_k + T, t_{k+1})$. Since switching has not been triggered yet, then $\sigma(t) = i$ for all $t \in [t_k + T, t_{k+1})$. Hence, from (35), $V(\tilde{x}, t) = \tilde{x}(t)'\tilde{X}_i\tilde{x}(t)$ and

$$\begin{aligned}\dot{V}(\tilde{x}, t) &= \dot{\tilde{x}}(t)'\tilde{X}_i\tilde{x}(t) + \tilde{x}(t)'\tilde{X}_i\dot{\tilde{x}}(t) \\ &= \tilde{x}(t)'(\tilde{A}_i'\tilde{X}_i + \tilde{X}_i\tilde{A}_i)\tilde{x}(t) \\ &\leq -\tilde{x}(t)'\left[\sum_{j\in\Omega_i}\pi_{i,j}(\tilde{Y}_{1,j} + \tilde{Y}_{2,j} - \tilde{X}_i) + \tilde{C}'\tilde{C} - \varepsilon\tilde{I}\right]\tilde{x}(t) \\ &\leq -\tilde{x}(t)'\tilde{C}'\tilde{C}\tilde{x}(t) + \varepsilon\tilde{x}(t)'\tilde{I}\tilde{x}(t) \\ &= -x(t)'C'Cx(t) + \varepsilon\end{aligned} \tag{37}$$

where the first inequality comes from (30), while the second inequality comes from (32b) in the switching rule.

Let us now consider the jumps of the Lyapunov function at the switching instants, i.e.,

$$\begin{aligned}\Delta V(\tilde{x}(t_{k+1}), t_{k+1}) &= V(\tilde{x}(t_{k+1}), t_{k+1}) - V(\tilde{x}(t_{k+1}^-), t_{k+1}^-) \\ &= \tilde{x}(t_{k+1})'\tilde{P}_j(t_{k+1})\tilde{x}(t_{k+1}) - \tilde{x}(t_{k+1})'\tilde{P}_i(t_{k+1}^-)\tilde{x}(t_{k+1}).\end{aligned}$$

Note that

$$\begin{aligned}\tilde{P}_j(t_{k+1}) &= e^{\tilde{A}_j' T}\tilde{X}_j e^{\tilde{A}_j T} + \int_0^T e^{\tilde{A}_j'\tau}\tilde{C}'\tilde{C}e^{\tilde{A}_j\tau}\mathrm{d}\tau \\ &= \tilde{Y}_{1,j} + \tilde{Y}_{2,j} \\ \tilde{P}_i(t_{k+1}^-) &= \tilde{X}_i\end{aligned}$$

where $\tilde{P}_j(t_{k+1})$ is obtained by backward integration of (31). Hence, replacing $\tilde{P}_j(t_{k+1})$ and $\tilde{P}_i(t_{k+1}^-)$, we obtain

$$\Delta V(\tilde{x}(t_{k+1}), t_{k+1}) = \tilde{x}(t_{k+1})'(\tilde{Y}_{1,j} + \tilde{Y}_{2,j} - \tilde{X}_i)\tilde{x}(t_{k+1}) < 0 \tag{38}$$

where the inequality comes from the switching law condition (32b) and the definition of the switching instant (33).

Thus, from (36)–(38), one has that $\dot{V}(\tilde{x}, t) \leq -x(t)'C'Cx(t) + \varepsilon$ for almost all $t \in [t_k, t_{k+1}]$ and $\Delta V(\tilde{x}(t), t) < 0$ at switching instants. Hence, iterating the above steps leads to

$$\dot{V}(\tilde{x}, t) \leq -x(t)'C'Cx(t) + \varepsilon \tag{39}$$

for all $t \geq t_0$. Thus, practical stability of the origin, as in Definition 1, is guaranteed by standard Lyapunov-based arguments. Finally, integrating both sides of (39), it holds that

$$J(x_0, t) = \int_{t_0}^{t} z(\tau)'z(\tau)\mathrm{d}\tau \leq V(x(t_0), t_0) + \varepsilon(t - t_0) \tag{40}$$

thus proving the statement. ■

It is worth highlighting that, in the case of linear switched systems, i.e., $b_i = 0$ for all $i \in \Omega$, the conditions of Theorem 3 are equivalent to those presented in Theorem 1 with $\varepsilon$ being arbitrarily small. Noticeably, similarly to Theorem 1, matrices $\tilde{P}_i(t)$ are time-varying. In fact, such matrices evolve according to the differential equation (31a) in the time interval $t \in [t_k, t_k + T)$, then they achieve $\tilde{P}_i(t) = \tilde{X}_i$ at $t = t_k + T$ and remain constant until the next switching instant. Therefore, while $\tilde{P}_i(t)$ has the block-diagonal form (29) in the interval $[t_k + T, t_{k+1})$, the same structure is not necessarily guaranteed, nor required, in the interval $[t_k, t_k + T)$.

*Remark 4.1 (Initialization of the switching signal):* Following the approach in the switched linear case, the switching signal can be initialized as $\sigma(t_0) = \mathrm{argmin}_\sigma V(x(t_0), t_0)$ to minimize the first term of the cost (40). ▽

*Remark 4.2 (Average cost ultimate bound):* From (40), it is clear that, in the long run

$$\lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} z(\tau)'z(\tau)\mathrm{d}\tau \leq \varepsilon\,. \quad ▽$$

The Lyapunov–Metzler condition (30) can be explicitly expressed as

$$\begin{bmatrix} A_i'X_i & 0_n \\ b_i'X_i & 0 \end{bmatrix} + \begin{bmatrix} X_iA_i & X_ib_i \\ 0_n' & 0 \end{bmatrix} + \sum_{j\in\Omega_i} \pi_{i,j} \left( \begin{bmatrix} e^{A_j'T}X_je^{A_jT} & e^{A_jT}X_jm_j \\ m_j'X_je^{A_jT} & m_j'X_jm_j + 1 \end{bmatrix} + \int_0^T \begin{bmatrix} e^{A'\tau}C'Ce^{A\tau} & e^{A'\tau}C'Cm_j \\ m_j'C'Ce^{A\tau} & m_j'C'Cm_j \end{bmatrix} \mathrm{d}\tau - \begin{bmatrix} X_i & 0_n \\ 0_n' & 1 \end{bmatrix} \right) + \begin{bmatrix} C'C & 0_n \\ 0_n' & 0 \end{bmatrix} < \varepsilon\tilde{I} \tag{41}$$

with

$$m_j := \int_t^{t+T} \mathrm{e}^{A_j(t+T-\tau)}b_j\mathrm{d}\tau = \int_0^T \mathrm{e}^{A_j(T-\tau)}b_j\mathrm{d}\tau.$$

Therefore, condition (30) is equivalent to finding symmetric positive-definite matrices $X_i \in \mathbb{R}^{n\times n}$ such that the extended Lyapunov–Metzler inequality (41) is satisfied.

In the following, we provide further intuition regarding the design of the switching rule (32).

*Remark 4.3 (Interpretation of the switching law):* The switching law presented in Theorem 3 relies on the comparison of the Lyapunov function of the active subsystem at the current time with the forecast of the Lyapunov functions of the remaining subsystems $T$ time instants forward in time plus the *cost-to-go* in the interval $[t, t + T]$. In fact, the switching law (32) can be rewritten as

$$\sigma(x(t), t) = i \quad \forall t \in [t_k, t_k + T] \tag{42a}$$

$$\sigma(x(t), t) = i \quad \forall t > t_k + T \tag{42b}$$
$$\text{if } V_j(x(t+T), t+T) + J_j(t, t+T) \geq V_i(x(t), t) \quad \forall j \in \Omega_i$$

$$\sigma(x(t_{k+1}), t_{k+1}) = \underset{j\in\Omega_i}{\mathrm{argmin}} V_j(x(t_{k+1}+T)) + J_j(t, t+T) \tag{42c}$$

where $V_j(x(t), t) := \tilde{x}'\tilde{X}_j\tilde{x}$

$$J_j(t, t+T) = \int_t^{t+T} x'(\tau)C'Cx(\tau)\mathrm{d}\tau = \int_0^T \tilde{x}'(t+\tau)\tilde{C}'\tilde{C}\tilde{x}(t+\tau)\mathrm{d}\tau = \tilde{x}'(t)\tilde{Y}_{2,j}\tilde{x}(t)$$

that is the cost over the $j$th subsystem. Note that the only way to evaluate conditions (42b) and (42c) is through the explicit computation of the Lyapunov functions and cost-to-go as presented in Theorem 3. ▽

## V. Parameter Tuning and Limit Cases

This section aims at providing some guidelines to apply the proposed approach under dwell-time constraints, even in the limit cases. The control strategies presented in the previous sections indeed require tuning some parameters, such as the Metzler matrix $\Pi$ and the scalar parameters $\epsilon$ and $\varepsilon$ in Corollary 2 and Theorem 3, respectively. The choice of these parameters must aim at minimizing the guaranteed cost upper bound. Hence, for Theorem 1, the optimal choice of matrices $\Pi$ and $X_i$ is the one obtained by solving the problem

$$\min_{\Pi\in\mathcal{M}, X_1>0,\ldots,X_M>0} \left\{ x(t_0)'P_{\sigma(t_0)}(t_0)x(t_0) \quad : \quad (7) \right\}$$

which is equivalent to

$$\min_{\Pi\in\mathcal{M}, X_1>0,\ldots,X_M>0} \left\{ x(t_0)' \left( e^{A_{\sigma(t_0)}'T}X_{\sigma(t_0)}e^{A_{\sigma(t_0)}T} + \int_0^T e^{A_{\sigma(t_0)}'\tau}C'Ce^{A_{\sigma(t_0)}\tau}\mathrm{d}\tau \right) x(t_0) \quad : \quad (7) \right\} \tag{43}$$

where $\sigma(t_0)$ is selected according to Remark 3.2.

In the case of affine systems addressed in Theorem 3, it is desirable to minimize the *persistent* cost provided by the term $\varepsilon$, that is

$$\min_{\Pi\in\mathcal{M},X_1>0,\ldots,X_M>0} \{\varepsilon \quad : \quad (30)\}. \tag{44}$$

Nevertheless, determining a numerical solution for the Lyapunov–Metzler inequalities concerning the variables $(\Pi, \{X_1,\ldots,X_M\})$ that solve problems (43) and (44) presents significant challenges and warrants further investigation. Due to the bilinear dependence between coefficients $\pi_{i,j}$ and matrices $X_i$, such inequalities are bilinear matrix inequalities, so that the primary difficulty arises from the nonconvex nature of the problem, which renders LMI solvers ineffective.

### A. Simplified Solution to Lyapunov–Metzler Inequalities

A potential strategy for further exploration is leveraging the specific structure of the inequalities, where $\pi_{i,j}$ are scalars, to develop an interactive method based on relaxation. Nevertheless, as shown in [4], a simpler, albeit more conservative, stability condition can be formulated using LMIs, making it solvable with existing techniques. The following theorem demonstrates that, by adapting Theorem 1 to the limit case with $T = 0$, and by focusing on a subclass of Metzler matrices, defined by identical diagonal elements, this objective can be achieved.

*Theorem 4:* Consider the switched linear system (2) and assume that there exist symmetric positive-definite matrices $X_i \in \mathbb{R}^{n\times n}$ and $\gamma > 0$ so that

$$A_i'X_i + X_iA_i + \gamma(X_j - X_i) + C'C < 0 \tag{45}$$

for any $i \in \Omega$, $j \in \Omega_i$. Then, considering $\sigma(t^-) = i$, the switching strategy

$$\sigma(t) = \operatorname*{argmin}_{j\in\Omega_i} x(t)'X_jx(t) \tag{46}$$

guarantees that the origin of system (2) is globally exponentially stable. Furthermore, the performance index (5) is bounded as

$$J(x_0,t) \le \sum_{i\in\Omega} x(t_0)'X_i(t_0)x(t_0). \tag{47}$$

*Proof:* The proof follows straightforwardly from [4, Thm. 4], and it is thus omitted. ■

Theorem 4 shows that the switching law presented in Theorem 1 is a generalization of [4, Thm. 4]. In fact, considering $T = 0$, condition (8) disappears and, considering $\pi_{i,i} = \gamma$ for all $i \in \Omega$, condition (7) reduces to (45). Finally, the first two terms of the switching law (9) are discarded for $T = 0$ while the third term reduces to (46).

As suggested in [4], the minimization problem (44) simplifies to

$$\min_{\gamma>0,X_1>0,\ldots,X_M>0} \left\{\sum_{i\in\Omega} x_0'X_ix_0 \quad : \quad (45)\right\} \tag{48}$$

which can be solved by using LMI and line search.

Remark 3.1 highlights that the results from Theorem 1 do not require the matrices $\{A_1,\ldots,A_M\}$ to be Hurwitz. Nevertheless, in the case each subsystem is asymptotically stable, the Lyapunov–Metzler condition (7) also holds for the choice of $\Pi = 0$, and the strategy proposed preserves stability. The same reasoning also applies for condition (30) in Theorem 3. In the case of a linear system (2) characterized by a set of matrices $\{A_1,\ldots,A_M\}$ being quadratically stable, it is well known that the Lyapunov–Metzler inequalities (7) admit a solution $X_1 = \cdots = X_M = X$ for which any switching law $\sigma(t) = i \in \Omega$ asymptotically stabilizes the switched linear system. Hence, Theorem 1 contains, as a particular case, the quadratic stability condition.

### B. Existence of a Hurwitz Average System

A further interesting result can also be obtained when the switched affine system possesses a convex Hurwitz linear combination and the dwell time characterizing the switching law is close to being null. To describe such a limit case, let us formalize the following property (see [3], [14], and [23]).

*Property 1:* Given the simplex $\Lambda := \{\lambda \in [0,1]^M \mid \sum_{i=1}^M \lambda_i = 1\}$, there exists $\lambda \in \Lambda$, such that

$$b_\lambda := \sum_{i\in\Omega} \lambda_i b_i = 0, \quad \text{and} \quad A_\lambda := \sum_{i\in\Omega} \lambda_i A_i \quad \text{is Hurwitz.} \tag{49}$$

Note that, Property 1, despite being a rather common requirement, is not a necessary condition for stabilization of switched affine systems; see [1, §3.4.2].

Finally, to extend the investigation to the case of switched affine systems, let

$$\tilde{A}_\lambda = \begin{bmatrix} A_\lambda & b_\lambda \\ 0_n' & 0 \end{bmatrix}.$$

*Proposition 5:* Consider the switched affine system (1), assume that Property 1 holds, and define $\pi_{i,j} = \frac{1}{T}\bar{\pi}_{i,j}$, with $\bar{\pi}_{i,j}$ being fixed entries of a matrix $\bar{\Pi} = [\bar{\pi}_{i,j}] \in \mathcal{M}$. Assume that there exist positive-definite matrices $\tilde{X}_i$ and positive scalar $\varepsilon$ such that the following Lyapunov–Metzler equalities hold:

$$\tilde{A}_i'\tilde{X}_i + \tilde{X}_i\tilde{A}_i + \frac{1}{T}\sum_{j\in\Omega_i} \bar{\pi}_{i,j}(\tilde{Y}_{1,j} + \tilde{Y}_{2,j} - \tilde{X}_i) + \tilde{C}'\tilde{C} = \varepsilon\tilde{I} \tag{50}$$

for any $i \in \Omega$. Then, as $T$ tends to 0, it follows that $\tilde{X}_j = \tilde{X}_i = \tilde{X}$ for all $i,j \in \Omega$ with $\tilde{X}$ satisfying the Lyapunov equality:

$$\tilde{A}_\lambda'\tilde{X} + \tilde{X}\tilde{A}_\lambda + \tilde{C}'\tilde{C} = 0.$$

Then, the switching law (32) makes the origin of system (1) a globally asymptotically stable switched equilibrium.

*Proof:* Let us consider the equality (50). Multiplying both sides by $T$, for $T \to 0$, it holds

$$\sum_{j\in\Omega_i} \bar{\pi}_{i,j}(\tilde{X}_j - \tilde{X}_i) = \sum_{j\in\Omega} \bar{\pi}_{i,j}\tilde{X}_j = 0_{(n+1)\times(n+1)} \quad \forall i \in \Omega \tag{51}$$

where the first equality derives from $\bar{\Pi} \in \mathcal{M}$. Letting $\zeta \in \mathbb{R}^{n+1}$, one can write

$$\sum_{j\in\Omega} \bar{\pi}_{i,j}\zeta'\tilde{X}_j\zeta = \sum_{j\in\Omega} \bar{\pi}_{i,j}\eta_j = 0 \quad \forall i \in \Omega \tag{52}$$

which in turn implies that $\bar{\Pi}\eta = 0$, with $\bar{\Pi} \in \mathcal{M}$ and $\eta \in \mathbb{R}^M$. By virtue of the Frobenius–Perron theorem, the eigenvector associated with the null eigenvalue of $\bar{\Pi}$ is strictly positive, leading to the conclusion that there always exists $\eta = \bar{\eta}1_M$, $\bar{\eta} > 0$, so that $\zeta'(\tilde{X}_i - \tilde{X}_j)\zeta = 0$, $\forall \zeta \in \mathbb{R}^{n+1}$ for $i, j \in \Omega$. This means that $\tilde{X}_j = \tilde{X}_i = \tilde{X}$ [4].

Note that, for $T \to 0$, we can set $\varepsilon = 0$. Now, multiplying the Lyapunov–Metzler equations (50) by the elements of the left Frobenius eigenvector $\lambda_i$, with $\lambda \in \Lambda$, substituting matrices $\tilde{X}_i$ with $\tilde{X}$ and summing over $i \in \Omega$, one has

$$\sum_{i\in\Omega} \lambda_i \left(\tilde{A}_i'\tilde{X} + \tilde{X}\tilde{A}_i + \tilde{C}'\tilde{C}\right) = 0$$

which implies

$$\left(\sum_{i\in\Omega} \lambda_i\tilde{A}_i'\right)\tilde{X} + \tilde{X}\left(\sum_{i\in\Omega} \lambda_i\tilde{A}_i\right) + \tilde{C}'\tilde{C} = \tilde{A}_\lambda'\tilde{X} + \tilde{X}\tilde{A}_\lambda + \tilde{C}'\tilde{C} = 0 \tag{53}$$

which, according to Property 1, in turn implies

$$A_\lambda'X + XA_\lambda + C'C = 0$$

where $X$ is adopted to construct $\tilde{X}$ as in (29).

Hence, taking the switching rule (32), by subtracting $\tilde{x}(t_k)'\tilde{X}_i\tilde{x}(t_k)$ to the argument of (32c) and dividing by $T$, for $T \to 0$, it reduces to

$$\sigma(t) = \underset{j\in\Omega}{\text{argmin}}\ \dot{v}(x(t), t)$$

where $v(x(t), t) = x'(t)Xx(t)$. By similar reasoning as in [4, Thm. 3], and exploiting the assumption that $b_\lambda = 0$, we have that

$$\dot{v}(x(t), t) \le -x'(t)C'Cx(t).$$

Since the Lyapunov function $v(x(t), t)$ is radially unbounded, then the equilibrium $x = 0$ of (2) is globally asymptotically stable. ∎

### C. Existence and Design of Metzler Matrix

Finally, a relevant point to be discussed now concerns the existence of a solution of the Lyapunov–Metzler inequalities (30) with respect to the variables $\Pi$ and $\{\tilde{X}_1, \dots, \tilde{X}_M\}$. It is not difficult to show through standard Kronecker calculus that, for fixed $\Pi \in \mathcal{M}$, a solution with respect to the remaining variables exists if and only if the $M \cdot (n+1)^2$-dimensional square matrix $\mathcal{J} = \mathcal{A} + \Upsilon(\Pi)$ is Hurwitz, with

$$\mathcal{A} = \begin{bmatrix} \tilde{A}_1' \oplus \tilde{A}_1' & 0 & \dots & 0 \\ 0 & \tilde{A}_2' \oplus \tilde{A}_2' & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \tilde{A}_M' \oplus \tilde{A}_M' \end{bmatrix}$$

and

$$\Upsilon(\Pi) = \Pi_d \otimes I_{(n+1)^2} + \left[(\Pi - \Pi_d) \otimes I_{(n+1)^2}\right] e^{\mathcal{A}T}$$

with $\Pi_d$ being the diagonal matrix with elements being those of the diagonal of $\Pi$. Hence, the existence of a solution to (30) reduces to the existence of $\Pi \in \mathcal{M}$ making matrix $\mathcal{J}$ asymptotically stable. A similar existence condition holds for the Lyapunov–Metzler inequality (7) where terms $\tilde{A}_i$ in $\mathcal{A}$, $i \in \Omega$, are replaced with terms $A_i$.

A possible approach to turn the optimization problems (43) and (44) into LMI is to fix matrix $\Pi$ and solve for matrices $X_i$. With regard to the choice of the matrix $\Pi$ in Theorem 1, Corollary 2, and Theorem 3, different approaches can be pursued, depending on the nature of the switched system. In general, two distinct families of systems can be identified: those for which Property 1 is satisfied, and those for which there does not exist such a convex combination satisfying condition (49). In the former case, indicating with $\lambda$ the vector that guarantees satisfaction of Property 1, matrix $\Pi \in \mathcal{M}$ can be chosen so that $\lambda'\Pi = 0$. Otherwise, if there does not exist any convex combination that satisfies Property 1, one can rely on the fact that the matrix $\Pi$ can be interpreted as the probability rate matrix of the Markov process associated with the switching sequence. This in turn implies that the elements $\pi_{i,j}$ of $\Pi$ are selected as the expected probability rate of switching to the $j$th subsystem when the $i$th subsystem is active, i.e., $\pi_{i,j} = \text{Prob}[\sigma(t + \text{d}t) = j \,|\, \sigma(t) = i]$, for $i \neq j$ so that

$$T_i = \frac{1}{\sum_{j\neq i} \pi_{i,j}}$$

is the average dwell time for the $i$th mode [31]. See, for instance, the congestion control problem in the next section. Noticeably, the latter approach requires knowledge of the expected desirable switching sequence.

## VI. Illustrative Examples

The switching strategies detailed in Sections III and IV are evaluated through various examples. The strategy formulated for switched linear systems, as outlined in Theorem 1, is applied to stabilize a switched system comprising two unstable linear subsystems. Subsequently, the switching strategy from Theorem 3 is implemented for the control of a boost–boost converter and to address a traffic congestion problem. Notably, while a convex combination of the subsystems exists for the boost–boost converter such that condition (49) is satisfied, no such convex combination exists for the traffic congestion scenario. Nevertheless, the proposed switching strategy manages to orchestrate the congestion scenario properly.

### A. Unstable Subsystems

Let us consider the switched linear system (2), with

$$A_1 = \begin{bmatrix} -2 & 0.3 \\ -2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 \\ -0.3 & -4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Both systems are unstable; however, it is easy to verify that the convex combination obtained with $\lambda_1 = \lambda_2 = 0.5$ generates a

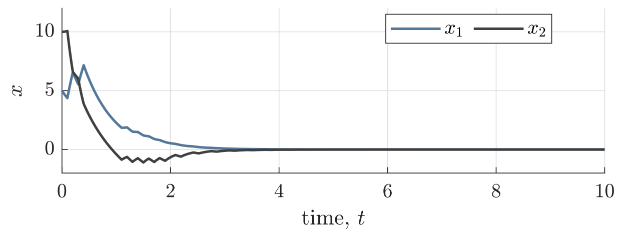


Fig. 1. Time evolution of the state $(x_1(t), x_2(t))$ for the switched linear system.

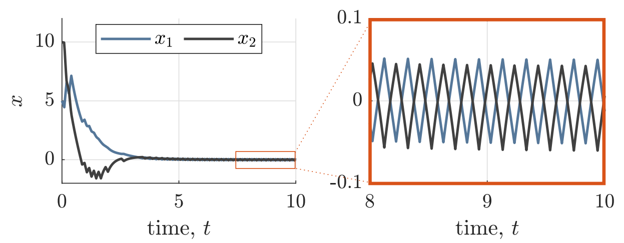


Fig. 2. Time evolution of the state $(x_1(t), x_2(t))$ for the affine switched system.

stable system. Then, matrix $\Pi$ can be chosen of the form

$$\Pi = \alpha \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \alpha > 0. \tag{54}$$

The simulation results, derived using the switching law in Theorem 1, with dwell time $T = 0.1$ and the initial condition $x_0 = [5\ 10]'$, are shown in Fig. 1. The figure demonstrates that the state trajectories converge asymptotically to the origin of the state space, despite the inherent instability of the two subsystems.

This numerical example can be extended by considering the switched affine system (1) with

$$b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

It is easy to see that the same convex combination as in the linear case satisfies condition (49); hence, the matrix $\Pi$ can be designed again as in (54). In this scenario, by applying the switching law from Theorem 3, with dwell time $T = 0.1$ and the initial condition $x_0 = [5\ 10]'$, the state trajectories converge toward a neighborhood of the origin and oscillate around it, as shown in Fig. 2. The switching signal alternates between $\sigma(t) = 1$ and $\sigma(t) = 2$, ultimately converging to a periodic solution (see Fig. 3).

### B. Boost–Boost Converter

The switching strategy proposed in Theorem 3 is particularly well suited for controlling dc–dc switched power converters. These converters naturally possess a switched structure, and the limited commutation frequency of their switching devices necessitates the design of a discontinuous control law that incorporates dwell time. In addition, most dc–dc switched power converters

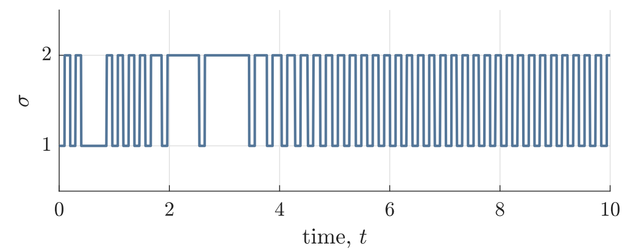


Fig. 3. Time evolution of the switching signal $\sigma(t)$ for the affine switched system.

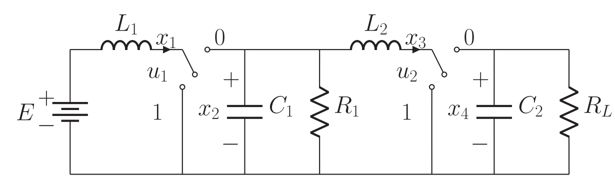


Fig. 4. Boost–boost converter topology.

can be modeled as switched affine systems. In this section, we focus on the control of the boost–boost converter, as discussed in [32, §2.10]. The boost–boost converter features two switching elements (see Fig. 4), resulting in four distinct configurations of the converter.

Therefore, such a converter can be modeled as a switched affine system of the form (1) with $\sigma \in \{1, 2, 3, 4\}$ and

$$A_1 = \begin{bmatrix} 0 & -\frac{1}{L} & 0 & 0 \\ \frac{1}{C_1} & -\frac{1}{R_1 C_1} & -\frac{1}{C_1} & 0 \\ 0 & \frac{1}{L_2} & 0 & -\frac{1}{L_2} \\ 0 & 0 & \frac{1}{C_2} & -\frac{1}{R_L C_2} \end{bmatrix}, \quad b_1 = \begin{bmatrix} \frac{E}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_1 C_1} & -\frac{1}{C_1} & 0 \\ 0 & \frac{1}{L_2} & 0 & -\frac{1}{L_2} \\ 0 & 0 & \frac{1}{C_2} & -\frac{1}{R_L C_2} \end{bmatrix}, \quad b_2 = b_1$$

$$A_3 = \begin{bmatrix} 0 & -\frac{1}{L} & 0 & 0 \\ \frac{1}{C_1} & -\frac{1}{R_1 C_1} & -\frac{1}{C_1} & 0 \\ 0 & \frac{1}{L_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_L C_2} \end{bmatrix}, \quad b_3 = b_1$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{R_1 C_1} & -\frac{1}{C_1} & 0 \\ 0 & \frac{1}{L_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{R_L C_2} \end{bmatrix}, \quad b_4 = b_1$$

where $x_1$ is the current through the inductor $L_1$, $x_2$ is the output voltage at the first stage of conversion, i.e., the voltage across capacitor $C_1$, $x_3$ represents the current through the inductor $L_2$, while $x_4$ indicates the output voltage at the second stage of conversion, i.e., the load voltage across capacitor $C_2$, or equivalently the load voltage. The switching signal $\sigma(t)$ denotes the configurations of the switch pair $(u_1, u_2)$, with $\sigma = 1$ corresponding to

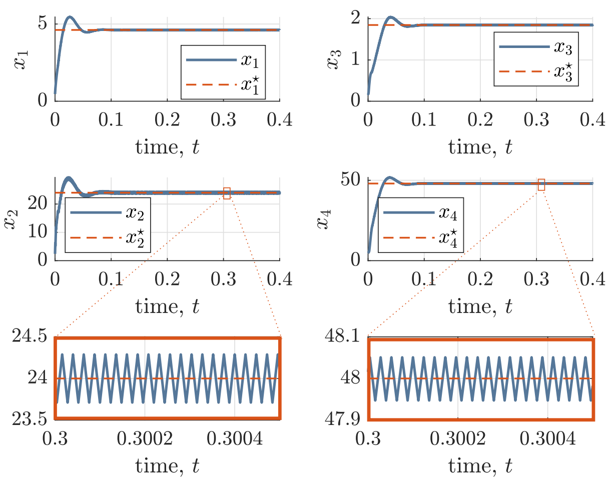


Fig. 5. Time evolution of the boost–boost converter states. Top left: current $x_1(t)$. Bottom left: first stage voltage $x_2(t)$. Top right: current $x_3(t)$. Bottom right: second stage voltage $x_4(t)$.

the configuration (0,0), $\sigma = 2$ to (1,0), $\sigma = 3$ to (0,1), and finally, $\sigma = 4$ to (1,1). The objective of the boost–boost converter is to regulate the voltages at the two output stages, specifically $x_2$ and $x_4$ to their respective reference values, $x_2^{\mathrm{e}}$ and $x_4^{\mathrm{e}}$. Hence, for given $x_2^{\mathrm{e}}$ and $x_4^{\mathrm{e}}$, a parameterization of the equilibrium point in terms of the steady-state output voltages is obtained as

$$x_1^{\mathrm{e}} = \frac{1}{R_1}\frac{x_2^{\mathrm{e}\,2}}{E} + \frac{1}{R_L}\frac{x_4^{\mathrm{e}\,2}}{E}, \quad x_3^{\mathrm{e}\,2} = \frac{1}{R_L}\frac{x_4^{\mathrm{e}\,2}}{x_2^{\mathrm{e}}}.$$

Therefore, the control objective is to regulate to zero the error variable $\breve{x} = x - x^{\mathrm{e}}, x^{\mathrm{e}} = [x_1^{\mathrm{e}}\ x_2^{\mathrm{e}}\ x_3^{\mathrm{e}}\ x_4^{\mathrm{e}}]'$, obeying the switched differential equation

$$\dot{\breve{x}}(t) = \breve{A}_{\sigma(t)}\breve{x}(t) + \breve{b}_{\sigma(t)}, \quad \breve{z}(t) = C\breve{x}(t) \tag{55}$$

where $\breve{A}_{\sigma(t)} = A_{\sigma(t)}$, $\breve{b}_{\sigma(t)} = A_{\sigma(t)}x^{\mathrm{e}} + b_{\sigma(t)}$, and $\breve{z} = z - Cx^{\mathrm{e}}$. The numerical values for the converter parameters are selected as specified in [32, §2.10]. Given the input voltage $E = 12$ V, with $x_2^{\mathrm{e}} = 24$ V and $x_4^{\mathrm{e}} = 48$ V, the convex combination obtained with $\lambda_i = 0.25$, $i = \{1, \ldots, 4\}$, results in a stable convex combination for system (55). The matrix $\Pi$ in (30) can be chosen as any matrix satisfying $\lambda'\Pi = 0$. Due to the high switching frequency required for dc–dc power converters, the dwell time is set to $T = 10^{-5}$ s, while the initial condition is chosen as $x(0) = [0.46\ 2.40\ 0.18\ 4.80]'$. The performance of the proposed switching law is depicted in Fig. 5, demonstrating that the states smoothly converge toward the desired reference values and oscillate around them.

The performance of the proposed switching law is then compared with the algorithms presented in [23] and [24]. While the former adopts a hybrid framework formulation to enforce a dwell-time switching, the latter relies on the definition of a periodic trajectory toward which the state is attracted asymptotically. The metric considered for the evaluation of the three algorithms is the cost functional (5) evaluated for the error variable $\breve{z}$ within the time interval $[0,\ 0.4]$, i.e.,

$$J(\breve{x}_0, t) = \int_0^{0.4} \breve{z}'(\tau)\breve{z}(\tau)\mathrm{d}\tau = \int_0^{0.4} \breve{x}'(\tau)C'C\breve{x}(\tau)\mathrm{d}\tau$$

where matrix $C$ has been chosen as

$$C = \begin{bmatrix} 0 & x_2^{\mathrm{e}^{-1}} & 0 & 0 \\ 0 & 0 & 0 & x_4^{\mathrm{e}^{-1}} \end{bmatrix}$$

to normalize the error variables with respect to the steady-state values. The values obtained for the chosen metric are given in Table I, where it is seen that our approach achieves performance comparable to the two considered literature algorithms.

TABLE I
COST EVALUATION FOR OUR PROPOSAL AND ALGORITHMS [23] AND [24]

| Theorem 3 | [23] | [24] |
|---|---|---|
| $8.966 \cdot 10^{-3}$ | $8.971 \cdot 10^{-3}$ | $8.809 \cdot 10^{-3}$ |

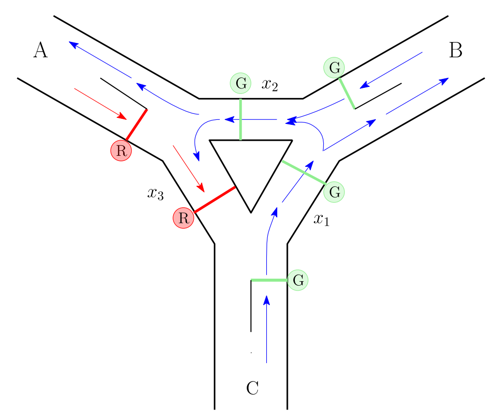


Fig. 6. Traffic control problem.

Nevertheless, determining the superiority of any of the three algorithms is challenging, as the results may vary depending on the specific values of their tuning parameters or the particular systems to which they are applied.

## C. Congestion Control

Consider a traffic management scenario at an intersection, as illustrated in Fig. 6. Three major roads (A, B, and C) merge into a "triangular junction" regulated by traffic signals. Three buffer variables, $x_1$, $x_2$, and $x_3$, indicate the number of vehicles waiting at the respective traffic signals within the triangular junction. The traffic signal configurations are assumed to be symmetric. In the first configuration, as shown in Fig. 6, the traffic signals corresponding to $x_1$, $x_2$, B, and C are green, while those corresponding to $x_3$ and A are red. Accordingly,

1) the buffer variable $x_3$ increases proportionally (this is described by the positive constant $\beta$) with respect to $x_2$;
2) the buffer variable $x_2$ remains approximately constant, receiving inflow from both road B and buffer $x_1$, while providing outflow to road A and buffer $x_3$;
3) the buffer variable $x_1$ exponentially decreases as the inflow from road C is directed entirely toward $x_2$ and road B (this

is due to $-\gamma < 0$). This exponential decrease accounts for the initial transient effect caused by traffic signal changes.

The other two configurations are obtained by a circular rotation of $x_1$, $x_2$, and $x_3$ (as well as of A, B, and C). The system description is completed by considering the effect of a constant input, i.e., the incoming traffic. Hence, the congestion control problem can be modeled as a switched affine system of the form (1), with

$$A_1 = \begin{bmatrix} -\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & \beta \\ 0 & -\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b_2 = b_1$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & -\gamma \end{bmatrix}, \quad b_3 = b_1$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $\gamma = 1$ and $\beta = 1.1$. In [11], it is demonstrated that there does not exist any convex combination $A_\lambda = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3$, such that $A_\lambda$ is Hurwitz, thereby failing to satisfy condition (49). Consequently, switching strategies that depend on the existence of a Hurwitz convex combination of the matrices $A_i$ (such as [14], [23], and [26]) cannot be applied to the congestion control problem. A periodic (open-loop) dwell-time switching law implementing the circular activation order of the traffic lights as $3, 2, 1, 3, 2, 1, \ldots$ is proposed in [11], and it is proved that this switching law stabilizes the switched affine system for dwell-time $T > 0.19$. This commutation sequence implies that the "red light" is imposed according to the circular order $3, 2, 1, 3, 2, 1, \ldots$. The same study shows that alternative switching sequences not only degrade system performance but can also render it unstable. By leveraging the insights provided in Section V, the matrix $\Pi$ is set according to the expected probability of switching from one operating mode to the next one. Hence, for the congestion control problem, $\Pi$ is set as

$$\Pi = \alpha \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \alpha > 0$$

with each row of $\Pi$ representing the probability of the Markov chain associated with the expected sequence $3, 2, 1, 3, 2, 1, \ldots$.

Similarly to the approach in [11], the proposed switching strategy is tested in simulation with $T = 2.1$ and $x(0) = [10\ 10\ 10]'$. The resulting state trajectories are presented in Fig. 7. As illustrated, the state converges to a limit cycle, while the switching signal $\sigma(t)$, generated by the switching strategy in Theorem 3, and shown in Fig. 8, autonomously enforces the sequence $3, 2, 1, 3, 2, 1, \ldots$, as proposed in [11]. A 3-D representation of the state trajectories is shown in Fig. 9 to further emphasize the cyclic behavior induced by the switching signal.

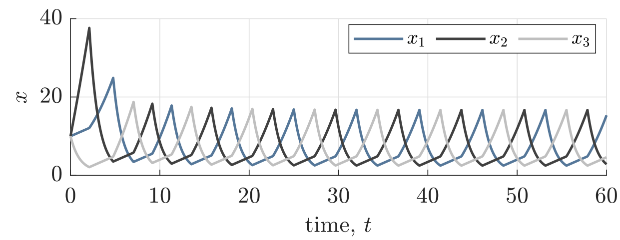


Fig. 7. Number of vehicles waiting at the three traffic lights inside the triangular loop.

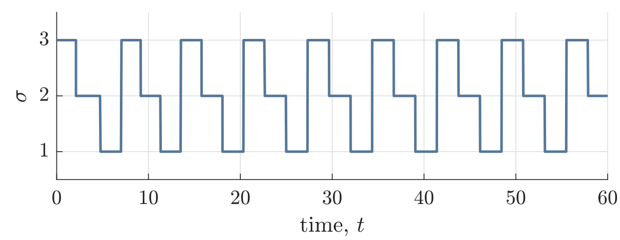


Fig. 8. Switching signal $\sigma(t)$ for the congestion control problem.

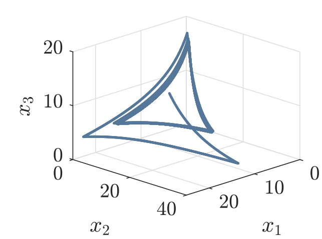


Fig. 9. State trajectory in $(x_1, x_2, x_3)$ space for the congestion control problem.

## VII. Conclusion

In this article, we have studied the stabilization of linear switched affine systems under dwell-time constraint about a neighborhood of the origin. The synthesis of two state-dependent switching rules is presented. First, the case of linear switched systems has been considered to regulate their state to the origin. Then, the case of switched affine systems has been addressed, proving practical stabilization of the systems' state in a vicinity of the origin. Both strategies rely on the solution of differential Lyapunov equalities and Lyapunov–Metzler inequalities, allowing us to guarantee a bound on the cost associated with the switching control laws. Special attention has been devoted to the minimization of such a bound in the case of affine systems, in order to guarantee suitable performance of the system response. Moreover, guidelines on the parameters tuning have been provided. The general validity of the proposed methods relies on their independence on the existence of a convex combination of subsystems that generates a Hurwitz average linear system. The theoretical results have been assessed in simulation and fairly compared with other methodologies in the literature.

An area of further work involves the application of these methods to the case of systems relying on polytopic Lyapunov functions [33]. Furthermore, for the sake of practical implementation, it would be of interest to adapt the proposed switching algorithm to the case of periodic and aperiodic sampled measurement of the state, as done in [17], where, however, the dwell-time constraint was not taken into account.

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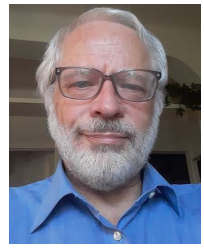

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