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Convex combinations of random variables stochastically dominate the parent for a new class of heavy tailed distributions*

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Abstract

Stochastic dominance of a random variable by a convex combination of its independent copies has recently been shown to hold within the relatively narrow class of distributions with concave odds function, and later extended to broader families of distributions. A simple consequence of this surprising result is that the sample mean can be stochastically larger than the underlying random variable. We show that a key property for this stochastic dominance result to hold is the subadditivity of the cumulative distribution function of the reciprocal of the random variable of interest, referred to as the inverted distribution. By studying relations and inclusions between the different classes for which the stochastic dominance was proved to hold, we show that our new class can significantly enlarge the applicability of the result, providing a relatively mild sufficient condition.

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1 Introduction

Stochastic dominance is a widely-used tool in probability, which expresses some notion of one random variable being larger than another from a distributional point of view (see Shaked and Shanthikumar [8]). The applications of this concept are numerous within different fields, such as statistics, economics, and finance, as is easily seen by the enormous references in the literature dealing with these concepts. Although the topic has been studied extensively, recent results, discussed below, have outlined some "surprising" behaviours of stochastic dominance, especially when we consider sums of random variables. While it may be intuitive, in a non-random setting, that summing the same quantity on both sides of some expression should not affect inequalities, and

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that a convex combination of points belongs to the convex hull, these basic principles are not generally true when random elements are involved. For example, in Pomatto et al. [7] it is shown that, under some conditions, the ordering between a pair of random variables can be obtained by summing an independent "noise" to both, while Chen et al. [2] proves that, within a given family of probability distributions, a random variable can be dominated by convex combinations of independent copies from it. In both cases, the results are related to the variability, or the tail-heaviness, of the random variables involved.

In this paper, we show that the dominance result recently obtained by Chen et al. [2], and then extended by Chen and Shneer [3] and Müller [6] to larger classes of distributions, can also hold under different, and in some cases broader, conditions. To be more specific, we search for conditions under which, given n i.i.d. copies of X, say $X_1, \ldots, X_n$, and weights $\theta_1, \ldots, \theta_n \geq 0$ such that $\theta_1 + \cdots + \theta_n = 1$, we have

$$X \leq_{st} \theta_1 X_1 + \cdots + \theta_n X_n. \tag{1.1}$$

where $\leq_{st}$ represents the standard stochastic dominance (see Definition 2.1 below to recall the formal definition). The implications of this result in terms of decision making under uncertainty, with meaningful applications in insurance and economic models, are quite remarkable, as it has been already explained in [2], and complemented by applications to economical and management problems in [3]. Using the same terminology as in [2], the relation in (1.1) represents an "unexpected" stochastic dominance result. Indeed, it is maybe intuitive to think that a convex combination is somewhere in between its components, which is actually the case for random variables with finite mean. In particular, if $\mathrm{E}\, X < \infty$, (1.1) holds trivially, with equality in distribution, if and only if exactly one coefficient is strictly positive. Differently, if $\mathrm{E}\, X < \infty$ and at least two coefficients are strictly positive, (1.1) does not hold, as follows from the fact that the random variables X and $\theta_1 X_1 + \cdots + \theta_n X_n$ have the same expectation and they are comparable in terms of variability, where X is more variable that $\theta_1 X_1 + \cdots + \theta_n X_n$ in terms of the convex order (see [8]), as we will discuss later. In [2] it is proved that (1.1) holds if X is an increasing convex transformation of a Pareto random variable with shape parameter 1, that these authors call a *super-Pareto* random variable. This property, as described later, is equivalent to the concavity of the odds function (the class of distributions defined through shape properties of the odds functions has recently been studied in Lando et al. [4]). However, the assumptions in [2] define a relatively narrow class of distributions, ruling out many important models for which stochastic dominance is still verified. Furthermore, the super-Pareto assumption implies that X is absolutely continuous, while it is reasonable to expect that (1.1) may hold even for some discrete models, as it can be seen in some special cases. The above examples justify the interest in finding weaker conditions for (1.1). As a consequence, Chen and Shneer [3] introduced new families of distributions for which (1.1) is verified, namely, the *super-Fréchet* and the *super-heavy-tailed* classes. The first one includes the super-Pareto, whereas the latter one does not, but it allows to extend (1.1) to distributions which are not in the super-Fréchet class. Even more recently, and using different arguments, Müller [6] shows that (1.1) holds for convex transformations of Cauchy variables, and denotes such class as *super-Cauchy*. In particular, he shows that the super-Cauchy class includes the super-Fréchet, which includes the super-Pareto. However, the super-heavy-tailed family does not include the super-Cauchy, and vice-versa. We remark, moreover, that the super-Cauchy property is still not compatible with discrete models.

In our main result, we show that (1.1) holds under the subadditivity of the inverted distribution, that is, the cumulative distribution function of the reciprocal of X (in the continuous case). This class, denoted as `InvSub`, includes the super-heavy-tailed class introduced in Chen and Shneer [3], and, under some conditions on the convex transformation, all the other classes discussed, except for the super-Cauchy family. On the other hand, we show, by examples, that the super-Cauchy class does not include our `InvSub` family. Finally, we note that, at the moment, the `InvSub` class seems the only one including some non degenerate discrete distributions.

2 Preliminaries and a new class of distributions

Let us start by introducing a few general concepts and terminology to be used in the sequel. Throughout this paper, "increasing" and "decreasing" are taken as "non-decreasing" and "non-increasing", respectively, and the generalised inverse of an increasing function v is denoted as $v^{-1}(u) = \sup\{x \in \mathbb{R} : v(x) \leq u\}$. Moreover, a function v is said to be subadditive if $v(x+y) \leq v(x) + v(y)$, for every x, y in the domain of v. The function v is called superadditive if the inequality is reversed. Finally, a function v defined in $[0, +\infty)$ is said to be star-shaped if $v(0) = 0$ and $\frac{v(x)}{x}$ is increasing. If $\frac{v(x)}{x}$ is decreasing, v is called anti-star-shaped.

Given a random variable X, we shall represent by F_X and $\overline{F}_X = 1 - F_X$ its cumulative distribution and survival functions, possibly using other subscripts if different such objects are under consideration. Moreover, as we will be dealing with some discrete distributions, we represent by $F_X^-(x) = \mathrm{P}(X < x)$ the left-continuous cumulative distribution function of the random variable. In general, we shall not be assuming the existence of densities. We shall also be referring to the odds function $\Lambda_X(x) = \frac{F_X(x)}{\overline{F}_X(x)}$, again possibly with different subscripts. We recall the definition of stochastic dominance.

Definition 2.1. *Given two random variables X and Y, we say that Y stochastically dominates X, denoted as $X \leq_{st} Y$, if $\overline{F}_X(x) \leq \overline{F}_Y(x)$, for every $x \in \mathbb{R}$.*

Note that we shall refer to the random variables or to their cumulative distribution functions, with the same notations, as is more convenient. In fact, the stochastic orders and the characterisations we will be discussing depend only on the cumulative distribution functions.

Bearing in mind that $F^-_{\frac{1}{X}}(x) = 1 - F_X(\frac{1}{x})$ is generally referred to as the *inverted distribution* of X, as this is the cumulative distribution function of $\frac{1}{X}$ in the continuous case, we introduce a new class of distributions that will be central to our main result.

Definition 2.2. *We say that a random variable X such that $F_X(0) = 0$ is* `InvSub` *(for "inverted subadditive") if $F^-_{\frac{1}{X}}$ is subadditive.*

We first present a simple characterisation of this class.

Lemma 2.3. *A random variable X is* `InvSub` *if and only if $F_X(0) = 0$ and*

$$F_X(\tfrac{x}{\theta}) + F_X(\tfrac{x}{1-\theta}) \leq F_X(x) + 1, \quad \forall x \geq 0,\, \theta \in (0,1). \tag{2.1}$$

Proof. The subadditivity of $1 - F_X(\frac{1}{x})$ is obviously equivalent to $1 - F_X(\frac{x}{\theta}) + 1 - F_X(\frac{x}{1-\theta}) \geq 1 - F_X(x)$, for every $x \geq 0$ and $\theta \in (0,1)$, which is clearly a rewriting of (2.1). □

The assumption $F_X(0) = 0$ is not essential either for Definition 2.2 or Lemma 2.3. However, it proves to be crucial for the proof of our main result, Theorem 3.1. Therefore, we define `InvSub` to be the class for which we can prove the stochastic dominance (1.1).

Example 2.4. A simple example of a class of distributions that are `InvSub` is obtained by considering random variables X with Fréchet distribution with shape parameter 1, that is, defined by the cumulative distribution function $\mathcal{H}(x) = e^{-1/x}$, for $x > 0$. In fact, given $\theta \in [0,1]$, $1 + \mathcal{H}(x) - \mathcal{H}(\frac{x}{\theta}) - \mathcal{H}(\frac{x}{1-\theta}) = (1 - e^{-\theta/x})(1 - e^{-(1-\theta)/x}) \geq 0$, so $\mathcal{H}$ satisfies (2.1). Moreover, note that it is easily seen that $\mathrm{E}\, X$ is infinite and the odds function is $\Lambda_{\mathcal{H}}(x) = \frac{e^{-1/x}}{1-e^{-1/x}}$, which is convex.

It is well-known that, for nonnegative functions, concavity implies subadditivity. Hence, when densities exist, a rather simple sufficient condition is available.

Proposition 2.5. *Assume the nonnegative random variable X has density f_X such that its hazard rate $r_X(x) = \frac{f_X(x)}{\overline{F}_X(x)} \leq \frac{1}{x}$, for $x > 0$. Then X is* `InvSub`*.*

Proof. Note that the derivative of $\frac{1}{x}(1 - F_X(\frac{1}{x}))$ has the same sign as $\frac{1}{x} f_X(\frac{1}{x}) + F_X(\frac{1}{x}) - 1$, which, according to the assumption, is easily seen to be negative. It follows that $1 - F_X(\frac{1}{x})$ is anti-star-shaped, as it vanishes at zero, therefore, it is subadditive. □

This very simple characterisation allows to show that a second family of distributions is `InvSub`.

Example 2.6. Consider nonnegative random variables Y_b, where $b \geq 0$, with survival function $\overline{F}_b(x) = \frac{1}{1+x^b \operatorname{Log}(x+1)}$, for $x > 0$. It is easily seen that these distributions satisfy the monotonicity assumption of Proposition 2.5 for $b \in (0,1)$ sufficiently small ($b \leq 0.7$, although this is not an optimal bound). Therefore Y_b, for these suitably small values of b are `InvSub`. Moreover, note that the odds function is $\Lambda_b(x) = x^b \operatorname{Log}(x+1)$, that can be checked to not be concave nor convex.

The existence of a density is not necessary. Indeed, the `InvSub` condition is also compatible with discrete models, as we illustrate in the next example.

Example 2.7. Let $F_X(x) = (1-p)^{\lceil \frac{1}{x} \rceil}$, be defined in the completed half line $(0, +\infty]$, where $p \in (0,1)$ and $\lceil \cdot \rceil$ denotes the ceiling function. This cumulative distribution function is a right-continuous step function, with jumps at points $\frac{1}{k}$, $k = 1, 2, 3, \ldots, +\infty$, and in particular, it assigns positive mass p to $+\infty$. This clearly implies that $\mathrm{E}\, X = +\infty$. Now, $F^-_{\frac{1}{X}}(x) = 1 - F_X(\frac{1}{x}) = 1 - (1-p)^{\lceil x \rceil}$ is the left-continuous version of the geometric cumulative distribution function, which can be seen to be subadditive.

One could ask about the existence of a proper discrete random variable, that is, a discrete random variable X such that $\mathrm{P}(X = +\infty) = 0$, in the `InvSub` class. The following result gives a negative answer to this question.

Proposition 2.8. *Let X be* `InvSub` *with discrete distribution. Then $\mathrm{P}(X = +\infty) > 0$.*

Proof. Assume that $\mathrm{P}(X = +\infty) = 0$. This implies that $F_{\frac{1}{X}}(0) = 0$ and is continuous at the origin. Obviously, the same holds for $F^-_{\frac{1}{X}}$. On the other hand, as $\frac{1}{X}$ is discrete, it follows that the jump points of $F^-_{\frac{1}{X}}$ have the origin as an accumulation point. Therefore, to characterize $F^-_{\frac{1}{X}}$ we denote the sequence of jumps points as $a_n \searrow 0$, and define $F^-_{\frac{1}{X}}(x) = d_n$ whenever $x \in (a_n, a_{n-1}]$. This sequence satisfies $d_n \leq d_{n-1}$ and, due to the continuity at the origin, $d_n \searrow 0$. Choose now $x \in (a_n, a_{n-1}]$ and $y \in (a_\ell, a_{\ell-1}]$, where we may assume that $\ell > n$. The subadditivity of $F^-_{\frac{1}{X}}$ implies that $F^-_{\frac{1}{X}}(x+y) \leq d_n + d_\ell$. Fix some $\varepsilon > 0$, take x such that $a_{n-1} - x < \varepsilon$, choose y such that $x + y \in (a_{n-1}, a_{n-2}]$ and $y < 2\varepsilon$. The subadditivity then implies that $d_{n-1} \leq d_n + d_\ell$, and we may take d_ℓ arbitrarily small. As ε may also be chosen arbitrarily small, it follows that $d_{n-1} \leq d_n$, hence, as this sequence is decreasing, $d_n = d_{n-1}$. Therefore, $F^-_{\frac{1}{X}}$ is constant throughout $[0, +\infty)$ and the only solution is $F^-_{\frac{1}{X}}(x) = 0$, for every $x \in [0, +\infty)$, hence an improper distribution. □

We close this section with a closure property about the class `InvSub`.

Theorem 2.9. *Let X be* `InvSub` *and h a continuous star-shaped function. Then $h(X)$ is* `InvSub`*.*

Proof. Note that, as h is continuous, $F_{h(X)}(x) = F_X(h^{-1}(x))$. Since h is star-shaped, its inverse, h^{-1}, is anti-star-shaped (see Lemma 4.1 in [1]), hence, given $\theta \in (0,1)$, it follows

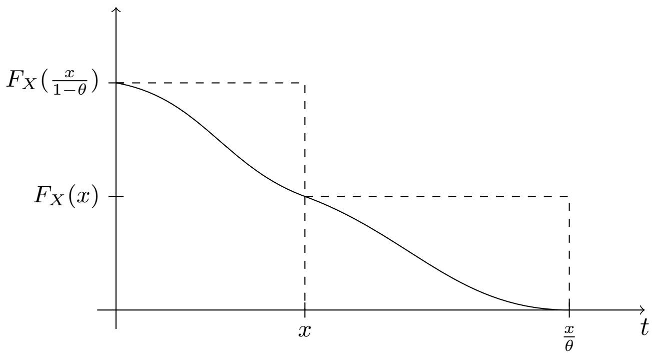

Figure 1: Upper bound for the integral in the initial induction step.

that $h^{-1}(\frac{x}{\theta}) \leq \frac{h^{-1}(x)}{\theta}$ and $h^{-1}(\frac{x}{1-\theta}) \leq \frac{h^{-1}(x)}{1-\theta}$. As every function considered is increasing, we have

$$\begin{aligned} & F_{h(X)}(\tfrac{x}{\theta}) + F_{h(X)}(\tfrac{x}{1-\theta}) \\ &\quad = \quad F_X\left(h^{-1}(\tfrac{x}{\theta})\right) + F_X\left(h^{-1}(\tfrac{x}{1-\theta})\right) \\ &\quad \leq \quad F_X(h^{-1}(x)) + 1 = F_{h(X)}(x) + 1, \end{aligned}$$

using (2.1) for the last inequality, thus, taking into account Lemma 2.3, the proof is concluded. □

3 Main result

We present our main result stating that the `InvSub` property implies the stochastic dominance between a random variable and a convex combination of its independent copies.

Theorem 3.1. *If X is* `InvSub`*, then the stochastic dominance (1.1) holds.*

Proof. We proceed by induction on the number of random variables. Conditioning, using the independence of the random variables, and remembering they are nonnegative, it is easily seen that, for every $x \geq 0$ and $\theta \in (0,1)$,

$$\mathrm{P}\left(\theta X_1 + (1-\theta)X_2 > x\right) = 1 - \int_0^{x/\theta} F_X(\tfrac{x-\theta t}{1-\theta})\, F_X(dt).$$

To find an upper bound for the integral, we consider the decomposition described in Figure 1, from which follows easily that

$$\int_0^{x/\theta} F_X(\tfrac{x-\theta t}{1-\theta})\, F_X(dt) \leq F_X(\tfrac{x}{1-\theta})F_X(x) + F_X(x)\left(F_X(\tfrac{x}{\theta}) - F_X(x)\right) \leq F_X(x),$$

using (2.1) for the last inequality. So, it follows that $\mathrm{P}\left(\theta X_1 + (1-\theta)X_2 > x\right) \geq 1 - F_X(x) = \mathrm{P}(X > x)$, so (1.1) holds for $n = 2$.

Assume now that (1.1) holds whenever considering $n-1$ random variables. Given

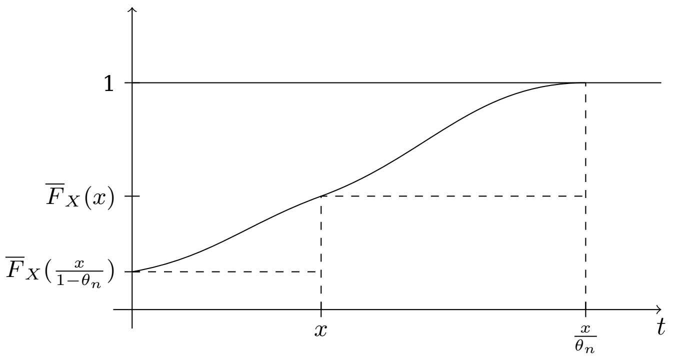

Figure 2: Lower bound for the integral in the induction step.

$\theta_1, \ldots, \theta_n > 0$ satisfying $\theta_1 + \cdots + \theta_n = 1$, we have that

$$\begin{aligned} &\mathrm{P}\left(\theta_1 X_1 + \cdots + \theta_n X_n > x\right) \\ &\quad = \mathrm{P}\left(X_n \geq \tfrac{x}{\theta_n}\right) + \mathrm{P}\left(\theta_1 X_1 + \cdots + \theta_n X_n \geq x, X_n \leq \tfrac{x}{\theta_n}\right) \\ &\quad = \overline{F}_X(\tfrac{x}{\theta_n}) + \int_0^{x/\theta_n} \mathrm{P}\left(\tfrac{\theta_1 X_1 + \cdots + \theta_{n-1} X_{n-1}}{1-\theta_n} \geq \tfrac{x-\theta_n t}{1-\theta_n}\right) F_X(dt) \\ &\quad \geq \overline{F}_X(\tfrac{x}{\theta_n}) + \int_0^{x/\theta_n} \overline{F}_X(\tfrac{x-\theta_n t}{1-\theta_n})\, F_X(dt), \end{aligned}$$

using the induction hypothesis. We need now to find a lower bound for this integral, which may achieved using the decomposition depicted in Figure 2, from which follows that

$$\begin{aligned} &\mathrm{P}\left(\theta_1 X_1 + \cdots + \theta_n X_n > x\right) \\ &\quad \geq \overline{F}_X(\tfrac{x}{\theta_n}) + \int_0^{x/\theta_n} \overline{F}_X(\tfrac{x-\theta_n t}{1-\theta_n})\, F_X(dt) \\ &\quad \geq \overline{F}_X(\tfrac{x}{\theta_n}) + \overline{F}_X(\tfrac{x}{1-\theta_n}) F_X(x) + \overline{F}_X(x)\left(F_X(\tfrac{x}{\theta_n}) - F_X(x)\right) \\ &\quad \geq \overline{F}_X(\tfrac{x}{\theta_n}) + \overline{F}_X(\tfrac{x}{1-\theta_n}) + \overline{F}_X(x)\left(\overline{F}_X(x) - \overline{F}_X(\tfrac{x}{\theta_n}) - \overline{F}_X(\tfrac{x}{1-\theta_n})\right). \end{aligned}$$

Finally, noting that the subadditivity assumption implies that the large parenthesis is negative, it follows that $\mathrm{P}\left(\theta_1 X_1 + \cdots + \theta_n X_n > x\right) \geq \overline{F}_X(\frac{x}{\theta_n}) + \overline{F}_X(\frac{x}{1-\theta_n}) \geq \overline{F}_X(x)$, using (2.1) written in terms of the survival function, thus concluding the proof. □

Without the assumption $F_X(0) = 0$ the lower bound for the induction step would be $\overline{F}_X(x) + F_X(0)(F_X(\frac{x}{\theta_n}) + \overline{F}_X(\frac{x}{1-\theta_n}))$, and it is easily seen that this term cannot be uniformly (with respect to θ_n) larger that $\overline{F}_X(x)$, thus preventing the conclusion of our proof.

Our Example 2.7 show that the stochastic dominance may be fulfilled by discrete distributions, although, according to Proposition 2.8, they need to assign some mass to $+\infty$. An example of a discrete distribution satisfying the stochastic dominance (1.1) is discussed in Section 3 of Müller [6], but takes a more extreme support allowing only for two points: the origin and $+\infty$.

It is easy to see that, if X has finite mean, then X is more variable then $\theta_1 X_1 + \cdots + \theta_n X_n$ in terms of the convex order, that is, for every convex function ϕ, $\mathrm{E}\,\phi(\theta_1 X_1 + \cdots + \theta_n X_n) \leq \mathrm{E}\,\phi(X)$ (this follows by using repeatedly the definition of convexity), meaning, for instance, that X has larger variance (when it is finite) than the convex combination. Differently, as already remarked in [2], the stochastic dominance stated in (1.1) is crucially linked to the fact that we are dealing with random variables with infinite means (see Proposition 2 in [2]). We present here another proof, using more elementary arguments.

Proposition 3.2. *Let $X_1, \ldots, X_n$ be independent random variables such that $\mathrm{P}(X_1 = \cdots = X_n) < 1$, and $\theta_1, \ldots, \theta_n > 0$ such that $\theta_1 + \cdots + \theta_n = 1$. If (1.1) holds, then X has infinite mean.*

Proof. It is enough to prove the case $n = 2$. Denote $Y = \theta X_1 + (1-\theta)X_2$, where X_1 and X_2 are independent and have the same distribution as X and are such that $\mathrm{P}(X_1 \neq X_2) > 0$. Moreover, assume $\mathrm{E}(X)$ is finite. As then follows that $\mathrm{E}(Y) = \mathrm{E}(X)$, both finite, this, together with (1.1), implies that X and Y have the same distribution. Therefore, $\mathrm{Var}\left(\sqrt{X}\right) = \mathrm{Var}\left(\sqrt{Y}\right)$, implying that $\mathrm{E}\left(\sqrt{X}\right) = \mathrm{E}\left(\sqrt{Y}\right)$. But this is not possible, as Jensen's inequality implies that $\mathrm{E}\left(\sqrt{Y}\right) = \mathrm{E}\left(\sqrt{\theta X_1 + (1-\theta)X_2}\right) > \theta\,\mathrm{E}\left(\sqrt{X}\right) + (1-\theta)\,\mathrm{E}\left(\sqrt{X}\right) = \mathrm{E}\left(\sqrt{X}\right)$, the inequality being strict because $\mathrm{P}(X_1 \neq X_2) > 0$. □

As a consequence of Theorem 3.1 and Proposition 3.2, the following result is immediate.

Proposition 3.3. *If X is* `InvSub`*, then $\mathrm{E}\,X$ is infinite.*

4 Comparing with earlier results

We will now address the relations between the `InvSub` class and the other families of distributions implying (1.1), discussed in the Introduction.

We shall represent the cumulative distribution function of the Pareto with shape parameter 1 by $\mathcal{P}(x) = 1 - \frac{1}{x}$, for $x \geq 1$. Note that it is straightforward to verify that $\mathcal{P}$ satisfies the subadditivity assumption in Theorem 3.1, that is, if Z is such that $F_Z = \mathcal{P}$ then Z is `InvSub`. Further, we shall denote the Cauchy distribution function by $\mathcal{C}(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$, for $x \in \mathbb{R}$. If a random variable T has cumulative distribution function $\mathcal{C}$ it can be verified that it is not in the class `InvSub`, however, its absolute value $|T|$ is `InvSub`.

4.1 The super-Pareto class

This family is closely linked to the Pareto distributions, as follows from its definition.

Definition 4.1 (Definition 1 in [2])**.** *A random variable Y is super-Pareto if $Y \stackrel{d}{=} h(Z)$, where $F_Z = \mathcal{P}$ and h is an increasing, convex and nonconstant function.*

As mentioned in [2], the super-Pareto family includes the generalised Pareto distributions, the Burr distributions and the log-logistic distribution. Therefore, it includes some common models either in economical applications or in extreme value theory. An adapted version of the main result in [2] is quoted below.

Theorem 4.2 (adapted version of Theorem 1 in [2])**.** *The conclusion of Theorem 3.1 holds if X is super-Pareto.*

Therefore, we are interested in relating the super-Pareto assumption with our `InvSub` assumption. We start by an equivalent characterisation of the super-Pareto family given below.

Proposition 4.3 (Proposition 1 in [2])**.** *A random variable X, with cumulative distribution function F_X, is super-Pareto if and only if $\frac{1}{\overline{F}_X(x)}$ is concave.*

Note that the concavity of $\Lambda_X(x) = \frac{F_X(x)}{\overline{F}_X(x)} = \frac{1}{\overline{F}_X(x)} - 1$ implies the differentiability of F_X at almost every point of the support, hence the concavity of Λ_X is equivalent to the decreasingness of the odds rate $\lambda_X(x) = \Lambda'_X(x) = \frac{f_X(x)}{\overline{F}^2_X(x)}$, considering side derivatives at the points of nondifferentiability. This family of distributions has been addressed in [4] or, more recently, in Arab et al. [1], being referred as the DOR family (for "*decreasing odds rate*"), that can be characterised using an appropriate stochastic order. We need some additional definitions to describe these relations more precisely.

Definition 4.4. *We say that a random variable X is* DOR if its odds function $\Lambda_X(x)$ is concave.

Definition 4.5. *Given two cumulative distribution functions F_1 and F_2, we say that, F_1 is smaller than F_2 in the convex transform order, represented by $F_1 \leq_c F_2$, if $F_2^{-1} \circ F_1$ is convex.*

The following result relates the shape of the odds function with the Pareto distribution through the convex transform order.

Proposition 4.6. F_X *is* DOR (or, equivalently, X is super-Pareto) if and only if $\mathcal{P} \leq_c F_X$. Analogously F_X has increasing odds rate if and only if $F_X \leq_c \mathcal{P}$.

Proof. Noting that the quantile of the Pareto is $\mathcal{P}^{-1}(u) = \frac{1}{1-u}$, we have $\mathcal{P}^{-1} \circ F_X = \frac{1}{\overline{F}_X} = \Lambda_X + 1$, so the result follows immediately. □

As mentioned in Example 2.4, the Fréchet class, with cumulative distribution functions $\mathcal{H}(x) = e^{-1/x}$ satisfies the assumption on Theorem 3.1. On other hand, its odds functions $\Lambda_{\mathcal{H}}$ is convex, hence $\mathcal{H} \leq_c \mathcal{P}$, so random variables with cumulative distribution function $\mathcal{H}$ are not super-Pareto. This provides an example where our main result Theorem 3.1 implies (1.1), while Theorem 4.2 is not applicable, as its assumptions are not satisfied. Further, by transitivity, note that if X is super-Pareto we have $\mathcal{H} \leq_c \mathcal{P} \leq_c F_X$.

We now present one example showing that shape conditions about the odds function are not the most appropriate way to find sufficient conditions for the stochastic dominance (1.1).

Example 4.7. The random variables Y_b introduced in Example 2.6 have odds function $\Lambda_b(x) = x^b \operatorname{Log}(x+1)$ that, as mentioned before, are not concave nor convex for $b < 1$. Nevertheless, as referred in Example 2.6, Y_b, for $b \in (0,1)$ suitably small, is InvSub.

Finally, we relate the super-Pareto class with the InvSub family.

Theorem 4.8. *If X is nonnegative super-Pareto then X is* InvSub.

Proof. According to Proposition 4.3, X is nonnegative DOR. Let Z be such that $F_Z = \mathcal{P}$ and $Y = Z - 1$, where $F_Y(x) = \frac{x}{x+1}$, for $x \geq 0$. Given the shift invariance of the convex transform order, X being super-Pareto is equivalent to $X \stackrel{d}{=} h(Y)$, where h is increasing convex and $h(0) = 0$, hence h is star-shaped. It is straightforward to verify that Y is InvSub, so the conclusion follows by applying Theorem 2.9. □

An extension of the super-Pareto family, also based on the convex transform order, was recently introduced in [3].

Definition 4.9 (adapted version of Definition 4 in [3])**.** *We say that a random variable X is super-Fréchet if $\mathcal{H} \leq_c F_X$.*

The super-Fréchet class is considered in [3] when studying the stochastic dominance for convex combinations of non identically distributed random variables. As mentioned above, it is easily seen that $\mathcal{H} \leq_c \mathcal{P}$, therefore it follows immediately that the super-Pareto is a subclass of the super-Fréchet.

4.2 The super-heavy-tailed class

In [3] the class of *super-heavy-tailed* distributions is introduced. As we shall see, this is closely related to a family of distributions that is well established in the literature.

Definition 4.10. *We say that a random variable X such that $F_X(0) = 0$ is*

1. ***(Definition 2 in [3])** super-heavy-tailed if $-\operatorname{Log} F_X(\frac{1}{x})$ is subadditive;*
2. `NWU` *if $\overline{F}_X(x)\overline{F}_X(y) \le \overline{F}_X(x+y)$, for every x and y.*

The characterisation of X being `NWU`, a well-known class of distributions, may be rewritten as $\operatorname{Log}\overline{F}_X$ is superadditive (see the initial notes in [8]), together with being nonnegative. Note that the nonnegativity is not strictly necessary for the above definitions.

It is readily seen that in the continuous case, X being super heavy-tailed is equivalent to $\frac{1}{X}$ being `NWU`. Note that, in the general case, $F_X(\frac{1}{x}) = 1 - F^-_{\frac{1}{X}}(x)$.

The following characterisation is straightforward.

Lemma 4.11. *A random variable X is super-heavy-tailed if and only if*

$$F_X(\tfrac{x}{\theta})F_X(\tfrac{x}{1-\theta}) \le F_X(x), \quad \forall x \ge 0,\, \theta \in (0,1). \tag{4.1}$$

We may now quote the stochastic dominance proved in [3].

Theorem 4.12 (adapted version of Theorem 1 in [3])**.** *The conclusion of Theorem 3.1 holds if X is super-heavy-tailed.*

We now discuss the relation between the super-heavy-tailed class and the `InvSub` we introduced.

Theorem 4.13. *If X is super-heavy-tailed, then X is* `InvSub`.

Proof. As noted after Definition 4.10, X being super-heavy-tailed means that $\operatorname{Log}(1 - F^-_{\frac{1}{X}}(x)) = \operatorname{Log} F_X(\frac{1}{x})$ is superadditive, that is $\operatorname{Log} F_X(\frac{1}{x+y}) \ge \operatorname{Log} F_X(\frac{1}{x}) + \operatorname{Log} F_X(\frac{1}{y})$, which implies that

$$F_X\left(\tfrac{1}{x+y}\right) \ge \exp\left(\operatorname{Log} F_X\left(\tfrac{1}{x}\right) + \operatorname{Log} F_X\left(\tfrac{1}{y}\right)\right) \ge F_X\left(\tfrac{1}{x}\right) + F_X\left(\tfrac{1}{y}\right) - 1,$$

as the exponential is superadditive. Finally, going to the complementary sets, this last inequality is just $\overline{F}_X(\frac{1}{x+y}) \le \overline{F}_X(\frac{1}{x}) + \overline{F}_X(\frac{1}{y})$, that is, $\overline{F}_X(\frac{1}{x})$ is subadditive, that is, X is `InvSub`. □

Of course, this result implies that Theorem 1 in [3] is contained in Theorem 3.1. Moreover, the following example shows that the `InvSub` class is strictly larger than the super-heavy-tailed family.

Example 4.14. Consider a random variable R_a with cumulative distribution functions of the form $a^{-1/x}\left(1 + \frac{1}{e^{2x-1}}\right)$. Choosing $a \in (0,1)$ sufficiently small (say, for example, $a = 0.5$), it can be verified that (2.1) is satisfied, by checking the assumption in Proposition 2.5, so R_a is `InvSub`. Differently, (4.1) is not, so R_a is not super-heavy-tailed.

4.3 The super-Cauchy class

In [6], in a similar way as to the "*super-something*" classes already discussed, the following class is considered.

Definition 4.15. *A random variable X is said super-Cauchy if $\mathcal{C} \le_c F_X$.*

Note that, unlike the previously defined classes, this is the only one that allows the support to be the whole $\mathbb{R}$. Therefore, no inclusion relationship is to be expected between the super-Cauchy and our InvSub class, unless, or course, we introduce some boundedness assumption about the support.

The following result is proved in [6].

Theorem 4.16 (Theorem 2.10 in [6])**.** *X is super-Pareto $\Rightarrow$ X is super-Fréchet $\Rightarrow$ X is super-Cauchy. Moreover, the conclusion of Theorem 3.1 holds if X is super-Cauchy.*

Theorem 2.14 in [6] shows that, if T has cumulative distribution function $\mathcal{C}$, then $|T|$ is super-Cauchy while T is not super-heavy-tailed. Nevertheless, $|T|$ can easily be seen to be InvSub. Moreover, to show that the super-Cauchy class does not contain the InvSub one, we provide the following example.

Example 4.17. Take the cumulative distribution function $V(x) = e^{-1/x^{.55}}\left(1 + e^{1-2x}\right)$, for $x > 0$. It can be seen that $1 - V(\frac{1}{x})$ is anti-star-shaped, so V is InvSub. Moreover, it is easily verified that $\mathcal{C}^{-1} \circ V$ is not concave nor convex, so this distribution is not super-Cauchy.

5 Characterisations based on a new inverted-subadditive order

Many relevant nonparametric families of distributions, such as the IHR, DHR, NBU, and NWU classes, discussed in the book of Marshall and Olkin [5], and many others, can be defined via a suitable stochastic order with respect to some reference distribution. This is also the case, for example, of the super-Pareto, super-Fréchet, and super-Cauchy families, which are defined through the convex transform order, and using the aforementioned distributions as reference points. Such types of characterisations are very useful, since they allow to determine a benchmark distribution and to establish relations between classes using transitivity and other properties of stochastic orders. We now show that also the super-heavy tailed and the InvSub classes can be defined again using the Fréchet and the Pareto distributions, respectively, as a benchmark. However, in this case the order is of a different nature, and it is introduced in the following definition.

Definition 5.1. *Given two random variables X and Y, we say that X is smaller than Y in the inverted-subadditive order, represented by $F_X \leq_{i-sb} F_Y$, if $S_{X,Y}(x) = (F^{-}_{\frac{1}{Y}})^{-1} \circ F^{-}_{\frac{1}{X}}(x)$ is subadditive.*

Taking into account that $F^{-}_{\frac{1}{Y}}(x) = 1 - F_Y(x)$, it follows that $(F^{-}_{\frac{1}{Y}})^{-1}(p) = \frac{1}{F_Y^{-1}(1-p)}$, hence $S_{X,Y}(x) = (F^{-}_{\frac{1}{Y}})^{-1} \circ F^{-}_{\frac{1}{X}}(\frac{1}{x}) = \frac{1}{F_Y^{-1} \circ F_X^{-}(\frac{1}{x})}$.

The inverted-subadditive is closely related to the superadditive order discussed by [8], defined through the superadditivity of the composition $F_Y^{-1} \circ F_X$. In our case, we use left-continuous cumulative distribution functions to include discrete models. Since subadditivity is preserved under composition, it is easy to see that the inverted-subadditive order is a proper stochastic order. Differently from other transform orders, such as the convex one, this new order is not shift-invariant. In fact, shifts may be crucial in this case, since we are dealing with inverted distributions. For the same reason, it is also clear that the inverted-subadditive order rules out distributions whose supports strictly include the origin.

The following result gives a complete description of the super-heavy-tailed class, avoiding the strict continuity difficulties.

Proposition 5.2. *A random variable X such that $F_X(0) = 0$ is super-heavy-tailed if and only if $F_X \leq_{i-sb} \mathcal{H}$.*

Proof. Note that $\mathcal{H}^{-1}(p) = -\frac{1}{\operatorname{Log} p}$, so $\frac{1}{\mathcal{H}^{-1} \circ F_X(\frac{1}{x})} = -\operatorname{Log}(F_X(\frac{1}{x}))$. □

We may obtain a similar characterisation for the `InvSub` class using the subadditive order and choosing the appropriate benchmark distribution.

Proposition 5.3. *A random variable X such that $F_X(0) = 0$ is* `InvSub` *if and only if $F_X \leq_{i-sb} \mathcal{P}$.*

Proof. As $\mathcal{P}^{-1}(p) = \frac{1}{1-p}$, we are reduced to verifying the subadditivity of $1 - F_X(\frac{1}{x})$. $\square$

Note that, by transitivity of the inverted-subadditive order, Propositions 5.2 and 5.3, together with $\mathcal{H} \leq_{i-sb} \mathcal{P}$, provide an alternative proof of Theorem 4.13.

It is interesting to note that choosing the Fréchet as a benchmark, compared to the Pareto, yields a larger class when we use the convex transform order, and a smaller one if we use the inverted-subadditive order, due to the different nature of these orders. In fact, the `InvSub` class contains $\mathcal{H}$, while the super-heavy-tailed family does not contain $\mathcal{P}$. However, it contains its shifted version of the Pareto distribution, $\mathcal{P}(x-1) = \frac{x}{1+x}$, $x > 0$ (see [3]). This means that the super-heavy-tailed family does not contain the super-Pareto class, but only its restriction to distributions whose supports start strictly at 0. Differently, all non-negative super-Pareto belong to the `InvSub` class, by Theorem 4.8.

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