Casimir energy and black hole pair creation in Schwarzschild-de Sitter spacetime

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Following the subtraction procedure for manifolds with boundaries, we calculate by variational methods, the Schwarzschild-de Sitter and the de Sitter space energy difference. By computing the one loop approximation for TT tensors we discover the existence of an unstable mode even for the non-degenerate case. This result seems to be in agreement with the sub-maximal black hole pair creation of Bousso-Hawking. The instability can be eliminated by the boundary reduction method. Implications on a foam-like space are discussed.

I. INTRODUCTION

An intriguing property of quantum physics is the particle creation generated by external fields, like a constant electric or magnetic field, or by quantum fluctuation of the vacuum. In this last case only virtual particles are involved. However when the energy scale is large enough virtual particles can be transformed into real. In line of principle the same mechanism can be shared by the gravitational field where virtual black holes [1,2] can be created and annihilated in analogy with particle physics. This particular phenomenon has been investigated in different contexts and in particular when a cosmological constant is introduced [1,3]. In this example the process is mediated by the corresponding gravitational instanton, and the semiclassical nucleation rate for a pair on a given background is given by

$$\Gamma = A \exp\left[-\left(I_{inst} - I_{back}\right)\right]. \tag{1}$$

I_{inst} is the classical action of the gravitational instanton mediating the pair creation, I_{back} is the action of the background field, and A is the prefactor containing quantum corrections. For the de Sitter (dS) space the quantum creation of black holes leads to the discovery of an unstable mode in the physical sector, when one-loop approximation is considered [4–6]. This quantum instability is related to the $S^2 \times S^2$ instanton responsible for the pair creation process. This instanton, termed the Nariai instanton [7], is nothing but the extreme Schwarzschild-de Sitter (SdS) solution written in another system of coordinates. This instability leads to spontaneous nucleation of black holes signaling a transition from a false vacuum to a true one [8]. This transition is possible when the energy stored in the boundaries is the same for both spaces [9]. However as remarked in Ref. [6], the nucleation appears with a temperature $T_{pair} = \frac{\sqrt{\Lambda_c}}{2\pi}$ different from the temperature of the heat bath, which is the dS space with $T_{dS} = \frac{1}{2\pi}\sqrt{\frac{\Lambda_c}{3}}$. This does not happen, for example, in the hot Minkowski space where the nucleated black hole has the same temperature as the heat bath [10]. The same situation holds even when we consider a negative cosmological constant, i.e. Anti-de Sitter (AdS) space. In fact to spontaneously nucleate a black hole, which has an intrinsic temperature T_{S-AdS}, the same temperature has to be imposed to the AdS space [11–13]. However in Ref. [14], we have shown that a semi-classical instability (WKB) appears for Minkowski space even at zero temperature, provided that boundary conditions be energy preserving. The same semi-classical instability appears also for the AdS space [15] with the same energy condition. An interesting common feature between these cases comes from the expression of the mixed Ricci tensor R_i^a computed in these different backgrounds, namely the Schwarzschild, S-AdS and SdS metrics respectively. Indeed in the first case, R_i^a has components:

$$R_i^a = \left\{-\frac{2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3}\right\}, \tag{2}$$

while the case with the cosmological constant is

$$R_i^a = \left\{-\frac{2MG}{r^3} \pm 2/b^2, \frac{MG}{r^3} \pm 2/b^2, \frac{MG}{r^3} \pm 2/b^2\right\} \tag{3}$$

where the upper case and the lower case are related to the SdS and the S-AdS metrics respectively and $b^2 = 3/\Lambda_c$. Λ_c is the positive cosmological constant. It is straightforward to note that the only difference between these different tensors is in the presence of the cosmological term. This means that the source of instability residing in the first

component appears even in the other cases. However, only at one loop is possible to reveal the presence or the absence of such an instability. We recall that one loop computations of the energy in this context represent a Casimir-like energy which measures vacuum fluctuations. Following Refs. [14,15], we will consider a constant time slice Σ of the SdS manifold $\mathcal{M}$[1], whose perturbations at Σ in absence of matter fields define quantum fluctuations of the Einstein-Rosen bridge. Indeed, as will follow in Section II even though the SdS is not asymptotically flat, the hypersurface Σ defines a wormhole with topology $S^2 \times I$, where $I \subset R$ is a sub-interval of R. This is a consequence of having a cosmological radius which sets an upper bound to the radial coordinate. To this purpose we will fix our attention on a Hamiltonian with boundary

$$H_T = H_\Sigma + H_{\partial\Sigma} = \int_\Sigma d^3x(N\mathcal{H}+N_i\mathcal{H}^i) + H_{\partial\Sigma}, \tag{4}$$

where N is called the *lapse* function, N_i is the *shift* function and

$$\begin{cases} \mathcal{H}=G_{ijkl}\pi^{ij}\pi^{kl}\left(\frac{16\pi G}{\sqrt{g}}\right) - \left(\frac{\sqrt{g}}{16\pi G}\right)\left(R^{(3)} - \frac{6}{b^2}\right) \\ \mathcal{H}^i = -2\pi^{ij}_{|j} \end{cases}. \tag{5}$$

$H_{\partial\Sigma}$ represents the energy stored into the boundary. The aim of this paper is the evaluation of

$$E^{SdS}(M,b) = E^{dS}(b) + \Delta E^{SdS}_{dS}(M,b)_{|classical} + \Delta E^{SdS}_{dS}(M,b)_{|1-loop}, \tag{6}$$

representing the total energy computed to one-loop in a SdS background. $E^{dS}(b)$ is the reference space energy, i.e. the de Sitter space. $\Delta E^{SdS}_{dS}(M,b)_{|classical}$ is the energy difference between the SdS and the dS metrics, stored in the boundaries and $\Delta E^{SdS}_{dS}(M,b)_{|1-loop}$ is the quantum correction to the classical term. The rest of the paper is structured as follows, in section II we compute the quasilocal energy for the SdS space, in section III we give some of the basic rules to perform the functional integration to evaluate the energy density of the Hamiltonian approximated up to second order in the SdS background, in section IV we look for stable modes of the spin-two operator acting on transverse traceless tensors, in section V we show the existence of only one negative mode under suitable conditions and we compute the energy density for stable modes, in section VI, we confirm the existence of one negative mode for the extreme SdS background, namely the Nariai metric, also in our framework and we give a computation for the stable part in analogy with its non-extreme sector, in section VII we find a critical radius below which we have a stabilization of the system. We summarize and conclude in section VIII.

II. QUASILOCAL ENERGY FOR THE SDS SPACE

In this section we fix our attention to the classical part of Eq.(6). We begin to define the line element

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2 d\Omega^2, \tag{7}$$

referred to the SdS metric, where

$$f(r) = \left(1 - \frac{2MG}{r} - \frac{r^2}{b^2}\right). \tag{8}$$

For $\Lambda_c = 0$ the metric describes the Schwarzschild metric, while for $M = 0$, we obtain the de Sitter metric (dS)

$$ds^2 = -\left(1 - \frac{r^2}{b^2}\right)dt^2 + \left(1 - \frac{r^2}{b^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{9}$$

The gravitational potential $f(r)$ admits three real roots. One is negative and it is located at

$$r_- = \frac{2}{\sqrt{3}} b \cos\left(\frac{\theta + 2\pi}{3}\right), \tag{10}$$

[1]In Appendix A, we will report the details concerning the Kruskal-Szekeres description of the SdS manifold.

while

$$r_+ = \frac{2}{\sqrt{3}} b \cos\left(\frac{\theta + 4\pi}{3}\right),\ r_{++} = \frac{2}{\sqrt{3}} b \cos\left(\frac{\theta}{3}\right) \tag{11}$$

are associated to the black hole and cosmological horizons respectively, with

$$\cos\theta = -3MG\sqrt{3}/b. \tag{12}$$

However in the wormhole language, we will say that r_+ is the inner throat and r_{++} is the outer throat. Note also that the hypersurface Σ is described by the three-dimensional wormhole whose metric is

$$ds^2 = f(r)^{-1} dr^2 + r^2 d\Omega^2, \tag{13}$$

where $f(r)$ is given by Eq.(8). A relation between the three roots is given by

$$\begin{cases} b^2 = r_+^2 + r_+ r_{++} + r_{++}^2 \\ 2Ml_p^2 b^2 = (r_+ r_{++})(r_+ + r_{++}) \\ 0 = r_- + r_+ + r_{++} \end{cases}. \tag{14}$$

Thus , we can write

$$f(r) = -\frac{1}{rb^2}(r - r_+)(r - r_{++})(r + r_+ + r_{++}), \tag{15}$$

with

$$r_+ \leq r \leq r_{++} \qquad \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]. \tag{16}$$

Since r_+ is a monotonic increasing function of θ, while r_{++} is a monotonic decreasing with

$$\begin{cases} r_+^{\uparrow} \in [0, b] \\ r_{++}^{\downarrow} \in [0, b] \end{cases}, \tag{17}$$

in order to have the inequality (16) preserved, we have to consider $\theta \in \left[\frac{\pi}{2}, \pi\right]$. Indeed when $\theta \in \left[\pi, \frac{3\pi}{2}\right]$ the inequality (16) is reversed and the meaning of the internal and external roots is exchanged. Thus the de Sitter region delimited by the bound of Eq.(16), at time fixed, can be represented as in Fig.1.

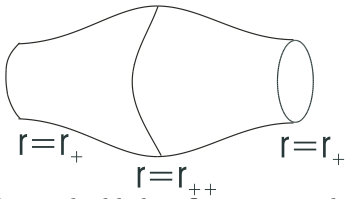

FIG. 1. The geometry of the constant time slice embedded in flat space with a polar angle suppressed. Isometric copies of this surface can be smoothly joined at the throats, producing a periodic $S^2 \times R^1$ spatial topology.

A common value is reached when $\theta = \pi$ where $r_+ = r_{++} = b/\sqrt{3}$ and the metric is termed extreme. This particular case will be discussed in section VI. However, instead of looking at the de Sitter region with the topology of Fig.1, we will look at the Einstein-Rosen bridge corresponding to the inner bifurcation surface depicted in Fig.2.

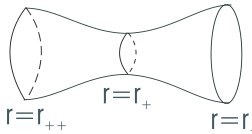

FIG. 2. The same representation of Fig.1 but with the cosmological "*wormhole mouths*" placed in antipodal region of the de Sitter universe with a periodic $S^2 \times R^1$ spatial topology.

At this point we can discuss the computation of the classical energy term

$$E^{SdS}(M,b) = E^{dS}(b) + \Delta E^{SdS}_{dS}(M,b)_{|classical}\,, \tag{18}$$

which can be computed by means of quasilocal energy. Quasilocal energy is defined as the value of the Hamiltonian that generates unit time translations orthogonal to the two-dimensional boundary,

$$\Delta E^{SdS}_{dS}(M,b)_{|classical} = \frac{1}{8\pi G}\int_{S^2} d^2x\sqrt{\sigma}\left(k-k^0\right), \tag{19}$$

where $|N|=1$ at S^2 and k^0 is the trace of the extrinsic curvature corresponding to the reference space, which in this case is the dS space. For practical purposes, however, it is convenient to embed both spaces (Sds and dS) into flat space and perform the subtraction procedure. To this purpose the radial coordinate x continuous on $\mathcal{M}$ is defined by

$$dx = \pm\frac{dr}{\sqrt{1-\frac{2MG}{r}-\frac{r^2}{b^2}}}, \tag{20}$$

where the plus sign is relative to Σ_+, while the minus sign is related to Σ_-. The surfaces located at r_+ and r_{++} are bifurcation surfaces denoted S^0_+ and S^0_{++}, respectively. When $M=0$, we obtain the embedding of dS space into flat space. In Σ_+ the evaluation of $\Delta E^{SdS}_{dS}(M,b)_{|classical}$ can be obtained as follows: first we consider the static Einstein-Rosen bridge associated to the SdS space [16,17]

$$ds^2 = -N^2(r)\,dt^2 + g_{xx}dx^2 + r^2(x)\,d\Omega^2, \tag{21}$$

where N, g_{xx}, and r are functions of x defined by Eq.(20). Second, we consider the boundary S^2_+, located at $x(r)=\bar{x}^+(R)$, and its associated normal $n^\mu = (h^{xx})^{\frac{1}{2}}\delta^\mu_y$. The expression of the trace

$$k = -\frac{1}{\sqrt{h}}\left(\sqrt{h}n^\mu\right)_{,\mu}, \tag{22}$$

gives for the SdS space

$$k^{SdS} = -2\frac{r_{,x}}{r}_{|SdS} = -2\frac{\sqrt{f(r)}}{r}_{|SdS} = -\frac{2}{r}\sqrt{1-\frac{2MG}{r}-\frac{r^2}{b^2}}. \tag{23}$$

Note that if we make the identification $N^2 = 1-\frac{2MG}{r}-\frac{r^2}{b^2}$, the line element (21) reduces to the SdS metric written in another form. The same applies to the dS metric by putting $M=0$. Nevertheless for our purposes the form of $N(r)$ can be left unspecified. Thus the computation of E_+ gives

$$\Delta E^{SdS}_{dS}(M,b)_{|classical} = \frac{1}{8\pi G}\int_{S^2} d\Omega^2 r^2\left[\frac{-2\sqrt{f(r)}}{r}+\frac{2\sqrt{f(r)_{|M=0}}}{r}\right]_{|r=R} \tag{24}$$

$$= -\frac{R}{G}\left[\sqrt{1-\frac{2MG}{R}-\frac{R^2}{b^2}}-\sqrt{1-\frac{R^2}{b^2}}\right], \tag{25}$$

where we have set $M=0$ in k^{SdS} to obtain the dS energy contribution. When $R\gg b$, $\Delta E^{SdS}_{dS}(M,b)_{|classical} \simeq -iMb/R$. Thus for every finite value of the boundary exceeding the cosmological radius, the classical energy acquires an imaginary component which will not be here considered[2]. In contrast, if we consider the approximation $R/b \ll 1$ and $2MG/R \ll 1$ [18], we obtain

[2]To deal with this case it is better to introduce the quasilocal mass, defined as

$$\Delta M^{SdS}_{dS}(M,b)_{|classical} = \frac{1}{8\pi G}\int_{S^2} d^2x N\sqrt{\sigma}\left(k-k^0\right)$$

$$\underset{R\gg b}{\simeq}\left(-iM\frac{b}{R}\right)\left(i\frac{R}{b}\right) = M.$$

$$\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}\simeq-\frac{R}{G}\left[\left(1-\frac{MG}{R}-\frac{R^2}{2b^2}\right)-\left(1-\frac{R^2}{2b^2}\right)\right]=M. \tag{26}$$

Thus the energy contribution limited to Σ_+ gives

$$E^{SdS}\left(M,b\right)=E^{dS}\left(b\right)+\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{+}=E^{dS}\left(b\right)+M, \tag{27}$$

that is the dS space cannot decay into the SdS space because the associated boundary energy is different. This is in complete analogy with the Schwarzschild and the S-AdS cases. However if we look at the whole hypersurface $\Sigma=\Sigma_+\cup\Sigma_-$ the total classical energy becomes

$$E^{SdS}\left(M,b\right)=E^{dS}\left(b\right)+E_{tot}\left(M,b\right)$$

$$=E^{dS}\left(b\right)+\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{+}+\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{-} \tag{28}$$

with

$$\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{+}=\frac{1}{8\pi G}\int_{S_+^2}d^2x\sqrt{\sigma}\left(k-k^0\right),$$

$$\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{-}=-\frac{1}{8\pi G}\int_{S_-^2}d^2x\sqrt{\sigma}\left(k-k^0\right). \tag{29}$$

Here the boundaries S_+^2 and S_-^2 are located in the two disconnected regions $\mathcal{M}_+$ and $\mathcal{M}_-$ respectively with coordinate values $x=\bar{x}^{\pm}$ and the trace of the extrinsic curvature in both regions is

$$k^{SdS}=\begin{cases}-2r_{,x}/r & on\ \Sigma_+\\ 2r_{,x}/r & on\ \Sigma_-\end{cases}. \tag{30}$$

Thus one gets

$$\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{\pm}=\begin{cases}M & on\ S_+^2\\ -M & on\ S_-^2\end{cases}, \tag{31}$$

where for E_- we have used the conventions relative to Σ_- and S_-^2. Therefore for every value of the boundary R, (provided we take symmetric boundary conditions with respect to the bifurcation surface), we have

$$E^{SdS}\left(M,b\right)=E^{dS}\left(b\right)+M+\left(-M\right)=E^{dS}\left(b\right), \tag{32}$$

namely the energy is conserved. As stressed in Ref. [16], since we have a spacetime with a bifurcation surface, the quantities $\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{+}$ and $\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{-}$ have the same relative sign, while the total energy is given by the sum $\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{+}+\Delta E_{dS}^{SdS}\left(M,b\right)_{|classical}^{-}$[3]. The energies associated to the boundaries are symmetric and they have the same relative sign while the total energy reflects the orientation reversal of the two boundaries. Since the total classical energy is conserved we can discuss the existence of an instability. To this aim we refer to the variational approach to compute the energy density to one-loop [14,15,19–21].

III. ENERGY DENSITY CALCULATION IN SCHRÖDINGER REPRESENTATION

In previous section we have fixed our attention to the classical part of Eq.(6). In this section, we apply the same calculation scheme of Refs. [14,15] to compute one loop corrections to the classical SdS term. Like the Schwarzschild and the S-AdS case, there appear two classical constraints

[3] In Ref. [16] we have a subtraction instead of a sum. This is due to conventions adopted.

$$\begin{cases} \mathcal{H} = 0 \\ \mathcal{H}^i = 0 \end{cases} \tag{33}$$

and two *quantum* constraints

$$\begin{cases} \mathcal{H}\tilde{\Psi} = 0 \\ \mathcal{H}^i\tilde{\Psi} = 0 \end{cases} \tag{34}$$

for the Hamiltonian respectively, which are satisfied both by the SdS and dS metric on shell. $\mathcal{H}\tilde{\Psi} = 0$ is known as the *Wheeler-DeWitt* equation (WDW). Our purpose is the computation of

$$\Delta E_{dS}^{SdS}(M,b)_{|1-loop} = \frac{\langle \Psi | H_{\Sigma}^{SdS} - H_{\Sigma}^{dS} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \tag{35}$$

where H_{Σ}^{SdS} and H_{Σ}^{dS} are the total Hamiltonians referred to the SdS and dS spacetimes respectively for the volume term [14] and Ψ is a wave functional obtained following the usual WKB expansion of the WDW solution[4]. In this context, the approximated wave functional will be substituted by a *trial wave functional* of the gaussian form according to the variational approach we shall use to evaluate $\Delta E_{dS}^{SdS}(M,b)_{|1-loop}$. To compute such a quantity we will consider on Σ deviations from the SdS metric spatial section of the form,

$$g_{ij} = \bar{g}_{ij} + h_{ij}, \tag{36}$$

where

$$\bar{g}_{ij}dx^i dx^j = \left(1 - \frac{2MG}{r} - \frac{r^2}{b^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{37}$$

is the spatial SdS background. By setting $M = 0$ in Eq.(8) on the same slice we will obtain perturbations also for the de Sitter metric. Thus the expansion of the three-scalar curvature $\int d^3x\sqrt{\bar{g}}R^{(3)}$ up to $o\left(h^2\right)$ gives

$$\int_{\Sigma} d^3x\sqrt{\bar{g}}\left[-\frac{1}{4}h\triangle h + \frac{1}{4}h^{li}\triangle h_{li} - \frac{1}{2}h^{ij}\nabla_l\nabla_i h^l_j + \frac{1}{2}h\nabla_l\nabla_i h^{li} - \frac{1}{2}h^{ij}R_{ia}h^a_j + \frac{1}{2}hR_{ij}h^{ij}\right]$$

$$+ \int_{\Sigma} d^3x\sqrt{\bar{g}}\left[\frac{1}{4}h^{li}\left(R^{(0)} - 6/b^2\right)h_{li} - \frac{1}{4}h^{li}\left(R^{(0)}\right)h_{li} + \frac{1}{4}h\left(R^{(0)}\right)h\right], \tag{38}$$

where $R^{(0)}$ is the three-scalar curvature on-shell. To explicitly make calculations, we need an orthogonal decomposition for both π_{ij} and h_{ij} to disentangle gauge modes from physical deformations. We define the inner product

$$\langle h,k\rangle := \int_{\Sigma} \sqrt{\bar{g}}G^{ijkl}h_{ij}(x)\,k_{kl}(x)\,d^3x, \tag{39}$$

by means of the inverse WDW metric G_{ijkl}, to have a metric on the space of deformations, i.e. a quadratic form on the tangent space at h, with

$$G^{ijkl} = (g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}). \tag{40}$$

The inverse metric is defined on co-tangent space and it assumes the form

$$\langle p,q\rangle := \int_{\Sigma} \sqrt{\bar{g}}G_{ijkl}p^{ij}(x)\,q^{kl}(x)\,d^3x, \tag{41}$$

so that

$$G^{ijnm}G_{nmkl} = \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right). \tag{42}$$

[4]See also [22–24] for other applications of the WKB approximation concerning black hole physics.

Note that in this scheme the "inverse metric" is actually the WDW metric defined on phase space. Now, we have the desired decomposition on the tangent space of 3-metric deformations [25,26]:

$$h_{ij} = \frac{1}{3} h g_{ij} + (L\xi)_{ij} + h^{\perp}_{ij} \tag{43}$$

where the operator L maps ξ_i into symmetric tracefree tensors

$$(L\xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} (\nabla \cdot \xi) . \tag{44}$$

Then the inner product between three-geometries becomes

$$\langle h, h \rangle := \int_{\Sigma} \sqrt{g} G^{ijkl} h_{ij}(x) h_{kl}(x) d^3x =$$

$$\int_{\Sigma} \sqrt{g} \left[-\frac{2}{3} h^2 + (L\xi)^{ij} (L\xi)_{ij} + h^{ij\perp} h^{\perp}_{ij} \right] . \tag{45}$$

With the orthogonal decomposition in hand we can define the trial wave functional

$$\Psi [h_{ij}(\overrightarrow{x})] = \mathcal{N} \exp \left\{ -\frac{1}{4 l_p^2} \left[\left\langle h K^{-1} h \right\rangle^{\perp}_{x,y} + \left\langle (L\xi) K^{-1} (L\xi) \right\rangle^{\parallel}_{x,y} + \left\langle h K^{-1} h \right\rangle^{Trace}_{x,y} \right] \right\} , \tag{46}$$

where $\mathcal{N}$ is a normalization factor. Since we are only interested in the perturbations of the physical degrees of freedom, we will only fix our attention on the TT (traceless and transverseless) tensor sector, therefore reducing the previous form into

$$\Psi [h_{ij}(\overrightarrow{x})] = \mathcal{N} \exp \left\{ -\frac{1}{4 l_p^2} \left\langle h K^{-1} h \right\rangle^{\perp}_{x,y} \right\} . \tag{47}$$

This restriction is motivated by the fact that if an instability appears this will be in the physical sector referred to TT tensors, namely a nonconformal instability. This choice seems to be corroborated by the action decomposition of [27], where only TT tensors contribute to the partition function[5]. To calculate the energy density, we need to know the action of some basic operators on $\Psi [h_{ij}]$ [19]. The action of the operator h_{ij} on $|\Psi\rangle = \Psi [h_{ij}]$ is realized by

$$h_{ij}(x) |\Psi\rangle = h_{ij}(\overrightarrow{x}) \Psi [h_{ij}] , \tag{48}$$

while the action of the operator π_{ij} on $|\Psi\rangle$, in general, is

$$\pi_{ij}(x) |\Psi\rangle = -i \frac{\delta}{\delta h_{ij}(\overrightarrow{x})} \Psi [h_{ij}] . \tag{49}$$

The inner product is defined by the functional integration

$$\langle \Psi_1 \mid \Psi_2 \rangle = \int [\mathcal{D} h_{ij}] \Psi_1^* \{h_{ij}\} \Psi_2 \{h_{kl}\} \tag{50}$$

and by applying previous functional integration rules, we obtain the expression of the one-loop-like Hamiltonian form for TT (traceless and transverseless) deformations [14,15]

$$H^{\perp} = \frac{1}{4 l_p^2} \int_{\mathcal{M}} d^3x \sqrt{g} G^{ijkl} \left[K^{-1\perp}(x,x)_{ijkl} + (\triangle_2)^a_j K^{\perp}(x,x)_{iakl} \right] . \tag{51}$$

The propagator $K^{\perp}(x,x)_{iakl}$ comes from a functional integration and it can be represented as

$$K^{\perp}(\overrightarrow{x}, \overrightarrow{y})_{iakl} := \sum_N \frac{h^{\perp}_{ia}(\overrightarrow{x}) h^{\perp}_{kl}(\overrightarrow{y})}{2 \lambda_N (p)} , \tag{52}$$

where $h^{\perp}_{ia}(\overrightarrow{x})$ are the eigenfunctions of $\triangle^a_{2j}$ and $\lambda_N (p)$ are infinite variational parameters.

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[5] See also [28] for another point of view.

IV. THE SCHWARZSCHILD-DE SITTER METRIC SPIN 2 OPERATOR AND THE EVALUATION OF THE ENERGY DENSITY

The Spin-two operator for the SdS metric is defined by

$$(\triangle_2)_j^a := -\triangle\delta_j^a + 2R_j^a - 6/b^2\delta_j^a \tag{53}$$

where $\triangle$ is the curved Laplacian (Laplace-Beltrami operator) on a SdS background and R_j^a is the mixed Ricci tensor whose components are:

$$R_i^a = \left\{-\frac{2MG}{r^3} + 2/b^2, \frac{MG}{r^3} + 2/b^2, \frac{MG}{r^3} + 2/b^2\right\}. \tag{54}$$

As stressed in the introduction, the form of the mixed Ricci tensor for the SdS space has the same dependence for the radial coordinate for both the Schwarzschild and the S-AdS spaces. Thus to evaluate the energy density, we are led to study the following eigenvalue equation

$$\left(-\triangle\delta_j^a + 2R_j^a - 6/b^2\delta_j^a\right) h_a^i = E^2 h_j^i \tag{55}$$

where E^2 is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation as modes of definite frequency, angular momentum and parity [29]. The quantum number corresponding to the projection of the angular momentum on the z-axis will be set to zero. This choice will not alter the contribution to the total energy since we are dealing with a spherical symmetric problem. In this case, Regge-Wheeler decomposition shows that the even-parity three-dimensional perturbation is

$$h_{ij}^{even}(r,\vartheta,\phi) = diag\left[H(r)\left(1 - \frac{2MG}{r} - \frac{r^2}{b^2}\right)^{-1}, r^2K(r), r^2\sin^2\vartheta K(r)\right] Y_{l0}(\vartheta,\phi). \tag{56}$$

In this representation $H(r)$ and $K(r)$ behave as they were scalar fields. For a generic value of the angular momentum L, one gets

$$\begin{cases} \left(-\triangle_l - \frac{4MG}{r^3} - \frac{2}{b^2}\right) H(r) = E_{l,H}^2 H(r) \\ \left(-\triangle_l + \frac{2MG}{r^3} - \frac{2}{b^2}\right) K(r) = E_{l,K}^2 K(r) \end{cases}, \tag{57}$$

where $E_{l,H}^2$ and $E_{l,K}^2$ are the eigenvalues for the $H(r)$ field and the $K(r)$ field respectively. The Laplacian restricted to Σ is

$$\triangle_l = \left(1 - \frac{2MG}{r} - \frac{r^2}{b^2}\right)\frac{d^2}{dr^2} + \left(\frac{2r - 3MG}{r^2} - 3\frac{r}{b^2}\right)\frac{d}{dr} - \frac{l(l+1)}{r^2}. \tag{58}$$

Defining reduced fields

$$H(r) = \frac{h(r)}{r}; \qquad K(r) = \frac{k(r)}{r}, \tag{59}$$

and passing to the proper geodesic distance from the *throat* of the bridge defined by Eq.(20), the system (57) becomes[6]

$$\begin{cases} -\frac{d^2}{dx^2}h(x) + \left(V_l^-(x) - \frac{3}{b^2}\right) h(x) = E_l^2 h(x) \\ -\frac{d^2}{dx^2}k(x) + \left(V_l^+(x) - \frac{3}{b^2}\right) k(x) = E_l^2 k(x) \end{cases} \tag{60}$$

with

$$V_l^{\mp}(x) = \frac{l(l+1)}{r^2(x)} \mp \frac{3MG}{r(x)^3}. \tag{61}$$

[6]The system is invariant in form if we make the minus choice in Eq.(20).

When $r \longrightarrow r_0 > r_+$

$$x(r) \simeq \sqrt{2\kappa_+ (r - r_+)} \qquad V_l^{\mp}(x) \longrightarrow \frac{l(l+1)}{r_0^2} \mp \frac{3MG}{r_0^3} = const, \tag{62}$$

where

$$\kappa_+ = \lim_{r \to r_+} \frac{1}{2} |g'_{00}(r)| = \frac{(r_+ - r_-)(r_{++} - r_+)}{2b^2 r_+} \tag{63}$$

is the "inner" surface gravity associated with the smallest root. The solution of (60) when $r \longrightarrow r_0 > r_+$ for both backgrounds is

$$h(px) = k(px) = \sqrt{\frac{2}{\pi}} \sin(px). \tag{64}$$

This choice is dictated by the requirement that

$$h(x), k(x) \to 0 \qquad \text{when} \qquad x(r) \to x(r_+) \simeq 0. \tag{65}$$

Thus the propagator becomes

$$K_{\pm}^{\perp}(x,y) = \frac{V}{2\pi^2} \int_0^{\infty} dpp^2 \frac{\sin(px)}{r(x)} \frac{\sin(py)}{r(y)} \frac{Y_{l0}(\vartheta,\phi) Y_{l'0}(\vartheta,\phi)}{\lambda_{\pm}(p)} \tag{66}$$

$\lambda_{\pm}(p)$ is referred to the potential function $V_l^{\pm}(x)$. Substituting Eq.(66) in Eq.(51) one gets (after normalization in spin space and after a rescaling of the fields in such a way as to absorb l_p^2)

$$E(M,b,\lambda) = \frac{V}{8\pi^2} \sum_{l=0}^{\infty} \sum_{i=1}^{2} \int_0^{\infty} dpp^2 \left[\lambda_i(p) + \frac{E_i^2(p,M,b,l)}{\lambda_i(p)} \right] \tag{67}$$

where

$$E_{1,2}^2(p,M,b,l) = p^2 + \frac{l(l+1)}{r_0^2} \mp \frac{3MG}{r_0^3} - \frac{3}{b^2}, \tag{68}$$

$\lambda_i(p)$ are variational parameters corresponding to the eigenvalues for a (graviton) spin-two particle in an external field and V is the volume of the system. By minimizing (67) with respect to $\lambda_i(p)$ one obtains $\overline{\lambda}_i(p) = \left[E_i^2(p,M,b,l)\right]^{\frac{1}{2}}$ and

$$E\left(M,b,\overline{\lambda}\right) = \frac{V}{8\pi^2} \sum_{l=0}^{\infty} \sum_{i=1}^{2} \int_0^{\infty} dpp^2 2\sqrt{E_i^2(p,M,b,l)} \tag{69}$$

with

$$p^2 + \frac{l(l+1)}{r_0^2} - \frac{3MG}{r_0^3} - \frac{3}{b^2} > 0.$$

For the SdS background we get

$$E(M,b) = \frac{V}{4\pi^2} \sum_{l=0}^{\infty} \int_0^{\infty} dpp^2 \left(\sqrt{p^2 + c_-^2} + \sqrt{p^2 + c_+^2} \right) \tag{70}$$

where

$$c_{\mp}^2 = \frac{l(l+1)}{r_0^2} \mp \frac{3MG}{r_0^3} - \frac{3}{b^2},$$

while when we refer to the dS space we put $M = 0$ and $c^2 = l(l+1)/r_0^2 - 3/b^2$. Here the meaning of the value r_0 is that of maintaining the same boundary conditions to correctly compute the Casimir-like energy. Then

$$E\left(b\right)=\frac{V}{4\pi^2}\sum_{l=0}^{\infty}\int_0^{\infty}dpp^2\left(2\sqrt{p^2+c^2}\right) \tag{71}$$

Now, we are in position to compute the difference between (70) and (71). Since we are interested in the ultraviolet limit, we have

$$\Delta E\left(M,b\right)=E\left(M,b\right)-E\left(b\right)$$

$$=\frac{V}{4\pi^2}\sum_{l=0}^{\infty}\int_0^{\infty}dpp^2\left[\sqrt{p^2+c_-^2}+\sqrt{p^2+c_+^2}-2\sqrt{p^2+c^2}\right]$$

$$=\frac{V}{4\pi^2}\sum_{l=0}^{\infty}\int_0^{\infty}dpp^3\left[\sqrt{1+\left(\frac{c_-}{p}\right)^2}+\sqrt{1+\left(\frac{c_+}{p}\right)^2}-2\sqrt{1+\left(\frac{c}{p}\right)^2}\right] \tag{72}$$

and for $p^2>>c_{\mp}^2,c^2$, we obtain

$$\frac{V}{4\pi^2}\sum_{l=0}^{\infty}\int_0^{\infty}dpp^3\left[1+\frac{1}{2}\left(\frac{c_-}{p}\right)^2-\frac{1}{8}\left(\frac{c_-}{p}\right)^4+1+\frac{1}{2}\left(\frac{c_+}{p}\right)^2-\frac{1}{8}\left(\frac{c_+}{p}\right)^4\right.$$

$$\left.-2-\left(\frac{c}{p}\right)^2+\frac{1}{4}\left(\frac{c}{p}\right)^4\right]=-\frac{V}{2\pi^2}\frac{c_M^4}{8}\int_0^{\infty}\frac{dp}{p}, \tag{73}$$

where $c_M^2=3MG/r_0^3$. We will use a cut-off Λ to keep under control the UV divergence

$$\int_0^{\infty}\frac{dp}{p}\sim\int_0^{\frac{\Lambda}{c_M}}\frac{dx}{x}\sim\ln\left(\frac{\Lambda}{c_M}\right), \tag{74}$$

where $\Lambda\leq m_p$. Thus $\Delta E\left(M,b\right)$ for high momenta becomes

$$\Delta E\left(M,b\right)\sim-\frac{V}{2\pi^2}\frac{c_M^4}{16}\ln\left(\frac{\Lambda^2}{c_M^2}\right)=-\frac{V}{32\pi^2}\left(\frac{3MG}{r_0^3}\right)^2\ln\left(\frac{r_0^3\Lambda^2}{3MG}\right). \tag{75}$$

and Eq.(6) to one loop is

$$\left(E^{SdS}\left(M,b\right)-E^{dS}\left(b\right)\right)_{r\simeq r_0>r_+}=-\frac{V}{32\pi^2}\left(\frac{3MG}{r_0^3}\right)^2\ln\left(\frac{r_0^3\Lambda^2}{3MG}\right)$$

$$=-\frac{V}{32\pi^2}\left(\frac{3\left(r_+r_{++}\right)\left(r_++r_{++}\right)}{2\left(r_+^2+r_+r_{++}+r_{++}^2\right)r_0^3}\right)^2\ln\left(\frac{2\left(r_+^2+r_+r_{++}+r_{++}^2\right)r_0^3\Lambda^2}{3\left(r_+r_{++}\right)\left(r_++r_{++}\right)}\right), \tag{76}$$

where we have used Eq.(14). On the other hand, when $r\longrightarrow r_{++}$

$$x\left(r\right)\simeq\sqrt{2\kappa_{++}\left(r_{++}-r\right)}\qquad V_l^{\mp}\left(x\right)\longrightarrow\frac{l\left(l+1\right)}{r_{++}^2}\mp\frac{3MG}{r_{++}^3}=const, \tag{77}$$

where

$$\kappa_{++}=\lim_{r\to r_{++}}\frac{1}{2}\left|g_{00}'\left(r\right)\right|=\frac{\left(r_{++}-r_-\right)\left(r_{++}-r_+\right)}{2b^2r_{++}} \tag{78}$$

is the "outer" surface gravity associated with the largest root. By repeating the steps going from Eq.(67) to Eq.(76) with

$$E^2_{1,2}\left(p,M,b,l\right)=p^2+\frac{l\left(l+1\right)}{r^2_{++}}\mp\frac{3MG}{r^3_{++}}-\frac{3}{b^2}, \tag{79}$$

we obtain

$$\left(E^{SdS}\left(M,b\right)-E^{dS}\left(b\right)\right)_{r\simeq r_{++}}=-\frac{V}{32\pi^2}\left(\frac{3MG}{r^3_{++}}\right)^2\ln\left(\frac{r^3_{++}\Lambda^2}{3MG}\right)$$

$$=-\frac{V}{32\pi^2}\left(\frac{3r_+\left(r_++r_{++}\right)}{2\left(r^2_++r_+r_{++}+r^2_{++}\right)r^2_{++}}\right)^2\ln\left(\frac{2\left(r^2_++r_+r_{++}+r^2_{++}\right)r^2_{++}\Lambda^2}{3r_+\left(r_++r_{++}\right)}\right). \tag{80}$$

Like the Schwarzschild and the S-AdS cases, we observe that

$$\lim_{M\to0}\lim_{r_0\to r_+}\Delta E\left(M,b\right)\neq\lim_{r_0\to r_+}\lim_{M\to0}\Delta E\left(M,b\right). \tag{81}$$

This behavior seems to confirm that quantum effects come into play when we try to reach the inner throat. Differently to the Schwarzschild and S-AdS cases, here we have two scales corresponding to the two throats (horizons) of the metric. This is an artifact of the approximation we have adopted to deal with the differential equations of the system (60). By defining a scale variable $x=3MG/\left(r^3_0\Lambda^2\right)$ and a scale variable $y=3MG/\left(r^3_{++}\Lambda^2\right)$, we obtain

$$E^{SdS}\left(M,b\right)-E^{dS}\left(b\right)=\left(E^{SdS}\left(M,b\right)-E^{dS}\left(b\right)\right)_{r\simeq r_0>r_+}+\left(E^{SdS}\left(M,b\right)-E^{dS}\left(b\right)\right)_{r\simeq r_{++}}$$

$$=-\frac{V}{32\pi^2}\left[\left(\frac{3MG}{r^3_0}\right)^2\ln\left(\frac{r^3_0\Lambda^2}{3MG}\right)+\left(\frac{3MG}{r^3_{++}}\right)^2\ln\left(\frac{r^3_{++}\Lambda^2}{3MG}\right)\right]$$

$$=\frac{V\Lambda^4}{32\pi^2}\left[x^2\ln x+y^2\ln y\right]=\Delta E\left(x,y\right). \tag{82}$$

A stationary point is reached for $x=y=0$, namely the dS space and another stationary point is in $x=y=e^{-\frac{1}{2}}$. This last one represents a minimum of $\Delta E\left(x,y\right)$. This means that there is a probability that the dS spacetime will be subjected to a topology change and it will produce a SdS wormhole with a black hole pair[7] generated on the hypersurface Σ. To see if this is really possible, we have to establish if there exist unstable modes.

V. SEARCHING FOR NEGATIVE MODES

In this paragraph we look for negative modes of the eigenvalue equation (53). To this purpose we restrict the analysis to the S wave. Indeed, in this state the centrifugal term is absent and this gives the function $V\left(x\right)$ a potential well form, which is different when the angular momentum $l\geq1$. Moreover the potential well appears only for the H component, whose eigenvalue equation is

$$\left(-\triangle-\frac{4MG}{r^3}-\frac{2}{b^2}\right)H\left(r\right)=-E^2_0H\left(r\right). \tag{83}$$

$\triangle$ is the operator $\triangle_l$ of Eq.(58) with $l=0$ and $E^2_0>0$. By defining the reduced field $h\left(r\right)=H\left(r\right)r$, Eq.(83) becomes

$$-\frac{d}{dr}\left(\sqrt{1-\frac{2MG}{r}-\frac{r^2}{b^2}}\frac{dh}{dr}\right)+\left(\frac{-3MG}{r^3}+\tilde{E}^2\right)\frac{h}{\sqrt{1-\frac{2MG}{r}-\frac{r^2}{b^2}}}=0, \tag{84}$$

[7] See also [30] for an approach of black hole pair creation based on the effective action.

where $\tilde{E}^2 = -3/b^2 + E_0^2$. By means of Eq.(20), one gets

$$-\frac{dx}{dr}\frac{d}{dx}\left(\sqrt{1-\frac{2MG}{r}-\frac{r^2}{b^2}}\frac{dh}{dx}\frac{dx}{dr}\right)+\left(-\frac{3MG}{r^3}+\tilde{E}^2\right)\frac{h}{\sqrt{1-\frac{2MG}{r}-\frac{r^2}{b^2}}}$$

$$=-\frac{d}{dx}\left(\frac{dh}{dx}\right)+\left(-\frac{3MG}{r^3}+\tilde{E}^2\right)h=0. \tag{85}$$

Near the throat

$$x\left(r\right)\simeq\pm\frac{\sqrt{2r_+}}{\sqrt{\kappa_+}}\sqrt{\left(\frac{r}{r_+}-1\right)}, \tag{86}$$

where we have used Eq.(63) defining the inner surface gravity. By defining the dimensionless variable $\rho = r/r_+$, we obtain $\rho \simeq 1 + y^2$ where

$$y=\sqrt{\kappa_+}x/\sqrt{2r_+}=\tilde{\kappa}_+x. \tag{87}$$

Eq.(85) becomes

$$-\frac{d}{dy}\left(\frac{dh}{dy}\right)\tilde{\kappa}_+^2+\left(-\frac{3MG}{r_+^3\rho^3\left(y\right)}+\tilde{E}^2\right)h$$

$$=-\frac{d^2h}{dy^2}+\left(-\frac{3MG}{\tilde{\kappa}_+^2r_+^3\left(1+y^2\right)^3}+\lambda\right)h=0, \tag{88}$$

where $\lambda = \tilde{E}^2/\tilde{\kappa}_+^2$. Expanding the potential around $y = 0$, one gets

$$-\frac{d^2h}{dy^2}+\left(-\frac{3MG}{\tilde{\kappa}_+^2r_+^3}\left(1-3y^2\right)+\lambda\right)h \tag{89}$$

$$=-\frac{d^2h}{dy^2}+\left(\omega^2y^2-\frac{3MG}{\tilde{\kappa}_+^2r_+^3}+\lambda\right)h=0, \tag{90}$$

with $\omega=\sqrt{9MG/\left(\tilde{\kappa}_+^2r_+^3\right)}$. In this approximation we have obtained the equation of a quantum harmonic oscillator equation whose spectrum is $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$. Since we are using natural units, $\hbar$ is set to one and

$$\lambda_n=3MG/\left(\tilde{\kappa}_+^2r_+^3\right)-\sqrt{9MG/\left(\tilde{\kappa}_+^2r_+^3\right)}\left(n+\frac{1}{2}\right). \tag{91}$$

After some algebraic calculation, with the help of relation (14), we obtain

$$\lambda_n=3\sqrt{\frac{2r_{++}\left(r_++r_{++}\right)}{\left(2r_++r_{++}\right)\left(r_{++}-r_+\right)}}\left(\sqrt{\frac{2r_{++}\left(r_++r_{++}\right)}{\left(2r_++r_{++}\right)\left(r_{++}-r_+\right)}}-\frac{1}{\sqrt{2}}\left(n+\frac{1}{2}\right)\right). \tag{92}$$

Let us examine the first two eigenvalues. The first one is

$$\lambda_0=3\sqrt{\frac{2r_{++}\left(r_++r_{++}\right)}{\left(2r_++r_{++}\right)\left(r_{++}-r_+\right)}}\left(\sqrt{\frac{2r_{++}\left(r_++r_{++}\right)}{\left(2r_++r_{++}\right)\left(r_{++}-r_+\right)}}-\frac{1}{2}\right). \tag{93}$$

Since the eigenvalue must be positive, the following inequality must hold

$$\sqrt{\frac{2r_{++}\left(r_++r_{++}\right)}{\left(2r_++r_{++}\right)\left(r_{++}-r_+\right)}}>\frac{1}{2}\Longrightarrow\frac{7}{4}r_+r_{++}+\frac{7}{4}r_{++}^2+\frac{1}{2}r_+^2>0, \tag{94}$$

which is verified $\forall\theta\in\left[\frac{\pi}{2},\pi\right)$. To proof that there is only one eigenvalue, we look at the second one

$$\lambda_1=3\sqrt{\frac{2r_{++}\left(r_++r_{++}\right)}{\left(2r_++r_{++}\right)\left(r_{++}-r_+\right)}}\left(\sqrt{\frac{2r_{++}\left(r_++r_{++}\right)}{\left(2r_++r_{++}\right)\left(r_{++}-r_+\right)}}-\frac{3}{2}\right). \tag{95}$$

This implies the inequality

$$18r_+^2>r_+r_{++}+r_{++}^2, \tag{96}$$

which is verified when $\theta\in(2.1708,\pi)$. Thus we have only one eigenvalue when $\theta\in\left[\frac{\pi}{2},2.1708\right)$. In terms of E^2 we get

$$E^2=-3/b^2-\frac{3r_{++}\left(r_++r_{++}\right)}{2b^2r_+^2}+\frac{3}{8b^2r_+^2}\sqrt{2r_{++}\left(r_++r_{++}\right)\left(2r_++r_{++}\right)\left(r_{++}-r_+\right)}. \tag{97}$$

Note that if we repeat the same calculation for the outer throat, i.e. r_{++}, we discover the absence of negative eigenvalues. Thus we can conclude that there is only **one eigenvalue** with the restriction that $\theta\in\left[\frac{\pi}{2},2.1708\right)$. According to Coleman [8], this is a signal of a transition from a false vacuum to a true one. It is evident that when we consider the limit where $r_{++}\to r_+$, an infinite number of eigenvalues enter in the discrete spectrum. This is a consequence of the approximated parabolic potential which is used to describe this extreme situation. To better deal with this problem, we introduce the Nariai metric.

VI. THE NARIAI METRIC SPIN 2 OPERATOR AND THE EVALUATION OF THE ENERGY DENSITY

When $r_+=r_{++}$, the metric becomes degenerate and the function $f(r)$ of Eq.(15) becomes

$$f_e(r)=-\frac{1}{rb^2}\left(r-\bar{r}\right)^2\left(r+2\bar{r}\right), \tag{98}$$

where $\bar{r}=b/\sqrt{3}$. Since $f_e(r)\leq 0$ everywhere r becomes a time coordinate and t becomes spatial. Nevertheless, this is an artifact of a poor coordinate choice. To see what happens, we follow Ref. [4] and we let $9M^2G^2\Lambda=1-3\varepsilon^2$ so that the limit $r_+\to r_{++}$ corresponds to $\varepsilon\to 0$. We define a new radial coordinate θ_1, and a new time coordinate ϕ_1, by

$$\cos\theta_1=\frac{\sqrt{3}}{b\varepsilon}\left(r-\bar{r}\right)\qquad\qquad\phi_1=t\frac{\sqrt{3}}{b\varepsilon}, \tag{99}$$

such that $r_+=\bar{r}-\varepsilon\sqrt{3}/b$ and $r_{++}=\bar{r}+\varepsilon\sqrt{3}/b$. To first order in ε the metric assumes the form

$$ds^2=-\frac{b^2}{3}\left(1+\frac{2}{3}\varepsilon\cos\theta_1\right)\sin^2\theta_1d\phi_1^2+\frac{b^2}{3}\left(1-\frac{2}{3}\varepsilon\cos\theta_1\right)d\theta_1^2$$

$$+\frac{b^2}{3}\left(1-2\varepsilon\cos\theta_1\right)d\Omega^2, \tag{100}$$

describing a nearly degenerate SdS metric with two distinct roots. The related surface gravities, to first order in ε assume the expressions [3]

$$\kappa_{+,++}=\frac{b}{\sqrt{3}}\left(1\mp\frac{2}{3}\varepsilon\cos\theta_1\right), \tag{101}$$

where the upper (lower) sign is for the cosmological (black hole) horizon. When $\varepsilon\to 0$, the line element is

$$ds^2=\frac{b^2}{3}\left(-\sin^2\theta_1d\phi_1^2+d\theta_1^2+d\Omega^2\right) \tag{102}$$

and the surface gravities degenerate into one with the value

$$\kappa=\frac{b}{\sqrt{3}}. \tag{103}$$

This is the Nariai metric [7] with topology $H^2 \times S^2$. Its Euclidean form has the well known topology $S^2 \times S^2$ and form

$$ds^2 = \frac{b^2}{3}\left(d\Omega_1^2 + d\Omega^2\right). \tag{104}$$

Every constant time section has the topology $S^1 \times S^2$ with the throats of the same size as illustrated in Fig.3

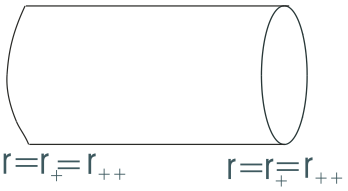

FIG. 3. The geometry of the constant time slice degenerate Schwarzschild-de Sitter (Nariai) spacetime with a polar angle suppressed. Isometric copies of this surface can be smoothly joined at the throats, producing a periodic $S^2 \times S^1$ spatial topology.

However we are interested for the Lorentzian version. In this case, Regge-Wheeler decomposition shows that the even-parity three-dimensional perturbation is

$$h_{ij}^{even}\left(\theta_1, \theta, \phi\right) = \frac{b^2}{3} diag\left[H\left(\theta_1\right), K\left(\theta_1\right), \sin^2\theta K\left(\theta_1\right)\right] Y_{l0}\left(\theta, \phi\right). \tag{105}$$

For a generic value of the angular momentum L, one gets

$$\begin{cases} \frac{3}{b^2}\left(-\partial_{\theta_1}^2 + l\left(l+1\right) - 2\right) H\left(\theta_1\right) = E_l^2 H\left(\theta_1\right) \\ \\ \frac{3}{b^2}\left(-\partial_{\theta_1}^2 + l\left(l+1\right)\right) K\left(\theta_1\right) = E_l^2 K\left(\theta_1\right) \end{cases}, \tag{106}$$

where the associated three dimensional mixed Ricci tensor is

$$R_i^j = \left\{0, \frac{3}{b^2}, \frac{3}{b^2}\right\}. \tag{107}$$

Also in this case the unstable mode appears for the $l = 0$ case leading to the eigenvalue equation

$$\left(-\partial_{\theta_1}^2 - 2\right) H\left(\theta_1\right) = -\lambda H\left(\theta_1\right), \tag{108}$$

where $\lambda = b^2 E^2/3$. The eigenvalue is easily determined and its value is

$$\lambda = 2, \tag{109}$$

with eigenfunction

$$H\left(\theta_1\right) = const. \tag{110}$$

Since the range of integration is finite due to the periodicity of the argument, the eigenfunction is normalizable. For the other component we have no solutions at all, because the operator is the same of a free particle having only a continuous spectrum. For completeness, we calculate the energy difference between the SdS background in the Nariai form and the dS metric in the stable sector. The total energy in the presence of the Nariai metric is

$$E_{Nariai}\left(b\right) = \frac{V}{2\pi^2}\frac{1}{2}\sum_{l=0}^{\infty}\int_0^{\infty} dpp^2\left(\sqrt{p^2 + \frac{3l\left(l+1\right)}{b^2} - 6/b^2} + \sqrt{p^2 + \frac{3l\left(l+1\right)}{b^2}}\right), \tag{111}$$

while for the pure de Sitter metric, we have

$$E_{dS}\left(b\right) = \frac{V}{2\pi^2}\frac{1}{2}\sum_{l=0}^{\infty}\int_0^{\infty} dpp^2 2\left(\sqrt{p^2 + \frac{3l\left(l+1\right)}{b^2} - 3/b^2}\right). \tag{112}$$

The Casimir-like energy becomes

$$\Delta E\left(b\right)=E_{Nariai}\left(b\right)-E_{dS}\left(b\right)$$

$$=\frac{V}{2\pi^{2}}\frac{1}{2}\sum_{l=0}^{\infty}\int_{0}^{\infty}dpp^{2}\left[\sqrt{p^{2}+\frac{3l\left(l+1\right)}{b^{2}}-6/b^{2}}+\sqrt{p^{2}+\frac{3l\left(l+1\right)}{b^{2}}}\right.$$

$$\left.-2\sqrt{p^{2}+\frac{3l\left(l+1\right)}{b^{2}}-3/b^{2}}\right] \tag{113}$$

When $p^{2}>>6/b^{2}$ we write

$$\Delta E\left(b\right)\simeq-\frac{V}{2\pi^{2}}\frac{9}{4b^{4}}\int_{0}^{\infty}\frac{dp}{p}. \tag{114}$$

Introducing an UV cut-off one gets

$$\int_{0}^{\infty}\frac{dp}{p}\sim\int_{0}^{\Lambda^{2}b^{2}}\frac{dx}{x}\sim\ln\left(\Lambda^{2}b^{2}\right) \tag{115}$$

and $\Delta E\left(b\right)$ for high momenta becomes

$$\Delta E\left(b\right)\sim-\frac{V}{2\pi^{2}}\frac{9}{16b^{4}}\ln\left(\Lambda^{2}b^{2}\right)=-\frac{V}{32\pi^{2}}\frac{9}{b^{4}}\ln\left(\Lambda^{2}b^{2}\right). \tag{116}$$

VII. BOUNDARY REDUCTION AND STABILITY

An equivalent approach to Eq.(84) can be set up by means of a variational procedure applied on a functional whose minimum represents the solution of the problem. Let us define

$$J\left(h,E^{2}\right)=\frac{1}{2}\int_{0}^{\bar{x}}dx\left[\left(\frac{dh\left(x\right)}{dx}\right)^{2}-\frac{3MG}{r^{3}\left(x\right)}h^{2}\left(x\right)\right]+\frac{\tilde{E}^{2}}{2}\int_{0}^{\bar{x}}h^{2}\left(x\right)dx,$$

where dx is given by Eq.(8). Eq.(84) is equivalent to find the minimum of

$$\tilde{E}^{2}=\frac{\int_{0}^{\bar{x}}dx\left[\left(\frac{dh(x)}{dx}\right)^{2}-\frac{3MG}{r^{3}(x)}h^{2}\left(x\right)\right]}{\int_{0}^{\bar{x}}h^{2}\left(x\right)dx}. \tag{117}$$

For future purposes, we use the boundary conditions

$$h\left(\bar{x}\right)=0. \tag{118}$$

When $2MG/r\ll1$ Eq.(85) becomes

$$-\frac{d^{2}h}{dx^{2}}+\tilde{E}^{2}h=0 \tag{119}$$

and the solution is

$$h\left(x\right)=A\exp\left(-\tilde{E}x\right)+B\exp\left(\tilde{E}x\right). \tag{120}$$

In the approximation of Eq.(119), we can suppose x so large that the increasing exponential has to be eliminated in such a way to take only $h\left(x\right)=A\exp\left(-\tilde{E}x\right)$. If we change the variables in a dimensionless form like Eq.(87), we get

$$\mu = \frac{\tilde{E}^2}{\tilde{\kappa}^2} = \frac{\int_0^{\bar{y}} dy \left[\left(\frac{dh(y)}{dy} \right)^2 - \frac{3MG}{r_+^3 \tilde{\kappa}_+^2 \rho^3(y)} h^2(y) \right]}{\int_0^{y(a)} dy h^2(y)}. \tag{121}$$

The asymptotic behaviour of $h(x)$ suggests to choose $h(\lambda, y) = \exp(-\lambda y)$ as a trial function, and Eq.(121) becomes

$$\mu(\lambda) = \lambda^2 - \frac{3MG}{r_+^3 \tilde{\kappa}_+^2} \frac{\int_0^{\bar{y}} \frac{dy}{\rho^3(y)} \exp(-2\lambda y)}{\frac{1-\exp(-2\lambda y)}{2\lambda}}. \tag{122}$$

Close to the throat $\exp(-2\lambda y) \simeq 1 - 2\lambda y$ and

$$\mu(\lambda) = \lambda^2 - \frac{3MG}{r_+^3 \tilde{\kappa}_+^2} + \frac{9MG}{r_+^3 \tilde{\kappa}_+^2} \left[\frac{\bar{y}}{2\lambda} + \bar{y}^2 \right]. \tag{123}$$

The minimum of $\mu(\lambda)$ is reached for $\bar{\lambda} = \left(\frac{9MG}{4 r_+^3 \tilde{\kappa}_+^2} \bar{y} \right)^{\frac{1}{3}}$ assuming therefore the value

$$\mu(\bar{\lambda}) = 3 \left(\frac{9}{4} D \bar{y} \right)^{\frac{2}{3}} - 3D + 3D\bar{y}^2, \tag{124}$$

where

$$D = \frac{MG}{r_+^3 \tilde{\kappa}_+^2} = \frac{2 r_{++} (r_+ + r_{++})}{(2r_+ + r_{++})(r_{++} - r_+)}$$

$$= \frac{2 \cos\left(\frac{\theta}{3}\right) \left(\cos\left(\frac{\theta + 4\pi}{3}\right) + \cos\left(\frac{\theta}{3}\right) \right)}{\left(2\cos\left(\frac{\theta+4\pi}{3}\right) + \cos\left(\frac{\theta}{3}\right) \right) \left(\cos\left(\frac{\theta}{3}\right) - \cos\left(\frac{\theta+4\pi}{3}\right) \right)}. \tag{125}$$

If we consider the value of θ such that only one eigenvalue appears, we obtain $D = 2.25$ and Eq.(124)becomes zero for $\bar{y}_c = .463\,68$ corresponding to $\bar{\rho}_c = 1.215$. This means that the unstable mode persists until the boundary radius $\bar{\rho}$ falls below $\bar{\rho}_c$.

VIII. SUMMARY AND CONCLUSIONS

In this paper we have extended the computation of the Casimir-like energy to the case of the Schwarzschild-de Sitter (SdS) background with the de Sitter (dS) space as a reference space. This evaluation has been done to one-loop in the TT (transverse, traceless) sector which is the gauge invariant part of the quantum fluctuation of the gravitational field. As stressed in the introduction to correctly compute the Casimir energy we need to subtract field configurations which have the same asymptotic properties and the same asymptotic boundary conditions. For the SdS metric and the dS metric, this is the case. In this context a lot of work has been done; for example in Refs. [4–6], it has been shown the existence of one negative mode in the TT sector when the saddle point approximation is considered: a clear sign of instability. In particular, the instability has been related to the probability of creating a black hole pair [6,31]. However this particular result has been obtained by looking at the partition function and therefore with the introduction of an equilibrium temperature, that in the case of the pure Schwarzschild and flat metric can be imposed to be equal, but in the present case (i.e. the SdS metric and the dS metric) it cannot. Therefore as stressed in Ref. [6], it is not very clear how these spaces having a different temperature (periodicity) can be compared. On the other hand if we adopt the hamiltonian approach we can avoid the introduction of a temperature and it is possible to build a scheme where the classical contribution is conserved; this point is fundamental to discuss the instability [9]. What we have found, in our framework, is the well known existence of an unstable mode in the S wave for the extreme SdS metric (Nariai)[8] but we have also found that an unstable mode exists also for the SdS metric. To interpret such

[8]For this last point a discussion can be found in Ref. [31].

an instability as a decay process nucleating a black hole pair, we need to show the existence of only "**one negative mode**" [8]. Unfortunately, in our approximation we have discovered a dependence of the number of negative modes on θ the variable defined in Eq.(12). In particular, we have seen that the closest is the approach to the extreme value $\theta = \pi$, the highest is the number of negative eigenvalues falling into the negative spectrum. This abundance of negative eigenvalues can be limited if we restrict the variability of θ into the interval $\left[\frac{\pi}{2}, 2.1708\right)$; in this range Coleman arguments can be applied. However the proliferation of negative modes in the nearly extremal SdS metric is a consequence of the approximated potential of the eigenvalue equation. Indeed by using nearly degenerate coordinates like those in Eq.(100), only one negative eigenvalue appears. It is interesting to observe that the appearance of a negative mode, even for the SdS metric, can be related with the production of a sub-maximal black hole pair [32]. What is interesting to observe is the existence of a critical radius ρ_c below which the instability disappears. This could open the possibility of an existing foam-like space composed by copies of bubbles in analogy with the model discussed in Ref. [33]. Even in this case, a dependence of the ultraviolet cut-off is present. This is principally due to the non renormalizability of quantum gravity. However, the fact that the same divergent behaviour appears also in this case it is a signal of a more general situation typically concerning the spherically symmetric metrics, that is every spherically symmetric metric describing a wormhole (black hole) can produce in the spectrum of quantum fluctuations in a semiclassical approximation one negative mode, provided the boundary conditions be energy conserving.

IX. ACKNOWLEDGEMENTS

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APPENDIX A: KRUSKAL-SZEKERES COORDINATES FOR SDS SPACETIME

We have defined the SdS line element in Eq.(7). To introduce the Kruskal-Szekeres [34–36] type coordinates we consider the following transformation

$$ds^2 = -\left(1 - \frac{2MG}{r} - \frac{r^2}{b^2}\right)\left[dt^2 - dr^{*2}\right] + r^2 d\Omega^2$$

$$= -\left(1 - \frac{2MG}{r} - \frac{r^2}{b^2}\right) dvdu + r^2\,(u,v)\, d\Omega^2, \tag{A1}$$

where $v = t + r^*$ is the ingoing radial null coordinate and $u = t - r^*$ is the outgoing radial null coordinate. The "tortoise coordinate" r^* is defined by

$$dr^* = -\frac{rb^2 dr}{(r - r_+)\,(r - r_{++})\,(r + r_+ + r_{++})} \tag{A2}$$

and by means of the surface gravity associated to each root, we write

$$\frac{1}{f} = \frac{1}{2\kappa_+\,(r - r_+)} + \frac{1}{2\kappa_{++}\,(r - r_{++})} + \frac{1}{2\kappa_-\,(r + r_+ + r_{++})}, \tag{A3}$$

where $f\,(r)$ has been defined by Eq.(8), while κ_{++} and κ_+ by Eq.(78) and Eq.(63), respectively. The last surface gravity associated with the negative root is

$$\kappa_- = \frac{(2r_+ + r_{++})\,(2r_{++} + r_+)}{2b^2\,(r_+ + r_{++})}. \tag{A4}$$

Thus

$$r^* = \frac{1}{2\kappa_+}\ln\left|\frac{r}{r_+} - 1\right| + \frac{1}{2\kappa_{++}}\ln\left|\frac{r}{r_{++}} - 1\right| + \frac{1}{2\kappa_-}\ln\left(\frac{r}{r_+ + r_{++}} + 1\right) \tag{A5}$$

To avoid singularities we can define Kruskal-Szekeres type coordinates

$$V^{++} = \exp\kappa_{++} v \qquad U^{++} = -\exp -\kappa_{++} u. \tag{A6}$$

These coordinates do not cover $r \leq r_+$ because of the coordinate singularity at $r = r_+$ (and $U^{++}V^{++}$ is complex for $r \leq r_+$), but $r = r_+$ and a similar four regions are covered by the (U^+, V^+) Kruskal-Szekeres-type coordinates to this case.

$$V^+ = \exp \kappa_+ v \qquad U^+ = -\exp -\kappa_+ u. \tag{A7}$$

For the ++ sign we have

$$U^{++}V^{++} = -\exp\left(\kappa_{++}\left(v-u\right)\right) = -\exp\left(2\kappa_{++}r^*\right) = -\left(\frac{r}{r_+}-1\right)^{\frac{\kappa_{++}}{\kappa_+}}\left(\frac{r}{r_+ + r_{++}}+1\right)^{\frac{\kappa_{++}}{\kappa_-}}\left(\frac{r}{r_{++}}-1\right) \tag{A8}$$

and the respective line element is

$$ds^2_{++} = -\frac{\left(r_+ + r_{++}\right) r_+ r_{++}}{b^2\kappa^2_{++} r}\left(\frac{r}{r_+ + r_{++}}+1\right)^{1-\frac{\kappa_{++}}{\kappa_-}}\left(\frac{r}{r_+}-1\right)^{1-\frac{\kappa_{++}}{\kappa_+}} dU^{++}dV^{++} + r^2\left(U^{++}, V^{++}\right)d\Omega^2$$

$$= -\frac{2MG}{\kappa^2_{++} r}\left(\frac{r}{r_+ + r_{++}}+1\right)^{1-\frac{\kappa_{++}}{\kappa_-}}\left(\frac{r}{r_+}-1\right)^{1-\frac{\kappa_{++}}{\kappa_+}} dU^{++}dV^{++} + r^2\left(U^{++}, V^{++}\right)d\Omega^2, \tag{A9}$$

while for the + sign we have

$$U^+V^+ = -\exp\left(\kappa_+\left(v-u\right)\right) = -\exp\left(2\kappa_+ r^*\right) = -\left(r-r_+\right)\left(r+r_+ + r_{++}\right)^{\frac{\kappa_+}{\kappa_-}}\left(r_{++}-r\right)^{\frac{\kappa_+}{\kappa_{++}}} \tag{A10}$$

and the associated line element is

$$ds^2_{+} = -\frac{\left(r_+ + r_{++}\right) r_+ r_{++}}{b^2\kappa^2_{+} r}\left(\frac{r}{r_+ + r_{++}}+1\right)^{1-\frac{\kappa_{+}}{\kappa_-}}\left(1-\frac{r}{r_{++}}\right)^{1-\frac{\kappa_{+}}{\kappa_{++}}} dU^{+}dV^{+} + r^2\left(U^{+}, V^{+}\right)d\Omega^2$$

$$= -\frac{2MG}{\kappa^2_{+} r}\left(\frac{r}{r_+ + r_{++}}+1\right)^{1-\frac{\kappa_{+}}{\kappa_-}}\left(1-\frac{r}{r_{++}}\right)^{1-\frac{\kappa_{+}}{\kappa_{++}}} dU^{+}dV^{+} + r^2\left(U^{+}, V^{+}\right)d\Omega^2. \tag{A11}$$

The conformal Penrose diagram of the SdS space is shown in Fig.4. Regions limited by $r_+ < r < r_{++}$ lie between the black hole and cosmological horizon. Regions $r > r_{++}$ correspond to an asymptotic de Sitter region and region $r_+ > r$ to the black hole interior.

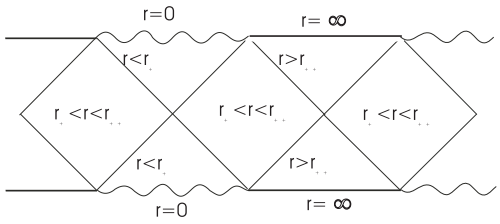

FIG. 4. Penrose diagram for the Schwarzschild-de Sitter spacetime.

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