# Entropy from the foam II

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A simple model of spacetime foam, made by two different types of wormholes in a semiclassical approximation, is taken under examination: one type is a collection of  $N_w$  Schwarzschild wormholes, while the other one is made by Schwarzschild-Anti-de Sitter wormholes. The area quantization related to the entropy via the Bekenstein-Hawking formula hints a possible selection between the two configurations. Application to the charged black hole are discussed.

# I. INTRODUCTION

The area-entropy relation, concerning black holes, has been proposed by J. Bekenstein [1] in the early seventies. Despite of its simple expression, it is still a central point of research in understanding the connection between General Relativity and Quantum Mechanics. In natural units one finds that the area-entropy becomes

$$S = \frac{A}{4G} = \frac{A}{4l_p^2},\tag{1}$$

where A is the area of the event horizon,  $G = l_p^2$  is the gravitational constant. From one side, we have a statistical mechanics problem and from the other side we have a pure geometrical problem. This is another aspect of Einstein's equations relating geometry and dynamics. However, the problem of understanding which kind of dynamics can give such a simple result has yet to come. One progress has been made in terms of string theory, where the entropy of a zero temperature black hole (extreme) has been calculated and shown to be identical to the Bekenstein-Hawking formula for the thermodynamical entropy [2]. This means that a count of states of the black holes in terms of string states reflects the quantum nature of a black hole. Another approach comes from the application of the Cardy's formula [3] in conformal field theory [4,5] without invoking the string theory framework. Therefore a further investigation from other points of view is as important as the string theory approach. Having in mind the Bekenstein's proposal

$$a_n = \alpha l_p^2 (n+\eta) \qquad \eta > -1 \qquad n = 1, 2, \dots,$$
 (2)

describing the quantization of the area for nonextremal black holes, in Refs. [6–8], we have proposed a model made by  $N_w$  wormholes, based on Wheeler's ideas of a foamy space-time [9]. In those papers, we have quantized the entropy of a Schwarzschild black hole assuming the validity of the area-entropy relation. The area quantization induced by the underlying foam background, whose quanta can be identified with wormholes of Planckian size, has immediately led to the quantization of the Schwarzschild black hole mass and of the cosmological constant. In this paper we wish to generalize the results obtained in Refs. [6–8] by looking at a foamy space composed by Schwarzschild-Anti-de Sitter wormholes (S-AdS). A selection of the different foamy constituents is suggested in terms of level spacings related to the Hawking radiation. An application to a Reissner-Nordström (RN) black hole is also given. The rest of the paper is structured as follows, in section II we briefly recall the results reported in Refs. [7,10] for the Schwarzschild and S-AdS wormholes respectively, in section III we compute the area spectrum for both foam configurations, in section IV the area quantization is applied to the RN black hole. We summarize and conclude in section V. Units in which  $\hbar = c = k = 1$  are used throughout the paper.

## II. SPACE-TIME FOAM: SCHWARZSCHILD OR SCHWARZSCHILD-ANTI-DE SITTER WORMHOLES?

We consider a complete manifold  $\mathcal{M}$ , divided in two wedges  $\mathcal{M}_+$  and  $\mathcal{M}_-$  by a bifurcation two-surface  $S_0$ , located in the right and left sectors of a Kruskal diagram. We consider a constant time hypersurface  $\Sigma$  crossing  $S_0$ , representing an Einstein-Rosen bridge with wormhole topology  $S^2 \times R^1$  such that  $\Sigma = \Sigma_+ \cup \Sigma_-$ . The line element we will consider is

$$ds^{2} = -N^{2}(r) dt^{2} + \frac{dr^{2}}{1 - \frac{b(r)}{r}} + r^{2} \left( d\theta^{2} + \sin^{2} \theta d\phi^{2} \right),$$
(3)

where N(r) is the lapse function and b(r) is the shape function such that

$$b(r) = \begin{cases} 2MG & Schwarzschild\\ 2MG - \frac{r^3}{b_{AdS}^2} & S - AdS \end{cases}$$
(4)

M is the wormhole mass, while  $b_{AdS}^2 = -3/\Lambda_{AdS}$  and  $\Lambda_{AdS}$  is the negative cosmological constant. Note that the throat is located at  $r_h = 2MG$  in the Schwarzschild case, while for the S-AdS case we get

$$1 - \frac{2MG}{r_h} + \frac{r_h^2}{b_{AdS}^2} = 0, (5)$$

where we have implicitly defined  $r_h$  in terms of M and  $b_{AdS}$ . The physical Hamiltonian defined on  $\Sigma$  assumes the form

$$H_P = H_{\Sigma} + H_{\partial \Sigma},\tag{6}$$

where

$$H_{\Sigma} = \frac{1}{16\pi G} \int_{\Sigma} d^3 x \left( N\mathcal{H} + N_i \mathcal{H}^i \right) \tag{7}$$

and

$$H_{\partial\Sigma} = \frac{1}{8\pi G} \int_{S_{+}} d^{2}x \sqrt{\sigma} \left(k - k^{0}\right) - \frac{1}{8\pi G} \int_{S_{-}} d^{2}x \sqrt{\sigma} \left(k - k^{0}\right).$$
(8)

The volume term  $H_{\Sigma}$  contains two constraints

$$\begin{cases} \mathcal{H} = G_{ijkl} \pi^{ij} \pi^{kl} \left( \frac{16\pi G}{\sqrt{g}} \right) - \left( \frac{\sqrt{g}}{16\pi G} \right) \left( R^{(3)} + \frac{6}{b_{AdS}^2} \right) = 0 \\ \mathcal{H}^i = -2\pi^{ij}_{|j} = 0 \end{cases},$$
(9)

both satisfied by the Schwarzschild and flat metric respectively when  $\Lambda_{AdS} = 0$   $(b_{AdS} \to \infty)$  and satisfied by the S-AdS and AdS metric when  $b_{AdS} < \infty$ . The supermetric is defined as  $G_{ijkl} = \frac{1}{2} (g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl})$  and  $R^{(3)}$  denotes the scalar curvature of the surface  $\Sigma$ . The boundary term  $H_{\partial\Sigma}$  (quasilocal energy [11,12]) is defined by means of a subtraction procedure whose purpose is the elimination of the asymptotic divergence, i.e.  $r \to +\infty$  coming from the curvature change [13]. In the present case, the Schwarzschild metric asymptotically tends to the flat metric, which naturally defines the reference space. The same happens for the S-AdS and the AdS spaces. In Refs. [10,15], we have shown that  $H_{\partial\Sigma} = 0$  provided that the boundary condition be symmetric with respect to the bifurcation surface<sup>1</sup>. Thus the Hamiltonian contribution comes from the off-shell volume term. Nevertheless the form of H is not satisfying, because the subtraction procedure appears only at the boundary level. However to be consistent with such a procedure, we extend the subtraction even at the volume term. Thus the final physical Hamiltonian will be of the form

$$\Delta H_P = H^{wormhole} - H^{No-wormhole} \tag{10}$$

and the change in energy can be computed with

$$\Delta E = \frac{\left\langle \Psi \left| H^{wormhole} - H^{No-wormhole} \right| \Psi \right\rangle}{\left\langle \Psi | \Psi \right\rangle},\tag{11}$$

by means of a variational approach, where the WKB functionals are substituted with trial wave functionals. This quantity is the natural extension to the volume term of the subtraction procedure for boundary terms and it is interpreted as the Casimir energy related to vacuum fluctuations. In practice, we consider perturbations at  $\Sigma$  of the type

<sup>&</sup>lt;sup>1</sup>See also Ref. [12].

$$g_{ij} = \bar{g}_{ij} + h_{ij},\tag{12}$$

where  $\bar{g}_{ij}$  is the spatial part of the Schwarzschild, Flat, S-AdS and AdS background in a WKB approximation. By restricting our attention to the graviton sector of the Hamiltonian approximated to second order, hereafter referred to as  $H_{|2}$ , we define

$$E_{|2} = \frac{\left\langle \Psi^{\perp} \left| H_{|2} \right| \Psi^{\perp} \right\rangle}{\left\langle \Psi^{\perp} \right| \Psi^{\perp} \right\rangle},\tag{13}$$

where

$$\Psi^{\perp} = \Psi\left[h_{ij}^{\perp}\right] = \mathcal{N}\exp\left\{-\frac{1}{4}\left[\left\langle \left(g-\bar{g}\right)K^{-1}\left(g-\bar{g}\right)\right\rangle_{x,y}^{\perp}\right]\right\}.$$
(14)

After having functionally integrated  $H_{|2}$ , we get

$$E_{|2} = \frac{1}{4} \int_{\Sigma} d^3x \sqrt{g} G^{ijkl} \left[ (16\pi G) K^{-1\perp} (x, x)_{ijkl} + (16\pi G)^{-1} (\triangle_2)^a_j K^{\perp} (x, x)_{iakl} \right].$$
(15)

Thus Eq.(11) becomes

$$\Delta E_{|2} = \frac{\left\langle \Psi^{\perp} \left| H_{|2}^{wormhole} \right| \Psi^{\perp} \right\rangle}{\left\langle \Psi^{\perp} | \Psi^{\perp} \right\rangle} - \frac{\left\langle \Psi^{\perp} \left| H_{|2}^{No-wormhole} \right| \Psi^{\perp} \right\rangle}{\left\langle \Psi^{\perp} | \Psi^{\perp} \right\rangle}.$$
(16)

The propagator  $K^{\perp}\left(x,x\right)_{iakl}$  can be represented as

$$K^{\perp}(\overrightarrow{x}, \overrightarrow{y})_{iakl} := \sum_{N} \frac{h_{ia}^{\perp}(\overrightarrow{x}) h_{kl}^{\perp}(\overrightarrow{y})}{2\lambda_{N}(p)},\tag{17}$$

where  $h_{ia}^{\perp}(\vec{x})$  are the eigenfunctions of

$$(\Delta_2)_j^a := -\Delta \delta_j^a + 2V_j^a. \tag{18}$$

This is the Lichnerowicz operator projected on  $\Sigma$  acting on traceless transverse quantum fluctuations and  $\lambda_N(p)$  are variational parameters.  $\Delta$  is the Laplacian in curved space

$$\Delta = \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \partial_j \right) \tag{19}$$

and  $V_j^a$  is a mixed tensor containing the mixed Ricci tensor whose components are:

$$V_j^a = \left\{ -\frac{2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3} \right\}$$
(20)

for the Schwarzschild case and

$$V_j^a = \left\{ -\frac{2MG}{r^3} + \frac{1}{b_{AdS}^2}, \frac{MG}{r^3} + \frac{1}{b_{AdS}^2}, \frac{MG}{r^3} + \frac{1}{b_{AdS}^2} \right\}$$
(21)

for the S-AdS case. The minimization with respect to  $\lambda$  and the introduction of a high energy cut-off  $\Lambda$  give to Eq.(16) the following form

$$\Delta E\left(M, b_{AdS}\right) = \Delta E\left(M\right) = -\frac{V}{32\pi^2} \left(\frac{3MG}{r_0^3}\right)^2 \ln\left(\frac{r_0^3\Lambda^2}{3MG}\right),\tag{22}$$

where  $r_0 > r_h$  and V is the volume strictly localized to the wormhole throat. Eq.(22) is valid even when  $b_{AdS} \to \infty$ . The minimum of  $\Delta E(M)$  is located at  $x_2 = e^{-\frac{1}{2}}$ , where  $x = 3MG/(r_0^3 \Lambda^2)$  and

$$\Delta E(x_2) = -\frac{V}{64\pi^2} \frac{\Lambda^4}{e}.$$
(23)

 $\Delta E(x_2)$  shows a shift of the minimum away from the expected one, namely  $x_1 = 0$  corresponding to flat space and the AdS space. The discrete spectrum contains exactly one mode. This gives the energy an imaginary contribution, namely we have discovered an unstable mode [7,10]. This system can be stabilized if we consider  $N_w$  wormholes in a semiclassical approximation and assume that there exists a covering of  $\Sigma$  such that  $\Sigma = \bigcup_{i=1}^{N_w} \Sigma_i$ , with  $\Sigma_i \cap \Sigma_j = \emptyset$  when  $i \neq i$ . Each  $\Sigma_i$  has topology  $S^2 \times R^1$  with boundaries  $\partial \Sigma^{\pm}$  with respect to each bifurgetion surface. The boundary is

 $i \neq j$ . Each  $\Sigma_i$  has topology  $S^2 \times R^1$  with boundaries  $\partial \Sigma_i^{\pm}$  with respect to each bifurcation surface. The boundary is located at  $R_{\pm}$  and it is reduced by a factor  $N_w$ , i.e.  $R_{\pm} \to R_{\pm}/N_w$ . On each surface  $\Sigma_i$ , the boundary Hamiltonian is

$$H_{\partial \Sigma_{i}^{\pm}} = \frac{1}{8\pi G} \int_{S_{i+}} d^{2}x \sqrt{\sigma} \left(k - k^{0}\right) - \frac{1}{8\pi G} \int_{S_{i-}} d^{2}x \sqrt{\sigma} \left(k - k^{0}\right).$$
(24)

Note that  $E_{\partial \Sigma_i^{\pm}}$  is zero for boundary conditions symmetric with respect to *each* bifurcation surface  $S_{0,i}$ . We are interested in a large number of wormholes, each of them contributing with a term of the type (11). The total semiclassical hamiltonian is

$$H_{tot}^{N_w} = H^1 + H^2 + \ldots + H^{N_w}$$
(25)

and the total trial wave functional is the product of  $N_w$  trial wave functionals

$$\Psi_{tot}^{\perp} = \Psi_1^{\perp} \otimes \Psi_2^{\perp} \otimes \dots \quad \Psi_{N_w}^{\perp} = \mathcal{N} \exp N_w \left\{ -\frac{1}{4} \left[ \left\langle (g - \bar{g}) K^{-1} \left( g - \bar{g} \right) \right\rangle_{x,y}^{\perp} \right] \right\}.$$
(26)

By repeating the same steps of the single wormhole, one gets

$$\Delta E_{N_w}(x,\Lambda) = N_w \frac{V}{32\pi^2} \Lambda^4 x^2 \ln x, \qquad (27)$$

where we have defined the usual scale variable  $x = 3MG/(r_0^3\Lambda^2)$ . Then at one loop the cooperative effects of wormholes behave as one *macroscopic single* field multiplied by  $N_w$ , but without the unstable mode. At the minimum,  $\bar{x} = e^{-\frac{1}{2}}$ 

$$\Delta E_{N_w}\left(\bar{x}\right) = -N_w \frac{V}{64\pi^2} \frac{\Lambda^4}{e} \tag{28}$$

valid in presence or in absence of  $\Lambda_{AdS}$ .

# III. AREA SPECTRUM AND ENTROPY

Here we briefly recall how the area quantization process of a black hole is obtained by means of the foam model. The area is measured by the quantity

$$A(S) = \int_{S} d^2x \sqrt{\sigma}.$$
(29)

 $\sigma$  is the two-dimensional determinant coming from the induced metric  $\sigma_{ab}$  on the boundary S. The evaluation of the mean value of the area

$$A(S) = \frac{\left\langle \Psi_F \left| \hat{A} \right| \Psi_F \right\rangle}{\left\langle \Psi_F \right| \Psi_F \right\rangle} = \frac{\left\langle \Psi_F \left| \int_S \widehat{d^2 x \sqrt{\sigma}} \right| \Psi_F \right\rangle}{\left\langle \Psi_F \left| \Psi_F \right\rangle},\tag{30}$$

is computed on the following state

$$|\Psi_F\rangle = \Psi_1^{\perp} \otimes \Psi_2^{\perp} \otimes \dots \oplus \Psi_{N_w}^{\perp}.$$
(31)

Each single wormhole contributes with the quantity

$$A(S) = \frac{\left\langle \Psi_F \left| \hat{A} \right| \Psi_F \right\rangle}{\left\langle \Psi_F \right| \Psi_F \right\rangle} = 4\pi \bar{r}^2.$$
(32)

Suppose to consider the mean value of the area A computed on a given *macroscopic* fixed radius R. On the basis of our foam model, we obtain  $A = \bigcup_{i=1}^{N} A_i$ , with  $A_i \cap A_j = \emptyset$  when  $i \neq j$ . Thus

$$A = 4\pi R^2 = \sum_{i=1}^{N} A_i = 4\pi l_p^2 \sum_{i=1}^{N} x_i^2 = 4\pi l_p^2 N \overline{x^2} = 4\pi l_p^2 N \alpha,$$
(33)

where a new scale  $x_i = \bar{r}_i/l_p$  has been introduced.  $\alpha$  represents how each single wormhole area is distributed with respect to the black hole area. Comparing Eq.(33) with the Bekenstein area spectrum proposal, we have

$$4\pi l_p^2 N\alpha = 4l_p^2 N\ln 2 \tag{34}$$

and  $\alpha$  is fixed to

$$\frac{\ln 2}{\pi} = \alpha. \tag{35}$$

The entropy is simply

$$S = \frac{A}{4l_p^2} = N\ln 2 \tag{36}$$

and for the Schwarzschild geometry we get

$$S = \frac{4\pi \left(2MG\right)^2}{4G} = 4\pi M^2 G = 4\pi M^2 l_p^2 = N \ln 2.$$
(37)

It is immediate to see that

$$M = \frac{\sqrt{N}}{2l_p} \sqrt{\frac{\ln 2}{\pi}},\tag{38}$$

namely the Schwarzschild black hole mass is *quantized* in terms of  $l_p$  which is in agreement with the results presented in Refs. [16–22]. This implies also that the level spacing of the transition frequencies is

$$\omega_0 = \Delta M = \left(8\pi M l_p^2\right)^{-1} \ln 2. \tag{39}$$

When we use Eq. (36) for the de Sitter geometry, we get

$$S = N \ln 2 = \frac{3\pi}{l_p^2 \Lambda_c} = \frac{A}{4l_p^2} = \frac{N 4 \pi l_p^2}{4l_p^2} = N\pi,$$
(40)

that is

$$\frac{3\pi}{l_p^2 N \ln 2} = \Lambda_c. \tag{41}$$

An interesting aspect appears when we put numbers in Eq.(41). When N = 1, the foam system is highly unstable and the cosmological constant assumes the value, in order of magnitude, of  $\Lambda_c \sim 10^{38} GeV^2$ . However the system becomes stable when the whole universe has been filled with wormholes of Planckian size and this leads to the huge number  $N = 10^{122}$  corresponding to the value of  $\Lambda_c \sim 10^{-84} GeV^2$  which is the order of magnitude of the cosmological constant of the space in which we now live.

#### A. S-AdS Space-Time Foam

In Section II, we have briefly reported how a model of space-time foam formed can be realized by S-AdS wormholes. Even if the computation has been done for a single wormhole, the procedure of a large  $N_w$  S-AdS wormhole approach can be realized straightforwardly in analogy with the Schwarzschild case<sup>2</sup>. To consider a large  $N_w$  approach to space-time foam even with S-AdS wormholes, one has to consider the following rescaling

$$\begin{cases}
R_{\pm} \to R_{\pm}/N_w \\
l_p^2 \to N_w l_p^2 \\
\Lambda_{AdS} \to \Lambda_{AdS}/N_w^2
\end{cases},$$
(42)

where  $R_{\pm}$  are the boundaries related to a single wormhole. This rescaling is a consequence of the boundary reduction related to the semiclassical superposition of wormholes wave functionals leading to a stable system. From Eq.(28), we see that both representations of the foam, i.e. S-AdS and Schwarzschild wormholes, have the same energy contribution. Thus we need one more information to select which representation seems to be the correct one. This information comes exactly from the area-entropy quantization applied to a S-AdS black hole. The procedure is simply a repetition of what has been done in the Schwarzschild case but with S-AdS wormholes. Let us consider a S-AdS black hole whose horizon is located at  $r_h$ . Then from Eq.(5), we write

$$M_{S-AdS} = \frac{r_h}{2l_p^2} \left( 1 + \frac{r_h^2}{b_{AdS}^2} \right) \tag{43}$$

which tends to the Schwarzschild mass M when  $b_{AdS} \to \infty$ . The application of Eq.(33) to the area of the horizon gives

$$r_h = l_p \sqrt{N \alpha_{AdS}}.\tag{44}$$

 $\alpha_{AdS}$  represents the size of each S-AdS wormhole inside the black hole horizon. However Eq.(43) does not uniquely define a black hole. To this purpose, we consider the value  $r_{h,m} = b_{AdS}/\sqrt{3}$  obtained by minimizing the surface gravity with respect to the horizon radius. Note that in the terminology of the black hole thermodynamics  $r_{h,m}$  corresponds to the unique black hole solution whose temperature reaches its minimum. Thus  $b_{AdS} = \sqrt{3}l_p\sqrt{N\alpha_{AdS}}$  and

$$M_{AdS} = \frac{2\sqrt{\alpha_{AdS}N}}{3l_p} \xrightarrow[b_{AdS} \to \infty]{} M_S = \frac{\sqrt{N}}{2l_p} \sqrt{\frac{\ln 2}{\pi}}.$$
(45)

This fixes  $\alpha_{AdS}$  to

$$\alpha_{AdS} = \frac{9\ln 2}{16\pi}.\tag{46}$$

A straightforward consequence of Eq.(46) is that the transition frequencies of the emitted radiation of a black hole can have a different spectrum. Indeed, if the foam is represented by Schwarzschild wormholes, the level spacing observed is given by Eq.(39) in both cases, i.e. the S-AdS and Schwarzschild black hole. Nevertheless, if the foam is represented by S-AdS wormholes, the level spacing of a Schwarzschild black hole is given by

$$\omega_0 = \Delta M_S^{AdS} = \left(8M_S l_p^2\right)^{-1} \alpha_{AdS} = \left(8M_S l_p^2\right)^{-1} \frac{9\ln 2}{16\pi} = \Delta M \frac{9}{16}$$
(47)

that it means that for a given Schwarzschild black hole of mass M, the S-AdS foam representation gives smaller frequencies.

# IV. THE REISSNER-NORDSTRÖM BLACK HOLE

In this section we would like to apply the foam covering to the case of a black hole with a charge. A charged black hole is described by

$$ds^{2} = -f(r) dt^{2} + f(r)^{-1} dr^{2} + r^{2} d\Omega^{2}, \qquad (48)$$

<sup>&</sup>lt;sup>2</sup>See Ref. [10] for details.

with

$$f(r) = \left(1 - \frac{2MG}{r} + \frac{G\left(Q_e^2 + Q_m^2\right)}{r^2}\right) = \left(1 - \frac{2MG}{r} + \frac{Q^2}{r^2}\right).$$
(49)

 $Q_e$  and  $Q_m$  are the electric and magnetic charge respectively. When Q = 0 the metric describes the Schwarzschild metric, while Q = M = 0 describe a flat metric. For  $Q \neq 0$ , we can distinguish three different cases:

a) MG > Q. In this case the gravitational potential f(r) admits two real distinct solutions located at

$$\begin{cases} r_{+} = MG + \sqrt{(MG)^{2} - Q^{2}} \\ r_{-} = MG - \sqrt{(MG)^{2} - Q^{2}} \end{cases},$$
(50)

with f(r) > 0 for  $r > r_+$  and  $0 < r < r_-$ .  $r_-$  is a Cauchy horizon and  $r_+$  is an event horizon. For each root there is a surface gravity defined by

$$\kappa_{\pm} = \lim_{r \to r_{\pm}} \frac{1}{2} \left| g'_{00}(r) \right|, \tag{51}$$

whose values are

$$\begin{cases} \kappa_{+} = (r_{+} - r_{-})/2r_{+}^{2} \\ \kappa_{-} = (r_{-} - r_{+})/2r_{-}^{2} \end{cases}$$
(52)

The *Hawking temperature* associated with the surface gravity of the event horizon is

$$T_H = \frac{\kappa_+}{2\pi}.\tag{53}$$

- b) MG = Q. This is the extreme case. The gravitational potential f(r) admits two real coincident solutions located at  $r_+ = r_- = r_e = MG$  and its form is  $f(r) = (1 MG/r)^2$ . Here we discover that  $\kappa_+ = \kappa_- = 0$  and  $T_H = 0$ .
- c) MG < Q. In this case the gravitational potential f(r) admits two complex conjugate solutions located at

$$\begin{cases} r_{+,i} = MG + i\sqrt{Q^2 - (MG)^2} \\ r_{-,i} = MG - i\sqrt{Q^2 - (MG)^2} \end{cases},$$
(54)

respectively.

Cases a) and b) imply Q = 0 when M = 0. We will consider the application of the area quantization to cases a) and b). The constant  $\alpha$ , describing the "fine structure", will be left unspecified for the whole computation. To have a matching with the Bekenstein proposal we express the mass M of the charged black hole in terms of the area of the event horizon and its charge Q. This is easily done by invoking the Christodoulou-Ruffini formula [23]

$$M = \left[\frac{A}{16\pi G^2} \left(1 + \frac{4\pi Q^2}{A}\right)^2\right]^{\frac{1}{2}}$$
(55)

obtained by inverting

$$A = 4\pi r_{+}^{2} = 4\pi \left( MG + \sqrt{\left(MG\right)^{2} - Q^{2}} \right)^{2}.$$
(56)

We observe that if Q = 0, then we recover the Schwarzschild case, while if  $A = 4\pi Q^2$ , we have the extreme one. The application of Eq.(33) to the area of the RN black hole gives

$$A = 4\pi \alpha l_p^2 N_2,\tag{57}$$

where  $N_2$  is the wormholes number used for the covering of the RN black hole area. However, from Eq.(56)we obtain that

$$Q^2 = \sqrt{N_2} \left( \sqrt{N_1} - \sqrt{N_2} \right) \alpha l_p^2, \tag{58}$$

where we have used Eq.(38). We immediately see that from the above equation we have  $N_1 \ge N_2$ , where the equality corresponds to the vanishing charge. Moreover we choose  $N_1$  and  $N_2$  in such a way that  $Q^2 = \alpha l_p^2 q$ , q = 0, 1, 2, ... This means that

$$\sqrt{N_2} \left( \sqrt{N_1} - \sqrt{N_2} \right) = q. \tag{59}$$

Note that if we put Eq.(58) into Eq.(55), we obtain always the Schwarzschild case. This means that the imposed condition of Eq.(59) reveals the physics of the charged black hole. By solving with respect to  $N_2$ , we get

$$N_2 = -q + \frac{N_1}{2} \pm \frac{\sqrt{N_1}\sqrt{N_1 - 4q}}{2}.$$
(60)

with  $N_1 \ge 4q$ . When q = 0, we recover the Schwarzschild case, namely  $N_2 = N_1$  which implies the plus choice in Eq.(60)<sup>3</sup>. On the other hand, when we consider  $N_1 = 4q$ , we get  $N_2 = q$  corresponding to the extreme case. Inserting Eqs.(57) and (59) into Eq.(55) we get

$$M = \frac{\sqrt{\alpha}}{2l_p} \sqrt{N_2} \left( 1 + \frac{q}{N_2} \right). \tag{61}$$

It is interesting to see what happens on the level spacing when we consider a transition in mass with a fixed charge<sup>4</sup>. From Eq. (56) we get

$$\omega_0 = \Delta M = \frac{\partial M}{\partial r_+} \frac{dr_+}{dN} \Delta N = \frac{\pi}{A} \left( r_+ - r_- \right) \alpha \tag{62}$$

with  $\Delta N = 1$ , which vanishes in the extreme limit [24,25]. It is also interesting to see that the wormhole model perfectly agrees with the fourth postulate of Ref. [27], asserting that:

If  $E_i$  and  $E_f$  are the initial and the final energies which may be extracted from the horizon, and  $n_iA_0$  and  $n_fA_0$  are the corresponding horizon areas, then

$$\int_{E_i}^{E_f} \frac{dE}{\kappa(E)} = \frac{\ln 2}{2\pi} \left( n_f - n_i \right),$$
(63)

where  $\kappa(E)$  is the surface gravity of the horizon. Let us apply Eq.(63) to the RN black hole. The left hand side becomes

$$\int_{M_i}^{M_f} \frac{dM}{\kappa (M,Q)} = \int_{M_i}^{M_f} \frac{r_+^2 dM}{\sqrt{(MG)^2 - Q^2}} = \left[ M^2 G + M \sqrt{(MG)^2 - Q^2} \right]_{M_i}^{M_f} = \left[ M r_+ \right]_{M_i}^{M_f}.$$
(64)

We now use Eqs.(56, 61) and Eq.(64) becomes

$$\left[\frac{\alpha N_2}{2} \left(1 + \frac{q}{N_2}\right)\right]_{N_i}^{N_f} = \frac{\alpha}{2} \left(N_{2,f} - N_{2,i}\right),\tag{65}$$

which is the right hand side of fourth postulate. Note that the value of  $\alpha$  which matches with the postulate is given by the Schwarzschild foam model. Indeed if the foam model is represented by the S-AdS wormholes we have a factor 9/16 that changes the postulate.

<sup>&</sup>lt;sup>3</sup>This is only Eq.(56) written in terms of quantum numbers.

<sup>&</sup>lt;sup>4</sup>See also Refs. [24,26]

### V. CONCLUSIONS

In this paper we have applied the model presented in Refs. [6-8] to a larger types of black holes including the negative cosmological constant and a charge. Even in this case, assuming the validity of the Bekenstein-Hawking relation, the entropy has been "quantized". Precisely, it is the area that it has been quantized; this is the effect of a space-time filled by a given integer number of disjoint non-interacting wormholes. Nevertheless, we have two possibility of reproducing a foamy space-time:

- 1. A foamy space-time made by Schwarzschild wormholes.
- 2. A foamy space-time made by Schwarzschild-Anti-de Sitter wormholes.

This means that we need a selection mechanism. This is given exactly by a possible observation of the spectrum of the Hawking radiation and in particular from the level spacings of the black hole under examination. We recall that the level spacings appear because the black hole mass has been quantized in terms of wormholes. Note that this quantization procedure is in agreement with the quantized area proposed heuristically by Bekenstein and reproduced by different authors [16-22]. The degeneracy factor  $(\ln 2)$  is here interpreted as an effect of the invariance of the orientation of the wormhole with respect to the black hole area, erroneously interpreted in Ref. [6] as an effect of having a model of an "*ideal Boltzmann gas*" of wormholes. The statistics of the foam wave function has never been introduced, but however the logarithmic factor appears due to the average distribution of the different radius sizes: it is a pure geometrical effect, that in the General Relativity language means dynamical effect. The covering procedure applied to the RN black hole introduces a second quantum number related to the charge. Nevertheless this number comes in because a different horizon with different forces has been considered. This means that even the charge quantum number could be related to a geometrical/dynamical effect [28]. Note that also the extreme case can be easily considered. Concerning this point there exists an open problem about the validity of the area-entropy relation<sup>5</sup>. Indeed, there exist well known indications that in the extreme RN black hole, the entropy is zero due to the strict relation between entropy and topology [30–33]. This conclusion comes from the following equation

$$S = \chi \frac{A}{4G},\tag{66}$$

where  $\chi$  is the Euler characteristic<sup>6</sup>. Since the value of  $\chi$  is zero for the extreme RN, the entropy is zero too. How the foam can approach this problem will be the subject of a future investigation.

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$$S = \chi \frac{A}{8G}.$$
(67)

This formula has been further generalized in Ref. [34], where the author considers the extension to topological black holes.

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 $<sup>^{5}</sup>$ There are arguments coming from a combination between quantum, thermodynamic and statistical arguments [29] leading to a failure of the entropy-area relation.

<sup>&</sup>lt;sup>6</sup>In Ref. [33], the formula proposed for the area-entropy relation has been modified in

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