Computers & Operations Research 164 (2024) 106519

Contents lists available at ScienceDirect

Computers and Operations Research

journal homepage: www.elsevier.com/locate/cor

A two-level Hierarchical Nested Cooperative Location model

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ARTICLE INFO

Keywords:
Facility location
Nested hierarchy
Cooperative covering
Joint coverage

ABSTRACT

This paper introduces the two-level Hierarchical Nested Cooperative Location (HNCL) model, extending the literature on discrete cooperative covering problems. We analyse the problem of locating facilities arranged in a two-level nested hierarchy and cooperating to maximize the covered demand. Facilities are characterized by different coverage decay functions according to the hierarchical level they belong to (upper and lower). Cooperation occurs between facilities at the same level (*intra-level cooperation*) and at different levels (*inter-level cooperation*) of the hierarchy according to mechanisms modelled through joint coverage functions. Two budget constraints are introduced for the total cost the decision-maker is willing to incur for locating facilities at each level. The HNCL problem is first formulated as a mixed-integer non-linear programming (MINLP) model; then, an equivalent mixed-integer linear program (MILP) is developed. We tested the HNCL model on instances randomly generated and extracted from a real-world network. The results are provided and discussed, highlighting the contribution of cooperative coverage compared to the individual one.

1. Introduction

Demand covering problems, initially introduced by Toregas et al. (1971), constitute a major class of problems in facility location analysis, with numerous applications in a wide range of domains. Their essence is to determine appropriate locations for a set of facilities that must offer some kind of services to a set of customers.

The wide applicability of covering models stimulated strong interest in the research community, with numerous publications over the last 50 years. According to Berman et al. (2010), there are three main assumptions in the early approaches to covering problems. Firstly, under the *binary (all or nothing) coverage* assumption, they assume that a customer is either fully covered if the distance to the nearest operating facility does not exceed a predetermined limit, referred to as the covering radius, or not covered at all if the nearest facility is beyond that limit. Moreover, the coverage of each customer is called *individual* since it depends only on the nearest established facility. In other words, demand sharing is not allowed among multiple facilities. Lastly, the covering radius is assumed to be a parameter under the direct control of the decision-maker (*fixed radius*).

As the range of demand covering applications expanded over the years, it was recognized that these assumptions were somewhat unrealistic and that the original models could not adequately represent realistic situations.

The first generalization concerned the concept of *gradual or partial* coverage, introduced by Church and Roberts (1983) to model situations where a customer is fully covered up to a certain distance from a facility, but the coverage decreases according to a decay function beyond that distance.

The second generalization was introduced by Berman et al. (2009a, 2011) to model situations where the individual coverage assumption may not be valid and *cooperation* among facilities is allowed. In this setting, it is assumed that each facility provides a coverage whose strength decays over distance according to some *coverage decay function* and the individual coverages are somehow aggregated, through a specified *aggregate coverage function*. In this case, a demand node is considered covered when the aggregated coverage exceeds a certain acceptable threshold.

Finally, the third generalization concerns the possibility of considering the *covering radius as a decision variable* of the problem instead of a fixed parameter (Berman et al., 2009b).

Here, we focus on the literature stream related to cooperative location problems. In this context, several applications have been investigated. Typical examples dealt with the provision of physical signals such as light, sound, and warning sirens. Herein, each target point receives the sum of all of the signals instead of just the strongest one (Berman et al., 2009a). However, the literature has also explored the problem of designing service networks where the coverage is related to the possibility of serving demand distributed over a study region and the contextual presence of several facilities increases the effectiveness

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https://doi.org/10.1016/j.cor.2023.106519
Received 31 December 2022; Received in revised form 14 November 2023; Accepted 19 December 2023
Available online 26 December 2023

of the service itself. Proposed applications concern the design of fire services (Wang et al., 2016), evacuation systems (Zhang et al., 2017), and relief trains (Bababeik et al., 2018). In some cases, cooperation also occurs among multiple types of facilities characterized by different features (Li et al., 2018) or facilities arranged in a hierarchical structure (Karatas, 2020).

In this work, we analyse the problem of multiple types of facilities arranged in a nested hierarchical structure and cooperating to cover the demand. Specifically, we propose the two-level Hierarchical Nested Cooperative Location (HNCL) model. The hierarchy is nested (or successively inclusive) in the sense that facilities of the upper level offer all services available at facilities of the lower level, plus at least one additional service (Narula, 1984). According to such definition, two sets of services are defined: (i) *exclusive services*, provided solely by upper-level facilities, and (ii) *shared services*, provided by facilities of both hierarchical levels.

As regards the exclusive services, the model intends to guarantee the coverage of all the demand according to the traditional coverage paradigm (binary and individual). This means that each demand point needs to find at least a facility within a fixed radius. Instead, as concerns the shared services, cooperation is allowed. In particular, we consider two distinct coverage decay functions, each associated with a hierarchical level representing the probability of coverage by facilities. Cooperation may occur between facilities at the same level (*intra-level cooperation*) and at different levels (*inter-level cooperation*).

The aim is to design a network by locating facilities at each hierarchical level to cover all the demand for exclusive services and maximize the covered demand for shared services. We assume a specific available budget for the installation of facilities of each level.

The proposed HNCL problem is first formulated as a Mixed Integer Non-Linear Programming (MINLP) model. Then, a linearizing approach is developed to provide an equivalent Mixed Integer Linear Programming (MILP) formulation.

We perform extensive computational experiments to compare the performance of MINLP and MILP formulations and investigate the effect of certain parameters on the optimal solution to instances of various sizes. The results are analysed, highlighting the contributions of the coverage mechanisms.

The aim of this paper is threefold. First, The HNCL aims to extend the available literature on cooperative problems. Indeed, to the best of our knowledge, no hierarchical cooperative problems have been previously introduced considering a nested hierarchy. Therefore, inter-level cooperation has never been taken into account.

A second contribution is given by the linearization method developed to provide an equivalent MILP formulation of the HNCL.

Finally, a further novelty is the proposal of a classification framework to measure the contribution of cooperation to the coverage, representing a valuable tool to analyse the scenarios produced by the model by varying crucial parameters.

This work may also open to many research perspectives as the problem can be effectively used to represent many real-world applications, especially in the logistics, retail and healthcare sectors.

The rest of the paper is organized as follows. In Section 2, we analyse the cooperative location literature by highlighting the gaps our work aims to bridge. In Section 3, we formulate the proposed HNCL model as a non-linear programming model and introduce a process for transforming it into an equivalent linear formulation. We then explain our experimentation plan in Section 4 and report our findings on a set of real and randomly generated instances in Section 5. Some managerial implications are then drawn in Section 6. Finally, in Section 7, we summarize our work in the concluding section and suggest some possibilities for future research.

2. Literature review

Cooperative location problems in a discrete location space were formally introduced by Berman et al. in 2011 and have received increasing attention in the following years, leading to a worthy literature body.

The majority of papers contributing to this class of problems consider a single type of facility (S) (see, e.g., Zhang et al., 2017; Davari, 2019; Drezner and Drezner, 2019; Wang et al., 2021).

Those considering multiple types of facilities (M) (e.g., Wang et al., 2016; Li et al., 2018; Karatas, 2018) generally considered distinct coverage decay functions depending on facilities' features (e.g., size, capacity, attributes) or provided services. For instance, Karatas and Eriskin (2021) analyzed the minimal covering location problem of undesirable capacitated facilities by assuming gradual and cooperative coverage. Facilities differ on the basis of their size, and the related coverage decay functions depend on such a parameter. Ceteris paribus, as the size grows, the coverage increases.

In some cases, the different types of facilities may be arranged in a hierarchical structure, which could be nested (Ne) or non-nested (NNe). Karatas and Dasci (2020) proposed a two-level non-nested hierarchical location and sizing problem for maximizing expected demand coverage while satisfying a budget constraint. Facilities of the upper and lower levels provide disjoint sets of services, and cooperation occurs only among lower-level facilities. A non-nested hierarchy is also considered by Karatas (2020) to address the two-level location problem for heterogeneous sensor networks in a cooperative setting. In particular, the hierarchy is modelled as a hub and spoke topology where spokes cooperate to provide a certain level of coverage to critical facilities that need protection against intruders. Hubs, on the other hand, serve as communication centres and collect data from assigned spokes.

In a cooperative framework, the individual coverages have been modelled through various decay functions, many of which borrowed from gradual covering literature: inverse quadratic (e.g., Karabulut et al., 2017), trapezoidal-shaped (e.g., Bababeik et al., 2018), linear (e.g., Berman et al., 2019), sigmoid (e.g., Ming et al., 2021), Fermi-type model (e.g., Karatas and Eriskin, 2021), exponential (e.g., Davari, 2019). Such coverages may assume different meanings. They may represent the *intensities* of physical signals (e.g., light or sound) dissipating over the distance according to given physical laws (Berman et al., 2011), or *probabilities* when they model the spatial interaction among demand and facilities (Zhang et al., 2017).

The mechanism ruling the aggregation of single coverages depends on the nature of the coverages themselves. When dealing with intensities, the aggregation mechanism is typically deterministic (D), and the overall coverage is obtained as the sum of individual coverages (see, e.g, Ashtiani et al., 2018; Ming et al., 2021). In contrast, when dealing with probabilities, the aggregation mechanism is probabilistic (P) and differently computed if the coverage events are assumed independent (I) or somehow correlated (C). On the one hand, when the events are perfectly correlated, the demand perceives only the maximum coverage among those by located facilities. Contrariwise, if the events are independent, all located facilities contribute to a demand node coverage, which is obtained through a function of the product of the single coverages. The case of partial correlation has been modelled through the (non-linear) *joint coverage function* by Drezner and Drezner (2014) and Berman et al. (2019), defined as a convex combination of the two extreme cases. The weight θ rules the combination and represents a correlation degree among coverage events. By varying θ, it is possible to capture the whole spectrum of dependencies.

The aggregate coverage function represents one of the peculiar features of the cooperative mechanism and its mathematical definition deeply affects the whole formulation of the problem. In a deterministic framework, the model is typically linear (L) in contrast to the probabilistic one, where the aggregate function leads to the insurgence of non-linearities (NL). Some approaches have been developed to linearize them. For instance, Karatas (2017) developed an approach to linearize the aggregate coverage function defined under conditions of independent coverage events. Álvarez-Miranda and Sinnl (2019) exploited the

Table 1
Positioning HNCL in cooperative literature.

Paper	Facility type	Hierarchy	Cooperative mechanism	Coverage events	Model formulation	Additional relaxations	Solution approach
Álvarez-Miranda and Sinnl (2019)	S	–	P	C	NL, L	G	E
Ashtiani et al. (2018)	M	–	D	–	L	–	–
Bababeik et al. (2018)	S	–	D	–	L	–	E
Bagherinejad et al. (2018)	S	–	D	–	NL	G	H
Berman et al. (2009a)	S	–	D	–	L	–	E, H
Berman et al. (2011)	S	–	D	–	L	–	H
Berman et al. (2019)	S	–	P	C	NL	G	H
Davari (2019)	S	–	D	–	NL, L	–	H
Drezner and Drezner (2014)	S	–	P	C	NL	G	H
Drezner and Drezner (2019)	S	–	D	–	NL	G	H
Karabulut et al. (2017)	M	–	D, P	I	NL, L	G	H
Karatas (2017)	S	–	P	I	NL, L	G, V	E
Karatas (2018)	M	–	P	I	NL, L	–	H
Karatas (2020)	M	Nn	P	I	NL, L	–	–
Karatas and Dasci (2020)	M	Nn	P	I	NL, L	G, V	–
Karatas and Eriskin (2021)	M	–	P	I	NL, L	G, V	–
Li et al. (2018)	M	–	D	–	L	–	–
Ming et al. (2021)	M	–	D	–	L	G	–
Wang et al. (2016)	M	–	D	–	NL	G	–
Wang et al. (2021)	S	–	D	–	L	–	–
Zhang et al. (2017)	S	–	P	I	NL	–	H
HNCL	M	Ne	P	C	NL, L	–	–

The features are: Facility type (S: Single, M: Multiple), Hierarchy (Ne: Nested, Nn: Non-Nested, –: Not present), Cooperative mechanism (D: Deterministic, P: Probabilistic), Coverage events (I: based on Independent events, C: based on possibly Correlated events, –: Not applicable), Model formulation (L: Linear, NL: Non-linear), Additional relaxations (G: Gradual coverage, V: Variable radius, –: None), Solution approach (E: Exact, H: Heuristic and Metaheuristic, –: Not presented).

submodularity features of the joint coverage function to linearize it in a single facility type setting.

In some models, the cooperative assumption is jointly considered with the other relaxations mentioned in Section 1, which are gradual coverage (G) and variable coverage radius (V). Bagherinejad et al. (2018) proposed a non-linear maximal covering location problem, assuming gradual and cooperative coverage.

As regards solution approaches, various heuristics and metaheuristics (H) as well as exact methods (E) have been developed. For instance, Bababeik et al. (2018) proposed a combined e-constraint method (AUGMECON) with a fuzzy-logic approach for solving a bi-objective model related to the problem of locating relief trains. Karabulut et al. (2017) propose three tabu search heuristics for solving the optimal cooperating sensor deployment problem.

In order to compare the HNCL with existing cooperative models, we summarized the main features of the cited papers in Table 1. It can be noticed that no previous studies consider multiple facilities that (i) are arranged in a nested hierarchy and (ii) cooperate (also among levels) to cover the demand while (iii) assuming possibly correlated coverage events.

The HNCL problem proposed in this work aims to bridge this gap by considering two-level nested hierarchical facilities where cooperation is not only between facilities at the same level (intra-level cooperation) but also at different levels (inter-level cooperation) of the hierarchy. Such cooperations are (non-linearly) modelled through three different joint coverage functions.

It is worth highlighting that inter-level cooperation implies further non-linearities in the model formulation compared to those already investigated in the literature. Therefore, another contribution of the present paper concerns the development of a linearizing approach for both intra-level and inter-level cooperations, and, hence, the extension of the linearization proposed by Karatas (2017) to provide an equivalent MILP of the HNCL.

In the following section, we present the HNCL problem, its MINLP formulation and the development of the equivalent MILP.

3. The Hierarchical Nested Cooperative Location (HNCL) model

In this section, we introduce the two-level Hierarchical Nested Cooperative Location (HNCL) model. We first define the problem framework an its main assumptions; then, in Section 3.1 we provide the MINLP formulation of the problem while the equivalent MILP formulation is presented in Section 3.2.

We assume a two-level nested hierarchy involving upper and lower-level facilities and two distinct sets of services. Specifically, exclusive services are complex services provided solely by upper-level facilities, and customers may be willing to travel a relatively long distance to access them. Whilst, shared services, provided by facilities of both hierarchical levels, are basic services that customers are expected to find within short distances.

While the coverage mechanism for exclusive services is binary and individual, in the sense that a demand node is covered if it finds at least one upper-level facility within a predefined (large) radius R, the cooperation may occur among facilities at the same level (intra-level cooperation) and at different levels (inter-level cooperation) to cover the demand for shared services.

In order to provide a clear picture of the HNCL problem setting, in Fig. 1, we compare different versions of hierarchical covering problems by referring to an example with 7 demand nodes and 5 facilities. In particular, we classify the problem according to (a) the typology of hierarchy (nested vs non-nested) and (b) the coverage mechanism for low-level services (individual vs cooperative). By combining the criteria, we consider four problem settings:

1. The *two-level hierarchical non-nested problem* (Fig. 1(a)), where facilities at the two levels provide disjoint sets of services and are able to cover the demand within different covering radii (depicted in blue and red for facilities at the lower and upper levels, respectively).
2. The *two-level hierarchical nested problem* (Fig. 1(b)), where facilities at the upper level provide, besides exclusive services, those provided by the lower level (shared services). Compared to Fig. 1(a), in Fig. 1(b), a different symbol has been used to represent upper-level facilities. Moreover, the latter are characterized by two radii, the blue one for the shared services and the red one for exclusive ones. Here, a demand node can be covered for shared services by an upper-level facility. See node n2 that in the previous case remained uncovered.
3. The *two-level hierarchical non-nested cooperative problem* (Fig. 1(c)). Lower-level facilities provide decay coverages (as shown in the figure through the gradient colour around them) that can be

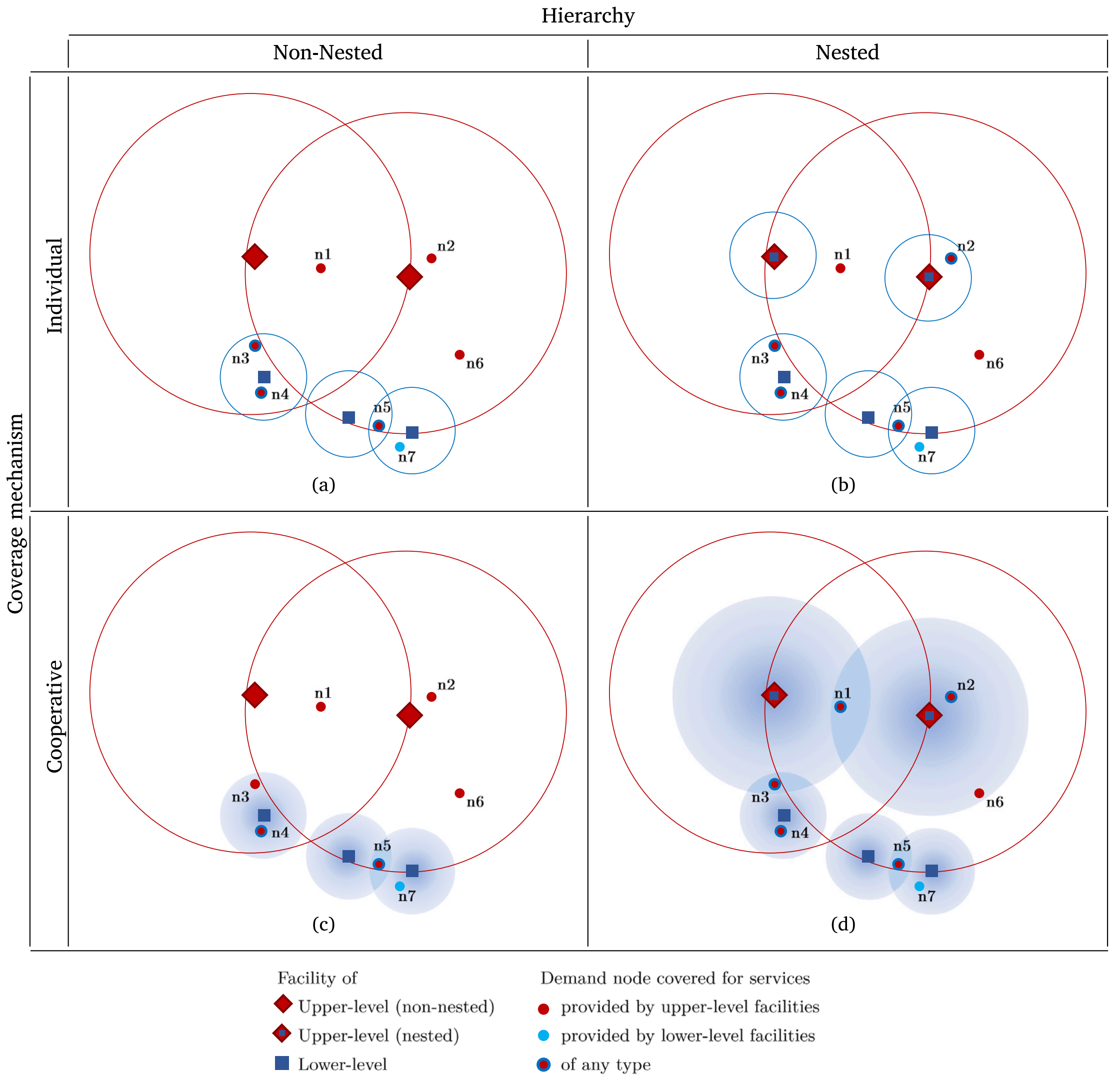


Fig. 1. Examples of coverage under different assumptions in a two-level hierarchy: non-nested vs nested hierarchy, classical coverage vs cooperative coverage.

aggregated to cover the demand. On the contrary, upper-level facilities do not contribute to the provision of low-level services. In this case, the node n5 is covered thanks to cooperation.

4. The *two-level HNCL* (our problem) that is nested and cooperative (Fig. 1(d)). The main novelty is that the coverage for shared services can be provided thanks to cooperation among facilities at the same level (lower-lower or upper-upper) and at different levels (lower-upper). See node n1 covered by cooperation between the two upper-level facilities and n3 by cooperation between one upper and one lower-level facility.

This last case has not been explored in the literature, as shown in Section 2, and is the focus of the work. Given this rough picture of the problem, let us formalize it.

Let I denote the set of demand nodes, $K = \{U, L\}$ the set of facility hierarchical levels – where U and L define the upper and the lower ones, respectively – and J^k the set of candidate sites for facilities of level $k \in K$.

We introduce two distinct *coverage decay functions* ϕ_{ij}^k for shared services, each associated with a hierarchical level $k \in K$, representing the coverage level provided by a k-level facility $j \in J^k$ to a demand node $i \in I$. The functions ϕ_{ij}^k are left-continuous and non-increasing over the distance d_{ij} from i to j ($\phi_{ij}^k = f_k(d_{ij})$), and explain the coverage probability ($0 \leq \phi_{ij}^k \leq 1$). We assume that, for each given distance, the coverage level of an upper-level facility is higher than that provided by a lower-level one.

All located facilities contribute to the aggregate coverage of each demand node for shared services. We use the joint coverage function (Drezner and Drezner, 2014) both for aggregating coverages from the same level and from different levels. Let us remind that the joint coverage is defined as a convex combination of two contributions: the first depicting the case of a *perfect dependency* in which the customers perceive only the maximum coverage, and the second one representing the case of *perfect independence* in which a function of the product of the single coverages gives the aggregate coverage. By varying the weights of the two contributions, it is possible to capture the whole spectrum of event dependencies. In our problem, we introduce three different aggregate coverage functions associated with intra-lower, intra-upper and inter-level cooperations:

(i) *Intra-level aggregate coverages* Φ_i^k. Each of these functions is associated with a hierarchical level $k \in K$ and combines the coverage levels provided by located facilities of level k to demand node $i \in I$:

$$\Phi_i^k = \theta^k \max_{j \in J^k} \left\{ \phi_{ij}^k x_j^k \right\} + \left(1 - \theta^k\right)\left(1 - Q_i^k\right) \quad \forall i \in I, k \in K \tag{1}$$

where x_j^k is equal to 1 if a facility of level k is located at candidate site $j \in J^k$ and 0 otherwise, and Q_i^k represents the overall non-coverage probability of a demand node i by all located facilities of level k under the assumption of independent coverage events, i.e. $Q_i^k = \prod_{j \in J^k} \left(1 - \phi_{ij}^k x_j^k\right)$. The weight θ^k depicts the correlation degree between coverages from facilities of level k. If $\theta^k = 1$, the coverage of i is given by the maximum coverage probability among located facilities (perfect dependence of coverage events). In contrast, if $\theta^k = 0$, the cooperation is the maximum possible, being the coverage events independent. Therefore, the degree of cooperation among facilities of level k may be controlled by setting the parameter θ^k depending on the specific application at hand;

(ii) *Inter-level aggregate coverage* Φ_i. This function combines the coverage level jointly from facilities of different hierarchical level to demand node $i \in I$:

$$\begin{aligned}\Phi_i &= \theta \max\left\{\Phi_i^U;\Phi_i^L\right\} + (1-\theta)\left[1 - \left(1-\Phi_i^U\right)\left(1-\Phi_i^L\right)\right] \\ &= \theta \max\left\{\Phi_i^U;\Phi_i^L\right\} + (1-\theta)\left(\Phi_i^U + \Phi_i^L - \Phi_i^U\Phi_i^L\right) \quad \forall i \in I \end{aligned} \tag{2}$$

Similarly, the parameter θ regulates the cooperation degree among facilities at distinct hierarchical levels. Therefore, $\theta = 1$ means that the coverage of demand node i is provided by facilities of that hierarchical level providing the maximum; conversely, the cooperation among facilities at different hierarchical levels is maximum if $\theta = 0$.

We assume that a demand node $i \in I$ is covered if the global inter-level aggregate coverage Φ_i exceeds a certain *threshold* T (binary coverage assumption, Berman et al., 2010).

The HNCL model aims to locate two types of facilities arranged in a nested hierarchy such that the demand covered for shared services is maximized whilst the coverage for exclusive services is guaranteed to each customer within a (large) fixed covering radius R. We introduce two budgets, each available to locate facilities at a specific level. Thereby, a decision-maker implementing the HNCL model may calibrate budgets depending on the investment they are willing to undertake for locating facilities of a specific hierarchical level characterized by different features.

3.1. HNCL: a mixed-integer non-linear programming (MINLP) formulation

In order to formulate the two-level HNCL model, we introduced the following notation:

Sets

I	Set of demand nodes;
K	Set of hierarchical levels. $K = \{U, L\}$ where U and L denote the upper and the lower levels, respectively;
J^k	Set of candidate sites of facilities of level $k \in K$;

Parameters

w_i	Demand associated with node $i \in I$;
c_j^k	Cost for locating a facility of level $k \in K$ at location $j \in J^k$;
B^k	Budget available for locating facilities of level $k \in K$;
d_{ij}	Distance between demand node $i \in I$ and candidate facility location $j \in \bigcup_{k\in K} J^k$;
R	Covering radius for exclusive services, solely provided by upper-level facilities;
ϕ_{ij}^k	Coverage level for shared services provided to demand node $i \in I$ by facility of level $k \in K$ located at $j \in J^k$;
θ^k	Intra-level correlation factor depicting the dependency among facilities at level $k \in K$;
θ	Inter-level correlation factor depicting the dependency among facilities at different levels;
T	The threshold for coverage for shared services.

Covering sets

N_i^U	Set of candidate sites of upper-level facilities $j \in J^U$ that are within the distance R from the demand node $i \in I$, i.e. $N_i^U = \{j \in J^U : d_{ij} \le R\}$.

Decision variables

x_j^k	Binary decision variable equal to 1 if a facility of level $k \in K$ is located at $j \in J^k$;
y_i	Binary decision variable equal to 1 if demand node $i \in I$ is covered for the shared services;
Φ_i^k	Intra-level aggregate coverage provided to node $i \in I$ by facilities located at the same level $k \in K$;
Φ_i	Inter-level Aggregate coverage provided to node $i \in I$ by facilities located at any level.

With this notation, we can formulate the two-level HNCL model as follows:

$$\max \sum_{i \in I} w_i y_i \tag{3a}$$

$$\text{s.t.} \sum_{j \in N_i^U} x_j^U \ge 1 \qquad \forall i \in I \tag{3b}$$

(1),(2)

$$\Phi_i \ge T y_i \qquad \forall i \in I \tag{3c}$$

$$\sum_{j \in J^k} c_j^k x_j^k \le B^k \qquad \forall k \in K \tag{3d}$$

$$x_j^k \in \{0,1\} \qquad \forall k \in K, j \in J^k \tag{3e}$$

$$y_i \in \{0,1\} \qquad \forall i \in I \tag{3f}$$

$$\Phi_i^U, \Phi_i^L, \Phi_i \ge 0 \qquad \forall i \in I \tag{3g}$$

The objective function (3a) maximizes the total covered demand for shared services provided by facilities of both upper and lower levels. Constraints (3b) guarantee that each demand node is covered within the distance R by at least one located upper-level facility. Constraints (1) functionally define the intra-level aggregate coverage from facilities located at the same level k for each demand node $i \in I$. The global (inter-level) coverage of each demand node $i \in I$ from all located facilities is defined by constraints (2). Constraints (3c) guarantee that a demand node is covered for shared services if the global coverage level exceeds the threshold T. The maximum budget to spend for locating facilities of each hierarchical level $k \in K$ is assigned by constraints (3d). Finally, constraints (3e)–(3g) define the nature of decision variables.

3.2. HNCL: a mixed-integer linear programming (MILP) formulation

In this section, we develop a linear reformulation of the MINLP (3) presented in Section 3.1. The non-linearities occur both in aggregating coverages at the same hierarchical level (constraints (1)) and in aggregating coverages among the hierarchical levels (constraints (2)). We first describe our linearizing approach; then, the equivalent MILP reformulation is provided in Section 3.2.1.

The max-parts in (1) and (2) are linearized by introducing the following auxiliary decision variables:

z_{ij}^k	Binary decision variable equal to 1 if the maximum coverage level at demand node i from facilities of level $k \in K$ is achieved at $j \in J^k$ (and is equal to ϕ_{ij}^k);
Φ_i^{max}	Maximum aggregate coverage provided to demand node $i \in I$ by single levels, i.e. $\Phi_i^{max} = \max\left\{\Phi_i^U;\Phi_i^L\right\}$;
s_i^k	Binary decision variable equal to 1 if $\Phi_i^k \ne \Phi_i^{max}$, $i \in I, k \in K$.

and adding the following constraints to the model:

$$\sum_{j \in J^k} z_{ij}^k \le 1 \qquad \forall i \in I, k \in K \tag{4a}$$

$$z_{ij}^k \le x_j^k \qquad \forall i \in I, k \in K, j \in J^k \tag{4b}$$

$$\sum_{j \in J^k} \phi_{ij}^k z_{ij}^k \ge \phi_{ij}^k x_j^k \qquad \forall i \in I, k \in K, j \in J^k \tag{4c}$$

$$\Phi_i^{max} \ge \Phi_i^k \qquad \forall i \in I, k \in K \tag{4d}$$

$$\Phi_i^{max} \le \Phi_i^k + s_i^k \qquad \forall i \in I, k \in K \quad (4e)$$
$$\sum_{k\in K} s_i^k \le 1 \qquad \forall i \in I \quad (4f)$$
$$z_{ij}^k \in \{0,1\} \qquad \forall i \in I, j \in J^k, k \in K \quad (4g)$$
$$s_i^k \in \{0,1\} \qquad \forall i \in I, k \in K \quad (4h)$$
$$\Phi_i^{max} \ge 0 \qquad \forall i \in I \quad (4i)$$

Constraints (4a)–(4c) assign each demand node $i \in I$ to the located facility of each level $k \in K$ providing the maximum coverage level. Specifically, each demand node i must be assigned to at most one (constraints (4a)), located (constraints (4b)) facility of level k providing the highest coverage level among all the located ones (constraints (4c)). Then, $\max_{j\in J^k}\{\phi_{ij}^k x_j^k\} = \sum_{j\in J^k}\phi_{ij}^k z_{ij}^k$. Constraints (4d)–(4f) guarantee that the variable Φ_i^{max} is equal to the maximum between Φ_i^U and Φ_i^L for each demand node i. Indeed, constraints (4d) set the lower bound of Φ_i^{max} to the larger between Φ_i^U and Φ_i^L. In contrast, constraints (4e) set the upper bound of Φ_i^{max} to the smaller between Φ_i^U and Φ_i^L plus the decision variable s_i^k. The variable s_i^k is equal to 1 only for at most one (constraints (4f)) inequality among those obtained by varying k in (4e) for each demand node. For constraints (4d)–(4f) to be simultaneously verified, the upper and the lower bound of Φ_i^{max} are identical and correspond to the maximum.

The prod-parts included in (1) and (2) are represented by Q_i^k and $\Phi_i^U\Phi_i^L$, respectively. In order to linearize Q_i^k, we implement the linearization process introduced by Karatas (2017) and the references therein (Morton et al., 2007; Salmerón, 2012). The main idea is to reduce each Q_i^k to a network flow problem. For the sake of simplicity, let us consider the overall non-coverage probability Q_i^L of node i by facilities of the lower level L. The linearizing approach maps a network crossed by a flow, as shown in Fig. 2.

By referring to Fig. 2, the network is designed as follows: (i) $|J^L|+1$ nodes (depicted by blue bullets) represent the candidate locations for facilities of level L plus a pseudo location (the $(|J^L|+1)$-th); (ii) each adjacent node pair, the j-th and $(j+1)$-th is linked through two arcs – one labelled positive (in dark green) and one negative (in light green) associated with auxiliary decision variables v_{ijL}^+ and v_{ijL}^-, respectively – that describe the flow; (iii) the labelled positive arc connecting j and $(j+1)$ is associated with a weight equal to $(1-\phi_{ij}^L)$; (iv) the variable x_j^L rules the flow outgoing from node j. The flow crossing the network is initialized to 1, i.e. the value of Q_i^L if no facilities of level L are located. Suppose a facility is located at j and, then, $x_j^L = 1$. In this case, the flow crosses the labelled positive arc from node j to $(j+1)$ and is scaled of a factor equal to $(1-\phi_{ij}^k)$. Otherwise, the flow crosses the labelled negative arc, remaining unchanged. Therefore, the flow entering the node $(j+1)$ of the network is equal to $\prod_{t=1}^{j}\left(1-\phi_{it}^k x_t^k\right)$. Finally, the flow entering the pseudo location $(|J^L|+1)$ and thus outgoing the network equals the non-coverage probability Q_i^L. See Karatas (2017) for a detailed description.

The approach by Karatas (2017) allows to linearize Q_i^k of a single hierarchical level $k \in K$, while linearizing the prod-part of the aggregation of coverages results in a further and more challenging task when more levels of facilities interact to cooperate. Indeed, additional products among decision variables are involved, as will be shown. Our main contribution to the linearizing process is extending the approach proposed by Karatas (2017) when cooperation occurs in a two-level hierarchy of facilities. The prod-part $\Phi_i^U\Phi_i^L$ is exploded, recalling their definition in (1), and replacing $\max_{j\in J^k}\{\phi_{ij}^k x_j^k\}$ with its equivalent $\sum_{j\in J^k}\phi_{ij}^k z_{ij}^k$, as follows:

$$\Phi_i^U\Phi_i^L = \theta^U\left(1-\theta^L\right)\sum_{j\in J^U}\phi_{ij}^U z_{ij}^U + \left(1-\theta^U\right)\theta^L\sum_{j\in J^L}\phi_{ij}^L z_{ij}^L + \tag{5a}$$
$$+\left(1-\theta^U\right)\left(1-\theta^L\right)\left(1-Q_i^U-Q_i^L\right)+$$
$$-\theta^U\left(1-\theta^L\right)Q_i^L\sum_{j\in J^U}\phi_{ij}^U z_{ij}^U + \tag{5b}$$
$$-\left(1-\theta^U\right)\theta^L Q_i^U\sum_{j\in J^L}\phi_{ij}^L z_{ij}^L + \tag{5c}$$
$$+\left(1-\theta^U\right)\left(1-\theta^L\right)Q_i^L Q_i^U + \tag{5d}$$
$$+\theta^U\theta^L\sum_{j\in J^L}\phi_{ij}^L z_{ij}^L\sum_{j\in J^U}\phi_{ij}^U z_{ij}^U \tag{5e}$$

See Appendix for a comprehensive description.

The extended expression of $\Phi_i^U\Phi_i^L$ is made up of the right-hand side of (5a), which is linear and will be defined by Λ_i for shortness in the following, and four non-linear contributions (5b)-(5e). By setting Δ_i^1 equal to the reverse sign of (5b), Δ_i^2 to the reverse sign of (5c), Δ_i^3 to (5d), and Δ_i^4 to (5e), the expression (5) can be equivalently written as:

$$\Phi_i^U\Phi_i^L = \Lambda_i - \Delta_i^1 - \Delta_i^2 + \Delta_i^3 + \Delta_i^4$$

Each non-linear contribution is given by a product of several factors. Our approach consists of introducing a new network flow problem for each of the non-linearities Δ_i^n $n \in \{1,2,3,4\}$ that linearizes one of that factors, which can be handled in a (Karatas, 2017) fashion (e.g., Q_i^L), and initializing the flow crossing the network to the rest of the non-linear contribution.

The networks we introduce are depicted in Fig. 3. Additional auxiliary variables δ_{ij}^{n+} and δ_{ij}^{n+} describe the flow crossing the arcs. The network in Fig. 3(a) is used for linearizing Δ_i^1, which contains Q_i^L. The network is designed to linearize Q_i^L, as in the previous case: the nodes represent the candidate sites of lower-level facilities; variables x_j^L direct the flow outgoing from j; the labelled positive arc from facility candidate sites j to $(j+1)$ is weighted by $(1-\phi_{ij}^L)$. However, in this case, the initial flow is set to the rest of the expression of Δ_i^1, i.e. $\theta^U\left(1-\theta^L\right)\sum_{j\in J^U}\phi_{ij}^U z_{ij}^U$. The outgoing flow after having crossed the whole network will equal $\Delta_i^1 = \theta^U\left(1-\theta^L\right)Q_i^L\sum_{j\in J^U}\phi_{ij}^U z_{ij}^U$.

The contribution Δ_i^2 is a mirror-like case of Δ_i^1 where this time Q_i^U is linearized through the network depicted in Fig. 3(b), which is made by: red nodes representing candidate sites of upper-level facilities; $(1-\phi_{ij}^U)$ is the weight associated with the labelled positive arc linking nodes j and $(j+1)$; the flow is initialized to $\left(1-\theta^U\right)\theta^L\sum_{j\in J^L}\phi_{ij}^L z_{ij}^L$ and is direct from j by the value of x_j^U.

The network related to Δ_i^3 (Fig. 3(c)) is still designed to linearize Q_i^U whilst the flow is initialized to Q_i^L, representing the output of the network in Fig. 2, multiplied by $\left(1-\theta^U\right)\left(1-\theta^L\right)$.

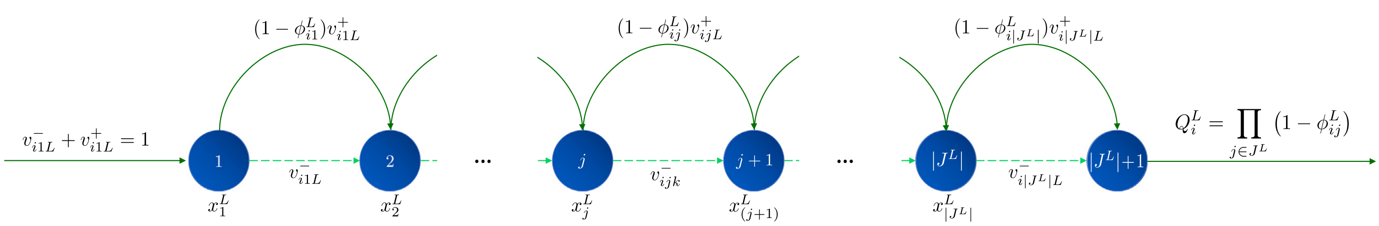


Fig. 2. Network related to demand node i and crossed by the flow – scaled by the located facilities of level L – representing the overall non-coverage probability Q_i^L. Each blue node represents a candidate location j of level L; labelled positive and labelled negative arcs are depicted by solid dark green lines and dashed light green lines, respectively. A generic node j is related to variable x_j^L, whose value directs the flow outgoing from j on the labelled positive arc ($x_j^L = 1$) and the labelled negative arc ($x_j^L = 0$).

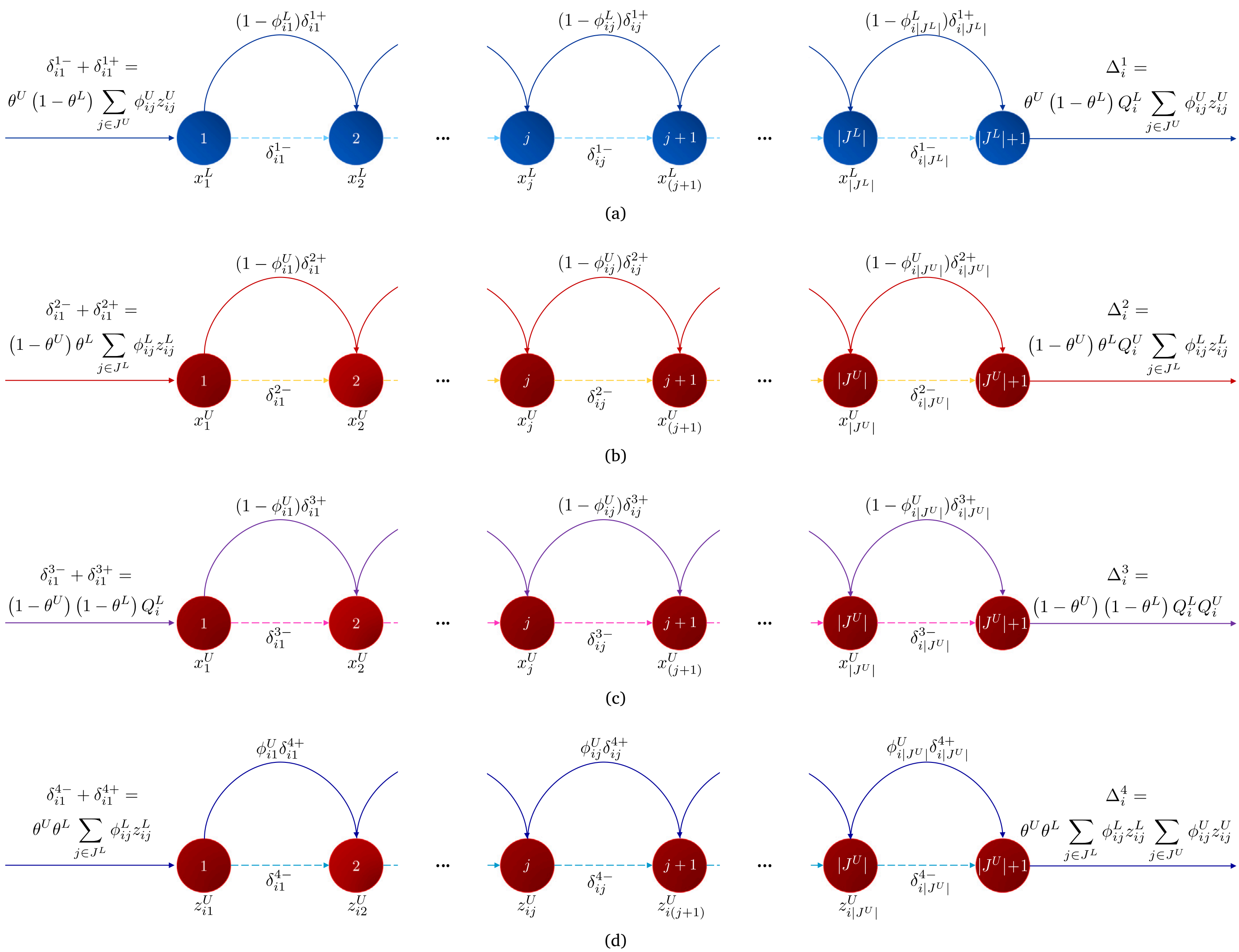


Fig. 3. The four networks related to demand node i and crossed by the flow representing the non-linear contributions: (a) Δ_i^1 equal to the reverse sign of (5b); (b)Δ_i^2 equal to the reverse sign of (5c); (c)Δ_i^3 equal to (5d); (d)Δ_i^4 equal to (5e). Blue and red nodes depict the candidate sites of facilities of the lower and the upper level, respectively. Labelled positive and labelled negative arcs are depicted by solid and dashed lines, respectively. The variable below each node directs the flow across the arcs.

The contribution Δ_i^4 is dissimilar since it does not involve any Q_i^k. However, a modified network flow problem may nevertheless be implemented, as shown in Fig. 3(d). Indeed, the network can be designed as follows: the nodes represent candidates sites of upper-level facilities to handle $\sum_{j\in J^U}\phi_{ij}^U z_{ij}^U$; the weight associated with the labelled positive arc linking nodes j and $j+1$ is equal to ϕ_{ij}^U; the variable z_{ij}^U is responsible for directing the flow outgoing from j. The flow entering the network is $\theta^U\theta^L\sum_{j\in J^L}\phi_{ij}^L z_{ij}^L$. Then, the flow will cross only one labelled positive arc corresponding to the located upper-level facility j that provides the maximum coverage ($z_{ij}^U = 1$) and all the others labelled negative arcs of the network. Therefore, our linearization approach holds since only one addend is positive in $\sum_{j\in J^U}\phi_{ij}^U z_{ij}^U$.

It is worth noticing that we preferred to design the network on the set J^U instead J^L, whenever possible. This way we reduce the number of auxiliary decision variables, reasonably assuming $|J^U| \ll |J^L|$ in real-case applications.

3.2.1. HNCL: the MILP mathematical model

In order to linearize the model (3), we introduce the following *auxiliary decision variables*:

- Q_i^k Probability of demand node $i \in I$ to be not covered by all located facilities of level $k \in K$ when their coverage probabilities are based on independent events, i.e. $Q_i^k = \prod_{j\in J^k}\left(1-\phi_{ij}^k x_j^k\right)$;
- Δ_i^n n-th non-linear contribution in the definition of $\Phi_i^U\Phi_i^L$, where $n\in\{1,2,3,4\}$. Δ_i^1 corresponds to the reverse sign of (5b), Δ_i^2 to the reverse sign of (5c), Δ_i^3 to (5d), and Δ_i^4 to (5e);
- v_{ijk}^+ Labelled positive flow of demand node $i \in I$ and facility candidate site $j \in J^k$ of level $k \in K$ related to Q_i^k;
- v_{ijk}^- Labelled negative flow of demand node $i \in I$ and facility candidate site $j \in J^k$ of level $k \in K$ related to Q_i^k;
- δ_{ij}^{n+} Labelled positive flow of demand node $i \in I$ and facility candidate site j related to Δ_i^n, where $n = 1$ if $j \in J^L$ and $n \in \{2,3,4\}$ if $j \in J^U$;
- δ_{ij}^{n-} Labelled positive flow of demand node $i \in I$ and facility candidate site $j \in J^k$ related to Δ_i^n, where $n = 1$ is $j \in J^L$ and $n \in \{2,3,4\}$ if $j \in J^U$.

The MILP alternative formulation of HNCL model is:

max (3a)

$$\text{s.t. } v_{i,1,k}^- + v_{i,1,k}^+ = 1 \qquad \forall i \in I, k \in K \quad (6a)$$

$$v^-_{i,j,k} + (1-\phi^k_{ij})v^+_{i,j,k} = v^-_{i,(j+1),k} + v^+_{i,(j+1),k} \qquad \forall i \in I, k \in K, j \in \{1,\dots,|J^k|-1\} \quad (6b)$$

$$v^-_{i,|J^k|,k} + (1-\phi^k_{i,|J^k|})v^+_{i,|J^k|,k} = Q^k_i \qquad \forall i \in I, k \in K \quad (6c)$$

$$x^k_j \ge v^+_{ijk} \qquad \forall i \in I, k \in K, j \in J^k \quad (6d)$$

$$1 - x^k_j \ge v^-_{ijk} \qquad \forall i \in I, k \in K, j \in J^k \quad (6e)$$

$$\Phi^k_i = \theta^k \sum_{j\in J^k} \phi^k_{ij} z^k_{ij} + (1-\theta^k)(1-Q^k_i) \qquad \forall i \in I, k \in K \quad (6f)$$

$$\delta^{1-}_{i1} + \delta^{1+}_{i1} = \theta^U\left(1-\theta^L\right)\sum_{j\in J^U}\phi^U_{ij}z^U_{ij} \qquad \forall i \in I \quad (6g)$$

$$\delta^{2-}_{i1} + \delta^{2+}_{i1} = \theta^L\left(1-\theta^U\right)\sum_{j\in J^L}\phi^L_{ij}z^L_{ij} \qquad \forall i \in I \quad (6h)$$

$$\delta^{3-}_{i1} + \delta^{3+}_{i11} = \left(1-\theta^U\right)\left(1-\theta^L\right)Q^L_i \qquad \forall i \in I \quad (6i)$$

$$\delta^{4-}_{i1} + \delta^{4+}_{i1} = \theta^U\theta^L\sum_{j\in J^L}\phi^L_{ij}z^L_{ij} \qquad \forall i \in I \quad (6j)$$

$$\delta^{1-}_{ij} + (1-\phi^L_{ij})\delta^{1+}_{ij} = \delta^{1-}_{i(j+1)} + \delta^{1+}_{i(j+1)} \qquad \forall i \in I, j \in \{1,\dots,|J^L|-1\} \quad (6k)$$

$$\delta^{n-}_{ij} + (1-\phi^U_{ij})\delta^{n+}_{ij} = \delta^{n-}_{i(j+1)} + \delta^{n+}_{i(j+1)} \qquad \forall i \in I, n \in \{2,3\}, j \in \{1,\dots,|J^U|-1\} \quad (6l)$$

$$\delta^{4-}_{ij} + \phi^U_{ij}\delta^{4+}_{ij} = \delta^{4-}_{i(j+1)} + \delta^{4+}_{i(j+1)} \qquad \forall i \in I, j \in \{1,\dots,|J^U|-1\} \quad (6m)$$

$$\delta^{1-}_{i|J^L|} + (1-\phi^L_{i|J^L|})\delta^{1-}_{i|J^L|} = \Delta^1_i \qquad \forall i \in I \quad (6n)$$

$$\delta^{n-}_{i|J^U|} + (1-\phi^U_{i|J^U|})\delta^{n+}_{i|J^U|} = \Delta^n_i \qquad \forall i \in I, n \in \{2,3\} \quad (6o)$$

$$\delta^{4-}_{i|J^U|} + \phi^U_{i|J^U|}\delta^{4+}_{i|J^U|} = \Delta^4_i \qquad \forall i \in I \quad (6p)$$

$$x^L_j \ge \delta^{1+}_{ij} \qquad \forall i \in I, j \in J^L \quad (6q)$$

$$x^U_j \ge \delta^{n+}_{ij} \qquad \forall i \in I, j \in J^U, n \in \{2,3\} \quad (6r)$$

$$z^U_{ij} \ge \delta^{4+}_{ij} \qquad \forall i \in I, j \in J^U \quad (6s)$$

$$1 - x^L_j \ge \delta^{1-}_{ij} \qquad \forall i \in I, j \in J^L \quad (6t)$$

$$1 - x^U_j \ge \delta^{n-}_{ij} \qquad \forall i \in I, n \in \{2,3\}, j \in J^U \quad (6u)$$

$$1 - z^U_{ij} \ge \delta^{4-}_{ij} \qquad \forall i \in I, j \in J^U \quad (6v)$$

$$\Phi_i = \theta\Phi^{max}_i + (1-\theta)\left(\Phi^U_i + \Phi^L_i - \Lambda_i + \Delta^1_i + \Delta^2_i - \Delta^3_i - \Delta^4_i\right) \qquad \forall i \in I \quad (6w)$$

$$v^+_{ijU}, v^-_{ijU}, \delta^+_{ij1}, \delta^+_{ij2}, \delta^+_{ij3}, \delta^-_{ij1}, \delta^-_{ij2}, \delta^-_{ij3} \ge 0 \qquad \forall i \in I, j \in J^U \quad (6x)$$

$$v^+_{ijL}, v^-_{ijL}, \delta^+_{ij4}, \delta^-_{ij4} \ge 0 \qquad \forall i \in I, j \in J^L \quad (6y)$$

$$Q^U_i, Q^L_i, \Delta^1_i, \Delta^2_i, \Delta^3_i, \Delta^4_i \ge 0 \qquad \forall i \in I \quad (6z)$$

(3b), (3c), (3d), (3e), (3f), (3g), (4)

In order to compute the non-coverage probability Q^k_i for each demand node $i \in I$ and each level k: constraints (6a) initialize the flow entering the network to 1; constraints (6b) balance the outgoing scaled flow from $j \in J^k$ to the incoming flow to $(j+1) \in J^k$; finally, Q^k_i is computed by (6c) as the scaled flow outgoing from the network, i.e. from $|J^k|$. Furthermore, technical constraints (6d) ensure that a facility of level k is located at candidate site $j \in J^k$ if v^+_{ijk} is positive. Conversely, if v^-_{ijk} is positive, a facility must not be located at j by constraints (6e).

The equivalent formulation of the intra-level aggregate coverage of facilities of level $k \in K$ expressed by (1) is provided by constraints (6f).

In order to compute $\Phi^U_i\Phi^L_i$ for each demand node $i \in I$, the four flows, each corresponding to a non-linear contribution in (5), are initialized by constraints (6g)–(6j). Then, the flow balance crossing the four networks is defined by constraints (6k)–(6m). Finally, the four non-linear contributions are computed as the outgoing flow from the networks (constraints (6n)–(6p)). Moreover, if δ^{n+}_{ij} is positive, a facility must be located at the candidate site $j \in J^L$ when $n = 1$ (constraints (6q)) or at the candidate site $j \in J^U$ when $n \in \{2,3\}$ (constraints (6r)) or the facility located at candidate site $j \in J^U$ provides the maximum coverage level among all located upper-level facilities when $n = 4$ (constraints (6s)). Instead, a facility is not located at the candidate site j when δ^-_{ijn} is positive (constraints (6t) and (6u)), neither it can provide the maximum coverage level (constraints (6v)).

The equivalent linear formulation of the global (inter-level) aggregate coverage defined by (2) for each demand node is provided by the set of constraints (6w).

Finally, constraints (6x)–(6z) declare the nature of the additional decision variables.

3.3. The value of the cooperation

In order to measure the contribution of intra-level and inter-level cooperations, we introduced the following framework classifying covered demand nodes according to the mechanism responsible for the coverage, as summarized in Table 2.

Let $x^{k*}_j, \Phi^{k*}_i, \Phi^*_i$ be the optimal solution of a HNCL model and ϕ^{max*}_i the maximum coverage provided to demand node i by a single located facility, regardless of its hierarchical level, i.e. $\phi^{max}_i = \max_{k\in K, j\in J^k}\phi^k_{ij}x^{k*}_j, i \in I$. If ϕ^{max*}_i exceeds the threshold T, the demand node i is nevertheless covered, regardless of any established cooperation. In this case, since locating (just) such a facility guarantees the covering condition $\Phi^*_i \ge T$, we can state that the involved mechanism is the *individual coverage*. Instead, if ϕ^{max*}_i is lower than the threshold T albeit the demand node i is covered $\left(\Phi^*_i \ge T\right)$, the *cooperation* is responsible for covering. Additionally, we assess the type of cooperation involved (intra-level or inter-level) by investigating the specific values of Φ^k_i in relation to the threshold T, as reported in Table 2. For instance, a demand node i is covered by intra-lower level cooperation if ϕ^{max*}_i

Table 2
Coverage classification of covered demand nodes, where $x^{k*}_j, \Phi^{k*}_i, \Phi^*_i$ is the optimal solution of an HNCL model (6), and $\phi^{max}_i = \max_{k\in K, j\in J^k}\phi^k_{ij}x^{k*}_j$.

Covered demand	Set of nodes covered	Percentage of covered demand
by Individual coverage	$I^{Ind} = \{i \in I : \phi^{max*}_i \ge T\}$	$\sum_{i\in I^{Ind}} w_i \Big/ \sum_{i\in I} w_i$ (%)
by Cooperation	$I^{Coop} = \{i \in I : \phi^{max*}_i < T \wedge \Phi^*_i \ge T\}$	$\sum_{i\in I^{Coop}} w_i \Big/ \sum_{i\in I} w_i$ (%)
Intra-Upper Level	$I^{UC} = \{i \in I^{Coop} : \Phi^{U*}_i \ge T \wedge \Phi^{L*}_i < T\}$	$\sum_{i\in I^{UC}} w_i \Big/ \sum_{i\in I} w_i$ (%)
Intra-Lower Level	$I^{LC} = \{i \in I^{Coop} : \Phi^{U*}_i < T \wedge \Phi^{L*}_i \ge T\}$	$\sum_{i\in I^{LC}} w_i \Big/ \sum_{i\in I} w_i$ (%)
Intra-Upper/Lower Level	$I^{ULC} = \{i \in I^{Coop} : \Phi^{U*}_i \ge T \wedge \Phi^{L*}_i \ge T\}$	$\sum_{i\in I^{ULC}} w_i \Big/ \sum_{i\in I} w_i$ (%)
Inter-Level	$I^{IC} = \{i \in I^{Coop} : \Phi^{U*}_i < T \wedge \Phi^{L*}_i < T\}$	$\sum_{i\in I^{IC}} w_i \Big/ \sum_{i\in I} w_i$ (%)

Table 3
Setting of the 5 randomly generated instances.

Random instances	
Set cardinalities	$\|I\| = 100$ $\|J^U\| = \|J^L\| = 50$
Fixed parameters	d_{ij} is the Euclidean distance between $i \in I$ and $j \in J^U \cup J^L$ $R = 1,000$ ϕ_{ij}^U, ϕ_{ij}^L given by (7) with $\alpha_1 = 1/6, \alpha_2 = 1/2$ $T = 0.7$ $w_i \sim U(10, 500)$ $c_j^U \sim U(800, 1000)$, $c_j^L \sim U(8, 10)$ $B^U = \min \left\{ \sum_{j \in J^U} c_j^U x_j^U \; : \; \sum_{j \in N_i^U} x_j^U \geq 1, i \in I, \quad x_j^U \in \{0,1\}, j \in J^U \right\}$

and Φ_i^{U*} are below the threshold T, which is, instead, exceeded by Φ_i^{L*}. Thus, the installation of lower-level facilities and their resulting cooperation is responsible for the coverage of i. Instead, if Φ_i^{U*} and Φ_i^{L*} are less than T while Φ_i^* exceeds T, the demand node i is covered by the cooperation among all types of facilities (inter-level cooperation).

Finally, based on the above node classification, the demand is aggregated to compute the percentage of covered demand by each mechanism, as shown in Table 2. Such percentages represent their contribution to the coverage of the demand.

4. Test instances and experimental design

In order to prove the capability of the model to provide meaningful solutions, the MINLP and MILP formulations of the HNCL model presented in Section 3 have been tested both on random and real instances.

The settings of the random instances are listed in Table 3. Specifically, for each instance, we randomly generated 100 demand nodes and 50 candidate location sites, assuming $J^U = J^L$, from a uniform distribution on a two-dimensional 1.000 × 1.000 square. Using the Euclidean distance metric, we computed d_{ij} for each demand node and candidate location site pair. The covering radius R within which demand nodes must have access to exclusive services provided by upper-level facilities is fixed at 1.000. In contrast, we defined two distinct coverage decay functions to model the coverages ϕ_{ij}^U and ϕ_{ij}^L provided by facilities at the upper and lower levels, respectively. They are expressed by (7) and depicted in Fig. 4.

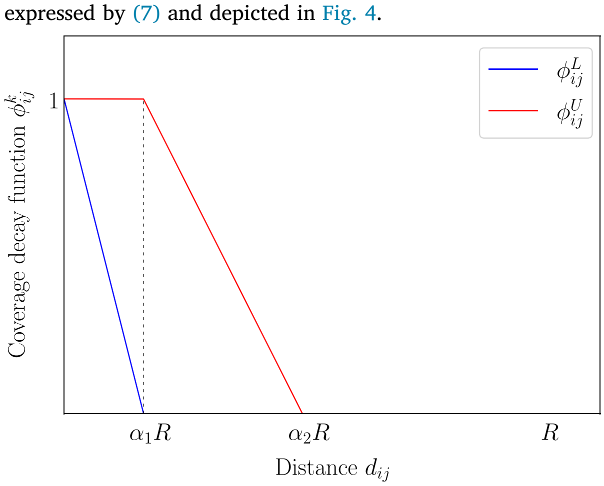


Fig. 4. Coverage decay functions ϕ_{ij}^U, ϕ_{ij}^L where $0 < \alpha_1 < \alpha_2 < 1$.

$$\phi_{ij}^U = \begin{cases} 1 & d_{ij} \leq \alpha_1 R \\ \dfrac{\alpha_2 R - d_{ij}}{(\alpha_2 - \alpha_1) R} & \alpha_1 R < d_{ij} \leq \alpha_2 R \\ 0 & d_{ij} > \alpha_2 R \end{cases} \tag{7a}$$

$$\phi_{ij}^L = \max \left\{ 0; \frac{\alpha_1 R - d_{ij}}{\alpha_1 R} \right\} \qquad 0 < \alpha_1 < \alpha_2 < 1 \tag{7b}$$

Specifically, we defined a *trapezoidal-shaped function* for the upper-level facilities and a *linear function* for the lower-level ones. We assume upper-level facilities to have a stronger coverage capability on an equal distance compared to lower-level ones. The parameters characterizing both functions are defined as proportional (of a value between 0 and 1) to the covering radius R (see Table 3). The threshold T for considering a node as covered is set equal to 0.7.

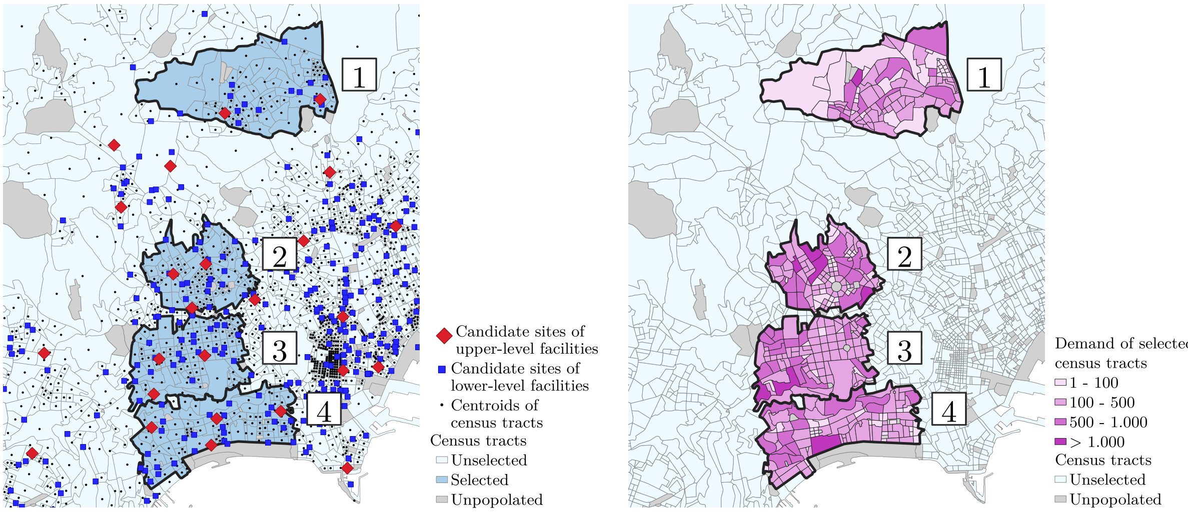
We have drawn the weight of each demand node w_i and the costs for locating facilities of both levels (c_j^U and c_j^L) from uniform distributions, whose parameters are reported in Table 3. Finally, the budget B^U has been set by solving a set covering problem aiming at locating upper-level facilities within R from each demand node at the minimum cost.

Furthermore, we applied the HNCL model to a real-world case study of a distribution network system operating in an urban context. Specifically, we considered a retail company outsourcing basic services by integrating the network of internal facilities (upper-level J^U) with external ones (lower-level J^L). We defined four real instances taken from four neighbourhoods in a city, as shown in Fig. 5 that reports the cardinality of the sets of demand nodes I and candidate sites J^U and J^L. In particular, for each instance, we discretized the demand space into nodes corresponding to the centroids of the populated census tracts of the neighbourhood (set of demand nodes I). The location and the demand nodes are depicted in Fig. 5(a). The set of upper-level facilities J^U (red diamonds) consists of existing facilities owned by the retail company; the lower-level facilities representing set J^L are chosen within the network of retail shops (blue squares). The weight of each demand node w_i represents the resident population of the census tract (Fig. 5(b)). The remaining parameters are set up as for the random instances.

For each real and random instance, we varied the intra-level correlation factors, assuming them identical $\theta^U = \theta^L$, and the inter-level correlation factor θ in order to compare solutions obtained with different degrees of cooperation at the same (by the value of θ^U, θ^L) and at different hierarchical levels (by the value of θ). In particular, we tested the correlation factors at their extreme values, i.e. 0 when maximum cooperation is allowed and 1 when no cooperation is allowed, and at intermediate values 0.3 and 0.7.

Furthermore, we are interested in investigating how lower-level facilities support the upper-level ones and how they interact in covering the demand for shared services. For this reason, we tested the HNCL on the random and real instances by varying the budget for locating lower-level facilities B^L, starting from 0 with step 10 until obtaining a value of objective function $z^* = \bar{z}$. Such $\bar{z}$ represents the maximum achievable covered demand with a given combination of $\theta^U, \theta^L, \theta$, obtained as the value of the objective function of a relaxed HNCL where B^L is not assigned and a lower-level facility is located in every candidate site.

We conducted a total of 2.509 experiments for each formulation (MINLP and MILP), of which 1.713 were for random instances and 796 for real instances.



(a) Location and demand space (b) Distribution of the demand

Real Instance	1	2	3	4
Set cardinality	$\|I\| = 86$ $\|J^U\| = 2$ $\|J^L\| = 13$	$\|I\| = 102$ $\|J^U\| = 3$ $\|J^L\| = 21$	$\|I\| = 97$ $\|J^U\| = 3$ $\|J^L\| = 23$	$\|I\| = 99$ $\|J^U\| = 4$ $\|J^L\| = 27$

Fig. 5. Representation and setting of the real instances.

5. Results

In this section, the results obtained by solving the HNCL for the test instances are presented and analysed. We first compare computational results obtained by solving the MINLP and the MILP formulations in Section 5.1. In Section 5.2, we present some illustrative solutions showing the contribution of cooperation in covering the demand. Finally, in Section 5.3 the obtained results for random instances are summarized and discussed.

5.1. Computational experiments

In order to evaluate the proposed linearization approach, we compare the computational results obtained by using both the MINLP and MILP formulations on all the random and real instances.

All experiments were conducted on a PC with an Intel Core i7 processor at 2.30 GHz with 16 GB RAM. The MILP model was solved using CPLEX 20.1 within the AIMMS modelling environment, whereas the original non-linear model (MINLP) was solved using the Knitro Solver within AMPL. A time limit of 10 min (600 s) was set for each solver.

Table 4 shows the computational results in terms of the minimum, the maximum and the average value of CPU time (in seconds) and optimality gap (%) for each formulation, also distinguishing for each set of instances (real and random).

Clearly, the linearized model produces much better solutions both in terms of quality and computation time. In the real instances, the non-linear model required, on average, nearly 28 s of CPU time, whereas the linear model took only around 1.5 s. With respect to solution quality, CPLEX was able to solve all real instances optimally, whereas Knitro returned optimality gaps up to 20%. The differences in the random instances are even more extreme. While CPLEX took an average of 37.68 s and managed to solve all instances to optimality, Knitro required an average of 400 s per instance and gave optimality gaps as large as 53.70% as it exceeded the maximum time limit (600 s) on several occasions.

5.2. Illustrative solutions: measuring the cooperation

In this section, illustrative solutions are presented, showing the contribution of each coverage mechanism according to the classification provided in Section 3.3.

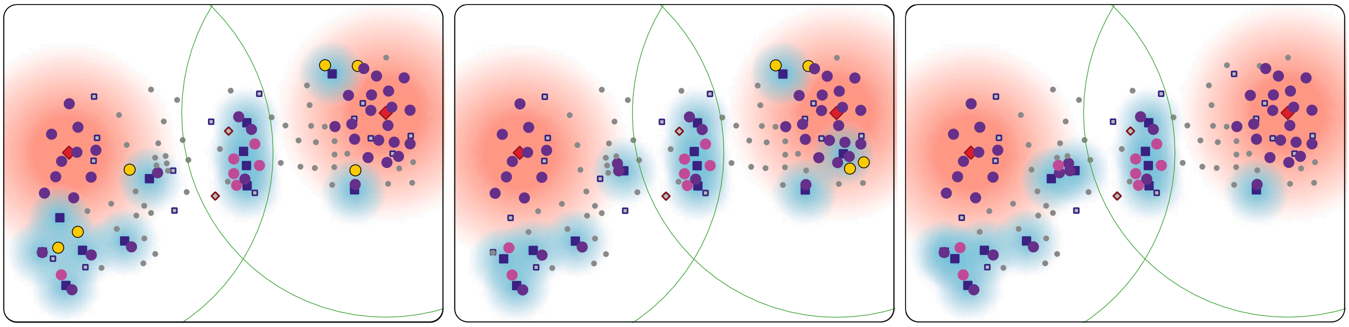
Fig. 6 depicts the maps of three illustrative solutions obtained for Real Instance 4 by varying the inter-level correlation factor $\theta \in \{0, 0.7, 1\}$ with fixed $B^L = 110$ and $\theta^U = \theta^L = 0$ (maximum intra-level cooperation). The demand nodes are coloured according to the mechanism responsible to their coverage.

Firstly, it can be noticed that by fixing the budget for locating upper-level facilities B^U in a set covering fashion, the individual coverages ϕ^U_{ij} provided by the 2 located upper-level facilities do not overlap; thus, intra-upper level cooperation does not take place.

Table 4
Computational results of MILP and MINLP formulations.

		CPU time (s)			Optimality gap (%)		
		Min	Max	Average	Min	Max	Average
MINLP	All Instances	0.02	600.00[a]	251.90	0.00%	53.70%	12.04%
	Real Instances	0.02	300.00	27.79	0.00%	20.00%	1.02%
	Random Instances	14.64	600.00[a]	400.48	0.00%	53.70%	19.35%
MILP	All Instances	0.11	455.92	23.27	0.00%	0.00%	0.00%
	Real Instances	0.11	5.64	1.52	0.00%	0.00%	0.00%
	Random Instances	0.12	455.92	37.68	0.00%	0.00%	0.00%

[a] Reached the time limit.



(a) $\theta = 0$, Covered demand 50.26% (b) $\theta = 0.7$, Covered demand 45.62% (c) $\theta = 1$, Covered demand 45.19%

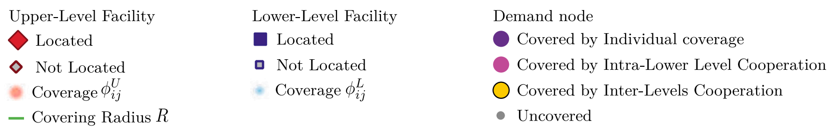
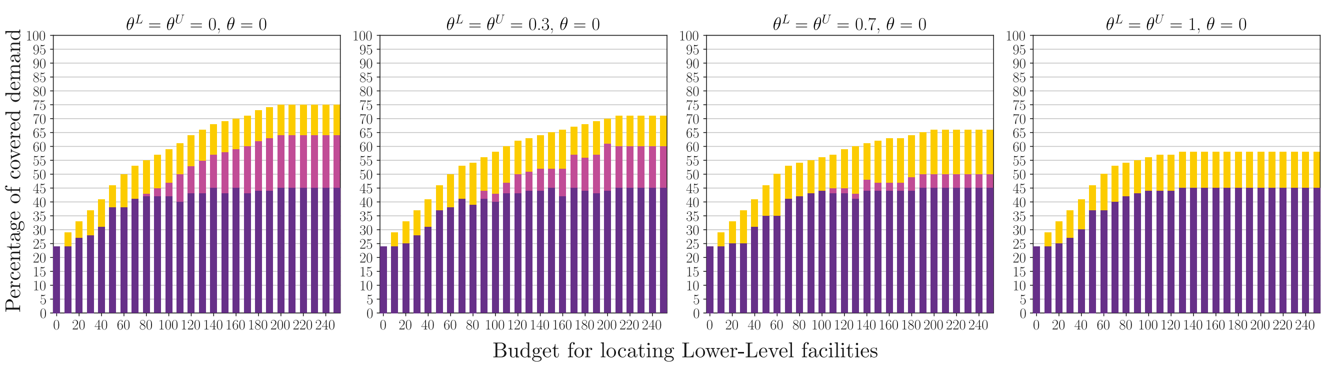


Fig. 6. Illustrative solutions for Real Instance 4 by varying θ with $B^L = 110, \theta^U = \theta^L = 0$.

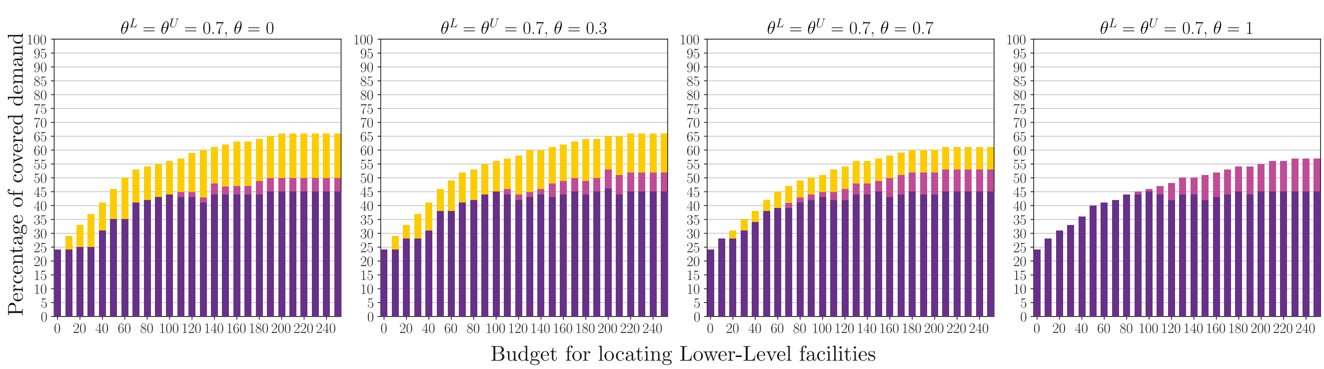
Secondly, the effect produced by increasing θ, i.e., restricting inter-level cooperation, may be appreciated by comparing maps in Figs. 6(a), 6(b), and 6(c). Demand nodes covered by inter-level cooperation (yellow dots) progressively decrease, from 6 to 4 to 0, as well as the overall covered demand, from 50.26% to 45.62% to 45.19%, when θ rises from 0 to 0.7 to 1.

In Fig. 7, we compare the percentage of covered demand in Random Instance 3 by varying the correlation factors $\theta^U = \theta^L, \theta$ and the budget for locating lower-level facilities B^L. The percentage of covered demand is split according to our coverage classification scheme: individual coverage is represented in purple, intra-lower level in pink, and inter-level in yellow. As in the solutions shown in Fig. 6, the



$\bar{z}^{\%} = \bar{z} / \sum_{i \in I} w_i (\%)$	75%	71%	66%	58%
$B^L(\bar{z})$	200	210	200	130

(a) Fixed $\theta = 0$, varied $\theta^U = \theta^L$



$\bar{z}^{\%} = \bar{z} / \sum_{i \in I} w_i (\%)$	66%	66%	61%	57%
$B^L(\bar{z})$	200	220	210	230

(b) Fixed $\theta^L = \theta^U = 0.7$, varied θ

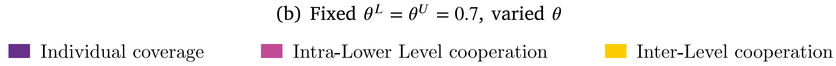


Fig. 7. Random instance 3: contribution of coverages by varying $B^L, \theta^L = \theta^U, \theta$.

cooperation among upper-level facilities does not occur, regardless of the value of θ^U. For each setting of $\theta^U = \theta^L$ and θ, we report the upper bound $\bar{z}^{\%}$ of the percentage of covered demand $\left(\bar{z}^{\%} = \bar{z} / \sum_{i \in I} w_i(\%)\right)$, and the minimum budget for lower-level facilities to achieve that percentage, $B^L(\bar{z})$. All solutions corresponding to larger budgets than $B^L(\bar{z})$ are identical to the solution obtained with $B^L = B^L(\bar{z})$.

By analysing the graphs in Fig. 7, the percentage of covered demand progressively increases by expanding the lower-level network, i.e., by increasing the budget B^L until reaching $B^L(\bar{z})$. However, different coverage mechanisms of various extents are involved. In particular, in Fig. 7(a), by fixing θ and increasing θ^L, thus limiting intra-lower level cooperation, the pink stripe (intra-lower level cooperation) tends to thin until it disappears for $\theta^U = \theta^L = 1$. Conversely, the yellow stripe tends to reduce when increasing θ (limiting inter-level cooperation) in Fig. 7(b). In both cases, a restriction in the cooperation degree, by increasing θ^L or θ, results in an overall reduction of the covered demand for a fixed budget B^L. It is worth noticing in Fig. 7(b) (fixed $\theta^L = 0.7$) that the upper bound of covered demand is invariant for an increased θ from 0 to 0.3. However, such coverage can be reached only through an increment of budget B^L equal to 20. In conclusion, Fig. 7 highlights how cooperation has a significant impact in covering the demand and that the correlation factors can be leveraged to rule the cooperation degree.

5.3. Summary results

In this section, summary results obtained for random instances are reported in Table 5 and discussed, referring to some specific representative combinations of correlation factors θ^L, θ. Since intra-upper level cooperation is not established, θ^U is irrelevant, and thus they are not reported. For each combination, we selected three representative budgets, i.e. small $\left(B^L = 50\right)$, medium $\left(B^L = 100\right)$ and large $\left(B^L = 300\right)$. In correspondence with each budget, the minimum, the maximum and the average percentage of covered demand among all random instances are indicated. Then, according to what is illustrated in Section 3.3, the percentage of covered demand is split into the contributions given by the coverage mechanisms, individual or cooperative. The latter is further subdivided according to the cooperation types.

If $\theta^L = \theta = 1$, no cooperation is allowed. The HNCL turns into a non-cooperative model, which nevertheless includes the decay in

Table 5
Summary of results for random instances.

$\theta^L = \theta = 1$	Min			Max			Average		
	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$
Covered demand%	32%	40%	45%	41%	50%	55%	37%	46%	50%
by Individual coverage%	32%	40%	45%	41%	50%	55%	37%	46%	50%
by Cooperation%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Intra-Lower Level	0%	0%	0%	0%	0%	0%	0%	0%	0%
Inter-Level	0%	0%	0%	0%	0%	0%	0%	0%	0%

$\theta^L = \theta = 0$	Min			Max			Average		
	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$
Covered demand%	40%	55%	69%	53%	67%	87%	46%	60%	77%
by Individual coverage%	32%	44%	48%	40%	44%	49%	34%	41%	45%
by Cooperation%	8%	11%	21%	13%	23%	38%	12%	19%	32%
Intra-Lower Level	0%	4%	13%	0%	7%	17%	1%	6%	17%
Inter-Level	8%	7%	8%	13%	16%	21%	11%	13%	15%

$\theta^L = \theta = 0.7$	Min			Max			Average		
	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$
Covered demand%	36%	46%	58%	45%	57%	65%	40%	52%	62%
by Individual coverage%	30%	38%	49%	41%	49%	54%	36%	44%	48%
by Cooperation%	6%	8%	9%	4%	8%	11%	3%	8%	14%
Intra-Lower Level	2%	3%	4%	0%	1%	4%	1%	2%	7%
Inter-Level	4%	5%	5%	4%	7%	7%	2%	6%	7%

$\theta^L = 1, \theta = 0$	Min			Max			Average		
	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$
Covered demand%	40%	52%	56%	52%	64%	71%	45%	57%	62%
by Individual coverage%	32%	40%	44%	40%	46%	52%	35%	43%	47%
by Cooperation%	8%	12%	12%	12%	18%	19%	10%	14%	14%
Intra-Lower Level	0%	0%	0%	0%	0%	0%	0%	0%	0%
Inter-Level	8%	12%	12%	12%	18%	19%	10%	14%	14%

$\theta^L = 0.7, \theta = 0.3$	Min			Max			Average		
	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$
Covered demand%	38%	52%	61%	49%	61%	74%	43%	55%	66%
by Individual coverage%	33%	44%	48%	39%	48%	53%	35%	43%	47%
by Cooperation%	5%	8%	13%	10%	13%	21%	7%	12%	19%
Intra-Lower Level	0%	2%	0%	0%	0%	3%	0%	1%	6%
Inter-Level	5%	6%	13%	10%	13%	18%	7%	11%	13%

$\theta^L = 0.3, \theta = 0.7$	Min			Max			Average		
	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$	$B^L = 50$	$B^L = 100$	$B^L = 300$
Covered demand%	37%	50%	61%	46%	61%	80%	41%	53%	70%
by Individual coverage%	37%	41%	46%	38%	46%	53%	35%	41%	47%
by Cooperation%	0%	9%	15%	8%	15%	27%	6%	12%	23%
Intra-Lower Level	0%	4%	10%	3%	10%	20%	2%	7%	17%
Inter-Level	0%	5%	5%	5%	5%	7%	3%	5%	5%

the coverage level provided by different hierarchical facilities and a minimum threshold for coverage. In this case, each demand node perceives only the maximum coverage by a single facility (individual coverage), regardless of its hierarchical level. If the budget $B^L = 300$, the covered demand is on average equal to 50% with an increase of 13% with respect to the case of $B^L = 50$. This slight improvement shows that lower-level facilities may contribute to a small extent when they cannot cooperate, given their individual coverage dissipates rapidly.

On the contrary, when the maximum intra and inter-level cooperations are allowed ($\theta^L = \theta = 0$), the coverage achieves the maximum value. For example, the average covered demand for $B^L = 300$ reaches 77%, +27% compared to the previous case, given the substantial contribution given by intra-lower level (17%) and inter-level (15%) cooperations that partially replaces the individual coverage, which goes from 50% to 45%.

The analysed settings represent the extreme coverage cases from the minimum ($\theta^L = \theta = 1$) to the maximum ($\theta^L = \theta = 0$) achievable covered demand, and any other solution obtained with any combination of θ^L and θ, presents a percentage of covered demand within the range thus defined. For instance, setting $\theta^L = \theta = 0.7$ and $B^L = 100$, on average, the demand covered by cooperation is 8%, of which 2% is intra-lower level and 6% is inter-level, compared to 19% if $\theta^L = \theta = 0$. The overall covered demand equals 52%, between 46% (if $\theta^L = \theta = 1$) and 60% (if $\theta^L = \theta = 0$).

A case of interest is the combination $\theta^L = 1$ and $\theta = 0$. In this case, each demand node perceives the maximal coverage provided by a single facility at the lower level ($\theta^L = 1$ and 0% of covered demand by intra-lower level cooperation) and a single facility at the upper level (intra-upper level cooperation is not established). These two coverages are combined to define the overall coverage with the maximum freedom of cooperation ($\theta = 0$). The inter-level cooperation gives discrete support to the overall demand coverage, i.e., on average, 10% with a small budget ($B^L = 50$) and 14% with a medium ($B^L = 100$) or large ($B^L = 300$) budget; individual coverage, in comparison, covers 35%, 43%, and 47% of demand, respectively.

Comparing the contributions of the cooperation mechanisms, if $\theta^L \geq \theta$, the inter-level cooperation generally covers a more extensive amount of demand than the cooperation among lower-level facilities. See, for instance, the results obtained by choosing $\theta^L = 0.7$ and $\theta = 0.3$. Considering the instance showing the maximum covered demand, the only contributing cooperation type with a small or medium budget ($B^L = 50$ and $B^L = 100$) is the inter-level one, with 10% and 13% of covered demand, respectively; intra-lower level cooperation does not occur. Even considering a large budget ($B^L = 300$), the inter-level cooperation reaches 18% of covered demand, whilst the intra-lower level cooperation contributes with a 3%.

Conversely, if $\theta^L < \theta$, the coverage provided by intra-level cooperation is typically more significant, as shown by results with $\theta^L = 0.3$ and $\theta = 0.7$. In this setting with a large budget, regarding the solution presenting the maximum covered demand, the intra-level cooperation provides 20% of covered demand, opposing the inter-level cooperation that contributes with a 7%.

As expected, the presented results show that solutions are highly sensitive to the correlation factors ruling the degree of cooperation and how, by establishing additional lower-level facilities, a greater extent of demand can be covered thanks to intra and inter-level cooperations.

6. Managerial implications

The formulated model can be effectively used to represent many real-world decision-making problems concerning the design of service networks.

Indeed, there are several application contexts where users' accessibility to a given service does not depend solely on the distance from their closest facility but on the number, density and characteristics of all the facilities located in their neighbourhood.

From a methodological point of view, this is testified by well-established approaches in the spatial accessibility-related literature, considering the simultaneous presence of facilities within a given distance from users. It is the case of the Floating Catchment Area (FCA) or the gravity-based methods (Luo and Wang, 2003). Also, in the location literature, many problems consider sophisticated spatial interaction models to determine the demand allocation or the attraction degree of located facilities.

From a practical perspective, many examples of interests involving cooperative and hierarchical systems may be provided.

For example, in the logistics sector, optimizing last-mile delivery is gaining momentum after the e-commerce boom, and new delivery solutions have been explored to meet this growing demand. In this context, the "Click & Collect" is a relatively new method of selling products according to which customers place their order online and then collect it at a specific location, such as parcel lockers, manned collection points, or shops. It allows consolidating deliveries (for providers) and saving fees on shipping or delivery (for customers). The Click & Collect network must be expanded to encourage customers to opt for self-collection instead of home delivery (Mangiaracina et al., 2019). For this purpose, a cooperative paradigm may effectively adapt. Indeed, the customers' acceptance of self-collection depends on the set of attractive options they may patronize within a limited distance. Such options are heterogeneous and can be organized according to hierarchical networks. For example, shops and collection points can be arranged in a nested hierarchy as both can be used to collect parcels, but shops also provide additional services.

Moreover, given the recent digitalization trend in the banking sector, financial institutions are restructuring their branch networks. The ultimate goal is to shrink the network and progressively transform the remaining branches into semi-digital facilities with staff dedicated to complex and added-value operations. In the new model, the physical provision of basic services (e.g., deposits, withdrawals and bill payments) is internally consolidated and outsourced to external facilities, usually located at regularly visited shops (e.g., supermarkets or tobacco shops). This strategy would allow banks to maintain a capillary presence across the territory and guarantee proximity to those customers who prefer in-person channels (Baldassarre et al., 2024). In this system, external facilities cooperate to provide coverage to users among them and also with the internal branches themselves.

Lastly, in the healthcare sector, users interact with various types of facilities to receive services and health facilities are often arranged in nested hierarchical structures (Mitropoulos et al., 2006). For example, diagnostic centres, primary-care centres, and hospitals that offer progressively complex services. All these facilities cooperate in a hierarchical fashion to ensure that their patients have adequate and fair availability and accessibility to the healthcare services they commonly provide.

In the above-described settings, the HCNL may be suitable for effectively designing nested hierarchical facilities networks, where cooperation at each level and between levels can be exploited as a mechanism to cover the demand better.

In the design of service networks, the coverage level provided by a single facility to a given demand node represents its ability to attract that node and depends on its characteristics and the distance from the demand (coverage decay function). The way users simultaneously perceive the coverage levels from all facilities in terms of overall attraction depends on the type of services under consideration and the users' preferences. Despite more options within the covering radius, there are cases where users perceive only that facility with the strongest attraction (individual coverage) and others where they start experiencing the effects of other facilities in their neighbourhood. This aspect is modelled through the aggregate coverage function. The choice of adopting the joint coverage function allows us to represent a wide spectrum of situations characterized by different degrees of cooperation. Additionally, by introducing the concept of inter-level cooperation, we assume a distinct function to aggregate coverages from different levels of facilities. This is significant as it enables us to represent situations where users perceived coverages differently from the two levels.

As demonstrated by the results, the scenarios produced by the model are significantly affected by the calibration of parameters ruling the aggregations (i.e., θ^U, θ^L and θ). Although tuning these parameters in real-world applications can be challenging, as they represent how users perceive the simultaneous presence of multiple facilities, it could be very insightful for the decision-maker to analyse the scenarios generated by the model by varying them. By doing so, indeed, the decision-maker may assess the robustness of the obtained solutions in terms of the network configuration. High robustness levels can increase the decision-maker's confidence in activating specific locations in contrast to solutions highly sensitive to the parameters mentioned. Moreover, the decision-maker may appreciate the value of cooperation and how this contribution is dependent on the hypothesis made on the aggregation mechanisms.

Finally, the model may help decision-makers to allocate the available budget optimally. Generally, the higher-level facilities have a stronger attraction but a higher activation cost, while lower-level facilities have lower costs and less attractiveness. Hence, the model helps design the network to find a trade-off between location costs and user coverage. By varying the budget for activating lower-level facilities, decision-makers can assess the marginal benefit in terms of additional covered demand achievable with an extra investment. Once the marginal benefit becomes low, the decision-maker may decide that it is not convenient to expand the lower-level facility network further.

7. Conclusions

In this paper, we presented a novel cooperative covering model, namely HNCL, aiming at maximizing the covered demand by locating facilities arranged in a two-level nested hierarchy. Facilities provide a decaying coverage to demand nodes according to defined functions. First, the two cooperations among facilities at the same hierarchical level are treated separately (intra-level cooperation) and modelled as joint coverage functions. Then, a further joint coverage function is introduced to aggregate the coverage provided by facilities at different hierarchical levels (inter-level cooperation). The three introduced joint coverage functions present three key parameters – $\theta^U, \theta^L, \theta$ – regulating the cooperation degrees among facilities at the same and different levels. A given threshold defines the minimum coverage level that must be provided to a demand node to be covered. Two distinct available budgets are considered to locate facilities of each level.

A first non-linear HNCL formulation has been presented. Then, we applied a linearizing approach to the HNCL developed by extending the linearization by Karatas (2017).

Both non-linear and linear formulations have been tested on instances randomly generated as well as extracted from real-world networks of a retail company. Several results have been obtained by varying the budget for locating facilities of the lower level and the parameters $\theta^U, \theta^L, \theta$. We first compared the two formulations from a computational point of view. The results have shown that our linearizing approach outperformed the non-linear formulation both in computing times and quality of solutions. Moreover, the results demonstrate the capability of the model to provide meaningful solutions and how decision-makers may levera the key parameters $\theta^U, \theta^L, \theta$ in order to regulate the cooperation degree depending on the specific real scenario at hand, where some sort of dependencies among potential facilities may occur.

Moreover, depending on the specific real context, the decay coverage functions should be carefully modelled in order to be able to shape the customers' perceptions with respect to service coverage in a realistic manner. A possible future research direction, indeed, can involve the analysis of a suitable service sector and the application of the HNCL model to a real network by considering decay coverage functions based on real customer data obtained, for instance, by means of surveys.

Finally, the HNCL may be easily extended by formulating a gradual HNCL model involving the possibility of partial coverage of the demand. However, measuring the cooperation as defined in Section 3.3 by subdividing the contribution of each covering mechanism is no longer possible since the threshold T is not considered. Furthermore, as the size of the instance grows the HNCL problem may be challenging to solve. Thus, a further research direction could involve the development of a solution approach.

CRediT authorship contribution statement

Silvia Baldassarre: Conceptualization, Data curation, Methodology, Validation, Visualization, Writing. **Giuseppe Bruno:** Conceptualization, Data curation, Methodology, Validation, Visualization, Writing. **Ioannis Giannikos:** Conceptualization, Data curation, Methodology, Validation, Visualization, Writing. **Carmela Piccolo:** Conceptualization, Data curation, Methodology, Validation, Visualization, Writing.

Data availability

Data will be made available on request.

Appendix. Exploding $\Phi_i^U \Phi_i^L$

In order to handle the prod-part $\Phi_i^U \Phi_i^L$ $(i \in I)$ included in the inter-level aggregate coverage Φ_i, let Φ_i^U and Φ_i^L be defined as in (1):

$$\Phi_i^U \Phi_i^L = \left[\theta^U \max_{j\in J^U}\{\phi_{ij}^U x_j^U\} + \left(1-\theta^U\right)\left(1-Q_i^U\right)\right] \cdot \left[\theta^L \max_{j\in J^L}\{\phi_{ij}^L x_j^L\} + \left(1-\theta^L\right)\left(1-Q_i^L\right)\right] \tag{A.1}$$

Let us recall the auxiliary decision variable z_{ij}^k that is equal 1 when the located facility $j \in J^k$ of level $k \in K$ provides the maximum coverage to i among all the located facilities of level k and 0 otherwise, as defined in Section 3.2. Therefore, $\max_{j\in J^k}\{\phi_{ij}^k x_j^k\}$ is equivalent to $\sum_{j\in J^k} \phi_{ij}^k z_{ij}^k$ and can be replaced by the latter in (A.1).

$$\Phi_i^U \Phi_i^L = \left[\theta^U \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U + \left(1-\theta^U\right)\left(1-Q_i^U\right)\right] \cdot \left[\theta^L \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L + \left(1-\theta^L\right)\left(1-Q_i^L\right)\right]$$

$$\Phi_i^U \Phi_i^L = \left[\theta^U \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U + \left(1-\theta^U\right) - \left(1-\theta^U\right) Q_i^U\right] \cdot \left[\theta^L \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L + \left(1-\theta^L\right) - \left(1-\theta^L\right) Q_i^L\right] \tag{A.2}$$

The expression (A.2) can be exploded as follows:

$$\begin{aligned}
\Phi_i^U \Phi_i^L = {} & \theta^U \theta^L \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L + \theta^U \left(1-\theta^L\right) \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U \\
& - \theta^U \left(1-\theta^L\right) Q_i^L \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U + \\
& + \left(1-\theta^U\right)\theta^L \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L + \left(1-\theta^U\right)\left(1-\theta^L\right) - \left(1-\theta^U\right)\left(1-\theta^L\right) Q_i^L + \\
& - \left(1-\theta^U\right)\theta^L Q_i^U \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L - \left(1-\theta^U\right)\left(1-\theta^L\right) Q_i^U + \\
& + \left(1-\theta^U\right)\left(1-\theta^L\right) Q_i^U Q_i^L
\end{aligned}$$

By handling the expression and separating all the linear contributions, we obtain:

$$\begin{aligned}\Phi_i^U \Phi_i^L = {} & \theta^U \left(1-\theta^L\right) \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U + \left(1-\theta^U\right) \theta^L \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L + \\ & + \left(1-\theta^U\right)\left(1-\theta^L\right) - \left(1-\theta^U\right)\left(1-\theta^L\right) Q_i^L - \left(1-\theta^U\right)\left(1-\theta^L\right) Q_i^U + \\ & + \theta^U \theta^L \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L - \theta^U \left(1-\theta^L\right) Q_i^L \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U + \\ & - \left(1-\theta^U\right) \theta^L Q_i^U \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L + \left(1-\theta^U\right)\left(1-\theta^L\right) Q_i^U Q_i^L\end{aligned}$$

$$\begin{aligned}\Phi_i^U \Phi_i^L = {} & \theta^U \left(1-\theta^L\right) \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U + \left(1-\theta^U\right) \theta^L \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L + \\ & + \left(1-\theta^U\right)\left(1-\theta^L\right)\left(1-Q_i^U-Q_i^L\right) + && \text{(A.3a)} \\ & \theta^U \theta^L \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L && \text{(A.3b)} \\ & - \theta^U \left(1-\theta^L\right) Q_i^L \sum_{j\in J^U} \phi_{ij}^U z_{ij}^U + && \text{(A.3c)} \\ & - \left(1-\theta^U\right) \theta^L Q_i^U \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L + \left(1-\theta^U\right)\left(1-\theta^L\right) Q_i^U Q_i^L && \text{(A.3d)}\end{aligned}$$

(A.3a) includes the linear contributions, which we denote by the label Λ_i, and the non-linear contributions are (A.3b)–(A.3d). The latter are separately linearized by extending the linearizing process by Karatas (2017), as described in Section 3.2. Let Δ_i^n denote the nth non-linear contribution. Then, (A.3) is given by:

$$\Phi_i^U \Phi_i^L = \Lambda_i - \Delta_i^1 - \Delta_i^2 + \Delta_i^3 + \Delta_i^4$$

where
$$\begin{aligned} \Delta_i^1 &= \theta^U \left(1-\theta^L\right) Q_i^L \textstyle\sum_{j\in J^U} \phi_{ij}^U z_{ij}^U, \\ \Delta_i^2 &= \theta^L \left(1-\theta^U\right) Q_i^U \textstyle\sum_{j\in J^L} \phi_{ij}^L z_{ij}^L, \\ \Delta_i^3 &= \left(1-\theta^U\right)\left(1-\theta^L\right) Q_i^U Q_i^L, \\ \Delta_i^4 &= \theta^U \theta^L \textstyle\sum_{j\in J^U} \phi_{ij}^U z_{ij}^U \sum_{j\in J^L} \phi_{ij}^L z_{ij}^L. \end{aligned}$$

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