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Optimal weighted Poincaré inequalities for the Barenblatt and uniform densities in $\mathbb{R}^n$

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Abstract
We prove weighted n-dimensional Poincaré inequalities for Barenblatt densities, which are stationary states of certain Fokker–Planck type equations with non-constant diffusion coefficients. The interest in Poincaré inequalities for Barenblatt densities stems from their deep connection with the trend-to-equilibrium problem for Fokker–Planck type equations that admit these densities as steady states.

Keywords Fokker–Planck type equations · Chernoff inequalities · Poincaré inequalities · Barenblatt densities · Cauchy densities

Mathematics Subject Classification 35Q84 · 37A30 · 39B62 · 60J60 · 82B21

1 Introduction

Let $a > 0$, and let $B_a = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < a\}$ be the n-dimensional ball of radius a. In this paper we deal with weighted Poincaré inequalities for the n-dimensional compactly supported Barenblatt densities

$$f_p(\mathbf{x}) = C_{a,p}(a^2 - |\mathbf{x}|^2)^{\frac{1}{p-1}}, \quad \mathbf{x} \in B_a, \tag{1.1}$$

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where $p > 1$ and the positive constant $C_{a,p}$ is chosen to fix the mass equal to unity. As $p \to \infty$, the limit function is the n-dimensional uniform density

$$f_\infty(\mathbf{x}) = \frac{1}{|B_a|}\chi_{B_a}(\mathbf{x})$$

where $|B_a|$ is the Lebesgue measure of the ball B_a. The Barenblatt densities appear as self–similar profiles of the Porous Medium Equation (PME) [4, 17]

$$\frac{\partial f}{\partial t}(\mathbf{x}, t) = \Delta(f^p(\mathbf{x}, t)), \quad \mathbf{x} \in \mathbb{R}^n, \ t > 0, \ p > 1, \tag{1.2}$$

and in this case $a = a(t)$.

In equation (1.2), $f = f(\mathbf{x}, t)$ is a nonnegative scalar function of space $\mathbf{x}$ and time t and Δ is the Laplacian acting only in the space variables. The equation can be posed in the whole space $\mathbb{R}^n$ or in a bounded domain, so it is generally complemented by an initial condition (and boundary conditions). This equation is one of the simplest physically relevant examples of nonlinear evolution equation of parabolic type, characterized by a slow diffusion.

The related counterpart, characterized by an exponent $p < 1$ is well–known as Fast Diffusion Equation (FDE) and in this case the self-similar profile is given by the so–called Cauchy–type densities [2, 7, 12]

$$\tilde{f}_p(\mathbf{x}) = \frac{C_{a,p}}{(a^2 + |\mathbf{x}|^2)^{\frac{1}{1-p}}}, \quad 1 - \frac{2}{n} < p < 1. \tag{1.3}$$

While less interesting from the physical point of view, fast diffusion equations are mathematically relevant and have been intensively studied in many aspects. Among others, various inequalities for the Cauchy–type densities, including weighted Poincaré inequalities, have been recently shown (cf. [3, 15] and the references therein).

On the contrary, weighted Poincaré inequalities for the Barenblatt densities do not seem to have been widely studied, even if we are aware that most of the results we will present are more or less known. In the case of Barenblatt and the limiting uniform densities, the weighted Poincaré inequalities have indeed been studied in many research fields, going from the Sobolev embeddings (see Theorem 3.1 in Opic-Gurka [14]) to John domains (see Theorem 3.1 in Drelichman-Durán [5]), to spectral analysis (see for example Chapter 5 in Dunkl-Xu [6]).

However, our goal is to highlight a method for obtaining Poincaré inequalities for densities that are stationary states of certain Fokker–Planck equations with non-constant diffusion coefficients. The interest in such inequalities lies in the fact that they can be used to prove the rate of convergence to equilibrium of the solutions to the corresponding Fokker-Planck equation as established in [8]. We have already developed the same methodology with the Cauchy-type densities, both in the one dimensional [10] and in the n-dimensional case [11].

The Barenblatt densities still were out of reach, and we present here an exhaustive discussion both of the one-dimensional and n-dimensional setting. We recall that, in

the one-dimensional case, weighted Poincaré inequalities for Barenblatt densities with an optimal constant have been recently obtained in [16].

In our framework, we are interested in the sharp weight in the Poincaré inequality and possibly in determining the explicit constant. This is not the usual approach of studying a given symmetric differential operator with an invariant measure and of determining the corresponding spectral gap, as for example in Dunkl-Xu [6], or the classical treatment of Bakry-Gentil-Ledoux [1]. We reverse this perspective: we start from the measure and we look for the best possible weight in a Poincaré inequality by resorting to a suitable Fokker-Planck equation which leads to that weight through a perfect balance between the diffusion and drift coefficients.

The connection between Fokker–Planck equations and functional inequalities also extends to logarithmic Sobolev inequalities associated with their stationary densities. On the one hand, these inequalities are fundamental tools for establishing convergence to equilibrium for solutions of Fokker–Planck equations; on the other hand, the corresponding evolution equations can be used to derive such inequalities. In particular, in [9], the authors derived a weighted logarithmic Sobolev inequality for one-dimensional Barenblatt densities and, more generally, for a family of Beta–type densities. A natural direction for further research could be to investigate whether an analogous weighted logarithmic Sobolev inequality holds for Barenblatt densities in dimensions $n > 1$, using a strategy similar to the one developed here.

The plan of the paper is as follows. In Sect. 2, we recall a general one-dimensional result, proved in [10], in which a weighted Poincaré inequality is closely related to the diffusion and drift coefficients of a suitable Fokker–Planck type equation. In Sect. 3, we present the weighted Poincaré inequality for the Barenblatt densities in dimension one (see Subsection 3.1) and in dimension n (see Subsection 3.2). In the one-dimensional case, Theorem 2 is a slight modification of the analogous result in [16], as we point out in Remark 3. The main novelty lies in the n-dimensional case and is contained in Lemma 6, which is combined with the classical result recalled in Lemma 8 in order to prove Theorem 5. In Sect. 4, we deduce weighted Poincaré inequalities for one-dimensional and n-dimensional uniform densities from the inequalities satisfied by the Barenblatt densities. Finally, in Sect. 5, we underline the sharpness of the weight functions with respect to their vanishing order at the boundary of the support.

2 A general result in dimension one

We start by recalling a result we proved in [10] which refers to a class of Fokker–Planck type equations with variable coefficients of diffusion and drift in the form

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\left(P(x)f(x,t)\right) + \frac{\partial}{\partial x}\left(Q(x)f(x,t)\right), \tag{2.1}$$

where $x \in \mathcal{I} = (i_-, i_+) \subseteq \mathbb{R}$, with $i_- < i_+$, coupled with no-flux boundary conditions

$$\frac{\partial}{\partial x}\left(P(x)f(x,t)\right) + Q(x)f(x,t)\Big|_{x=\{i_-,i_+\}} = 0,$$

which ensure mass conservation. In equation (2.1) the coefficients of the diffusion P and the drift Q are smooth functions, and $P(x) > 0$ on $\mathcal{I}$. In addition, P and Q are such that, for any $\alpha \in \mathcal{I}$, the steady state

$$f_\infty(x) = \frac{C_\alpha}{P(x)} \exp\left\{-\int_\alpha^x \frac{Q(y)}{P(y)}\, dy\right\}$$

is a probability density supported in $\mathcal{I}$ for a given value of the constant C_α. Clearly, the steady state of the Fokker–Planck (2.1) solves the first order differential equation

$$\frac{\partial}{\partial x}\left(P(x) f_\infty(x)\right) + Q(x) f_\infty(x) = 0. \tag{2.2}$$

Starting from this Fokker–Planck equation it is always possible to prove a weighted Poincaré inequality for f_∞ where the weight is the ratio P/Q'.

Theorem 1 [10] *Let X be a random variable distributed with density $f_\infty(x)$, $x \in \mathcal{I} = (i_-, i_+) \subseteq \mathbb{R}$. Let us suppose moreover that f_∞ satisfies (2.2), where P and Q are C^1 functions, $P(x) > 0$, $Q'(x) > 0$ on $\mathcal{I}$ and*

$$\lim_{x \to i_-} Q(x) < 0, \quad \lim_{x \to i_+} Q(x) > 0.$$

Let w be defined by

$$w(x) = \frac{P(x)}{Q'(x)}, \quad x \in \mathcal{I}.$$

Then, for any smooth function ϕ on $\mathcal{I}$, and $\phi(X)$ with finite variance, it holds

$$Var[\phi(X)] \le E\left\{w(X)[\phi'(X)]^2\right\},$$

that is

$$\int_{\mathcal{I}} \left(\phi(x) - \left(\int_{\mathcal{I}} \phi(y) f_\infty(y)\, dy\right)\right)^2 f_\infty(x)\, dx \le \int_{\mathcal{I}} w(x) \left(\phi'(x)\right)^2 f_\infty(x)\, dx.$$

3 Poincaré inequalities for Barenblatt type densities

For given $a > 0$ and $p > 1$, we consider the class of Barenblatt densities supported in the sphere $B_a = \{|\mathbf{x}| < a\} \subset \mathbb{R}^n$, $n \ge 1$, in the form

$$f_p(\mathbf{x}) = C_{a,p}(a^2 - |\mathbf{x}|^2)^{\frac{1}{p-1}}, \quad \mathbf{x} \in B_a,$$

where the constant $C_{a,p}$ is chosen to fix the mass equal to unity.

3.1 The result on the 1–dimensional Barenblatt density

We first consider the case $n = 1$. In this case $\mathcal{I} = (-a, a)$ in the notations of Theorem 1. The result is as follows.

Theorem 2 (Weighted Poincaré for 1–d Barenblatt density) *Let X be a random variable distributed with density $f_p(x)$, for all $p > 1$, where*

$$f_p(x) = C_{a,p}(a^2 - x^2)^{\frac{1}{p-1}}, \quad x \in \mathcal{I} = (-a, a)$$

with the constant $C_{a,p}$ chosen to fix the mass equal to unity. Then, for any smooth function ϕ on $\mathcal{I}$, and $\phi(X)$ with finite variance, it holds

$$\int_{\mathcal{I}} \left(\phi(x) - \left(\int_{\mathcal{I}} \phi(y) f_p(y)\, dy\right)\right)^2 f_p(x)\, dx \leq \frac{1}{2\left(\frac{1}{p-1} + \frac{1}{2}\right) a^2} \int_{\mathcal{I}} (a^2 - x^2)^2 \left(\phi'(x)\right)^2 f_p(x)\, dx.$$

Proof Let us fix $p > 1$. For all $\alpha \in \mathbb{R}$ it is quite straightforward to show that f_p is a stationary state of the following family of Fokker–Planck equations

$$\partial_t f_p = \partial_x^2 \left((a^2 - x^2)^\alpha f_p\right) + \partial_x \left(2\left(\frac{1}{p-1} + \alpha\right) \frac{x}{(a^2 - x^2)^{1-\alpha}} f_p\right), \quad x \in \mathcal{I}. \tag{3.1}$$

We denote

$$P_{p,\alpha}(x) = (a^2 - x^2)^\alpha, \quad Q_{p,\alpha}(x) = 2\left(\frac{1}{p-1} + \alpha\right) \frac{x}{(a^2 - x^2)^{1-\alpha}}, \quad x \in \mathcal{I}. \tag{3.2}$$

In order to consider only bounded diffusion coefficients, we will assume $\alpha \geq 0$. Moreover, for $\alpha \leq 1$ the coefficients $P_{p,\alpha}$ and $Q_{p,\alpha}$ satisfy the following properties:

(1) $P_{p,\alpha}$ and $Q_{p,\alpha}$ belong to $C^1(\mathcal{I})$;
(2) $P_{p,\alpha} > 0$ on $\mathcal{I}$ and $\lim_{x \to -a^+} Q_{p,\alpha} < 0$, $\lim_{x \to a^-} Q_{p,\alpha} > 0$;
(3) Since

$$Q'_{p,\alpha}(x) = 2\left(\frac{1}{p-1} + \alpha\right) \frac{a^2 + x^2(1 - 2\alpha)}{(a^2 - x^2)^{2-\alpha}}$$

we have $Q'_{p,\alpha}(x) > 0$ on $(-a, a)$ for $0 \leq \alpha \leq 1$.

From now on we will then consider

$$0 \leq \alpha \leq 1.$$

So, we can apply Theorem 1 and we have to estimate $w_{p,\alpha} = \dfrac{P_{p,\alpha}}{Q'_{p,\alpha}}$ where

$$\frac{P_{p,\alpha}(x)}{Q'_{p,\alpha}(x)} = \frac{(a^2-x^2)^2}{2\left(\frac{1}{p-1}+\alpha\right)(a^2+x^2(1-2\alpha))}.$$

- For $0 \le \alpha \le \frac{1}{2}$ we have

$$\frac{P_{p,\alpha}(x)}{Q'_{p,\alpha}(x)} \le \frac{(a^2-x^2)^2}{2\left(\frac{1}{p-1}+\alpha\right)a^2}; \tag{3.3}$$

- for $\frac{1}{2} \le \alpha < 1$ we have

$$\frac{P_{p,\alpha}(x)}{Q'_{p,\alpha}(x)} \le \frac{(a^2-x^2)^2}{4\left(\frac{1}{p-1}+\alpha\right)(1-\alpha)a^2}; \tag{3.4}$$

- For $\alpha = 1$ it is not possibile to bound P/Q' with a quadratic term $(a^2-x^2)^2$ but only

$$\frac{P_{p,1}(x)}{Q'_{p,1}(x)} = \frac{a^2-x^2}{2\left(\frac{1}{p-1}+1\right)}. \tag{3.5}$$

We remark that for all $\alpha \in [0,1)$ we obtain the weight $(a^2-x^2)^2$ and the best possible coefficient in front of the weight is obtained for $\alpha = \frac{1}{2}$, namely

$$\frac{P_{p,\frac{1}{2}}(x)}{Q'_{p,\frac{1}{2}}(x)} \le \frac{(a^2-x^2)^2}{2\left(\frac{1}{p-1}+\frac{1}{2}\right)a^2}.$$

This achieves the proof. □

Remark 3 We would like to remark that we have chosen to bound the weight P/Q' with a quadratic term $(a^2-x^2)^2$ which is the smallest one near the boundary of B_a among all the powers $(a^2-x^2)^\alpha$ which bound P/Q' (of course with appropriate coefficients in front). In fact, since for $|x| \le a$ we have $(a^2-x^2)^2 \le a^2(a^2-x^2)$, a weighted Poincaré inequality with $(a^2-x^2)^2$ weight will automatically imply an inequality with smaller powers. On the contrary, in [16] a linear weight with respect to the distance from the boundary has been preferred, since it was coherent with the choice of a more standard Fokker–Planck equation with linear drift. We underline that the linear weight is the only one for which the constant does not depend on the size of the support. We will come back to the sharpness of the quadratic choice in the weight at the end of the paper.

Remark 4 The Poincaré inequalities as in Theorem 2 can be used to establish the rate of convergence to equilibrium for the family of Fokker-Planck equations (3.1) (see

[8]). The particular case $\alpha = 1$ in (3.1) has been detailed in [8] (see also [11] for the n-dimensional framework).

3.2 The result on the *n*-dimensional Barenblatt density

Let us come to the weighted Poincaré inequality satisfied by the n–dimensional Barenblatt densities. The result is as follows.

Theorem 5 (Weighted Poincaré for n-dimensional Barenblatt density) *Let $n > 1$ and $\mathbf{X}$ be a random variable distributed with a n-dimensional Barenblatt density*

$$f_p(\mathbf{x}) = C_{a,p}\left(a^2 - |\mathbf{x}|^2\right)^{\frac{1}{p-1}} \quad a > 0, \quad p > 1,$$

where $C_{a,p} > 0$ is chosen to fix the mass equal to unity. There exists an explicit positive constant C depending on n, a and p such that for any smooth scalar function $\phi(\mathbf{x})$ on the ball $B_a = \{x : |\mathbf{x}| < a\}$ with finite variance, it holds

$$\int_{B_a}\left(\phi(\mathbf{x}) - \int_{B_a}\phi(\mathbf{y})\, f_p(\mathbf{y})\, d\mathbf{y}\right)^2 f_p(\mathbf{x})\, d\mathbf{x} \le C\int_{B_a}(a^2 - |\mathbf{x}|^2)^2\, |\nabla\phi(\mathbf{x})|^2\, f_p(\mathbf{x})\, d\mathbf{x}.$$

In this case we consider probability densities which are stationary states for Fokker–Planck equations of the form

$$\frac{\partial f(\mathbf{x},t)}{\partial t} = \nabla\cdot\left[\nabla(P(|\mathbf{x}|)f(\mathbf{x},t)) + \frac{\mathbf{x}}{|\mathbf{x}|}Q(|\mathbf{x}|)f(\mathbf{x},t)\right], \quad \mathbf{x}\in\mathcal{I}\subset\mathbb{R}^n.$$

For the n–dimensional Barenblatt densities f_p as in (1.1), it is easy to verify that we can consider the same coefficients P and Q as in (3.2) (where $|\mathbf{x}|$ replaces x). If we consider the 1–dimensional functions $f_p(\rho)$, where $\rho = |\mathbf{x}|$ (which are not probability densities on $(0,a)$), we remark that $f_p(\rho)$ are stationary states of the following family of Fokker–Planck equations

$$\partial_t f_p = \partial_\rho^2\left((a^2-\rho^2)^\alpha f_p\right) + \partial_\rho\left(2\left(\alpha + \frac{1}{p-1}\right)\frac{\rho}{(a^2-\rho^2)^{1-\alpha}}f_p\right), \quad \rho\in(0,a). \tag{3.6}$$

Let us consider

$$g_p(\rho) = \rho^{n-1} f_p(\rho).$$

Since f_p satisfies equation (3.6), then g_p satisfies the corresponding stationary equation

$$\partial_\rho\left((a^2-\rho^2)^\alpha g_p\right) + \left(2\left(\alpha+\frac{1}{p-1}\right)\frac{\rho}{(a^2-\rho^2)^{1-\alpha}} - \frac{n-1}{\rho}(a^2-\rho^2)^\alpha\right)g_p = 0. \tag{3.7}$$

In view of the previous one-dimensional result, we assume once and forall $\alpha = \frac{1}{2}$.

In order to prove a weighted Poincaré inequality for the n–dimensional Barenblatt densities, we want once again to exploit the Fokker–Planck structure. First we prove a Lemma which takes into account a test function which vanishes inside a suitable ball contained in B_a.

Lemma 6 *Let $p > 1$ and f_p the Barenblatt density on a ball $B_a \subset \mathbb{R}^n$, $n > 1$, defined in (1.1) as*

$$f_p(\mathbf{x}) = C_{a,p}(a^2 - |\mathbf{x}|^2)^{\frac{1}{p-1}}, \quad \mathbf{x} \in B_a.$$

Let

$$R_p = R(n, a, p) = a\frac{1}{\sqrt{1 + \frac{1}{n-1}\left(\frac{1}{2} + \frac{1}{p-1}\right)}}. \tag{3.8}$$

For any smooth function ϕ, supported in $B_a \setminus B_{R_p}$ where $B_{R_p} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < R_p\}$ we have

$$\int_{|\mathbf{x}|<a} \phi^2(\mathbf{x}) f_p(\mathbf{x})\, d\mathbf{x} \le \frac{1}{\left(\frac{1}{p-1} + \frac{1}{2}\right) a^2} \int_{|\mathbf{x}|<a} (a^2 - |\mathbf{x}|^2)^2 |\nabla\phi(\mathbf{x})|^2 f_p(\mathbf{x})\, d\mathbf{x}.$$

Proof Let us consider $p > 1$ as fixed. Denote $P_{p,\frac{1}{2}} = P$ and $Q_{p,\frac{1}{2}} = Q$ for the reader's convenience, where as in (3.2)

$$P_{p,\frac{1}{2}}(\rho) = \sqrt{a^2 - \rho^2}, \quad Q_{p,\frac{1}{2}} = \left(1 + \frac{2}{p-1}\right)\frac{\rho}{\sqrt{a^2 - \rho^2}}, \quad \rho \in [0, a).$$

Let ϕ be a smooth scalar function defined on B_a and such that $\phi(\mathbf{x}) = 0$ for $|\mathbf{x}| < R_p$ where R_p is as in (3.8). We have

$$\int_{|\mathbf{x}|<a} \phi^2(\mathbf{x}) f_p(\mathbf{x})\, d\mathbf{x} = \int_{R_p \le |\mathbf{x}|<a} (\phi(\mathbf{x}) - \phi(0))^2 f_p(\mathbf{x})\, d\mathbf{x}$$

and in hyperspherical coordinates

$$\int_{R_p \le |\mathbf{x}|<a} (\phi(\mathbf{x}) - \phi(0))^2 f_p(\mathbf{x})\, d\mathbf{x} = \int_{S^{n-1}} \int_{R_p}^{a} (\phi(\rho, \Theta) - \phi(\mathbf{0}))^2 \rho^{n-1} f_p(\rho)\, d\rho\, d\Theta.$$

Now, due to the properties of the function Q detailed in (1), (2), and (3) we can perform a change of variables and let

$$\phi(\rho, \Theta) = \psi(Q(\rho), \Theta). \tag{3.9}$$

Since $Q(0) = 0$ we get

$$\phi(\mathbf{0}) = \phi(0, \Theta) = \psi(Q(0), \Theta) = \psi(0, \Theta)$$

and so

$$\phi(\rho,\Theta)-\phi(\mathbf{0})=\psi(Q(\rho),\Theta)-\psi(0,\Theta).$$

Thanks to the gradient theorem, for the (smooth) function $\psi(s,\Theta)$ we have

$$\psi(Q(\rho),\Theta)-\psi(0,\Theta)=\int_0^{Q(\rho)}\frac{\partial}{\partial s}\psi(s,\Theta)\,ds=\int_0^1\frac{\partial\psi}{\partial s}(t\,Q(\rho),\Theta)\,Q(\rho)\,dt. \tag{3.10}$$

Equality (3.10) shows that the angles θ_i, $i=1,2,\dots,n-1$, appear as constants, so that the gradient theorem, for any given Θ, takes a scalar form involving only a variation in the radius ρ. By Jensen's inequality, (3.10) implies

$$(\psi(Q(\rho),\Theta)-\psi(0,\Theta))^2\le\int_0^1\left|\frac{\partial\psi}{\partial s}(t\,Q(\rho),\Theta)\right|^2\,Q(\rho)^2\,dt$$

and so

$$\begin{aligned}&\int_{S^{n-1}}\int_{R_p}^{a}(\phi(\rho,\Theta)-\phi(\mathbf{0}))^2\,\rho^{n-1}f_p(\rho)\,d\rho\,d\Theta\\&=\int_{S^{n-1}}\int_{R_p}^{a}(\psi(Q(\rho),\Theta)-\psi(0,\Theta))^2\rho^{n-1}f_p(\rho)\,d\rho\,d\Theta\\&\le\int_{S^{n-1}}\int_{R_p}^{a}\left[Q(\rho)\rho^{n-1}f_p(\rho)\right]\left(Q(\rho)\int_0^1\left|\frac{\partial\psi}{\partial s}(t\,Q(\rho),\Theta)\right|^2\,dt\right)d\rho\,d\Theta.\end{aligned}$$

Now, thanks to equation (3.7), recalling $g_p(\rho)=\rho^{n-1}f_p(\rho)$ we have

$$Q(\rho)\rho^{n-1}f_p(\rho)=-\frac{\partial}{\partial\rho}\left(P(\rho)\rho^{n-1}f_p(\rho)\right)+\frac{n-1}{\rho}P(\rho)\rho^{n-1}f_p(\rho).$$

Now the choice of R_p as in (3.8) implies

$$\frac{n-1}{R_p}P(R_p)=\frac{1}{2}Q(R_p)$$

and, for $\rho>R_p$, we have

$$\frac{n-1}{\rho}P(\rho)\le\frac{1}{2}Q(\rho).$$

So

$$Q(\rho)\rho^{n-1}f_p(\rho)\le-2\frac{\partial}{\partial\rho}\left(P(\rho)\rho^{n-1}f_p(\rho)\right)$$

and we get

$$\int_{S^{n-1}}\int_{R_p}^{a}\left[Q(\rho)\rho^{n-1}f_p(\rho)\right]\left(Q(\rho)\int_0^1\left|\frac{\partial\psi}{\partial s}(tQ(\rho),\Theta)\right|^2 dt\right)d\rho\, d\Theta \le$$
$$-2\int_{S^{n-1}}\int_{R_p}^{a}\frac{\partial}{\partial\rho}\left(P(\rho)\rho^{n-1}f_p(\rho)\right)\left(Q(\rho)\int_0^1\left|\frac{\partial\psi}{\partial s}(tQ(\rho),\Theta)\right|^2 dt\right)d\rho\, d\Theta$$
$$=\int_{S^{n-1}}\left[-2P(\rho)\rho^{n-1}f_p(\rho)\left(Q(\rho)\int_0^1\left|\frac{\partial\psi}{\partial s}(tQ(\rho),\Theta)\right|^2 dt\right)\right]_{\rho=R_p}^{a}$$
$$+2\int_{R_p}^{a}P(\rho)\rho^{n-1}f_p(\rho)\frac{\partial}{\partial\rho}\left(Q(\rho)\int_0^1\left|\frac{\partial\psi}{\partial s}(tQ(\rho),\Theta)\right|^2 dt\right)d\rho\, d\Theta.$$

Now, we remark that for $\rho = R_p$ we have

$$\int_0^1\left|\frac{\partial\psi}{\partial s}(tQ(R_p),\Theta)\right|^2 dt = 0. \tag{3.11}$$

Indeed we know that $\phi(\rho,\Theta) = 0$ for $\rho \le R_p$, which implies $\frac{\partial\phi}{\partial\rho}(\rho,\Theta) = 0$ for $\rho \le R_p$. Since by (3.9) we have

$$\phi(\rho,\Theta) = \psi(Q(\rho),\Theta)$$

and

$$0 = \frac{\partial\phi}{\partial\rho}(\rho,\Theta) = \frac{\partial}{\partial\rho}\left(\psi(Q(\rho),\Theta)\right) = \frac{\partial\psi}{\partial s}(Q(\rho),\Theta)Q'(\rho)$$

with

$$Q'(\rho) = \left(\frac{2}{p-1}+1\right)\frac{a^2}{(a^2-\rho^2)^{\frac{3}{2}}} \neq 0,$$

we get

$$\frac{\partial\psi}{\partial s}(Q(\rho),\Theta) = 0, \quad \rho \le R_p.$$

Now, for all $t \in [0,1]$ we show that there exists $\xi = \xi(t) \in [0, R_p]$ so that $tQ(R_p) = Q(\xi)$. Indeed Q is a one-to-one transformation from $[0, R_p]$ to $[0, Q(R_p)]$ and so, since for $t \in [0,1]$ we have $tQ(R_p) \in [0, Q(R_p)]$ of course there is $\xi \in [0, R_p]$ so that $Q(\xi) = tQ(R_p)$. This implies that

$$\frac{\partial\psi}{\partial s}(tQ(R_p),\Theta) = \frac{\partial\psi}{\partial s}(Q(\xi),\Theta) = 0, \quad t \in [0,1]$$

which entails (3.11). Since the contribution of the boundary terms is non positive, we get

$$\begin{aligned}
&\int_{S^{n-1}} \left[-2P(\rho)\rho^{n-1} f_p(\rho) \left(Q(\rho) \int_0^1 \left| \frac{\partial \psi}{\partial s}(tQ(\rho), \Theta) \right|^2 dt \right) \right]_{\rho=R_p}^{a} \\
&\quad + 2 \int_{R_p}^{a} P(\rho)\rho^{n-1} f_p(\rho) \frac{\partial}{\partial \rho} \left(Q(\rho) \int_0^1 \left| \frac{\partial \psi}{\partial s}(tQ(\rho), \Theta) \right|^2 dt \right) d\rho\, d\Theta \\
&\leq 2 \int_{S^{n-1}} \int_{R_p}^{a} P(\rho)\rho^{n-1} f_p(\rho) \frac{\partial}{\partial \rho} \left(Q(\rho) \int_0^1 \left| \frac{\partial \psi}{\partial s}(tQ(\rho), \Theta) \right|^2 dt \right) d\rho\, d\Theta \\
&= 2 \int_{S^{n-1}} \int_{R_p}^{a} P(\rho)\rho^{n-1} f_p(\rho) Q'(\rho) \frac{\partial}{\partial y} \left(\int_0^1 y \left| \frac{\partial \psi}{\partial s}(ty, \Theta) \right|^2 dt \right)_{y=Q(\rho)} d\rho\, d\Theta.
\end{aligned}$$

The proof is completed by observing that, for any given function $G(yt)$ one has the identity

$$\frac{\partial}{\partial y} \left(y\, G(yt) \right) = \frac{\partial}{\partial t} \left(t\, G(yt) \right). \tag{3.12}$$

Clearly, (3.12) implies

$$\begin{aligned}
&\frac{\partial}{\partial y} \left(\int_0^1 y \left| \frac{\partial \psi}{\partial s}(ty, \Theta) \right|^2 dt \right) = \int_0^1 \frac{\partial}{\partial y} \left(y \left| \frac{\partial \psi}{\partial s}(ty, \Theta) \right|^2 \right) dt = \\
&\int_0^1 \frac{\partial}{\partial t} \left(t \left| \frac{\partial \psi}{\partial s}(ty, \Theta) \right|^2 \right) dt = \left| \frac{\partial \psi}{\partial s}(y, \Theta) \right|^2
\end{aligned}$$

and for $y = Q(\rho)$ we get

$$\left(\left| \frac{\partial \psi}{\partial s}(y, \Theta) \right|^2 \right)_{y=Q(\rho)} = \left| \frac{\partial \psi}{\partial s}(Q(\rho), \Theta) \right|^2.$$

Now,

$$\frac{\partial \phi}{\partial \rho}(\rho, \Theta) = \frac{\partial \psi}{\partial s}(Q(\rho), \Theta) Q'(\rho)$$

so

$$\frac{\partial \psi}{\partial s}(Q(\rho), \Theta) = \frac{1}{Q'(\rho)} \frac{\partial \phi}{\partial \rho}(\rho, \Theta)$$

and we get

$$
\begin{aligned}
&2\int_{S^{n-1}}\int_{R_p}^{a} P(\rho)\rho^{n-1} f_p(\rho) Q'(\rho)\frac{\partial}{\partial y}\left(\int_0^1 y\left|\frac{\partial\psi}{\partial y}(ty,\Theta)\right|^2 dt\right)_{y=Q(\rho)} d\rho\, d\Theta \\
&\quad = 2\int_{S^{n-1}}\int_{R_p}^{a}\frac{P(\rho)}{Q'(\rho)}\left|\frac{\partial\phi}{\partial\rho}(\rho,\Theta)\right|^2 \rho^{n-1} f_p(\rho)\, d\rho\, d\Theta \\
&\quad \le 2\int_{S^{n-1}}\int_{R_p}^{a}\frac{P(\rho)}{Q'(\rho)}\left|\nabla\phi(\rho,\Theta)\right|^2 \rho^{n-1} f_p(\rho)\, d\rho\, d\Theta.
\end{aligned}
$$

Going back to cartesian coordinates gives the result for ϕ

$$
\begin{aligned}
\int_{|\mathbf{x}|<a}\phi^2(\mathbf{x}) f_p(\mathbf{x})\, d\mathbf{x} &\le 2\int_{S^{n-1}}\int_{R_p}^{a}\frac{P(\rho)}{Q'(\rho)}\left|\nabla\phi(\rho,\Theta)\right|^2 \rho^{n-1} f_p(\rho)\, d\rho\, d\Theta \\
&= 2\int_{|\mathbf{x}|<a}\frac{P(|\mathbf{x}|)}{Q'(|\mathbf{x}|)}\left|\nabla\phi(\mathbf{x})\right|^2 f_p(|\mathbf{x}|)\, d\mathbf{x}.
\end{aligned}
$$

Now we go back to inequality (3.4), with $|\mathbf{x}|$ instead of x and $\alpha = \frac{1}{2}$, getting

$$
\frac{P(|\mathbf{x}|)}{Q'(|\mathbf{x}|)} \le \frac{1}{2\left(\frac{1}{p-1}+\frac{1}{2}\right)a^2}(a^2-|\mathbf{x}|^2)^2.
$$

This completes the proof. □

Remark 7 We remark that it is possible to choose R independent of p since for all $p > 1$ we have $R_p \le a\dfrac{1}{\sqrt{1+\frac{1}{2(n-1)}}}$. From now on, we take

$$
R = a\frac{1}{\sqrt{1+\frac{1}{2(n-1)}}}. \tag{3.13}
$$

In addition, the following local Poincaré inequality holds true for any probability density supported on a ball. For a proof, see for instance [13].

Lemma 8 *Let us denote $\Omega = B_R$ the ball of $\mathbb{R}^n$ with center 0 and radius $R > 0$, and let us consider $\nu \in \mathrm{P}(\Omega)$ a probability measure such that (abusing notations) $\nu, \frac{1}{\nu} \in L^\infty(\Omega)$. Then for any (smooth) function f, there holds*

$$
\int_\Omega\left(f-\int_\Omega f d\nu\right)^2 d\nu \le k(R)\int_\Omega|\nabla f|^2 d\nu
$$

with

$$
k(R) = 2\,\mathrm{diam}(\Omega)^2\|\nu\|_\infty\left\|\frac{1}{\nu}\right\|_\infty.
$$

Therefore

$$\int_\Omega f^2 dv \le k(R) \int_\Omega |\nabla f|^2 dv + \left(\int_\Omega f dv \right)^2 .$$

Let us come to the proof of Theorem 5.

Proof of Theorem 5 Without loss of generality, we can assume

$$\int_{B_a} \phi(\mathbf{y}) f_p(\mathbf{y})\, d\mathbf{y} = 0 \tag{3.14}$$

and so, for any $0 < \epsilon < 1$ we get one of the crucial points of this proof, namely

$$\int_{|\mathbf{y}| \le a(1-\epsilon)} \phi(\mathbf{y}) f_p(\mathbf{y})\, d\mathbf{y} = - \int_{a(1-\epsilon) < |\mathbf{y}| \le a} \phi(\mathbf{y}) f_p(\mathbf{y})\, d\mathbf{y}. \tag{3.15}$$

Let us denote $R = a h_n$, with $h_n = \frac{1}{\sqrt{1+\frac{1}{2(n-1)}}}$ as in (3.13), and

$$K(|\mathbf{x}|) = (a^2 - |\mathbf{x}|^2)^2.$$

For $\epsilon \in (0, 1 - h_n)$ let us introduce a n-dimensional radial cut off function

$$T(\mathbf{x}) = \begin{cases} 1 & \text{on } |\mathbf{x}| < R = a h_n \\ \dfrac{\log\left(\frac{a-|\mathbf{x}|}{a\epsilon}\right)}{\log\left(\frac{1-h_n}{\epsilon}\right)} & \text{on } a h_n \le |\mathbf{x}| \le a(1-\epsilon) \\ 0 & \text{on } a(1-\epsilon) < |\mathbf{x}| < a. \end{cases} \tag{3.16}$$

We remark that

$$0 \le T(\mathbf{x}) \le 1, \quad |\mathbf{x}| < a. \tag{3.17}$$

Moreover, for $|\mathbf{x}| < a$, $|\mathbf{x}| \ne a h_n$, $|\mathbf{x}| \ne a(1-\epsilon)$ we have

$$|\nabla T(\mathbf{x})| \le \frac{1}{a - |\mathbf{x}|} \frac{1}{\log\left(\frac{1-h_n}{\epsilon}\right)} \tag{3.18}$$

and $\nabla T(\mathbf{x})$ is supported in the annulus $a h_n \le |\mathbf{x}| \le a(1-\epsilon)$. Let us decompose ϕ as $\phi = \phi_i + \phi_o$, where

$$\phi_i(\mathbf{x}) = \phi(\mathbf{x}) T(\mathbf{x}), \quad \phi_o(\mathbf{x}) = \phi(\mathbf{x})(1 - T(\mathbf{x})).$$

Of course ϕ_i is supported in $|\mathbf{x}| \le a(1-\epsilon)$ and ϕ_o is supported in $a h_n \le |\mathbf{x}| < a$. By the assumption (3.14) we get

$$\int_{B_a} \phi_i(\mathbf{y}) f_p(\mathbf{y})\, d\mathbf{y} = - \int_{B_a} \phi_o(\mathbf{y}) f_p(\mathbf{y})\, d\mathbf{y}.$$

We have to prove that there is an explicit positive constant C, depending on a, n and p such that

$$\int_{B_a} \phi^2(\mathbf{x})\, f_p(\mathbf{x})\, d\mathbf{x} \le C \int_{B_a} K(|x|)\, |\nabla \phi(\mathbf{x})|^2\, f_p(\mathbf{x})\, d\mathbf{x}.$$

Thanks to the trivial inequality $(a+b)^2 \le 2(a^2+b^2)$, let us decompose

$$\int_{B_a} \phi(\mathbf{x})^2 f_p(\mathbf{x}) d\mathbf{x} \le 2 \left[\underbrace{\int_{B_a} \phi_i(\mathbf{x})^2 f_p(\mathbf{x}) d\mathbf{x}}_{(1)} + \underbrace{\int_{B_a} \phi_o(\mathbf{x})^2 f_p(\mathbf{x}) d\mathbf{x}}_{(2)} \right]. \tag{3.19}$$

Let us begin with the term (1). Since ϕ_i is supported in $B_{a(1-\epsilon)}$, we can apply a local Poincaré inequality as in Lemma 8. Moreover, for $\mathbf{x} \in B_{a(1-\epsilon)}$ we have $|\phi_i(\mathbf{x})| \le |\phi(\mathbf{x})|$ and so we get

$$\begin{aligned}
\int_{B_a} \phi_i(\mathbf{x})^2 f_p(\mathbf{x}) d\mathbf{x} &= \int_{B_{a(1-\epsilon)}} \phi_i(\mathbf{x})^2 f_p(\mathbf{x}) d\mathbf{x} \\
&\le \int_{B_{a(1-\epsilon)}} \phi(\mathbf{x})^2 f_p(\mathbf{x}) d\mathbf{x} \\
&\le k(\epsilon) \int_{B_{a(1-\epsilon)}} |\nabla \phi(\mathbf{x})|^2 f_p(\mathbf{x}) d\mathbf{x} \\
&\quad + \frac{1}{m(B_{a(1-\epsilon)})} \left(\int_{B_{a(1-\epsilon)}} \phi(\mathbf{x})\, f_p(\mathbf{x}) d\mathbf{x} \right)^2,
\end{aligned}$$

where

$$k(\epsilon) = 8a^2 \frac{(1-\epsilon)^2}{\epsilon^{\frac{1}{p-1}} (2-\epsilon)^{\frac{1}{p-1}}}$$

and $m(B_{a(1-\epsilon)}) = \int_{B_{a(1-\epsilon)}} f_p(\mathbf{x}) d\mathbf{x}$. By (3.15) and Cauchy–Schwarz inequality we get

$$\begin{aligned}
&\frac{1}{m(B_{a(1-\epsilon)})} \left(\int_{B_{a(1-\epsilon)}} \phi(\mathbf{x})\, f_p(\mathbf{x}) d\mathbf{x} \right)^2 \\
&= \frac{1}{m(B_{a(1-\epsilon)})} \left(\int_{a(1-\epsilon) \le |\mathbf{x}| < a} \phi(\mathbf{x})\, f_p(\mathbf{x}) d\mathbf{x} \right)^2 \\
&\le \frac{1}{m(B_{a(1-\epsilon)})} \left(\int_{a(1-\epsilon) \le |\mathbf{x}| < a} \phi^2(\mathbf{x})\, f_p(\mathbf{x}) d\mathbf{x} \right) \left(\int_{a(1-\epsilon) \le |\mathbf{x}| < a} f_p(\mathbf{x}) d\mathbf{x} \right) \\
&\le \frac{m(B_a) - m(B_{a(1-\epsilon)})}{m(B_{a(1-\epsilon)})} \int_{|\mathbf{x}| < a} \phi^2(\mathbf{x})\, f_p(\mathbf{x}) d\mathbf{x}
\end{aligned}$$

where

$$\frac{m(B_a) - m(B_{a(1-\epsilon)})}{m(B_{a(1-\epsilon)})} = \frac{\int_{a(1-\epsilon)\le|\mathbf{x}|<a} f_p(\mathbf{x})d\mathbf{x}}{\int_{|\mathbf{x}|<a(1-\epsilon)} f_p(\mathbf{x})d\mathbf{x}}.$$

We recall that

$$\begin{aligned} m(B_{a(1-\epsilon)}) &= C_{a,p}\int_{|x|<a(1-\epsilon)} (a^2 - |x|^2)^{\frac{1}{p-1}}dx \\ &= C_{a,p}S_{n-1}\int_0^{a(1-\epsilon)} (a^2 - t^2)^{\frac{1}{p-1}} t^{n-1}dt \end{aligned}$$

where S_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. Now, with the change of variable $u = \left(\frac{t}{a}\right)^2$ we obtain

$$m(B_{a(1-\epsilon)}) = C_{a,p}\frac{S_{n-1}}{2}(a^2)^{\frac{1}{p-1}+\frac{n}{2}}\int_0^{(1-\epsilon)^2} (1-u)^{\frac{1}{p-1}} u^{\frac{n}{2}-1}du.$$

In a similar way we get

$$m(B_a) - m(B_{a(1-\epsilon)}) = C_{a,p}\frac{S_{n-1}}{2}(a^2)^{\frac{1}{p-1}+\frac{n}{2}}\int_{(1-\epsilon)^2}^1 (1-u)^{\frac{1}{p-1}} u^{\frac{n}{2}-1}du.$$

So we get

$$\frac{m(B_a) - m(B_{a(1-\epsilon)})}{m(B_{a(1-\epsilon)})} \le \frac{1-(1-\epsilon)^n}{(1-\epsilon)^n} \to 0, \quad \epsilon \to 0.$$

We have obtained

$$\begin{aligned} &\int_{B_a} \phi_i(\mathbf{x})^2 f_p(\mathbf{x})d\mathbf{x} \\ &\le \int_{B_{a(1-\epsilon)}} \phi(\mathbf{x})^2 f_p(\mathbf{x})d\mathbf{x} \\ &\le k(\epsilon)\sup_{B_{a(1-\epsilon)}} \frac{1}{K(|\mathbf{x}|)}\int_{B_a} K(|\mathbf{x}|)|\nabla\phi(\mathbf{x})|^2 f_p(|\mathbf{x}|)d\mathbf{x} + \frac{1-(1-\epsilon)^n}{(1-\epsilon)^n}\int_{B_a}\phi(\mathbf{x})^2 f_p(\mathbf{x})d\mathbf{x} \\ &\le C(\epsilon)\int_{B_a} K(|\mathbf{x}|)|\nabla\phi(\mathbf{x})|^2 f_p(|\mathbf{x}|)d\mathbf{x} + \frac{1-(1-\epsilon)^n}{(1-\epsilon)^n}\int_{B_a}\phi(\mathbf{x})^2 f_p(\mathbf{x})d\mathbf{x} \end{aligned} \tag{3.20}$$

with

$$\begin{aligned} C(\epsilon) &= k(\epsilon)\sup_{B_{a(1-\epsilon)}} \frac{1}{K(|\mathbf{x}|)} \\ &= 8a^2\frac{(1-\epsilon)^2}{\epsilon^{\frac{1}{p-1}}(2-\epsilon)^{\frac{1}{p-1}}}\frac{1}{(a^2 - a^2(1-\epsilon)^2)^2} \\ &= \frac{8}{a^2}\frac{(1-\epsilon)^2}{\epsilon^{2+\frac{1}{p-1}}(2-\epsilon)^{2+\frac{1}{p-1}}} \end{aligned}$$

Let us come to the term (2) in (3.19). The ϕ_o function is smooth and it is equal to 0 for $|\mathbf{x}| \leq R$ (we come back to the notation R instead of ah_n which here is more compact). So we can apply Lemma 6 and we get

$$\int_{B_a} \phi_o(\mathbf{x})^2 f_p(\mathbf{x}) d\mathbf{x} \leq \frac{1}{\left(\frac{1}{p-1} + \frac{1}{2}\right) a^2} \int_{R \leq |\mathbf{x}| \leq a} K(|x|) |\nabla \phi_o(\mathbf{x})|^2 f_p(|\mathbf{x}|) d\mathbf{x}. \quad (3.21)$$

Let us expand the gradient $\nabla \phi_o$ as

$$\nabla \phi_o(\mathbf{x}) = \nabla \phi(\mathbf{x})(1 - T(\mathbf{x})) - \phi(\mathbf{x}) \nabla T(\mathbf{x}).$$

So again since $(a+b)^2 \leq 2(a^2+b^2)$, by (3.17) we get

$$\begin{aligned}
&\int_{R \leq |\mathbf{x}| < a} K(|x|) |\nabla \phi_o(\mathbf{x})|^2 f_p(|\mathbf{x}|) d\mathbf{x} \\
&\quad \leq 2 \left[\int_{R \leq |\mathbf{x}| < a} K(|x|) |\nabla \phi(\mathbf{x})(1 - T(\mathbf{x}))|^2 f_p(|\mathbf{x}|) d\mathbf{x} \right. \\
&\qquad \left. + \int_{R \leq |\mathbf{x}| < a} K(|x|) |\phi(\mathbf{x}) \nabla T(\mathbf{x})|^2 f_p(|\mathbf{x}|) d\mathbf{x} \right] \\
&\quad \leq 2 \left[\int_{|\mathbf{x}| < a} K(|x|) |\nabla \phi(\mathbf{x})|^2 f_p(|\mathbf{x}|) d\mathbf{x} \right. \\
&\qquad \left. + \int_{R \leq |\mathbf{x}| < a} K(|x|) |\phi(\mathbf{x}) \nabla T(\mathbf{x})|^2 f_p(|\mathbf{x}|) d\mathbf{x} \right]. \qquad (3.22)
\end{aligned}$$

Since the term $\phi^2 |\nabla T|^2 f_p$ in the second term is supported in the annulus $R \leq |\mathbf{x}| \leq a(1-\epsilon)$ by the definition (3.16) of the cut–off function T and the estimate (3.18) we get

$$\begin{aligned}
&\int_{R \leq |\mathbf{x}| \leq a(1-\epsilon)} K(|x|) |\phi(\mathbf{x}) \nabla T(\mathbf{x})|^2 f_p(|\mathbf{x}|) d\mathbf{x} \\
&\quad \leq (2a)^2 \frac{1}{\log^2\left(\frac{1-h_n}{\epsilon}\right)} \int_{R \leq |\mathbf{x}| \leq a(1-\epsilon)} |\phi(\mathbf{x})|^2 f_p(|\mathbf{x}|) d\mathbf{x} \qquad (3.23) \\
&\quad \leq (2a)^2 \frac{1}{\log^2\left(\frac{1-h_n}{\epsilon}\right)} \int_{B_a} |\phi(\mathbf{x})|^2 f_p(|\mathbf{x}|) d\mathbf{x}
\end{aligned}$$

where

$$(2a)^2 \frac{1}{\log^2\left(\frac{1-h_n}{\epsilon}\right)} \to 0, \quad \epsilon \to 0.$$

Gathering estimates (3.19), (3.20), (3.21), (3.22) and (3.23) we have obtained the following inequality

$$\begin{aligned}
&\int_{B_a} \phi^2(\mathbf{x}) f_p(\mathbf{x})\, d\mathbf{x} \\
&\quad \le 2\left(C(\epsilon) + \frac{2}{\left(\frac{1}{p-1}+\frac{1}{2}\right)a^2}\right)\int_{B_a} K(|\mathbf{x}|)|\nabla\phi(\mathbf{x})|^2 f_p(|\mathbf{x}|)d\mathbf{x} \\
&\qquad + 2\left(\frac{1-(1-\epsilon)^n}{(1-\epsilon)^n} + \frac{8}{\left(\frac{1}{p-1}+\frac{1}{2}\right)\log^2\left(\frac{1-h_n}{\epsilon}\right)}\right)\int_{B_a} \phi(\mathbf{x})^2 f_p(\mathbf{x})d\mathbf{x}
\end{aligned}$$

where

$$\lim_{\epsilon\to 0} 2\left(\frac{1-(1-\epsilon)^n}{(1-\epsilon)^n} + \frac{8}{\left(\frac{1}{p-1}+\frac{1}{2}\right)\log^2\left(\frac{1-h_n}{\epsilon}\right)}\right) = 0.$$

So, for ϵ small enough we have

$$2\left(\frac{1-(1-\epsilon)^n}{(1-\epsilon)^n} + \frac{8}{\left(\frac{1}{p-1}+\frac{1}{2}\right)\log^2\left(\frac{1-h_n}{\epsilon}\right)}\right) \le \frac{1}{2} \tag{3.24}$$

and this implies

$$\frac{1}{2}\int_{B_a} \phi^2(\mathbf{x}) f_p(\mathbf{x})\, d\mathbf{x} \le 2\left(C(\epsilon) + \frac{2}{\left(\frac{1}{p-1}+\frac{1}{2}\right)a^2}\right)\int_{B_a} K(|\mathbf{x}|)|\nabla\phi(\mathbf{x})|^2 f_p(|\mathbf{x}|)d\mathbf{x}$$

which completes the proof of Theorem 5 with

$$C = 4\left(\frac{8}{a^2}\frac{(1-\epsilon)^2}{\epsilon^{2+\frac{1}{p-1}}(2-\epsilon)^{2+\frac{1}{p-1}}} + \frac{2}{\left(\frac{1}{p-1}+\frac{1}{2}\right)a^2}\right) \tag{3.25}$$

(the dependence on n is hidden in the choice of ϵ as in (3.24)). □

Remark 9 The condition (3.24) gives an explicit value of ϵ and this implies an explicit value of the Poincaré constant. Since for all $p > 1$ we have

$$\frac{8}{\left(\frac{1}{p-1}+\frac{1}{2}\right)\log^2\left(\frac{1-h_n}{\epsilon}\right)} \le \frac{16}{\log^2\left(\frac{1-h_n}{\epsilon}\right)},$$

one can choose ϵ independent of p so that

$$2\left(\frac{1-(1-\epsilon)^n}{(1-\epsilon)^n}+\frac{16}{\log^2\left(\frac{1-h_n}{\epsilon}\right)}\right)=\frac{1}{2} \tag{3.26}$$

where, as in (3.13)

$$h_n=\frac{1}{\sqrt{1+\frac{1}{2(n-1)}}}$$

and so the constant C in (3.25) is explicitly determined.

4 Weighted Poincaré inequality for the uniform density

Let us come to the classical Poincaré inequality, namely the inequality satisfied by the uniform density on a ball $B_a \subseteq \mathbb{R}^n$ for $a > 0$ and $n \geq 1$

$$f_\infty(\mathbf{x})=\frac{1}{|B_a|}\chi_{B_a}(\mathbf{x})$$

where

$$|B_a|=\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}a^n.$$

The uniform density can be seen as the limit for $p \to +\infty$ of the Barenblatt densities

$$f_p(\mathbf{x})=C_{a,p}(a^2-|\mathbf{x}|^2)^{\frac{1}{p-1}}, \quad \mathbf{x}\in B_a$$

and for all $n \geq 1$ and $\alpha \in \mathbb{R}$ is a stationary state of the family of Fokker–Planck equations

$$\frac{\partial f(\mathbf{x},t)}{\partial t}=\nabla\cdot\left[\nabla((a^2-|\mathbf{x}|^2)^\alpha f(\mathbf{x},t))+\frac{2\alpha\mathbf{x}}{(a^2-|\mathbf{x}|^2)^{1-\alpha}}f(\mathbf{x},t)\right],$$
$$\mathbf{x}\in B_a\subset\mathbb{R}^n, \quad t>0.$$

We can repeat word by word the proofs for the Barenblatt densities and obtain the weighted version for the classical Poincaré inequality. Alternatively, since the parameter ϵ in (3.26) does not depend on p, it is possible to pass to the limit in the Poincaré inequality for the Barenblatt densities both in the one and n dimensional cases in order to obtain the following.

Theorem 10 (Weighted Poincaré for 1–d uniform density) *Let $a > 0$ and $\mathbf{X}$ be a random variable distributed with a 1-dimensional uniform density*

$$f_\infty(x)=\frac{1}{2a}\chi_{(-a,a)}(x).$$

Then, for any smooth function ϕ on $(-a, a)$, and $\phi(X)$ with finite variance, it holds

$$\int_{-a}^{a} \left(\phi(x) - \frac{1}{2a} \left(\int_{-a}^{a} \phi(y)\, dy \right) \right)^2 dx \leq \frac{1}{a^2} \int_{-a}^{a} (a^2 - x^2)^2 \left(\phi'(x) \right)^2 dx.$$

Theorem 11 (Weighted Poincaré for n–d uniform density) *Let $n > 1$, $a > 0$ and $\mathbf{X}$ be a random variable distributed with a n-dimensional uniform density*

$$f_\infty(\mathbf{x}) = \frac{1}{|B_a|} \chi_{B_a}(\mathbf{x})$$

where

$$|B_a| = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} a^n.$$

For any smooth scalar function $\phi(\mathbf{x})$ on the ball $B_a = \{x : |\mathbf{x}| < a\}$ with finite variance it holds

$$\int_{B_a} \left(\phi(\mathbf{x}) - \frac{1}{|B_a|} \int_{B_a} \phi(\mathbf{y})\, d\mathbf{y} \right)^2 d\mathbf{x} \leq \frac{16}{a^2} \left(\frac{2\,(1-\epsilon)^2}{\epsilon^2 (2-\epsilon)^2} + 1 \right) \int_{B_a} (a^2 - |x|^2)^2 \, |\nabla \phi(\mathbf{x})|^2 \, d\mathbf{x}$$

with ϵ satisfying (3.26).

5 Optimality of the vanishing order at the boundary

We are going to show the optimality of the vanishing order at the boundary of the weight function in the Poincaré inequality for the Barenblatt (and uniform) densities. The reason lies in the nature of both the weight and the Barenblatt functions, which are in fact function of the distance to the boundary. Let us set $a = 1$ for the sake of simplicity. First let us consider the one dimensional case, with

$$f_p(x) = C_p (1 - x^2)^{\frac{1}{p-1}}, \quad |x| < 1.$$

Let us consider a smooth one dimensional bump function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

a) $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$
b) $\phi(x) = 0$ for $|x| > \frac{1}{2}$

and for $\epsilon \in (0, 1)$ small enough let us define $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ as follows:

$$\phi_\epsilon(x) = \begin{cases} \phi\left(\dfrac{x - (1-\epsilon)}{\epsilon} \right) & x \geq 0 \\ -\phi\left(\dfrac{-x - (1-\epsilon)}{\epsilon} \right) & x < 0 \end{cases}$$

and so in the Poincaré inequality the mean term vanishes. Let us suppose that for $\eta \geq 0$ the following Poincaré inequality holds true

$$\int_{-1}^{1} \phi_\epsilon^2(x) f_p(x) dx \leq C \int_{-1}^{1} (1-x^2)^{2+\eta} \left(\phi_\epsilon'(x)\right)^2 f_p(x) dx.$$

Since the function ϕ_ϵ^2 is even, by a change of variable we can estimate the left hand side as follows:

$$\begin{aligned}
&\int_{-1}^{1} \phi_\epsilon^2(x) f_p(x) dx = 2 \int_{0}^{1} \phi_\epsilon^2(x) f_p(x) dx \\
&= 2C_p \int_0^1 \phi^2 \left(\frac{x-(1-\epsilon)}{\epsilon}\right) (1-x^2)^{\frac{1}{p-1}} dx \\
&= \epsilon^{\frac{1}{p-1}} 2C_p \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi^2(t) (1-t)^{\frac{1}{p-1}} (2-\epsilon+\epsilon t)^{\frac{1}{p-1}} \epsilon dt \\
&\geq \epsilon^{\frac{1}{p-1}+1} 2C_p \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi^2(t) (1-t)^{\frac{1}{p-1}} dt.
\end{aligned}$$

On the right hand side we estimate

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2)^{2+\eta} \left(\phi_\epsilon'(x)\right)^2 f_p(x) dx \\
&= 2C_p \int_0^1 (1-x^2)^{2+\eta} \left(\frac{d}{dx}\left(\phi\left(\frac{x-(1-\epsilon)}{\epsilon}\right)\right)\right)^2 (1-x^2)^{\frac{1}{p-1}} dx \\
&= 2C_p \int_0^1 (1-x^2)^{2+\eta} \left(\phi'\left(\frac{x-(1-\epsilon)}{\epsilon}\right)\right)^2 \frac{1}{\epsilon^2} (1-x^2)^{\frac{1}{p-1}} dx \\
&\leq \epsilon^{\eta+\frac{1}{p-1}+1} 2C_p 2^{2+\eta+\frac{1}{p-1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} (1-t)^{2+\eta+\frac{1}{p-1}} \left(\phi'(t)\right)^2 dt.
\end{aligned}$$

For $\eta > 0$ this is a contradiction, since the right hand side vanishes at an order greater than the left hand side. Of course this is true also for the limiting case $p \to +\infty$, so for the uniform density. For the n-dimensional case, it is enough to consider a radially symmetric bump function $\phi : \mathbb{R}^n \to \mathbb{R}$ supported in the ball centered in the origin and of radius $\frac{1}{2}$.Then for $\epsilon \in (0,1)$ and $\mathbf{e}_1 = (1,0,0,\ldots,0)$ we define

$$\phi_\epsilon(\mathbf{x}) = \phi\left(\frac{\mathbf{x}-(1-\epsilon)\mathbf{e}_1}{\epsilon}\right) - \phi\left(\frac{\mathbf{x}-(-1+\epsilon)\mathbf{e}_1}{\epsilon}\right).$$

In this case too the mean value of ϕ_ϵ with respect to the measure $f_p(\mathbf{x})d\mathbf{x}$ vanishes and we can repeat the same argument as before in order to show that it is not possible to prove a weighted Poincaré inequality with a weight as $(1-|x|^2)^{2+\eta}$ for $\eta > 0$.

Indeed the change of variable

$$\mathbf{y} = \frac{\mathbf{x} - (1-\epsilon)\mathbf{e}_1}{\epsilon}$$

gives $d\mathbf{x} = \epsilon^n d\mathbf{y}$; moreover we have $(1 - |\mathbf{x}|^2) \sim \epsilon$ on the support of the bump function and this leads to the comparison between $\epsilon^{n+\frac{1}{p-1}}$ on the left hand side and $\epsilon^{n+\frac{1}{p-1}+\eta}$ to the right hand side, which is a contradiction for $\eta > 0$. This remark should be applied also to the Cauchy type densities (1.3), which have the form

$$\tilde{f}_p(\mathbf{x}) = \frac{C_p}{(1+|\mathbf{x}|^2)^{\frac{1}{1-p}}}, \quad p > 1 - \frac{2}{n}, \ \mathbf{x} \in \mathbb{R}^n.$$

Also in this case the optimal weighted Poincaré inequality has been proved at least in [3], [2] and in [11] and is as follows. For all ϕ smooth enough

$$\int_{\mathbb{R}^n} \left(\phi(\mathbf{x}) - \int_{\mathbb{R}^n} \phi(\mathbf{y}) \tilde{f}_p(\mathbf{y}) d\mathbf{y} \right)^2 \tilde{f}(\mathbf{x}) d\mathbf{x} \leq C \int_{\mathbb{R}^n} \left(1 + |x|^2\right) |\nabla \phi(\mathbf{x})|^2 \tilde{f}_p(\mathbf{x}) d\mathbf{x}.$$

where the constant depends (*a priori*) on n and p. Also in this case both the weight and the Cauchy densities have the same form and describe the distance to the boundary at infinity of the domain as $|\mathbf{x}|$. In order to prove that for $\eta > 0$ an inequality of the form

$$\int_{\mathbb{R}^n} \left(\phi(\mathbf{x}) - \int_{\mathbb{R}^n} \phi(\mathbf{y}) \tilde{f}_p(\mathbf{y}) d\mathbf{y} \right)^2 \tilde{f}(\mathbf{x}) d\mathbf{x} \leq C \int_{\mathbb{R}^n} \left(1 + |\mathbf{x}|^2\right)^{1-\eta} |\nabla \phi(\mathbf{x})|^2 \tilde{f}_p(\mathbf{x}) d\mathbf{x}$$

cannot be true, one can choose (in dimension one) a family of test functions for $R > 0$ of the form

$$\phi_R(x) = \begin{cases} \phi\left(\dfrac{x-R}{R}\right) & x \geq 0 \\ -\phi\left(\dfrac{-x-R}{R}\right) & x < 0 \end{cases}$$

where ϕ is a smooth one dimensional bump function as before. One can repeat the same proof as in the bounded Barenblatt case in order to prove that $\eta > 0$ implies a contradiction and we can extend the example to the n-dimensional case as before.

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Declarations

Conflict of interest The authors state that there is no conflict of interest.

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