Mathematische Zeitschrift (2024) 308:52
https://doi.org/10.1007/s00209-024-03615-4
Mathematische Zeitschrift

Single radius spherical cap discrepancy on compact two-point homogeneous spaces

L. Brandolini[1] · B. Gariboldi[1] · G. Gigante[1] · A. Monguzzi[1]

Received: 4 June 2024 / Accepted: 3 September 2024

Abstract

In this note we study estimates from below of the single radius spherical discrepancy in the setting of compact two-point homogeneous spaces. Namely, given a d-dimensional manifold $\mathcal{M}$ endowed with a distance ρ so that $(\mathcal{M}, \rho)$ is a two-point homogeneous space and with the Riemannian measure μ, we provide conditions on r such that if D_r denotes the discrepancy of the ball of radius r, then, for an absolute constant $C > 0$ and for every set of points $\{x_j\}_{j=1}^N$, one has $\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) \geqslant CN^{-1-\frac{1}{d}}$. The conditions on r that we have depend on the dimension d of the manifold and cannot be achieved when $d \equiv 1 \pmod 4$. Nonetheless, we prove a weaker estimate for such dimensions as well.

Keywords Discrepancy · Lower bounds · Irregularities of distribution

Mathematics Subject Classification 11K38 · 43A85 · 33C45

1 Introduction

In this work we provide some estimates from below for the single radius spherical discrepancy in the setting of compact two-point homogeneous spaces. To provide some context, let us start by recalling a result of Beck. In [1], but see also [2, Corollary 24c, pg.182], Beck proved that it is not possible to distribute a finite sequence of points on the unit sphere $\mathcal{S}^d$ so that such distribution of points is regular with respect to spherical caps. Namely, for $x \in \mathcal{S}^d$ and $h \in [-1, 1]$, let $B(x, h)$ be the spherical cap defined as

$$B(x,h) = \left\{ y \in \mathcal{S}^d : x \cdot y \geq h \right\}.$$

✉ G. Gigante
giacomo.gigante@unibg.it

L. Brandolini
luca.brandolini@unibg.it

B. Gariboldi
biancamaria.gariboldi@unibg.it

A. Monguzzi
alessandro.monguzzi@unibg.it

1 Dipartimento di Ingegneria Gestionale, dell'Informazione e della Produzione, Università degli Studi di Bergamo, Viale G. Marconi 5, 24044 Dalmine, BG, Italy

Given a finite sequence of points $\{x_j\}_{j=1}^N$, let μ be the normalized surface measure of $\mathcal{S}^d$ and let the discrepancy $D(B(x,h))$ of $B(x,h)$ be defined as

$$D(B(x,h)) = \frac{1}{N}\sum_{j=1}^{N} \chi_{B(x,h)}(x_j) - \mu(B(x,h)).$$

Then, one has the following result ([1]). There exists a constant $c > 0$ such that for any finite sequence of point $\{x_j\}_{j=1}^N$ it holds

$$\int_{-1}^{1}\int_{\mathcal{S}^d} |D(B(x,h))|^2 \, d\mu(x) dh \geq cN^{-1-\frac{1}{d}}. \tag{1}$$

An important feature to point out in Beck's estimate is that in the left-hand side of (1) one averages over all the possible values of h. It is still unclear, and a matter of investigation, if such averaging ploy is necessary or not to obtain large discrepancy estimates. In regard of this matter, Montgomery [19, 20] proved, for instance, that in order to have large discrepancy for discs in the two dimensional torus, it is enough to average on two discs only, one of radius $1/2$ and the other of radius $1/4$. His strategy is based on an inequality on exponential sums, now known as Cassels-Montgomery inequality [20, Theorem 5.12], which can be naturally extended to higher dimensions, and has been proved to hold for eigenfunctions of the Laplacian on compact Riemannian manifolds [8, 16]. This strategy has proven to be very versatile, allowing to obtain several estimates from below of the discrepancy with respect to different testing families such as rectangles or convex sets [7, 10–12], or in different ambient spaces other than the d-dimensional torus, such as the sphere itself or compact two-point homogeneous spaces [3, 9].

In particular, in [9] the first three authors of this paper proved a similar "two radius" estimate for the discrepancy with respect to balls in the setting of compact two-point homogeneous spaces. However, before stating such result it is necessary to recall Skriganov's generalization of Beck's result [26].

Let $\mathcal{M}$ be a compact d-dimensional Riemannian manifold endowed with metric ρ so that $(\mathcal{M}, \rho)$ is a two-point homogeneous space (the definition of such spaces is recalled later). Let μ be the normalized Riemannian measure on $\mathcal{M}$ and let $B_r(x) = \{y \in \mathcal{M} : \rho(x,y) < r\}$. For a given finite sequence of points $\left\{x_j\right\}_{j=1}^N \subseteq \mathcal{M}$ and positive weights $\left\{a_j\right\}_{j=1}^N$ such that $a_1 + a_2 + \cdots + a_N = 1$ we define the discrepancy of the ball $B_r(x)$ by

$$D_r(x) = \sum_{j=1}^{N} a_j \chi_{B_r(x)}(x_j) - \mu(B_r(x)). \tag{2}$$

In the case of equal weights, it is proved in [26, Theorem 2.2] that if η is a positive, locally integrable function on $(0, \pi)$ which satisfies a suitable integrability condition, then there exists $c > 0$ such that for every finite sequence of N points the estimate

$$\int_0^{\pi}\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x)\eta(r) dr \geq cN^{-1-\frac{1}{d}} \tag{3}$$

holds. Such estimate is known to be optimal. Indeed, the existence of distributions with quadratic discrepancy bounded above by $cN^{-1-1/d}$ has been proved (with probabilistic arguments) even in more general settings than compact two-point homogeneous spaces, including all compact d-dimensional Riemannian manifolds. See for example [5, Corollary 8.2], [25, Theorem 1.1] or [27, Theorem 2.1]. In fact, in the case of compact two-point homogeneous spaces, Skriganov [26, Corollary 2.1] proved a deterministic result, showing that for well-distributed optimal cubature formulas one has

$$\int_0^{\pi} \int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x)\eta(r)dr \leq cN^{-1-\frac{1}{d}}. \tag{4}$$

The existence of such cubature formulas for the sphere has been proved by Bondarenko, Radchenko and Vyazovska in [4], whereas for more general Riemannian manifolds it has been proved in [13, 15].

Going back to the issue of single radius estimates, notice that, similarly to Beck's result, also in the left-hand side of (3) one is averaging over all the possible radii. However, the following two radii result is proved in [9]. If $d \not\equiv 1 (\mathrm{mod}\, 4)$, for any $0 < r < \frac{\pi}{2}$, one has the sharp estimate

$$\int_{\mathcal{M}} \left(|D_r(x)|^2 + |D_{2r}(x)|^2\right) d\mu(x) \geq cN^{-1-\frac{1}{d}}.$$

If $d \equiv 1 (\mathrm{mod}\, 4)$ the technique used in [9], which goes back to Montomery's aforementioned works, fails. It fails also if one is willing to average on any finite number of radii. At the cost of losing the sharpness, a single radius estimate, which holds for any dimension, is also proved in [9]. Namely, for every $\varepsilon > 0$ and for almost every $0 < r < \pi$ there exists a constant $c > 0$ such that, for every finite sequence $\{x_j\}_{j=1}^N \subseteq \mathcal{M}$, one has

$$\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) \geq cN^{-1-\frac{3}{d}-\varepsilon}. \tag{5}$$

In a recent work of Bilyk, Mastrianni and Steinerberger a sharp single radius estimate is proved for some suitable radii in the case of $\mathcal{M} = \mathcal{S}^d$ with $d \not\equiv 1$ (mod 4). Precisely, in [3, Theorem 5] it is proved that if $\mathcal{M} = \mathcal{S}^d$ with $d \not\equiv 1$ (mod 4) and if $\cos r$ is $(d+1)/2-$gegenbadly approximable, then

$$\int_{\mathcal{M}} |D_r(x)|^2 d\mu(x) \geq cN^{-1-\frac{1}{d}}.$$

We recall that $x \in (-1, 1)$ is λ-gegenbadly approximable, $\lambda > 0$, if there exists a constant $c_x > 0$ such that, for all $m \in \mathbb{N}$,

$$|P_m^{\lambda}(x)| \geq c_x m^{\lambda-1},$$

where P_m^{λ} denotes the Gegenbauer polynomial of degree m (see [3, Section 1]). In particular, a necessary and sufficient condition to check if a number is λ-gegenbadly approximable or not is also given in [3]. See also Corollary 3.6 here.

Our first result is in the same spirit as the one by Bilyk–Mastrianni–Steinerberger. However, before stating it, we need to precisely introduce the setting in which we are working. A d-dimensional Riemannian manifold $\mathcal{M}$ with distance ρ is said to be a two-point homogeneous space if given four points $x_1, x_2, y_1, y_2 \in \mathcal{M}$ such that $\rho(x_1, y_1) = \rho(x_2, y_2)$, then there exists an isometry g of $\mathcal{M}$ such that $gx_1 = x_2$ and $gy_1 = y_2$. Compact connected two-point homogeneous spaces have been completely characterized by Wang [30]. Namely, it turns out that $\mathcal{M}$ is isometric to one of the following compact rank 1 symmetric spaces:

(i) the Euclidean sphere $\mathcal{S}^d = SO(d+1)/SO(d) \times \{1\}, d \geqslant 1$;
(ii) the real projective space $P^n(\mathbb{R}) = O(n+1)/O(n) \times O(1), n \geqslant 2$;
(iii) the complex projective space $P^n(\mathbb{C}) = U(n+1)/U(n) \times U(1), n \geqslant 2$;
(iv) the quaternionic projective space $P^n(\mathbb{H}) = Sp(n+1)/Sp(n) \times Sp(1), n \geqslant 2$;

(v) the octonionic projective plane $P^2(\mathbb{O})$.

From now on we will always assume that $\mathcal{M}$ is one of the above symmetric spaces with a metric ρ normalized so that $\operatorname{diam}(\mathcal{M}) = \pi$ and the Riemannian measure μ normalized so that $\mu(\mathcal{M}) = 1$. If d denotes the real dimension of $P^n(\mathbb{F})$, then $d = nd_0$, where $d_0 = 1, 2, 4, 8$ according to the real dimension of $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ respectively. In the case of $\mathcal{S}^d$ it will be convenient to set $d_0 = d$. See [14, pp. 176-178], see also [17], [26] and [31]. Recall that $D_r(x)$ denotes the discrepancy for balls defined as in (2). The following is our extension to compact two point homogeneous spaces of the result in [3].

Theorem 1.1 *Let p and q be coprime integers, $0 < p < q$, such that*

$$\frac{d + d_0 + 2}{4} p - \frac{d-1}{4} q \notin \mathbb{Z}. \tag{6}$$

Then, there exists a constant $C > 0$ such that for every set of points $\left\{x_j\right\}_{j=1}^N \subseteq \mathcal{M}$ and positive weights $\left\{a_j\right\}_{j=1}^N$ satisfying $a_1 + a_2 + \cdots + a_N = 1$ we have

$$\int_{\mathcal{M}} \left|D_{p\pi/q}(x)\right|^2 d\mu(x) \geq C N^{-1-\frac{1}{d}}.$$

We point out that condition (6) can always be achieved for particular choices of p and q, except when $\mathcal{M}$ is the Euclidean sphere or the real projective space with dimension $d \equiv 1 \pmod 4$. Observe that in all the other spaces we are interested in, the dimension is even, and therefore $d \not\equiv 1 \pmod 4$. In this case, indeed, condition (6) is simply satisfied by choosing, for example, $p \in 4\mathbb{Z}$ and q odd. Again this result is optimal in view of Corollary 8.2 in [5]. Moreover, with the same technique used by Skriganov to prove the estimate from above for cubature formulas (4), it is possible to prove a uniform estimate in the radius r,

$$\int_{\mathcal{M}} |D_r(x)|^2 d\mu(x) \leq c N^{-1-\frac{1}{d}},$$

for suitable choices of weights $\{a_j\}$ and points $\{x_j\}$. See [9, Theorem 15] for a precise statement.

When $d \equiv 1 \pmod 4$ the technique we use to prove Theorem 1.1 fails since the estimate of Theorem 3.4 does not hold. In the case of the d-dimensional torus $\mathbb{T}^d$ it is known that for such values of d the discrepancy can actually be a bit smaller than the expected value $N^{-1-\frac{1}{d}}$. See [22, Theorem 3.1]. See also [6], [18] and [21]. We remind that when $d \equiv 1 \pmod 4$, $\mathcal{M}$ can only be the Euclidean sphere or the real projective space. In this case we can prove the following result (cf. with [9, Theorem 2]).

Theorem 1.2 *Let $d \equiv 1 \pmod 4$. Let $\{q_n\}$ be the sequence of primes in increasing order and let $\{p_n\}$ be a sequence of positive integers such that for some $\delta > 0$ and for every n we have $\delta \leq p_n/q_n \leq 1 - \delta$. Then, for every $N \geq 3$, for every choice of points $\{x_j\}_{j=1}^N$ and positive weights $\{a_j\}_{j=1}^N$ with $\sum_{j=1}^N a_j = 1$ there exists $n \leq c \log N / \log\log N$ such that*

$$\int_{\mathcal{M}} \left|D_{p_n\pi/q_n}(x)\right|^2 d\mu(x) \geq C N^{-1-\frac{1}{d}} \frac{\log\log N}{\log^4 N}.$$

In the next section we recall some preliminaries necessary to work in two-point homogeneous spaces, whereas in Section 3 we prove our main results.

With the notation $A \approx B$, we mean the fact that there exist two constant c_1 and c_2 independent of the involved variables such that $c_1 A \leq B \leq c_2 A$.

2 Harmonic analysis preliminaries in two-point homogeneous spaces

The preliminaries contained in this section are essentially taken from the literature and from [9]. Hence, we omit most of the proofs and we invite the reader to refer to [9] and the references therein.

In the following, to keep notation simple, we will use

$$a = \frac{d-2}{2}, \qquad b = \frac{d_0 - 2}{2}.$$

Recall that if $\mathcal{M} = P^n(\mathbb{F})$, then $d = nd_0$, where $d_0 = 1, 2, 4, 8$ according to the real dimension of $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ respectively. Instead, if $\mathcal{M} = \mathcal{S}^d$ it is convenient to set $d = d_0$.

If o is a fixed point in $\mathcal{M}$, then $\mathcal{M}$ can be identified with the homogeneous space G/K, where G is the group of isometries of $\mathcal{M}$ and K is the stabilizer of o. We will also identify functions $F(x)$ on $\mathcal{M}$ with right K-invariant functions $f(g)$ on G by setting $f(g) = F(x)$ when $go = x$. If μ is the Riemannian measure on $\mathcal{M}$ normalized so that $\mu(\mathcal{M}) = 1$, then μ is invariant under the action of G, in other words, for every $g \in G$,

$$\int_{\mathcal{M}} F(gx)d\mu(x) = \int_{\mathcal{M}} F(x)d\mu(x)$$

Definition 2.1 A function F on $\mathcal{M}$ is a zonal function (with respect to o) if for every $x \in \mathcal{M}$ and every $k \in K$ we have $F(kx) = F(x)$. We will say that F is zonal with respect to y if $F(gx)$ is a zonal function (with respect to o) and $y = go$.

Lemma 2.2 *Let F be a zonal function. Then $F(x)$ depends only on $\rho(x, o)$. Furthermore, defining F_0 so that $F(x) = F_0(\rho(x, o))$ we have*

$$\int_{\mathcal{M}} F(x)d\mu(x) = \int_0^{\pi} F_0(r)A(r)dr, \tag{7}$$

where

$$A(r) = c(a, b)\left(\sin\frac{r}{2}\right)^{2a+1}\left(\cos\frac{r}{2}\right)^{2b+1}$$

and

$$c(a, b) = \left(\int_0^{\pi}\left(\sin\frac{r}{2}\right)^{2a+1}\left(\cos\frac{r}{2}\right)^{2b+1}dr\right)^{-1} = \frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)}.$$

Let Δ be the Laplace-Beltrami operator on $\mathcal{M}$, let $\lambda_0, \lambda_1, \ldots,$ be the distinct eigenvalues of $-\Delta$ arranged in increasing order, let $\mathcal{H}_m$ be the eigenspace associated with the eigenvalue λ_m, and let d_m its dimension. It is well known that

$$L^2(\mathcal{M}) = \bigoplus_{m=0}^{+\infty}\mathcal{H}_m. \tag{8}$$

If $F(x) = F_0(\rho(x, o))$ is a zonal function on $\mathcal{M}$, then

$$\Delta F(x) = \frac{1}{A(t)}\frac{d}{dt}\left(A(t)\frac{d}{dt}F_0(t)\right)\Bigg|_{t=\rho(x,o)} \tag{9}$$

(see (4.16) in [14]).

Definition 2.3 The zonal spherical function of degree $m \in \mathbb{N}$ with pole $x \in \mathcal{M}$ is the unique function $Z_x^m \in \mathcal{H}_m$, given by the Riesz representation theorem, such that for every $Y \in \mathcal{H}_m$

$$Y(x) = \int_{\mathcal{M}} Y(y) Z_x^m(y) d\mu(y).$$

The next lemma summarizes the main properties of zonal functions. The case $\mathcal{M} = \mathcal{S}^d$ is discussed in detail in [28], whereas we refer to [9, Lemma 6] for the general case.

Lemma 2.4 *i) If $Y_m^1, \ldots, Y_m^{d_m}$ is an orthonormal basis of $\mathcal{H}_m \subset L^2(\mathcal{M})$, then*

$$Z_x^m(y) = \sum_{\ell=1}^{d_m} \overline{Y_m^\ell(x)} Y_m^\ell(y).$$

ii) *Z_x^m is real valued and $Z_x^m(y) = Z_y^m(x)$.*
iii) *If $g \in G$, then $Z_{gx}^m(gy) = Z_x^m(y)$.*
iv) $\forall x \in \mathcal{M},\ \|Z_x^m\|_\infty = Z_x^m(x) = \|Z_x^m\|_2^2 = d_m$.
v) *$Z_o^m(x)$ is a zonal function and*

$$Z_o^m(x) = \frac{d_m}{P_m^{a,b}(1)} P_m^{a,b}(\cos(\rho(x, o))) \tag{10}$$

where $P_m^{a,b}$ are the Jacobi polynomials.
vi) *$\left\{d_m^{-1/2} Z_o^m\right\}_{m=0}^{+\infty}$ is an orthonormal basis of the subspace of $L^2(\mathcal{M})$ of zonal functions.*
vii) *Let $\mathbb{P}_m$ denote the orthogonal projection of $L^2(\mathcal{M})$ onto $\mathcal{H}_m$. Then for every zonal function f,*

$$\mathbb{P}_m f(x) = d_m^{-1} \int_{\mathcal{M}} f(y) Z_o^m(y) d\mu(y) Z_o^m(x).$$

viii) $\lambda_m = m(m + a + b + 1)$.
ix) *We have*

$$\begin{aligned} d_m &= (2m + a + b + 1) \frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(a+b+2)} \frac{\Gamma(m+a+b+1)}{\Gamma(m+b+1)} \frac{\Gamma(m+a+1)}{\Gamma(m+1)} \\ &\approx m^{d-1}. \end{aligned}$$

3 Proof of the main results

In this section we prove our main results. In order to do so we need some preliminary results taken from [9]. In the following lemma we obtain an identity for the L^2 discrepancy which separates the contribution of the geometry of the balls $B_r(x)$ from the one of the sequence $\{x_j\}_{j=1}^N$ and of the weights $\{a_j\}_{j=1}^N$.

Lemma 3.1 *Let $D_r(x)$ be as in (2). Then*

$$\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) = \sum_{m=1}^{+\infty} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^N a_j Y_m^\ell(x_j) \right|^2 d_m^{-2} \left| \int_{B_r(o)} Z_o^m(y) d\mu(y) \right|^2.$$

Proof By (8) we have

$$\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) = \sum_{m=0}^{+\infty} \int_{\mathcal{M}} |\mathbb{P}_m D_r(x)|^2 d\mu(x)$$

Since o is arbitrary and $\chi_{B_r(x_j)}$ is zonal with respect to x_j, applying (vii) of Lemma 2.4 with o substituted by x_j, we have

$$\mathbb{P}_m \left(\chi_{B_r(\cdot)}(x_j)\right)(x) = \mathbb{P}_m \left(\chi_{B_r(x_j)}\right)(x) = d_m^{-1} \int_{\mathcal{M}} \chi_{B_r(x_j)}(y) Z_{x_j}^m(y) d\mu(y) Z_{x_j}^m(x).$$

Overall, applying (iii) of Lemma 2.4 we obtain

$$\mathbb{P}_m D_r(x) = \sum_{j=1}^{N} a_j d_m^{-1} \int_{B_r(o)} Z_o^m(y) d\mu(y) Z_{x_j}^m(x) - \delta_0(m)\mu(B_r(o)),$$

where $\delta_0(m)$ is the Kronecker delta. In particular $\mathbb{P}_0 D_r(x) = 0$ and for $m > 0$

$$\begin{aligned} \mathbb{P}_m D_r(x) &= d_m^{-1} \sum_{j=1}^{N} a_j \int_{B_r(o)} Z_o^m(y) d\mu(y) Z_{x_j}^m(x) \\ &= d_m^{-1} \sum_{\ell=1}^{d_m} \left(\sum_{j=1}^{N} a_j \overline{Y_m^\ell(x_j)} \right) \int_{B_r(o)} Z_o^m(y) d\mu(y) Y_m^\ell(x). \end{aligned}$$

Finally,

$$\int_{\mathcal{M}} |D_r(x)|^2 d\mu(x) = \sum_{m=1}^{+\infty} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 d_m^{-2} \left| \int_{B_r(o)} Z_o^m(y) d\mu(y) \right|^2 .$$

□

It is clear from the above lemma that we now need to estimate the quantities

$$\sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \quad \text{and} \quad d_m^{-2} \left| \int_{B_r(o)} Z_o^m(y) d\mu(y) \right|^2 . \tag{11}$$

The first of these quantities is controlled with a Cassels-Montgomery-type estimate in the following proposition.

Proposition 3.2 *There exist $C_0, C_1 > 0$ such that for every $L \geq M > 0$ we have*

$$\sum_{m=M}^{L} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \geq C_1 \sum_{j=1}^{N} a_j^2 L^d - C_0 M^d.$$

Proof By the Cassels-Montgomery inequality for manifolds (see Theorem 1 in [8] or Theorem 9 in [16]) along with the fact that

$$\sum_{m=0}^{L} d_m \approx \sum_{m=0}^{L} m^{d-1} \approx L^d,$$

we have

$$\sum_{m=0}^{L}\sum_{\ell=1}^{d_m}\left|\sum_{j=1}^{N} a_j Y_m^{\ell}(x_j)\right|^2 \geq C_1 \sum_{j=1}^{N} a_j^2 L^d.$$

Since, by Lemma 2.4 (iv) $Z_{x_j}^m(x_k) \leq d_m$,

$$\sum_{m=0}^{M-1}\sum_{\ell=1}^{d_m}\left|\sum_{j=1}^{N} a_j Y_m^{\ell}(x_j)\right|^2 = \sum_{m=0}^{M-1}\sum_{j,k=1}^{N} a_j a_k Z_{x_j}^m(x_k) \leq \sum_{m=0}^{M-1} d_m \leq C_0 M^d,$$

and the proposition follows. □

Moving now to the second quantity in (11), in the next lemma we obtain an identity for the integral of zonal spherical functions on a ball.

Lemma 3.3 *For any $0 \leq r \leq \pi$ and for any $m \geq 1$ we have*

$$\int_{B_r(o)} Z_o^m(x)d\mu(x) = \frac{c(a,b)d_m}{m P_m^{a,b}(1)} P_{m-1}^{a+1,b+1}(\cos r)\left(\sin\frac{r}{2}\right)^{2a+2}\left(\cos\frac{r}{2}\right)^{2b+2}.$$

Proof By (7) and (10),

$$\begin{aligned}\int_{B_r(o)} Z_o^m(x)d\mu(x) &= \frac{d_m}{P_m^{a,b}(1)}\int_0^r P_m^{a,b}(\cos t)A(t)dt\\ &= \frac{c(a,b)d_m}{2^{a+b+1}P_m^{a,b}(1)}\int_{\cos r}^1 P_m^{a,b}(x)(1-x)^a(1+x)^b dx,\end{aligned}$$

and the thesis follows applying Rodrigues' formula (see [29, (4.3.1)])

$$P_m^{a,b}(x)(1-x)^a(1+x)^b = -\frac{1}{2m}\frac{d}{dx}\left(P_{m-1}^{a+1,b+1}(x)(1-x)^{a+1}(1+x)^{b+1}\right).$$

□

By Lemma 3.1 we need to estimate

$$\left|\int_{B_r(o)} Z_o^m(y)d\mu(y)\right|^2. \tag{12}$$

However, it is clear from the previous lemma that this quantity vanishes when $\cos r$ is a zero of the Jacobi polynomial $P_{m-1}^{a+1,b+1}$. On the other hand, since (see [29, formula (4.3.3)])

$$\begin{aligned}&\int_0^{\pi} P_m^{\alpha,\beta}(\cos r)^2\left(\sin\frac{r}{2}\right)^{2\alpha+1}\left(\cos\frac{r}{2}\right)^{2\beta+1}dr\\ &\quad= \frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)\Gamma(m+1)\Gamma(m+\alpha+\beta+1)} \geq \frac{C}{m},\end{aligned}$$

we would expect that, on average, $|P_m^{\alpha,\beta}(\cos r)| \geq Cm^{-1/2}$. The following theorem identifies the values of r for which this relation holds.

Theorem 3.4 *Let $\alpha > -1$ and $\beta > -1$, and call $\gamma = (\alpha+\beta+1)/2$ and $\delta = -(2\alpha-1)/4$. Let $r \in \mathbb{R} \setminus \pi\mathbb{Z}$. There exist positive constants C and m_0 such that for $m \geq m_0$,*

$$\left|P_m^{\alpha,\beta}(\cos r)\right| \geq Cm^{-\frac{1}{2}}$$

if and only if $r/\pi = p/q \in \mathbb{Q} \setminus \mathbb{Z}$*, with* p *and* q *coprime, and*

$$\gamma p + \delta q \notin \mathbb{Z}. \tag{13}$$

Notice that if $r \in \pi\mathbb{Z}$, then the situation is somewhat particular. More precisely $|P_m^{\alpha,\beta}(\cos r)| = |P_m^{\alpha,\beta}(\pm 1)|$ and this is $|\binom{m+\alpha}{m}| \approx m^\alpha$ or $|(-1)^m\binom{m+\beta}{m}| \approx m^\beta$.

Proof Remember that for $r \notin \pi\mathbb{Z}$ (see [29, Theorem 8.21.8])

$$\begin{aligned} &P_m^{\alpha,\beta}(\cos r)\\ &= m^{-\frac{1}{2}}\pi^{-\frac{1}{2}}\left(\sin\frac{r}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{r}{2}\right)^{-\beta-\frac{1}{2}}\cos\left(\left(m+\frac{\alpha+\beta+1}{2}\right)r-\frac{2\alpha+1}{4}\pi\right)+O(m^{-\frac{3}{2}}). \end{aligned}$$

We are now looking for values of r for which there exists $\eta > 0$ and $m_0 \geq 1$ such that

$$\left|\cos\left(\left(m+\frac{\alpha+\beta+1}{2}\right)r-\frac{2\alpha+1}{4}\pi\right)\right| \geq \eta \tag{14}$$

for all integer values of $m \geq m_0$. If $r/\pi \notin \mathbb{Q}$ then $\{mr : m \geq m_0\}$ is dense mod π in $[0, \pi]$ and condition (14) cannot be achieved. Assume now that $r/\pi = p/q \in \mathbb{Q}$ with p and q coprime. Then condition (14) is equivalent to asking that, for all $m = 1, \ldots, q$,

$$\left(m+\frac{\alpha+\beta+1}{2}\right)\frac{p}{q}-\frac{2\alpha+1}{4}+\frac{1}{2} \notin \mathbb{Z},$$

(notice that the above condition is q-periodic in m) or equivalently

$$Q = pm + \frac{\alpha+\beta+1}{2}p - \frac{2\alpha-1}{4}q = pm + \gamma p + \delta q \quad \text{is not a multiple of } q. \tag{15}$$

Also, (15) is equivalent to require that

$$H = \gamma p + \delta q \notin \mathbb{Z}.$$

Indeed, if H is not an integer, then Q is not an integer either and (15) holds. Conversely, assume that H is an integer. Then, since p and q are coprime, the equation $pm + qj = -H$ has integer solutions, and (15) does not hold. □

According to the values of γ and δ, it may be the case that condition (13) holds for any possible choice of coprime p and q, for particular values of p and q or for no values of p and q. The following proposition shows all the possible different cases.

Proposition 3.5 *Let* $p/q \in \mathbb{Q} \setminus \mathbb{Z}$*. Then we have the following.*

- *(i) Suppose* $1, \gamma, \delta$ *linearly independent over* $\mathbb{Q}$*. Then* (13) *holds for any choice of coprime* p *and* q*.*
- *(ii) Suppose* γ *and* δ *irrational, there exist integers* j_1*,* j_2 *and* j_3 *with no common divisors such that* j_1 *and* j_2 *have a nontrivial common divisor, and* $j_1\gamma + j_2\delta + j_3 = 0$*. Then* (13) *holds for any choice of coprime* p *and* q*.*
- *(iii) Suppose* γ *and* δ *irrational, there exist integers* j_1*,* j_2 *and* j_3 *such that* j_1 *and* j_2 *have no nontrivial common divisors and* $j_1\gamma + j_2\delta + j_3 = 0$*. Then* (13) *holds for any choice of coprime* p *and* q*, except for* $p/q = j_1/j_2$*.*
- *(iv) Suppose* γ *rational and* δ *irrational, or viceversa. Then* (13) *holds for any choice of coprime* p *and* q*.*
- *(v) Suppose* γ *and* δ *are integers. Then* (13) *does not hold for any choice of coprime* p *and* q*.*

(vi) Suppose that both γ and δ are rational, but at least one of them is not integer. Then there exist coprime p and q such that (13) is achieved.

Proof If $1, \gamma, \delta$ are linearly independent over the rationals, then $\gamma p + \delta q$ is not an integer and therefore condition (13) is achieved for any choice of p/q.

Let us now study the case $1, \gamma, \delta$ linearly dependent over the rationals. In particular, there exist three integers j_1, j_2, j_3, not all equal to 0, such that

$$j_1\gamma + j_2\delta + j_3 = 0.$$

Assume without loss of generality that j_1, j_2 and j_3 do not have a nontrivial common divisor.

Suppose first that both γ and δ are irrational. Then $j_1 \neq 0$ and $j_2 \neq 0$, and

$$\delta = -\frac{j_1}{j_2}\gamma - \frac{j_3}{j_2}$$

so that

$$\gamma p + \delta q = \gamma p + \left(-\frac{j_1}{j_2}\gamma - \frac{j_3}{j_2}\right) q = \left(\frac{p}{q} - \frac{j_1}{j_2}\right) q\gamma - \frac{j_3}{j_2}q.$$

If $p/q \neq j_1/j_2$, then $\gamma p + \delta q$ is not rational and (13) is achieved.

Assume now $p/q = j_1/j_2$.

Suppose first that j_1 and j_2 are coprime. Then $q = \pm j_2$ and since

$$\gamma p + \delta q = -\frac{j_3}{j_2}q = \pm j_3,$$

it follows that (13) is not achieved.

Suppose now that $j_1 = hp$ and $j_2 = hq$ for some integer $h \neq \pm 1$. Of course h does not divide j_3. Then

$$\gamma p + \delta q = -\frac{j_3}{j_2}q = -\frac{j_3}{h} \notin \mathbb{Z},$$

and (13) is achieved.

Suppose now γ irrational and δ rational. Since $p \neq 0$, then $\gamma p + \delta q$ is not rational and condition (13) is achieved.

Similarly, if δ is irrational and γ is rational, since $q \neq 0$, then $\gamma p + \delta q$ is not rational and condition (13) is achieved.

If $\gamma, \delta \in \mathbb{Z}$, then (13) is not achieved for any value of p and q.

Finally, if γ and δ are both rational, but, say, δ is not integer, then it suffices to let p be any integer such that $\gamma p \in \mathbb{Z}$, and q any integer prime with p such that $\delta q \notin \mathbb{Z}$. □

Since the Gegenbauer polynomials are particular cases of the Jacobi polynomials,

$$P_m^\lambda(x) = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)\Gamma(m + 2\lambda)}{\Gamma(2\lambda)\Gamma\left(m + \lambda + \frac{1}{2}\right)} P_m^{\lambda-\frac{1}{2},\lambda-\frac{1}{2}}(x)$$

($\lambda > -1/2$), we can recover the result in [3] on λ-gegenbadly approximable numbers.

Corollary 3.6 *Let $\lambda > -1/2$. Let $r \in \mathbb{R} \setminus \pi\mathbb{Z}$. There exist positive constants C and m_0 such that for $m \geq m_0$,*

$$\left|P_m^\lambda(\cos r)\right| \geq C m^{\lambda-1}$$

if and only if $r/\pi = p/q \in \mathbb{Q} \setminus \mathbb{Z}$, *with* p *and* q *coprime, and*

$$\lambda p - \frac{\lambda - 1}{2} q \notin \mathbb{Z}. \tag{16}$$

In particular, the following three cases are quickly treated.

(i) Suppose λ *is irrational. Then* (16) *holds for any choice of coprime* p *and* q*, except for* $p/q = 1/2$.

(ii) Suppose λ *is an odd integer. Then* (16) *does not hold for any choice of coprime* p *and* q.

(iii) Suppose that $\lambda \in \mathbb{Q}$*, but* λ *is not an odd integer. Then there exist coprime* p *and* q *such that* (16) *is achieved.*

Proof It suffices to apply Theorem 3.4 and Proposition 3.5, and notice that the cases (i), (ii) and (iv) of Proposition 3.5 do not occur. □

We are now ready to apply Theorem 3.4 and Proposition 3.5 to estimate (12) from below.

Lemma 3.7 *Let* $0 < r < \pi$. *There exist positive constants* C *and* m_0 *such that for* $m \geq m_0$,

$$\left| \int_{B_r(o)} Z_o^m(x) d\mu(x) \right| \geq C d_m m^{-a-3/2}$$

if and only if $r/\pi = p/q$*, where* p *and* q *are coprime and*

$$\frac{d + d_0 + 2}{4} p - \frac{d-1}{4} q \notin \mathbb{Z}.$$

In particular, in this case, $d \not\equiv 1 \pmod 4$.

Observe that the above condition can easily be satisfied, for example take $p \in 4\mathbb{Z}$ and q odd.

Proof By Lemma 3.3, for any $0 \leq r \leq \pi$ and for any $m \geq 1$ we have

$$\int_{B_r(o)} Z_o^m(x) d\mu(x) = \frac{c(a,b) d_m}{m P_m^{a,b}(1)} P_{m-1}^{a+1,b+1}(\cos r) \left(\sin \frac{r}{2}\right)^{2a+2} \left(\cos \frac{r}{2}\right)^{2b+2}.$$

We can now apply Theorem 3.4 with $\alpha = a + 1 = d/2$ and $\beta = b + 1 = d_0/2$, so that $\gamma = (\alpha + \beta + 1)/2 = (d + d_0 + 2)/4$ and $\delta = -(2\alpha - 1)/4 = -(d-1)/4$. The result now follows immediately, recalling that $P_m^{a,b}(1) = \binom{m+a}{m} \approx m^a$. □

We can now prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 Let m_0 as in Lemma 3.7 and let $L \geq m_0$. Then by Lemma 3.1, Proposition 3.2 and Lemma 3.7, we have

$$\begin{aligned}
&\|D_{p\pi/q}\|_2^2 \\
&= \sum_{m=1}^{+\infty} d_m^{-2} \left| \int_{B_{p\pi/q}(o)} Z_o^m(x) d\mu(x) \right|^2 \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \\
&\geq \min_{m_0 \leq m \leq L} \left(d_m^{-2} \left| \int_{B_{p\pi/q}(o)} Z_o^m(x) d\mu(x) \right|^2 \right) \sum_{m=m_0}^{L} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \\
&\geq C \left(\min_{m_0 \leq m \leq L} m^{-2a-3} \right) \left(C_1 L^d \sum_{j=1}^{N} a_j^2 - C_0 m_0^d \right) \geq C L^{-2a-3} \left(C_1 L^d \sum_{j=1}^{N} a_j^2 - C_0 m_0^d \right).
\end{aligned}$$

Applying the Cauchy-Schwarz inequality to $\sum_{j=1}^{N} a_j = 1$ gives

$$\sum_{j=1}^{N} a_j^2 \geq \frac{1}{N}.$$

Let $N \geq N_0 = C_1/(2C_0)$. Then, setting $L = \lfloor m_0(2C_0C_1^{-1}N)^{1/d} \rfloor + 1$, we have $L \geq m_0$,

$$C_1 L^d \sum_{j=1}^{N} a_j^2 - C_0 m_0^d \geq C_1 \frac{L^d}{2N}$$

and

$$\| D_{p\pi/q} \|_2^2 \geq C N^{-1-\frac{1}{d}}. \tag{17}$$

Let now $N < N_0$ and let us consider the points and weights

$$\widetilde{x}_j = \begin{cases} x_j & 1 \leq j \leq N-1, \\ x_N & N \leq j \leq N_0, \end{cases} \qquad \widetilde{a}_j = \begin{cases} a_j & 1 \leq j \leq N-1, \\ \dfrac{a_N}{N_0 - N + 1} & N \leq j \leq N_0. \end{cases}$$

Since the discrepancy $\widetilde{D}_r$ of the points $\{\widetilde{x}_j\}_{j=1}^{N_0}$ and weights $\{\widetilde{a}_j\}_{j=1}^{N_0}$ coincides with the discrepancy D_r of the points $\{x_j\}_{j=1}^{N}$ and weights $\{a_j\}_{j=1}^{N}$, applying (17) to $\widetilde{D}_r$ gives

$$\| D_{p\pi/q} \|_2^2 \geq C N_0^{-1-\frac{1}{d}} \geq C N_0^{-1-\frac{1}{d}} N^{-1-\frac{1}{d}}$$

also when $1 \leq N < N_0$. □

Proof of Theorem 1.2 Let H and L be positive integers that will be fixed later. By Lemma 3.1,

$$\begin{aligned} \sum_{n=1}^{H} \frac{1}{H} \| D_{p_n\pi/q_n} \|_2^2 &= \sum_{m=1}^{+\infty} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \sum_{n=1}^{H} \frac{1}{H} \left| \int_{B_{p_n\pi/q_n}(o)} d_m^{-1} Z_o^m(y) d\mu(y) \right|^2 \\ &\geq \sum_{m=1}^{L} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \sum_{n=1}^{H} \frac{1}{H} \left| \int_{B_{p_n\pi/q_n}(o)} d_m^{-1} Z_o^m(y) d\mu(y) \right|^2. \end{aligned}$$

By Lemma 3.3 and [29, Theorem 8.21.8], uniformly in $\varepsilon \leq r \leq \pi - \varepsilon$,

$$\begin{aligned} &\int_{B_r(o)} d_m^{-1} Z_o^m(x) d\mu(x) = \frac{c(a,b)}{m P_m^{a,b}(1)} P_{m-1}^{a+1,b+1}(\cos r) \left(\sin \frac{r}{2}\right)^{2a+2} \left(\cos \frac{r}{2}\right)^{2b+2} \\ &= \frac{c(a,b)\pi^{-\frac{1}{2}}}{m^{\frac{3}{2}} P_m^{a,b}(1)} \left(\sin \frac{r}{2}\right)^{a+\frac{1}{2}} \left(\cos \frac{r}{2}\right)^{b+\frac{1}{2}} \cos\left(\left(m + \frac{a+b+1}{2}\right) r - \left(a + \frac{3}{2}\right)\frac{\pi}{2}\right) + O(m^{-a-\frac{5}{2}}). \end{aligned}$$

Since $d = 4S + 1$ for some positive integer S, then $\mathcal{M}$ is either $\mathcal{S}^d$ or $P^d(\mathbb{R})$ and therefore $d_0 = d$ or $d_0 = 1$. Also $a = (d-2)/2$ and $b = (d_0 - 2)/2$. Hence, whenever $m + S + (d_0 - 1)/4 \notin q_n\mathbb{Z}$, since $q_n \leq 2n \log(n+1)$ (see [24, formula (3.13)]), we have

$$\begin{aligned} &\left| \cos\left(\left(m + \frac{d + d_0 - 2}{4}\right) \frac{p_n}{q_n}\pi - \frac{d+1}{4}\pi\right) \right| = \left| \cos\left(\left(m + \frac{4S + d_0 - 1}{4}\right) \frac{p_n}{q_n}\pi - \frac{4S+2}{4}\pi\right) \right| \\ &= \left| \sin\left(\left(m + S + \frac{d_0 - 1}{4}\right) \frac{p_n}{q_n}\pi\right) \right| \geq \left| \sin\left(\frac{\pi}{q_n}\right) \right| \geq \frac{2}{q_n} \geq \frac{1}{n \log(n+1)}. \end{aligned}$$

Now, for each m such that

$$m + S + \frac{d_0 - 1}{4} < q_1 \cdots q_H,$$

there exists $1 \leq n \leq H$ such that $m + S + (d_0 - 1)/4 \notin q_n\mathbb{Z}$. Hence, if

$$L < q_1 \cdots q_H - S - \frac{d_0 - 1}{4},$$

then, for $m \leq L$,

$$\sum_{n=1}^{H} \frac{1}{H} \left| \int_{B_{p_n\pi/q_n}(o)} d_m^{-1} Z_o^m(y) d\mu(y) \right|^2 \geq \frac{1}{H} \left(\frac{1}{H^2 \log^2 H} \frac{c_1}{m^{2a+3}} - \frac{c_2}{m^{2a+5}} \right),$$

and, for $m \geq c_3 H \log H$ with c_3 a suitable constant (say, $c_3 = (2c_2/c_1)^{1/2}$),

$$\sum_{n=1}^{H} \frac{1}{H} \left| \int_{B_{p_n\pi/q_n}(o)} d_m^{-1} Z_o^m(y) d\mu(y) \right|^2 \geq \frac{1}{H^3 \log^2 H} \frac{c_1}{2m^{2a+3}}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{H} \frac{1}{H} \left\| D_{p_n\pi/q_n} \right\|_2^2 &\geq \sum_{m=1}^{L} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \sum_{n=1}^{H} \frac{1}{H} \left| \int_{B_{p_n\pi/q_n}(o)} d_m^{-1} Z_o^m(y) d\mu(y) \right|^2 \\ &\geq \sum_{m=c_3 H \log H}^{L} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \frac{1}{H^3 \log^2 H} \frac{c_1}{2m^{2a+3}}. \end{aligned}$$

By Proposition 3.2,

$$\begin{aligned} \sum_{n=1}^{H} \frac{1}{H} \left\| D_{p_n\pi/q_n} \right\|_2^2 &\geq \frac{1}{H^3 \log^2 H} \frac{c_1}{2L^{2a+3}} \sum_{m=c_3 H \log H}^{L} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 \\ &\geq \frac{1}{H^3 \log^2 H} \frac{c_1}{2L^{d+1}} \left(C_1 \frac{L^d}{N} - C_0 c_3^d H^d \log^d H \right). \end{aligned}$$

If $H \geq 13$, by [23, Theorem 4], then $q_1 \cdots q_H > e^{H \log H}$. Since we need $L < q_1 \cdots q_H - S - (d_0 - 1)/4$, it suffices to have $S + (d_0 - 1)/4 \leq L \leq e^{H \log H}/2$. Let us choose

$$\begin{aligned} L &= L_N := \kappa N^{1/d} \log N, \\ H &= H_L := \tau \frac{\log(2L)}{\log\log(2L)} = \tau \frac{\frac{1}{d} \log N + \log(2\kappa) + \log\log N}{\log(\frac{1}{d} \log N + \log(2\kappa) + \log\log N)}, \end{aligned}$$

for some $\kappa, \tau \geq 1$ to be fixed later. Then

$$e^{H \log H} = e^{\frac{\tau \log(2L)}{\log\log(2L)} \log\left(\frac{\tau \log(2L)}{\log\log(2L)}\right)} = e^{\tau \log(2L)\left(1 + \frac{\log \tau}{\log\log(2L)} - \frac{\log\log\log(2L)}{\log\log(2L)}\right)} \geq 2L$$

for all $L > e/2$ if τ is large enough (say, $\tau \geq 2$). Also, if $N \geq e$ and $2L \geq e^e$,

$$\begin{aligned} C_0 c_3^d H^d \log^d H &= C_0 c_3^d \left(\tau \frac{\log(2L)}{\log\log(2L)} \log\left(\tau \frac{\log(2L)}{\log\log(2L)} \right) \right)^d \\ &= C_0 c_3^d \left(\tau \log(2L) \left(\frac{\log \tau}{\log\log(2L)} + 1 - \frac{\log\log\log(2L)}{\log\log(2L)} \right) \right)^d \\ &\leq C_0 c_3^d \tau^d \left(\log(2\kappa) + \log\log N + \frac{1}{d} \log N \right)^d (\log \tau + 1)^d \\ &\leq C_0 c_3^d \tau^d (\log \tau + 1)^d \left(\log(2\kappa) + 1 + \frac{1}{d} \right)^d \log^d N \\ &\leq \frac{C_1}{2} \kappa^d \log^d N = \frac{C_1}{2} \frac{L^d}{N}, \end{aligned}$$

if

$$C_0^{1/d} c_3 \tau (\log \tau + 1) \left(\log(2\kappa) + 1 + \frac{1}{d} \right) \leq \left(\frac{C_1}{2} \right)^{\frac{1}{d}} \kappa.$$

If $2\kappa \geq e^2$, then

$$\log(2\kappa) + 1 + \frac{1}{d} \leq 2 \log(2\kappa),$$

so that we need

$$\frac{\kappa}{2 \log(2\kappa)} \geq \frac{(2C_0)^{1/d} c_3 \tau (\log \tau + 1)}{C_1^{1/d}} =: \gamma.$$

Assuming, as we may, $\gamma \geq e/4$, it suffices

$$\kappa \geq 4\gamma \log(4\gamma).$$

Hence, if $\tau = 13$ and $\kappa \geq \max\{S + (d_0 - 1)/4,\ e^e/2,\ 4\gamma \log(4\gamma)\}$, then for all $N \geq 3$,

$$\sum_{n=1}^{H} \frac{1}{H} \left\| D_{p_n \pi / q_n} \right\|_2^2 \geq C_1 c_1 \frac{1}{H^3 \log^2 H} \frac{1}{L} \frac{1}{4N} \geq C N^{-1-1/d} \frac{\log\log N}{\log^4 N}.$$

Since the function $x \mapsto x/\log(x)$ is increasing in $x \geq e$, it follows that

$$H_L \leq \tau \frac{(2 + \log(2\kappa)) \log N}{\log\left((2 + \log(2\kappa) \log N\right)} \leq \tau (2 + \log(2\kappa)) \frac{\log N}{\log(\log N)}.$$

Thus, in the statement of the theorem, we can say that for all $N \geq 3$,

$$n \leq \tau (2 + \log(2\kappa)) \frac{\log N}{\log(\log N)}.$$

□

Acknowledgements All the authors are partially supported by the Indam–Gnampa project CUP_E53C23001670001 and by the PRIN 2022 project "TIGRECO—TIme-varying signals on Graphs: REal and COmplex methods" funded by the European Union—Next Generation EU, Grant_20227TRY8H, CUP_F53D23002630001.

Funding Open access funding provided by Università degli studi di Bergamo within the CRUI-CARE Agreement.

References

1. Beck, J.: Sums of distances between points on a sphere–an application of the theory of irregularities of distribution to discrete geometry. Mathematika **31**(1), 33–41 (1984)
2. Beck, J., Chen, W.W.L.: Irregularities of distribution, volume 89 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2008. Reprint of the 1987 original
3. Bilyk, D., Mastrianni, M., Steinerberger, S.: Single radius spherical cap discrepancy via gegenbadly approximable numbers. Adv. Math. **452**, 109812 (2024)
4. Bondarenko, A., Radchenko, D., Viazovska, M.: Optimal asymptotic bounds for spherical designs. Ann. Math. **178**(2), 443–452 (2013)
5. Brandolini, L., Chen, W.W.L., Colzani, L., Gigante, G., Travaglini, G.: Discrepancy and numerical integration on metric measure spaces. J. Geom. Anal. **29**(1), 328–369 (2019)
6. Brandolini, L., Colzani, L., Gigante, G., Travaglini, G.: L^p and weak-L^p estimates for the number of integer points in translated domains. Math. Proc. Cambridge Philos. Soc. **159**(3), 471–480 (2015)
7. Brandolini, L., Colzani, L., Travaglini, G.: Irregularities of distribution for bounded sets and half-spaces. Mathematika **69**(1), 68–89 (2023)
8. Brandolini, L., Gariboldi, B., Gigante, G.: On a sharp lemma of Cassels and Montgomery on manifolds. Math. Ann. **379**(3–4), 1807–1834 (2021)
9. Brandolini, L., Gariboldi, B., Gigante, G.: Irregularities of distribution on two-point homogeneous spaces. https://arxiv.org/abs/2207.10695 (2023) To appear in: The Mathematical Heritage of Guido Weiss, Edited by E. Hernandez, M. M. Peloso, F. Ricci, F. Soria, A. Tabacco. Appl. Numer. Harmon. Anal., Springer Cham (2025)
10. Brandolini, L., Gariboldi, B., Gigante, G., Monguzzi, A.: On a discrete approach to lower bounds in discrepancy theory. https://arxiv.org/abs/2312.10668 (2023)
11. Brandolini, L., Travaglini, G.: Irregularities of distribution and geometry of planar convex sets. Adv. Math. **396**, 40 (2022)
12. Chen, W.W.L., Srivastav, A., Travaglini, G. (eds.): A panorama of discrepancy theory. Lecture Notes in Mathematics. Springer, Cham (2014)
13. Ehler, M., Etayo, U., Gariboldi, B., Gigante, G., Peter, T.: Asymptotically optimal cubature formulas on manifolds for prefixed weights. J. Approx. Theory **271**, 24 (2021)
14. Gangolli, R.: Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters. Ann. Inst. H. Poincaré Sect. B (N.S.) **3**, 121–226 (1967)
15. Gariboldi, B., Gigante, G.: Optimal asymptotic bounds for designs on manifolds. Anal. PDE **14**(6), 1701–1724 (2021)
16. Gariboldi, B., Gigante, G.: Almost positive kernels on compact Riemannian manifolds. Math. Z. **302**(2), 783–801 (2022)
17. Helgason, S.: Groups and geometric analysis, volume 83 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000. Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original
18. Konyagin, S.V., Skriganov, M.M., Sobolev, A.V.: On a lattice point problem arising in the spectral analysis of periodic operators. Mathematika **50**(1–2), 87–98 (2003)
19. Montgomery, H.L.: Irregularities of distribution by means of power sums. In Proceedings of the Congress on Number Theory (Spanish) (Zarauz, 1984), pages 11–27. Univ. País Vasco-Euskal Herriko Unib., Bilbao, (1989)
20. Montgomery, H.L.: Ten lectures on the interface between analytic number theory and harmonic analysis, volume 84 of CBMS Regional Conference Series in Mathematics. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, (1994)
21. Parnovski, L., Sidorova, N.: Critical dimensions for counting lattice points in Euclidean annuli. Math. Model. Nat. Phenom. **5**(4), 293–316 (2010)

22. Parnovski, L., Sobolev, A.V.: On the Bethe-Sommerfeld conjecture for the polyharmonic operator. Duke Math. J. **107**(2), 209–238 (2001)
23. Robin, G.: Estimation de la fonction de Tchebychef θ sur le k-ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n. Acta Arith. **42**(4), 367–389 (1983)
24. Rosser, J.B., Schoenfeld, L.: Approximate formulas for some functions of prime numbers. Illinois J. Math. **6**, 64–94 (1962)
25. Skriganov, M.M.: Point distributions in compact metric spaces. Mathematika **63**(3), 1152–1171 (2017)
26. Skriganov, M.M.: Point distributions in two-point homogeneous spaces. Mathematika **65**(3), 557–587 (2019)
27. Skriganov, M.M.: Bounds for L_p-discrepancies of point distributions in compact metric measure spaces. Constr. Approx. **51**(2), 413–425 (2020)
28. Stein, E.M., Weiss, G.: Introduction to Fourier analysis on Euclidean spaces. Princeton University Press, Princeton, NJ (1971)
29. Szegő, G.: Orthogonal polynomials. American Mathematical Society Colloquium Publications, Providence, RI (1975)
30. Wang, H.C.: Two-point homogeneous spaces. Ann. Math. **2**(55), 177–191 (1952)
31. Wolf, J.A.: Spaces of constant curvature. AMS Chelsea Publishing, Providence RI (2011)