Eur. Phys. J. C (2023) 83:369
https://doi.org/10.1140/epjc/s10052-023-11464-4

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Effects of an electric charge on Casimir wormholes: changing the throat size

Remo Garattini[1,2,a]

[1] Dipartimento di Ingegneria e Scienze Applicate, Università degli Studi di Bergamo, Viale Marconi 5, 24044 Dalmine, Bergamo, Italy
[2] I.N.F.N.-sezione di Milano, Milan, Italy

Received: 8 February 2023 / Accepted: 1 April 2023

Abstract In this paper we continue the investigation of the connection between Casimir energy and the traversability of a wormhole. In addition to the negative energy density obtained by a Casimir device, we include the effect of an electromagnetic field generated by an electric charge. This combination defines an electrovacuum source which has an extra parameter related to the size of the throat. Even if the electromagnetic energy density is positive, the null energy condition is still violated. The main reason is that the electromagnetic field satisfies the property $\rho = -p_r$. As a consequence, the traversable wormhole throat can be changed as a function of the electric charge. This means that the throat is no longer Planckian and the traversability is slightly less in principle but slightly greater in practice.

1 Introduction

Casimir wormholes are traversable wormholes (TW) obtained by solving the semiclassical Einstein field equations (EFE)

$$G_{\mu\nu} = \kappa \left\langle T_{\mu\nu} \right\rangle^{\text{Ren}} \quad \kappa = \frac{8\pi G}{c^4} \tag{1}$$

with a source of the form

$$\rho_C\left(d\right) = -\frac{\hbar c \pi^2}{720 d^4}, \tag{2}$$

$$P\left(d\right) = \frac{F\left(d\right)}{S} = -3\frac{\hbar c \pi^2}{720 d^4}, \tag{3}$$

representing the energy density and the pressure, respectively; d is the plate separation [1], and $\left\langle T_{\mu\nu} \right\rangle^{\text{Ren}}$ describes the renormalized stress–energy tensor of some matter fields, which in this specific case is obtained by the zero-point energy (ZPE) contribution of the electromagnetic field. The two key ingredients useful for forming a Casimir wormhole are in the relationship $P\left(d\right)/\rho_C\left(d\right) = 3$. This number is the cornerstone of a Casimir wormhole. In addition, the plate separation d has been promoted to be the radial coordinate r considered as a variable. For this reason the pressure $P\left(d\right)$ will be interpreted as a radial pressure $p_r\left(r\right)$. To build a Casimir wormhole we need to introduce the following spacetime metric

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 d\Omega^2, \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the unit sphere, and $\Phi(r)$ and $b(r)$ are two arbitrary functions of the radial coordinate $r \in [r_0, +\infty)$, denoted as the redshift function and the shape function, respectively [2–4]. With the help of the metric (4), the EFE, written in an orthonormal frame, are

$$\frac{b'\left(r\right)}{r^2} = \kappa\rho\left(r\right), \tag{5}$$

$$\frac{2}{r}\left(1 - \frac{b\left(r\right)}{r}\right)\Phi'(r) - \frac{b\left(r\right)}{r^3} = \kappa p_r\left(r\right) \tag{6}$$

and

$$\left\{\left(1 - \frac{b\left(r\right)}{r}\right)\left[\Phi''(r) + \Phi'(r)\left(\Phi'(r) + \frac{1}{r}\right)\right] - \frac{b'\left(r\right)r - b\left(r\right)}{2r^2}\left(\Phi'(r) + \frac{1}{r}\right)\right\} = \kappa p_t(r), \tag{7}$$

where $\rho\left(r\right)$ is the energy density,[1] $p_r\left(r\right)$ is the radial pressure, and $p_t\left(r\right)$ is the lateral pressure. The line element (4) represents a spherically symmetric and static wormhole, and r_0 is the location of the throat. We can complete the EFE with

[a] e-mail: remo.garattini@unibg.it (corresponding author)

[1] However, if $\rho\left(r\right)$ represents the mass density, then we have to replace $\rho\left(r\right)$ with $\rho\left(r\right)c^2$

the expression of the conservation of the stress–energy tensor which can be written in the same orthonormal reference frame

$$p_r'(r) = \frac{2}{r}(p_t(r) - p_r(r)) - (\rho(r) + p_r(r))\Phi'(r). \quad (8)$$

If we assume that an equation of state (EoS) $p_r(r) = \omega\rho_C(r)$ is imposed, then we find the following solution to the semi-classical EFE

$$\Phi(r) = \frac{1}{2}(\omega - 1)\ln\left(\frac{r\omega}{(\omega r + r_0)}\right) \quad (9)$$

$$b(r) = \left(1 - \frac{1}{\omega}\right)r_0 + \frac{r_0^2}{\omega r}, \quad (10)$$

where we have used the energy density $\rho_C(r)$ of Eq. (2) and the radial pressure $p_r(r)$ described by Eq. (3) as a source, with d replaced by r. A fundamental property of a wormhole is that a flaring out condition of the throat, given by $(b - b'r)/b^2 > 0$, must be satisfied as well as the condition that $1 - b(r)/r > 0$. Furthermore, at the throat, $b(r_0) = r_0$ and the condition $b'(r_0) < 1$ is imposed to have wormhole solutions. It is also fundamental that there are no horizons present, which are identified as the surfaces with $e^{2\Phi(r)} \to 0$, so that $\Phi(r)$ must be finite everywhere. The procedure used to obtain a Casimir wormhole can be extended to include the electromagnetic field as an additional source. The key point is in the following observation: the algebraic structure of stress–energy tensors for electromagnetic fields is determined by [5,6]

$$T_0^0 = T_1^1 \quad (11)$$

which means

$$\rho = -p_r. \quad (12)$$

As a warm-up exercise we can consider a pure spherically symmetric electromagnetic field without the contribution of the Casimir energy, to see whether it is possible to build a TW even if the energy density is positive. The stress–energy tensor (SET) we will consider is generated by a spherically symmetric electromagnetic field, namely

$$E_r = E_1(r) = cF_{01} = -cF_{10}. \quad (13)$$

All other components are zero, since there are no currents or magnetic monopoles. This means that the electric field can only have a radial component. Also, this radial component cannot depend on θ or ϕ. With these assumptions, in an orthonormal frame, we can write

$$T_{\mu\nu}^{EM} = \frac{1}{\mu_0}\left(g_{\nu\alpha}F_{\mu\gamma}F^{\alpha\gamma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) = \frac{Q^2}{2(4\pi)^2\varepsilon_0 r^4}\mathrm{diag}(1, -1, 1, 1), \quad (14)$$

which is conserved and traceless. From the energy density ρ of the SET, it is possible to obtain the following shape function

$$b(r) = r_0 + \frac{r_2^2}{r_0} - \frac{r_2^2}{r}; \quad r_2^2 = \frac{GQ^2}{4\pi c^4\varepsilon_0}, \quad (15)$$

which has the correct properties. Indeed,

$$b'(r_0) = \frac{r_2^2}{r_0^2} < 1 \Longrightarrow r_2 < r_0 \quad (16)$$

and

$$1 - \frac{b(r)}{r} = \frac{(r - r_0)(rr_0 - r_2^2)}{r_0 r} > 0 \quad \text{if } r > r_0;\ r > \frac{r_2^2}{r_0} \quad (17)$$

$$\frac{b - b'r}{b^2} = \frac{r_0 r^2(r_0^2 + r_2^2)}{(r_0^2 r + r_2^2 r - r_2^2 r_0)^2} > 0. \quad (18)$$

Equation (6), together with the shape function (15), allows the computation of the redshift function

$$\Phi(r) = \frac{1}{2}\ln\left(\frac{(r - r_0)(rr_0 - r_2^2)}{r^2}\right) + C, \quad (19)$$

where C is an integration constant.[2] As expected, we cannot adopt the strategy of Ref. [1] because the $\ln(r - r_0)$ never disappears for any choice of r_0, even if property (12) is satisfied. One could insist by imposing an inhomogeneous EoS of the form

$$\omega(r) = -\frac{b(r)}{rb'(r)} = \frac{r_0 r_2^2 - (r_0^2 + r_2^2)r}{r_2^2 r_0} \quad (22)$$

making it possible to fix $\Phi(r) = 0$. Nevertheless, Eq. (22) must be compatible with

$$\omega(r) = \frac{p_r(r)}{\rho(r)} = -1. \quad (23)$$

This leads to

$$\omega(r_0) = -1 \quad (24)$$

which is incompatible with the flare-out condition (16), because one gets

$$r_0 = r_2. \quad (25)$$

[2] Actually, because of the property (12), from Eq. (6) one can write

$$\frac{2}{r}\left(1 - \frac{b(r)}{r}\right)\Phi'(r) - \frac{b(r)}{r^3} + \frac{b'(r)}{r^2} = 0 \quad (20)$$

which has the following solution

$$\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{b(r)}{r}\right) + C, \quad (21)$$

which is a signature of a black hole.

This means that a pure electromagnetic field cannot support a TW even if $\rho + p_r = 0$. It is necessary to have $\rho + p_r < 0$. For this reason, we are led to consider the superposition of the Casimir source with the electromagnetic field. Such a combination could potentially produce a different result thanks to the property (12) which defines an electrovacuum source. Note that such an electrovacuum source was investigated earlier in Ref. [7] even in the context of generalized uncertainty principle (GUP) distortions. Note also that the idea of including an electric charge or an electromagnetic field in a TW configuration is not new. Indeed, this was first proposed by Kim and Lee [8], who considered a combination of the Morris–Thorne wormhole and the Reissner–Nordström spacetime. Balakin et al. discussed a nonminimal Einstein–Maxwell model [9]. Kuhfittig [10] considered a modification of the Kim and Lee charged wormhole to enable compatibility with the quantum inequality of Ford and Roman [11]. The purpose of this paper is to repeat the procedure which led to the original Casimir wormhole spacetime to see whether it is possible to obtain new EFE solutions with an additional electric field. The rest of the paper is structured as follows. In Sect. 2 we continue the investigation to determine whether the Casimir energy density with an additional electric charge can be considered as a source for a traversable wormhole. In Sect. 3 we examine the features of the Casimir wormhole with the additional electric charge, and in Sect. 4 we consider the Casimir energy and the additional electric charge with the plate separation regarded as a parameter instead of a variable. We summarize and conclude in Sect. 5.

2 The Casimir traversable wormhole with an additional electric charge

In this section we assume that our exotic matter will be represented by the Casimir energy density (2). Following Ref. [1], we promote the constant plate separation d to a radial coordinate r. In addition to the Casimir source we include the contribution of an electric field generated by a point charge. Since it is the null energy condition (NEC) that must be violated, the inequality $\rho(r) + p_r(r) < 0$ must hold. We want to point out to the reader that, thanks to the property (12), we can write

$$\rho(r) + p_r(r) = \rho_C(r) + p_{r,C}(r) + \rho_E(r) + p_{r,E}(r) = -\frac{4\hbar c\pi^2}{720 r^4} < 0, \tag{26}$$

where

$$\rho_C(r) = -\frac{\hbar c\pi^2}{720 r^4}; \quad p_{r,C}(r) = -\frac{3\hbar c\pi^2}{720 r^4};$$
$$\rho_E(r) = \frac{Q^2}{2(4\pi)^2 \varepsilon_0 r^4}; \quad p_{r,E}(r) = -\frac{Q^2}{2(4\pi)^2 \varepsilon_0 r^4}. \tag{27}$$

In this context, the total energy density is represented by

$$\rho(r) = \rho_C(r) + \rho_E(r) = -\frac{\hbar c\pi^2}{720 r^4} + \frac{Q^2}{2(4\pi)^2 \varepsilon_0 r^4} = -\frac{r_1^2}{\kappa r^4} + \frac{r_2^2}{\kappa r^4}, \tag{28}$$

where

$$r_1^2 = \frac{\pi^3 l_p^2}{90}, \tag{29}$$
$$r_2^2 = \frac{G Q^2}{4\pi c^4 \varepsilon_0}. \tag{30}$$

Thus

$$\begin{array}{lll} \rho(r) < 0 & & r_1 > r_2 \\ \rho(r) = 0 & \text{when} & r_1 = r_2\,. \\ \rho(r) > 0 & & r_1 < r_2 \end{array} \tag{31}$$

The shape function $b(r)$ can be obtained plugging $\rho(r)$ of Eq. (28) into Eq. (5), whose solution leads to

$$b(r) = r_0 + \left(r_2^2 - r_1^2\right)\int_{r_0}^{r} \frac{dr'}{r'^2} = r_0 - \frac{r_1^2 - r_2^2}{r_0} + \frac{r_1^2 - r_2^2}{r}, \tag{32}$$

where the throat condition $b(r_0) = r_0$ has been imposed. The redshift function can be obtained by solving Eq. (6) with the help of the shape function (32). One finds

$$\frac{2}{r}\left(1 - \frac{r_0}{r} + \frac{r_1^2 - r_2^2}{r_0 r} - \frac{r_1^2 - r_2^2}{r^2}\right)\Phi'(r) - \frac{r_0}{r^3} + \frac{r_1^2 - r_2^2}{r_0 r^3} - (1-\omega)\frac{r_1^2 - r_2^2}{r^4} = 0, \tag{33}$$

where we have used an EoS of the form $p_r(r) = \omega\rho(r)$. The solution can be written as

$$\Phi(r) = \left(\frac{r_0^2 - \omega\left(r_1^2 - r_2^2\right)}{2\left(r_0^2 + r_1^2 - r_2^2\right)}\right)\ln(r - r_0) - \left(\frac{\omega r_0^2 - r_1^2 + r_2^2}{2\left(r_0^2 + r_1^2 - r_2^2\right)}\right)\ln\left(r r_0 + r_1^2 - r_2^2\right) + \frac{\omega - 1}{2}\ln(r) + C. \tag{34}$$

If $Q = 0$, one recovers the familiar form of the Casimir wormhole redshift function. It is possible to eliminate the horizon if we constrain ω to be

$$\omega = \omega_0 = \frac{r_0^2}{r_1^2 - r_2^2} \tag{35}$$

and the redshift function (34) becomes

$$\Phi(r) = -\left(\frac{\omega - 1}{2}\right)\ln\left(\frac{\omega r r_0 + r_0^2}{\omega r}\right) + C. \tag{36}$$

By assuming that $\Phi(r) \to 0$ for $r \to \infty$, then we find

$$\Phi(r) = \frac{\omega - 1}{2} \ln\left(\frac{\omega r}{\omega r + r_0}\right), \tag{37}$$

which is formally the same as the result of Ref. [1]. The shape function (32) can be rearranged to obtain the familiar form of the Casimir wormhole [1]

$$b(r) = r_0 - \frac{r_1^2 - r_2^2}{r_0} + \frac{r_1^2 - r_2^2}{r} = r_0\left(1 - \frac{1}{\omega}\right) + \frac{r_0^2}{\omega r}. \tag{38}$$

On the other hand, since the ratio in Eq. (35) is not constrained, we can use the ratio $p_r(r)/\rho(r)$ to extract information about the size of the throat. It can immediately be seen that for

$$\omega = \frac{p_r(r)}{\rho(r)} = \frac{3r_1^2 + r_2^2}{r_1^2 - r_2^2}; \quad r_1 \neq r_2, \tag{39}$$

the EoS is satisfied and is not dependent on r. Plugging the value of ω in Eq. (39) into Eq. (35), one finds

$$r_0 = \sqrt{3r_1^2 + r_2^2}. \tag{40}$$

Note that r_2 can be variable, while r_1 cannot. It is convenient to take r_1 as a reference scale. Thus, if we introduce a dimensionless variable

$$x = \frac{r_2}{r_1} = \sqrt{\frac{90 G Q^2}{4\pi c^4 \varepsilon_0 \pi^3 l_p^2}} = \sqrt{\frac{90}{\pi^3} n^2 \frac{e^2}{4\pi\varepsilon_0 \hbar c}} = \frac{3n}{\pi}\sqrt{\frac{10}{\pi}\alpha}, \tag{41}$$

one finds

$$\omega = \frac{3 + x^2}{1 - x^2}; \quad r_0 = r_1\sqrt{3 + x^2}; \quad x \neq 1, \tag{42}$$

where $Q = ne$, e is the electron charge, α is the fine structure constant, and n is the total number of electron charges. Note that for

$$Q \to 0, \quad r_2 \to 0 \Longrightarrow x \to 0, \tag{43}$$

one recovers the shape function of Ref. [1] with $\omega = 3$. On the other hand, when

$$Q \to \infty, \quad r_2 \to \infty \Longrightarrow x \to \infty \tag{44}$$

and $\omega \to -1$. In conclusion, we can write

$$\omega \in (3, +\infty); \quad x \in \left(0, 1^-\right) \tag{45}$$

$$\omega \in (-\infty, -1); \quad x \in \left(1^+, +\infty\right). \tag{46}$$

A comment on the limit (44) is in order. Indeed, when $\omega \to -1$, the shape function assumes the form

$$b(r) = 2r_0 - \frac{r_0^2}{r} \tag{47}$$

and the redshift function is

$$\Phi(r) = \ln\left(\frac{r - r_0}{r}\right) \tag{48}$$

no longer representing a TW, but a black hole. However, $\omega \to -1$ is a limiting value which will never be reached. This means that Q can be arbitrarily large but finite. Note that in this range, $\omega < -1$. This means that the pure electromagnetic field, in this context, acts as a phantom energy source. For completeness, we report the expression of the transverse pressure which, in terms of ω, is identical to $p_t(r)$ of Ref. [1]. One finds

$$p_t(r) = \frac{r_0^2}{\kappa\omega r^4}\left[\omega + \frac{r_0\left(1 - \omega^2\right)}{4\omega\,(\omega r + r_0)}\right] = \omega_t(r)\left(\frac{r_0^2}{\kappa\omega r^4}\right) = \omega_t(r)\,\rho(r), \tag{49}$$

where we have introduced an inhomogeneous EoS on the transverse pressure with

$$\omega_t(r) = -\left(\omega + \frac{r_0\left(1 - \omega^2\right)}{4\omega\,(\omega r + r_0)}\right), \tag{50}$$

and the final form of the SET is

$$T_{\mu\nu} = \frac{r_0^2}{\kappa\omega r^4}\left[\mathrm{diag}\left(-1, -\omega, \omega_t(r), \omega_t(r)\right)\right]. \tag{51}$$

The conservation of the SET is satisfied but a comparison with the SET source shows that

$$T_{\mu\nu} = T_{\mu\nu}^{\text{Source}} - \frac{1}{\kappa r^4}\Big[\mathrm{diag}\,(0, 0, \omega_t(r) - \left(r_1^2 + r_2^2\right), \omega_t(r) - \left(r_1^2 + r_2^2\right))\Big], \tag{52}$$

namely a discrepancy in the transverse pressure with respect to the SET source is present. We recall that the SET source is defined by

$$T_{\mu\nu}^{\text{Source}} = T_{\mu\nu}^{\text{Casimir}} + T_{\mu\nu}^{EM} = \frac{1}{\kappa r^4}\left[\mathrm{diag}\left(-r_1^2 + r_2^2, -\left(3r_1^2 + r_2^2\right), r_1^2 + r_2^2, r_1^2 + r_2^2\right)\right]. \tag{53}$$

It is important to note that there exists another interesting value for ω, namely when $\omega = 1$. For this special choice, one finds that the line element reduces to the Ellis–Bronnikov (EB) wormhole [12,13], that is,

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{r_0^2}{r^2}} + r^2\, d\Omega^2 \tag{54}$$

whose associated SET is

$$T_{\mu\nu}^{EB} = \frac{r_1^2 - r_2^2}{\kappa r^4}\left[\mathrm{diag}\,(-1, -1, 1, 1)\right]. \tag{55}$$

Nevertheless, $\omega = 1$ is incompatible with the relationship (39), and therefore this option will be discarded.

2.1 Special case $r_1^2 = r_2^2$

In the special case

$$r_1^2 = r_2^2 = r_e^2, \tag{56}$$

we find that the energy density vanishes. Therefore, the wormhole shape function is

$$b(r) = r_0. \tag{57}$$

On the other hand, the redshift function appears to be non-trivial. Indeed, from the EFE (6), we find

$$\frac{2}{r}\left(1-\frac{r_0}{r}\right)\Phi'(r) - \frac{r_0}{r^3} + \frac{4r_e^2}{r^4} = 0 \tag{58}$$

which can be rearranged to give

$$\Phi'(r) = \frac{r_0 r - 4r_e^2}{2(r-r_0)r^2}. \tag{59}$$

The solution is

$$\Phi(r) = -\frac{\ln(r)}{2} + \frac{2\ln(r)\, r_e^2}{r_0^2} - \frac{2r_e^2}{r_0 r} + \frac{\ln(r-r_0)}{2}\left(1-\frac{4r_e^2}{r_0^2}\right) + C. \tag{60}$$

It can immediately be seen that for

$$r_0 = 2r_e \tag{61}$$

the horizon disappears and

$$\Phi(r) = -\frac{r_0}{2r}, \tag{62}$$

where we have assumed that $\Phi(r) \to 0$ for $r \to \infty$. Therefore, in this special case we still have a TW with the following line element

$$ds^2 = -\exp\left(-\frac{r_0}{r}\right) dt^2 + \frac{dr^2}{1-r_0/r} + r^2 d\Omega^2, \tag{63}$$

which is traversable in principle but not in practice, because the throat is Planckian. To complete this special case, we compute the transverse pressure and we find

$$\left\{\frac{r_0^2}{r^4} - \frac{r_0^3}{4r^5}\right\} = \kappa p_t(r). \tag{64}$$

Note that for this special case, we cannot impose an EoS of the form $p_r(r) = \omega\rho(r)$ because $\rho(r)$ is vanishing. In the next section, we will explore some of the features of the TW obtained by the Casimir source and the electromagnetic field.

3 Properties of the Casimir wormhole with an additional electromagnetic field

In Sect. 2, we introduced the shape function (32) or (38) obtained by the Casimir energy plus the electromagnetic field. Here we want to discuss some of its properties. The first quantity we will analyze is the proper radial distance, defined by

$$l(r) = \pm\int_{r_0}^{r} \frac{dr'}{\sqrt{1-\frac{b(r')}{r'}}}. \tag{65}$$

In this specific case, plugging Eq. (38) into Eq. (65), one gets

$$\begin{aligned} l(r) &= \pm\int_{r_0}^{r} \frac{dr'}{\sqrt{1-\frac{r_0}{r'}\left(1-\frac{1}{\omega}\right)-\frac{r_0^2}{\omega r'^2}}} \\ &= \pm\left(\frac{\sqrt{r-r_0}\,\sqrt{\omega r+r_0}}{\sqrt{\omega}}\right. \\ &\quad\left. + r_0\frac{\omega-1}{2\omega}\ln\left(\frac{r_0+(2r-r_0)\,\omega+2\sqrt{r-r_0}\,\sqrt{\omega r+r_0}\,\sqrt{\omega}}{(\omega+1)\,r_0}\right)\right). \end{aligned} \tag{66}$$

We find

$$l(r) \simeq \begin{cases} \pm\left(r+\frac{r_0(\omega-1)}{2\omega}\left(\ln\left(\frac{4\omega r}{r_0(\omega+1)}\right)-1\right)+O\left(\frac{1}{r}\right)\right) & r\to\infty \\ \pm 2\sqrt{\frac{r_0\omega}{\omega+1}}\,(r-r_0)+O\left((r-r_0)^{\frac{3}{2}}\right) & r\to r_0 \end{cases}, \tag{67}$$

where the $\pm$ depends on the wormhole side we are on. The proper radial distance is an essential tool to estimate the possible time trip in going from one station located in the lower universe, say at $l = -l_1$, and ending up in the upper universe station, say at $l = l_2$. Following Ref. [2], we shall locate l_1 and l_2 at a value of the radius such that $l_1 \simeq l_2 \simeq 10^4 r_0$, which means that $1-b(r)/r \simeq 1$. Assuming that the traveler has a radial velocity $v(r)$ as measured by a static observer positioned at r, one may relate the proper distance traveled dl, radius traveled dr, coordinate time elapsed dt, and proper time elapsed as measured by the observer $d\tau$, by the following relationships

$$\begin{aligned} v &= e^{-\Phi(r)}\frac{dl}{dt} \\ &= e^{-\Phi(r)}\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}\frac{dr}{dt} \end{aligned} \tag{68}$$

and

$$\begin{aligned} v\gamma &= \frac{dl}{d\tau} = \mp\left(1-\frac{b(r)}{r}\right)^{-\frac{1}{2}}\frac{dr}{d\tau}; \\ \gamma &= \left(1-\frac{v^2(r)}{c^2}\right)^{-\frac{1}{2}} \end{aligned} \tag{69}$$

respectively. If the traveler travels at constant speed v, then the total time is given by

$$\Delta t = \int_{r_0}^{r} \frac{e^{-\Phi(r')}dr'}{v\sqrt{1-\frac{b(r')}{r'}}} = \omega^{1-\frac{\omega}{2}} \int_{r_0}^{r} \frac{r'^{\frac{3}{2}-\frac{\omega}{2}} \left(\omega r' + r_0\right)^{-1+\frac{\omega}{2}}}{v\sqrt{r'-r_0}} dr' \tag{70}$$

while the proper total time is

$$\Delta\tau = \int_{r_0}^{r} \frac{dr'}{v\sqrt{1-\frac{b(r')}{r'}}} = \frac{\sqrt{r-r_0}\,\sqrt{\omega r + r_0}}{v\sqrt{\omega}} + r_0 \frac{\omega-1}{2\omega} \ln\left(\frac{r_0 + (2r - r_0)\,\omega + 2\sqrt{r-r_0}\,\sqrt{\omega r + r_0}\,\sqrt{\omega}}{(\omega+1)\,r_0}\right). \tag{71}$$

To evaluate Δt we can proceed with the following approximations. Close to the throat, one finds

$$\Delta t \simeq \omega^{1-\frac{\omega}{2}} \int_{r_0}^{r} \frac{r'^{\frac{3}{2}-\frac{\omega}{2}} \left(\omega r' + r_0\right)^{-1+\frac{\omega}{2}}}{v\sqrt{r'-r_0}} dr' \simeq \frac{2\sqrt{r_0}}{v} \left(\frac{\omega}{\omega+1}\right)^{1-\frac{\omega}{2}} \sqrt{r-r_0} \tag{72}$$

and, in this range, $\Delta t \simeq \Delta\tau$ except for the value $\omega = -1$, where the TW turns into a black hole. On the other hand, when $r \to \infty$, one finds

$$\Delta t \simeq \int_{r_0}^{r} \frac{r'^{\frac{1}{2}}}{v\sqrt{r'-r_0}} dr' \simeq \frac{r_0}{v} \times \left(1 - \frac{1}{\omega}\right) \ln\left(\frac{r}{r_0}\right) + \frac{r}{v} \tag{73}$$

and even with this approximation, the leading term is the same of $\Delta\tau$. On the same grounds, we can compute the embedded surface, which is defined by

$$z\,(r) = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{r'}{b(r')} - 1}} \tag{74}$$

and in the present case we find

$$z\,(r) = \pm \int_{r_0}^{r} \frac{\sqrt{r_0}\,\sqrt{(\omega-1)\,r' + r_0}}{\sqrt{\omega r' + r_0}\,\sqrt{r'-r_0}} dr' \pm \frac{2r_0}{\omega^2} \left(F\left(\frac{\sqrt{r-r_0}\,\sqrt{\omega}}{\sqrt{\omega r + r_0}}, \frac{1}{\omega}\right) + \Pi\left(\frac{\sqrt{r-r_0}\,\sqrt{\omega}}{\sqrt{\omega r + r_0}}, 1, \frac{1}{\omega}\right)\left(\omega^2 - 1\right)\right), \tag{75}$$

where $F\,(\varphi, k)$ is the elliptic integral of the first kind and $\Pi\,(\varphi, n, k)$ is the elliptic integral of the third kind. Close to the throat, one can write

$$z\,(r) \simeq \pm 2\sqrt{r_0}\sqrt{\frac{\omega}{1+\omega}}\sqrt{r-r_0}. \tag{76}$$

It is interesting to note the singularity appearing when $\omega = -1$, showing the presence of a black hole. To further investigate the properties of the shape function (38), we consider the computation of the total gravitational energy for a wormhole [14], defined as

$$E_G\,(r) = \int_{r_0}^{r} \left[1 - \sqrt{\frac{1}{1 - b\,(r')\,/r'}}\right] \rho\left(r'\right) dr' r'^2 + \frac{r_0}{2G} = \left(M - M^P_{\pm}\right) c^2, \tag{77}$$

where M is the total mass M and M^P is the proper mass, respectively. Even in this case, the $\pm$ depends on the wormhole side we are on. In particular,

$$M = \int_{r_0}^{r} \frac{4\pi}{c^2} \rho\left(r'\right) r'^2 dr' + \frac{r_0}{2Gc^2} = \frac{c^2}{2G}\left(r_0\left(1 - \frac{1}{\omega}\right) + \frac{r_0^2}{\omega r} - r_0\right) + \frac{r_0 c^2}{2G} = \frac{c^2}{2G}\left(-\frac{r_0}{\omega} + \frac{r_0^2}{\omega r}\right) + \frac{r_0}{2Gc^2} \underset{r\to\infty}{\simeq} -\frac{r_0 c^2}{2G\omega} + \frac{r_0 c^2}{2G} = \frac{r_0 c^2}{2G}\left(1 - \frac{1}{\omega}\right) \tag{78}$$

and

$$M^P_{\pm} = \pm \frac{4\pi}{c^2} \int_{r_0}^{r} \frac{\rho\left(r'\right) r'^2}{\sqrt{1 - b\,(r')\,/r'}} dr' = \pm \frac{c^2}{2G} \int_{r_0}^{r} \frac{b'\left(r'\right)}{\sqrt{1 - b\,(r')\,/r'}} dr' \mp \frac{r_0 c^2}{4G\sqrt{\omega}}\left(\pi - 2\arctan\left(\frac{r\omega - r + 2r_0}{2\sqrt{r\omega + r_0}\,\sqrt{r-r_0}}\right)\right) \underset{r\to\infty}{\simeq} \mp \frac{r_0 c^2}{4G\sqrt{\omega}}\left(\pi - 2\arctan\left(\frac{\omega - 1}{2\sqrt{\omega}}\right)\right). \tag{79}$$

Thus, at infinity one finds

$$E_G\,(r) \underset{r\to\infty}{\simeq} \frac{r_0 c^2}{2G}\left[\left(1 - \frac{1}{\omega}\right) \mp \frac{1}{2\sqrt{\omega}}\left(\pi - 2\arctan\left(\frac{\omega-1}{2\sqrt{\omega}}\right)\right)\right]. \tag{80}$$

An important traversability condition is that the acceleration felt by the traveler should not exceed Earth's gravity $g_\oplus \simeq 980\ \text{cm/s}^2$. In an orthonormal basis of the traveler's proper reference frame, we can find

$$|\mathbf{a}| = \left|\sqrt{1 - \frac{b\,(r)}{r}}\, e^{-\Phi(r)} \left(\gamma e^{\Phi(r)}\right)'\right| \le \frac{g_\oplus}{c^2}. \tag{81}$$

If we assume a constant speed and $\gamma \simeq 1$, then we can write

$$|\mathbf{a}| = \left| \sqrt{1 - \frac{r_0}{r}\left(1 - \frac{1}{\omega}\right) - \frac{r_0^2}{\omega r^2}} \frac{(\omega - 1) r_0}{2r(\omega r + r_0)} \right| \leq \frac{g_\oplus}{c^2}. \tag{82}$$

We can see that in the proximity of the throat, the traveler has a vanishing acceleration. Always following Ref. [2], we can estimate the tidal forces by imposing an upper bound represented by $g_\oplus$. The radial tidal constraint

$$\left| \left(1 - \frac{b(r)}{r}\right) \left[\Phi''(r) + \left(\Phi'(r)\right)^2 - \frac{b'(r) r - b(r)}{2r(r - b(r))} \Phi'(r) \right] \right| \times c^2 \left| \eta^{\hat{1}'} \right| \leq g_\oplus, \tag{83}$$

constrains the redshift function, and the lateral tidal constraint

$$\left| \frac{\gamma^2 c^2}{2r^2} \left[\frac{v^2(r)}{c^2} \left(b'(r) - \frac{b(r)}{r}\right) + 2r(r - b(r)) \Phi'(r) \right] \right| \times \left| \eta^{\hat{2}'} \right| \leq g_\oplus, \tag{84}$$

constrains the velocity with which the observers traverse the wormhole. $\eta^{\hat{1}'}$ and $\eta^{\hat{2}'}$ represent the size of the traveler. In Ref. [2], they are fixed approximately equal, at the symbolic value of 2 m. Close to the throat, the radial tidal constraint (83) becomes

$$\begin{aligned} &\left| \left[\frac{b(r) - b'(r) r}{2r^2} \Phi'(r) \right] \right| \\ &= \frac{((\omega - 1) r + 2r_0) r_0^2 (\omega - 1)}{4\omega(\omega r + r_0) r^4} \bigg|_{r \to r_0} \\ &= \frac{1}{6r_0^2} \leq \frac{g_\oplus}{c^2 \left| \eta^{\hat{1}'} \right|} \\ &\Longrightarrow 10^8 m \lesssim r_0. \end{aligned} \tag{85}$$

For the lateral tidal constraint, we find

$$\begin{aligned} &\frac{v^2 r_0}{2r^4} \left| \frac{r(\omega - 1) + 2r_0}{\omega} \right| \left| \eta^{\hat{2}'} \right| \lesssim g_\oplus \Longrightarrow \\ &v \lesssim r_0 \sqrt{\left| \frac{\omega}{\omega + 1} \right| g_\oplus} \Longrightarrow v \lesssim 3.13 r_0 \sqrt{\left| \frac{\omega}{\omega + 1} \right|}\, m/s. \end{aligned} \tag{86}$$

If the observer has a vanishing v, then the tidal forces are null. We can use these last estimates to complete the evaluation of the crossing time, which is approximately

$$\Delta t \simeq \frac{\Delta l}{3.13 r_0 \sqrt{\left|\frac{\omega}{\omega+1}\right|}} \times \left(\frac{\omega}{\omega + 1}\right)^{-\frac{\omega}{2}} \simeq 6.4 \times 10^3 \left(\left|\frac{\omega}{\omega + 1}\right|\right)^{-\frac{1}{2}} \left(\frac{\omega}{\omega + 1}\right)^{-\frac{\omega}{2}} s, \tag{87}$$

in agreement with the estimates found in Ref. [2], even for $\omega \rightarrow \pm\infty$. The last property we will discuss is the "total amount" of averaged NEC (ANEC)-violating matter in the spacetime [15], which is described by

$$I_V = \int [\rho(r) + p_r(r)] \mathrm{d}V \tag{88}$$

and for the line element (4), one can write

$$I_V = \frac{1}{\kappa} \int (r - b(r)) \times \left[\ln \left(\frac{\mathrm{e}^{2\Phi(r)}}{1 - \frac{b(r)}{r}} \right) \right]' \mathrm{d}r, \tag{89}$$

where the measure dV has been changed into $r^2 \mathrm{d}r$. For the metric (38), one obtains

$$I_V = -\frac{1}{\kappa} \int_{r_0}^{\infty} \frac{(\omega + 1) r_0^2}{\omega r^2} \mathrm{d}r = -\frac{(\omega + 1) r_0}{\omega \kappa}. \tag{90}$$

Even in this case, I_V is finite everywhere. The reason is that the structure of the shape function and of the redshift function are equal to the pure Casimir wormhole [1]. Therefore, we can conclude that in the proximity of the throat, the ANEC can be arbitrarily small.

3.1 Properties of the Casimir wormhole with an additional electromagnetic field for the special case $r_1^2 = r_2^2$

In Sect. 2.1, we considered the special case in which the negative Casimir energy is compensated by the positive electromagnetic field with the assumption that $r_1^2 = r_2^2$. We want to discuss some of its properties, even if the size of this TW is Planckian. Repeating the same steps of Sect. 3, we find that the proper radial distance is

$$l(r) = \pm \int_{r_0}^{r} \frac{\mathrm{d}r'}{\sqrt{1 - \frac{r_0}{r'}}} = \pm \left(\sqrt{r - r_0} \sqrt{r} + \frac{r_0}{2} \ln \left(1 + 2 \frac{\sqrt{r - r_0} \left(\sqrt{r} + \sqrt{r - r_0} \right)}{r_0} \right) \right) \tag{91}$$

and its asymptotic behavior becomes

$$l(r) \simeq \begin{cases} \pm\left(r + \frac{r_0}{2}\left(\ln\left(\frac{4r}{r_0}\right) - 1\right) + O\left(\frac{1}{r^2}\right)\right) & r \to \infty \\ \pm 2\sqrt{r_0(r-r_0)} + O\left((r-r_0)^{\frac{3}{2}}\right) & r \to r_0 \end{cases}, \quad (92)$$

where the $\pm$ depends on the wormhole side we are on. From Eqs. (68) and (69), we can compute the total time Δt and the proper total time $\Delta\tau$, respectively. The total time is

$$\Delta t = \int_{r_0}^{r} \frac{\sqrt{r'}\exp\left(-\frac{r_0}{2r'}\right)}{v\sqrt{r'-r_0}} dr' \quad (93)$$

and it is bounded by the following inequality chain

$$\frac{1}{v\sqrt{e}}\int_{r_0}^{r}\frac{\sqrt{r'}}{\sqrt{r'-r_0}}dr' \le \Delta t \le \frac{1}{v}\int_{r_0}^{r}\frac{\sqrt{r'}}{\sqrt{r'-r_0}}dr'. \quad (94)$$

On the other hand, the proper total time is simply

$$\Delta\tau = \frac{\Delta l}{v}. \quad (95)$$

On the same grounds, we can compute the embedded surface, which is defined by

$$z(r) = \pm\int_{r_0}^{r}\frac{dr'}{\sqrt{\frac{r'}{b(r')}-1}} = \pm 2\sqrt{r_0}\sqrt{r-r_0} \quad (96)$$

Note that in this special case, the total gravitational energy for a wormhole [14] is vanishing. As regards the acceleration felt by the traveler, the relationship (81) becomes

$$|\mathbf{a}| = \left|\sqrt{1-\frac{r_0}{r}}\frac{r_0}{2r^2}\right| \le \frac{g_\oplus}{c^2}, \quad (97)$$

where we have assumed a constant speed and $\gamma \simeq 1$. Even in this case, in the proximity of the throat, the traveler has a vanishing acceleration. On the other hand, the radial tidal constraint (83) and the lateral tidal constraint (84) become respectively on the throat

$$\frac{r_0^2}{4r^4} \underset{r\to r_0}{=} \frac{1}{4r_0^2} \le \frac{g_\oplus}{c^2\left|\eta^{\hat{1}'}\right|}\frac{10^8 m}{2g_\oplus} \le r_0 \Longrightarrow 10^8 m \lesssim r_0 \quad (98)$$

and

$$\frac{v^2}{2r^2}\left|-\frac{r_0}{r}\right|\left|\eta^{\hat{2}'}\right| \lesssim g_\oplus \Longrightarrow$$
$$v \lesssim r_0\sqrt{2g_\oplus} \Longrightarrow v \lesssim 4.43 r_0\ m/s, \quad (99)$$

where we have assumed that the size of the traveler, described by $\eta^{\hat{1}'}$ and $\eta^{\hat{2}'}$, is fixed at the symbolic value of 2 m. If the observer has a vanishing v, then the tidal forces are null. We can use these last estimates to complete the evaluation of the crossing time, which is approximately

$$\Delta t \simeq \frac{\Delta l}{4.43 r_0} \simeq 4.5\times 10^3 s, \quad (100)$$

where we have assumed that $\Delta l \simeq 2\times 10^4 r_0$ as in Sect. 3. It is important to observe that since r_0 has a Planckian value, then one finds $\left|\eta^{\hat{1}'}\right| \lesssim 2.1\times 10^{-43}$ m. This means that with a Planckian wormhole, nothing can traverse it. The last property we will discuss is the "total amount" of ANEC-violating matter in the spacetime [15], which is described by Eq. (89), and for the present case, one finds

$$I_V = \frac{1}{\kappa}\int (r-r_0)\left[\ln\left(\frac{e^{-r0/r}}{1-\frac{r_0}{r}}\right)\right]' dr$$
$$= -\frac{1}{\kappa}\int_{r_0}^{\infty}\frac{r_0^2}{r^2}dr = -\frac{r_0}{\kappa}, \quad (101)$$

Even in this case, I_V is finite everywhere, and this corresponds to taking the limit $\omega \to \infty$ in Eq. (90).

4 The Casimir traversable wormhole with an additional electric charge: the constant plate separation case

In this section we consider the following setting for the energy density and radial pressure

$$\rho_C(d) = -\frac{\hbar c\pi^2}{720 d^4};$$
$$p_{r,C}(d) = -\frac{3\hbar c\pi^2}{720 d^4}; \quad \rho_E(r) = \frac{Q^2}{2(4\pi)^2\varepsilon_0 r^4};$$
$$p_{r,E}(r) = -\frac{Q^2}{2(4\pi)^2\varepsilon_0 r^4}, \quad (102)$$

namely that our exotic matter will be represented by the Casimir energy density (2) and only the electric field is variable, with a radial coordinate r. Of course, even in this case we find that the NEC is violated, namely

$$\rho(r) + p_r(r) = \rho_C(d) + p_{r,C}(d) + \rho_E(r)$$
$$+ p_{r,E}(r) = -\frac{4\hbar c\pi^2}{720 d^4} < 0. \quad (103)$$

The total energy density is represented by

$$\rho(r) = \rho_C(d) + \rho_E(r)$$
$$= -\frac{\hbar c\pi^2}{720 d^4} + \frac{Q^2}{2(4\pi)^2\varepsilon_0 r^4}. \quad (104)$$

It is interesting to observe that, in contrast to Eq. (28), the energy density (104) vanishes when

$$r = \bar{r} = d\sqrt{\frac{r_2}{r_1}} = \frac{d\sqrt{n}}{\pi}\sqrt[4]{90\pi\alpha}, \quad (105)$$

where $Q = ne$, e is the electron charge, α is the fine structure constant, and n is the total number of the electron charges.

In particular, we find that

$$\rho(r) \gtreqless 0, \quad \text{when} \quad \begin{cases} r_0 \le r < \bar{r} \\ r = \bar{r} \\ r > \bar{r} \end{cases}. \tag{106}$$

The shape function $b(r)$ can be obtained by plugging $\rho(r)$ (28) into Eq. (5), whose solution leads to

$$\begin{aligned} b(r) &= r_0 + \frac{GQ^2}{2\pi c^4 \varepsilon_0} \int_{r_0}^{r} \frac{dr'}{r'^2} \\ &\quad - \frac{\pi^3}{90 d^4} \left(\frac{\hbar G}{c^3}\right) \int_{r_0}^{r} r'^2 dr' \\ &= r_0 + \frac{r_2^2}{r_0} - \frac{r_2^2}{r} - \frac{r_1^2}{3d^4}\left(r^3 - r_0^3\right), \end{aligned} \tag{107}$$

where r_1 and r_2 have the same meaning as in the previous section. We know that the shape function (107) does not represent a TW because it is not asymptotically flat. Moreover, for large r, $b(r)$ becomes negative. This means that there exists $\tilde{r}$ such that $b(\tilde{r}) = 0$. However, instead of discarding $b(r)$ of Eq. (107), we can try to establish whether there is a way to obtain a TW from Eq. (107). One important property is the flare-out condition described by

$$b'(r_0) < 1 \Longleftrightarrow \frac{r_2^2 d^4 - r_0^4 r_1^2}{r_0^2 d^4} < 1, \tag{108}$$

which is satisfied when

$$r_0 > \frac{\sqrt{2}d}{2r_1}\sqrt{-d^2 + \sqrt{d^4 + 4r_1^2 r_2^2}}. \tag{109}$$

Another property that has to be satisfied is the absence of a horizon for the redshift function. From Eqs. (6) and (107) one finds

$$\Phi'(r) = \frac{\left(3d^4 r_0^2 + 3d^4 r_2^2 + r_0^4 r_1^2\right) r - 6r_2^2 r_0 d^4 - 10 r^4 r_1^2 r_0}{2r_0 r_1^2 r^5 + 6d^4 r_0 r^3 + \left(-6d^4 r_0^2 - 6d^4 r_2^2 - 2r_0^4 r_1^2\right) r^2 + 6d^4 r_0 r_2^2 r}. \tag{110}$$

Close to the throat, the r.h.s. can be approximated by

$$\Phi'(r) = \frac{\left(3d^4 r_0^2 + 3d^4 r_2^2 + r_0^4 r_1^2\right) r_0 - 6r_2^2 r_0 d^4 - 10 r_0^5 r_1^2}{10 r_0^5 r_1^2 + 18 d^4 r_0^3 + 2\left(-6d^4 r_0^2 - 6d^4 r_2^2 - 2r_0^4 r_1^2\right) r_0 + 6r_2^2 r_0 d^4} (r - r_0)^{-1} + \mathrm{O}(1). \tag{111}$$

It can be clearly seen that a horizon will be present unless we impose the condition that

$$\left(3d^4 r_0^2 + 3d^4 r_2^2 + r_0^4 r_1^2\right) r_0 - 6r_2^2 r_0 d^4 - 10 r_0^5 r_1^2 = 0. \tag{112}$$

We have four solutions, but only two are real. They are represented by

$$r_0 = \frac{\sqrt{6}\sqrt{d^2 \pm \sqrt{d^4 - 12r_1^2 r_2^2}}\, d}{6r_1}. \tag{113}$$

The first one is

$$r_0 = \frac{\sqrt{6}d}{6r_1}\sqrt{d^2 + \sqrt{d^4 - 12r_1^2 r_2^2}} \underset{d \gg r_1}{\simeq} \frac{\sqrt{3}d^2}{3r_1} + O\left(\frac{1}{d^2}\right), \tag{114}$$

which is independent of r_2 and therefore of the electric field. The result has a dependence on d similar to the one obtained in Refs. [16,17]. The other interesting solution is

$$r_0 = \frac{\sqrt{6}d}{6r_1}\sqrt{d^2 - \sqrt{d^4 - 12r_1^2 r_2^2}} \underset{d \gg r_1}{\simeq} r_2 + O\left(\frac{1}{d^4}\right), \tag{115}$$

which is independent of the plate separation but is dependent on the electric field. However, the two solutions can be merged into one if

$$d = \sqrt[4]{3}\sqrt{2r_1 r_2}. \tag{116}$$

It is straightforward to verify that

$$\frac{\sqrt{6}\sqrt{d^2 \pm \sqrt{d^4 - 12r_1^2 r_2^2}}\, d}{6r_1} > \frac{\sqrt{2}d}{2r_1}\sqrt{-d^2 + \sqrt{d^4 + 4r_1^2 r_2^2}} \tag{117}$$

even when the condition (116) is used. This means that the throat location is compatible with the flare-out condition. Note that Eq. (116) establishes a relationship between the plate separation and the charge quantity that can be used. In particular, if d is of the order of nm $\left(d \simeq 10^{-9} m\right)$ and r_1 is Planckian, one obtains

$$\frac{d}{r_1} \simeq \frac{10^{-9}m}{10^{-35}m} \simeq \frac{n}{\pi}\sqrt{\frac{180}{137\pi}} \simeq 0.205\,85 n \Longrightarrow n \simeq 10^{26}, \tag{118}$$

where we have used the fine structure constant introduced in (41). Note that for silver, the average number of conduction electrons is $5.8 \cdot 10^{28}/m^3$. As we can see, the relationship (116) constrains the plate separation to be a function of the electric charge. This also implies that the throat (113) reduces to

$$r_0 = \sqrt{2} r_2. \tag{119}$$

If we put numbers in (119), one finds

$$r_0 = \frac{180n^2}{137\pi^3} r_1 \simeq \left(4.2374 \times 10^{-2} n^2\right) \times \left(1.6 \times 10^{-37} m\right) \simeq \left(10^{26}\right)^2 6.7798 \times 10^{-39} m \simeq 10^{13} m. \tag{120}$$

Note that the equation connecting the throat with the plate separation and the fine structure constant can also be obtained with the help of an EoS of the form $p_r(r) = \omega(r)\rho(r)$ with

$$\omega(r) = -\frac{b(r)}{rb'(r)} = rd^4 \left(\frac{r_0 + \frac{r_2^2}{r_0} - \frac{r_2^2}{r} - \frac{r_1^2}{3d^4}\left(r^3 - r_0^3\right)}{r_1^2 r^4 - r_2^2 d^4} \right) \tag{121}$$

allowing us to fix $\Phi'(r) = 0$. On the other hand, from $p_r(r)$ and $\rho(r)$ defined in (102), one can also obtain

$$\omega(r) = \frac{p_r(r)}{\rho(r)} = \frac{3r_1^2 r^4 + r_2^2 d^4}{r_1^2 r^4 - r_2^2 d^4}. \tag{122}$$

If we impose the condition that $\omega(r)$ in Eq. (121) must be equal to $\omega(r)$ in Eq. (122), one finds that the only solution is represented by Eq. (113). Nevertheless, outside of the throat, the function $\omega(r)$ in Eq. (121) is no longer equal to the one of Eq. (122). Plugging the value of r_0 in (119) into the shape function (107), one finds

$$b(r) = \frac{14}{9} r_0 - \frac{r_0^2}{2r} - \frac{r^3}{18r_0^2}. \tag{123}$$

In this reduced form, it is easier to see that there exists

$$\tilde{r} = 2.9208 r_0 \quad \text{where} \quad b(\tilde{r}) = 0. \tag{124}$$

For $r > \bar{r}$, $b(r) < 0$. Therefore, to avoid negative values, we can cut off the region where $r > \bar{r}$. For this reason, the range of the wormhole must be constrained to be very close to the throat. A possible profile comes from the following setup

$$\begin{cases} b(r) = \frac{14}{9} r_0 - \frac{r_0^2}{2r} - \frac{r^3}{18r_0^2} & r_0 \le r \le \tilde{r} \quad b(r) = 0 \quad r \ge \tilde{r} \\ \Phi(r) = 0. \end{cases} \tag{125}$$

Note that in the region $\bar{r} \le r \le \tilde{r}$, the constant Casimir source becomes relevant. One can be tempted to classify such a TW as an absurdly-benign traversable wormhole (ABTW) defined by [16]

$$\begin{cases} b(r) = r_0 \left(1 - \mu (r - r_0)\right)^2 & r_0 \le r \le r_0 + \frac{1}{\mu}; \quad b(r) = 0 \quad r \ge r_0 + \frac{1}{\mu} \\ \Phi(r) = 0. \end{cases} \tag{126}$$

Even if, outside the region $r = \bar{r}$, $\rho(r)$ and $p_r(r)$ tend to a constant value, one has to realize that this behavior cannot be extended to the whole space; rather it is likely that $\rho(r)$ and $p_r(r)$ vanish in the proximity of the external part of the plates, as in the model introduced by Visser, who proposed considering the following SET [4]

$$T_\sigma^{\mu\nu} = \sigma \hat{t}^\mu \hat{t}^\nu \left[\delta(z) + \delta(z - a)\right] + \Theta(z)\,\Theta(a - z) \frac{\hbar c \pi^2}{720 a^4} \left[\eta^{\mu\nu} - 4\hat{z}^\mu \hat{z}^\nu\right], \tag{127}$$

where $\hat{t}^\mu$ is a unit time-like vector, $\hat{z}^\mu$ is a normal vector to the plates, and σ is the mass density of the plates.

5 Conclusions

In this paper, we have extended the study begun in Ref. [1] by including an electromagnetic source. Since the electromagnetic field satisfies the property (12), the NEC is still violated, and it seems to be independent of the strength of the electromagnetic field. Repeating the same strategy adopted in Ref. [1], we have found another Casimir wormhole with a different ω, as it should be. We would like to point out to the reader that the additional electric field is a part of the source and not a feature of the TW. The most important consequence is that the wormhole throat becomes directly dependent on the charge in an additive way, even if under the square root

$$r_0 = \sqrt{3r_1^2 + r_2^2}. \tag{128}$$

If this result seems encouraging, on the other hand we have two aspects that must be explored. The first one is that for $r_2 \gg r_1$, the energy density becomes positive. At this stage of the analysis, we do not know whether the TW ceases to exist or not. A possible answer could come from a back reaction investigation of the electromagnetic and gravitational fields together, which is beyond the scope of this paper. The second aspect is that, always in the range where $r_2 \gg r_1$, $\omega \to -1$ and a horizon seems to be appear. However, this is the result of a limiting procedure, and the value $\omega = -1$ can never be reached. For this purpose, we have to recall that it is the NEC that must be violated. This means that with the help of *phantom energy*, $\rho(r) > 0$ [18–20]. Indeed, the following relationship

$$p_r(r) = \omega\rho(r), \Longrightarrow p_r(r) + \rho(r) < 0 \iff (1 + \omega)\rho(r) < 0, \tag{129}$$

allows us to keep $\rho(r) > 0$, provided that $\omega < -1$. However, generally speaking, it is not known how to build and manipulate such *phantom energy*. The electrovacuum example we have discussed in this paper seems to be encouraging, because in this context the electromagnetic field appears to behave exactly like a phantom source. Note that this is not true when the electromagnetic field is the only source. To further proceed, note that there is an essential discontinuity when $r_2 = r_1$ into the relationship (39). For this reason, this case has been examined separately in paragraph 2.1. It is important to observe that even in this case, a TW can exist at zero density with a throat of Planckian size. A different behavior appears when one considers the mixed source case, namely that only the electromagnetic field has a variable radius, while the plate separation has been considered as a parameter. In this framework, one finds that it is possible to avoid the creation of a horizon if the throat satisfies Eq. (113). Because we have two solutions, it is possible to have only one solution if we impose the constraint (116). This constraint creates a relationship between the plate separation, the Planck length which is not modifiable, and the quantity of charge introduced which can be changed. Unfortunately, even in this case the main limitation comes from the plate separation, which also has consequences with respect to the throat size. However, such a limitation is not so stringent as in Refs. [16,17] because of the presence of the square root in Eq. (116). Indeed, if it were possible to push the plate separation at a distance of the order of pm, one would find that the throat could be of the order of $10^9 m$: a gain of a factor 10^2 with respect to what we found in Ref. [16]. Note that the constraint (116) implies that we have a throat radius directly proportional to the charge of the electromagnetic field. For this reason, in contrast to the pure Casimir source, the introduction of an electromagnetic field seems to go towards a traversable wormhole which is slightly less traversable in principle and slightly more traversable in practice. It is also important to observe that the shape function (107) and subsequently the shape function (123) can be promoted to be a traversable wormhole shape function if we assume that there is a smooth transition between the curved space and flat space expressed by Eq. (124). Alternatively, one could use the *cut-and-paste* technique and glue the shape function (123) with a Schwarzschild profile. Of course, the whole analysis can be generalized to include even a magnetic field, and this will be examined in a future publication. It is interesting to observe that, contrary to the pure Casimir wormhole, with the additional electromagnetic field, we avoid a TW with a throat of Planckian size. Needless to say, things could drastically change with the inclusion of quantum corrections along the line of Refs. [21–25].

Acknowledgements This work has been supported by the project " Traversable Wormholes and Warp Drive: A Road To Interstellar Exploration" of Texas A & M Engineering Experiment Station (TEES), The Texas A & M University System. LSI-GF-2022-00.

Data availability statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data has been listed.]

Funded by SCOAP[3]. SCOAP[3] supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

1. R. Garattini, Eur. Phys. J. C **79**, 11, 951 (2019). arXiv:1907.03623 [gr-qc]
2. M.S. Morris, K.S. Thorne, Am. J. Phys. **56**, 395 (1988)
3. M.S. Morris, K.S. Thorne, U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988)
4. M. Visser, *Lorentzian Wormholes: From Einstein to Hawking* (American Institute of Physics, New York, 1995)
5. I. Dymnikova, Class. Quantum Gravity **21**, 4417–4429 (2004). arXiv:gr-qc/0407072
6. I. Dymnikova, E. Galaktionov, J. Phys. Conf. Ser. **2103**, 012078 (2021)
7. D. Samart, T. Tangphati, P. Channuie, Nucl. Phys. B **980**, 115848 (2022). arXiv:2107.11375 [gr-qc]
8. S.W. Kim, H. Lee, Phys. Rev. D **63**, 064014 (2001). arXiv:gr-qc/0102077
9. A.B. Balakin, J.P.S. Lemos, A.E. Zayats, Phys. Rev. D **81**, 084015 (2010). arXiv:1003.4584 [gr-qc]
10. P.K.F. Kuhfittig, Cent. Eur. J. Phys. **9**, 1144 (2011). arXiv:1104.4662 [gr-qc]
11. L.H. Ford, T.A. Roman, Phys. Rev. D **53**, 5496 (1996). arXiv:gr-qc/9510071
12. H. Ellis, J. Math. Phys. **14**, 104 (1973)
13. K.A. Bronnikov, Acta Phys. Pol. B **4**, 251 (1973)
14. K.K. Nandi, Y.Z. Zhang, R.G. Cai, A. Panchenko, Phys. Rev. D **79**, 024011 (2009). arXiv:0809.4143 [gr-qc]
15. M. Visser, S. Kar, N. Dadhich, Phys. Rev. Lett. **90**, 201102 (2003). arXiv:gr-qc/0301003
16. R. Garattini, Generalized absurdly beningn traversable wormholes powered by Casimir energy. Eur. Phys. J. C **80**, 12, 1172 (2020). arXiv:2008.05901 [gr-qc] **(e-Print)**
17. R. Garattini, Eur. Phys. J. C **81**, 9, 824 (2021). arXiv:2107.09276 [gr-qc]
18. S.V. Sushkov, Phys. Rev. D **71**, 043520 (2005). arXiv:gr-qc/0502084
19. F.S.N. Lobo, Phys. Rev. D **71**, 084011 (2005). arXiv:gr-qc/0502099
20. F.S.N. Lobo, Phys. Rev. D **71**, 124022 (2005). arXiv:gr-qc/0506001
21. R. Garattini, Class. Quantum Gravity **22**, 1105 (2005). arXiv:gr-qc/0501105
22. R. Garattini, Class. Quantum Gravity **24**, 1189 (2007). arXiv:gr-qc/0701019
23. R. Garattini, F.S.N. Lobo, Class. Quantum Gravity **24**, 2401 (2007). arXiv:0701020 [gr-qc]
24. R. Garattini, F.S.N. Lobo, Phys. Lett. B **671**, 146 (2009). arXiv:0811.0919 [gr-qc]
25. R. Garattini, F.S.N. Lobo, Phys. Rev. D **85**, 024043 (2012). arXiv:1111.5729 [gr-qc]