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The L^p-regularity problem for the Bergman projection of two-dimensional Rudin ball quotients ☆

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ABSTRACT

We solve the L^p-regularity problem of the Bergman projection of two-dimensional Rudin ball quotients.

1. Introduction and main results

Let D be a domain in $\mathbb{C}^n$, i.e., a non-empty, open and connected subset of $\mathbb{C}^n$. The Bergman space of D is defined as

$$A^2(D) := L^2(D) \cap \mathcal{O}(D),$$

where $L^2(D)$ denotes the space of square-integrable functions on D (with respect to the Lebesgue measure), and $\mathcal{O}(D)$ is the space of holomorphic functions on D. It is well known that $A^2(D)$ is a closed subspace of $L^2(D)$.

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The orthogonal projection $P_D : L^2(D) \to A^2(D)$ is called the *Bergman projection* associated with D and can be represented as an integral operator with kernel $K_D(z,w)$, the *Bergman kernel* of D.

While P_D is bounded on $L^2(D)$ by construction, understanding its behavior on other L^p spaces is a subtle and challenging problem. More precisely, one seeks to determine for which $p \in [1, +\infty]$ the estimate

$$\|P_D f\|_{L^p} \le C\|f\|_{L^p} \qquad \forall\, f \in L^2(D) \cap L^p(D) \tag{1}$$

holds with a constant $C > 0$. This question is often referred to as the *L^p-regularity problem for the Bergman projection*, and its resolution depends sensitively on the geometry of the domain D. For instance, it was proved in [12] that (1) fails for $p = 1$ whenever D is a smooth domain. However, it is not known if the same result is true for a generic bounded domain. Classical results reveal a highly variable picture: for various classes of smooth pseudoconvex domains [8,18,26,29,30,32,33], the range of p is known to be the whole interval $(1, \infty)$. However, there are many examples where p is restricted to a proper subinterval of $(1, \infty)$ [15,16,23,31], or even to the trivial case $p = 2$ only [2,17,24,25,37]. For a detailed historical account and further references, see the recent survey [38].

Despite the extensive literature on the subject, the precise relationship between the geometry of the domain and the regularity of the Bergman projection remains poorly understood. In recent years, this question has been investigated in increasingly general settings, in the hope of uncovering deeper structural principles governing its L^p regularity. In this paper we are interested in the setting of certain *quotient domains*, equivalently, to certain classes of singular domains that can be covered, in the sense of ramified coverings, by simpler domains, e.g., by polydiscs or balls. Progresses for the L^p-regularity problem for some families of such domains have been made in [5,9–11,21,22,28]. Other related papers worth mentioning are [1,7,13,14,19,20,34]. Here we focus in particular on some domains obtained by taking the quotient of the unit ball B_2 of $\mathbb{C}^2$ with respect to the action of a finite unitary reflection group. These domains are sometimes called *Rudin ball quotients* because of the seminal paper [36].

Throughout the paper we use the notation c to denote a positive constant, whose value may change from one occurrence to another.

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1.1. Rudin ball quotients

Let G be a finite unitary reflection group (from now on, f.u.r.g.), that is, a finite subgroup of the unitary group $U(n)$ generated by reflections. A unitary reflection is an element of $U(n)$ of finite order that fixes pointwise exactly a hyperplane. We refer to [27] for the theory of f.u.r.g.'s. These groups are characterized among finite subgroups of $U(n)$ by the property that their ring of invariants,

$$\mathbb{C}[z_1, \ldots, z_n]^G = \big\{P \in \mathbb{C}[z_1, \ldots, z_n] : P(g.z) = P(z) \quad \forall g \in G\big\},$$

is generated by n algebraically independent homogeneous polynomials $P_1, \ldots P_n$. Here, and from now on, we use the notation $g.z$ to denote the action of g in G on $\mathbb{C}^2$.

For any f.u.r.g. G and any choice of generators $P_1, \ldots, P_n$ as above we define a *G-orbit map* $\pi : \mathbb{C}^n \to \mathbb{C}^n$ as

$$\pi(z) = \big(P_1(z), \ldots P_n(z)\big), \quad z \in \mathbb{C}^n.$$

Such maps have the following significant property.

Proposition 1.1 *([36, Theorem 1.6], [11, Proposition 1.7]). Let G be a f.u.r.g. and let π be a G-orbit map as above. Let B_n be the unit ball of $\mathbb{C}^n$. Then, $D := \pi(B_n)$ is a bounded domain and $\pi : B_n \to D$ is a proper holomorphic mapping.*

Rudin actually proved in [36] that any proper holomorphic map defined on the unit ball and satisfying a mild boundary regularity condition is, up to biholomorphisms of the domain and of the target, equivalent to a G-orbit map. Moreover, any *proper map* $\pi : B_n \to D$ actually defines a *ramified covering* (see, e.g., [11, Remark 1.2]). Recall that π is a proper map if $\pi^{-1}(K)$ is compact whenever $K \subseteq D$ is compact, whereas by ramified covering we mean that there exist two proper complex analytic subsets $V \subseteq B_n$ and $W \subseteq D$ such that $\pi : B_n \backslash V \to D \backslash W$ is a finite-sheeted covering map in the usual sense (see, e.g., Chapter III of [6]) that is also holomorphic. In this case we say that D is covered by the ball B_n via the map π. It is clear that a G-orbit map π is constant on the orbit of G acting on B_n, hence we briefly write $\pi(B_n) = D = B_n/G$. From now on, when we refer to a *Rudin ball quotient*, we mean a domain D constructed in this manner.

To every ramified covering $\pi : B_n \to D$ we associate its *covering group*

$$G = \big\{g \in \mathrm{Aut}(B_n \setminus V) \ : \ \pi(g.z) = \pi(z) \ \ \forall z \in B_n \setminus V\big\}.$$

The covering is called *normal* if G acts transitively on the fibers of π, that is,

$$\pi(z_1) = \pi(z_2) \quad \iff \quad \exists\, g \in G \text{ with } g.z_1 = z_2.$$

In particular, if $\pi : B_n \to D$ is a G-orbit map, then it is a *normal ramified covering with covering group G* ([11, Proposition 1.7]).

We are interested in the L^p-regularity of the Bergman projection on $D = \pi(B_n)$. Our approach is based on lifting the L^p-regularity problem along a ramified covering. The transformation formulas underlying this procedure go back to the pioneering work of Bell [3,4] and have subsequently been used in several settings involving proper holomorphic coverings. In [9], this approach was applied to domains covered by the polydisc, yielding, in particular, a non-sharp range of L^p-regularity for the symmetrized polydisc. The boundedness range for the symmetrized bidisk was later improved in [10] and its sharpness was subsequently established in [21]. In [5] and [11], the lifting procedure is formulated in terms of the covering group and the associated group action on the lifted operator, a perspective that makes possible cancellation effects more transparent. While the boundedness arguments in [9] and [10] rely primarily on weighted estimates, such cancellation is explicitly exploited in [21]. The interplay between the covering group, the lifted operator, and cancellation is also a central feature of the present work. We conclude the present section with an example of an infinite family of finite unitary reflection groups.

Example 1.2 *([27, Chapter 2]).* Let $m, n \geq 1$ and assume that ℓ is a positive divisor of m. Let θ be a primitive m-th root of unity. Then, the f.u.r.g. $G(m, \ell, n)$ consists of the linear transformations of the form

$$(z_1, \ldots, z_n) \mapsto (\theta^{\nu_1} z_{\tau(1)}, \ldots, \theta^{\nu_n} z_{\tau(n)})$$

where τ is a permutation of $\{1, \ldots, n\}$ and the ν_j's are integers whose sum is divisible by ℓ.

An orbit map for $G(m, \ell, 2)$ is given by

$$\pi(z_1, z_2) = \big(z_1^m + z_2^m, (z_1 z_2)^{\frac{m}{\ell}}\big). \tag{2}$$

1.2. Ramified coverings and L^p-regularity

We now state a result illustrating how the L^p-regularity problem can be lifted along a ramified covering. We present the result specifically for Rudin ball quotients, although it holds in greater generality. No proofs are provided here; for full details, we refer the reader to [5, Section 4] and [11, Section 2]. The result is a consequence of a theorem of S. Bell [3, Theorem 1].

Proposition 1.3. *Let G be a f.u.r.g., let π be a G-orbit map, and set $D = \pi(B_n)$. Fix $p \in [1, +\infty)$ and $C > 0$. Then the estimate*

$$\int_D |P_D u|^p \le C \int_D |u|^p \qquad \forall\, u \in L^2(D) \cap L^p(D)$$

is equivalent to the weighted estimate

$$\int_{B_n} |Q_{B_n,G} f|^p \,\sigma \le C \int_{B_n} |f|^p \,\sigma \qquad \forall\, f \in L^2(B_n) \cap L^p(B_n, \sigma), \tag{3}$$

where $\sigma = |J(\pi)|^{2-p}$. Here $Q_{B_n,G}$ is the integral operator with kernel

$$K_G(z, w) = \frac{1}{|G|} \sum_{g \in G} K_{B_n}(z, g.w)\, \overline{J(g)}, \tag{4}$$

where $K_{B_n}(z, w)$ denotes the Bergman kernel of the unit ball B_n and $J(g)$ is the determinant of the complex jacobian matrix of g.

We point out that in the statement above, and in the rest of the paper, integrals are always with respect to the Lebesgue measure, which is therefore omitted from the notation.

1.3. Main results

By an elementary manipulation, it is clear that the weighted estimate (3) is equivalent to an unweighted estimate for the integral operator with kernel

$$K_{G,p}(z, w) = |J\pi(z)|^{\frac{2}{p}-1} K_G(z, w) |J\pi(w)|^{1-\frac{2}{p}}, \tag{5}$$

where $J(\pi)$ is the determinant of the complex jacobian matrix of π. Our main theorem provides a bound for the size of this kernel in dimension 2.

Theorem 1.4. *Let p be in $(1, +\infty)$. There exists a constant $c > 0$ such that*

$$|K_{G,p}(z, w)| \leqslant c \sum_{g \in G} |K_{B_2}(g.z, w)| \tag{6}$$

for every (z, w) in $B_2 \times B_2$.

As a consequence we solve the L^p-regularity problem for two-dimensional Rudin ball quotients.

Corollary 1.5. *Let D be a Rudin ball quotient in $\mathbb{C}^2$. Then, there exists a constant $c > 0$ such that*

$$\|P_D f\|_{L^p} \leqslant c \|f\|_{L^p} \qquad \forall f \in L^2(D) \cap L^p(D),$$

if and only if $p \in (1, +\infty)$.

We conclude the introduction with a brief explanation of why our results are established only in dimension 2. Let us first fix some notation.

Since G is a f.u.r.g., by definition it is generated by reflections. Given a reflection $r \in G$, let Y_r denote its reflecting hyperplane,

$$Y_r := \{z \in \mathbb{C}^n : r.z = z\},$$

and let $\mathcal{R}_G$ be the set of all reflecting hyperplanes of reflections in G. If $Y_r \in \mathcal{R}_G$, any unit vector ρ orthogonal to Y_r is called a *root* of the reflection r. To each $Y \in \mathcal{R}_G$, we associate the subgroup

$$G_Y := \{g \in G : g.v = v \quad \forall v \in Y\},$$

which consists of the identity together with all reflections $r \in G$ such that $Y_r = Y$. This subgroup G_Y is cyclic of some finite order m_Y, which is called the *multiplicity* of Y.

We recall the following classical result, which illustrates how the analytic description of a G-orbit map reflects the underlying geometry of the action of the group G.

Proposition 1.6 *([27, Theorem 9.8]). Choose a root ρ_Y for each reflecting hyperplane Y of G. Then the Jacobian of a G-orbit map $\pi : \mathbb{C}^n \to \mathbb{C}^n$ is given by*

$$J(\pi)(z) = c_\pi \prod_{Y \in \mathcal{R}_G} \langle z, \rho_Y \rangle^{m_Y - 1}, \tag{7}$$

where $c_\pi \in \mathbb{C}$ is a nonzero constant depending on the choice of generators, and $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian product of $\mathbb{C}^n$.

It follows that the union of the reflecting hyperplanes of G is precisely the zero set of $J(\pi)$. Thus, to estimate $K_{G,p}$, it is essential to understand how the zeros of $J(\pi)$ interact with K_G.

In dimension 2, we can take advantage of the fact that two distinct hyperplanes intersect only at the origin. Consequently, if $z \neq 0$, two factors in (7) cannot vanish simultaneously. In higher dimensions, the geometry is more intricate: distinct hyperplanes intersect in positive-dimensional subspaces, and a deeper understanding of the arrangement of the hyperplanes of a f.u.r.g. is required.

2. Preliminary results

Unless strictly necessary, we will state and prove our results for the n-dimensional ball B_n, specializing to the two-dimensional case only when it cannot be avoided. To obtain our main estimate, we decompose $\overline{B_n} \times \overline{B_n}$ in several regions, which depend on the geometry of the given f.u.r.g. G. For each g in G and for each $\varepsilon > 0$ set

$$\mathcal{U}_g(\varepsilon) = \left\{(z,w) \in \overline{B_n} \times \overline{B_n} : d(z, bB_n) + d(w, bB_n) + |g.z - w| < \varepsilon\right\}.$$

The next elementary lemma shows why the regions $\mathcal{U}_g(\varepsilon)$'s are useful to localize the singularities of the kernel

$$K_G(z,w) = \frac{1}{|G|}\sum_{g\in G} K_{B_n}(z, g.w)\,\overline{J(g)} = \frac{1}{|G|}\sum_{g\in G} K_{B_n}(g.z, w)\,J(g) = \frac{1}{|G|}\sum_{g\in G} \frac{J(g)}{(1 - \langle g.z, w\rangle)^{n+1}}.$$

Lemma 2.1. *Let $\varepsilon > 0$ and let g in G be fixed. Set*

$$\mathcal{S}_g(\varepsilon) = \left\{(z,w) \in \overline{B_n} \times \overline{B_n} : |1 - \langle g.z, w\rangle| < \varepsilon\right\}.$$

Then,

$$\mathcal{U}_g(\varepsilon) \subseteq \mathcal{S}_g(3\varepsilon) \subseteq \mathcal{U}_g(12\varepsilon).$$

Proof. Let (z, w) be in $\mathcal{U}_g(\varepsilon)$. Then,

$$|1 - \langle g.z, w\rangle| \leqslant (1 - |w|^2) + |\langle g.z - w, w\rangle| \leqslant 2d(w, bB_n) + |g.z - w||w| < 3\varepsilon.$$

Hence, the pair (z, w) belongs to $\mathcal{S}_g(3\varepsilon)$. Now,

$$3\varepsilon > |1 - \langle g.z, w\rangle| \geqslant 1 - |g.z||w| \geqslant 1 - |w| = d(w, bB_n),$$

where we used that g is a unitary matrix. Similarly, we obtain

$$3\varepsilon > |1 - \langle g.z, w\rangle| \geqslant 1 - |g.z| = 1 - |z| = d(z, bB_n).$$

Then,

$$|g.z - w| \leqslant (1 - |g.z|) + (1 - |w|) < 6\varepsilon.$$

In conclusion we get that $\mathcal{S}_g(3\varepsilon)$ is a subset of $\mathcal{U}_g(12\varepsilon)$. □

It is crucial to understand how the regions $\mathcal{U}_g(\varepsilon)$ interact with one another as g varies.

Lemma 2.2. *Let $n = 2$. There exists $\varepsilon > 0$ such that, for all $g, \ell \in G, g \neq \ell$, with $\ell^{-1}g$ not a reflection, one has*

$$\mathcal{U}_g(\varepsilon) \cap \mathcal{U}_\ell(\varepsilon) = \emptyset.$$

Proof. Let g and ℓ such that $\ell^{-1}g$ is not a reflection be fixed. Assume that such ε does not exist. Then, for every positive ε_k there exists (z_k, w_k) in $\mathcal{U}_g(\varepsilon_k) \cap \mathcal{U}_\ell(\varepsilon_k)$, that is,

$$d(z_k, bB_2) + d(w_k, bB_2) + |g.z_k - w_k| < \varepsilon_k$$

and

$$d(z_k, bB_2) + d(w_k, bB_2) + |\ell.z_k - w_k| < \varepsilon_k.$$

Let us consider a sequence $\{\varepsilon_k\}_k$ such that $\varepsilon_k \to 0^+$ as $k \to +\infty$ and the associated sequence $\{(z_k, w_k)\}_k$. Since the latter is a bounded sequence, it admits a limit point (z, w). From the above we deduce that, necessarily, it holds

$$(z, w) \in bB_2 \times bB_2$$

and

$$g.z = w = \ell.z \iff \ell^{-1}g.z = z.$$

Thus, z is a fixed point of $\ell^{-1}g$. However, since $\ell^{-1}g$ is not a reflection and we are in dimension 2, its only fixed point is the origin, but z belongs to bB_2. This is a contradiction. Therefore, there exists $\varepsilon > 0$ such that $\mathcal{U}_g(\varepsilon) \cap \mathcal{U}_\ell(\varepsilon) = \emptyset$. The conclusion follows by letting vary g and ℓ and observing that they vary in a finite group. □

The situation changes significantly if, on the contrary, $\ell^{-1}g$ is a reflection.

Lemma 2.3. *Let g, ℓ be in G and assume that $\ell^{-1}g$ is a reflection. Then, for every $\varepsilon > 0$ it holds*

$$\mathcal{U}_g(\varepsilon) \cap \mathcal{U}_\ell(\varepsilon) \neq \emptyset.$$

Proof. Let $\varepsilon > 0$ be fixed. Set $Y_{\ell^{-1}g}$ to be the reflecting hyperplane of $\ell^{-1}g$. Let z be in $Y_{\ell^{-1}g}$ such that $d(z, bB_n) < \varepsilon/2$ and consider the pair $(z, \ell.z)$. Then, since ℓ is unitary and $z \in Y_{\ell^{-1}g}$,

$$d(z, bB_n) + d(\ell.z, bB_n) + |g.z - \ell.z| = d(z, bB_n) + d(z, bB_n) + |\ell^{-1}g.z - z| < \varepsilon,$$

since $\ell^{-1}g.z = z$. Hence, $(z, \ell.z)$ is in $\mathcal{U}_g(\varepsilon)$. Similarly,

$$d(z, bB_n) + d(\ell.z, bB_n) + |\ell.z - \ell.z| = d(z, bB_n) + d(z, bB_n) < \varepsilon,$$

that is, $(z, \ell.z)$ is in $\mathcal{U}_\ell(\varepsilon)$. Hence, the conclusion follows. □

Remark 2.4. The situation becomes more delicate when examining the intersection of three or more regions, so we restrict to the case $n = 2$. In this setting, such an intersection may or may not be empty for sufficiently small ε, depending on the orders of the reflections involved.

Assume that for every $\varepsilon > 0$,

$$\mathcal{U}_g(\varepsilon) \cap \mathcal{U}_h(\varepsilon) \cap \mathcal{U}_\ell(\varepsilon) \neq \emptyset.$$

Arguing as in the previous lemmas, this would yield the existence of (z, w) in $bB_2 \times bB_2$ such that

$$w = g.z = h.z = \ell.z.$$

Thus, since we are in dimension 2, z must lie on the reflecting hyperplanes of the (possible) reflections $\ell^{-1}h$, $\ell^{-1}g$, and $g^{-1}h$. Since in $\mathbb{C}^2$ two distinct hyperplanes intersect only at the origin, this can only occur if these three reflections share the same reflecting hyperplane. However, if G contains only reflections of order 2, the intersection must necessarily be the origin, as there cannot be two distinct reflections in G with the same reflecting hyperplane.

On the other hand, if r in G is a reflection of order $m \geq 3$, then $Y_{r^j} = Y_r$ for every $j = 1, \ldots, m-1$. Take $z \in Y_r$ such that $d(z, bB_2) < \frac{\varepsilon}{2}$ and consider the pair $(z, r^j.z)$. Then

$$d(z, bB_2) + d(r^j.z, bB_2) + |r^s.z - r^j.z| = d(z, bB_2) + d(z, B_2) + |r^{s-j}.z - z| \leq \varepsilon,$$

so that, for every $\varepsilon > 0$, the pair $(z, r^j.z)$ belongs to each region $\mathcal{U}_{r^s}(\varepsilon)$ for $s = 0, \ldots, m-1$.

We conclude the section with a general observation on the symmetry properties of the kernel $K_{G,p}$ defined in (5). A stronger statement was proved in [11, Lemma 4.1], but for our purposes a simpler fact will suffice. With the notation introduced in Proposition 1.6, we state the following classical result.

Proposition 2.5 *([27, Lemma 9.10]). Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be a skew polynomial, i.e.,*

$$P(g.z) = J(g)^{-1}P(z) \qquad \forall g \in G.$$

Then $\prod_{Y \in \mathcal{R}_G} \langle z, \rho_Y \rangle^{m_Y - 1}$ divides P.

Notice that, by Proposition 1.6, the function $\prod_{Y\in\mathcal{R}_G}\langle z,\rho_Y\rangle^{m_Y-1}$ coincides, up to a nonzero multiplicative constant, with the Jacobian $J(\pi)$ of any G-orbit map π.

Recall that

$$K_{G,p}(z,w)=|J\pi(z)|^{\frac{2}{p}-1}|J\pi(w)|^{1-\frac{2}{p}}K_G(z,w),$$

where $\pi:\mathbb{C}^n\to\mathbb{C}^n$ is a G-orbit map and

$$K_G(z,w)=\frac{1}{|G|}\sum_{g\in G}J(g)K_{B_n}(g.z,w)=\frac{1}{|G|}\frac{Q(z,w)}{\prod_{\ell\in G}(1-\langle \ell.z,w\rangle)^{n+1}}$$

with

$$Q(z,w)=\sum_{m\in G}J(m)\prod_{\ell\neq m}(1-\langle \ell.z,w\rangle)^{n+1}.$$

Notice that $P(z,w)=Q(z,\overline{w})$ is a holomorphic polynomial as a function of z and w. Also, $P(z,w)$ is a skew polynomial with the respect to the natural action of the f.u.r.g. $G\oplus\overline{G}$. Indeed, for $(g,\overline{h})\in G\oplus\overline{G}$, we have

$$\begin{aligned}P(g.z,\overline{h}.w)=Q(g.z,h.\overline{w})&=\sum_{m\in G}J(m)\prod_{\ell\neq m}(1-\langle h^{-1}\ell g.z,\overline{w}\rangle)^{n+1}\\&=\sum_{m\in G}J(m)\prod_{\ell\neq h^{-1}mg}(1-\langle \ell.z,\overline{w}\rangle)^{n+1}=\sum_{hkg^{-1}\in G}J(h)J(k)\overline{J(g)}\prod_{\ell\neq k}(1-\langle \ell.z,\overline{w}\rangle)^{n+1}\\&=J(h)\overline{J(g)}Q(z,\overline{w})=J(g\oplus\overline{h})^{-1}P(z,w).\end{aligned}$$

By Proposition 2.5, it follows that $J\pi(z)\overline{J\pi(\overline{w})}$ divides $P(z,w)$ for any G-orbit map π. Namely,

$$Q(z,\overline{w})=P(z,w)=J\pi(z)\overline{J\pi(\overline{w})}M(z,w),\tag{8}$$

where $M(z,w)$ is a holomorphic polynomial in z and w. In conclusion, we get

$$\begin{aligned}|K_{G,p}(z,w)|&=|J\pi(z)|^{\frac{2}{p}-1}|J\pi(w)|^{1-\frac{2}{p}}|K_G(z,w)|\\&=|J\pi(z)|^{\frac{2}{p}}|J\pi(w)|^{2-\frac{2}{p}}\frac{|M(z,\overline{w})|}{|\prod_{\ell\in G}(1-\langle \ell.z,w\rangle)^{n+1}|}.\end{aligned}\tag{9}$$

Notice that the powers of the jacobian factors in (9) are positive for every $p\in(1,\infty)$. The problem is now to understand how the zero sets of $M(z,\overline{w})$ and of the denominator interact with each other.

3. Bounds on $K_{G,p}$

In this section we bound the kernel $K_{G,p}$ by decomposing $\overline{B_n}\times\overline{B_n}$ using the regions $\mathcal{U}_g(\varepsilon)$'s. From now on, ε will always denote a sufficiently small positive constant, chosen so that Lemma 2.2 applies.

Lemma 3.1. *Let $n=2$, let G be a f.u.r.g. and fix g in G. There exists a constant $c>0$ such that, for every (z,w) in*

$$\mathcal{I}_g(\varepsilon):=\mathcal{U}_g(\varepsilon)\setminus\Bigg(\bigcup_{\substack{\ell\in G,\\ \ell^{-1}g\ ref}}\Big(\mathcal{U}_g(\varepsilon)\cap\mathcal{U}_\ell(\varepsilon)\Big)\Bigg),$$

we have $|K_{G,p}(z,w)|\leqslant c|K_{B_2}(g.z,w)|$.

Proof. By (9) we know that

$$|K_{G,p}(z,w)| = |J\pi(z)|^{\frac{2}{p}}|J\pi(w)|^{2-\frac{2}{p}}\frac{|M(z,\overline{w})|}{|\prod_{\ell\in G}(1-\langle \ell.z,w\rangle)^{n+1}|}.$$

Since $(z,w)\in\mathcal{U}_g(\varepsilon)$, by Lemma 2.1, we have

$$|1-\langle g.z,w\rangle|<3\varepsilon.$$

Let now $\ell\neq g$ be in G such that $\ell^{-1}g$ is not a reflection. If ε is sufficiently small so that $\mathcal{U}_g(12\varepsilon)\cap\mathcal{U}_\ell(12\varepsilon)=\varnothing$ (see Lemma 2.2), then

$$|1-\langle \ell.z,w\rangle\,|\geqslant 3\varepsilon,\quad \forall (z,w)\in\mathcal{I}_g(\varepsilon).$$

Otherwise, by Lemma 2.1, we would have (z,w) in $\mathcal{U}_g(12\varepsilon)\cap\mathcal{U}_\ell(12\varepsilon)\neq\varnothing$. If $\ell^{-1}g$ is a reflection, then, for every (z,w) in $\mathcal{I}_g(\varepsilon)$, we have $|1-\langle \ell.z,w\rangle\,|\geqslant\varepsilon/4$. Otherwise, again by Lemma 2.1, (z,w) would belong to $\mathcal{U}_\ell(\varepsilon)\cap\mathcal{U}_g(\varepsilon)$ but this cannot be the case by assumption since $(z,w)\in\mathcal{I}_g(\varepsilon)$. In conclusion,

$$|1-\langle \ell.z,w\rangle|\geqslant\varepsilon/4,\quad \forall (z,w)\in\mathcal{I}_g(\varepsilon),\forall \ell\in G,\ell\neq g.$$

Therefore, for every (z,w) in $\mathcal{I}_g(\varepsilon)$, it holds

$$|K_{G,p}(z,w)|\leqslant c\frac{|J\pi(z)|^{\frac{2}{p}}|J\pi(w)|^{2-\frac{2}{p}}}{|\prod_{\ell\in G}(1-\langle \ell.z,w\rangle)^{n+1}|}\leqslant\frac{c}{|1-\langle g.z,w\rangle|^{n+1}}=c|K_{B_n}(g.z,w)|.\quad\square$$

We now turn to estimating the kernel $K_{G,p}$ on the region $\mathcal{U}_g(\varepsilon)\cap\mathcal{U}_\ell(\varepsilon)$ when $\ell^{-1}g$ is a reflection. This requires some intermediate steps and auxiliary results. The next two lemmas are adaptations of [11, Theorem A.1], originally proved for the case of a group generated by a single reflection of order 2. We show that the same strategy can be implemented for reflections of arbitrary finite order, with suitable modifications. We include the proofs to make the argument self-contained. For clarity, we restrict to the two-dimensional setting, pointing out that an analogous reasoning applies in higher dimensions as well.

Lemma 3.2. *Let $G\subseteq U(2)$ be the f.u.r.g.*

$$G=\left\{\begin{pmatrix} e^{\frac{2\pi ij}{m}} & 0\\ 0 & 1\end{pmatrix}, j=0,\dots,m-1\right\}.$$

Set

$$\mathcal{G}=\left\{(z,w)\in B_2\times B_2:|z_1\overline{w_1}|\geqslant\tfrac{1}{2}|1-z_2\overline{w_2}|\right\}. \tag{10}$$

If $(z,w)\in\mathcal{G}^c$, we have the identity

$$K_G(z,w)=\frac{1}{m}\sum_{j=0}^{m-1}\frac{e^{\frac{2\pi ij}{m}}}{(1-e^{\frac{2\pi ij}{m}}z_1\overline{w}_1-z_2\overline{w}_2)^3}=\frac{(z_1\overline{w}_1)^{m-1}}{(1-z_2\overline{w}_2)^{m+2}}A(z,w), \tag{11}$$

where $A(z,w)$ is a bounded function on $\mathcal{G}^c$.

Proof. The proof follows from a straightforward computation. For (z,w) in $\mathcal{G}^c$ we have

$$\frac{1}{(1-e^{\frac{2\pi ij}{m}}z_1\overline{w_1}-z_2\overline{w_2})^3}=\frac{1}{(1-z_2\overline{w_2})^3}\sum_{k=0}^{+\infty}\frac{(k+2)(k+1)}{2}\left(\frac{e^{\frac{2\pi ij}{m}}z_1\overline{w_1}}{1-z_2\overline{w_2}}\right)^k.$$

So that,

$$\begin{aligned}K_G(z,w)&=\frac{1}{m(1-z_2\overline{w_2})^3}\sum_{j=0}^{m-1}e^{\frac{2\pi ij}{m}}\sum_{k=0}^{+\infty}\frac{(k+2)(k+1)}{2}\left(\frac{e^{\frac{2\pi ij}{m}}z_1\overline{w_1}}{1-z_2\overline{w_2}}\right)^k\\&=\frac{1}{m(1-z_2\overline{w_2})^3}\sum_{k=0}^{+\infty}\frac{(k+2)(k+1)}{2}\left(\frac{z_1\overline{w_1}}{1-z_2\overline{w_2}}\right)^k\sum_{j=0}^{m-1}e^{\frac{2\pi ij}{m}(k+1)}.\end{aligned}$$

The inner sum is nonzero if and only if $k\equiv -1 \mod m$. Hence,

$$\begin{aligned}K_G(z,w)&=\frac{1}{(1-z_2\overline{w_2})^3}\sum_{\ell=1}^{+\infty}\frac{((\ell m-1)+2)((\ell m-1)+1)}{2}\left(\frac{z_1\overline{w_1}}{1-z_2\overline{w_2}}\right)^{\ell m-1}\\&=\frac{1}{(1-z_2\overline{w_2})^3}\left(\frac{z_1\overline{w_1}}{1-z_2\overline{w_2}}\right)^{m-1}\sum_{\ell=0}^{+\infty}\frac{(m(\ell+1)+1)(m(\ell+1))}{2}\left(\frac{z_1\overline{w_1}}{1-z_2\overline{w_2}}\right)^{\ell m}.\end{aligned}$$

The last series converges absolutely since in $\mathcal{G}^c$ we have $|z_1\overline{w}_1|/|1-z_2\overline{w}_2|<1/2$. □

Lemma 3.3. *Let $G\subseteq U(2)$ be a f.u.r.g. generated by a single reflection of any finite order. Then, there exists a constant $c>0$ such that*

$$|K_{G,p}(z,w)|\leqslant c\sum_{g\in G}|K_{B_2}(g.z,w)|,\quad \forall (z,w)\in B_2\times B_2.$$

Proof. First note that if the inequality above holds for a f.u.r.g. $G\subseteq U(2)$, it also holds for each conjugate f.u.r.g. H as well. Suppose that $G=uHu^{-1}=\{uhu^{-1}:h\in H\}$ for a $u\in U(2)$, then formula (4) shows that $K_H(z,w)=K_G(uz,uw)$. The orbit maps π_G and π_H can be taken to be related by $\pi_H=\pi_G\circ u$, so that $|J\pi_H(z)|=|J\pi_G(u(z))|$, and consequently $K_{H,p}(z,w)=K_{G,p}(uz,uw)$. Therefore assuming that the result holds for the group G we have

$$\begin{aligned}|K_{H,p}(z,w)|=|K_{G,p}(uz,uw)|\leqslant c\sum_{g\in G}|K_{B_2}(g.uz,uw)|&=c\sum_{g\in G}|K_{B_2}(u^{-1}g.uz,w)|\\&=c\sum_{h\in H}|K_{B_2}(h.z,w)|,\end{aligned}$$

so that the result also holds for the conjugate group H (with the same constant). Therefore, after a unitary conjugation, we may assume that

$$G=\left\{\begin{pmatrix}e^{\frac{2\pi ij}{m}}&0\\0&1\end{pmatrix}:j=0,\dots,m-1\right\}.$$

For this group one has

$$K_{G,p}(z,w)=\frac{1}{m}\left(\frac{|z_1|}{|w_1|}\right)^{(m-1)(\frac{2}{p}-1)}\sum_{j=0}^{m-1}\frac{e^{\frac{2\pi ij}{m}}}{\left(1-e^{\frac{2\pi ij}{m}}z_1\overline{w_1}-z_2\overline{w_2}\right)^3}.$$

Let $\mathcal{G}$ be the region defined in (10). If $(z,w) \in \mathcal{G}^c$, then by Lemma 3.2 we obtain

$$K_{G,p}(z,w) = \left(\frac{|z_1|}{|w_1|}\right)^{(m-1)(\frac{2}{p}-1)} \frac{(z_1\overline{w}_1)^{m-1}}{(1-z_2\overline{w}_2)^{m+2}} A(z,w).$$

Therefore,

$$|K_{G,p}(z,w)| \leqslant c\frac{|z_1|^{(m-1)\frac{2}{p}}\,|w_1|^{(m-1)(2-\frac{2}{p})}}{|1-z_2\overline{w_2}|^{m+2}}.$$

Observe that

$$|z_1|^2 \leqslant 1-|z_2|^2 = (1+|z_2|)(1-|z_2|) \leqslant 2(1-|z_2|) \leqslant 2|1-z_2\overline{w}_2|,$$

and a similar estimate holds for $|w_1|$. Hence,

$$|K_{G,p}(z,w)| \leqslant c\frac{1}{|1-z_2\overline{w}_2|^3}.$$

Since $(z,w) \in \mathcal{G}^c$, we further have

$$|1-z_1\overline{w}_1 - z_2\overline{w}_2| \leqslant \tfrac{3}{2}|1-z_2\overline{w}_2|,$$

and thus

$$|K_{G,p}(z,w)| \leqslant c\,|K_{B_2}(z,w)|. \tag{12}$$

On the other hand, if $(z,w) \in \mathcal{G}$, then

$$\frac{|z_1|}{|w_1|} = \frac{|z_1|^2}{|w_1||z_1|} \leqslant 2\frac{|z_1|^2}{|1-z_2\overline{w}_2|} \leqslant 2\frac{1-|z_2|^2}{1-|z_2||\overline{w}_2|} \leqslant 4,$$

and the same bound holds for $\frac{|w_1|}{|z_1|}$. Consequently,

$$|K_{G,p}(z,w)| \leqslant c\sum_{g\in G} |K_{B_2}(g.z,w)|. \tag{13}$$

The claim follows by combining (12) and (13). $\square$

Our next goal is to prove the following major step toward the proof of Theorem 1.4.

Proposition 3.4. *Let G be a f.u.r.g. There exists a constant $c>0$ such that, for every reflection $r \in G$,*

$$|K_{G,p}(z,w)| \leqslant c\sum_{\ell\in G} |K_{B_2}(\ell.z,w)|, \qquad \forall (z,w) \in \mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon).$$

Here id *denotes the identity element of the group G.*

We need two elementary lemmas.

Lemma 3.5. *Let $G \subseteq U(n)$ be a f.u.r.g. Then there are constants C_1 and C_2 such that, for each reflection $r \in G$ and for each $z \in \mathbb{C}^n$,*

$$C_1 \left|\langle z, \rho\rangle\right| \leq |r.z - z| \leq C_2 \left|\langle z, \rho\rangle\right|, \tag{14}$$

where ρ is a root of the reflection r.

Proof. Denote by Π the orthogonal projection from $\mathbb{C}^n$ onto Y_r, the reflecting hyperplane of r. Then

$$r.z = \Pi z + e^{i\theta_r} \langle z, \rho\rangle \rho$$

where $0 < \theta_r \leq \pi$. Since $z = \Pi z + \langle z, \rho\rangle \rho$ we see that

$$z - r.z = (1 - e^{i\theta_r}) \langle z, \rho\rangle \rho,$$

so that

$$|z - r.z|^2 = 4 \sin^2 \frac{\theta_r}{2} \cdot \left|\langle z, \rho\rangle\right|^2,$$

so that the result follows, with $C_1 = 2 \min\limits_{r\in G} \sin \frac{\theta_r}{2}$ and $C_2 = 2 \max\limits_{r\in G} \sin \frac{\theta_r}{2}$ $\square$

Remark 3.6. Observe that the unitary change of coordinates

$$\begin{cases} \zeta = h.z, \\ \omega = w \end{cases}$$

ensures that $(z, w) \in \mathcal{U}_g(\varepsilon) \cap \mathcal{U}_h(\varepsilon)$ if and only if $(\zeta, \omega) \in \mathcal{U}_{gh^{-1}}(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$, where id denotes the identity element of the group. Moreover, if $h^{-1}g$ is a reflection, then so is gh^{-1}, since

$$gh^{-1} = h(h^{-1}g)h^{-1},$$

that is, gh^{-1} is obtained by conjugation of a reflection. Consequently, it suffices to focus on regions of the form $\mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$ with r a reflection. Estimates on the more general regions $\mathcal{U}_g(\varepsilon) \cap \mathcal{U}_h(\varepsilon)$ with $h^{-1}g$ a reflection will then follow by a unitary change of coordinates.

Lemma 3.7. *Let G be a f.u.r.g. Then there exists a constant $c > 0$ such that, for each $r \in G$,*

$$\mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon) \subseteq \{(z, w) : |\langle z, \rho\rangle| \leq c\varepsilon, |\langle w, \rho\rangle| \leq c\varepsilon\},$$

where ρ is a root of r.

Proof. Let $(z, w) \in \mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$. By (14) and the definition of $\mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$ we immediately get

$$|\langle z, \rho\rangle| \leq \frac{1}{C_1} |r.z - z| \leqslant \frac{1}{C_1} (|r.z - w| + |z - w|) < \frac{2\varepsilon}{C_1}. \tag{15}$$

Notice that $(z, w) \in \mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$ if and only if $(w, z) \in \mathcal{U}_{r^{-1}}(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$. Then (15) shows that we must have $|\langle w, \rho\rangle| \leq c\varepsilon$ as well. $\square$

We are ready to prove Proposition 3.4. Here it is essential that we are working in dimension $n = 2$.

Proof Proposition 3.4. Recall that Y_r is the reflecting hyperplane of r, that is,

$$Y_r = \Big\{ z : \langle z, \rho_r \rangle = 0 \Big\}, \quad |\rho_r| = 1.$$

By Lemma 3.7, if $(z,w) \in \mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$, then both z and w are uniformly close to the hyperplane Y_r. Hence, this and the fact that two hyperplanes in $\mathbb{C}^2$ only intersect at the origin, assure that for (z,w) in $\mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$, the linear forms $\langle z, \rho_g \rangle, \langle w, \rho_g \rangle$ are bounded from below for every $g \neq r$, g reflection in G. Here ρ_g denotes a root of the hyperplane Y_g. Hence, recalling that $J(\pi)$ is given by (7), we get that, for every $(z,w) \in \mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$,

$$|K_{G,p}(z,w)| \leqslant c |\langle z, \rho_r \rangle|^{(\frac{2}{p}-1)(m_r-1)} |\langle w, \rho_r \rangle|^{(1-\frac{2}{p})(m_r-1)} |K_G(z,w)|, \tag{16}$$

where m_r is the order of the reflection r.

Let now $H = \langle r \rangle$ be the f.u.r.g. generated by the single reflection r and consider the lateral decomposition of G with respect to this subgroup H, that is,

$$G = \bigsqcup_{g \in G/H} gH.$$

Then, if we denote by g^* the dual operator of g with respect to the Hermitian inner product of $\mathbb{C}^2$, we get

$$\begin{aligned} K_G(z,w) &= \frac{1}{|G|} \sum_{g \in G/H} J(g) \sum_{\ell \in H} J(\ell) K_{B_2}(g\ell.z, w) = \frac{|H|}{|G|} \sum_{g \in G/H} J(g) K_H(z, g^*.w) \\ &= c\Bigg(K_H(z,w) + \sum_{\substack{g \in G/H \\ g \neq \mathrm{id}}} J(g) K_H(z, g^*.w) \Bigg) =: c\Big(K_H(z,w) + B(z,w) \Big). \end{aligned} \tag{17}$$

By Lemma 3.3 we know that

$$|\langle z, \rho_r \rangle|^{(\frac{2}{p}-1)(m_r-1)} |\langle w, \rho_r \rangle|^{(1-\frac{2}{p})(m_r-1)} |K_H(z,w)| \leqslant c |K_{B_2}(z,w)| \tag{18}$$

for every (z,w) in $B_2 \times B_2$. Let us now focus on

$$B(z,w) = \sum_{\substack{g \in G/H \\ g \neq \mathrm{id}}} J(g) K_H(z, g^*.w) = \sum_{\substack{g \in G/H \\ g \neq \mathrm{id}}} J(g) \frac{\sum_{\ell \in H} J(\ell) \prod_{q \neq \ell} (1 - \langle q.z, g^*.w \rangle)^3}{\prod_{\ell \in H} (1 - \langle \ell.z, g^*.w \rangle)^3}.$$

From (8) and Proposition 2.5 we obtain that

$$B(z,w) = \sum_{\substack{g \in G/H \\ g \neq \mathrm{id}}} J(g) \frac{(\langle z, \rho_r \rangle \overline{\langle g^*.w, \rho_r \rangle})^{m_r - 1} P_g(z,w)}{\prod_{\ell \in H} (1 - \langle \ell.z, g^*.w \rangle)^3},$$

where $P_g(z,w)$ is a holomorphic polynomial in z and antiholomorphic in w. We conclude that

$$B(z,w) = \langle z, \rho_r \rangle^{m_r - 1} \frac{P_H(z,w)}{\prod_{\substack{g \in G \\ g \notin H}} (1 - \langle g.z, w \rangle)^3}, \tag{19}$$

where $P_H(z,w)$ is a holomorphic polynomial as a function of z and an antiholomorphic polynomial as a function of w. We use the notation P_H with the subscript H to emphasize that such polynomial is determined by the subgroup H. Recall also that

$$B(z,w)=\frac{1}{|H|}\bigg(\sum_{g\in G}\frac{J(g)}{(1-\langle g.z,w\rangle)^3}-\sum_{\ell\in H}\frac{J(\ell)}{(1-\langle \ell.z,w\rangle)^3}\bigg).$$

Hence, $B(z,w)=\overline{B(w,z)}$ since the right-hand side of the above identity satisfies the same property. Also, setting

$$L(z,w)=\prod_{g\in G\setminus H}(1-\langle g.z,w\rangle)^3,$$

we notice that $L(z,w)=\overline{L(w,z)}$ as well. Indeed,

$$\begin{aligned}L(z,w)&=\prod_{g\in G\setminus H}(1-\langle g.z,w\rangle)^3=\overline{\prod_{g\in G\setminus H}(1-\langle w,g.z\rangle)^3}\\&=\overline{\prod_{g\in G\setminus H}(1-\langle g^*.w,z\rangle)^3}=\overline{\prod_{g\in G\setminus H}(1-\langle g.w,z\rangle)^3}\\&=\overline{L(w,z)},\end{aligned}$$

where the second-last equality holds true since $g\notin H \iff g^*=g^{-1}\notin H$ being H a subgroup. Therefore,

$$\begin{aligned}B(z,w)=\overline{B(w,z)}&\iff \langle z,\rho_r\rangle^{m_r-1}\,\frac{P_H(z,w)}{L(z,w)}=\overline{\langle w,\rho_r\rangle}^{m_r-1}\,\frac{\overline{P_H(w,z)}}{\overline{L(w,z)}}\\&\iff \langle z,\rho_r\rangle^{m_r-1}\,P_H(z,w)=\overline{\langle w,\rho_r\rangle}^{m_r-1}\,\overline{P_H(w,z)}.\end{aligned}$$

Now, we see that whenever the antiholomorphic linear form $\overline{\langle w,\rho_r\rangle}$ vanishes the antiholomorphic polynomial (as a function of w) $P_H(z,w)$ vanishes as well. Therefore in the polynomial ring $\mathbb{C}[z_1,\dots,z_n,\overline{w_1},\dots,\overline{w_n}]$ in $2n$ variables, the polynomial p given by $p(z_1,\dots,\dots,z_n,\overline{w_1},\dots,\overline{w_n})=\overline{\langle w,\rho_r\rangle}$ divides $P_H(z,w)$: this follows by either directly from the Nullstellensatz, or by a linear change of variables in the polynomial ring in which $\overline{\langle w,\rho_r\rangle}$ is taken to be one of the new variables. Repeating this argument for each ρ_r, we conclude that

$$\langle z,\rho_r\rangle^{m_r-1}\,P_H(z,w)=\langle z,\rho_r\rangle^{m_r-1}\,\overline{\langle w,\rho_r\rangle}^{m_r-1}Q_H(z,w),$$

where Q_H is a holomorphic polynomial as a function of z and antiholomorphic as a function of w. In conclusion,

$$B(z,w)=\frac{\langle z,\rho_r\rangle^{m_r-1}\,\overline{\langle w,\rho_r\rangle}^{m_r-1}Q_H(z,w)}{L(z,w)}.$$

Hence,

$$\left(\frac{|\,\langle z,\rho_r\rangle\,|}{|\,\langle w,\rho_r\rangle\,|}\right)^{\left(\frac{2}{p}-1\right)(m_r-1)}|B(z,w)|\leqslant |\,\langle z,\rho_r\rangle\,|^{\frac{2}{p}(m_r-1)}|\,\langle w,\rho_r\rangle\,|^{\left(2-\frac{2}{p}\right)(m_r-1)}\frac{|Q_H(z,w)|}{|L(z,w)|}\leqslant c. \tag{20}$$

The last estimate follows from the fact that $Q_H(z,w)$ is a polynomial, hence bounded in $\overline{B_2}\times\overline{B_2}$, and that in $\mathcal{U}_r(\varepsilon)\cap\mathcal{U}_{\mathrm{id}}(\varepsilon)$ we have $1/|L(z,w)|$ bounded as well. Indeed, for $g\in G\setminus H$, in order to have

$$|1-\langle z,g^*.w\rangle\,|=|1-\langle g.z,w\rangle\,|<\varepsilon/4$$

we would need (z,w) in $\mathcal{U}_g(\varepsilon)$ by Lemma 2.1. However, we already have that $(z,w) \in \mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$ and, by Remark 2.4, we know that $\mathcal{U}_g(\varepsilon) \cap \mathcal{U}_r(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$ is necessarily empty unless $g \in H$. By (18) and (20) we conclude the proof. □

We now prove the same estimate as in Proposition 3.4, but in more general regions.

Proposition 3.8. *Let G be a f.u.r.g. There exists $c > 0$ such that*

$$|K_{G,p}(z,w)| \leqslant c \sum_{\ell \in G} |K_{B_2}(\ell.z, w)|,$$

for every (z,w) in $\mathcal{U}_g(\varepsilon) \cap \mathcal{U}_h(\varepsilon)$ with $h^{-1}g$ a reflection.

Proof. We have that $(z,w) \in \mathcal{U}_g(\varepsilon) \cap U_h(\varepsilon)$ if and only if $(\zeta,\omega) \in \mathcal{U}_{gh^{-1}}(\varepsilon) \cap \mathcal{U}_{\mathrm{id}}(\varepsilon)$ where

$$\begin{cases} \zeta = h.z \\ \omega = w. \end{cases}$$

Recall that gh^{-1} is a reflection whenever $h^{-1}g$ is (Remark 3.6). Then,

$$\begin{aligned} |K_{G,p}(z,w)| = |K_{G,p}(h^{-1}.\zeta,\omega)| &= |J(\pi)(h^{-1}.\zeta)|^{\frac{2}{p}-1} |J(\pi)(\omega)|^{1-\frac{2}{p}} \frac{1}{|G|} \left| \sum_{g \in G} \frac{J(g)}{(1 - \langle gh^{-1}.\zeta, \omega\rangle)^3} \right| \\ &= |J(\pi)(\zeta)|^{\frac{2}{p}-1} |J(\pi)(\omega)|^{1-\frac{2}{p}} \frac{|J(h)|}{|G|} \left| \sum_{g \in G} \frac{J(g)}{(1 - \langle g.\zeta, \omega\rangle)^3} \right| \\ &= |K_{G,p}(\zeta,\omega)|. \end{aligned}$$

Notice that we used the identity $|J\pi(h^{-1}.\zeta)| = |J\pi(\zeta)|$. This follows from the fact that if ρ_r is a root of the hyperplane Y_r associated to the reflection r, then $h.\rho_r$ is a root for the reflecting hyperplane associated to the reflection hrh^{-1}. Thus, by Proposition 3.4, we obtain

$$|K_{G,p}(z,w)| = |K_{G,p}(\zeta,\omega)| \leqslant c \sum_{g \in G} |K_{B_2}(g.\zeta,\omega)| \leqslant c \sum_{g \in G} |K_{B_2}(gh.z,w)| \leqslant c \sum_{g \in G} |K_{B_2}(g.z,w)|. \quad \square$$

The proof of Theorem 1.4 is now immediate.

Proof of Theorem 1.4. By Lemma 3.1 and Proposition 3.8, we already have the desired estimate for $|K_{G,p}|$ on the region $\bigcup_{g \in G} \mathcal{U}_g(\varepsilon)$. On the complement of this region in $B_2 \times B_2$, we can directly use the identity (9) to see that $|K_{G,p}|$ remains bounded.

Combining these observations, we thus obtain the uniform estimate

$$|K_{G,p}(z,w)| \leq c \sum_{g \in G} |K_{B_2}(g.z,w)|, \qquad \forall (z,w) \in B_2 \times B_2,$$

as claimed. □

At last, we solve the L^p-regularity problem for every Rudin ball quotients in $\mathbb{C}^2$ as promised.

Proof of Corollary 1.5. If $p \in (1,+\infty)$ the result follows from the main estimate (6) and the well-known L^p-regularity of the positive Bergman projection of the unit ball, that is, the integral operator associated to the kernel $|K_{B_2}(z,w)|$ (see, e.g., [35, Chapter 7]). The unboundedness for $p = 1, +\infty$ follows from adapting a known argument for the unboundedness of the Bergman projection of the unit ball; see [35] and [11, Theorem 1.16] $\square$

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