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Untangling temporal graphs of bounded degree ☆,☆☆

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ABSTRACT

In this contribution we consider a variant of the vertex cover problem in temporal graphs that has been recently introduced to summarize timeline activities in social networks. The problem is NP-hard, even when the time domain considered consists of two timestamps. We further analyze the complexity of this problem, focusing on temporal graphs of bounded degree. We prove that the problem is NP-hard when (1) each vertex has degree at most one in each timestamp and (2) each vertex is connected with at most three neighbors, has degree at most two in each timestamp and the time domain consists of three timestamps. On the other hand, we prove that the problem is in P when each vertex is connected with at most two neighbors. Then we present a fixed-parameter algorithm for the restriction where we bound the number of interactions in each timestamp and the length of the interval where a vertex has incident temporal edges.

1. Introduction

Network analysis has recently extended several classic problems on static graphs to *temporal graphs*. Temporal graphs describe dynamics of edge activity in a discrete time domain [14,12,17], while the vertex set is not changing. Several problems for finding paths and analyzing graph connectivity has been considered in the literature [14,19,20,5,8,21,9,4,15,1,7]. Recently, one of the most relevant problem in graph theory and theoretical computer science, Vertex Cover, has been extended to temporal graphs [2,18]. In this paper we analyze a variant of Vertex Cover, called Network Untangling, that has been introduced in [18] for discovering event timelines and summarizing temporal networks. Given a sequence of interactions between entities (for example users of a social network platform), the proposed problem looks for an explanation of the observed interactions with few (and short) activity *covering intervals* of entities, such that each interaction is covered by at least one of the two entities involved (at least one of the two entities is active when an interaction between them is observed). This can be seen as a variant of Vertex Cover, where the temporal edges have to be covered with vertex activities of minimum length, called span. The *span* of a vertex is defined as the difference between the ending and starting interval endpoints where the vertex is defined to be active. A consequence of this definition is that when a vertex is active in a single timestamp, it has a span equal to 0.

Four formulations of the problem have been considered in [18], depending on the fact that a vertex activity is defined as a single interval or $k \geq 2$ intervals and that the objective function is the minimization of the sum of vertex spans or the minimization of the maximum vertex span. In this paper we consider the formulation, denoted by MinTimelineCover, that asks for the definition of a covering interval for each vertex, so that (1) for each temporal edge $\{u, v, t\}$ at least one of the

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covering intervals of u and v includes t and (2) the overall span is minimized. Given a temporal graph, MinTimelineCover looks for a cover of the temporal edges that has minimum span and such that each vertex is active in one time interval.

Other variants of Vertex Cover in temporal graphs have been introduced in [1]. A first variant asks for the minimum number of timestamps where vertices are defined to be active, such that each (non-temporal) edge $e = \{u, v\}$ is temporally covered, that is there exists a timestamp t where e is defined and one of (u, t) and (v, t) belongs to the cover. A second variant asks for each temporal edge to be temporally covered at least once for every interval of a given length. The two variants are NP-hard, also in very restricted cases [1]. Further results on the problem variants, including their approximability, have been given in [1,11].

The MinTimelineCover problem is known to be NP-hard [18], even in the restriction where the time domain consists of two timestamps [10] (when the time domain consists of a single timestamp, the problem is trivially in P, since any solution of the problem has span 0). For this restriction, MinTimelineCover is fixed-parameter tractable, when parameterized by the span of the solution [10]. The work in [10] has analyzed the parameterized complexity of the variants of Network Untangling proposed in [18], considering as parameters the number of vertices of the temporal graph, the length of the time domain, the number of intervals of vertex activity and the span of a solution.

In this paper, we consider the complexity of the MinTimelineCover problem when we bound the local degree (maximum number of interactions of a vertex in a timestamp) and the total degree (maximum number of neighbors of a vertex in the overall time domain). We prove in Section 3 that the problem is NP-hard even when there exists at most one interaction in each timestamp, and thus the local degree is bounded by 1. In Section 4 we consider the complexity of MinTimelineCover when we bound the total (and possibly the local) degree of the temporal graph. We show that, while the problem is in P when the total degree is bounded by two, it is NP-hard when the total degree is bounded by three, the local degree is bounded by two and the time domain consists of three timestamps. Finally, in Section 5 we prove that the problem is fixed parameter tractable when the parameter is the size of the time window that bounds (1) the number of vertices with temporal edges in a timestamp (thus also the local degree) and (2) the length of the interval where a vertex has incident temporal edges. The idea of considering a time window has been applied before on temporal graphs, for example a sliding time window has been considered in the context of graph coloring [16] and for the graph covering formulation defined in [2].

We conclude the paper with some open problems in Section 6. In Section 2 we present some definitions and we formally define the MinTimelineCover problem.

2. Preliminaries

We start this section by defining the discrete time domain over which is defined a temporal graph.

Definition 1. A discrete *time domain* $\mathcal{T} = [1, \ldots, t_{max}]$ is a sequence of timestamps. An *interval* $T = [t_i, t_j]$ over $\mathcal{T}$, where $t_i, t_j \in \mathcal{T}$ and $t_i \leq t_j$, is the sequence of timestamps between t_i and t_j.

Two intervals $T_1 = [t_{a,1}, t_{b,1}]$, $T_2 = [t_{a,2}, t_{b,2}]$ are *disjoint* if they do not share any timestamp, that is $t_{a,1} \leq t_{b,1} < t_{a,2} \leq t_{b,2}$ or $t_{a,2} \leq t_{b,2} < t_{a,1} \leq t_{b,1}$.

Given a set of pairwise disjoint intervals $T_1 = [t_{a,1}, t_{b,1}]$, $T_2 = [t_{a,2}, t_{b,2}]$, $\ldots$, $T_q = [t_{a,q}, t_{b,q}]$, where $t_{a,1} \leq t_{b,1} < t_{a,2} \leq t_{b,2} < \cdots < t_{a,q} \leq t_{b,q}$, and $t_{b,i} = t_{a,i+1} - 1$, $1 \leq i \leq q-1$, we can define the *concatenation* of these intervals:

$$T_1 \cdot T_2 \cdot \cdots \cdot T_q = [t_{a,1}, t_{b,q}].$$

We present now the definition of temporal graph. The vertex set is not changing in the time domain, that is the vertex set is identical in each timestamp (see the example of Fig. 1).

Definition 2. A *temporal graph* $G = (V, E, \mathcal{T})$ consists of

1. A set V of vertices
2. A time domain $\mathcal{T}$
3. A set $E = \{\{u, v, t\} : u, v \in V, t \in \mathcal{T}\}$.

A temporal graph $G = (V, E, \mathcal{T})$ is associated with a static graph, called *union graph* $G_u = (V, E_u)$, over the same vertex set V, where the set of edges E_u is defined as follows (see the example of Fig. 1):

$$E_u = \{\{u, v\} : \{u, v, t\} \in E, \text{ for some timestamp } t \text{ of the time domain } \mathcal{T}\}.$$

Notice that a slightly different definition of temporal graphs is sometimes considered in the literature (for example in [2,11]). A temporal graph is defined as a labeled graph, more precisely as a pair (G, λ), where $G = (V, E)$ is the underlying graph (G_u in our notation) and λ is a labeling function that associates with every edge the timestamps where it is defined. So if $\{u, v, t\}$ is a temporal edge, in this notation the edge $\{u, v\}$ will be labeled by t.

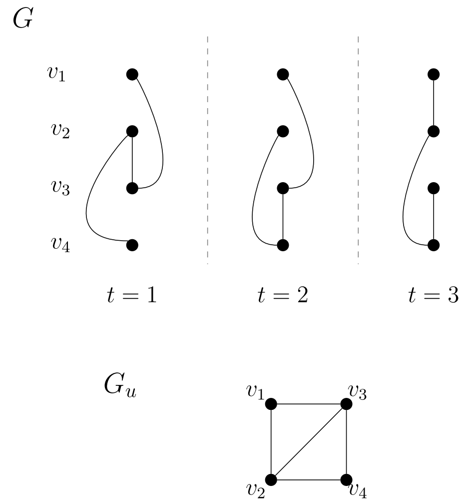

Fig. 1. An example (in the upper part) of a temporal graph G consisting of four vertices (v_1, v_2, v_3, v_4) and three timestamps (1, 2, 3) and the corresponding underlying static graph G_u (in the lower part). For each timestamp, the active temporal edges of G are represented. For example for $t = 1$, the active edges are $\{v_1, v_3, 1\}$, $\{v_2, v_3, 1\}$, $\{v_2, v_4, 1\}$. Notice that the local degree $\Delta^L_G = 2$, while the total degree $\Delta^T_G = 3$ (v_2 and v_3 have degree three in G_u).

Given an interval I of $\mathcal{T}$, $E(I)$ denotes the set of active edges in the timestamps of I, that is:

$$E(I) = \{\{u, v, t\} | \{u, v, t\} \in E \wedge t \in I\}.$$

$E(t) = E([t, t])$ denotes the set of active edges in timestamp $t \in \mathcal{T}$.

Given a vertex $v \in V$, a *covering interval* of v is defined as an interval $I_v = [l_v, r_v]$, with $1 \leq l_v \leq r_v \leq t_{max}$, of the time domain where v is defined to be *active*; an edge $\{v, u, t\}$, with $l_v \leq t \leq r_v$, is covered by I_v, that is a vertex v active in a covering interval $I_v = [l_v, r_v]$ covers the temporal edges incident in v and defined in a timestamp between l_v and r_v. In any timestamp not in I_v, v is considered inactive and does not cover temporal edges incident in it. Notice that if $I_v = [l_v, r_v]$ is a covering interval of v, there may exist temporal edges $\{u, v, t\}$, with $t < l_v$ or $t > r_v$ (see the example in Fig. 2). An activity timeline $\mathcal{A}$ is a set of covering intervals, one for each vertex of the temporal graph, defined as follows:

$$\mathcal{A} = \{I_v : v \in V\}.$$

Given a temporal graph $G = (V, E, \mathcal{T})$, an *activity timeline* $\mathcal{A}$ covers G if for each temporal edge $\{u, v, t\} \in E$, t belongs to I_u or to I_v, where I_u (I_v, respectively) is the covering interval of u (of v, respectively) defined by $\mathcal{A}$. It follows that, for a temporal edge $\{u, v, t\}$, at least one of u, v is active in an interval that includes t.

The *span* of an interval $I_v = [l_v, r_v]$, with $v \in V$, is defined as follows:

$$s(I_v) = |r_v - l_v|.$$

Notice that for a covering interval $I_v = [l_v, r_v]$ consisting of a single timestamp, that is where $l_v = r_v$, the span is equal to 0, that is it holds that $s(I_v) = 0$. The overall span of an activity timeline $\mathcal{A}$ is equal to

$$s(\mathcal{A}) = \sum_{I_v \in \mathcal{A}} s(I_v).$$

Now, we are ready to define the problem we are interested into (see the example of Fig. 2).

Problem 1. (MinTimelineCover)
Input: A temporal graph $G = (V, E, \mathcal{T})$.
Output: An activity timeline of minimum span that covers G.

Next, we introduce the concept of local and total degree (illustrated in Fig. 1). Given a temporal graph $G = (V, E, \mathcal{T})$ and a vertex $v \in V$, the *local degree* of v in a timestamp t, denoted by $\Delta^L_G(v, t)$, is defined as follows:

$$\Delta^L_G(v, t) = |\{\{v, u, t\} \in E\}|,$$

that is the number of temporal edges $\{v, u, t\}$. The *total degree* of a vertex $v \in V$, denoted by $\Delta^T_G(v)$, is the degree of v in the underlying static graph G_u, that is

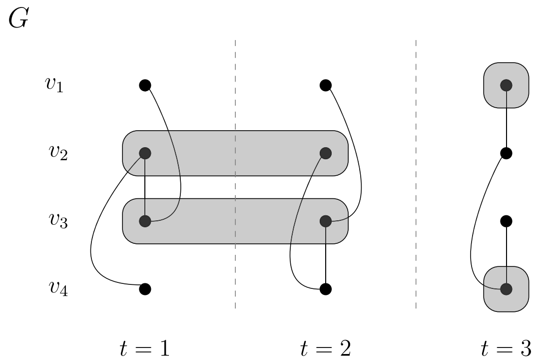

Fig. 2. An example of MinTimelineCover on a temporal graph G, where the gray rectangles represent the covering intervals of each vertex; v_1 is active in timestamp 3, with span equal to 0, v_2 and v_3 are active in interval $[1, 2]$, each one with span equal to 1, v_4 is active in timestamp 3, with span equal to 0. The activity timeline defined by the vertex covering intervals covers each temporal edge of the temporal graph with total span equal to 2.

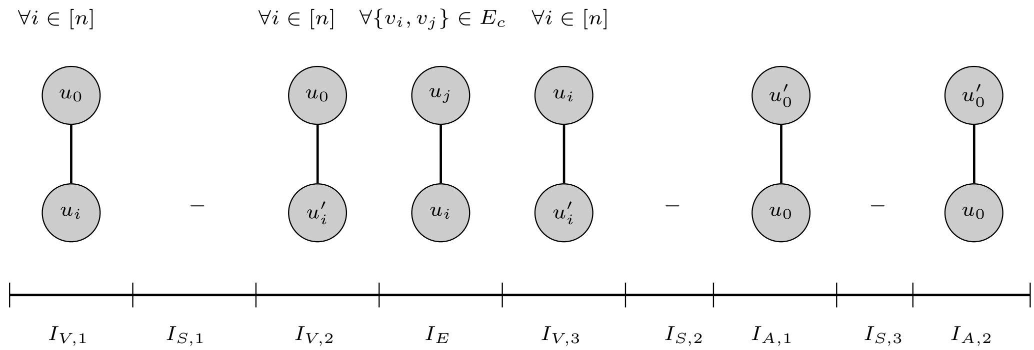

Fig. 3. A sketch of the time domain $\mathcal{T}$ built by the reduction. For each of the nine disjoint intervals of $\mathcal{T}$, we present in the upper part a temporal edge defined in that interval or − when no temporal edge is defined.

$$\Delta_G^T(v) = |\{\{v, x\} \in E_u\}|.$$

The local degree Δ_G^L of G is the maximum over v and t of $\Delta_G^L(v, t)$; the total degree Δ_G^T of G is the maximum over v of $\Delta_G^T(v)$.

Given an interval I of the time domain $\mathcal{T}$, the *time window* associated with I (denoted by $W(I)$) is defined as

$$W(I) = \{(v, t) : \{v, u, t\} \in E \wedge t \text{ in } I\}$$

that is the set of pairs consisting of vertices $v \in V$ and timestamps t of I, such that there exists a temporal edge incident in v in timestamp t. We define a temporal graph to be (w, h)-window-constrained if (1) the temporal edges incident in each vertex belong to an interval of length at most w and (2) in each timestamp there are at most h vertices with incident temporal edges.

3. Hardness of MinTimelineCover for bounded local degree

In this section we consider the MinTimelineCover problem when each timestamp contains at most a single active edge, and thus the local degree Δ_G^L of G is also bounded by 1. We denote this restriction by 1-MinTimelineCover. We prove that 1-MinTimelineCover is NP-hard by giving a reduction from the Vertex Cover problem. Next, we recall the definition of Vertex Cover:

Problem 2. (Vertex Cover)
Input: A graph $G_c = (V_c, E_c)$.
Output: A minimum cardinality set $V_c' \subseteq V_c$ such that for each $\{u, v\} \in E_c$, $u \in V_c'$ or $v \in V_c'$.

Consider an instance $G_c = (V_c, E_c)$ of Vertex Cover, where $|V_c| = n$ and $|E_c| = m$, we define a corresponding instance of MinTimelineCover, that is a temporal graph $G = (V, E, \mathcal{T})$, as follows (a sketch of the temporal graph is given in Fig. 3). We assume in what follows that m is large enough so that $m^4 > n(m^2 + m + 2n)$.

We start by defining the time domain $\mathcal{T}$ that consists of $2m^4 + m^2 + m + n + 2$ timestamps and is defined starting from the following disjoint intervals:

- $I_{V,1} = [1, n]$ $I_{V,1}$ consists of n timestamps
- $I_{S,1} = [n + 1, m^2 + n]$ $I_{S,1}$ consists of m^2 timestamps
- $I_{V,2} = [m^2 + n + 1, m^2 + 2n]$ $I_{V,2}$ consists of n timestamps
- $I_E = [m^2 + 2n + 1, m^2 + m + 2n]$ I_E consists of m timestamps
- $I_{V,3} = [m^2 + m + 2n + 1, m^2 + m + 3n]$ $I_{V,3}$ consists of n timestamps
- $I_{S,2} = [m^2 + m + 3n + 1, m^4 + m^2 + m + 3n]$ $I_{S,2}$ consists of m^4 timestamps
- $I_{A,1} = [m^4 + m^2 + m + 3n + 1, m^4 + m^2 + m + 3n + 1]$ $I_{A,1}$ consists of 1 timestamp
- $I_{S,3} = [m^4 + m^2 + m + 3n + 2, 2m^4 + m^2 + m + 3n + 1]$ $I_{S,3}$ consists of m^4 timestamps
- $I_{A,2} = [2m^4 + m^2 + m + 3n + 2, 2m^4 + m^2 + m + 3n + 2]$ $I_{A,2}$ consists of 1 timestamp

The time domain $\mathcal{T}$ is the concatenation of the disjoint intervals defined previously:

$$\mathcal{T} = I_{V,1} \cdot I_{S,1} \cdot I_{V,2} \cdot I_E \cdot I_{V,3} \cdot I_{S,2} \cdot I_{A,1} \cdot I_{S,3} \cdot I_{A,2}$$

The set V is defined as follows:

$$V = \{u_i, u'_i : v_i \in V_c, 1 \le i \le n\} \cup \{u_0\} \cup \{u'_0\}$$

Now, we define the set E of temporal edges in each interval of the time domain $\mathcal{T}$. In each interval $I_{S,x}$, $x \in \{1, 2, 3\}$, no temporal edge is active. We assume that there are no self-loops and that the edges of G_c are ordered as follows: for two edges $\{v_i, v_j\}, \{v_x, v_y\} \in E_c$, where $i < j$ and $x < y$, $\{v_i, v_j\} < \{v_x, v_y\}$ if $i < x$, or $i = x$ and $j < y$. We refer to the p-th edge of G_c as the edge in position p based on this order. Recall that $E(I)$ denotes the set of temporal edges active in interval I. Next, we define the sets of temporal edges in $I_{V,1}$, $I_{V,2}$, I_E, $I_{V,3}$, $I_{A,1}$ and $I_{A,2}$:

- $E(I_{V,1}) = \{\{u_i, u_0, t\} : 1 \le t \le n\}$
- $E(I_{V,2}) = \{\{u'_i, u_0, t\} : t = m^2 + n + i, 1 \le i \le n\}$
- $E(I_E) = \{\{u_i, u_j, t\} : t = m^2 + 2n + p, 1 \le p \le m, \{v_i, v_j\}$ is the p-edge of $G_c\}$
- $E(I_{V,3}) = \{\{u_i, u'_i, t\} : t = m^2 + m + 2n + i, 1 \le i \le n\}$
- $E(I_{A,1}) = \{\{u_0, u'_0, t\} : t = m^4 + m^2 + m + 3n + 1\}$
- $E(I_{A,2}) = \{\{u_0, u'_0, t\} : t = 2m^4 + m^2 + m + 3n + 2\}$.

We start by proving some properties of G. First, notice that by construction in each timestamp there exists at most one active temporal edge, thus G is an instance of 1-MinTimelineCover. Now, we present a property of vertices u_0 and u'_0.

Lemma 1. *Given an instance G_c of Vertex Cover, let G be the corresponding instance of 1-MinTimelineCover. Let $\mathcal{A}$ be a solution of 1-MinTimelineCover on instance G of span at most $n(m^2 + m + 2n)$, then each of the vertices u_0 and u'_0 is active in exactly one of the timestamps of intervals $I_{A,1}$ and $I_{A,2}$.*

Proof. Consider a solution $\mathcal{A}$ of 1-MinTimelineCover on instance G of span at most $n(m^2 + m + 2n)$. Notice that, by construction, each of u_0, u'_0 is active in exactly one of the intervals $I_{A,1}$ and in $I_{A,2}$, as otherwise $\mathcal{A}$ has a span of at least $m^4 > n(m^2 + m + 2n)$. Since temporal edges incident in u'_0 are defined only in the timestamps of $I_{A,1}$ and $I_{A,2}$, we can assume that u'_0 is active in a timestamp of $I_{A,1}$ or of $I_{A,2}$. Now, consider vertex u_0. There exist temporal edges incident in u_0 in intervals $I_{A,1}$, $I_{A,2}$, $I_{V,1}$ and $I_{V,2}$. Since u_0 has to be active in one of $I_{A,1}$, $I_{A,2}$, if u_0 is active in a timestamp of $I_{V,1}$ or $I_{V,2}$, it has a span of at least $m^4 > n(m^2 + m + 2n)$, hence $\mathcal{A}$ has a span of at least $m^4 > n(m^2 + m + 2n)$. Hence $\mathcal{A}$ defines u_0 active only in a timestamp of $I_{A,1}$ or $I_{A,2}$, thus concluding the proof. □

We show next how to relate a vertex cover of G_c and a solution of MinTimelineCover on G.

Lemma 2. *Consider an instance G_c of Vertex Cover, let G be the corresponding instance of 1-MinTimelineCover. If there exists a vertex cover of G_c consisting of k vertices, then there exists a solution of 1-MinTimelineCover on instance G of span at most $k(m^2 + m + 2n) + (n - k)(n + m)$.*

Proof. Given a vertex cover $V'_c \subseteq V_c$ of G, with $|V'_c| = k$, we define a solution $\mathcal{A}$ of 1-MinTimelineCover as follows:

- Vertex u_0 (u'_0, respectively), is active in the unique timestamp of interval $I_{A,1}$ (of interval $I_{A,2}$, respectively); u_0 and u'_0 have a span equal to 0.
- For each vertex $v_i \in V_c \setminus V'_c$, $1 \le i \le n$, vertex u_i is active in timestamp i and has a span equal to 0, vertex u'_i is active in interval $[m^2 + n + i, m^2 + m + 2n + i]$, and has a span equal to $m + n$.
- For each vertex $v_i \in V'_c$, $1 \le i \le n$, u_i is active in interval $[i, m^2 + m + 2n + i]$, and has a span equal to $m^2 + m + 2n$; vertex u'_i is active in timestamp $m^2 + n + i$, and has a span equal to 0.

Consider the activity timeline $\mathcal{A}$, we show that it covers every temporal edge of G. Indeed, the temporal edges of $I_{A,1}$ and $I_{A,2}$ are covered by u_0 and u'_0, respectively. The temporal edges in $I_{V,1}$ are covered by vertices u_i, $1 \leq i \leq n$, since u_i is active in timestamp i. The temporal edges of $I_{V,2}$ are covered by u'_i, $1 \leq i \leq n$, since u'_i is active in timestamp m^2+n+i. Consider a temporal edge $\{u_i, u_j, t\}$ defined in interval I_E. Since V'_c is a vertex cover of G_c, by construction u_i is active in interval $I_{u_i} = [i, m^2+m+n+i]$ or u_j is active in interval $I_{u_j} = [j, m^2+m+2n+j]$; both intervals include I_E, thus at least one of I_{u_i} or I_{u_j} covers the temporal edge $\{u_i, u_j, t\}$. Finally, the edges of $I_{V,3}$ are covered either by u_i, if $v_i \in V'_c$, since u_i in this case is active in interval $[i, m^2+m+2n+i]$, or by u'_i, if $v_i \in V_c \setminus V'_c$, since u'_i in this case is active in interval $[m^2+n+i, m^2+m+2n+i]$. It follows that all the temporal edges are covered, thus $\mathcal{A}$ covers G.

Now, consider the span of $\mathcal{A}$. Since k vertices u_i, with $v_i \in V'_c$, have a span of m^2+m+2n and $n-k$ vertices u'_i, with $v_i \in V_c \setminus V'_c$, have a span of $n+m$, the overall span of $\mathcal{A}$ is $k(m^2+m+2n)+(n-k)(n+m)$, thus concluding the proof. □

Based on Lemma 1, we can prove the following result.

Lemma 3. *Given an instance G_c of Vertex Cover, consider the corresponding instance G of 1-MinTimelineCover. Let $\mathcal{A}$ be a solution of 1-MinTimelineCover on instance G having span $k(m^2+m+2n)+(n-k)(n+m)$, then there exists a vertex cover of G_c consisting of at most k vertices.*

Proof. Consider a solution $\mathcal{A}$ of 1-MinTimelineCover on instance G that has a span $k(m^2+m+2n)+(n-k)(n+m) \leq n(m^2+m+2n))$. By Lemma 1, we can assume that each of u_0 and u'_0 is active in one of the timestamps of $I_{A,1}$, $I_{A,2}$. This implies that in any solution of 1-MinTimelineCover on instance G, each vertex u_i, with $1 \leq i \leq n$, must be active in timestamp i and each vertex u'_i, with $1 \leq i \leq n$, must be active in timestamp m^2+n+i. Since each temporal edge $\{u_i, u_j, t\}$ in I_E must be covered by one of u_i, u_j, it follows that $\mathcal{A}$ defines a span of at least m^2+n for a subset $U \subseteq \{u_i : 1 \leq i \leq n\}$ of vertices. Furthermore, for each i with $1 \leq i \leq n$, the temporal edge $\{u_i, u'_i, m^2+m+2n+i\}$ of interval $I_{V,3}$ must be covered by u_i or by u'_i.

Now, we claim that, for each vertex $u_i \in U$, u_i covers the temporal edge $\{u_i, u'_i, m^2+m+2n+i\}$. Assume it is not the case, then the temporal edge $\{u_i, u'_i, m^2+m+2n+i\}$ is covered by u'_i. Then u_i has a span of at least m^2+n, while u'_i has a span of $m+n$. If we modify the solution $\mathcal{A}$ so that u_i is active in interval $[i, m^2+m+2n+i]$ and thus it covers also the temporal edge $\{u_i, u'_i, m^2+m+2n+i\}$, while u'_i is active only in timestamp m^2+n+i, we obtain that u_i has a span of m^2+m+2n, while u'_i has a span of 0, thus we obtain a solution that does not increase the span with respect to $\mathcal{A}$.

Since $\mathcal{A}$ has a span of $k(m^2+m+2n)+(n-k)(m+n)$, it follows that the set U contains at most k vertices. Define the following subset V'_c of vertices of V_c:

$$V'_c = \{v_i : u_i \in U\}.$$

By construction, it holds that $|V'_c| \leq k$. Furthermore, we claim that V'_c is vertex cover of G_c. Indeed, assume that there exists an edge $\{v_i, v_j\} \in E_c$ such that $v_i \notin V'_c$ and $v_j \notin V'_c$. Then, by construction $u_i, u_j \notin U$, thus the temporal edge $\{u_i, u_j, t\}$ of I_E would not be covered by $\mathcal{A}$, thus leading to a contradiction. It follows that V'_c is a vertex cover of G_c of size at most k, thus concluding the proof. □

Now, we are able to prove the main result of this section.

Theorem 1. *1-MinTimelineCover is NP-hard and the decision version of 1-Min TimelineCover is NP-complete.*

Proof. It follows from Lemma 2 and Lemma 3 that we have designed a polynomial-time reduction from Vertex Cover to 1-MinTimelineCover. Since Vertex Cover is NP-hard [13], it follows that also 1-MinTimelineCover is NP-hard.

Notice that the decision version of 1-MinTimelineCover is in NP, since giving a solution of 1-MinTimelineCover we can check in polynomial time that it covers G and that has at most a given span. □

4. MinTimelineCover for bounded total degree

In this section we analyze another restriction of the MinTimelineCover problem, in particular we consider a bound on the total degree of the input temporal graph. We start by showing that MinTimelineCover is polynomial-time solvable when the total degree Δ_G^T is bounded by 2. Then we show that the problem is NP-hard when the total degree Δ_G^T is equal to 3 and, furthermore, the following restrictions hold: (1) the local degree $\Delta_G^L = 2$ and (2) the time domain consists of three timestamps.

4.1. Total degree Δ_G^T bounded by 2

We start by showing that when $\Delta_G^T \leq 2$, MinTimelineCover is solvable in polynomial time. Consider the underlying static graph G_u associated with G and let $H = (V_H, E_H)$ be a connected component of G_u. Since $\Delta_G^T \leq 2$, it follows that $H =$

(V_H, E_H) is either a simple cycle (that is, it contains no chord) or a path. We present a polynomial algorithm that defines an activity timeline $\mathcal{A}$ of minimum span when H is a cycle (if H is a path then the algorithm can be easily adapted).

First, define a vertex of H as h_1, and define vertices $h_2, \ldots, h_z$ according to a depth first search of H that starts from h_1. Notice that, since H is a simple cycle, h_z is adjacent to h_1 in H. Now, we consider two timestamps a, b, with $1 \le a \le b \le t_{max}$, where h_1 must be active and we define function $D_{a,b}[i, t_{i,1}, t_{i,2}]$ as follows. $D_{a,b}[i, t_{i,1}, t_{i,2}]$ is the minimum span of an activity timeline $\mathcal{A}$ of vertices $h_1, \ldots h_i$ that covers each temporal edge between vertices $h_1, \ldots, h_i$ such that h_i is defined to be active in interval $[t_{i,1}, t_{i,2}]$ and vertex h_1 is defined to be active in interval $[a, b]$.

Next, we present a recurrence to compute $D_{a,b}[i, t_{i,1}, t_{i,2}]$. For $2 \le i \le z$, $D_{a,b}[i, t_{i,1}, t_{i,2}]$ is defined as follows:

$$D_{a,b}[i, t_{i,1}, t_{i,2}] = \min_{t_{i-1,1}, t_{i-1,2}:\, t_{i-1,1} \le t_{i-1,2}} D_{a,b}[i, t_{i-1,1}, t_{i-1,2}] + t_{i,2} - t_{i,1} \quad (1)$$

such that the following conditions hold:

1. Each temporal edge between h_{i-1} and h_i is defined in a timestamp contained in interval $[t_{i-1,1}, t_{i-2,2}]$ or in interval $[t_{i,1}, t_{i,2}]$.
2. If $i = z$, each temporal edge between h_1 and h_z is defined in a timestamp that belongs to interval $[t_{z,1}, t_{z,2}]$ or $[a, b]$.

If condition 1 and 2 defined above do not hold, $D_{a,b}[i, t_{i,1}, t_{i,2}] = -\infty$.

For $i = 1$, $D_{a,b}[1, t_{1,1}, t_{1,2}]$ is defined as follows:

$$D_{a,b}[1, t_{1,1}, t_{1,2}] = t_{1,2} - t_{1,1} \quad (2)$$

if $t_{i,1} = a$ and $t_{i,2} = b$, else $D_{a,b}[1, t_{1,1}, t_{1,2}] = -\infty$.

Next, we prove the correctness of the recurrence (Equation (1) and Equation (2)).

Lemma 4. *Given a connected component H of G_u, there exists an activity timeline $\mathcal{A}$ for vertices $h_1, \ldots, h_i$, with $1 \le i \le z$, such that (1) it covers each temporal edge between vertices $h_1, \ldots, h_i$ (2) it has a span of k and (3) it defines h_i active in interval $[t_{i,1}, t_{i,2}]$ and h_1 active in interval $[a, b]$ if and only if $D_{a,b}[i, t_{i,1}, t_{i,2}] = k$.*

Proof. We prove the lemma by induction on $i \ge 1$. If $i = 1$, since there is no temporal edge connecting h_1 with itself, it follows by the definition of $D_{a,b}[1, t_{1,1}, t_{1,2}]$ that $D_{a,b}[1, t_{1,1}, t_{1,2}] = t_{1,2} - t_{1,1} = k$ if and only if there exists an activity timeline $\mathcal{A}$ that has a span of $b - a = k$ and defines h_1 active in interval $[a, b]$.

Now, assume that the lemma holds for $i - 1$, we show that it holds for i. Assume that $D_{a,b}[i, t_{i,1}, t_{i,2}] = k$, then it follows that there exist two timestamps $t_{i-1,1}$, $t_{i-1,2}$, with $t_{i-1,1} \le t_{i-1,2}$ such that $D_{a,b}[i, t_{i-1,1}, t_{i-1,2}] = k - (t_{i,2} - t_{i,1})$. By induction hypothesis there exists an activity timeline $\mathcal{A}'$ for the vertices $h_1, \ldots, h_{i-1}$ such that (1) it covers each temporal edge between vertices $h_1, \ldots, h_{i-1}$ (2) it has a span of $k - (t_{i,2} - t_{i,1})$ and (3) it defines h_{i-1} active in interval $[t_{i-1,1}, t_{i-1,2}]$ and h_1 active in interval $[a, b]$. By the definition of $D_{a,b}[i, t_{i,1}, t_{i,2}]$, it follows that the activity timeline $\mathcal{A}$ defined from $\mathcal{A}'$ by adding h_i active in interval $[t_{i,1}, t_{i,2}]$ covers each temporal edge between vertices h_{i-1} and h_i, since by definition of $D_{a,b}[i, t_{i,1}, t_{i,2}]$ every temporal edge between h_i and h_{i-1} must belong to a timestamp in interval $[t_{i,1}, t_{i,2}]$ or $[t_{i-1,1}, t_{i-1,2}]$. Furthermore if $i = z$, by the definition of $D_{a,b}[z, t_{z,1}, t_{z,2}]$, every temporal edge between h_1 and h_z is defined in a timestamp that belongs to interval $[a, b]$ or $[t_{z,1}, t_{z,2}]$. Finally, $\mathcal{A}$ has a span of k and defines h_i active in interval $[t_{i,1}, t_{i,2}]$ and h_1 active in interval $[a, b]$.

Assume that there exists an activity timeline $\mathcal{A}$ of minimum span for vertices $h_1, \ldots, h_i$ such that (1) it covers each temporal edge between vertices $h_1, \ldots, h_i$ (2) it has a span of k and (3) it defines h_i active in interval $[t_{i,1}, t_{i,2}]$ and h_1 active in interval $[a, b]$. Consider the activity timeline $\mathcal{A}'$ for vertices $h_1, \ldots, h_{i-1}$ such that (1) it covers each temporal edge between vertices $h_1, \ldots, h_{i-1}$ (2) it has a span of k_1 and (3) it defines h_{i-1} active in interval $[t_{i-1,1}, t_{i-1,2}]$ and h_1 active in interval $[a, b]$. By induction hypothesis, $D_{a,b}[i-1, t_{i-1,1}, t_{i-1,2}] = k_1$. Consider the covering interval defined for vertex h_i as $[t_{i,1}, t_{i,2}]$. Notice that it covers each temporal edge between vertices h_{i-1} and h_i not covered by vertex h_{i-1}. Furthermore if $i = z$, every temporal edge between h_1 and h_z is defined in a timestamp that belongs to interval $[a, b]$ or $[t_{z,1}, t_{z,2}]$. Thus $D_{a,b}[i, t_{i,1}, t_{i,2}] = D_{a,b}[i-1, t_{i-1,1}, t_{i-1,2}] + t_{i,2} - t_{i,1} = k$. □

Lemma 5. *MinTimelineCover is solvable in polynomial time when $\Delta_G^T \le 2$.*

Proof. MinTimelineCover can be solvable in polynomial time when $\Delta_G^T \le 2$ by applying the dynamic programming recurrence $D_{a,b}[z, t_{z,1}, t_{z,2}]$ on each connected component of G_u. The correctness of the recurrence follows from Lemma 4. In particular, consider a connected component H of G_u consisting of vertices $h_1, \ldots, h_z$. There exists an activity timeline $\mathcal{A}$ that covers the temporal edges of H such that it has a span of k and it defines h_z active in interval $[t_{z,1}, t_{z,2}]$ and h_1 active in interval $[a, b]$ if and only if $D_{a,b}[z, t_{z,1}, t_{z,2}] = k$.

The time complexity of the algorithm on each connected component $H = (V_H, E_H)$, with $|V_H| = z$, is $O(z\, t_{max}^6)$. Indeed, fixed two timestamps a and b, there are $O(z\, t_{max}^2)$ entries of $D_{a,b}[i, t_{i,1}, t_{i,2}]$, as $1 \le i \le z$ and $1 \le t_{i,1} \le t_{i,2} \le t_{max}$. Each entry $D_{a,b}[i, t_{i,1}, t_{i,2}]$ can be computed in $O(t_{max}^2)$ time, since we have to consider two possible timestamps $t_{i-1,1}$, $t_{i-1,2}$,

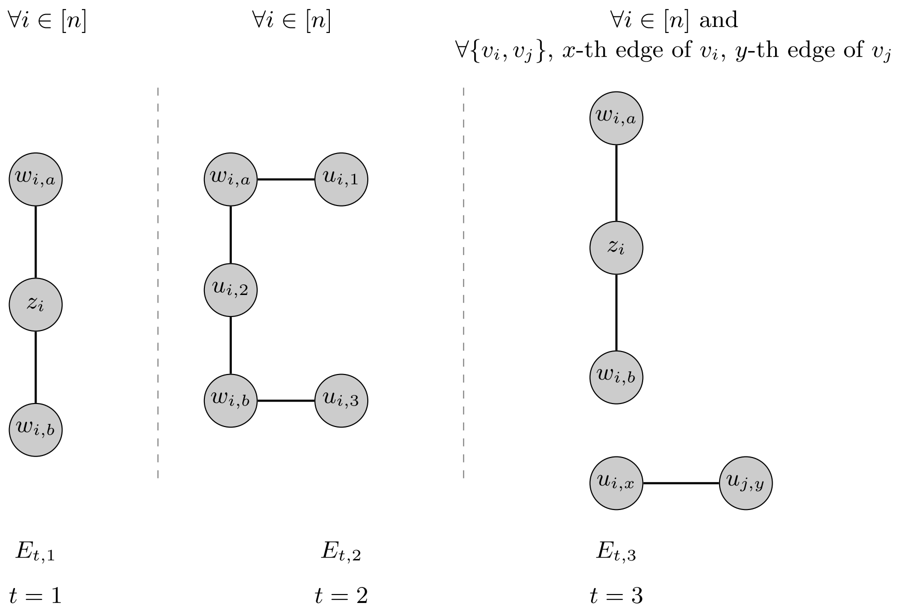

Fig. 4. The structure of the temporal graph G builds by the reduction from Cubic Vertex Cover. $E_{t,1}$, $E_{t,2}$, $E_{t,3}$ are the sets of temporal edges defined in the three timestamps 1, 2 and 3, respectively.

with $1 \le t_{i-1,1} \le t_{i-1,2} \le t_{max}$. Thus the entries of $D_{a,b}[i, t_{i,1}, t_{i,2}]$ can be computed in $O(z\ t^4_{max})$ time. Since there are $O(t^2_{max})$ choices of values a and b, the overall time complexity is $O(z\ t^6_{max})$ for each connected component H. The overall time complexity of the algorithm on input $G = (V, E, \mathcal{T})$ is $O(|V| t^6_{max})$. □

4.2. Total degree $\Delta^T_G = 3$

We prove that MinTimelineCover is NP-hard when the total degree $\Delta^T_G = 3$, the time domain consists of three timestamps and the local degree $\Delta^L_G = 2$, by giving a reduction from Vertex Cover on cubic graphs (a variant of Vertex Cover denoted by Cubic Vertex Cover). We recall that a graph is cubic when each of its vertex has degree equal to three.

Given an instance $G_c = (V_c, E_c)$ of Cubic Vertex Cover, we build a corresponding temporal graph $G = (V, E, \mathcal{T})$ as follows (a sketch of G is given in Fig. 4). First, the time domain $\mathcal{T} = [1, 2, 3]$.

The vertex set V is defined as follows. For each $v_i \in V_c$ ($1 \le i \le |V_c|$), define the set U_i of vertices:

$$U_i = \{u_{i,1}, u_{i,2}, u_{i,3}, w_{i,a}, w_{i,b}, z_i : v_i \in V_c\}.$$

The set V is then defined as:

$$V = \bigcup_{i=1}^{|V_c|} U_i.$$

The set E consists of three subsets $E_{t,1}$ ($E_{t,2}$, $E_{t,3}$, respectively) representing temporal edges active in timestamp 1 (2, 3, respectively). As in the reduction of Section 3, we assume that the edges of G_c are ordered based on lexicographic order. Since G_c is cubic, based on this order we refer to the edges incident on a vertex $v \in V_c$ as the first (second, third, respectively) edge of v. The set $E_{t,1}$, $E_{t,2}$, $E_{t,3}$ are defined as follows:

$$E_{t,1} = \{\{w_{i,a}, z_i, 1\}, \{w_{i,b}, z_i, 1\} : v_i \in V_c, 1 \le i \le |V_c|\}$$

$$E_{t,2} = \{\{u_{i,1}, w_{i,a}, 2\}, \{u_{i,2}, w_{i,a}, 2\}, \{u_{i,2}, w_{i,b}, 2\}, \{u_{i,3}, w_{i,b}, 2\} : v_i \in V_c, 1 \le i \le |V_c|\}$$

$$E_{t,3} = \{\{w_{i,a}, z_i, 3\}, \{w_{i,b}, z_i, 3\} : v_i \in V_c\} \cup$$

$$\{\{u_{i,x}, u_{j,y}, 3\} : \{v_i, v_j\} \in E_c \text{ and } \{v_i, v_j\} \text{ is the } x\text{-th edge of } v_i$$

$$\text{and the } y\text{-th edge of } v_j, 1 \le i \le |V_c|, 1 \le x, y \le 3\}$$

By construction, G is defined over a time domain consisting of three timestamps. We prove now that G has total degree equal to three and local degree equal to two.

Lemma 6. *Given an instance G_c of Cubic Vertex Cover, let G be the corresponding instance of MinTimelineCover. Then the local degree Δ^L_G of G is equal to 2 and the total degree Δ^T_G of G is equal to three.*

Proof. We start by proving that the local degree Δ^L_G of G is equal to 2. By construction, for the set $E_{t,1}$ of temporal edges, each vertex is incident in two temporal edges (vertex z_i) or one temporal edge (vertices $w_{i,a}$, $w_{i,b}$), for each i with $1 \le i \le |V_c|$. For the set $E_{t,2}$ of temporal edges, each vertex is incident in two temporal edges (vertices $w_{i,a}$, $w_{i,b}$, $u_{i,2}$) or one temporal edge (vertex $u_{i,1}$, $u_{i,3}$), for each $1 \le i \le |V_c|$. For the set $E_{t,3}$ of temporal edges, each vertex is incident in at most two edges (vertex z_i) or one edge (vertex $u_{i,1}$, $u_{i,2}$, $u_{i,3}$, $w_{i,a}$, $w_{i,b}$), for each i with $1 \le i \le |V_c|$.

Now, we consider the total degree Δ^T_G of G. Each vertex z_i, $1 \le i \le |V_c|$, is adjacent to vertices $w_{i,a}$, $w_{i,b}$, hence it has a total degree of two. Vertices $w_{i,a}$, $w_{i,b}$, $1 \le i \le |V_c|$, are adjacent to three vertices in G_u and in G (that is z_i, $u_{i,x}$, $u_{i,y}$, for some $x, y \in \{1, 2, 3\}$). Finally, each vertex $u_{i,x}$, with $1 \le i \le |V_c|$ and $1 \le x \le 3$, is adjacent to at most three vertices in G_u and in G ($w_{i,a}$, $w_{i,b}$ and $u_{j,y}$, for some j with $1 \le j \le |V_c|$, $y \in \{1, 2, 3\}$). □

Next, we prove that we can restrict ourselves to solutions where each vertex z_i, $1 \le i \le |V_c|$, is active only in timestamp 3.

Lemma 7. *Given an instance G_c of Cubic Vertex Cover, let G be the corresponding instance of MinTimelineCover. Then, given a solution $\mathcal{A}$ of MinTimelineCover on instance G, we can compute in polynomial time a solution $\mathcal{A}'$ of MinTimelineCover on instance G such that each vertex z_i, $1 \le i \le |V_c|$, is active only in timestamp 3 and the span of $\mathcal{A}'$ is at most the span of $\mathcal{A}$.*

Proof. Consider the solution $\mathcal{A}$ of MinTimelineCover on instance G. Assume that $\mathcal{A}$ defines z_i, for some i with $1 \le i \le |V_c|$, active in interval $[1, 3]$ (notice that the temporal edges incident in z_i are defined in timestamps 1 and 3). Then the temporal edges $\{z_i, w_{i,a}, 1\}$, $\{z_i, w_{i,b}, 1\}$, $\{z_i, w_{i,a}, 3\}$, $\{z_i, w_{i,b}, 3\}$ are covered by z_i, hence we can assume that $w_{i,a}$, $w_{i,b}$ are active only in timestamp 2. It follows that z_i has a span of 2, while $w_{i,a}$, $w_{i,b}$ have a span of 0. We can define an activity timeline $\mathcal{A}'$ so that $w_{i,a}$, $w_{i,b}$ are both active in interval $[1, 2]$ (each one having span 1) and z_i is active only in timestamp 3 (with span 0), thus the span of $w_{i,a}$, $w_{i,b}$, z_i in $\mathcal{A}'$ is not increased with respect to $\mathcal{A}$.

Assume that z_i, $1 \le i \le |V_c|$, is active only in timestamp 1, it follows that $w_{i,a}$, $w_{i,b}$ must be active in timestamp 3 (and possibly in timestamp 2). Then we can define an activity timeline $\mathcal{A}'$ where z_i is active only in timestamp 3, $w_{i,a}$, $w_{i,b}$ are active in timestamp 1 (and possibly in timestamp 2 if they are defined active in timestamp 2 by $\mathcal{A}$). In this case the span of $w_{i,a}$, $w_{i,b}$, z_i in $\mathcal{A}'$ is not increased with respect to $\mathcal{A}$, thus concluding the proof. □

Now, we show how to relate a solution of Cubic Vertex Cover on G_c and a solution of MinTimelineCover on G.

Lemma 8. *Given an instance G_c of Cubic Vertex Cover, let G be the corresponding instance of MinTimelineCover. Let V'_c be a solution of Cubic Vertex Cover, with $|V'_c| = k$, then there exists a solution of MinTimelineCover on instance G of span at most $|E_c| + k - |V_c|$.*

Proof. Consider a solution V'_c of Cubic Vertex Cover on instance G, we define a solution $\mathcal{A}$ of MinTimelineCover on instance G as follows. For each set U_i, $1 \le i \le |V_c|$, associated with $v_i \in V_c \setminus V'_c$, $\mathcal{A}$ is defined as follows:

- $w_{i,a}$, $w_{i,b}$ are active in interval $[1, 2]$, each one with span 1
- For each $\{v_i, v_j\} \in E$, which is the p-th edge of v_i and the q-th edge of v_j, $u_{i,p}$ is active in timestamp 3 with span 0
- Vertex z_i is active in timestamp 3, with span 0.

For each set U_i, $1 \le i \le |V_c|$, associated with $v_i \in V'_c$, $\mathcal{A}$ is defined as follows:

- Vertices $w_{i,a}$, $w_{i,b}$ are active in timestamp 1, each one with span 0
- Each vertex $u_{i,p}$, $1 \le p \le 3$, is active in timestamp 2 (the span of u_i depends on the next point)
- For each $\{v_i, v_j\} \in E_c$, with $v_i, v_j \in V'_c$, which is the p-th edge of v_i and the q-th edge of v_j: if $i < j$, then $u_{i,p}$ is active in interval $[2, 3]$ (with span 1), else $u_{j,q}$ is active in interval $[2, 3]$ (with span 1).
- Vertex z_i is active in timestamp 3, with span 0.

By construction, $\mathcal{A}$ covers each temporal edge of G. Indeed, for each i with $1 \le i \le |V_c|$, the following properties hold. The temporal edges in $E_{t,1}$ are covered by $w_{i,a}, w_{i,b}$. The temporal edges in $E_{t,2}$ are either covered by $w_{i,a}, w_{i,b}$, if $v_i \in V_c \setminus V'_c$, or by $u_{i,1}$, $u_{i,2}$, $u_{i,3}$, if $v_i \in V'_c$. The temporal edges $\{w_{i,a}, z_i, 3\}$ and $\{w_{i,b}, z_i, 3\}$ of $E_{t,3}$ are covered by z_i; the temporal edges $\{u_{i,x}, u_{j,y}, 3\}$ are covered by one of $u_{i,x}$, $u_{j,y}$.

Now, the span of $\mathcal{A}$ is 1 for each $\{v_i, v_j\} \in E_c$, where $v_i, v_j \in V'_c$. Consider now an edge $\{v_i, v_j\} \in E_c$, where either $v_i \in V_c \setminus V'_c$ or $v_j \in V_c \setminus V'_c$ (notice that both v_i, v_j cannot be in $V_c \setminus V'_c$) assume without loss of generality that $v_i \in V_c \setminus V'_c$. Then $\mathcal{A}$ has a span of 2 for the set U_i (both $w_{i,a}$ and $w_{i,b}$ have a span 1) for the three edges incident in v_i in G_c. Hence the overall span of $\mathcal{A}$ is $|E_c| - |V_c \setminus V'_c| = |E_c| + k - |V_c|$, thus concluding the proof. □

Based on Lemma 7, we can prove the following result.

Lemma 9. *Given an instance G_c of Cubic Vertex Cover let G be the corresponding instance of MinTimelineCover. If there exists a solution of MinTimelineCover on instance G of span $|E_c| + k - |V_c|$, then there exists a solution of Cubic Vertex Cover on instance G_c of size at most k.*

Proof. Let $\mathcal{A}$ be a solution of MinTimelineCover on instance G. First, by Lemma 7, each vertex z_i, with $1 \le i \le |V_c|$, is defined active in $\mathcal{A}$ only in timestamp 3 (with span 0). This implies that $w_{i,a}$, $w_{i,b}$ must be active in timestamps 1.

Consider the vertices $w_{i,a}$ or $w_{i,b}$, with $1 \le i \le |V_c|$, that are active in interval $[1, 2]$. First, assume that exactly one of $w_{i,a}$, $w_{i,b}$ is active in timestamp 2, without loss of generality $w_{i,a}$. Then $w_{i,a}$ has a span 1, and $u_{i,2}$, $u_{i,3}$ must be active in timestamp 2, in order to cover temporal edges $\{u_{i,2}, w_{i,b}\}$, $\{u_{i,3}, w_{i,b}\}$. Now, we can modify the solution $\mathcal{A}$, by defining $u_{i,1}$ active in timestamp 2, thus increasing its span by at most 1, and defining $w_{i,a}$ active only in timestamp 1, thus decreasing the span of $\mathcal{A}$ by 1. We can conclude that we can compute in polynomial time a solution $\mathcal{A}'$ of MinTimelineCover on instance G such that either $w_{i,a}$, $w_{i,b}$ are both covering interval $[1, 2]$ or they are both active only in timestamp 1.

Now, consider the case that $w_{i,a}$, $w_{i,b}$ are both active in interval $[1, 2]$, since all the temporal edges defined in timestamp 2 that are incident in vertices $u_{i,1}$, $u_{i,2}$, $u_{i,3}$ are covered by $w_{i,a}$ and $w_{i,b}$, we can assume that $u_{i,1}$, $u_{i,2}$, $u_{i,3}$ are active in timestamp 3 (each one with span 0).

Now, assume that, for some v_i, v_j with $\{v_i, v_j\} \in E_c$, $w_{i,a}$, $w_{i,a}$, $w_{j,a}$, $w_{j,b}$ are all active in interval $[1, 2]$. Consider the temporal edge $\{u_{i,p}, u_{j,q}, 3\}$ and notice that it is covered by both $u_{i,p}$ and $u_{j,q}$. Now, we modify $\mathcal{A}$ by making $w_{j,a}$, $w_{j,b}$ active only in timestamp 1 (thus decreasing the span of each of $w_{j,a}$, $w_{j,b}$ by 1), by making $u_{j,p}$ active in timestamp 2 (with span of 0) and by making the other two vertices of U_j, active in timestamp 2 and possibly 3 (thus increasing the span of U_j by at most 2). Notice that the span of $\mathcal{A}$ is not increased. Thus we obtain a solution of MinTimelineCover on G such that, for each $\{v_i, v_j\} \in E_c$, if $w_{i,a}$, $w_{i,b}$ are active in timestamps 1 and 2, then $w_{j,a}$, $w_{j,b}$ are active only in timestamp 1.

Now, we construct a vertex cover V'_c of G_c as follows:

- For each i with $1 \le i \le |V_c|$, if $w_{i,a}$, $w_{i,a}$ are active only in timestamp 1, define $v_i \in V'_c$

By construction, for each $\{v_i, v_j\} \in E_c$, at most one of the sets $\{w_{i,a}, w_{i,b}\}$ $\{w_{j,a}, w_{j,b}\}$ is active in timestamp $[1, 2]$. It follows that V'_c covers each edge in E. Now, we claim that $|V'_c| \le k$. Indeed, each temporal edge $\{u_{i,p}, u_{j,q}, t\}$ is covered by one of $u_{i,p}$, $u_{j,q}$ with span equal to 1, except when $w_{i,a}$, $w_{i,b}$ are both active in timestamps 1 and 2. In this case, three edges are covered by $w_{i,a}$, $w_{i,b}$ with a span equal to one for each of these vertices. Since the solution $\mathcal{A}$ has a cost of $|E_c| + k - |V_c|$, it follows that there exist $|V_c| - k$ sets $\{w_{i,a}, w_{i,b}\}$ active in interval $[1, 2]$, thus by construction $|V'_c| \le |V_c| - (|V_c| - k) = k$. □

Now, we can prove the main result of this section.

Theorem 2. *MinTimelineCover is NP-hard (and the decision version of MinTimelineCover is NP-complete) when the total degree Δ^T_G is equal to 3, the local degree Δ^L_G is equal to 2 and the time domain consists of three timestamps.*

Proof. The reduction from Cubic Vertex Cover defines a temporal graph G on a time domain of three timestamps and such that, by Lemma 6, its total degree Δ^T_G is equal to 3 and its local degree Δ^L_G is equal to 2.

By Lemma 8 and Lemma 9, it follows that we have designed a polynomial-time reduction from Cubic Vertex Cover to MinTimelineCover. Since Cubic Vertex Cover is NP-hard [3], it follows that also MinTimelineCover is NP-hard when the total degree Δ^T_G is equal to 3, the local degree Δ^L_G is equal to 2 and the time domain consists of three timestamps.

Notice that the decision version of MinTimelineCover is in NP, since giving a solution of MinTimelineCover we can check in polynomial time that it covers G and that has at most a given span. □

5. Bounding the time window

In this section we consider the parameterized complexity of MinTimelineCover for (w, h)-window constrained temporal graphs, when the parameters is the product of w and h (the size of the time window). Notice that when exactly one of w, h is the parameter, the problem is not in the class XP (in particular if $h = 2$ for the result in Section 3, if $w = 2$ for the hardness of MinTimelineCover on a time domain of two timestamps [10]).

We denote by W_j a time window $W([j - w + 1, j])$ of length w that ends in a timestamp j and that consists of the set of pairs (v, t) such that $v \in V$ has a temporal edge defined in some timestamp t, with $j - w + 1 \le t \le j$.

An *activity assignment* F_j for a time window W_j is a function that establishes, for each pair (v, t) of W_j, if v is active in timestamp t. Formally, an *activity assignment* F_j is a function

$$F_j : W_j \to \{0, 1\}$$

such that, for each pair (v, t), it holds that:

1. v is active in timestamp t if and only if $F_j(v,t)=1$ (thus v is not active in timestamp t if and only if $F_j(v,t)=0$)
2. if $F_j(v,t_1)=1$ and $F_j(v,t_2)=1$, with $j-w+1 \le t_1 \le t_2 \le j$, then $F_j(v,t)=1$ for each t with $t_1 \le t \le t_2$.

The span of F_j, denoted by $s(F_j)$, is the span induced by the activity assignment F_j.

Consider two time windows W_j and W_i, with $1 \le i-w+1 \le i < j \le t_{max}$, and two assignment functions F_j and F_i. F_j and F_i are in *agreement* if the following holds:

$$F_j(v,t) = F_i(v,t) \text{ for each } (v,t) \in W_j \cap W_i.$$

An activity timeline $\mathcal{A}$ is in agreement with an activity assignment F_j if the activity defined by $\mathcal{A}$ for the pairs in W_j is identical to F_j.

Next, we describe a dynamic programming algorithm to compute a solution of MinTimelineCover parameterized by w and h. Given two assignment functions F_j and F_{j-1} that are in agreement, define the value $D(F_j,F_{j-1})$ as the span added by F_j (in timestamp j) with respect to F_{j-1}. Formally, $D(F_j,F_{j-1})$ is defined as follows:

$$D(F_j,F_{j-1}) = |\{v : (v,j-1) \in W_{j-1} \wedge (v,j) \in W_j \wedge F_j(v,j) = F_{j-1}(v,j) = 1\}| \quad (3)$$

Given an activity assignment F_j of a time window W_j, define the function $C[F_j]$ as the minimum span of an activity timeline of the temporal graph G on interval $[1,j]$, such that:

1. The activity of vertices in the time window W_j is defined by F_j
2. Each temporal edge $\{u,v,t\}$ of G, with $1 \le t \le j$, is covered
3. Each vertex active in timestamp j is not active in interval $[1,j-w]$ (since G is (w,h)-window constrained)

Now, $C[F_j]$ is computed with the following recurrence:

– If $j > w$, then $C[F_j]$ is the minimum, over F_{j-1} in agreement with F_j, of

$$C[F_{j-1}] + D(F_j,F_{j-1})$$

– If $j = w$, $C[F_w] = s(F_w)$, that is the span of F_w.

Next, we prove the correctness of the recurrence.

Lemma 10. *$C[F_j]=q$, if and only if there exists an activity timeline that covers G in interval $[1,j]$ and that has span q.*

Proof. We prove the lemma by induction on j. In the base cases, when $j=w$, if $C[F_w]=q$, then $C[F_w]=s(F_w)$ and F_w defines an activity timeline that covers G on interval $[1,w]$ of span q. Given an activity timeline $\mathcal{A}$ that covers G on interval $[1,w]$ and has span q, then by considering the activity assignment F_w that is in agreement with $\mathcal{A}$, it follows that $C[F_w]=s(F_w)=q$.

Now, we prove that the lemma holds for $j>w$, assuming by induction hypothesis that it holds for $j-1$. Consider an activity timeline $\mathcal{A}$ that covers G on interval $[1,j]$ such that $\mathcal{A}$ is an activity timeline of minimum span q in agreement with F_j. We prove that $C[F_j]=q$. Since $\mathcal{A}$ is an activity timeline of G on interval $[1,j]$, it follows that it covers also the temporal edges active in $[1,j-1]$, with span $q' \le q$, and that $\mathcal{A}$ defines an activity of vertices in $[j-w,j-1]$ defined by an assignment function F_{j-1} that is in agreement with F_j. Then, by induction hypothesis $C[F_{j-1}]=q'$ and by construction $D(F_j,F_{j-1})=q-q'$. By the definition of recurrence, since F_{j-1} and F_j agree and $\mathcal{A}$ is an activity timeline of minimum span in agreement with F_j, it follows that $C[F_j]=q$.

Now, assume that $C[F_j]=q$. By definition of the recurrence, it follows that there exists an activity assignment F_{j-1} such that F_{j-1} and F_j are in agreement and $C[F_{j-1}]=q'$, with $q' \le q$ and $q-q'=D(F_j,F_{j-1})$. Since $C[F_{j-1}]=q'$, by induction hypothesis there exists an activity timeline $\mathcal{A}'$ that covers G on interval $[1,j-1]$ such that (1) $\mathcal{A}'$ has span q' and (2) the activity timeline of vertices in $[j-w,j-1]$ specified by $\mathcal{A}'$ agrees with F_{j-1}. Now, by definition of $C[F_j]$, it follows that by adding the activity specified by F_j in timestamp j to $\mathcal{A}'$, we obtain an activity timeline $\mathcal{A}$ of G on interval $[1,j]$ that covers each temporal edge of G defined in interval $[1,j]$. Notice that $\mathcal{A}$ defines for each vertex one covering interval. Indeed, this holds by induction for $\mathcal{A}'$. Now, if a vertex v is defined active by F_j in position j, then since F_j and F_{j-1} are in agreement, either v is defined active also by F_{j-1} in position $j-1$, or v is defined not active by F_{j-1} in all positions i with $j-w \le i \le j-1$ and thus also in each positions in $[1,j-1]$, since there is no temporal edge incident in v defined in interval $[1,j-w-1]$. Finally, $\mathcal{A}$ has span q, since $D(F_j,F_{j-1})=q-q'$, thus concluding the proof. □

Based on Lemma 10, we can prove the main result of this section.

Theorem 3. *A solution of MinTimelineCover on instance G can be computed in $O(2^{h(w+1)} h\, t_{max})$ time.*

Proof. The correctness of the recurrence follows from Lemma 10, thus the span of an optimal solution of MinTimelineCover on instance G is computed in entry $C[F_{|\mathcal{T}|,w}]$.

Next, we consider the time complexity of the algorithm. The base case $C[F_w]$ can be computed in $O(2^{hw})$ time, as there exist at most h vertices with active temporal edges in each timestamp of interval $[1, w]$. Now, we consider the case $j > w$. First, notice that there exist $O(h\ t_{max})$ values of j and w. For each j and w, there exists at most 2^{hw} activity assignments F_j. Each entry $C[F_j]$ is computed starting from the values $C[F_{j-1}]$ (where F_j and F_{j-1} are in agreement) This requires the definition of the activity timeline of vertices in timestamp j (recall that at most h vertices have temporal edges in each timestamp) in $O(2^h)$ time and, for each of them, the computation of $D(F_j, F_{j-1})$, which requires $O(h)$ time. Hence the overall time complexity of the dynamic programming algorithm is $O(2^{h(w+1)}h\ t_{max})$. □

6. Conclusion

We have considered a variant of Vertex Cover, called MinTimelineCover, on temporal graphs that has been recently introduced to deal with event summarization. We have shown that the problem is NP-hard even when: (1) the local degree is bounded by one and (2) the total degree is three, the local degree is two and the time domain consists of three timestamps. On the other hand, we have shown that MinTimelineCover is in P for total degree equal to 2. Moreover, we have shown that MinTimelineCover is fixed-parameter tractable when the size of the time window is the parameter.

There are several interesting future directions related to MinTimelineCover. First, the parameterized complexity of MinTimelineCover is open when the problem is parameterized by the span of the solution (recall that in this case the problem is fixed-parameter tractable when the time domain consists of two timestamps [10]). It would be interesting to study the complexity of the problem when the local degree is equal to one and the global degree is bounded by a constant and the approximation complexity of the problem. Moreover, it would be interesting to study the complexity of deciding whether there exists an activity timeline that covers the set of temporal edges when every vertex has a bounded span.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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