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Effective action, massive gravitons and the Cosmological Constant

Remo Garattini

Abstract. The one loop effective action in a Schwarzschild background is here used to compute the cosmological constant in presence of massive gravitons. It is shown that the expression of the Zero Point Energy (ZPE) is equivalent to the one computed by means of a variational approach. To handle with ZPE divergences, we use the zeta function regularization. The regularization is closely related to the subtraction procedure appearing in the computation of Casimir energy in a curved background. A renormalization procedure is introduced to remove the infinities together with a renormalization group equation.

Università degli Studi di Bergamo, Facoltà di Ingegneria,
Viale Marconi 5, 24044 Dalmine (Bergamo) ITALY.
INFN - sezione di Milano, Via Celoria 16, Milan, Italy

E-mail: remo.garattini@unibg.it

The path integral approach to quantum gravity

$$Z=\int\mathcal{D}\left[g_{\mu\nu}\right]\exp iS_{g}\left[g_{\mu\nu}\right] \tag{1}$$

is a powerful method to study the quantization of the gravitational field, especially in the context of a WKB approximation on a given background. Indeed, if one considers a background $\bar{g}_{\mu\nu}$, the gravitational field splits into $g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}$, where $h_{\mu\nu}$ is a quantum fluctuation around the background field and Eq. (1) becomes

$$\int\mathcal{D}g_{\mu\nu}\exp iS_{g}\left[g_{\mu\nu}\right]\simeq\exp iS_{g}\left[\bar{g}_{\mu\nu}\right]\int\mathcal{D}h_{\mu\nu}\exp iS_{g}^{(2)}\left[h_{\mu\nu}\right], \tag{2}$$

where $S_{g}^{(2)}\left[h_{\mu\nu}\right]$ is the action approximated to second order. Since the second order action is quadratic in $h_{\mu\nu}$, the integration in Eq.(2) is straightforward. In this context, one is able to compute the averaged energy-momentum tensor

$$\langle T_{\mu\nu}\rangle=-\frac{2}{\sqrt{-g}}\frac{\delta\ln Z}{\delta g^{\mu\nu}}, \tag{3}$$

at least to one loop. The computation of the averaged energy-momentum tensor is particularly interesting for the cosmological constant problem. Indeed, if ρ is the energy density, then we can write $T_{\mu\nu}=-\langle\rho\rangle g_{\mu\nu}$ and the classical cosmological constant Λ_c becomes $\Lambda_{\text{eff}}=8\pi G\rho_{\text{eff}}=$

$\Lambda_c + 8\pi G\langle\rho\rangle$. In particular, we fix our attention on the gravitational field itself. In this way Λ_c is computed by the quantum fluctuations of the gravitational field itself. With the help of Eq.(3), the energy density becomes

$$\langle\rho\rangle = -i\int \frac{d^4k}{(2\pi)^4} \ln\left(\lambda^2\right)^{TT}, \tag{4}$$

where λ^{TT} are the eigenvalues of the following second order differential operator

$$O^{ikjl} h_{kl} = -\left(\lambda^2\right)^{TT} h^{ij} \tag{5}$$

and

$$O^{ikjl} = \triangle_L^{ikjl} + 4R^{ij}g^{kl} + \frac{1}{N^2}\frac{\partial^2}{\partial t^2} g^{ik}g^{jl}. \tag{6}$$

N is the lapse function and $\triangle_L$ is the Lichnerowicz operator. It is clear that the previous expression is written in a $3+1$ spacetime. Performing the Wick rotation and integrating over the temporal component, finally we arrive to the following familiar expression

$$\Lambda_{\text{eff}} = \langle\rho\rangle = \rho_1 + \rho_2 = \int \frac{d^3k}{(2\pi)^3}\left[\left(\lambda_{k,1}^2\right)^{TT} + \left(\lambda_{k,2}^2\right)^{TT}\right]. \tag{7}$$

If Λ_{eff} is considered as an eigenvalue of a Sturm-Liouville problem, we get the same expression[1]. The meaning of $\left(\lambda_{k,i}^2\right)^{TT}$ $i = 1, 2$ will be clear in a due course. This expression is the core of our evaluation of the cosmological constant and it represents the Zero Point Energy (ZPE) when massive (massless) gravitons are taken under examination. A very crude estimate of Eq.(7) with a cutoff at the Planck scale gives $E_{ZPE} \approx 10^{71} GeV^4$, while recent estimates on Λ_c give an order of $10^{-47} GeV^4$, with a difference of about 118 orders[2]. Eq.(7) is valid even in presence of massive gravitons provided that the Pauli-Fierz mass term[3]

$$S_{P.F.} = \frac{m_g^2}{8\kappa}\int d^4x\sqrt{-g^{(4)}}\left[h^{\mu\nu}h_{\mu\nu} - h^2\right] \tag{8}$$

is modified in such a way that only three dimensional gravitons are massive, namely we add a term of the type[4, 5]

$$S_m = \frac{m_g^2}{8\kappa}\int d^4x\sqrt{-\hat{g}}\left[h^{ij}h_{ij}\right]. \tag{9}$$

Here m_g is the graviton mass, $\kappa = 8\pi G$ with G the Newton constant. The Pauli-Fierz mass term breaks the gauge symmetry

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}, \tag{10}$$

but does not introduce ghosts. On the other hand, S_m satisfies the symmetry (10). Boulware and Deser tried to include a mass in the general framework and not simply in the linearized theory. They discovered that the theory is unstable and produce ghosts[6]. Another problem appearing when one consider a massive graviton in Minkowski space is the limit $m_g \to 0$: the analytic expression in the massive and in the mass-less limit does not coincide. This is known as van Dam-Veltman-Zakharov (vDVZ) discontinuity[7]. Other than the appearance of a discontinuity in the mass-less limit, they showed that a comparison with experiment, led the graviton to be rigorously mass-less. Actually, we know that there exist bounds on the graviton rest mass that put the upper limit on a value less than $10^{-62} - 10^{-66} g$[8]. When graviton are massive, to compute Eq.(7) in practice, we refer to a Schwarzschild background. Using Regge-Wheeler[9], the three-dimensional gravitational perturbation in its even-parity form becomes

$$\left(h^{even}\right)^i_j(r,\vartheta,\phi) = diag\left[H(r), K(r), L(r)\right] Y_{lm}(\vartheta,\phi) \tag{11}$$

and the spatial part of the operator (6) can be written as

$$-\triangle_S \left(h^{TT}\right)_i^j + \frac{6}{r^2}\left(1-\frac{2MG}{r}\right)\left(h^{TT}\right)_i^j + 2\left(Rh^{TT}\right)_i^j + \left(m_g^2 h^{TT}\right)_i^j, \tag{12}$$

where $\triangle_S$ is the scalar curved Laplacian, whose form is

$$\triangle_S = \left(1-\frac{2MG}{r}\right)\frac{d^2}{dr^2} + \left(\frac{2r-3MG}{r^2}\right)\frac{d}{dr} - \frac{L^2}{r^2} \tag{13}$$

and R_j^a is the mixed Ricci tensor whose components are:

$$R_i^a = \left\{-\frac{2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3}\right\}. \tag{14}$$

Thus $\left(\lambda^2_{k,i}\right)^{TT}$ $i = 1, 2$ in Eq.(7) becomes $(r \equiv r(x))$

$$\begin{cases} \left(\lambda^2_{k,1}\right)^{TT} = k^2 + m_g^2 + U_1(r) = k^2 + m_g^2 + m_1^2(r,M) - m_2^2(r,M) \\ \left(\lambda^2_{k,2}\right)^{TT} = k^2 + m_g^2 + U_2(r) = k^2 + m_g^2 + m_1^2(r,M) + m_2^2(r,M) \end{cases}. \tag{15}$$

$m_1^2(r,M) \to 0$ when $r \to \infty$ or $r \to 2MG$ and $m_2^2(r,M) = 3MG/r^3$. Note that, while $m_2^2(r)$ is constant in sign, $m_1^2(r)$ is not. Indeed, for the critical value $\bar{r} = 5MG/2$, $m_1^2(\bar{r}) = m_g^2$ and in the range $(2MG, 5MG/2)$ for some values of m_g^2, $m_1^2(\bar{r})$ can be negative. It is interesting therefore concentrate in this range, where $m_1^2(r,M)$ vanishes when compared to $m_2^2(r,M)$. So, in a first approximation we can write

$$\begin{cases} m_1^2(r) \simeq m_g^2 - m_2^2(r_0,M) = m_g^2 - m_0^2(M) \\ m_2^2(r) \simeq m_g^2 + m_2^2(r_0,M) = m_g^2 + m_0^2(M) \end{cases}, \tag{16}$$

where we have defined a parameter $r_0 > 2MG$ and $m_0^2(M) = 3MG/r_0^3$. The main reason for introducing a new parameter resides in the fluctuation of the horizon that forbids any kind of approach. It is now possible to explicitly evaluate Eq.(7) in terms of the effective mass. Including an additional 4π coming from the angular integration and introducing the zeta function regularization, we get

$$\Lambda = \rho_1 + \rho_2 = -\frac{\kappa}{16\pi^2}\sum_{i=1}^{2}\int_0^{+\infty} k_i^2\sqrt{k_i^2 + m_i^2(r)}dk_i, \longrightarrow \frac{1}{16\pi^2}\mu^{2\varepsilon}\sum_{i=1}^{2}\int_0^{+\infty} dk_i \frac{k_i^2}{\left(k_i^2 + m_i^2(r)\right)^{\varepsilon-\frac{1}{2}}}, \tag{17}$$

where we have introduced the additional mass parameter μ in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. Note that this procedure is completely equivalent to the subtraction procedure of the Casimir energy computation where the zero point energy (ZPE) in different backgrounds with the same asymptotic properties is involved. The integration has to be meant in the range where $k_i^2 + m_i^2(r) \geq 0$. One gets

$$\rho_i(\varepsilon) = \kappa \frac{m_i^2(r)}{256\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{m_i^2(r)}\right) + 2\ln 2 - \frac{1}{2}\right], \tag{18}$$

$i = 1, 2$. To handle with the divergent energy density we extract the divergent part of Λ, in the limit $\varepsilon \to 0$ and we set

$$\Lambda^{div} = \frac{G}{32\pi\varepsilon}\left(m_1^4(r) + m_2^4(r)\right). \tag{19}$$

Thus, the renormalization is performed via the absorption of the divergent part into the re-definition of the bare classical constant Λ

$$\Lambda \to \Lambda_0 + \Lambda^{div}. \tag{20}$$

The remaining finite value for the cosmological constant reads

$$\frac{\Lambda_0}{8\pi G} = \frac{1}{256\pi^2}\left\{m_1^4(r)\left[\ln\left(\frac{\mu^2}{\left|m_1^2(r)\right|}\right) + 2\ln 2 - \frac{1}{2}\right]\right.$$

$$\left. + m_2^4(r)\left[\ln\left(\frac{\mu^2}{m_2^2(r)}\right) + 2\ln 2 - \frac{1}{2}\right]\right\} = \left(\rho_1(\mu) + \rho_2(\mu)\right) = \rho_{eff}^{TT}(\mu, r). \tag{21}$$

The quantity in Eq.(21) depends on the arbitrary mass scale μ. It is appropriate to use the renormalization group equation to eliminate such a dependence. To this aim, we impose that[10]

$$\frac{1}{8\pi G}\mu\frac{\partial\Lambda_0^{TT}(\mu)}{\partial\mu} = \mu\frac{d}{d\mu}\rho_{eff}^{TT}(\mu, r). \tag{22}$$

Solving it we find that the renormalized constant Λ_0 should be treated as a running one in the sense that it varies provided that the scale μ is changing

$$\Lambda_0(\mu, r) = \Lambda_0(\mu_0, r) + \frac{G}{16\pi}\left(m_1^4(r) + m_2^4(r)\right)\ln\frac{\mu}{\mu_0}. \tag{23}$$

Substituting Eq.(23) into Eq.(21) we find

$$\frac{\Lambda_0(\mu_0, r)}{8\pi G} = -\frac{1}{256\pi^2}\left\{\left(m_g^2 - m_0^2(M)\right)^2\left[\ln\left(\frac{\left|m_g^2 - m_0^2(M)\right|}{\mu_0^2}\right) - 2\ln 2 + \frac{1}{2}\right]\right.$$

$$\left. + \left(m_g^2 + m_0^2(M)\right)^2\left[\ln\left(\frac{m_g^2 + m_0^2(M)}{\mu_0^2}\right) - 2\ln 2 + \frac{1}{2}\right]\right\}. \tag{24}$$

We can now discuss three cases: 1) $m_g^2 \gg m_0^2(M)$, 2) $m_g^2 = m_0^2(M)$, 3) $m_g^2 \ll m_0^2(M)$ .In case 1), we can rearrange Eq.(24) to obtain

$$\frac{\Lambda_0(\mu_0, r)}{8\pi G} \simeq -\frac{m_g^4}{128\pi^2}\left[\ln\left(\frac{m_g^2}{4\mu_M^2}\right) + \frac{1}{2}\right], \tag{25}$$

where we have introduced an intermediate scale defined by

$$\mu_M^2 = \mu_0^2 \exp\left(-\frac{3m_0^4(M)}{2m_g^4}\right). \tag{26}$$

With the help of Eq.(26), the computation of the minimum of Λ_0 is more simple. Indeed, if we define

$$x=\frac{m_g^2}{4\mu_M^2}\qquad\Longrightarrow\qquad\Lambda_{0,M}\left(\mu_0,x\right)=-\frac{G\mu_M^4}{\pi}x^2\left[\ln\left(x\right)+\frac{1}{2}\right].\tag{27}$$

As a function of x, $\Lambda_{0,M}\left(\mu_0,x\right)$ vanishes for $x=0$ and $x=\exp\left(-\frac{1}{2}\right)$ and when $x\in\left[0,\exp\left(-\frac{1}{2}\right)\right]$, $\Lambda_{0,M}\left(\mu_0,x\right)\geq 0$. It has a maximum for

$$\bar{x}=\frac{1}{e}\qquad\Longleftrightarrow\qquad m_g^2=\frac{4\mu_M^2}{e}=\frac{4\mu_0^2}{e}\exp\left(-\frac{3m_0^4\left(M\right)}{2m_g^4}\right)\tag{28}$$

and its value is

$$\Lambda_{0,M}\left(\mu_0,\bar{x}\right)=\frac{G\mu_M^4}{2\pi e^2}=\frac{G\mu_0^4}{2\pi e^2}\exp\left(-\frac{3m_0^4\left(M\right)}{m_g^4}\right)\tag{29}$$

or

$$\Lambda_{0,M}\left(\mu_0,\bar{x}\right)=\frac{G}{32\pi}m_g^4\exp\left(\frac{3m_0^4\left(M\right)}{m_g^4}\right).\tag{30}$$

In case 2), Eq.(24) becomes

$$\frac{\Lambda_0\left(\mu_0,r\right)}{8\pi G}\simeq\frac{\Lambda_0\left(\mu_0\right)}{8\pi G}=-\frac{m_g^4}{128\pi^2}\left[\ln\left(\frac{m_g^2}{4\mu_0^2}\right)+\frac{1}{2}\right]\tag{31}$$

or

$$\frac{\Lambda_0\left(\mu_0\right)}{8\pi G}=-\frac{m_0^4\left(M\right)}{128\pi^2}\left[\ln\left(\frac{m_0^2\left(M\right)}{4\mu_0^2}\right)+\frac{1}{2}\right].\tag{32}$$

Again we define a dimensionless variable

$$x=\frac{m_g^2}{4\mu_0^2}\qquad\Longrightarrow\qquad\frac{\Lambda_{0,0}\left(\mu_0,x\right)}{8\pi G}=-\frac{G\mu_0^4}{\pi}x^2\left[\ln\left(x\right)+\frac{1}{2}\right].\tag{33}$$

The formal expression of Eq.(33) is very close to Eq.(27) and indeed the extrema are in the same position of the scale variable x, even if the meaning of the scale is here different. $\Lambda_{0,0}\left(\mu_0,x\right)$ vanishes for $x=0$ and $x=4\exp\left(-\frac{1}{2}\right)$. In this range, $\Lambda_{0,0}\left(\mu_0,x\right)\geq 0$ and it has a minimum located in

$$\bar{x}=\frac{1}{e}\qquad\Longrightarrow\qquad m_g^2=\frac{4\mu_0^2}{e}\tag{34}$$

and

$$\Lambda_{0,0}\left(\mu_0,\bar{x}\right)=\frac{G\mu_0^4}{2\pi e^2}\tag{35}$$

or

$$\Lambda_{0,0}\left(\mu_0,\bar{x}\right)=\frac{G}{32\pi}m_g^4=\frac{G}{32\pi}m_0^4\left(M\right).\tag{36}$$

Finally the case 3) leads to

$$\frac{\Lambda_0\left(\mu_0,r\right)}{8\pi G}\simeq-\frac{m_0^4\left(M\right)}{128\pi^2}\left[\ln\left(\frac{m_0^2\left(M\right)}{4\mu_m^2}\right)+\frac{1}{2}\right],\tag{37}$$

where we have introduced another intermediate scale

$$\mu_m^2=\mu_0^2\exp\left(-\frac{3m_g^4}{2m_0^4\left(M\right)}\right).\tag{38}$$

By repeating the same procedure of previous cases, we define

$$x=\frac{m_0^2(M)}{4\mu_m^2}\qquad\Longrightarrow\qquad\Lambda_{0,m}(\mu_0,x)=-\frac{G\mu_m^4}{\pi}x^2\left[\ln(x)+\frac{1}{2}\right].\tag{39}$$

Also this case has a maximum for

$$\bar{x}=\frac{1}{e}\qquad\Longrightarrow\qquad m_0^2(M)=\frac{4\mu_m^2}{e}=\frac{4\mu_0^2}{e}\exp\left(-\frac{3m_g^4}{2m_0^4(M)}\right).\tag{40}$$

and

$$\Lambda_{0,m}(\mu_0,\bar{x})=\frac{G\mu_m^4}{2\pi e^2}=\frac{G\mu_0^4}{2\pi e^2}\exp\left(-\frac{3m_g^4}{m_0^4(M)}\right)\tag{41}$$

or

$$\Lambda_{0,M}(\mu_0,\bar{x})=\frac{G}{32\pi}m_0^4(M)\exp\left(\frac{3m_g^4}{m_0^4(M)}\right).\tag{42}$$

Remark Note that in any case, the maximum of Λ corresponds to the minimum of the energy density.

A quite curious thing comes on the estimate on the "square graviton mass", which in this context is closely related to the cosmological constant. Indeed, from Eq.(34) applied on the square mass, we get

$$m_g^2\propto\mu_0^2\simeq 10^{32}GeV^2=10^{50}eV^2,\tag{43}$$

while the experimental upper bound is of the order

$$\left(m_g^2\right)_{exp}\propto 10^{-48}-10^{-58}eV^2,\tag{44}$$

which gives a difference of about $10^{98}-10^{108}$ orders. This discrepancy strongly recall the difference of the cosmological constant estimated at the Planck scale with that measured in the space where we live.

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