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High Energy Physics – Theory

Traversable wormholes supported by holographic dark energy with a modified equation of state

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ABSTRACT

Inspired by holographic dark energy models, we explore various energy density profiles as potential sources for creating traversable wormholes. Because these energy densities are all positive, we introduce an equation of state in the form $p_r(r) = \omega_r(r)\rho(r)$. We find that achieving Zero Tidal Forces requires $\omega_r(r)$ to become infinite as r approaches infinity. To solve this issue, we introduce appropriate modifications on the function $\omega_r(r)$ in such a way to obtain a finite result everywhere. These modifications do not affect the behavior of the equation of state near the wormhole's throat. Surprisingly, all dark energy profiles end up in the phantom region. Among the profiles we consider, only one does not require changes to $\omega_r(r)$ to achieve Zero Tidal Forces. This profile is also consistent with the presence of a Global Monopole.

1. Introduction

The null energy condition (NEC) establishes that for any null vector $T_{\mu\nu}k^\mu k^\nu \geq 0$. $T_{\mu\nu}$ represents the Stress-Energy Tensor (SET). In terms of the energy density ρ and pressures p_i, the NEC can be written as

$$\rho + p_i \geq 0 \qquad i = 1, 2, 3. \tag{1}$$

A Traversable Wormhole (TW) is a solution of the Einstein Field Equations (EFE) with the property of connecting two distant regions by means of a tunnel violating the NEC. In particular, if we indicate with p_r the radial pressure, the following inequality

$$\rho + p_r \leq 0 \tag{2}$$

must hold. The first steps towards a traversable wormhole (TW) analysis were described by Ludwig Flamm [1] and subsequently by Einstein and Rosen [3], where the Schwarzschild solution was cast into a form to describe a bridge-like solution: the so-called Einstein-Rosen (ER) bridge. We had to wait for Morris and Thorne [4,5] to have the modern formulation for a TW [2]. It is interesting to note that the Schwarzschild solution represents a wormhole which is not traversable. Also, note that the violation of the Null Energy Condition (NEC) is related to the existence of "exotic" matter. Hence, the semiclassical theory (or perhaps a possible quantum

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theory) of gravity may be a good tool to tackle the underlying descriptions of a TW. The Casimir energy is the most likely candidate of matter forms that can be used to stabilize a TW. It is one type of vacuum energy which has been confirmed by experiment. An extension underlying a so-called Generalized Uncertainty Principle (GUP) [6] including electric charge [7,11] exists. Moreover, related investigations on the Casimir effect have been noticed in the literature [8–10,12]. Casimir wormholes have also been studied in the context of modified theories of gravity, see e.g., [13–15].

Models containing a component with an arbitrary equation of state $\omega = p/\rho$ are known as dark energy. Values of $\omega < -1/3$ are required for cosmic acceleration. For $\omega < -1$, one enters into the realm of phantom energy. Although many models and theories have been proposed to understand the nature of dark energy, they are still far from satisfying. However, among numerous dynamical dark energy models, a promising approach comes from the "holographic principle" [16–21], see also the review [22]. It refers to the duality between theories of the bulk and its boundary. The application of the holographic principle to study the nature of dark energy has been widely regarded as an attractive approach. It is noted that the holographic principle is inspired by the investigations of quantum properties of black holes and sheds some light on the cosmological and dark energy problems. In the holographic principle (HP), the conjecture proposed in Ref. [21] establishes a connection between energy density and length scale. In principle, one can identify the dark energy density as the vacuum energy density of the underlying effective field theory and propose a dynamical expression as $\rho_\Lambda \propto S/L^4$, where S is the entropy of the cosmological horizon and the infrared cutoff L, relevant to the dark energy, is the size of the event horizon. Accounting for the holographic principle, the observed density of dark energy, ρ_Λ, might be possibly explained, see for example [23]. Holographic dark energy models take this a step further by considering different scenarios for the infrared cutoff L. For instance, the Hubble scale, the particle horizon, and the future event horizon are common choices. Each of these choices leads to different implications for the evolution of the universe. One notable feature of these models is that they can potentially explain the coincidence problem, which questions why the dark energy density and matter density are of the same order of magnitude at the present time. Additionally, in some holographic dark energy models, the equation of state parameter w (which is the ratio of pressure to energy density) can evolve dynamically and even cross the phantom divide, where $w = -1$. This brings us into the regime of phantom energy. Phantom energy refers to a type of dark energy where the equation of state parameter w is less than -1. This leads to peculiar and significant cosmological consequences, such as a future scenario where the universe could end in a Big Rip, a state where the expansion of the universe accelerates so rapidly that all bound structures are torn apart. Therefore, holographic dark energy models provide a framework where the properties and behavior of dark energy can be connected to fundamental principles of quantum gravity and holography, offering rich and varied dynamics that include the possibility of phantom energy regimes. In Ref. [24], a proposal for the Holographic Dark Energy (HDE) model of the form

$$\rho_{\mathrm{Holo}} = 3\frac{c^2 M_p^2}{L^2} \tag{3}$$

has been done, where c is a numerical factor. A generalization of the previous proposal has also been considered in Refs. [25–27]. In case of black hole (BH) physics, it is proportional to the BH surface area, [28–35]. In Ref. [36], it has been explored the possibility of generating the standard holographic dark energy (SHDE) from the laws of horizon thermodynamics. In particular, the authors of Ref. [36] identify the infrared cut-off L with the apparent horizon r_A allowing the introduction of a horizon temperature of the form

$$T_H = \frac{1}{2\pi r_A}. \tag{4}$$

With this identification, the apparent horizon can be promoted to a radial coordinate. Therefore, the steps we will consider are the following

$$\rho_\Lambda \propto \frac{S}{L^4} \Longrightarrow \rho_\Lambda \propto \frac{S}{r_A^4} \Longrightarrow \rho_\Lambda \propto \frac{S}{r^4}. \tag{5}$$

Another form of ρ_Λ considered in Ref. [36] is the following

$$\rho_\Lambda \propto T\frac{dS}{dV}. \tag{6}$$

Therefore, each HDE model becomes dependent on the corresponding entropy assumption. Each of these energy density profiles can be considered as a potential source for traversable wormholes (TWs) with the help of an appropriate equation of state (EoS). These inspiring energy density profiles are not in contrast with the presence of a horizon. The authors of Ref. [39] investigated six specific solutions of statically and spherically symmetric traversable wormhole supported by the RDE fluids and analyzed the physical characteristics and properties of the RDE traversable wormholes. It is important to note that by utilizing astrophysical observations, they have constrained the parameters of the RDE model, narrowed down the number of viable models for wormhole research, and theoretically reduced the number of wormholes corresponding to different parameters within the RDE models. Indeed, in the case of a cosmological constant producing a cosmological horizon, TW models have been taken into consideration [37,38]. The rest of the paper is organized as follows: in Section 2, we introduce the general scheme for a TW, in Section 3, we introduce the profiles of the HDE which will be considered, in Section 4, we will examine the Bekenstein-Hawking HDE, in Section 5, we will examine the Moradpour energy density profile, in Section 6, we will examine the Standard Renyi HDE, and in Section 7, we will examine the Mixed Energy Density profile. We summarize and conclude in Section 8. Units in which $\hbar = c = k = 1$ are used throughout the paper and will be reintroduced whenever it is necessary.

2. General setup

We consider a static and spherically symmetric Morris-Thorne traversable wormhole in the Schwarzschild coordinates given by [4]

$$ds^2=-e^{2\Phi(r)}dt^2+\frac{dr^2}{1-\frac{b(r)}{r}}+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right), \tag{7}$$

in which $\Phi(r)$ and $b(r)$ are the redshift and shape functions, respectively. In the wormhole geometry, the redshift function $\Phi(r)$ should be finite in order to avoid the formation of an event horizon. Moreover, the shape function $b(r)$ determines the wormhole geometry, with the following condition $b(r_0)=r_0$, in which r_0 is the radius of the wormhole throat. Consequently, the shape function must satisfy the flaring-out condition [4]:

$$\frac{b(r)-rb'(r)}{b^2(r)}>0, \tag{8}$$

in which $b'(r_0)<1$ must hold at the throat of the wormhole. With the help of the line element (7), we obtain the following set of equations written in an orthonormal frame ($\kappa=8\pi G$)

$$\frac{b'(r)}{r^2}=\kappa\rho(r), \tag{9}$$

$$2\left(1-\frac{b(r)}{r}\right)\frac{\Phi'(r)}{r}-\frac{b(r)}{r^3}=\kappa p_r(r) \tag{10}$$

and

$$\left(1-\frac{b(r)}{r}\right)\left[\Phi''(r)+\Phi'(r)\left(\Phi'(r)+\frac{1}{r}\right)\right] \\ -\frac{b'(r)r-b(r)}{2r^2}\left(\Phi'(r)+\frac{1}{r}\right)=\kappa p_t(r), \tag{11}$$

where $p_t=p_\theta=p_\phi$. We can complete the EFE with the expression of the conservation of the Stress-Energy Tensor (SET) which can be written in the same orthonormal reference frame

$$p_r'(r)=\frac{2}{r}\left(p_t(r)-p_r(r)\right)-\left(\rho(r)+p_r(r)\right)\Phi'(r). \tag{12}$$

If Zero Tidal Forces (ZTF) are considered, we can impose the following Equation of State (EoS)

$$p_r(r)=\omega_r(r)\rho(r) \tag{13}$$

and write $\Phi(r)=0$. This is equivalent to

$$\omega_r(r)=-\frac{b(r)}{rb'(r)}. \tag{14}$$

Solving the previous equation with respect to $b(r)$, one finds the well know profile

$$b(r)=r_0\exp\left[-\int_{r_0}^{r}\frac{d\bar{r}}{\omega_r(\bar{r})\bar{r}}\right] \tag{15}$$

which must be consistent with the result obtained by solving Eq. (9). It is straightforward to see that some energy density profiles produce a divergent $\omega_r(r)$ to keep the validity of ZTF. For instance, Casimir wormholes [8] and Yukawa-Casimir wormholes [10] fall in this case, while the Ellis-Bronnikov TW does not. Indeed, the Ellis-Bronnikov is described by the following shape function

$$b(r)=\frac{r_0^2}{r} \tag{16}$$

and $\omega_r(r)=1$, while the Casimir wormhole has a shape function

$$b(r)=\frac{2}{3}r_0+\frac{r_0^2}{3r}, \tag{17}$$

and the associated $\omega_r(r)$ is such that

$$\omega_r(r)=\left(\frac{2r}{r_0}+1\right)\underset{r\to\infty}{\to}\infty. \tag{18}$$

A similar behavior holds also for Yukawa-Casimir wormholes. In this paper we would like to apply a modification of the original inhomogeneous EoS in such a way to convert divergent $\omega_r(r)$ for large r into a convergent function. In particular, we are interested in HDE profiles which will be described in the next section.

3. Wormhole geometries with holographic dark energy densities

We are going to apply the general setup presented in the previous section to some specific energy density profiles inspired by HDE. They are:

1. Bekenstein-Hawking (BH) HDE with an energy density of the form
$$\rho_1(r) = \frac{C\pi}{r^2}, \tag{19}$$
2. Moradpour et al.'s proposal with the following energy density profile
$$\rho_2(r) = \frac{C}{4\pi r^2\left(\pi\lambda r^2+1\right)} = \frac{C}{4\pi}\left(\frac{1}{r^2} - \frac{\pi\lambda}{\pi\lambda r^2+1}\right), \tag{20}$$
3. Standard Renyi HDE with the following energy density form
$$\rho_3(r) = \frac{C}{\lambda r^4}\ln\left(1+\pi\lambda r^2\right), \tag{21}$$
4. Mixed Energy density
$$\rho_4(r) = \frac{3C_M^2}{8\pi^2}\left[\frac{\pi}{r^2} - \pi^2\lambda\ln\left(1+\frac{1}{\pi\lambda r^2}\right)\right]. \tag{22}$$

As already mentioned in the previous section, it is likely that some profiles can produce a divergent $\omega_r(r)$ when $r \to \infty$. In such a case the choice (14) must be modified in an appropriate manner. For instance, if

$$\omega_r(r) \sim r^\alpha, \text{ for } r\to\infty \text{ and } \alpha > 0, \tag{23}$$

then we can define

$$\omega_r(r) = -\frac{r_0^\alpha b(r)}{r^{1+\alpha}b'(r)}, \tag{24}$$

which is now convergent for $r \to \infty$. Of course, with such a modification the behavior on the throat is unaffected and the ZTF cannot be imposed. Rather we have to determine the form of the redshift function. To this purpose, plugging Eq. (24) into Eq. (10), we obtain

$$\left[2\left(1-\frac{b(r)}{r}\right)\frac{\Phi'(r)}{r} - \frac{b(r)}{r^3}\right] = -\frac{r_0^\alpha b(r)}{r^{3+\alpha}}, \tag{25}$$

which can be rearranged to give

$$\Phi'(r) = \frac{b(r)\left(r^\alpha - r_0^\alpha\right)}{2r^{\alpha+1}(r-b(r))} = \frac{b(r)\left(r-r_0\right)}{2r^{\alpha+1}(r-b(r))}\sum_{i=0}^{\alpha-1}(r^{\alpha-i-1}r_0^i). \tag{26}$$

Close to the throat, we can use the following approximation

$$b(r) \simeq r_0 + B\left(r-r_0\right) + O\left(\left(r-r_0\right)^2\right), \tag{27}$$

where $B = b'(r_0)$. Then Eq. (26) becomes

$$\Phi'(r) \simeq \frac{r_0}{2r^{\alpha+1}(1-B)}\sum_{i=0}^{\alpha-1}(r^{\alpha-i-1}r_0^i). \tag{28}$$

For the energy density profiles we are going to examine, it will be sufficient to consider $\alpha = 1$. Then Eq. (24) reduces to

$$\omega_r(r) = -\frac{r_0 b(r)}{r^2 b'(r)}, \tag{29}$$

while Eq. (28) reduces to

$$\Phi'(r) = \frac{b(r)\left(r-r_0\right)}{2r^2(r-b(r))}. \tag{30}$$

Finally, plugging Eq. (29) and Eq. (30) into Eq. (12), one finds

$$\omega_t(r) = \frac{\left(r^2\left(r+r_0\right)b(r) - 2r^3 r_0\right)b'(r) - r_0\left(\left(r_0-5r\right)b(r) + 4r_0r^2\right)b(r)}{4b'(r)r^3(r-b(r))}, \tag{31}$$

where we have also used the additional EoS

$$p_t(r) = \omega_t(r)\,\rho(r). \tag{32}$$

Now we have all the elements to examine the different energy density profiles. We begin to examine the BH HDE. As we will see, this is the only example under examination which does not need a modification of the EoS.

Remark. The modification (24) and its special case (29) can be taken under consideration because $p_r(r)$ and $p_t(r)$ are not fixed. Only the energy density has a known profile. Indeed for the Casimir wormhole, this procedure cannot be applied, because the original SET is known.

4. Bekenstein-Hawking (BH) HDE

As a potential source for TW, we consider the Bekenstein-Hawking HDE density, whose profile is described by Eq. (19), which here we report [36]

$$\rho_1(r) = \frac{C\pi}{r^2}. \tag{33}$$

C is positive with dimensions of $\left[L^{-2}\right]$. Plugging the energy density (19) into Eq. (9), one finds

$$b(r) = r_0 + \beta\left(r - r_0\right) = r_0(1-\beta) + \beta r, \tag{34}$$

where $\beta = \kappa C\pi$. For $r \to \infty$, $b(r) \to \infty$. The flare-out condition demands that

$$b'(r_0) < 1, \tag{35}$$

namely $\beta < 1$. This means that $b(r)$ never vanishes. The line element (7) can be written in the following way

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{(1-\beta)\left(1-\frac{r_0}{r}\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{36}$$

with $\beta \neq 1$. This is in agreement with what has been found in Ref. [40], where the ZTF case has been discussed leading to $\Phi(r) = 0$, even if nothing has been said about a possible EoS allowing such a choice.

4.1. Homogeneous radial EoS

In this subsection, we adopt the strategy used in Refs. [8,10] and consider Eq. (10)

$$\left[\frac{2}{r}\left((1-\beta)\left(1-\frac{r_0}{r}\right)\right)\Phi'(r) - (1-\beta)\frac{r_0}{r^3} - \frac{\beta}{r^2}\right] = \kappa p_r(r), \tag{37}$$

where we have used Eq. (34) and the EoS $p_r(r) = \omega_r \rho(r)$. Solving with respect to $\Phi(r)$, one gets

$$\Phi(r) = \frac{\ln\left(r - r_0\right)}{2(1-\beta)}\left(1 + \beta\omega_r\right) - \frac{\ln(r)}{2} + K. \tag{38}$$

When $r \to r_0$, a horizon is present. It is straightforward to see that if we impose that

$$\omega_r = -\frac{1}{\beta}, \tag{39}$$

then the redshift function is regular for $r = r_0$ and one gets

$$\Phi(r) = \frac{1}{2}\ln\left(r_0/r\right), \tag{40}$$

where we have assumed that $\Phi\left(r_0\right) = 0$. However, such a choice is not complete, because the behavior of $\Phi(r)$ for $r \to +\infty$ has not been determined yet. From Eq. (38), one gets

$$\Phi(r) \underset{r\to\infty}{\simeq} \frac{\ln(r)}{2}\left[\frac{\beta\left(1+\omega_r\right)}{1-\beta}\right] + K, \tag{41}$$

that it means that

$$\omega_r = -1 \tag{42}$$

to have a finite result. The line element (36) becomes

$$ds^2 = -\frac{r_0}{r}dt^2 + \frac{dr^2}{(1-\beta)\left(1-\frac{r_0}{r}\right)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad r \to r_0, \tag{43}$$

$$ds^2 = -dt^2 + \frac{dr^2}{1-\beta} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad r \to \infty. \tag{44}$$

Note that for $r \to \infty$, we are in presence of a *Global Monopole* [8]. Since $\beta < 1$, we have an excess of the solid angle for the line element (44) which can be cast in the following way

$$ds^2 = -dt^2 + d\tilde{r}^2 + (1-\beta)\tilde{r}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{45}$$

Finally we consider Eq. (11) which reduces to

$$p_t(r) = \frac{1-\beta}{4\kappa r^2} \qquad r \to r_0, \tag{46}$$

$$p_t(r) = \frac{(1-\beta)r_0}{2\kappa r^3} \qquad r \to \infty. \tag{47}$$

To summarize, the Stress Energy Tensor (SET) becomes

$$T_{\mu\nu} = \frac{1}{\kappa r^2}\left[diag\left(\beta, -1, \frac{1-\beta}{4}, \frac{1-\beta}{4}\right)\right] \qquad r \to r_0, \tag{48}$$

$$T_{\mu\nu} = \frac{1}{\kappa r^2}\left[diag\left(\beta, -\beta, \frac{(1-\beta)r_0}{2r}, \frac{(1-\beta)r_0}{2r}\right)\right] \qquad r \to \infty. \tag{49}$$

Note that this SET cannot be traceless, because this should imply $\beta = -1/3$ for $r \to r_0$. This is not possible since $C > 0$. On the other hand, for $r \to \infty$, $\beta = 0$ which is inconsistent with the original energy density profile. Since in the homogeneous case, ω_r must assume two distinct values in two different spatial regions, we are going to examine the inhomogeneous EoS to see if there exists a unique choice for the EoS.

4.2. Inhomogeneous radial EoS

If we consider the relationship (14), we can set $\Phi(r) = 0$ everywhere and

$$\omega_r(r) = -\frac{r_0(1-\beta) + \beta r}{\beta r}. \tag{50}$$

We observe that

$$\omega_r(r_0) = -\frac{1}{\beta} \qquad \text{and} \qquad \omega_r(r) \underset{r\to\infty}{=} -1, \tag{51}$$

which is consistent with what has been found in section 4.1. Note that, since $0 < \beta < 1$, $\omega_r(r)$ tells us that we are in presence of phantom energy. The line element now becomes

$$ds^2 = -dt^2 + \frac{dr^2}{(1-\beta)\left(1-\frac{r_0}{r}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{52}$$

which is the same one which has been found in Ref. [40]. With the help of the shape function (15) and $\omega_r(r)$, it is possible to write the SET in its general representation, namely

$$\begin{aligned} T_{\mu\nu} &= \frac{r_0}{\kappa r^3} diag\left(-\frac{1}{\omega_r(r)}, -1, \frac{1}{2\omega_r(r)} + \frac{1}{2}, \frac{1}{2\omega_r(r)} + \frac{1}{2}\right)\exp\left[-\int_{r_0}^{r}\frac{d\bar{r}}{\omega_r(\bar{r})\bar{r}}\right] \\ &= -\frac{b(r)}{\kappa r^3\omega_r(r)} diag\left(1, \omega_r(r), -\frac{1}{2} - \frac{\omega_r(r)}{2}, -\frac{1}{2} - \frac{\omega_r(r)}{2}\right) \\ &= \rho(r)\, diag\left(1, \omega_r(r), -\frac{1}{2} - \frac{\omega_r(r)}{2}, -\frac{1}{2} - \frac{\omega_r(r)}{2}\right). \end{aligned} \tag{53}$$

4.3. SET conservation

With the help of the Equations of State (13) and (32), the SET conservation described by Eq. (12), becomes

$$\frac{d}{dr}\left(\omega_r(r)\rho(r)\right) = \frac{2}{r}\left(\omega_t(r) - \omega_r(r)\right)\rho(r) - \left(1 + \omega_r(r)\right)\rho(r)\Phi'(r). \tag{54}$$

Isolating $\Phi'(r)$, one gets

$$\Phi'(r) = -\frac{\omega_r'(r)}{1+\omega_r(r)} - \frac{\omega_r(r)\rho'(r)}{\left(1+\omega_r(r)\right)\rho(r)} + \frac{2}{r}\frac{\left(\omega_t(r)-\omega_r(r)\right)}{1+\omega_r(r)}. \tag{55}$$

In the case of BH HDE, Eq. (55) leads to the following equation for the redshift function

$$\Phi'(r) = \frac{-\omega_r'(r)\, r + 2\omega_t(r)}{r\left(\omega_r(r)+1\right)}, \tag{56}$$

where the profile (19) has been used. Differently from the other profiles we will investigate, this one can offer solutions without approximations. Before doing this we need to verify the consistency with Eq. (14). To this purpose, Eq. (56) can be rearranged to give

$$\frac{d}{dr}\left(\Phi(r) + \ln\left(\omega_r(r)+1\right)\right) = \frac{2\omega_t(r)}{r\left(\omega_r(r)+1\right)}. \tag{57}$$

If we assume that

$$\omega_t(r) = -\frac{\omega_r(r)+1}{2}, \tag{58}$$

then we find

$$\frac{d}{dr}\left(\Phi(r) + \ln\left(\left|\omega_r(r)+1\right|\right) + \ln(r)\right) = 0, \tag{59}$$

namely

$$\Phi(r) + \ln\left(r\left(\left|\omega_r(r)+1\right|\right)\right) = K. \tag{60}$$

The assumption in Eq. (58) is suggested by the SET in Eq. (53). The consistency with Eq. (14) is guaranteed if $\Phi(r) = 0$. On the other hand to obtain other redshift profiles, we can plug Eq. (56) into Eq. (10) and we find

$$2\left(1-\frac{b(r)}{r}\right)\frac{-\omega_r'(r)\, r + 2\omega_t(r)}{r^2\left(\omega_r(r)+1\right)} - \frac{b(r)}{r^3} - \frac{\omega_r(r)\, b'(r)}{r^2} = 0, \tag{61}$$

where we have used the EoS (13). With the help of the shape function (34) and the relationship (58), Eq. (61) becomes

$$\frac{d}{dr}\omega_r(r) = -\frac{\left(\omega_r(r)\beta r - (\beta-2) r - (1-\beta) r_0\right)\left(\omega_r(r)+1\right)}{2r(1-\beta)\left(r-r_0\right)}, \tag{62}$$

whose solution is

$$\omega_r(r) = -\frac{\sqrt{r-r_0}\sqrt{r} + \left(\left(r-r_0\right)\beta + r_0\right) C_1}{\sqrt{r}\left(C_1\beta\sqrt{r} + \sqrt{r-r_0}\right)}, \tag{63}$$

where C_1 is an arbitrary constant. $\omega_r(r)$ is such that

$$\omega_r\left(r_0\right) = -\frac{1}{\beta} \qquad \textit{and} \qquad \omega_r(r) \underset{r\to\infty}{\longrightarrow} -1. \tag{64}$$

As a consequence, from the relationship (58),

$$\omega_t\left(r_0\right) = \frac{1-\beta}{2\beta} \qquad \textit{and} \qquad \omega_t(r) \underset{r\to\infty}{\longrightarrow} 0. \tag{65}$$

Note that the integration constant C_1 is not determined. Plugging Eq. (63) into Eq. (56), one finds

$$\frac{d}{dr}\Phi(r) = \frac{r_0}{2r\left(C_1\beta\sqrt{r} + \sqrt{r-r_0}\right)\sqrt{r-r_0}}. \tag{66}$$

It is immediate to see that for $C_1 = 0$, Eq. (66) develops a horizon. Therefore this option will be discarded. The general solution of Eq. (66) is

$$\Phi(r) = \frac{1}{2}\ln\left(\frac{C_1\beta\sqrt{r} + \sqrt{r-r_0}}{C_1\beta\sqrt{r} - \sqrt{r-r_0}}\right) + \frac{1}{2}\ln\left(\frac{\left(C_1^2\beta^2 - 1\right) r + r_0}{C_1^2\beta^2 r}\right), \tag{67}$$

where we have assumed that $\Phi\left(r_0\right) = 0$.

5. Moradpour energy density

In this section, we are going to examine the following profile [36]

$$\rho_2(r) = \frac{C}{4\pi r^2\left(\pi\lambda r^2+1\right)} = \frac{C}{4\pi}\left(\frac{1}{r^2} - \frac{\pi\lambda}{\pi\lambda r^2+1}\right). \tag{68}$$

$\rho_2(r)$ vanishes for $r \to \infty$, as well as for $\lambda \to \infty$. For this asymptotic cases $b(r) = r_0$. On the other hand when $\lambda = 0$, we obtain the energy density described by Eq. (19) with an additional 4π term at the denominator. Note that C and λ have dimensions $\left[L^{-2}\right]$. Plugging $\rho_2(r)$ into the first EFE (9), we find

$$b(r) = r_0 + \frac{\kappa C}{4\pi\sqrt{\pi\lambda}}\left(\tan^{-1}\left(\sqrt{\pi\lambda}r\right) - \tan^{-1}\left(\sqrt{\pi\lambda}r_0\right)\right), \tag{69}$$

which is always positive. This shape function is such that

$$b(r) \underset{r\to\infty}{=\to} r_0 + \frac{\kappa C}{4\pi\sqrt{\pi\lambda}}\left(\pi/2 - \tan^{-1}\left(\sqrt{\pi\lambda}\, r_0\right)\right) = b_{M,\infty}, \tag{70}$$

which reduces to the value r_0 when $\lambda \to \infty$, as it should be. Since $\lambda \geq 0$, the flare-out condition, represented by

$$b'(r_0) = \frac{\kappa C}{4\pi\left(\pi\lambda r_0^2+1\right)} < 1, \tag{71}$$

is always satisfied. To gain enough information on the $\Phi(r)$, we examine the original inhomogeneous EoS to see if a modification is necessary. Such an EoS, if satisfied, allows us to impose ZTF and set $\Phi(r) = 0$ everywhere. To this purpose, we need to compute Eq. (14) which is represented by

$$\omega_r(r) = -\frac{1+\pi\lambda r^2}{\kappa C\sqrt{\pi\lambda}r}\left(4\pi\sqrt{\pi\lambda}r_0 + \kappa C\left(\tan^{-1}\left(\sqrt{\pi\lambda}r\right) - \tan^{-1}\left(\sqrt{\pi\lambda}r_0\right)\right)\right). \tag{72}$$

On the throat we find

$$\omega_r(r_0) = -\frac{4\pi}{\kappa C}\left(1+\pi\lambda r_0^2\right) < 0, \tag{73}$$

while for $r \to \infty$, one gets

$$\omega_r(r) \underset{r\to\infty}{\simeq} = -\left(\frac{8\pi\sqrt{\pi\lambda}r_0 + \kappa C\left(\pi - 2\tan^{-1}\left(\sqrt{\pi\lambda}r_0\right)\right)}{2\kappa C}\right)\sqrt{\pi\lambda}r \to \infty. \tag{74}$$

Since the quantity inside the round brackets never vanishes, $\omega_r(r)$ diverges for $r \to \infty$. Therefore the ZTF cannot be imposed. However, we can use the modification (29) to obtain

$$\omega_r(r) = -\frac{r_0\left(1+\pi\lambda r^2\right)}{\kappa C\sqrt{\pi\lambda}r^2}\left(4\pi\sqrt{\pi\lambda}r_0 + \kappa C\left(\tan^{-1}\left(\sqrt{\pi\lambda}r\right) - \tan^{-1}\left(\sqrt{\pi\lambda}r_0\right)\right)\right), \tag{75}$$

and this time, for $r \to \infty$, one finds

$$\omega_r(r) \underset{r\to\infty}{\simeq} = -\frac{r_0\sqrt{\pi\lambda}}{2\kappa C}\left(8\pi\sqrt{\pi\lambda}r_0 + \kappa C\left(\pi - 2\tan^{-1}\left(\sqrt{\pi\lambda}r_0\right)\right)\right). \tag{76}$$

The redshift function is described by Eq. (30) which, in this particular case, becomes

$$\Phi'(r) \simeq \frac{r_0}{2(1-B)r^2}, \tag{77}$$

where we have used the approximation (27) and where B is represented by Eq. (71). If we assume that $\Phi(r_0) = 0$, then one finds

$$\Phi(r) \simeq \frac{1}{2(1-B)}\left(1 - \frac{r_0}{r}\right). \tag{78}$$

On the other hand, when $r \to \infty$, we can use the asymptotic behavior of $b(r)$ described in (70) to obtain

$$\Phi'(r) \simeq \frac{b_{M,\infty}}{2r^2} \qquad \Longrightarrow \qquad \Phi(r) \simeq -\frac{b_{M,\infty}}{2r}. \tag{79}$$

To have consistency between $\Phi(r)$, the SET equation and the third EFE, we will use Eq. (31). Close to the throat, we can write

$$\omega_t(r) = \frac{\left(-r^3+4r^2r_0-6rr_0^2+r_0^3\right)B^2 + \left(-3r^2r_0+10rr_0^2-2r_0^3\right)B - 4rr_0^2 + r_0^3}{4B\,r^3\,(B-1)}, \tag{80}$$

which implies

$$\omega_t\left(r_0\right)=\frac{3-2B}{4B}, \tag{81}$$

where we have used the approximation (27). Note that, since $0<B<1$, $\omega_t\left(r_0\right)>0$. On the other hand, when $r\to\infty$, one gets

$$\omega_t\left(r\right)\simeq r_0\left(\frac{\pi\kappa C\sqrt{\pi\lambda}-2\sqrt{\pi\lambda}\kappa C\arctan\left(\sqrt{\pi\lambda}r_0\right)+8\pi^2r_0\lambda}{2\kappa C}\right)+O\left(\frac{1}{r^3}\right). \tag{82}$$

The same results hold also for the third EFE as it should be. We can observe that, although the original $\omega_r\left(r\right)$ of Eq. (72) is divergent when $r\to\infty$, the radial pressure given by Eq. (13) is not. This is due to the action of the energy density that decreases like $1/r^2$ leading to a radial pressure that decreases like $1/r$. The modification (29) allows not only to have a convergent $\omega_r\left(r\right)$, but also a radial pressure that goes at infinity like $1/r^2$. Regarding the transverse pressure, we can see that Eq. (32) and the approximation (82) tell us that $p_t\left(r\right)\simeq 1/r^2$ when $r\to\infty$.

6. Standard Renyi HDE

Here the Renyi HDE density from the CKN bound takes the form [36]

$$\rho_3\left(r\right)=\frac{C}{\lambda r^4}\ln\left(1+\pi\lambda r^2\right). \tag{83}$$

Note that C and λ have dimensions $\left[L^{-2}\right]$ and are positive. $\rho_3\left(r\right)$ vanishes for $r\to\infty$, as well as for $\lambda\to\infty$. On the other hand for $\lambda\to 0$, $\rho_3\left(r\right)\to\rho_1\left(r\right)$ described by Eq. (19). Plugging Eq. (83) into Eq. (9), one finds

$$\begin{aligned}b(r)&=r_0+\frac{\kappa C}{\lambda}\left(\frac{\ln\left(\pi\lambda r_0^2+1\right)}{r_0}-\frac{\ln\left(\pi\lambda r^2+1\right)}{r}\right)\\&+\frac{2\kappa C\sqrt{\pi}}{\sqrt{\lambda}}\left(\tan^{-1}\left(\sqrt{\pi\lambda}r\right)-\tan^{-1}\left(\sqrt{\pi\lambda}r_0\right)\right).\end{aligned} \tag{84}$$

Note that even for this profile when $\lambda\to\infty$, $b(r)=r_0$. This shape function is such that

$$b(r)\underset{r\to\infty}{\to}r_0+\frac{\pi^{3/2}C\kappa}{\sqrt{\lambda}}+\frac{C\kappa\ln\left(\pi\lambda r_0^2+1\right)}{\lambda r_0}-\frac{2\sqrt{\pi}C\kappa\tan^{-1}\left(\sqrt{\pi\lambda}r_0\right)}{\sqrt{\lambda}}=b_{R,\infty}. \tag{85}$$

Since $\lambda>0$, the flare-out condition, described by the following inequality

$$b'(r_0)=\frac{C\kappa\ln\left(\pi\lambda r_0^2+1\right)}{\lambda r_0^2}<1, \tag{86}$$

is always satisfied. Even for this profile, we will try to see if it is possible to set $\Phi\left(r\right)=0$ everywhere by means of the relationship (14). One gets

$$\omega_r\left(r\right)=1+\frac{r}{\ln\left(\pi\lambda r^2+1\right)r_0}\left(2r_0\sqrt{\lambda\pi}\arctan\left(\sqrt{\pi\lambda}r_0\right)-\frac{\lambda r_0^2}{\kappa C}-2r_0\sqrt{\lambda\pi}\arctan\left(\sqrt{\pi\lambda}r\right)-\ln\left(\pi\lambda r_0^2+1\right)\right). \tag{87}$$

This means that, even in this case, we find a divergent inhomogeneous $\omega_r\left(r\right)$. Of course, we can impose ZTF, but at the price of having a divergent inhomogeneous $\omega_r\left(r\right)$ for large values of r. Therefore the ZTF case will be discarded like for the Moradpour profile. However, following the same procedure of the previous section, we can modify the form of $\omega_r\left(r\right)$ in such a way to compensate the divergent behavior. Indeed, from Eq. (87), one finds that $\omega_r\left(r\right)\sim r$ when $r\to\infty$. Thus, if we adopt Eq. (31) also for the Renyi profile, we find

$$\omega_r\left(r\right)=\frac{r_0}{r}+\frac{1}{\ln\left(\pi\lambda r^2+1\right)}\left(2r_0\sqrt{\lambda\pi}\arctan\left(\sqrt{\pi\lambda}r_0\right)-\frac{\lambda r_0^2}{\kappa C}-2r_0\sqrt{\lambda\pi}\arctan\left(\sqrt{\pi\lambda}r\right)-\ln\left(\pi\lambda r_0^2+1\right)\right). \tag{88}$$

Now $\omega_r\left(r\right)\to 0$ when $r\to\infty$ and, on the throat, we get the same expression of Eq. (87). It is clear that the redshift function obeys the differential equation (30) whose solution close to the throat is represented by Eq. (78). Only the value of $B=b'(r_0)$ represented by Eq. (86) is different as it should be. On the other hand, when $r\to\infty$, we can use the asymptotic expression of $b\left(r\right)$ leading to the same analytic form of (79), but with $b_{M,\infty}$ replaced by $b_{R,\infty}$. It is easy to check that also for the Renyi profile, $\omega_t\left(r\right)$ assumes the same analytic expression of Eq. (31). This means that on the throat we will obtain the same value described in Eq. (81). Only the value of B will be different as it should be. On the other hand, when $r\to\infty$, one gets

$$\omega_t(r) \simeq \frac{C\kappa\pi\sqrt{\pi\lambda}r_0 - 2C\kappa\sqrt{\pi\lambda}r_0 \arctan\left(\sqrt{\pi\lambda}r_0\right) + \kappa C \ln\left(\pi\lambda r_0^2+1\right) + \lambda r_0^2}{C\kappa\ln\left(\pi\lambda r^2\right)}. \tag{89}$$

The same results hold also for the third EFE as it should be. Even for the Renyi profile, we can observe that, although the original $\omega_r(r)$ of Eq. (87) is divergent when $r \to \infty$, the radial pressure given by Eq. (13) is not. The reason is the same of the previous section: the behavior of the energy density when $r \to \infty$ is $\rho(r) \sim 1/r^2$ leading to a radial pressure that decreases like $1/r$. The modification (88) allows not only to have a convergent $\omega_r(r)$, but also a radial pressure that goes at infinity like $1/r^2$. Regarding the transverse pressure, we can see that Eq. (32) and the approximation (82) tell us that $p_t(r) \simeq 1/r^2$ when $r \to \infty$.

7. Mixed energy density

In this section we consider a combination of the form

$$\rho_4(r) = \frac{3C_M^2}{8\pi^2}\left[\frac{\pi}{r^2} - \pi^2\lambda\ln\left(1+\frac{1}{\pi\lambda r^2}\right)\right]. \tag{90}$$

Note that, C_M has dimensions $\left[L^{-1}\right]$ while λ has dimensions $\left[L^{-2}\right]$ and both are positive. For $r \to \infty$ and $\lambda \to \infty$, $\rho_4(r) \to 0$. Therefore for $\lambda \to \infty$, $b(r) = r_0$ represents a solution. For $\lambda \to 0$, the energy density reduces to the Bekenstein-Hawking (BH) HDE profile of section 4 for an appropriate choice of the constant C_M. Plugging Eq. (90) into Eq. (9), one finds

$$b(r) = r_0 + \frac{\kappa C_M^2}{8}\left[\pi r_0^3\lambda\ln\left(\frac{1+\pi\lambda r_0^2}{\lambda\pi r_0^2}\right) - \frac{2\arctan\left(\sqrt{\pi\lambda}r_0\right)}{\sqrt{\pi\lambda}} + \frac{2\arctan\left(\sqrt{\pi\lambda}r\right)}{\sqrt{\pi\lambda}} - r^3\lambda\pi\left(\ln\left(\frac{\pi\lambda r^2+1}{\lambda\pi r^2}\right)\right) + \left(r-r_0\right)\right]. \tag{91}$$

This shape function is such that

$$b(r) \underset{r\to\infty}{\to} r_0 + \frac{\kappa C_M^2}{8}\left[\pi r_0^3\lambda\ln\left(\frac{1+\pi\lambda r_0^2}{\lambda\pi r_0^2}\right) - \frac{2\arctan\left(\sqrt{\pi\lambda}r_0\right)}{\sqrt{\pi\lambda}} + \frac{\sqrt{\pi}}{\sqrt{\lambda}} + r_0\right] = b_{Mix,\infty}, \tag{92}$$

while the flare-out condition is described by the following inequality

$$b'(r_0) = \frac{3\kappa C_M^2}{8}\left[1 - r_0^2\lambda\pi\left(\ln\left(\frac{\pi\lambda r_0^2+1}{\lambda\pi r_0^2}\right)\right)\right] < 1. \tag{93}$$

It is easy to see that the previous inequality is always satisfied. Now, we need to know if the ZTF can be imposed. To this purpose, we compute $\omega_r(r)$, like in Eq. (14). Since the expression

$$\omega_r(r) = -\frac{\left\{r_0 + \frac{\kappa C_M^2}{8}\left[\pi r_0^3\lambda\ln\left(\frac{1+\pi\lambda r_0^2}{\lambda\pi r_0^2}\right) - \frac{2\arctan\left(\sqrt{\pi\lambda}r_0\right)}{\sqrt{\pi\lambda}} + \frac{2\arctan\left(\sqrt{\pi\lambda}r\right)}{\sqrt{\pi\lambda}} - r^3\lambda\pi\left(\ln\left(\frac{\pi\lambda r^2+1}{\lambda\pi r^2}\right)\right) + \left(r-r_0\right)\right]\right\}}{\frac{3\kappa C_M^2}{8}r\left[1 - r^2\lambda\pi\left(\ln\left(\frac{\pi\lambda r^2+1}{\lambda\pi r^2}\right)\right)\right]}. \tag{94}$$

On the throat, we find

$$\omega_r\left(r_0\right) = -\frac{8}{3\kappa C_M^2\left[1 - r_0^2\lambda\pi\left(\ln\left(\pi\lambda r_0^2+1\right) - \ln\left(\pi\lambda r_0^2\right)\right)\right]}. \tag{95}$$

Note that for $\lambda \to \infty$, $\omega_r\left(r_0\right) \to -\infty$, as it should be. This can be easily understood by looking at the behavior of the energy density $\rho_4(r)$ in the same limit. On the other hand when $r \to \infty$, one finds

$$\omega_r(r) \underset{r\to\infty}{\to} \frac{2\sqrt{\pi\lambda}}{3}\left(-\pi^{\frac{3}{2}}\ln\left(1+\frac{1}{r_0^2\lambda\pi}\right)\lambda^{\frac{3}{2}}r_0^3 + r_0\sqrt{\pi\lambda} - \pi + 2\arctan\left(r_0\sqrt{\pi\lambda}\right)\right)r - \frac{16r_0\pi\lambda}{3C^2\kappa}r + O(1), \tag{96}$$

namely $\omega_r(r)$ is linearly divergent for $r \to \infty$. Like in the previous sections, we are going to modify the construction of $\omega_r(r)$ in such a way that

$$\omega_r(r) = -\frac{r_0 b(r)}{r^2 b'(r)}. \tag{97}$$

We know that $\omega_r\left(r_0\right)$ does not change, while for $r \to \infty$, we get

$$\omega_r(r) \underset{r\to\infty}{\to} \frac{2r_0\sqrt{\pi\lambda}}{3}\left(-\pi^{\frac{3}{2}}\ln\left(1+\frac{1}{r_0^2\lambda\pi}\right)\lambda^{\frac{3}{2}}r_0^3+r_0\sqrt{\pi\lambda}-\pi+2\arctan\left(r_0\sqrt{\pi\lambda}\right)\right)-\frac{16r_0^2\pi\lambda}{3C^2\kappa}. \tag{98}$$

Even for the mixed case, the redshift function obeys the differential equation (30) whose solution close to the throat is represented by Eq. (78). Only the value of $B = b'(r_0)$ represented by Eq. (93) is different as it should be. On the other hand, when $r \to \infty$, we can use the asymptotic expression of $b(r)$ leading to the same analytic form of (79), but with $b_{M,\infty}$ replaced by $b_{Mix,\infty}$. Even with this profile, we have to check if $\Phi(r)$ satisfies the SET equation and the third EFE. It is easy to check that also for the mixed profile $\omega_t(r)$ assumes the same analytic expression of Eq. (31). This means that on the throat we will obtain the same value described in Eq. (81). Only the value of B will be different as it should be. On the other hand, when $r \to \infty$, one gets

$$\omega_t(r) \simeq \frac{C\kappa\pi\sqrt{\pi\lambda}r_0 - 2C\kappa\sqrt{\pi\lambda}r_0\arctan\left(\sqrt{\pi\lambda}r_0\right)+\kappa C\ln\left(\pi\lambda r_0^2+1\right)+\lambda r_0^2}{C\kappa\ln\left(\pi\lambda r^2\right)}. \tag{99}$$

The same results also hold for the third EFE, as it should. Additionally, for the mixed energy density profile, we observe that although the original $\omega_r(r)$ of Eq. (96) is divergent when $r \to \infty$, the radial pressure given by Eq. (13) is not. The reason is the same as in the previous two sections: the behavior of the energy density when $r \to \infty$ is $\rho(r) \sim 1/r^2$, leading to a radial pressure that decreases like $1/r$. The modification (98) allows not only for a convergent $\omega_r(r)$ but also for a radial pressure that goes to infinity like $1/r^2$. Regarding the transverse pressure, we can see that Eq. (32) and the approximation (99) tell us that $p_t(r) \simeq 1/\left(\ln(r)r^2\right)$ as $r \to \infty$.

8. Concluding remarks

In this paper, we have considered different energy density profiles inspired by holographic dark energies as possible sources needed to have traversable wormhole solutions. Since in each profile the energy density is positive and since it is the NEC that must be violated, we are forced to introduce an EoS of the form (13). This implies that

$$\rho(r) + p_r(r) = \left(1+\omega_r(r)\right)\rho(r) \le 0 \tag{100}$$

which implies $\omega_r(r) < -1$. This means that our energy density profiles are of the "phantom" type. We noticed that as a concrete instance, the authors of Ref. [39] studied the Ricci dark energy (RDE) traversable wormholes. They particularly discovered that the null energy condition (NEC) is violated when the effective equation of state parameter $\omega_X < -1$ similar to our present findings. They further investigated six specific solutions of statically and spherically symmetric traversable wormhole supported by the RDE fluids and analyzed the physical characteristics and properties of the RDE traversable wormholes. It is important to note that by utilizing astrophysical observations, they have constrained the parameters of the RDE model, narrowed down the number of viable models for wormhole research, and theoretically reduced the number of wormholes corresponding to different parameters within the RDE models. With such an EoS given in Eq. (100), we have tried to impose ZTF, meaning that $\Phi(r)$ can assume a constant value or it can be vanishing. For the benefit of the reader, we emphasize that the significance of Zero Tidal Forces (ZTF) lies in their ability to nullify differential gravitational forces, resulting in no stretching or compressing effects on a body. In general relativity, tidal forces arise from spacetime curvature, causing different parts of an object to experience varying gravitational accelerations. When tidal forces are zero, it indicates a highly symmetric spacetime configuration. This symmetry simplifies the analysis of cosmological models, particularly those involving dark energy, by making the field equations more tractable and enabling straightforward derivations of the Equation of State (EoS) parameters. Such simplification is crucial for accurately modeling the nature and dynamics of dark energy with reduced computational complexity.. Unfortunately, as a size effect, the function $\omega_r(r) \to \infty$ when $r \to \infty$. To overcome this problem, we have modified the $\omega_r(r)$ function in such a way that the behavior at infinity is convergent, while on the throat remains unchanged and well defined. With this modification, every profile admits a solution describing a TW. However, only one energy density proposal needs no modification. This is represented by the Bekenstein-Hawking energy density proposal, which has a regular behavior at infinity and on the throat from the beginning. Moreover, such a profile leads to a shape function that is in agreement with that proposed in Ref. [40]. It is worth mentioning that the Bekenstein-Hawking profile is characterized by its regular behavior both near the throat of the wormhole and at infinity. Unlike some other energy density profiles, the Bekenstein-Hawking profile does not diverge as the distance from the throat approaches infinity. This regularity ensures that the radial pressure associated with this profile remains finite throughout the spacetime region of interest, making it an attractive candidate for constructing traversable wormhole solutions. The significance of the Bekenstein-Hawking energy density profile lies in its ability to meet the conditions for Zero Tidal Forces (ZTF) without requiring any modifications. This makes it unique among the energy density profiles considered in wormhole theory, as it stands out as the only profile that naturally satisfies the requirements for ZTF. It is also interesting to note that every energy density proposal is in the "phantom" energy regime. Additionally, along with the present study, one can consider an extension of the HDE wormhole solutions by introducing a Yukawa deformation, see e.g., [10,41]. This way allows us to have the possibility of building a new family of solutions. Moreover, the stability analyses of the traversable wormhole solutions are of great interest. To this end, we juts perform a linear stability analysis by introducing small perturbations to the wormhole solution and studying their evolution. We can straightforwardly follow the work of Ref. [42]. However, this topic lies beyond the scope of the present work.

CRediT authorship contribution statement

Remo Garattini: Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Phongpichit Channuie:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

Data availability

No data was used for the research described in the article.

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