# Fredkin Gates for Finite–valued Reversible and Conservative Logics

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#### Abstract

The basic principles and results of *Conservative Logic* introduced by Fredkin and Toffoli in [FT, 82] on the basis of a seminal paper of Landauer [La, 61] are extended to d-valued logics, with a special attention to three-valued logics. Different approaches to d-valued logics are examined in order to determine some possible universal sets of logic primitives. In particular, we consider the typical connectives of Lukasiewicz and Gödel logics, as well as Chang's MV-algebras. As a result, some possible three-valued and d-valued universal gates are described which realize a functionally complete set of fundamental connectives.

### 1 Introduction

The present paper is based on two different research areas which have been developed in the last years: *Conservative Logic* and *Many-valued Logics*. Conservative logic is a model of computation introduced by Fredkin and Toffoli in [FT, 82] on the basis of the seminal paper of Landauer [La, 61] (see also [Be, 73]) to improve the efficiency and performance of computing processes in terms of dissipated energy. The model is based on the *Fredkin gate*, a universal Boolean gate which is both conservative and reversible.

On the other hand, many-valued logics are extensions of classical twovalued (i.e., Boolean) logic which have a great diffusion due to their ability to manage incomplete and/or uncertain knowledge. These two main subjects are briefly described in the next sections.

<sup>\*</sup>This work has been supported by MIUR\COFIN project "Formal Languages and Automata: Theory and Application"

In this paper we propose an extension of conservative logic in order to include the main features of many-valued logics with a finite number of truth values. As a result we define some d-valued universal gates which have the properties required by the conservative and many-valued paradigms.

# 2 Reversibility, Conservativeness, and Conditional Control of Boolean Gates

Computational models are usually based upon Boolean logic, and use some universal set of primitive connectives such as, for example, {AND, NOT}.

From a general point of view, a (classical deterministic) n-inputs/moutputs gate (where n, m are positive integers) is a special-purpose computer schematized as a device able to compute (Boolean) logical functions  $G : \{0,1\}^n \to \{0,1\}^m$ . Any  $\vec{x} = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$  (resp.,  $\vec{y} =$  $(y_1, y_2, \ldots, y_m) \in \{0,1\}^m$ ) is called an input (resp., output) configuration. For every  $i \in \{1, 2, \ldots, n\}$  (resp.,  $j \in \{1, 2, \ldots, m\}$ ), called the input (resp., output) bit of position i (resp., j), the Boolean value  $x_i \in \{0,1\}$  (resp.,  $y_j \in \{0,1\}$ ) is said to be the state of bit i (resp., j) with respect to configuration  $\vec{x}$  (resp.,  $\vec{y}$ ). Finally, in the sequel we denote by  $\lambda_f$  the generic configuration belonging to the range of G.

The action of the multi-output map G on an input configuration  $\vec{x}$ produces the output configuration  $G(\vec{x}) = (G_1(\vec{x}), G_2(\vec{x}), \dots, G_m(\vec{x}))$  determined by the component *logical truth functions* (single-output maps)  $G_j : \{0,1\}^n \to \{0,1\}$ , for any  $j = 1, 2, \dots, m$ , with a possible *parallel* implementation drawn in Figure 1.

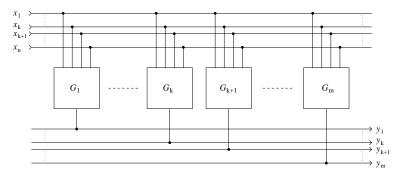


Figure 1: Standard parallel architecture of an n-inputs/m-outputs gate

*Conservative logic* is a theoretical model of computation whose principal aim is to compute with zero *internal power dissipation*. This goal is reached by basing the model upon *reversible* and *conservative primitives*, which reflect physical principles such as the reversibility of microscopic dynamical laws and the conservation of certain *physical quantities*, such as the energy of the physical system used to perform the computations.

**Reversibility.** Most of the times, computational models lack of *reversibility*; that is, one cannot in general deduce the input values of a gate from its output values. For example, knowing that the output of an AND gate is the logical value 0 one cannot deduce the input values that generated it. The original motivation for the study of reversibility in classical computing came from the observation that heat dissipation is one of the major obstacles for miniaturization of classical computers and the fact that the second law of thermodynamics implies that irreversible state changes during computation must dissipate heat. "Thus, in the more abstract context of computing, the laws of "conservation of information" may play a role analogous to those of conservation of energy and momentum in physics." [To, 80].

Lack of reversibility means that during the computation some information is lost. As shown by R. Landauer [La, 61] (see also C.H. Bennett [Be, 88] which can be found in [LR, 90]), a loss of information implies a loss of energy and therefore any computational model based on irreversible primitives is necessarily *informationally dissipative*. This is nowadays known (see [Be, 98]) as:

Landauer's principle. To erase a bit of classical information within a computer, 1 bit of entropy must be expelled into the computer's environment, typically in the form of waste heat. Thus logical irreversibility is associated with physical irreversibility and requires a minimal heat generation, per machine cycle, typically of the order of kT for each irreversible operation.

In practice the heat dissipation per bit processed by (irreversible) computers in use today is some orders of magnitude greater than the theoretical lower bound  $1kT \ln 2$  given by Landauer's principle. However, if computer hardware continues to shrink in size as so far, then the only feasible option to beat Landauer's lower bound seems to be reversible computation.<sup>1</sup>

"In the customary approach, this transition [from the irreversibility of the given computing process to the reversibility of the physical laws, NdA] occurs at a very low level and is hidden — so to speak — in the "physics" of the individual gate; as a consequence of this approach, the details of the work-to-heat conversion process are put beyond the reach of the conceptual model of computation that is used.

<sup>&</sup>lt;sup>1</sup> In modern computers heat dissipation is about  $kT10^8$  per logical operation. The heat must be removed by external means, for example, by constant cooling of all components by the thermal coupling of the circuits to a heat reservoir, i.e., air.

On the other hand, it is possible to formulate a more general conceptual model of computation such that the gap between the irreversibility of the desired behavior and the reversibility of a given underlying mechanism is bridged *in an explicit way* within the model itself." [To, 80, p. 3].

Let us make these considerations less informal by considering as a first example the logical function L of Table 1 computed by a three–inputs/three–outputs gate and discussed by Landauer in [La, 61].

$x_1$	$x_2$	$x_3$	$\rightarrow$	$y_1$	$y_2$	$y_3$
0	0	0		0	0	0
0	0	1		1	1	0
0	1	0		0	0	0
0	1	1		1	1	0
1	0	0		0	0	0
1	0	1		1	1	0
1	1	0		0	0	1
1	1	1		1	1	1

Table 1: The Landauer three-inputs/three-outputs gate

Following Landauer "There are eight possible initial states, and in thermal equilibrium they will occur with equal probability. How much entropy reduction will occur in a machine cycle? States (111) and (001) occur with a probability 1/8 each; states (110) and (000) have a probability of occurrence of 3/8 each. The initial entropy was:

$$S_i(3) = -k \sum_{\vec{x}} P(\vec{x}) \log_e P(\vec{x})$$
  
=  $-k \sum_{\vec{x}} \frac{1}{8} \log_e \frac{1}{8} = 3 k \log_e 2$ 

The final entropy is

$$S_f(L) = -k \sum_{\lambda_f} P_L(\lambda_f) \log_e P_L(\lambda_f)$$
  
=  $-k \left(\frac{1}{8} \log_e \frac{1}{8} + \frac{1}{8} \log_e \frac{1}{8} + \frac{3}{8} \log_e \frac{3}{8} + \frac{3}{8} \log_e \frac{3}{8}\right)$ 

The difference  $S_i(3) - S_f(L)$  is 0.82 k. The minimum dissipation, if the initial state has no useful information, is therefore  $E_i(3) - E_f(L) = (S_i(3) - S_f(L))T = 0.82 k T$ ."

More precisely, for any admissible output  $\lambda_f = (y_1, y_2, y_3) \in Im(L)$  we can introduce the set

$$M_L(\lambda_f) := L^{-1}(\lambda_f) = \{ (x_1, x_2, x_3) \in \{0, 1\}^3 : L(x_1, x_2, x_3) = \lambda_f \}$$

whose cardinality  $|M_L(\lambda_f)|$  expresses the *indistinguishability degree* of the output  $\lambda_f$ , i.e., the total number of possible inputs which cannot be distinguished by L with respect to the output  $\lambda_f$ . Then the above probabilities can be expressed as

$$P_L(\lambda_f) = \frac{|M_L(\lambda_f)|}{|\{0,1\}^3|} = \frac{1}{8} |M_L(\lambda_f)|$$

We want now to extend these considerations in order to compare the dissipation of *informational energy* in the case of devices whose number of output lines is not necessarily equal to the number of input lines. To this end, let us denote by  $\mathcal{F}(\{0,1\},n,m) = (\{0,1\}^m)^{\{0,1\}^n}$  the collection of all Boolean gates  $G: \{0,1\}^n \to \{0,1\}^m$ , and by  $\mathcal{F}(\{0,1\},n,\mathbb{N}) = \bigcup_{m \in \mathbb{N}} \mathcal{F}(\{0,1\},n,m)$  the collection of all Boolean gates with n fixed and m ranging in  $\mathbb{N}$ . For instance,  $\mathcal{F}(\{0,1\},2,\mathbb{N})$  contains both the gate AND :  $\{0,1\}^2 \to \{0,1\}$ , associating to the Boolean pair  $(x_1, x_2)$  the Boolean value AND $(x_1, x_2) = x_1 \cdot x_2$ , and the two–inputs/four–outputs gate defined by Table 2.

$x_1$	$x_2$	$\mapsto$	$y_1$	$y_2$	$y_3$	$y_4$
0	0		0	0	0	0
0	1		0	1	1	0
1	0		0	1	1	1
1	1		1	0	0	1

Table 2: Example of a two-inputs/four-outputs reversible gate

In  $\mathcal{F}(\{0,1\}, n, \mathbb{N})$ , owing to the assumption that in thermal equilibrium all possible inputs  $\vec{x}$  will occur with equal probability  $P(\vec{x})$ , the *input infor*mation entropy is independent from the particular gate and equal to:

$$S_i(n) := -k \sum_{\vec{x}} P(\vec{x}) \log_e P(\vec{x})$$
$$= -k \sum_{\vec{x}} \frac{1}{2^n} \log_e \frac{1}{2^n} = n k \log_e 2$$

What depends on the gate  $G \in \mathcal{F}(\{0,1\}, n, \mathbb{N})$  is the set of  $\lambda_f$ -indistinguishable input configurations, where  $\lambda_f \in Im(G)$  is any admissible output configuration of G:

$$M_G(\lambda_f) := G^{-1}(\lambda_f)$$
  
= {(x<sub>1</sub>, x<sub>2</sub>, ..., x<sub>n</sub>) ∈ {0,1}<sup>n</sup> : G(x<sub>1</sub>, x<sub>2</sub>, ..., x<sub>n</sub>) =  $\lambda_f$ }

Let us notice that the collection  $\{M_G(\lambda_f) : \lambda_f \in Im(G)\}$  of all such subsets constitutes a partition of  $\{0,1\}^n$ . Borrowing some terminology from axiomatic quantum mechanics, elements  $\lambda_f$  from Im(G) can be called *eigen*values (possible output values) of G, Im(G) is the spectrum of G, the set  $M_G(\lambda_f)$  is the eigenspace (set of possible inputs) associated to the eigenvalue  $\lambda_f$ , and the characteristic function  $\chi_{M_G(\lambda_f)}$  (= 1 if  $\vec{x} \in M_G(\lambda_f)$ , and 0 otherwise) is the spectral projection associated to the eigenspace. The collection of all spectral projections of G, for  $\lambda_f$  ranging on the spectrum of G, is a spectral identity resolution of G:

$$Id = \sum_{\lambda_f \in Im(G)} \chi_{M_G(\lambda_f)}$$
$$G = \sum_{\lambda_f \in Im(G)} \lambda_f \, \chi_{M_G(\lambda_f)}$$

The indistinguishability degree of the admissible output configuration  $\lambda_f \in Im(G)$  is defined as  $|M_G(\lambda_f)|$ , and the probability of occurrence of  $\lambda_f$  as:

$$P_G(\lambda_f) = \frac{1}{2^n} |M_G(\lambda_f)|$$

with corresponding *output information entropy*:

$$S_f(G) := -k \sum_{\lambda_f \in Im(G)} P_G(\lambda_f) \log_e P_G(\lambda_f)$$
(1a)  
$$k \sum_{i=1}^k |M_i(\lambda_i)| = \log_e |M_i(\lambda_i)| + C_i(n)$$
(1b)

$$= -\frac{k}{2^n} \sum_{\lambda_f \in Im(G)} |M_G(\lambda_f)| \cdot \log_e |M_G(\lambda_f)| + S_i(n)$$
(1b)

Hence, the *information energy dissipation* of G is:

$$\Delta E(G) = (S_i(n) - S_f(G)) \cdot T$$
$$= \frac{kT}{2^n} \sum_{\lambda_f \in Im(G)} |M_G(\lambda_f)| \cdot \log_e |M_G(\lambda_f)|$$

In particular, the information energy loss by the AND gate is  $\Delta E(\text{AND}) = \frac{3kT}{4} \log_e 3 \approx 0.82 \, k \, T$  whereas the gate of Table 2 (owing to its reversibility) has no information energy dissipation.

From (1) it follows immediately that the output information entropy is bounded by:

$$0 \le S_f(G) \le S_i(n)$$

Of course, a generic gate  $G : \{0,1\}^n \to \{0,1\}^m$  is reversible (one-to-one mapping) iff n = m and every element  $\lambda_f$  of  $\{0,1\}^n$  is an admissible output; in this case the corresponding  $|M_G(\lambda_f)|$  is equal to 1 which leads to  $S_i(n) - S_f(G) = 0$ , and thus also  $E_i(n) - E_f(G) = 0$ . Precisely, the following Proposition holds.

**Proposition 2.1.** Let G be any n-inputs Boolean gate. Then the information energy dissipation is bounded by

$$0 \le \Delta E(G) \le T S_i(n) = nkT \log_e 2$$

Moreover:

- 1. Im(G) is a singleton if and only if  $\Delta E(G) = T S_i(n)$ ;
- 2. the gate is reversible (one-to-one) if and only if  $\Delta E(G) = 0$ .

Quoting Toffoli:

"Using invertible logic gates, it is ideally possible to build a sequential computer with zero internal power dissipation. The only source of power dissipation arises outside the circuit, typically at the input/output interface, if the user chooses to connect input or output lines to nonreversible digital circuitry. Even in this case, power dissipation is at most proportional to the number of argument/result lines, rather than to the number of logic gates (as in ordinary computers), and is thus independent of the "complexity" of the function being computed. This constitutes the central result of the present paper." [To, 80, p. 32]

Let us stress that in the case of an n-inputs/n-outputs gate realizing the logical function  $G : \{0,1\}^n \to \{0,1\}^n$  the reversibility condition corresponds to the fact that G is a permutation of the set  $\{0,1\}^n$ . For instance, a 2-inputs/2-outputs reversible gate computes a permutation of the set  $\{0,0,1,10,11\}$ . Table 3 shows an example of a gate of this kind.

$x_1$	$x_2$	$\mapsto$	$y_1$	$y_2$
0	0		1	1
0	1		1	0
1	0		0	1
1	1		0	0

Table 3: Example of a two-inputs/two-outputs reversible gate, i.e., a permutation of the set  $\{0,1\}^2$ 

**Conservativeness.** This condition is usually modelled by the property that each output  $(y_1, y_2, \ldots, y_n)$  of the gate is a permutation of the corresponding input  $(x_1, x_2, \ldots, x_n)$ . We call this condition *strict conservativeness* of the gate. Trivially a gate of this kind must necessarily have the same number of input and output lines. In Table 4 an example of a (strictly) conservative 2-inputs/2-outputs gate is presented.

$x_1$	$x_2$	$\mapsto$	$y_1$	$y_2$
0	0		0	0
0	1		1	0
1	0		1	0
1	1		1	1

Table 4: Example of a two-inputs/two-outputs conservative gate, i.e., in each row the output pattern is a permutation of the input pattern

"Some conservative (but not reversible) circuits using complementary signal streams were discussed by von Neumann [vN, 56] as early as 1952. More recently, Kinoshita and associates [Ki, 76] worked out a classification of logic elements that "conserve" 0's and 1's; their work, motivated by research in magnetic–bubble circuitry, mentions the possibility of more energy–efficient computation, but has apparently little concern for reversibility." [To, 80, p. 33]

The importance of conservativeness is further on stressed by Toffoli in [To, 80]:

"In a conservative logic circuit, the number of 1's, which is conserved in the operation of the circuit, is the sum of the number of 1's in different parts of the circuit. Thus, this quantity is additive, and can be shown to play a formal role analogous to that of energy in physical systems. [...]

In conclusion, conservative logic represents a substantial step in the development of a model of computation that adequately reflects the basic laws of physics." [To, 80, p. 32]

In general, in concrete devices the Boolean values 0 and 1 are realized by impulses of energy  $\varepsilon_0$  and  $\varepsilon_1$  respectively, with  $0 < \varepsilon_0 < \varepsilon_1$ .

"In the classical realization the bit, which, for example could be imagined to be just a mechanical switch, is a system which is designed to have two distinguishable states; there should be a sufficiently large energy barrier between them that no spontaneous transition, which would evidently be detrimental, can occur between the two states." [BZ, 00].

In the case of a generic (non necessarily conservative) gate which computes a logical function  $G : \{0,1\}^n \to \{0,1\}^n$ , a transition  $\vec{x} = (x_1, x_2, \ldots, x_n) \to G(\vec{x}) = \vec{y} = (y_1, y_2, \ldots, y_n)$  corresponding to a

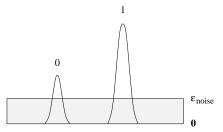


Figure 2: Realization of Boolean values 0 and 1 by impulses of energy  $\varepsilon_0$ and  $\varepsilon_1$ , with  $0 < \varepsilon_{\text{noise}} < \varepsilon_0 < \varepsilon_1$ 

row of the tabular definition of the Boolean function produces a variation of the internal energy whose amount is

$$\Delta U(\vec{x}, \vec{y}) = (\varepsilon_{y_1} + \varepsilon_{y_2} + \ldots + \varepsilon_{y_n}) - (\varepsilon_{x_1} + \varepsilon_{x_2} + \ldots + \varepsilon_{x_n})$$

Therefore, the total *internal energy dissipation* of G is

$$\Delta U(G) = \sum_{\vec{x} \in \{0,1\}^n} \Delta U(\vec{x}, G(\vec{x}))$$

Conservativeness of the gate G trivially implies no internal energy dissipation  $(\Delta U(G) = 0)$ .

"From the viewpoint of a physical implementation, where signals are encoded in some form of energy, each constant input entails the supply of energy of predictable form, or *work*, and each garbage output entails the removal of energy of unpredictable form, or *heat*. In this context, a realization with fewer source and sink lines might point the way to a physical implementation that dissipates less energy." [To, 80, p. 13]

**Conclusions on Reversibility and Conservativeness.** Up to now the loss of energy due to irreversibility and nonconservativeness of logical primitives was irrelevant compared to the energy dissipated by an electronic device implementing logical gates. But the problems rising from an extreme miniaturization in electronics have led to the investigation of new ways of implementing circuits, borrowing the knowledge of quantum mechanics. These new research areas introduce the possibility of reversible and conservative computations based on reversible and conservative physical behavior, encouraging the definition of new computational models.

Let us stress that there are gates which are reversible but non conservative (for instance, the gate of Table 3 whose transition  $00 \rightarrow 11$  is non

conservative) and gates which are conservative and non reversible (for instance, the gate of Table 4 where both inputs 01 and 10 are mapped into the same output 10).

A simple example of reversible and conservative two-inputs/two-outputs gate is the realization of the *exchange* logical function EXC :  $\{0,1\}^2 \rightarrow \{0,1\}^2$  whose tabular representation is given in Table 5. In each row the

$x_1$	$x_2$	$\mapsto$	$y_1$	$y_2$
0	0		0	0
0	1		1	0
1	0		0	1
1	1		1	1

Table 5: The EXC two-inputs/two-outputs reversible and conservative gate

output pair  $(y_1, y_2)$  is a permutation of the corresponding input pair  $(x_1, x_2)$ , and the map EXC is a (global) permutation of the set  $\{0, 1\}^2 = \{00, 01, 10, 11\}$ .

**Conditional Control Gates.** Let us consider the Boolean two–inputs/two– outputs reversible non–conservative gate  $G^{(CN)}$  :  $\{0,1\}^2 \rightarrow \{0,1\}^2$  whose component maps are the following:

$$G_1^{(CN)} : \{0,1\}^2 \to \{0,1\}, \quad G_1^{(CN)}(x_1,x_2) := x_1$$
  
$$G_2^{(CN)} : \{0,1\}^2 \to \{0,1\}, \quad G_2^{(CN)}(x_1,x_2) := x_1 \oplus x_2$$

The corresponding truth table is given in Table 6.

$x_1$	$x_2$	$\mapsto$	$y_1$	$y_2$
0	0		0	0
0	1		0	1
1	0		1	1
1	1		1	0

Table 6: The Controlled–NOT reversible non–conservative gate

We can describe the behavior of this gate by considering the information  $x_1$  as a *control input* which is left unchanged but which determines the action of a prescribed operation on the *target input*  $x_2$ , transforming it into the output  $y_2$ . To be precise, if the control input is 1 then the value of the target line is negated (i.e., the gate NOT acts on  $x_2$  when  $x_1 = 1$ ), otherwise it is left unchanged (i.e., the gate Id acts on  $x_2$  when  $x_1 = 0$ ). Formally, this is realized by a direct connection of the first input line with the first output line, whereas the action on the input of the second line is described by two

maps  $\delta_{x_1}^{(CN)}$ :  $\{0,1\} \to \{0,1\}$ , where  $\delta_{x_1}^{(CN)} = G_2^{(CN)}(x_1, \cdot)$  for  $x_1 \in \{1,0\}$ . Precisely,

$$\delta_0^{(CN)} := G_2^{(CN)}(0, \cdot) = \text{Id} \text{ and } \delta_1^{(CN)} := G_2^{(CN)}(1, \cdot) = \text{NOT}$$

The input value of the control unit  $x_1$  selects the map  $\delta_{x_1}^{(CN)}$  (either the identity or the NOT map) which acts on the input value  $x_2$  of the second line. For this reason this gate is called the *Controlled*-NOT (usually abbreviated with CNOT) gate.

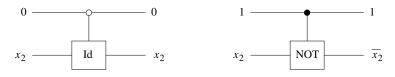


Figure 3: The conditional action of the Controlled–NOT gate

From the general viewpoint, the *Conditional Control* method applies to the cases in which the n-inputs/n-outputs gate can be divided into two parts: a *control unit* and an *target* (also *operating*) *unit* (see Figure 4).

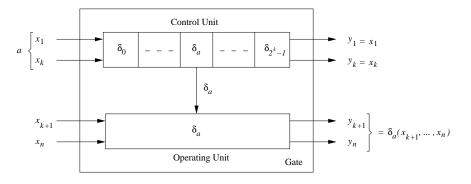


Figure 4: Ideal realization of a generic Conditional Control gate: the gate is divided into a *control unit* and an *operating unit*. The input values of the control unit are left unchanged and select a prescribed function to be applied to the input values of the operating unit

The control unit has in general k input and k output lines, while the target unit has (n - k) input and (n - k) output lines. Thus any configuration  $x_1, \ldots, x_k, x_{k+1}, \ldots, x_n$  can be split into two parts: the control configuration  $x_1, \ldots, x_k$  and the operating (sometimes also called *target*) configuration  $x_{k+1}, \ldots, x_n$ . Any of the  $2^k$  possible control configurations  $x_1, \ldots, x_k$  is labelled by the integer number  $a = \sum_{t=1}^k x_t 2^{t-1}$ . Moreover,  $2^k$  functions  $\delta_0, \delta_1, \ldots, \delta_{2^k-1}$  of the kind  $\{0, 1\}^{n-k} \to \{0, 1\}^{n-k}$  are stored in

the memory of the control unit, the function  $\delta_a$  being bijectively associated to the configuration labelled by the integer number  $a \in \{0, \ldots, 2^k - 1\}$ .

When a configuration  $x_1, \ldots, x_k$  (labelled by a) is fed as input to the control lines two things happen:

- 1. the control configuration  $x_1, \ldots, x_k$  is returned unchanged into the output lines of the control unit;
- 2. the function  $\delta_a$  bijectively associated to the control configuration is selected and applied to the input configuration  $x_{k+1}, \ldots, x_n$  of the operating unit, producing the output configuration  $\delta_a(x_{k+1}, \ldots, x_n)$ .

We can look at a controlled gate as a finite automaton. The original space  $\{0,1\}^n$  on which a controlled gate G acts can be split in the set  $\mathcal{A} := \{0,1\}^k$ , called the *alphabet* of the gate, and the set  $Q := \{0,1\}^{n-k}$ , called the *phase* space of the gate; elements of  $\mathcal{A}$  are symbols of the alphabet and elements of Q are states of the gate. Hence, the gate can be represented as a mapping  $G : \mathcal{A} \times Q \to \mathcal{A} \times Q$ , associating with any symbol–state pair  $(\vec{a}, \vec{s})$  a new symbol-state pair  $G(\vec{a}, \vec{s}) := (\vec{a}, \delta_a(\vec{s}))$ . Therefore, if we put the gate in cascade with the trivial decoder (according to [To, 80])  $\pi_Q : \mathcal{A} \times Q \to Q$  associating with any pair (a, s) the single state  $\pi_Q(a, s) := s$  one obtains a deterministic finite automaton  $\mathbb{A}^G = \langle \mathcal{A}, Q, \delta \rangle$  with (finite) alphabet  $\mathcal{A}$ , set of states Q, and next state (also transition) function  $\delta := (\pi_Q \circ G) : \mathcal{A} \times Q \to Q$  associating with any letter–state pair  $(\vec{a}, \vec{s}) := \pi_Q(G(\vec{a}, \vec{s})) = \delta_a(\vec{s})$ .

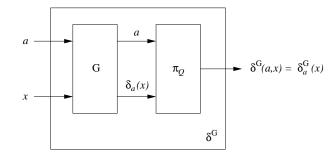


Figure 5: Automaton generated by a controlled gate

This automaton can be equivalently described by the pair  $\langle Q, \{\delta_0, \delta_1, \ldots, \delta_{2^k-1}\}\rangle$  consisting of the phase space  $Q = \{0, 1\}^{n-k}$  and the collection of  $2^k$  transformations of the phase space  $\delta_a : Q \to Q$ , for a running in  $\{0, 1, \ldots, 2^k - 1\}$ .

Vice versa, any (finite) automaton  $\mathbb{A} = \langle \mathcal{A}, Q, \delta \rangle$  consisting of the (finite) alphabet  $\mathcal{A}$ , the (finite) phase space Q, and the next state function  $\delta : \mathcal{A} \times Q \to Q$  can be equivalently described by the pair  $\langle Q, \{\delta_0, \delta_1, \ldots, \delta_{|\mathcal{A}|-1}\} \rangle$ 

based on the phase space Q and the (finite) collection of phase space transformations  $\delta_a : Q \to Q$  (for  $a \in \{0, 1, \dots, |\mathcal{A}| - 1$ ) associating with any state  $\vec{s}$  the next state  $\vec{s}' = \delta_a(\vec{s}) := \delta(\vec{a}, \vec{s})$ . This automaton generates a controlled gate  $G^{\delta} : \mathcal{A} \times Q \to \mathcal{A} \times Q$  associating with the symbol-state input pair  $(\vec{a}, \vec{s})$  the symbol-state output pair  $G^{\delta}(\vec{a}, \vec{s}) := (\vec{a}, \delta(\vec{a}, \vec{s}))$ . Trivially, if  $|\mathcal{A}| = 2^k$  and  $|Q| = 2^h$  by a suitable binary representations of each symbol  $\vec{a}$  and each state  $\vec{s}$  this conditional control gate is realized by a mapping  $G^{\delta} : \{0, 1\}^n \to \{0, 1\}^n$ , with n = h + k.

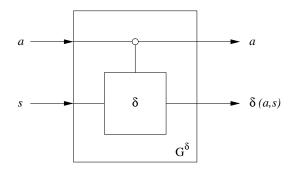


Figure 6: Controlled gate generated by an automaton

In conclusion, the class of Boolean conditional control gates is categorically equivalent to the class of (deterministic, finite) automata in which both the alphabet and the phase space have a power of 2 cardinality.

The reversible and conditional controlled gate generated by a nonreversible gate. If a Boolean gate  $G : \{0,1\}^n \to \{0,1\}^m$  is not reversible, it is always possible to construct a corresponding reversible gate  $G^r : \{0,1\}^{m+n} \to \{0,1\}^{m+n}$  associating to the input pair  $(\vec{a}, \vec{s}) \in \{0,1\}^n \times \{0,1\}^m$  the output pair  $(\vec{a}, \vec{s} \oplus G(\vec{a})) \in \{0,1\}^n \times \{0,1\}^m$ . This gate is reversible (and generally non conservative). Moreover it is a controlled gate, that is a finite automaton with respect to the alphabet  $\mathcal{A} = \{0,1\}^n$ , the phase space  $Q = \{0,1\}^m$ , and the set of next state functions  $\delta_{\vec{a}}$  (for  $\vec{a} \in \{0,1\}^n$ ) associating to any state  $\vec{s} \in \{0,1\}^m$  the next state  $\vec{s'} = \delta_{\vec{a}}(\vec{s}) :=$  $\vec{s} \oplus G(\vec{a}) \in \{0,1\}^m$  (see Figure 7).

How to transform a reversible and non conservative Boolean gate into a reversible and conservative one. If  $G^r$  is a non conservative reversible gate, we can extend it to a conservative gate  $G^{rc}$  by adding some new input and output lines, and maintaining the original reversibility. Let  $\{\vec{x}\}_1 = \sum_{i=1}^{n+m} x_i$  be the number of ones contained into the input  $\vec{x}$ ; analogously, let  $\{G^r(\vec{x})\}_1 = \sum_{i=1}^{n+m} G_i^r(\vec{x})$  be the number of ones contained into the corresponding output  $G^r(\vec{x})$ . We denote with  $E(\vec{x})$  the quantity  $\{G^r(\vec{x})\}_1 - \{\vec{x}\}_1$ . Clearly  $E(\vec{x})$  is an integer number from the interval [-(n+m), n+m]. It is immediately seen that if  $G^r$  would be conservative then it would hold  $E(\vec{x}) = 0$  for every  $\vec{x} \in \{0, 1\}^{n+m}$ . On the other hand,

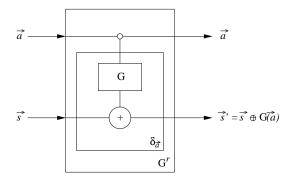


Figure 7: Reversible gate generated by a nonreversible one

since we have supposed that  $G^r$  is a non conservative gate, there exists an  $\vec{x} \in \{0,1\}^{n+m}$  such that  $E(\vec{x}) \neq 0$ .

For the moment, let us suppose that  $E(\vec{x}) > 0$ . Then there exists an  $\vec{x}' \in \{0,1\}^{n+m}$  such that  $E(\vec{x}') < 0$ . In fact we can express the quantity  $\sum_{\vec{x} \in \{0,1\}^{n+m}} E(\vec{x})$  as follows:

$$\sum_{\vec{x}\in\{0,1\}^{n+m}} E(\vec{x}) = \sum_{\vec{x}\in\{0,1\}^{n+m}} \left( \{G^r(\vec{x})\}_1 - \{\vec{x}\}_1 \right)$$
$$= \sum_{\vec{x}\in\{0,1\}^{n+m}} \{G^r(\vec{x})\}_1 - \sum_{\vec{x}\in\{0,1\}^{n+m}} \{x\}_1$$
(2)

Since  $G^r$  is reversible, it is a permutation over the set  $\{0,1\}^{n+m}$ . This means that the two sums in (2) are over the same elements, and thus:

$$\sum_{\vec{x}\in\{0,1\}^{n+m}}E(\vec{x})=0$$

As a consequence, if  $E(\vec{x}) > 0$  there must exist an  $\vec{x}' \in \{0,1\}^{n+m}$  such that  $E(\vec{x}') < 0$ . In a completely analogous way we can show that if  $E(\vec{x}) < 0$  then there exists an  $\vec{x}' \in \{0,1\}^{n+m}$  such that  $E(\vec{x}') > 0$ .

For the considerations above, if we define  $\ell = -\min_{\vec{x}} E(\vec{x})$  and  $h = \max_{\vec{x}} E(\vec{x})$ , and the gate  $G^r$  is non conservative, then  $\ell$  and h are positive integers. For any  $\vec{x} \in \{0,1\}^{n+m}$  such that  $E(\vec{x}) < 0$ , let  $E_{\ell}(\vec{x})$  be the string  $\underbrace{1,\ldots,1}_{-E(\vec{x})}, 0,\ldots, 0$  of length  $\ell$  (if  $\ell = 0$  we obtain the empty string); analo-

gously, whenever  $E(\vec{x}) > 0$  we define  $E_h^c(\vec{x})$  as the string  $\underbrace{0, \ldots, 0}_{E(\vec{x})}, 1, \ldots, 1$ 

of length h.

To extend  $G^r$  to a reversible and conservative gate  $G^{rc}$  we can use  $\ell$  ancillae lines (that we briefly indicate with  $\vec{y}$ ) to provide  $-E(\vec{x})$  ones whenever  $E(\vec{x}) < 0$ , and h ancillae lines (that we indicate with  $\vec{z}$ ) to remove  $E(\vec{x})$ 

ones whenever  $E(\vec{x}) > 0$ . More precisely, we define  $G^{rc}: \{0,1\}^{n+m+\ell+h} \to \{0,1\}^{n+m+\ell+h}$  as follows:

$$\forall \vec{x} \in \{0,1\}^{n+m}, \forall \vec{y} \in \{0,1\}^{\ell}, \forall \vec{z} \in \{0,1\}^{h}$$

$$\begin{cases} (G^{r}(\vec{x}), E_{\ell}(\vec{x}), \vec{1}_{h}) & \text{if } E(\vec{x}) < 0, \ \vec{y} = \vec{0} \text{ and } \vec{z} = \vec{1} & \text{i} ) \\ (\vec{k}, \vec{0}_{\ell}, \vec{1}_{h}) & \text{if } G^{r}(\vec{k}) = \vec{x}, \ E(\vec{k}) < 0, \\ \vec{u} - E_{\ell}(\vec{k}) \text{ and } \vec{z} - \vec{1} & \text{ii} ) \end{cases}$$

$$(G^{r}(\vec{x}) | \vec{0}_{\ell} | \vec{1}_{\ell}) \qquad \text{if } E(\vec{x}) = 0 \quad \vec{u} = \vec{0} \text{ and } \vec{z} = \vec{1} \qquad \text{if}$$

$$G^{rc}(\vec{x}, \vec{y}, \vec{z}) := \begin{cases} (G^{r}(\vec{x}), \vec{0}_{\ell}, \mathbf{1}_{h}) & \text{if } E(\vec{x}) = 0, \ \vec{y} = 0 \text{ and } \vec{z} = 1 \\ (G^{r}(\vec{x}), \vec{0}_{\ell}, E^{c}_{h}(\vec{x})) & \text{if } E(\vec{x}) > 0, \ \vec{y} = \vec{0} \text{ and } \vec{z} = \vec{1} \end{cases}$$
 iv)

$$\begin{array}{ccc} (\vec{k}, \vec{0}_{\ell}, \vec{1}_{h}) & \text{ if } G^{r}(\vec{k}) = \vec{x}, \, E(\vec{k}) > 0, \\ \vec{y} = \vec{0} \text{ and } \vec{z} = E^{c}_{h}(\vec{k}) & \text{ v} ) \end{array}$$

$$(\vec{x}, \vec{y}, \vec{z})$$
 otherwise vi

A direct inspection of  $G^{rc}$  shows that the map  $G^r$  is obtained in the first n + m output lines when the ancillae lines  $\vec{y}$  and  $\vec{z}$  are fixed respectively with the input values  $\vec{0}$  and  $\vec{1}$ . Notice that the rules ii) and v) are designed in order to provide the inverses of the tuples produced by rules i) and iv), respectively. On the other hand, the tuples produced by rule iii) can be inverted by computing the inverse of the first n + m components through the inverse of the map  $G^r$ . Finally, rule vi) makes the gate behave as the identity when none of the previous rules are satisfied: as a consequence, the corresponding tuples can be trivially inverted. Summarizing, the inverse of  $G^{rc}$  is obtained by substituting rule iii) in the analytic expression of  $G^{rc}$  with the following:

$$(\vec{k}, \vec{0}_{\ell}, \vec{1}_h)$$
 if  $G^r(\vec{k}) = \vec{x}, E(\vec{k}) = 0, \ \vec{y} = \vec{0}$  and  $\vec{z} = \vec{1}$ 

Reconstruction of the original gate from the reversible and conservative induced gate. Following Toffoli ([To, 80]), the original arbitrary Boolean gate G can be recovered by means of the just constructed reversible and conservative gate  $G^{rc}$  in the following way.

In more general mathematical parlance, a *realization* of a function G consists in a new function  $G^{rc}$  together with two mappings  $\mu$  and  $\pi_Q$  (respectively, the *encoder* and the *decoder*) such that  $G = \pi_Q \circ G^{rc} \circ \mu$ . In this context, our plan is to obtain a realization  $\pi_Q \circ G^{rc} \circ \mu$  of G such that  $G^{rc}$  is invertible [i.e., reversible] and conservative, and the mappings  $\mu$  and  $\pi_Q$  are essentially independent of G and contain as little "computing power" as possible.

More precisely, though the form of  $\mu$  and  $\pi_Q$  must obviously reflect the number of input and output components of G, and thus the format of G's truth table, we want them to be otherwise independent of the particular contents of such truth table as G is made to range over the set of all combinatorial functions.

In the present case, the encoder is realized by the mapping  $\mu : \{0,1\}^n \to \{0,1\}^{n+m+\ell+h}$  associating to the input  $\vec{a} \in \{0,1\}^n$  the output 4-tuple  $\mu(\vec{a}) := (\vec{a}, \vec{0}_m, \vec{0}_\ell, \vec{1}_h) \in \{0,1\}^n \times \{0,1\}^m \times \{0,1\}^\ell \times \{0,1\}^h$  (independent of the particular form of G). The decoder is realized by the projection mapping  $\pi_Q : \mathcal{A} \times Q \times \{0,1\}^\ell \times \{0,1\}^h \to Q$ . Trivially, for any  $\vec{a} \in \{0,1\}^n$  one gets  $(\pi_Q \circ G^{rc} \circ \mu)(\vec{a}) = (\pi_Q \circ G^{rc})(\vec{a}, \vec{0}_m, \vec{0}_\ell, \vec{1}_h) = \pi_Q(\vec{a}, G(\vec{a}), \vec{y}, \vec{z}) = G(\vec{a})$ .

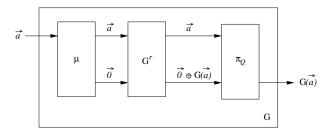


Figure 8: Original nonreversible gate obtained by its reversible extension

The FAN–OUT gate as a cloning procedure induced by the Controlled–Not gate. A very important connective in reversible computing is FAN–OUT :  $L \rightarrow L^2$ , defined by the law FAN–OUT(x) = (x, x). In other words, the FAN–OUT function simply clones the input value. When dealing with classical circuits, the FAN–OUT function is implemented by sticking two output wires to an existing input wire. The Controlled–NOT gate (see Table 6) provides a possible realization of the FAN–OUT function by a two–inputs/two–outputs reversible gate. Indeed, if the operating line is fixed with the input value  $x_2 = 0$ , then the control input is cloned realizing in this way a classical FAN–OUT (see Figure 9).

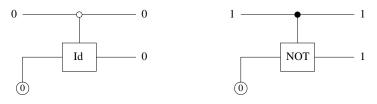


Figure 9: Realization of the FAN–OUT function with the Controlled–NOT gate

### 3 The Conservative and Reversible Fredkin Gate

(F1) One of the paradigmatic conservative and reversible primitive is the

Fredkin gate, a three–inputs/three–outputs gate that computes the following function FG :  $\{0, 1\}^3 \rightarrow \{0, 1\}^3$ :

$$y_1 = x_1$$
  

$$y_2 = (x_1 \land x_2) \lor (\neg x_1 \land x_3)$$
  

$$y_3 = (\neg x_1 \land x_2) \lor (x_1 \land x_3)$$

In tabular notation it is presented in Table 7.

$x_1$	$x_2$	$x_3$	$\mapsto$	$y_1$	$y_2$	$y_3$
0	0	0		0	0	0
0	0	1		0	1	0
0	1	0		0	0	1
0	1	1		0	1	1
1	0	0		1	0	0
1	0	1		1	0	1
1	1	0		1	1	0
1	1	1		1	1	1

Table 7: The Fredkin reversible and conservative gate

(F2) The Fredkin gate is self-reversible, i.e., the inverse of function FG is FG itself. This is a particularly desirable feature for the construction of the quantum version of a reversible circuit, since the part of the circuit which "undoes" the computation (in order to disentangle input/output lines and the so-called ancillae lines) is thus completely symmetrical to the part which computes the output value.

Note that self-reversibility implies the reversibility property. The converse is not generally true: if  $f: L^m \to L^m$  is reversible then it is a permutation of  $L^m$  and, as it is well known, in general the composition of a permutation with itself does not give the identity as a result. In particular, it is immediate to see that only those permutations which are expressible as the composition of disjoint cycles of length 2 (and fixed points) are self-reversible.

(F3) Looking at Table 7, it follows immediately that the Fredkin gate is conservative. This property allows for the realization of the Fredkin gate in the framework of "billiard ball" computing, and led to the following observation concerning the physical meaning of conservativeness:

"In conservative logic, all signal processing is ultimately reduced to *conditional routing* of signals. Roughly speaking, signals are treated as unalterable objects that can be moved around in the course of a computation but never created or destroyed." ([FT, 82], page 227). The billiard ball model, developed by Fredkin and Toffoli in [FT, 82], is an excellent example of a toy scientific model of no immediate practical application but of large scientific impact. Balls of radius 1 travel on a unit grid in two directions. The direction of their movements can be changed either by an elastic collision, or by a reflection at a "mirror". Using this model it is possible to implement a *switch gate*; the Toffoli gate can then be implemented with four of them.

(F4) If the first of the inputs is set to 0 then the Fredkin gate exchanges the second input with the third one, whereas if the first input is set to 1 it returns all the inputs unchanged, as it is shown in Figure 10.



Figure 10: The Fredkin gate as a conditional switch

Therefore, the Fredkin gate is a Conditional Control gate with  $x_1$  as control input,  $\delta_0 = \text{EXC}$  and  $\delta_1 = \text{Id}$ .

(F5) From the Fredkin gate we can obtain some classical unary and binary connectives by setting respectively two and one of the input lines to a constant value (that is, either 0 or 1). For example,

- by fixing x<sub>3</sub> = 0 in the input, the second output becomes y<sub>2</sub> = (x<sub>1</sub> ∧ x<sub>2</sub>) ∨ (¬x<sub>1</sub> ∧ 0) = x<sub>1</sub> ∧ x<sub>2</sub>, i.e., y<sub>2</sub> gives the logical AND between x<sub>1</sub> and x<sub>2</sub>. In this case the outputs y<sub>1</sub> and y<sub>3</sub> are called garbage;
- by fixing  $x_2 = 1$  and  $x_3 = 0$  the input  $x_1$  is negated in the output  $y_3 = \neg x_1$ , with garbage  $y_1$  and  $y_2$ . In this way we realize the NOT connective.

(F6) Differently from the realization of the FAN–OUT gate by the reversible nonconservative Controlled–NOT gate, it is easy to see that it is impossible to realize the FAN–OUT function by a conservative two–inputs/two–outputs gate. Such a realization requires at least three input lines and three output lines, even when working with Boolean logic. The Fredkin gate supplies one of these possible realizations:

• if we fix  $x_2 = 1$  and  $x_3 = 0$  then the first input is cloned in the first and second outputs, i.e., we obtain the FAN-OUT function, with the output  $y_3$  as garbage.

Summarizing, the Fredkin gate has the following properties:

- F-1 ) it is a three–inputs/three–outputs gate, where each input/output assumes values in  $\{0, 1\}$ ;
- F-2 ) it is reversible, that is a bijective mapping from  $\{0,1\}^3$  onto  $\{0,1\}^3;$
- F-2') it is *self-reversible*, that is  $FG^2 = FG \circ FG = Id$  (the identity function on  $\{0, 1\}^3$ );
- F-3 ) it is *conservative*, in the sense that the number of 0 and 1 in the input is the same as in the output;
- F-4) it is a controlled gate, that is  $x_1$  is a control input which is left unchanged but which determines a transformation of the target input  $(x_2, x_3)$  into the output  $(y_2, y_3)$  by the gate EXC if  $x_1 = 0$ and by the identity gate if  $x_1 = 1$ ;
- F-5 ) it is a universal primitive, that is, from the configurations of the gate we can obtain the classical logical connectives AND, OR, IM-PLICATION, NOT which constitute a "functionally complete" set of connectives for the Boolean logic, that is a set of primitive truth functions with which all the possible truth functions (i.e., all the functions  $\{0,1\}^n \to \{0,1\}$  for n ranging in N) can be realized;
- F-6 ) it realizes the FAN–OUT connective, which plays a central role in reversible computations since it clones a given input signal.

Our aim is to extend this computational Boolean framework based on the Fredkin gate to include the main features of many-valued logics, when a finite number of truth values are involved. In the next section we give a brief summary of the main aspects of this subject.

# 4 Many–valued Logics

The simplest extension of classical two-valued logic consists in the introduction of a third "intermediate", or "neutral" or "indeterminate" value. Lukasiewicz developed this idea in [Lu, 20]. In such paper he introduced a third truth value to take into account propositions which are neither true nor false, defining in this way a three-valued logic. This logic was then extended to deal with d truth values as well as with an infinite number of truth values, in particular the  $\aleph_0$  and  $\aleph_1$  cardinalities.

Let us begin with a brief exposition of the main features of the many– valued logics of Lukasiewicz; the definition and the properties of the operators are the same for the finite and the infinite–valued cases, unless otherwise stated.

Technically speaking, truth values of a logical system are defined just as syntactic labels, with no numerical meaning. In a subsequent step, it is possible to give an interpretation of the logical system in terms of an algebraic structure; only during such a process, the truth values are associated with elements of the structure, which can be more abstract mathematical objects than real or integer numbers. Indeed, all the notions here exposed can be restated in such a formal way; however, for our purposes it will be convenient to deal with the following sets of truth values, treated as numerical sets equipped with the standard total order relation induced by  $\mathbb{R}$ :

- $L_d = \left\{0, \frac{1}{d-1}, \frac{2}{d-1}, \dots, \frac{d-2}{d-1}, 1\right\}$ , with  $d \ge 2$ , for *d*-valued logics;
- $L_{\aleph_0} = [0,1] \cap \mathbb{Q}$ , that is the set of rational in the interval [0,1], for infinite-valued logics with  $\aleph_0$  truth values;
- $L_{\aleph_1} = [0, 1]$ , that is the set of real values in the interval [0, 1], for infinite-valued logics with  $\aleph_1$  truth values.

The numbers of  $L_{\alpha}$ ,  $\alpha \in \{d, \aleph_0, \aleph_1\}$  are interpreted, after Lukasiewicz, as the possible truth values which the logical sentences can be assigned to. As usually done in literature, the values 1 and 0 denote respectively truth and falseness, whereas all the other values are used to indicate different degrees of indefiniteness. With the introduction of the new truth values, the propositional connectives of Boolean logic must be redefined. Accordingly, many-valued logics represent strong generalizations of bivalent (i.e., classical) logic.

#### 4.1 Lukasiewicz approach

The **Lukasiewicz** system on the totally ordered numerical set of truth values  $L_{\alpha}$ , with  $\alpha \in \{d, \aleph_0, \aleph_1\}$ , considers as primitive the *implication*  $(\rightarrow_L)$  connective, which is defined by the following equation:

$$\begin{aligned} x \to_L y &:= \min\{1, 1 - x + y\} \qquad \text{(Lukasiewicz implication)} \\ &= \begin{cases} 1 - x + y & \text{if } y < x\\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

In the system  $\langle L_{\alpha}, \rightarrow_L \rangle$  a *negation*  $(\neg)$  connective is derived according to the rule:

 $\neg x := x \to_L 0 \qquad (\text{diametrical negation})$ = 1 - x

Using these two connectives Łukasiewicz defines some other derived ones as:

$x \lor y := (x \to_L y) \to_L y$	(Lukasiewicz disjunction)
$x \wedge y := \neg (\neg x \vee \neg y)$	(Lukasiewicz conjunction)
$x \leftrightarrow_L y := (x \to_L y) \land (y \to_L x)$	(Lukasiewicz equivalence)

the former two being the algebraic realizations of the logical connectives OR and AND respectively.

From these definitions it is easy to see that the following equalities hold:

$$x \lor y = \max\{x, y\}$$
 and  $x \land y = \min\{x, y\}$ 

where max and min are the lub and glb of the pair of numbers x, y with respect to the standard total order of  $L_{\alpha}$ , which can also be expressed in the form:

$$x \leq y$$
 iff  $x \to_L y = 1$ 

One important feature of all many–valued connectives now presented is that they are equal to the analogous Boolean connectives when only 0 or 1 are involved.

Zawirski in [Za, 34] for the first time considered as primitive connective on  $L_{\alpha}$  (instead of the Lukasiewicz implication) the binary operation of *truncated sum* defined as follows:

$$x \oplus y := \min\{1, x + y\}$$
 (truncated sum)  
= 
$$\begin{cases} x + y & \text{if } x + y < 1\\ 1 & \text{otherwise} \end{cases}$$

The two systems based on the numerical set of truth values  $L_{\alpha}$ , the Lukasiewicz one  $\langle L_{\alpha}, \rightarrow_L \rangle$  and the Zawirski one  $\langle L_{\alpha}, \oplus \rangle$ , are mutually equivalent owing to the "translation" rules:

$$x \oplus y = \neg x \to_L y \quad \text{and} \quad x \to_L y = \neg x \oplus y$$
 (3)

Furthermore, the following binary operation can be defined in the Zawirski  $\langle L_{\alpha}, \oplus \rangle$ -system:

$$x \odot y := \neg(\neg x \oplus \neg y) = \max\{0, x + y - 1\}$$
$$= \begin{cases} x + y - 1 & \text{if } 1 < x + y \\ 0 & \text{otherwise} \end{cases}$$

In some semantical interpretations,  $\oplus$  and  $\odot$  are considered as algebraic realizations of the logical connectives VEL and ET respectively, and they are also called the *disjunction* and *conjunction* MV–connectives.

Let us stress that on the basis either of the Łukasiewicz system or of the Zawirski one it is always possible to derive a structure  $\langle L_{\alpha}, \wedge, \vee, \neg \rangle$  of distributive lattice with a nonstandard negation. The lattice join and meet operations, algebraic realizations of the logical connectives OR and AND, can be defined in the two systems respectively as follows:

$$x \lor y = \max\{x, y\} = (x \odot \neg y) \oplus y = \neg(\neg x \oplus y) \oplus y$$
(4a)

$$x \wedge y = \min\{x, y\} = (x \oplus \neg y) \odot y = \neg[\neg(x \oplus \neg y) \oplus \neg y]$$
(4b)

Note that the excluded middle law holds in the case of the VEL connective  $(\forall x \in L_{\alpha}: x \oplus \neg x = 1)$ , whereas in general this law does not hold for the OR connective  $(\forall x \in L_{\alpha} \setminus \{0,1\}: x \vee \neg x \neq 1)$ . A similar result is verified with respect to the non-contradiction law  $(\forall x \in L_{\alpha}: x \odot \neg x = 0 \text{ and } \forall x \in L_{\alpha} \setminus \{0,1\}: x \wedge \neg x \neq 0)$ . However, the desirable law  $x \vee x \rightarrow_L x = 1$  holds relatively to the OR connective, but for every  $x \neq 0, 1$  one has that  $x \oplus x \rightarrow_L x \neq 1$ . In the Zawirski context, the standard ordering on  $L_{\alpha}$  assumes now the form:

$$x \leq y$$
 iff  $\neg x \oplus y = 1$ 

Two modal connectives, *possibility* ( $\diamondsuit$ ) and *necessity* ( $\Box$ ), can be introduced on  $L_{\alpha}$  according to the following definitions:

$\Diamond x = \begin{cases} 0\\1 \end{cases}$	$\begin{array}{ll} \text{if} & x = 0 \\ \text{if} & x \neq 0 \end{array}$	(possibility)
$\Box x = \begin{cases} 0\\1 \end{cases}$	$\begin{array}{ll} \text{if} & x \neq 1 \\ \text{if} & x = 1 \end{array}$	(necessity)

Note that these two modal connectives are mutually interdefinable, owing to the following relationships:

$$\Box x = \neg \Diamond \neg x \quad \text{and} \quad \Diamond x = \neg \Box \neg x \tag{5}$$

Moreover, the restriction of both connectives to the Boolean values coincides with the identity function (these modalities are meaningless in the Boolean environment).

Besides the diametrical negation  $(\neg)$  two other negation connectives can be defined as many-valued extensions of the standard Boolean negation: the *intuitionistic negation* (also *impossibility*  $\sim$ ) and the *anti-intuitionistic negation* (also *contingency*  $\flat$ ) defined as:

 $\sim$ 

$$\begin{aligned} x &:= \neg \Diamond x & \text{(impossibility)} \\ &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} & \text{(intuitionistic negation)} \end{aligned}$$

$$bx := \neg \Box x \qquad (contingency)$$
$$= \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases} \qquad (anti-intuitionistic negation)$$

In agreement with the intuitionistic propositional logic of Brouwer and Heyting, the intuitionistic negation "impossibility" fails the excluded middle law  $(\forall x \in L_{\alpha} \setminus \{0, 1\}: x \lor \sim x = x \oplus \sim x = x \neq 1)$ , but does not fail the law of noncontradiction ( $\forall x \in L_{\alpha}: x \land \sim x = x \odot \sim x = 0$ ). Note that the restriction of the three negations to the two Boolean values collapses in a unique (standard) negation ( $\forall x \in \{0, 1\}: \neg x = \sim x = bx = 1 - x$ ). Trivially, also these two negation connectives are mutually interdefinable according to:

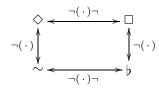
$$bx = \neg \sim \neg x \quad \text{and} \quad \sim x = \neg b \neg x \tag{6}$$

The intuitionistic negation is a primitive one, together with the diametrical negation, in *BZ*-lattice structures, of which the system  $\langle L_{\alpha}, \wedge, \vee, \neg, \rangle$  is a standard model. Also in this case, the modal connectives can be recovered from the two involved negations according to the following:

$$\Box x = \neg \sim x \quad \text{and} \quad \Diamond x = \sim \neg x \tag{7}$$

For further information on BZ structures see [CN, 89].

In conclusion, in the algebraic approach to many-valued logics we have considered as primitive two mutually interdefinable (according to (3)) systems, the Łukasiewicz one  $\langle L_{\alpha}, \rightarrow_L \rangle$  and the Zawirski one  $\langle L_{\alpha}, \oplus \rangle$ . An new system of distributive lattice with diametrical negation  $\langle L_{\alpha}, \wedge, \vee, \neg \rangle$  can always be induced. Moreover, the set of unary connectives  $\{\Box, \diamondsuit, \sim, \flat\}$  (two modalities and two negations) are mutually interdefinable making use of the diametrical negation  $(\neg)$  according to the following diagram:



#### 4.1.1 The finite-valued case

In the three-valued logic  $L_3$  one has:

$$\diamond x = \neg x \to_L x = x \oplus x \Box x = \neg (x \to_L \neg x) = \neg (\neg x \oplus \neg x) = x \odot x$$

and

Let us stress that in the three valued case  $L_3$  the above definition of "it is possible that x" coincides with "if not x then x" ( $\Diamond x = \neg x \rightarrow_L x$ ); in [Lu, 30] Lukasiewicz mentioned that Tarski, a student of him, in 1921 proposed this as the definition of possibility. Therefore in this particular case we can derive the modal connectives from the system  $\langle L_3, \rightarrow_L \rangle$  (equivalently,  $\langle L_3, \oplus \rangle$ ) which is thus sufficient to generate all the connectives introduced above. In particular, the two BZ negations have the form  $\sim x = \neg(\neg x \rightarrow_L x)$  $x) = \neg(x \oplus x)$  and  $\flat x = x \rightarrow_L \neg x = \neg x \oplus \neg x$ .

In the more general finite d-valued case the link between possibility and VEL connectives is extended by the following identity which is true for every  $x \in L_d$ :

$$\Diamond x := \underbrace{x \oplus x \oplus \ldots \oplus x}_{(d-1)-\text{times}}$$

Thus owing to this result and the relation

$$\Box x = \neg \Diamond \neg x = \underbrace{x \odot x \odot \ldots \odot x}_{(d-1)-\text{times}}$$

also in any finite-valued case the modal connectives of possibility and necessity can be both derived inside the system  $\langle L_d, \oplus \rangle$ .

We observe that, for infinite-valued logics, it is not possible to derive from  $\rightarrow_L$  and  $\neg$  the modal operators  $\Box$  and  $\diamond$  and the intuitionistic and anti-intuitionistic negations  $\sim$  and  $\flat$  as we have just done for the finitevalued case. In fact, in [Mc, 51] it has been proved the following theorem.

**Theorem 4.1.** Let  $L \in \{L_{\aleph_0}, L_{\aleph_1}\}$ . A function  $f : L^m \to L$  is expressible as a formula containing only the operators  $\to_L$  and  $\neg$  if and only if it is continuous.

#### 4.2 Gödel approach

The extension of classical connectives to many-valued logics is not unique. For example, different types of implications have been defined in literature; one of these, which is often used, is the implication  $\rightarrow_G$  defined by Gödel:

$$x \to_G y := \begin{cases} y & \text{if } y < x \\ 1 & \text{otherwise} \end{cases}$$
(Gödel implication)

Note that, if the use of the constant value 0 is allowed, we can obtain the intuitionistic negation as  $\sim x = x \rightarrow_G 0$ . Moreover, in the three-valued case Gödel's implication differs from  $\rightarrow_L$  only for the input pair  $(\frac{1}{2}, 0)$ : in fact,  $\frac{1}{2} \rightarrow_L 0 = \frac{1}{2}$  whereas  $\frac{1}{2} \rightarrow_G 0 = 0$ .

# 5 Functional Completeness of Finite–valued Calculus

We face now the problem whether any conceivable function  $f: L_d^n \to L_d$ , for *n* ranging in  $\mathbb{N}$ , is constructible using only the operators  $\neg$  and  $\rightarrow_L$ , i.e., the *functional completeness problem* on  $L_d$  of the pair of connectives  $\{\neg, \rightarrow_L\}$ .

The following result, originally due to Jerzy Słupecki (see, for example, [RT, 52]), gives a negative answer.

**Theorem 5.1.** The *d*-valued (with  $d \ge 3$ ) propositional calculus of Lukasiewicz based on operators  $\neg$  and  $\rightarrow_L$  is not functionally complete. That is, there exist functions  $f: L^n_d \to L_d$  which are not expressible as a composition of the logical functions  $\neg$  and  $\rightarrow_L$  (from which we stress that it is possible to derive the logical functions  $\lor$ ,  $\land$ ,  $\oplus$ ,  $\odot$ ,  $\diamondsuit$ ,  $\square$ ,  $\flat$ ,  $\sim$ ,  $\leftrightarrow_L$ ).

*Proof.* The result follows directly from the fact that every function built up using only  $\neg$  and  $\rightarrow_L$  gives a result in  $\{0, 1\}$  when its arguments are assigned with values in this set. As a consequence we cannot represent, for example, the constant function which is identically equal to  $\frac{1}{d-1}$ .

To make the *d*-valued sentential calculus *functionally complete* Słupecki introduced a new unary connective, called *tertium*, which is defined by the constant function  $\mathcal{T}: L_d \to L_d$ :

$$\forall x \in L_d \qquad \mathcal{T}(x) := \frac{1}{d-1}$$

In fact, the following theorem holds.

**Theorem 5.2.** The *d*-valued (with  $d \ge 2$ ) propositional calculus of Lukasiewicz is functionally complete with respect to the set of primitive truth functions  $\{\neg, \rightarrow_L, \mathcal{T}\}$ .

The proof of this theorem is constructive and uses the following d functions:

$$j_k: L_d \to L_d, \quad \text{for } k \in L_d$$

defined as:

$$j_k(x) = \begin{cases} 1 & \text{if } x = k \\ 0 & \text{otherwise} \end{cases}$$

We do not give here the proof of the theorem above mentioned due to its length. The interested reader can find it, for example, in [RT, 52]. Observe that functions  $j_k$  can also be defined as:

$$j_k(x) = \Box(x \leftrightarrow_L k) \tag{8}$$

where the constant values  $k \in L_d$  are directly involved (besides the connectives  $\Box$  and  $\leftrightarrow_L$ , both definable using the pair  $\{\neg, \rightarrow_L\}$ ). Using these functions  $\{j_k : L_d \rightarrow L_d \mid k \in L_d\}$  it is immediate to obtain the tertium function  $\mathcal{T}$  as follows:

$$\mathcal{T}(x) = \frac{1}{d-1} \sum_{k \in L_d} j_k(x) \tag{9}$$

On the other hand, the *d*-valued  $(d \geq 3)$  propositional calculus of Lukasiewicz with the tertium function  $\{\neg, \rightarrow_L, \mathcal{T}\}$  is functionally complete, and thus owing to (8) and (9) also  $\langle L_d, \rightarrow_L \rangle$  is functionally complete (recall that  $\neg x = x \rightarrow_L 0$ ) according to the following definition:

• A collection of primitive truth functions  $L_d^n \to L_d$  and a set of constants from  $L_d$  is *universal* or (according to [RT, 52]) functionally complete if and only if all possible truth functions  $L_d^n \to L_d$ , with  $n \in \mathbb{N}$ , are constructible by combining these primitive functions and assigned constants.

This means that it is functionally equivalent to assume the tertium function or the presence of constants to the original set of primitives  $\{\neg, \rightarrow_L\}$ .

#### 5.1 Generalization of the disjunctive normal form (GDNF)

Let us consider the following function from  $L_d^{2n}$  to  $L_d$ :

$$M(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_n) := \bigwedge_{i=1}^n j_{c_i}(x_i)$$

It is easily verified that for every possible choice of  $x_1, x_2, \ldots, x_n, c_1, c_2, \ldots, c_n$ in  $L_d$  it holds:

$$M(x_1, x_2, \dots, x_n, c_1, c_2, \dots, c_n) = \begin{cases} 1 & \text{if } \forall i \in \{1, \dots, n\}, \ x_i = c_i \\ 0 & \text{otherwise} \end{cases}$$

We can thus state the following theorem.

**Theorem 5.3 (Generalization of the disjunctive normal form).** Let  $f: L_d^n \to L_d$  be a function. For every choice of  $(x_1, \ldots, x_n)$  in  $L_d^n$  it holds:

$$f(x_1, \dots, x_n) = \bigvee_{\substack{f(c_1, \dots, c_n) \neq 0}} [M(x_1, \dots, x_n, c_1, \dots, c_n) \wedge f(c_1, \dots, c_n)]$$
(10)

Another generalization of the disjunctive normal form which is analogous to the one presented here can be found in [Ur, 86](Lemma 2.9).

Notice that if we let  $E = \{0, 1\}$  then expression (10) can be expanded as:

$$f(x_1, \dots, x_n) = \bigvee_{\substack{f(c_1, \dots, c_n) \notin E}} [M(x_1, \dots, x_n, c_1, \dots, c_n) \land f(c_1, \dots, c_n)] \lor$$
$$\bigvee_{\substack{f(c_1, \dots, c_n) = 1}} M(x_1, \dots, x_n, c_1, \dots, c_n)$$

and eventually simplified by observing that  $\Box(x \leftrightarrow_L 1) = \Box x$  and  $\Box(x \leftrightarrow_L 0) = \sim x$ .

#### 5.2 Generalization of the conjunctive normal form (GCNF)

Analogously to the disjunctive form, we introduce the function  $S: L_d^{2n} \to L_d$  defined as:

$$S(x_1,\ldots,x_n,c_1,\ldots,c_n) := \bigvee_{i=1}^n h_{c_i}(x_i)$$

where

$$h_k(x) := \flat(x \leftrightarrow_L k) \quad \text{for } k \in L_d$$

The following lemma holds.

**Lemma 5.1.** For every choice of  $x_1, \ldots, x_n, c_1, \ldots, c_n$  in  $L_d$  it holds:

$$S(x_1, \dots, x_n, c_1, \dots, c_n) = \begin{cases} 1 & \text{if } \exists i \in \{1, \dots, n\} \text{ s.t. } x_i \neq c_i \\ 0 & \text{if } \forall i \in \{1, \dots, n\}, x_i = c_i \end{cases}$$

*Proof.* Since  $h_k(x) = \flat(x \leftrightarrow_L k) = \neg \Box(x \leftrightarrow_L k) = \neg j_k(x)$ , it holds:

$$h_k(x) = \begin{cases} 1 & \text{if } x \neq k \\ 0 & \text{if } x = k \end{cases}$$

The proof of the claim follows immediately from the fact that S is a disjunction of the functions  $h_k(x)$ .

We can thus state the following theorem.

**Theorem 5.4 (Generalization of the conjunctive normal form).** Let  $f: L_d^n \to L_d$  be a function. For every choice of  $(x_1, \ldots, x_n)$  in  $L_d^n$  it holds:

$$f(x_1, \dots, x_n) = \bigwedge_{\substack{f(c_1, \dots, c_n) \neq 1}} [S(x_1, \dots, x_n, c_1, \dots, c_n) \lor f(c_1, \dots, c_n)]$$
(11)

Notice that if we let  $E=\{0,1\}$  then expression (11) can be expanded as:

$$f(x_1, \dots, x_n) = \bigwedge_{\substack{f(c_1, \dots, c_n) \notin E}} [S(x_1, \dots, x_n, c_1, \dots, c_n) \lor f(c_1, \dots, c_n)] \land$$
$$\bigwedge_{\substack{f(c_1, \dots, c_n) = 0}} S(x_1, \dots, x_n, c_1, \dots, c_n)$$

and eventually simplified by observing that  $\flat(x \leftrightarrow_L 1) = \flat x$  and  $\flat(x \leftrightarrow_L 0) = \diamondsuit x$ .

#### 5.3 Clay's representation

Another way to represent the functions from  $L_d^n$  to  $L_d$  is given in the following theorem, taken from [Cl, 62].

**Theorem 5.5.** Let  $f : L_d^n \to L_d$  be a function. For every choice of  $(x_1, \ldots, x_n)$  in  $L_d^n$  it holds:

$$f(x_1, \dots, x_n) = \bigwedge_{\substack{f(c_1, \dots, c_n) \neq 1}} [M(x_1, \dots, x_n, c_1, \dots, c_n) \to_L f(c_1, \dots, c_n)]$$
(12)

*Proof.* It suffices to notice that, from the definition of M, for every choice of  $x_1, \ldots, x_n, c_1, \ldots, c_n$  and y in  $L_d$  it holds:

$$M(x_1, \dots, x_n, c_1, \dots, c_n) \to_L y = \begin{cases} y & \text{if } \forall i \in \{1, \dots, n\}, x_i = c_i \\ 1 & \text{otherwise} \end{cases}$$

Since  $x \to_L 0 = \neg x$ , expression (12) can be simplified as:

$$f(x_1,\ldots,x_n) = \bigwedge_{\substack{f(c_1,\ldots,c_n)\notin E}} [M(x_1,\ldots,x_n,c_1,\ldots,c_n) \to_L f(c_1,\ldots,c_n)] \land$$
$$\bigwedge_{f(c_1,\ldots,c_n)=0} \neg M(x_1,\ldots,x_n,c_1,\ldots,c_n)$$

or, alternatively, as:

$$f(x_1, \dots, x_n) = \bigwedge_{\substack{f(c_1, \dots, c_n) \notin E}} [M(x_1, \dots, x_n, c_1, \dots, c_n) \to_L f(c_1, \dots, c_n)] \land$$
$$\bigwedge_{\substack{f(c_1, \dots, c_n) = 0}} S(x_1, \dots, x_n, c_1, \dots, c_n)$$

## 6 Finite-valued Conservative Logics

In this section we extend conservative logic to include the main features of d-valued logics, with a particular attention towards three-valued logics. Since conservative logic is based on the Fredkin gate, we will extend it in order to deal with d possible truth values on its input and output lines.

First of all we restrict our attention to gates having the same number of input and output lines. For brevity, we denote by (n, d)-gate an ninputs/n-outputs gate whose input and output lines may assume values from  $L_d$ . Thus, an (n, d)-gate computes a function  $f : L_d^n \to L_d^n$ , where  $L_d^n = \underbrace{L_d \times \ldots \times L_d}_{n \text{ times}}$ . Any finite sequence  $(string) \ (x_1, x_2, \ldots, x_n) \in L_d^n$  is

called a *configuration* or *pattern* of size n.

**Reversibility** The extension of the *reversibility* property is simple: an (n,d)-gate is reversible if and only if the function computed by the gate is one-to-one (or, in other words, a permutation of the set  $L_d^n$ ). A similar argument holds for *self-reversibility*: a gate is self-reversible if and only if the corresponding function applied twice is the identity function. As noted above, this happens if and only if the function is a permutation which can be expressed as the composition of disjoint cycles of length two (plus, possibly, some fixed points).

**Conservativeness** More complex is the case of *conservativeness*. A gate is *strictly conservative* if and only if each output configuration is a permutation of the input one. This definition reflects perfectly the observation made by Fredkin and Toffoli in [FT, 82], cited above on page 17.

Notice that the permutation of the input values is not fixed, but varies depending on the pattern of values presented to the input lines; an example can be seen in Figure 10, where two possible permutations are chosen according to the value fed to the first input of the Fredkin gate.

Clearly the two-valued Fredkin gate is strictly conservative, and in our first efforts to make an extension of this gate to the finite-valued case we tried to preserve this property. Unfortunately, if the number n of in-put/output lines of a strictly conservative gate for a d-valued logic is not greater than d, then it is impossible to realize in its configurations the FAN–OUT function, as stated in the following proposition.

**Proposition 6.1.** If n and d are two integer numbers such that  $0 < n \leq d$  then there is no function  $f : L_d^n \to L_d^n$  which corresponds to a strictly conservative gate realizing in its configurations the FAN-OUT gate.

*Proof.* If n = 1 then the gate has one output, and thus it cannot realize the FAN–OUT function. So, assume that  $1 < n \leq d$ , and that there exists a strictly conservative gate realizing FAN–OUT and corresponding to a function  $f: L_d^n \to L_d^n$ . In the gate configuration realizing the FAN–OUT function, one input line is fed with a variable value and n - 1 input lines are fed with constant values. Since  $n-1 \leq d-1$ , there exists at least one truth value  $\ell \in L_d$  which does not appear in the fixed constant input values. When the variable value of the input is set to  $\ell$ , both the following properties should hold:

- the output configuration should be a permutation of the input configuration (since the gate is strictly conservative), and
- $\ell$  should appear twice in the output values (as the gate realizes the FAN–OUT function),

which is clearly impossible.

If the condition  $n \leq d$  in Proposition 6.1 is relaxed, then it is not difficult to see that FAN–OUT can be realized through gates which are both reversible and strictly conservative: see, for example, the Fredkin gate, where n = 3 and d = 2.

Weak Conservativeness An alternative approach is to weaken the conservativeness property in order to obtain some reasonable gate that computes the FAN–OUT function. Thus we say that a gate is *weakly conservative* if and only if the sum of output values is always equal to the sum of input values. It is clear that if a gate is strictly conservative then it is also weakly conservative, while the converse is not generally true.

For example, if the input of a gate is  $(\lambda, 0, 1)$  and the corresponding output is  $(0, 1, \lambda)$  then the gate is both strictly conservative and weakly conservative for this input/output pair, regardless of the numerical value associated to  $\lambda \in L_d$ . On the other hand, if the corresponding output is  $(\lambda, \lambda, \lambda)$  then the gate is weakly conservative if and only if we associate to  $\lambda$ the numerical value  $\frac{1}{2}$ , while it is not strictly conservative, whatever is the numerical value associated to  $\lambda$ . Indeed it is easy to see that, for a given pattern of input values, the set of admissible patterns for output prescribed by the weak conservativeness property varies depending upon the numerical values associated to the truth values.

Assuming  $L_d$  as the set of truth values, we propose a possible physical interpretation of the weak conservativeness property. To produce a given pattern of input values for a gate we need some amount of energy. A "conservative" gate has to build the output pattern in such a way that this energy is preserved; in other words, the output produced must have the property that, if built from scratch, it requires the same amount of energy which was required to build the input. The simplest way to satisfy this property is to produce a permutation of the input values, as strictly conservative gates do.

Now, let us suppose to encode the d truth values on a physical system which has the energy levels that are equally spaced and ordered according to the numerical value associated to the truth values. Thus, to switch from a given truth value, say  $\frac{k}{d-1}$ , to the next, that is  $\frac{k+1}{d-1}$ , we need to provide a

fixed amount  $\Delta E$  of energy. Analogously, when passing from a given truth value to the previous, the same amount  $\Delta E$  of energy is released.

For a gate to be conservative, it must build the output pattern without requiring energy from an external source nor dissipating energy towards the environment; this means that it can switch a line from a truth value  $\frac{k_1}{d-1}$ to a higher value  $\frac{k_2}{d-1}$  if and only if the energy needed (which is equal to  $(k_2 - k_1) \cdot \Delta E$ ) becomes available by lowering of the same amount the truth value stored in some other line. This is clearly equivalent to requiring that the sum of the values on the output lines be equal to the sum of the values on the input lines.

**0 and 1-Regularity** We now define two other properties of the Fredkin gate. They are not fundamental properties but characterize, for dvalued logics, three-inputs/three-outputs gates that have a behavior which is similar to the two-valued Fredkin gate. We recall that the Fredkin gate exchanges the second input with the third one when the first input is set to 0, and it gives as outputs the inputs unchanged when the first input is set to 1. According to this point of view, let  $G : L_d^3 \to L_d^3$ be the function computed by a (3,d)-gate; we say that the gate is 0regular if and only if  $G(0, x_2, x_3) = (0, x_3, x_2)$  for every possible choice of  $x_2, x_3$  in  $L_d$ . Analogously, we say that the gate is 1-regular if and only if  $G(1, x_2, x_3) = (1, x_2, x_3)$  for every possible choice of  $x_2, x_3$  in  $L_d$ .

**Functional Completeness** The last fundamental property satisfied by the Fredkin gate is *universality* (or *functional completeness*). Indeed, according to the definition given above, with the *d* valued extensions of the Fredkin gate we will discuss in the next sections it is possible to realize two universal sets for *d*-valued logics, either  $\{\neg, \rightarrow_L, \mathcal{T}\}$  or  $\{\neg, \oplus, \mathcal{T}\}$ .

**Conclusions** In the next sections we look for universal gates for d-valued logics which preserve as many of the following properties as possible:

- F-1 ) it is a (3, d)–gate, that is a three–inputs/three–outputs gate where each input and each output line may assume one of the values in  $L_d = \left\{0, \frac{1}{d-1}, \frac{2}{d-1}, \dots, \frac{d-2}{d-1}, 1\right\};$
- F-2 ) it is reversible;
- F-2') it is self–reversible;
- F-3) it is weakly–conservative;
- F-3') it is strictly–conservative;
- F-4 ) it is a universal gate, that is, from the configurations of the gate a universal set of connectives is obtained, included FAN–OUT;
- F-5 ) it is 0–regular;
- F-6 ) it is 1–regular;
- F-7 )  $y_1 = x_1$ , that is, the first output is always equal to the first input (conditional control condition);

F-8 ) when feeded with Boolean input triples it behaves as the classical Fredkin gate.

Properties F-5)–F-8) are not essential from the point of view of conservative logic, but nonetheless are desirable, since they [characterize] the Fredkin gate.

# 7 Three–valued Universal Gates

In order to devise a universal gate for a three–valued logic, the first idea that comes to mind is to take the equations which define the input/output behavior of the Fredkin gate and to interpret  $\neg$ ,  $\lor$  and  $\land$  respectively as the Lukasiewicz negation, disjunction and conjunction. However this approach does not work, as the mapping from  $L_3^3$  to  $L_3^3$  thus obtained is not even a bijection. As a consequence, we have to look for gates which are universal and preserve as many properties from F-1) – F-8) as possible.

The next table presents all the binary three-valued connectives that we are interested to realize with our three-valued universal gates: the Łukasie-wicz implication  $\rightarrow_L$ , the Gödel implication  $\rightarrow_G$ , the Łukasiewicz disjunction  $\lor$ , the Lukasiewicz conjunction  $\land$ , the VEL-disjunction  $\oplus$  and the ET-conjunction  $\odot$ :

x	y	$\rightarrow_L$	$\rightarrow_G$	$\wedge$	$\vee$	$\oplus$	$\odot$
0	0	1	1	0	0	0	0
0	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0
0	1	1	1	0	1	1	0
$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{\frac{1}{2}}{\frac{1}{2}}$	1	0
$\frac{\frac{1}{2}}{\frac{1}{2}}$	1	1	1	$\frac{\frac{1}{2}}{\frac{1}{2}}$	1	1	$\frac{1}{2}$
1	0	0	0		1	1	0
1	$\frac{1}{2}$	$\begin{array}{c} 0\\ \frac{1}{2}\\ 1 \end{array}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{\frac{1}{2}}{\frac{1}{2}}$
1	1	1	1	1	1	1	1

The unary connectives here considered are, besides the trivial identity connective Id, the negation connectives  $\neg$ ,  $\sim$ ,  $\flat$  and the modal connectives  $\diamondsuit$  and  $\Box$  depicted in the following table:

x		2	þ	$\diamond$	
0	1	1	1	0	0
$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	0
1	0	0	0	1	1

It is important to stress that besides unary and binary connectives we must consider the FAN–OUT gate which plays a fundamental role for reversible computations. Due to Proposition 6.1, the presence of this gate forbids the strict conservativeness of a universal (3,3)–gate.

The first three-valued gate that we introduce  $(F_1)$  allows to obtain from its configurations all the main connectives of the Lukasiewicz logic  $\langle L_3, \rightarrow_L \rangle$ , as well as the Gödel implication. The truth table of the gate is given in Table 8; as it can be seen, the gate is self-reversible (and thus reversible), 0-regular and 1-regular. Moreover, it satisfies properties F-7) and F-8).

$x_1x_2x_3 \vdash$	$\rightarrow y_1 y_2 y_3$	$x_1 x_2 x_3$	$\mapsto$	$y_1 y_2 y_3$	$x_1x_2x_3 \vdash$	$\rightarrow y_1 y_2 y_3$
0 0 0	0 0 0	$\frac{1}{2}$ 0 0		$\frac{1}{2}$ 0 0	1 0 0	$1 \ 0 \ 0$
$0 \ 0 \ \frac{1}{2}$	$0 \frac{1}{2} 0$	$\frac{1}{2} \ 0 \ \frac{1}{2}$		$\frac{1}{2} \ 0 \ \frac{1}{2}$	$1 \ 0 \ \frac{1}{2}$	$1 \ 0 \ \frac{1}{2}$
$0 \ 0 \ 1$	$0 \ 1 \ 0$	$\frac{1}{2}$ 0 1		$\frac{1}{2}$ 0 1	$1 \ 0 \ 1$	$1 \ 0 \ 1$
$0 \frac{1}{2} 0$	$0 \ 0 \ \frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 0		$\frac{1}{2}$ $\frac{1}{2}$ 0	$1 \frac{1}{2} 0$	$1 \frac{1}{2} 0$
$0 \frac{1}{2} \frac{1}{2}$	$0 \frac{1}{2} \frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	**	$\frac{1}{2}$ 1 0	$1 \frac{1}{2} \frac{1}{2}$	$1 \frac{1}{2} \frac{1}{2}$
$0 \frac{1}{2} 1$	$0 \ 1 \ \frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 1		$\frac{1}{2}$ 1 $\frac{1}{2}$	$1 \frac{1}{2} 1$	$1 \frac{1}{2} 1$
$0 \ 1 \ 0$	$0 \ 0 \ 1$	$\frac{1}{2}$ 1 0	**	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$1 \ 1 \ 0$	$1 \ 1 \ 0$
$0 \ 1 \ \frac{1}{2}$	$0 \frac{1}{2} 1$	$\frac{1}{2}$ 1 $\frac{1}{2}$		$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $1$	$1 \ 1 \ \frac{1}{2}$	$1 \ 1 \ \frac{1}{2}$
$0 \ 1 \ 1$	$0\ 1\ 1$	$\frac{1}{2}$ 1 1		$\frac{1}{2}$ 1 1	1 1 1	$1 \ 1 \ 1$

Table 8: Truth table of gate  $F_1$ 

Table 9 shows all the relevant connectives which can be obtained from the gate by fixing one or two of its input lines with constant values from  $L_3$ ;  $\Pr_1$  and  $\Pr_2$  are the projectors connectives defined as  $\Pr_1(x_1, x_2) = x_1$ and  $\Pr_2(x_1, x_2) = x_2$  respectively. We can observe that this gate realizes two negations (the diametrical and the intuitionistic one) and both the Lukasiewicz and Gödel implications introduced in Section 4; as a consequence, the universality property F-4) is satisfied for both kinds of threevalued logic. On the other hand, the necessity modal connective, the antiintuitionistic negation and both the binary MV-connectives are not realized.

Due to Proposition 6.1, gate  $F_1$  cannot be strictly conservative, as it realizes the FAN–OUT function. More precisely, strict conservativeness is lost in the two table rows marked with (\*\*). However, for these rows the gate is weakly conservative, and therefore the entire gate is weakly conservative.

The next two gates that we introduce are part of the results of an exhaustive search — performed with a program written on purpose — over all three–valued gates having the following properties:

F-1 ) it is a (3,3)-gate;

F-2') it is self–reversible;

Connective	Inputs	Constants	Outputs	Garbage
FAN-OUT	$x_1$	$x_2 = 1, x_3 = 0$	$y_1, y_2$	$y_3$
$Pr_1$	$x_2, x_3$	$x_1 = 0$	$y_3$	$y_1, y_2$
$\Pr_2$	$x_2, x_3$	$x_1 = 0$	$y_2$	$y_1, y_3$
$\rightarrow_L$	$x_1, x_3$	$x_2 = 1$	$y_3$	$y_1, y_2$
$\rightarrow_G$	$x_1, x_2$	$x_3 = 1$	$y_2$	$y_1, y_3$
$\vee$	$x_1, x_3$	$x_2 = 1$	$y_2$	$y_1,y_3$
$\wedge$	$x_1, x_2$	$x_3 = 0$	$y_2$	$y_1,y_3$
Id	$x_1$	$x_2 = 0, x_3 = 0$	$y_1$	$y_2,y_3$
_	$x_1$	$x_2 = 1, x_3 = 0$	$y_3$	$y_1, y_2$
$\sim$	$x_1$	$x_2 = 0, x_3 = 1$	$y_2$	$y_1, y_3$
$\diamond$	$x_1$	$x_2 = 0, x_3 = 1$	$y_3$	$y_1,y_2$

Table 9: The operators obtained through gate  $F_1$ 

- F-3 ) it is weakly conservative;
- F-8 ) when feeded with Boolean input triples it behaves as the Fredkin gate.

The first of the two obtained gates (F<sub>2</sub>) is substantially equivalent to F<sub>1</sub>; its truth table is given in Table 10. As we can see, this gate differs from F<sub>1</sub> only for the input triples  $0\frac{1}{2}\frac{1}{2}$  and  $\frac{1}{2}0\frac{1}{2}$ . It is only 1–regular and it has not the property F-7).

$x_1x_2x_3 \mapsto$	$y_1 y_2 y_3$	$x_1x_2x_3 \vdash$	$\rightarrow y_1 y_2 y_3$	$x_1x_2x_3 \vdash$	$\rightarrow y_1 y_2 y_3$
0 0 0	0 0 0	$\frac{1}{2}$ 0 0	$\frac{1}{2}$ 0 0	1 0 0	$1 \ 0 \ 0$
$0 \ 0 \ \frac{1}{2}$	$0 \frac{1}{2} 0$	$\frac{1}{2} \ 0 \ \frac{1}{2}$		$1 \ 0 \ \frac{1}{2}$	$1 \ 0 \ \frac{1}{2}$
$0 \ 0 \ 1$	$0 \ 1 \ 0$	$\frac{1}{2}$ 0 1	$\frac{1}{2}$ 0 1	$1 \ 0 \ 1$	$1 \ 0 \ 1$
$0 \frac{1}{2} 0$	$0 \ 0 \ \frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 0	$\frac{1}{2}$ $\frac{1}{2}$ 0	$1 \frac{1}{2} 0$	$1 \frac{1}{2} 0$
$0 \frac{1}{2} \frac{1}{2}$	$\frac{1}{2}$ 0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ 1 0	$1 \frac{1}{2} \frac{1}{2}$	$1 \frac{1}{2} \frac{1}{2}$
$0 \frac{1}{2} 1$	$0 \ 1 \ \frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{1}{2}$ 1 $\frac{1}{2}$	$1 \frac{1}{2} 1$	$1 \frac{1}{2} 1$
$0 \ 1 \ 0$	$0 \ 0 \ 1$	$\frac{1}{2}$ 1 0	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$1 \ 1 \ 0$	$1 \ 1 \ 0$
$0 \ 1 \ \frac{1}{2}$	$0 \frac{1}{2} 1$	$\frac{1}{2}$ 1 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 1	$1 \ 1 \ \frac{1}{2}$	$1 \ 1 \ \frac{1}{2}$
0 1 1	0 1 1	$\frac{1}{2}$ 1 1	$\frac{1}{2}$ 1 1	1 1 1	1 1 1

Table 10: Truth table of gate  $F_2$ 

Table 11 shows all the relevant connectives which can be obtained from the gate by fixing one or two of its input lines with constant values from  $L_3$ . We can observe that the set of connectives is the same as  $F_1$ 's with

the exception of the modal connective  $\Box$ , which is not present in the first gate. Thus, the deficiencies with respect to gate  $F_1$  concerning the properties enjoyed by the gate are balanced with a richer set of realized connectives. As it does happen with gate  $F_1$ , the set of connectives realized by the gate  $F_2$  satisfies condition F-4) of universality.

Connective	Inputs	Constants	Outputs	Garbage
FAN-OUT	$x_1$	$x_2 = 1, x_3 = 0$	$y_1, y_2$	$y_3$
$\Pr_1$	$x_2, x_3$	$x_1 = 1$	$y_2$	$y_1, y_3$
$\Pr_2$	$x_2, x_3$	$x_1 = 1$	$y_3$	$y_1, y_2$
$\rightarrow_L$	$x_1, x_3$	$x_2 = 1$	$y_3$	$y_1, y_2$
$\rightarrow_G$	$x_1, x_2$	$x_3 = 1$	$y_2$	$y_1,y_3$
$\vee$	$x_1, x_3$	$x_2 = 1$	$y_2$	$y_1, y_3$
$\wedge$	$x_1, x_2$	$x_3 = 0$	$y_2$	$y_1,y_3$
Id	$x_1$	$x_2 = 0, x_3 = 0$	$y_1$	$y_2, y_3$
<b>–</b>	$x_1$	$x_2 = 1, x_3 = 0$	$y_3$	$y_1, y_2$
$\sim$	$x_1$	$x_2 = 0, x_3 = 1$	$y_2$	$y_1, y_3$
$\diamond$	$x_1$	$x_2 = 0, x_3 = 1$	$y_3$	$y_1, y_2$
	$x_1$	$x_2 = 0, x_3 = \frac{1}{2}$	$y_1$	$y_2, y_3$

Table 11: The operators obtained through gate  $F_2$ 

The last gate  $(F_3)$  we introduce allows one to realize the MV–connectives of the 3–valued case; its truth table is given in Table 12. Besides properties F-1), F-2'), F-3) and F-8), used by our program as the criteria for the exhaustive search, this gate satisfies property F-7) of conditional control; moreover, it is 0–regular and 1–regular.

$x_1x_2x_3 \vdash$	$\rightarrow y_1 y_2 y_3$	$x_1x_2x_3 \vdash$	$\rightarrow y_1 y_2 y_3$	$x_1x_2x_3 \vdash$	$\rightarrow y_1 y_2 y_3$
0 0 0	0 0 0	$\frac{1}{2}$ 0 0	$\frac{1}{2}$ 0 0	1 0 0	$1 \ 0 \ 0$
$0 \ 0 \ \frac{1}{2}$	$0 \frac{1}{2} 0$	$\frac{1}{2}$ 0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 0	$1 \ 0 \ \frac{1}{2}$	$1 \ 0 \ \frac{1}{2}$
$0 \ 0 \ 1$	$0 \ 1 \ 0$	$\frac{1}{2}$ 0 1	$\frac{1}{2}$ 0 1	$1 \ 0 \ 1$	$1 \ 0 \ 1$
$0 \frac{1}{2} 0$	$0 \ 0 \ \frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 0	$\frac{1}{2} \ 0 \ \frac{1}{2}$	$1 \frac{1}{2} 0$	$1 \frac{1}{2} 0$
$0 \frac{1}{2} \frac{1}{2}$	$0 \frac{1}{2} \frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ 1 0	$1 \frac{1}{2} \frac{1}{2}$	$1 \frac{1}{2} \frac{1}{2}$
$0 \frac{1}{2} 1$	$0 \ 1 \ \frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 1	$\frac{1}{2}$ $\frac{1}{2}$ 1	$1 \frac{1}{2} 1$	$1 \frac{1}{2} 1$
$0 \ 1 \ 0$	$0 \ 0 \ 1$	$\frac{1}{2}$ 1 0	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$1 \ 1 \ 0$	$1 \ 1 \ 0$
$0 \ 1 \ \frac{1}{2}$	$0 \frac{1}{2} 1$	$\frac{1}{2}$ 1 $\frac{1}{2}$	$\frac{1}{2}$ 1 $\frac{1}{2}$	$1 \ 1 \ \frac{1}{2}$	$1 \ 1 \ \frac{1}{2}$
$0\ 1\ 1$	$0\ 1\ 1$	$\frac{1}{2}$ 1 1	$\frac{1}{2}$ 1 1	1 1 1	$1 \ 1 \ 1$

Table 12: Truth table of gate  ${\rm F}_3$ 

Table 13 shows all the relevant connectives which can be obtained from the gate. Given the correspondences between the operators of the 3-valued Zawirski system  $\langle L_3, \oplus \rangle$  and those of the Łukasiewicz one  $\langle L_3, \to_L \rangle$  expressed by equation (3), we have that the set of connectives realized by the gate F<sub>3</sub> satisfies condition F-4) of universality.

Connective	Inputs	Constants	Outputs	Garbage
FAN-OUT	$x_1$	$x_2 = 1, x_3 = 0$	$y_1, y_2$	$y_3$
$\Pr_1$	$x_2, x_3$	$x_1 = 0$	$y_3$	$y_1, y_2$
$\Pr_2$	$x_2, x_3$	$x_1 = 0$	$y_2$	$y_1, y_3$
$\oplus$	$x_1, x_3$	$x_2 = 1$	$y_2$	$y_1, y_3$
$\odot$	$x_1, x_2$	$x_3 = 0$	$y_2$	$y_1, y_3$
Id	$x_1$	$x_2 = 0, x_3 = 0$	$y_1$	$y_2, y_3$
-	$x_1$	$x_2 = 1, x_3 = 0$	$y_3$	$y_1, y_2$
$\sim$	$x_1$	$x_2 = 0, x_3 = 1$	$y_2$	$y_1, y_3$
$\diamond$	$x_1$	$x_2 = 0, x_3 = 1$	$y_3$	$y_1, y_2$
	$x_3$	$x_1 = \frac{1}{2}, x_2 = 0$	$y_3$	$y_1, y_2$

Table 13: The operators obtained through gate  $F_3$ 

It is worth noting that, as a consequence of Proposition 6.1, none of the gates presented in this section is strictly conservative.

We conclude this section with the following proposition.

**Proposition 7.1.** For  $d \geq 3$ , there is no (3,d)-gate satisfying properties F-2), F-3) and F-8) which is able to realize the Lukasiewicz connectives  $(\wedge, \vee, \rightarrow_L)$ , the Gödel implication  $(\rightarrow_G)$  and the MV-connectives  $(\oplus, \odot)$ .

*Proof.* The only configurations that allow one to realize the classical implication with a Boolean Fredkin gate are  $x_2 = 1$  and  $x_3 = 1$ . Thus, if we impose property F-8) to our (3, d)-gate we get the following two possibilities to implement  $\rightarrow_L$  and  $\rightarrow_G$ :

Connective	Inputs	Constant	Output	Garbage
$\begin{array}{c} \rightarrow_L \\ \rightarrow_G \end{array}$	$\begin{array}{c} x_1, x_3 \\ x_1, x_2 \end{array}$	$\begin{aligned} x_2 &= 1\\ x_3 &= 1 \end{aligned}$	$egin{array}{c} y_3 \ y_2 \end{array}$	$egin{array}{c} y_1,y_2\ y_1,y_3 \end{array}$
$\begin{array}{c} \rightarrow_G \\ \rightarrow_L \end{array}$	$\begin{array}{c} x_1, x_3 \\ x_1, x_2 \end{array}$	$\begin{aligned} x_2 &= 1\\ x_3 &= 1 \end{aligned}$	$egin{array}{c} y_3 \ y_2 \end{array}$	$egin{array}{c} y_1,y_2\ y_1,y_3 \end{array}$

However, in both cases there is no configuration that allows one to realize  $\oplus$ . In the next table we explore all the configurations and, for each case, we give a short proof of the incompatibility.

Connective	Inputs	Constant	Output	Garbage	Proof
$\oplus$	$x_1, x_3$	$x_2 = 1$	$y_2$	$y_1, y_3$	(1)
$\oplus$	$x_1, x_2$	$x_3 = 1$	$y_3$	$y_1, y_2$	(2)
$\oplus$	$x_2, x_3$	$x_1 = \lambda$	$y_1$	$y_2, y_3$	(3)
$\oplus$	$x_2, x_3$	$x_1 = \lambda$	$y_2$	$y_1, y_3$	(4)
$\oplus$	$x_2, x_3$	$x_1 = \lambda$	$y_3$	$y_1, y_2$	(5)
$\oplus$	$x_1, x_3$	$x_2 = \lambda$	$y_1$	$y_2, y_3$	(6)
$\oplus$	$x_1, x_3$	$x_2 = \lambda$	$y_2$	$y_1,y_3$	(7)
$\oplus$	$x_1, x_3$	$x_2 = \lambda$	$y_3$	$y_1, y_2$	(8)
$\oplus$	$x_1, x_2$	$x_3 = \lambda$	$y_1$	$y_2, y_3$	(9)
$\oplus$	$x_1, x_2$	$x_3 = \lambda$	$y_2$	$y_1, y_3$	(10)
$\oplus$	$x_1, x_2$	$x_3 = \lambda$	$y_3$	$y_1, y_2$	(11)

where  $\lambda \in L_d \setminus \{0, 1\}$ . In what follows,  $\gamma$  is an unspecified element of  $L_d$ .

(1) for  $\lambda \leq \frac{1}{2}$ , all the triples  $\lambda 1(1 - \lambda)$  are mapped to 011, thus violating F-2);

(3) triples  $\lambda 11$  are mapped to 111, thus violating F-3);

(4) for the triples  $\lambda 01$ , it should be  $0 \oplus 1 = 1$  on  $y_2$ , and also  $\lambda \to_L 0 = 1 - \lambda$  or  $\lambda \to_G 0 = 0$  on the same output;

(6) triples  $0\lambda 1$  are mapped to  $11\gamma$ , thus violating F-3);

(7) for the triples  $1\lambda 1$ , it should be  $1 \oplus 1 = 1$  on  $y_2$ , and also  $1 \to_L \lambda = 1 - \lambda$  or  $1 \to_G \lambda = \lambda$  on the same output;

(9) triples  $01\lambda$  are mapped to  $1\gamma 1$ , thus violating F-3);

(10) for the triples  $11\lambda$ , it should be  $1 \oplus 1 = 1$  on  $y_2$ , and also  $1 \to_L \lambda = 1 - \lambda$  or  $1 \to_G \lambda = \lambda$  on the same output;

(2),(5),(8),(11) can be obtained respectively from (1), (4), (7), (10) by exchanging the second and third input/output lines of the gate.

If a strictly conservative gate that realizes all the three-valued connectives mentioned above is needed then, due to Propositions 6.1 and 7.1, it is necessary to look for (n,3)-gates with  $n \ge 4$ . In a forthcoming paper, a (4,3)-gate which has all the required properties will be presented.

## 8 Finite–valued Universal Gates

After the discovery of the generalizations of the Fredkin gate to three–valued logics exposed in the previous section, we obviously tried to generalize further to *d*–valued logics.

The approach followed in the previous section, that is making an exhaustive search in the space of truth tables of all (3, d)-gates, is clearly not feasible to find a solution which is valid for every value of d. As a consequence, we looked for some analytic expressions which define the new

reversible and conservative gates independently of the cardinality of the set of truth values.

#### 8.1 A gate for Łukasiewicz and Gödel *d*-valued logics

The first function  $\underline{f}_d^1: L_d^3 \to L_d^3$  we define is:

$$\forall \underline{x} = (x_1, x_2, x_3) \in L_d^3$$

$$\begin{cases} (x_1, x_3, x_2) & \text{if } x_1 = 0 \text{ and } x_2 \neq x_3 & \text{i} \\ (x_1, x_2, x_2) & \text{if } 0 < x_1 < x_2 < 1 \text{ and } x_2 = 1 & \text{ii} \end{cases}$$

$$\begin{array}{cccc} (x_1, x_3, x_2) & \text{if } 0 < x_1 \leq x_3 < 1 \text{ and } x_2 = 1 & \text{if } ) \\ (x_1, x_3, x_2) & \text{if } 0 < x_1 \leq x_2 < 1 \text{ and } x_3 = 1 & \text{iii} ) \end{array}$$

$$(x_1, x_1, 1 - x_1 + x_3)$$
 if  $x_3 < x_1 < 1$  and  $x_2 = 1$  iv)

$$\underline{f}_{d}^{1}(\underline{x}) := \begin{cases} (x_{1}, 1, x_{3} + x_{1} - 1) & \text{if } x_{1} < 1, x_{2} = x_{1}, x_{3} + x_{1} \ge 1 \\ & \text{and } x_{3} < 1 \end{cases}$$
 v)

$$\begin{array}{cccc} (x_1, x_1, x_2 - x_1) & \text{if } 0 < x_1 < x_2 < 1 \text{ and } x_3 = 0 & \text{vi}) \\ (x_1, x_3 + x_1, 0) & \text{if } 0 < x_1, \, x_2 = x_1, \, x_3 + x_1 < 1 & \\ & \text{and } x_3 > 0 & \text{vii}) \\ (x_1, x_2, x_3) & \text{otherwise} & \text{viii}) \end{array}$$

A direct inspection of the definition allows to conclude that the function  $\underline{f}_d^1$  is well defined; that is, each triple  $(x_1, x_2, x_3)$  of  $L_d^3$  is associated by  $\underline{f}_d^1$  with a single triple  $(y_1, y_2, y_3)$  of  $L_d^3$ .

Let us see some properties of  $f_d^{\tilde{1}}$ 

**Proposition 8.1.**  $f_d^1$  is self-reversible.

*Proof.* We have to prove that  $\forall \underline{x} \in L_d^3$ ,  $\underline{f}_d^1(\underline{f}_d^1(\underline{x})) = \underline{x}$ . We can proceed by dividing the elements of the domain as in rules i), ii), ..., viii).

Let a and b be two arbitrary elements of  $L_d$ . Considering the above rules it holds:

i)  $f_d^1(f_d^1(0,a,b)) = f_d^1(0,b,a) = (0,a,b).$ 

ii) Let  $\underline{x} = (a, 1, b)$  with  $0 < a \le b < 1$ . Therefore  $\underline{y} = \underline{f}_d^1(a, 1, b) = (a, b, 1)$ .  $\underline{y}$  fulfills iii) then  $\underline{f}_d^1(a, b, 1) = (a, 1, b) = \underline{x}$ .

iii) Let  $\underline{x} = (a, b, 1)$  with  $0 < a \le b < 1$ .  $\underline{y} = \underline{f}_d^1(a, b, 1) = (a, 1, b)$  that fulfills ii) thus  $\underline{f}_d^1(a, 1, b) = (a, b, 1) = \underline{x}$ .

iv) Let  $\underline{x} = (a, 1, b)$  with b < a < 1.  $\underline{y} = \underline{f}_d^1(a, 1, b) = (a, a, 1 - a + b)$ . Since  $0 \le b < a < 1$ ,  $1 - a + b + a \ge 1$  and 1 - a + b < 1. Therefore  $\underline{y}$  fulfills v) thus  $\underline{f}_d^1(a, a, 1 - a + b) = (a, 1, 1 - a + b + a - 1) = (a, 1, b) = \underline{x}$ .

v) Let  $\underline{x} = (a, a, b)$  with a < 1,  $b + a \ge 1$  and b < 1.  $\underline{y} = \underline{f}_d^1(a, a, b) = (a, 1, b + a - 1)$ . Since  $b + a \ge 1$  and b < 1,  $0 \le b + a - 1 < a < 1$ . As a consequence we have that  $\underline{y}$  satisfies iv), so  $\underline{f}_d^1(a, 1, b + a - 1) = (a, a, 1 - a + b + a - 1) = (a, a, b) = \underline{x}$ .

vi) Let  $\underline{x} = (a, b, 0)$  with 0 < a < b < 1.  $\underline{y} = \underline{f}_d^1(a, b, 0) = (a, a, b - a)$ . b-a+a < 1 and b-a > 0. This implies that  $\underline{y}$  satisfies vii), and consequently  $\underline{f}_d^1(a, a, b - a) = (a, b, 0) = \underline{x}$ .

vii) Let  $\underline{x} = (a, a, b)$  with 0 < a, b + a < 1 and b > 0.  $\underline{y} = \underline{f}_d^1(a, a, b) = (a, b + a, 0)$ . We have that  $\underline{y}$  satisfies vi), therefore  $\underline{f}_d^1(a, b + a, 0) = (a, a, b) = \underline{x}$ .

viii) obvious.

The proof of the previous proposition shows the method used to build the function  $\underline{f}_d^1$ . Rules i) and viii) allow the function to behave as the Fredkin gate when the inputs are restricted to  $\{0, 1\}$ . Rules ii) and iv) have been introduced in order to allow the gate to generate the Łukasiewicz implication on the third output line and the Łukasiewicz disjunction on the second output line when the second input line is set to 1. Rules iii) and v) are the converse of rules ii) and iv); this is done in order to guarantee the selfreversibility of the gate. Rules vi) and viii) realize the Łukasiewicz conjunction on the second output line when the third input line is set to 0, whereas vii) and viii) are the converse. Rule viii) uses the simplest self-reversible function (the identity function) to deal with the cases not considered by other rules.

Properties F-5), F-6), F-7) and F-8) are trivially satisfied by  $\underline{f}_d^1$ . Moreover, each rule was written in order to verify the property of weak conservativeness. In fact, the following proposition holds, whose proof is straightforward, and thus it is omitted.

## **Proposition 8.2.** $f_d^1$ is weakly conservative.

It is also easy to see that  $\underline{f}_d^1$  is a universal function. In fact, as it is shown in Table 14, using suitable configurations of constants in the input lines we obtain a set of connectives which suffices to generate, besides the FAN–OUT gate, all the operators of Lukasiewicz and Gödel *d*–valued logics.

It is important to underline that, as said before, the properties of the gate do not depend on the number of truth values involved. Moreover, when d = 3 the function  $\underline{f}_d^1$  behaves just like the gate  $F_1$  presented in the previous section.

#### 8.2 A family of functions which realize necessity

Since  $\underline{f}_d^1$  does not allow one to realize the modal operator  $\Box$ , we propose the following family of functions. Let  $\lambda$  be any value from the set  $L_d \setminus \{0, 1\}$ ; the family of functions  $\underline{f}_{d,\lambda}^2 : L_d^3 \to L_d^3$ , parameterized with respect to  $\lambda$ , is defined as follows:

Connective	Inputs	Constants	Outputs	Garbage
FAN-OUT	$x_1$	$x_2 = 1, x_3 = 0$	$y_1, y_2$	$y_3$
$Pr_1$	$x_2, x_3$	$x_1 = 0$	$y_3$	$y_1, y_2$
$\Pr_2$	$x_2, x_3$	$x_1 = 0$	$y_2$	$y_1, y_3$
$\rightarrow_L$	$x_1, x_3$	$x_2 = 1$	$y_3$	$y_1, y_2$
$\rightarrow_G$	$x_1, x_2$	$x_3 = 1$	$y_2$	$y_1, y_3$
$\vee$	$x_1, x_3$	$x_2 = 1$	$y_2$	$y_1, y_3$
$\wedge$	$x_1, x_2$	$x_3 = 0$	$y_2$	$y_1, y_3$
Id	$x_1$	$x_2 = 0, x_3 = 0$	$y_1$	$y_2, y_3$
<b>–</b>	$x_1$	$x_2 = 1, x_3 = 0$	$y_3$	$y_1, y_2$
$\sim$	$x_1$	$x_2 = 0, x_3 = 1$	$y_2$	$y_1, y_3$
$\diamond$	$x_1$	$x_2 = 0, x_3 = 1$	$y_3$	$y_1,y_2$

Table 14: The operators obtained through function  $\underline{f}_d^1$ 

$$\begin{split} \forall \underline{x} &= (x_1, x_2, x_3) \in L_d^3 \\ \underbrace{f_{(x_2, x_1, x_3)}^2}_{(x_2, x_1, x_3)} & \text{if } x_1 = 0, \, 0 < x_2 < 1 \text{ and } x_3 = \lambda \qquad \text{i)} \\ \underbrace{f_{(x_2, x_1, x_3)}^2}_{(x_1, x_3, x_2)} & \text{if } 0 < x_1 < 1, \, x_2 = 0 \text{ and } x_3 = \lambda \qquad \text{ii)} \\ \underbrace{f_{(x_1, x_3, x_2)}^2}_{(x_1, x_3, x_2)} & \text{if } 0 \le x_1 \le x_3 < 1 \text{ and } x_2 \neq x_3 \quad \text{iii)} \\ \underbrace{f_{(x_1, x_3, x_2)}^2}_{(x_1, x_1, 1 - x_1 + x_3)} & \text{if } x_3 < x_1 < 1 \text{ and } x_2 = 1 \qquad \text{vi)} \\ \underbrace{f_{(x_1, x_1, x_2 - x_1)}^2}_{(x_1, x_1, x_2 - x_1)} & \text{if } 0 \le x_1 < x_2 < 1 \text{ and } x_3 = 1 \qquad \text{vii)} \\ \underbrace{f_{(x_1, x_2, x_3)}^2}_{(x_1, x_2 + x_1, x_3 + x_1 > 1 & \text{and } x_3 < 1 & \text{vii)} \\ \underbrace{f_{(x_1, x_2, x_3)}^2}_{(x_1, x_2 - x_1)} & \text{if } 0 \le x_1 < x_2 < 1 \text{ and } x_3 = 0 \qquad \text{viii)} \\ \underbrace{f_{(x_1, x_2, x_3)}^2}_{(x_1, x_2 - x_1)} & \text{if } 0 \le x_1, \, x_2 = x_1, \, x_3 + x_1 < 1 & \text{and } x_3 > 0 & \text{ix}) \\ \underbrace{f_{(x_1, x_2, x_3)}^2}_{(x_1, x_2, x_3)} & \text{otherwise} & x \end{aligned}$$

For each fixed value of  $\lambda$  we get a function which realizes the connectives exposed in Table 15. As it can be seen, the price we pay to realize the modal connective  $\Box$  together with all the connectives of  $\underline{f}_d^1$  is that the functions  $\underline{f}_{d,\lambda}^2$  loose 0-regularity in 2d - 5 input/output pairs and property F-7) in 2d - 4 input/output pairs. Now, let us see some properties of functions  $\underline{f}_{d,\lambda}^2$ . The proofs of the next two properties are similar to the once of Deconstitutes 8.1 and 8.2, and

Now, let us see some properties of functions  $\underline{f}_{d,\lambda}^2$ . The proofs of the next two propositions are similar to the ones of Propositions 8.1 and 8.2, and thus they are omitted.

**Proposition 8.3.** For each fixed value of  $\lambda$  in  $L_d \setminus \{0,1\}$ , the function  $\underline{f}_{d,\lambda}^2$  is self-reversible.

**Proposition 8.4.** For each fixed value of  $\lambda$  in  $L_d \setminus \{0,1\}$ , the function  $\underline{f}_{d,\lambda}^2$  is weakly conservative.

Properties F-6) and F-8) are trivially satisfied by functions  $\underline{f}_{d,\lambda}^2$ . Table 15 reports the operators that can be obtained with the functions  $\underline{f}_{d,\lambda}^2$  by fixing one or two input lines with constant values from  $L_d$ . Such configurations have been chosen on the example of the gate F<sub>2</sub> presented in the previous section.

We observe that, also in this case, for a fixed  $\lambda$  the constants involved in such configurations are independent of d.

Connective	Inputs	Constants	Outputs	Garbage
FAN-OUT	$x_1$	$x_2 = 1, x_3 = 0$	$y_1,y_2$	$y_3$
$\Pr_1$	$x_2, x_3$	$x_1 = 1$	$y_2$	$y_1,y_3$
$Pr_2$	$x_2, x_3$	$x_1 = 1$	$y_3$	$y_1, y_2$
$\rightarrow_L$	$x_1, x_3$	$x_2 = 1$	$y_3$	$y_1, y_2$
$\rightarrow_G$	$x_1, x_2$	$x_3 = 1$	$y_2$	$y_1, y_3$
$\vee$	$x_1, x_3$	$x_2 = 1$	$y_2$	$y_1, y_3$
$\wedge$	$x_1, x_2$	$x_3 = 0$	$y_2$	$y_1, y_3$
Id	$x_1$	$x_2 = 0, x_3 = 0$	$y_1$	$y_2, y_3$
<b>—</b>	$x_1$	$x_2 = 1, x_3 = 0$	$y_3$	$y_1, y_2$
$\sim$	$x_1$	$x_2 = 0, x_3 = 1$	$y_2$	$y_1,y_3$
$\diamond$	$x_1$	$x_2 = 0, x_3 = 1$	$y_3$	$y_1, y_2$
	$x_1$	$x_2 = 0, x_3 = \lambda$	$y_1$	$y_2, y_3$

Table 15: The operators obtained through functions  $f_{d,\lambda}^2$ 

Note that, when d = 3, the function  $\underline{f}_{d,\frac{1}{2}}^2$  behaves just like the gate  $F_2$  presented in the previous section.

### 8.3 A gate for MV–connectives

None of the gates just presented generates the MV connectives showed in Section 4. This fact led us to build the function  $\underline{m}_d : L_d^3 \to L_d^3$  defined as follows:

$$\forall \underline{x} = (x_1, x_2, x_3) \in L_d^3$$

$$= (x_1, x_2, x_3) \in L_d^3$$

$$= \begin{cases} (x_1, x_3, x_2) & \text{if } x_1 = 0 \text{ and } x_2 \neq x_3 & \text{i} ) \\ (x_1, x_1 + x_3, 1 - x_1) & \text{if } x_1 > 0, x_2 = 1 \text{ and } x_1 + x_3 < 1 & \text{ii} ) \\ (x_1, 1, x_2 - x_1) & \text{if } 0 < x_1 \leq x_2 < 1 \text{ and } x_3 = 1 - x_1 & \text{iii} ) \\ (x_1, x_1 + x_2 - 1, & & & \\ 1 - x_1) & \text{if } x_1 < 1, x_2 < 1, x_3 = 0 \text{ and} & & \\ x_1 + x_2 > 1 & & & \text{iv} ) \\ (x_1, x_2 + x_3, 0) & \text{if } 0 < x_2 < x_1 < 1 \text{ and } x_3 = 1 - x_1 & & \\ (x_1, x_3, x_2) & & & \text{if } 0 < x_1, x_2 > 0, x_3 = 0 \text{ and} & & \\ x_1 + x_2 \leq 1 & & & \\ (x_1, x_3, x_2) & & & & \text{if } 0 < x_1, x_2 = 0, x_3 > 0 \text{ and} & & \\ x_1 + x_3 \leq 1 & & & \\ \end{array}$$

$$(x_1, x_2, x_3)$$
 otherwise viii)

In order to find this gate we used the technique previously shown: first we looked at the gate  $F_3$  exposed in the previous section in order to know which configurations give rise to the operators  $\oplus$  and  $\odot$ ; successively, we wrote their inverses. Thus it is no wonder that, for d = 3, the function  $\underline{m}_d$ behaves like the gate  $F_3$  presented in the previous section.

As for the previous functions, we can state the following properties.

**Proposition 8.5.**  $\underline{m}_d$  is self-reversible.

#### **Proposition 8.6.** $\underline{m}_d$ is weakly conservative.

Moreover, properties F-5), F-6), F-7) and F-8) are trivially satisfied by  $\underline{m}_d$ .

Table 16 reports the operators that can be obtained from function  $\underline{m}_d$  by fixing one or two input lines with constant values from  $L_d$ . As we can see,  $\underline{m}_d$  is a gate providing functional completeness of finite-valued calculus, regardless of the value assumed by d.

## 9 Conclusions and Directions for Future Work

We presented some generalizations of the Fredkin gate for d-valued reversible and conservative logics, notably d-valued Lukasiewicz and d-valued Gödel logics. In particular, we introduced three gates for three-valued logics and three possible extensions of such gates for d-valued logics; one of the extensions was specifically designed to realize the MV-connectives. Moreover

Connectives	Inputs	Constants	Outputs	Garbage
FAN-OUT	$x_1$	$x_2 = 1, x_3 = 0$	$y_1, y_2$	$y_3$
$\Pr_1$	$x_2, x_3$	$x_1 = 0$	$y_3$	$y_1, y_2$
$\Pr_2$	$x_2, x_3$	$x_1 = 0$	$y_2$	$y_1, y_3$
$\oplus$	$x_1, x_3$	$x_2 = 1$	$y_2$	$y_1, y_3$
$\odot$	$x_1, x_2$	$x_3 = 0$	$y_2$	$y_1, y_3$
Id	$x_1$	$x_2 = 0, x_3 = 0$	$y_1$	$y_2, y_3$
_	$x_1$	$x_2 = 1, x_3 = 0$	$y_3$	$y_1, y_2$
$\sim$	$x_1$	$x_2 = 0, x_3 = 1$	$y_2$	$y_1, y_3$
$\diamond$	$x_1$	$x_2 = 0, x_3 = 1$	$y_3$	$y_1, y_2$
	$x_3$	$x_1 = \frac{1}{d-1}, x_2 = 0$	$y_3$	$y_1, y_2$

Table 16: The operators obtained through function  $\underline{m}_d$ 

we showed how to realize, with such gates, the operators that characterize some modal logics.

One of the purposes of our work was to show that the framework of reversible and conservative computation can be extended toward some non classical "reasoning environments", originally proposed to deal with propositions which embed imprecise and uncertain information, that are usually based upon many-valued and modal logics.

It remains open the question on how it is possible to extend further the framework towards infinite-valued logics, such as fuzzy logics, both with  $\aleph_0$  and  $\aleph_1$  truth values. We feel that in such settings many new and interesting questions arise; here we propose just a few of them. For example: since reversible circuits need no more to have the same number of input and output lines, and moreover we can encode on a single input (or output) as much information as we want, what are the computational properties of such circuits? What are the differences with respect to reversible and conservative circuits for *d*-valued logics? How can we characterize the set of functions computed by such circuits?

Moreover, it is not difficult to extend Proposition 6.1 to deal with an infinite number of truth values. A direct consequence is that there are no possible extensions of the Fredkin gate to infinite-valued logics which compute the FAN-OUT function and at the same time are strictly conservative. How does this change the notion of conservativeness? In this paper we proposed the alternative notion of weak conservativeness, together with a possible physical interpretation; however, when dealing with an infinite number of energy levels there are two possibilities: either the energy levels extend over an unlimited range, so that to switch from a given level to another it could be necessary an infinite amount of energy, or the levels become increasingly close to each other. In the latter case, an infinite precision on the amount of energy can be required to switch from one level to another; in this situation, when the energy gap between the levels becomes smaller than the underlying thermal noise the computing physical system goes out of control. The above observations lead naturally to the following question: are the circuits for infinite–valued logics physically realizable? On the other hand, do we really need them?

# 10 Appendix: The abstract algebraic approaches to many-valued logics

#### **10.1** *BZW* algebras

The set of numbers  $L_{\alpha}$  ( $\alpha \in \{d, \aleph_0, \aleph_1\}$ ), interpreted as possible truth values of propositional sentences, equipped with the connectives  $\rightarrow_L$ ,  $\neg$ , and  $\sim$  are standard models of an abstract system, called Brouwer–Zadeh–Wajsberg (*BZW*) algebra, which can be considered a useful algebraic environment of many–valued logics.

Wajsberg (W) algebras were introduced by Wajsberg in order to give an algebraic axiomatization to many valued logics [Wa, 31, Wa, 35]. Fundamental aspect of W algebras is the usage of Lukasiewicz implication as a primitive operator. Brouwer–Zadeh (BZ) lattices, on the other hand, involve an intuitionistic negation  $\sim$ , besides a fuzzy one  $\neg$  [CN, 89]. Moreover, by suitable compositions of the two negations, it is possible to define the two basic modal operators, *necessity* ( $\Box$ ) and *possibility* ( $\diamond$ ). Thus, by a pasting of the two structures one obtains BZW algebras. These algebras also result to be a general classical "unsharp environment" for an abstract introduction to rough approximation spaces.

**Definition 10.1.** A Brouwer Zadeh Wajsberg (BZW) algebra is a system  $\langle A, \rightarrow, \neg, \sim, 1 \rangle$ , where A is a nonempty set, 1 is a constant element,  $\neg$  and  $\sim$  are unary operators, and  $\rightarrow$  is a binary operator, obeying the following axioms:

- (BZW1)  $1 \rightarrow x = x$
- (BZW2)  $(x \to y) \to ((y \to z) \to (x \to z)) = 1$
- (BZW3)  $(x \to y) \to y = (y \to x) \to x$
- (BZW4)  $(\neg x \rightarrow \neg y) \rightarrow (y \rightarrow x) = 1$
- $(BZW5) \neg \sim x \rightarrow \sim \sim x = 1$
- $(BZW6) \ (\neg x \to \sim \sim x) \to \sim \sim x = 1$
- $(BZW7) \neg \sim ((x \rightarrow y) \rightarrow y) = (\neg \sim x \rightarrow \sim \sim y) \rightarrow \sim \sim y$

A de Morgan BZW, shortly a  $BZW^{dM}$ , algebra is a BZW algebra in which axiom (BZW7) is replaced by the following:

$$(BZW7') \sim \neg[(\neg x \to \neg y) \to \neg y] = (\neg \sim \sim x \to \neg \sim \sim y) \to \neg \sim \sim y$$

Let us note that the substructure  $\langle A, \rightarrow, \neg, 1 \rangle$  induced by a BZW algebra is a W algebra, just characterized by axioms (BZW1)-(BZW4) [Wa, 31, Wa, 35]. Now, from any BZW algebra it is possible to obtain in a canonical way a structure of BZ distributive lattice according to the following result. **Theorem 10.1.** Let  $\langle A, \rightarrow, \neg, \sim, 1 \rangle$  be a BZW algebra. Let us introduce a new constant and two derived operators according to the following:

$$0 := \neg 1 \tag{13}$$

$$x \lor y := (x \to y) \to y \tag{14}$$

$$x \wedge y := \neg((\neg x \to \neg y) \to \neg y) \tag{15}$$

Then the structure  $\langle A, \wedge, \vee, \neg, \sim, 0 \rangle$  is a distributive BZ lattice. In other words:

 A is a distributive lattice with respect to the join and the meet operations ∨, ∧ defined by (14) and (15), respectively. The partial order relation induced by these operations is:

$$x \le y \quad iff \quad x \to y = 1. \tag{16}$$

A is bounded by the least element 0 and the greatest element 1:

$$\forall x \in A, \quad 0 \le x \le 1$$

- (2) The unary operation ¬ : A → A is a Kleene (or Zadeh) orthocomplementation. In other words the following hold:
  - $\begin{array}{ll} (K1) & \neg(\neg x) = x \\ (K2) & \neg(x \lor y) = \neg x \land \neg y \\ (K3) & x \land \neg x \le y \lor \neg y. \end{array}$
- (3) The unary operation  $\sim: A \mapsto A$  is a Brouwer orthocomplementation. That is, it satisfies the following properties:
  - $(B1) \quad x \land \sim x = x$   $(B2) \quad \sim (x \lor y) = \sim x \land \sim y$  $(B3) \quad x \land \sim x = 0$
- (4) The two orthocomplementations are linked by the following interconnection rule:

(in)  $\neg \sim x = \sim \sim x$ 

Let us note that under condition (K1) the *de Morgan* law (K2) is equivalent to the dual de Morgan law " $\neg(x \land y) = \neg x \land \neg y$ " and to the Kleene *contraposition* law " $x \le y$  implies  $\neg y \le \neg x$ ". In general neither the *noncontradiction* law " $\forall x : x \land \neg x = 0$ " nor the *excluded-middle* law " $\forall x : x \lor \neg x = 1$ " hold from this negation, even if for some elements e (for instance e = 0, 1) it may happen that  $e \land \neg e = 0$  and  $e \lor \neg e = 1$ .

As to the Brouwer negation, we have that under condition (B1) the de Morgan law (B2) is equivalent to the Brouwer contraposition law " $x \leq y$  implies  $\sim y \leq \sim x^{"}$ , but not to the dual de Morgan law. In this case the intuitionistic *noncontradiction* law is verified, but the excluded middle law in general is not required to hold.

On the other hand, if the structure in Theorem 10.1 is a  $BZW^{dM}$  algebra then the Brouwer negation satisfies also the dual de Morgan law "~  $(x \land y) = ~ x \lor ~ y$ ". However, in both cases (either BZW or  $BZW^{dM}$ ) in general the Brouwer negation satisfies the *weak double negation* law (B1), also written as " $\forall x : x \leq \sim x$ ", which does not forbid that for some special elements e (for instance e = 0, 1)  $e = \sim e$  holds.

A third kind of complementation, called *anti-intuitionistic orthocomple*mentation, can be defined in any *BZW* algebra.

**Definition 10.2.** Let  $\mathcal{A} = \langle A, \rightarrow, \neg, \sim, 1 \rangle$  be a BZW algebra. The antiintuitionistic complementation is the unary operation  $\flat : A \mapsto A$  defined as follows:

$$bx := \neg \sim \neg x$$

One can easily show that  $\flat$  satisfies the following conditions:

- (AB1)  $bbx \le x;$
- (AB2)  $\flat x \lor \flat y = \flat (x \land y)$  [equivalently,  $x \le y$  implies  $\flat y \le \flat x$ ];
- (AB3)  $x \lor \flat x = 1.$

As we have said at the beginning of this Appendix, the structure  $\langle L_{\alpha}, \rightarrow_L, \neg, \sim, 1 \rangle$  based on the set of truth values  $L_{\alpha}$  from the real unit interval, is a model of  $BZW^{dM}$  algebraic structure with respect to the Lukasiewicz implication connective  $\rightarrow_L$ , the diametrical negation  $\neg$ , and the impossibility negation  $\sim$  introduced in Section 4.

#### 10.2 Modal operators in BZW algebras

Modal operators can be naturally introduced in any Brouwer Zadeh lattice (hence in any BZW algebra). The *necessity operator*  $\Box$  and the *possibility operator*  $\diamond$  are defined in terms of Zadeh and Brouwer complementations.

**Definition 10.3.** For any element x of a Brouwer Zadeh lattice A, the necessity and the possibility of x are defined as follows:

- (n) The necessity:  $\Box(x) := \sim \neg x$ .
- (p) The possibility:  $\diamondsuit(x) := \neg \Box(\neg x)$ .

As a consequence, one obtains:

$$\diamondsuit(x) = \neg \sim x \qquad \Box(\diamondsuit(x)) = \sim \sim x \\ \sim x = \Box(\neg x) = \neg(\diamondsuit x) \qquad \diamondsuit x = \neg(\Box x) = \diamondsuit(\neg x)$$

On this basis, similarly to the modal interpretation of intuitionistic logic, the Brouwer complementation  $\sim$  can be interpreted as the negation of possibility or *impossibility* (also the necessity of a negation). Analogously, the anti-Brouwer complementation  $\flat$  can be interpreted as the negation of necessity or *contingency*.

Our modal operators  $\Box$  and  $\diamond$  turn out to have an  $S_5$ -like behavior based on a Kleene algebra, rather than on a Boolean one. Since  $\neg$  represents here a *fuzzy* (i.e., *Kleene*) negation on a distributive lattice, the result will be a *fuzzy* (i.e., *Kleene*)  $S_5$  modal situation.

**Theorem 10.2.** In any BZ lattice the following conditions hold:

(1)

 $\Box(x) \le x \le \diamondsuit(x)$ 

In other words: necessity implies actuality and actuality implies possibility (a characteristic principle of the modal system T [Ch, 88]).

(2)

$$\Box(\Box(x)) = \Box(x)$$
$$\diamondsuit(\diamondsuit(x)) = \diamondsuit(x)$$

Necessity of necessity is equal to necessity; similarly for possibility (a characteristic  $S_4$ -principle [Ch, 88]).

(3)

$$x \le \Box(\Diamond(x))$$

Actuality implies necessity of possibility (a characteristic *B*-principle [Ch, 88]).

(4)

$$\diamondsuit(x) = \Box(\diamondsuit(x))$$
$$\Box(x) = \diamondsuit(\Box(x))$$

Possibility is equal to the necessity of possibility; analogously, necessity is equal to the possibility of necessity (a characteristic  $S_5$ -principle [Ch, 88]).

On this basis, the definition of BZW algebras admits of a natural modal translation. It is worthwhile noticing that the modal translation of axiom (BZW6)  $(\neg x \rightarrow \Box(\diamondsuit(x)) = \Box(\diamondsuit(x)))$  asserts a weak (modal) version of the *consecutio mirabilis* principle. As it is well known, the strong *consecutio mirabilis* principle  $(((\neg x \rightarrow x) \rightarrow x) = 1)$  is not generally valid in the case of BZW algebras.

#### 10.3 Rough approximation spaces in BZW algebras

As stated in Theorem 10.2, in general the order chain  $\Box(x) \leq x \leq \Diamond(x)$ holds. Clearly, this is a fuzzy situation. In a crisp environment we have no difference among necessity, actuality and possibility, i.e., we are interested to those elements for which  $e = \Diamond(e)$  (equivalently,  $e = \Box(e)$ ). This leads one to define the substructure of all *M*-sharp (exact, crisp) elements, denoted by  $A_{e,M}$ , as follows:

$$A_{e,M} := \{ e \in A : \Diamond(e) = e \} = \{ e \in A : \Box(e) = e \}$$

However, this is not the only way to define sharp elements. In fact, since in general  $x \wedge \neg x \neq 0$  (equivalently,  $x \vee \neg x \neq 1$ ) it is possible to consider as *Kleene sharp* (K–sharp) the elements which satisfy the non contradiction (or, equivalently, the excluded middle) law with respect to the Kleene negation:

$$A_{e,\neg} := \{ e \in A : e \land \neg e = 0 \} = \{ e \in A : e \lor \neg e = 1 \}$$

Alternatively, considering the Brouwer negation we have that the *weak double negation* law holds ( $\forall x \in A, x \leq \sim \sim x$ ) whereas the double negation law fails. So we can introduce a further definition of *Brouwer sharp* (B–sharp) elements as follows:

$$A_{e,\sim} := \{e \in A : \sim e = e\} = \{e \in A : b \models e = e\}$$

Finally, as stated before the property  $\neg x \rightarrow (\Box(\Diamond(x))) = (\Box(\Diamond(x)))$  holds but in general it is not true that  $\neg x \rightarrow x = x$ . As a consequence, the  $\rightarrow$ -sharp elements are:

$$A_{e,\to} := \{ e \in A : \neg e \to e = e \}$$

The relation among all these different substructures of exact elements is figured out in the following proposition.

Proposition 10.1. Let A be a BZW algebra. Then

$$A_{e,\sim} = A_{e,M} \subseteq A_{e,\rightarrow} = A_{e,\neg}$$

Let A be a  $BZW^{dM}$  algebra. Then

$$A_{e,\sim} = A_{e,M} = A_{e,\rightarrow} = A_{e,-}$$

Consequently, in the case of  $BZW^{dM}$  algebras we simply talk of *sharp* elements and write  $A_e$ . Otherwise, in the more general case of BZW algebras we distinguish between *B*-sharp elements, i.e., elements in  $A_{e,\sim}$  (=  $A_{e,M}$ ), and *K*-sharp elements, i.e., elements belonging to  $A_{e,\neg}$  (=  $A_{e,\rightarrow}$ ).

As we have seen, in any BZW algebra it is possible, through the composition of the two negations, to introduce the modal operators  $\Box$  and  $\diamond$ .

These operators can be used to give a rough approximation of any element  $x \in A$  by B-sharp definable elements. In fact,  $\Box(x)$  (resp.,  $\diamond(x)$ ) turns out to be the best approximation from the bottom (resp., top) of x by B-sharp elements. To be precise, for any element  $x \in A$  the following holds:

- (I1)  $\Box(x)$  is B-sharp ( $\Box(x) \in A_{e,\sim}$ )
- (I2)  $\Box(x)$  is an *inner* (*lower*) approximation of x ( $\Box(x) \le x$ )
- (I3)  $\Box(x)$  is the best inner approximation of x by B-sharp elements (let  $e \in A_{e,\sim}$  be such that  $e \leq x$ , then  $e \leq \Box(x)$ )

Analogously,

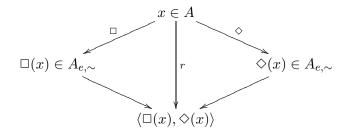
- (O1)  $\diamondsuit(x)$  is B-sharp  $(\diamondsuit(x) \in A_{e,\sim})$
- (O2)  $\diamond(x)$  is an outer (upper) approximation of  $x \ (x \leq \diamond(x))$
- (O3)  $\diamond(x)$  is the best outer approximation of x by B-sharp elements (let  $f \in A_{e,\sim}$  be such that  $x \leq f$ , then  $\diamond(x) \leq f$ )

**Definition 10.4.** Given a BZW algebra  $\langle A, \rightarrow, \neg, \sim, 1 \rangle$ , the induced rough approximation space is the structure  $\langle A, A_{e,\sim}, \Box, \diamond \rangle$  consisting of the set Aof all the elements which can be approximated, the set  $A_{e,\sim}$  of all definable (or *B*-sharp) elements, and the inner (resp., outer) approximation map  $\Box$ :  $A \rightarrow A_{e,\sim}$  (resp.,  $\diamond : A \rightarrow A_{e,\sim}$ ).

For any element  $x \in A$ , its rough approximation is defined as the pair of B-sharp elements:

$$r(x) := \langle \Box(x), \Diamond(x) \rangle \quad [with \quad \Box(x) \le x \le \Diamond(x)]$$

drawn in the following diagram:



So the mapping  $r : A \to A_{e,\sim} \times A_{e,\sim}$  approximates an unsharp (fuzzy) element by a pair of B-sharp (crisp, exact) ones representing its inner and outer sharp approximation, respectively. Clearly, B-sharp elements are characterized by the property that they coincide with their rough approximations:

$$e \in A_{e,\sim}$$
 iff  $r(e) = \langle e, e \rangle$ .

#### **10.4** *BZW* algebras and *BZMV* algebras

In Section 4 we have seen that in each  $L_{\alpha}$  the identities  $x \to_L y = \neg x \oplus y$ and  $x \oplus y = \neg x \to_L y$  show that there is no essential difference of expressive power between the implication connective  $\to_L$  and the additive one  $\oplus$ , owing to their mutual interdefinability. This suggests to introduce an algebraic structure, called BZMV algebra (see [CDG, 98], [CGP, 99]), based on the primitive connective  $\oplus$ , and to show that BZMV algebras and BZWalgebras are categorically equivalent. First of all we recall the definition of BZMV algebras.

**Definition 10.5.** A BZMV algebra is a system  $\langle A, \oplus, \neg, \sim, 0 \rangle$ , where A is a nonempty set, 0 is a constant,  $\neg$  and  $\sim$  are unary operators, and  $\oplus$  is a binary operator, obeying the following axioms:

- $(BZMV1) \ (x \oplus y) \oplus z = (y \oplus z) \oplus x$
- $(BZMV2) \ x \oplus 0 = x$
- $(BZMV3) \neg (\neg x) = x$
- $(BZMV4) \neg (\neg x \oplus y) \oplus y = \neg (x \oplus \neg y) \oplus x$
- $(BZMV5) \sim x \oplus \sim \sim x = \neg 0$
- (BZMV6)  $x \oplus \sim x = \sim x$
- $(BZMV7) \neg \sim [(\neg(\neg x \oplus y) \oplus y)] = \neg(\sim x \oplus \sim \sim y) \oplus \sim \sim y$

Now we introduce in a BZW algebra two new operators:

$$x \oplus y := \neg x \to y \tag{17}$$

$$x \odot y := \neg (x \to \neg y) \tag{18}$$

As to the relationship between the two structures of BZW and BZMV algebras we have the following theorem.

#### Theorem 10.3.

- 1. Let  $\mathcal{A} = \langle A, \rightarrow, \neg, \sim, 1 \rangle$  be a BZW algebra. Then putting  $x \oplus y := \neg x \to y$  and  $0 := \neg 1$ , the corresponding system  $\mathcal{A}_{BZMV} = \langle A, \oplus, \neg, \sim, 0 \rangle$  is a BZMV algebra.
- 2. Let  $\mathcal{A} = \langle A, \oplus, \neg, \sim, 0 \rangle$  be a BZMV algebra. Then putting  $x \to y := \neg x \oplus y$  and  $1 := \neg 0$ , the corresponding system  $\mathcal{A}_{BZW} = \langle A, \to, \neg, \sim, 1 \rangle$  is a BZW algebra.
- 3. (a) Let A be any BZW algebra, then A = (A<sub>BZMV</sub>)<sub>BZW</sub>
  (b) Let A be any BZMV algebra, then A = (A<sub>BZW</sub>)<sub>BZMV</sub>

Thus we have that under a suitable definition of the involved operators, BZW and BZMV algebras are categorically equivalent. Being no difference between the expressive power of the two structures, one can choose the algebra that best fits his/her analysis.

In particular, in the context of BZMV algebras the partial order relation defined by (16), taking into account (17), assumes the form:

$$x \le y \quad \text{iff} \quad \neg x \oplus y = 1 \tag{19}$$

Moreover, the induced structure of BZ distributive lattice  $\langle A, \vee, \wedge, \neg \sim, 0 \rangle$  of Theorem 10.1 in the present case is obtained from the operations of meet and join, intrinsically defined using the operators  $\oplus$  and  $\neg$  as follows:

$$\begin{aligned} x \lor y &:= (x \odot \neg y) \oplus y = \neg (\neg x \oplus y) \oplus y \\ x \land y &:= (x \oplus \neg y) \odot y = \neg [\neg (x \oplus \neg y) \oplus \neg y] \end{aligned}$$

As to the Kleene sharp elements we have now the following identifications:

$$A_{e,\neg} = \{ e \in A : e \oplus e = e \} = \{ e \in A : e \odot e = e \}$$

In the case of BZMV algebras, the set  $A_{e,\neg}$  of all Kleene sharp elements is closed under the operations  $\oplus$ ,  $\odot$ ,  $\neg$ , and  $\sim$ ; moreover on this set the lattice operations  $\lor$  and  $\land$  coincide with the MV algebra operations  $\oplus$  and  $\odot$  (see [CDG, 98, CGP, 99]) :

$$\forall e, f \in A_{e,\neg}, \quad e \oplus f = e \lor f \quad \text{and} \quad e \odot f = e \land f$$

The structure  $\langle A_{e,\neg}, \oplus, \odot, \neg, \sim, 0 \rangle$  is the largest BZMV subalgebra of A which is at the same time a Boolean BZ lattice with respect to the operations  $\vee (= \oplus), \ \wedge (= \odot), \ \neg \text{ and } \sim$ . That is,  $A_{e,\neg}$  is a BZ distributive lattice such that the substructure  $\langle A_{e,\neg}, \vee, \wedge, \neg, 0 \rangle$  is a Boolean (rather than just a Kleene) lattice.

Relatively to the Brouwer sharp elements, the set  $A_{e,\sim}$  is closed under the operations  $\oplus$ ,  $\odot$ ,  $\neg$ , and  $\sim$ . Moreover,

$$\begin{aligned} \forall e, f \in A_{e,\sim}, \quad e \oplus f = e \lor f \quad \text{and} \quad e \odot f = e \land f \\ \forall e \in A_{e,\sim}, \quad \neg e = \sim e \end{aligned}$$

Analogously, the structure  $\langle A_{e,\neg}, \oplus, \neg, 0 \rangle$  is the largest MV subalgebra of A which is at the same time a Boolean lattice with respect to the same operations  $\vee (= \oplus)$ ,  $\wedge (= \odot)$ , and  $\neg$ .

#### 10.5 Chang and Wajsberg many-valued algebras

MV-algebras are algebraic structures introduced by C.C. Chang in order to provide an algebraic proof of the completeness theorem for the infinite many-valued logic of Lukasiewicz (see [Ch, 58] and [Ch, 59]). A privileged model of this logic is based on the set  $L_{\aleph_1}$  of truth values, which gives rise to a totally ordered MV-algebra. Here we present a definition of MV-algebra which is simpler than the axiomatization proposed by Mangani in [Ma, 73].

**Definition 10.6.** An MV-algebra is a structure  $\langle L, \oplus, \neg, 0 \rangle$  where L is a nonempty set, 0 is a constant element of L,  $\oplus$  is a binary operation on L, and  $\neg$  is a unary operator on L, satisfying the following axioms:

P1) 
$$(x \oplus y) \oplus z = (y \oplus z) \oplus x$$
  
P2)  $x \oplus 0 = x$   
P3)  $x \oplus \neg 0 = \neg 0$   
P4)  $\neg (\neg 0) = 0$   
P5)  $\neg (\neg x \oplus y) \oplus y = \neg (x \oplus \neg y) \oplus x$ 

Axioms P1) – P5) are independent, as it is shown in [CL, 98]. In [CDG, 98] the following result is proved.

**Proposition 10.2.** Let  $\langle A, \oplus, \neg, \sim, 0 \rangle$  be a BZMV algebra. Then the substructure  $\langle A, \oplus, \neg, 0 \rangle$  is an MV algebra.

Using  $\oplus$  and  $\neg$ , in this algebraic context we can define the derived operations:

$$\begin{split} &1 := \neg 0 \\ &x \odot y := \neg (\neg x \oplus \neg y) \\ &x \lor y := (x \odot \neg y) \oplus y = \neg (\neg x \oplus y) \oplus y \\ &x \land y := (x \oplus \neg y) \odot y = \neg [\neg (x \oplus \neg y) \oplus \neg y] \end{split}$$

obtaining a structure  $(L, \oplus, \odot, \lor, \land, \neg, 0, 1)$  which satisfies the following conditions, assumed as axioms by Chang in his original definition:

(C1)	$x \oplus y = y \oplus x$	(C1')	$x\odot y=y\odot x$
(C2)	$x\oplus (y\oplus z)=(x\oplus y)\oplus z$	(C2')	$x \odot (y \odot z) = (x \odot y) \odot z$
(C3)	$x \oplus \neg x = 1$	(C3')	$x \odot \neg x = 0$
(C4)	$x \oplus 1 = 1$	(C4')	$x \odot 0 = 0$
(C5)	$x \oplus 0 = x$	(C5')	$x \odot 1 = x$
(C6)	$\neg(x\oplus y)=\neg x\odot \neg y$	(C6')	$ eg (x \odot y) = \neg x \oplus \neg y$
(C7)	$\neg(\neg x) = x$	(C8)	$\neg 0 = 1$
(C9)	$x \vee y = y \vee x$	(C9')	$x \wedge y = y \wedge x$
(C10)	$x \lor (y \lor z) = (x \lor y) \lor z$	(C10')	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(C11)	$x\oplus (y\wedge z)=(x\oplus y)\wedge (x\oplus z)$	(C11')	$x \odot (y \lor z) = (x \odot y) \lor (x \odot z)$

Notice that these conditions stress the fact that an MV–algebra represents a particular weakening of a Boolean algebra, where  $\oplus$  and  $\odot$  are generally non idempotent. Also in any MV–algebra a partial order relation can be induced making use of (19).

**Theorem 10.4.** The structure  $(L, \wedge, \vee, \neg, 0, 1)$  is a Kleene lattice (that is a bounded involutive distributive lattice satisfying the Kleene condition).

As a general consequence of this result, since in the many-valued case  $\oplus$  and  $\odot$  together with  $\neg$  can express  $\lor$  and  $\land$ , the (additive) operations  $\oplus$  and  $\odot$  are regarded as more fundamental than the lattice operations. Every MV-algebra is a subdirect product of totally ordered MV-algebras and an equation holds in the class of all MV-algebras if and only if it holds in the MV-algebra based on  $L_{\aleph_1}$  ([Ch, 58]). Actually the proof of completeness of finite valued logics needs stronger structures; for this purpose R. Grigolia ([Gr, 77]) introduced MV<sub>d</sub>-algebras which are particular kinds of MV-algebras.

Let us recall that a Wajsberg (W) algebra is a system  $\langle A, \rightarrow_L, \neg, 1 \rangle$  where L is a nonempty set, 1 is a constant element,  $\rightarrow_L$  is a binary operation, and  $\neg$  is a unary operation, satisfying conditions (BZW1)-(BZW4) of Definition 10.1. The two structures of Chang and of Wajsberg many-valued algebras are categorically equivalent according to the following straightforward result.

#### Theorem 10.5.

- 1. Let  $\mathbf{L} = \langle L, \oplus, \neg, 0 \rangle$  be a Chang MV-algebra. Then putting  $x \to_L y = \neg x \oplus y$  and  $1 = \neg 0$  the corresponding structure  $\mathbf{L}^W = \langle L, \to_L, \neg, 1 \rangle$  is a Wajsberg MV-algebra.
- 2. Let  $\mathbf{A} = \langle A, \rightarrow_L, \neg, 1 \rangle$  be a Wajsberg MV-algebra. Then putting  $x \oplus y = \neg x \to_L y$  and  $0 = \neg 1$  the corresponding structure  $\mathbf{A}^C = \langle A, \oplus, \neg, 0 \rangle$  is a Chang MV-algebra.
- 3. Let **L** be a Chang MV-algebra then  $(\mathbf{L}^W)^C = \mathbf{L}$  and let **A** be a Wajsberg MV-algebra then  $(\mathbf{A}^L)^C = \mathbf{A}$ .

## References

- [Be, 73] C. H. Bennett, Logical reversibility of computation, IBM J. Res. Dev., 17, (1973) 525–532.
- [Be, 88] C. H. Bennett, Notes on the history of reversible computation, IBM J. Res. Dev., 32, (1988) 16–23.
- [Be, 98] C. H. Bennett, Quantum information, Technical Report, Tutorial, MFCS '98, (1998).
- [BZ, 00] D. Bouwmeester, A. Zeilinger, The physics of quantum information: basic concepts, in "The Physics of Quantum Information", D. Bouwmeester and A. Ekert (eds.), Springer-Verlag, Berlin (2000).
- [CDG, 98] G. Cattaneo, M. L. Dalla Chiara, R. Giuntini, Some algebraic structures for many-valued logics, *Tatra Mountains Mathematical Publication*, 15, (1998) 173–196.
- [CGP, 99] G. Cattaneo, R. Giuntini, R. Pilla, BZMV<sup>dM</sup> algebras and stonian MV-algebras (applications to fuzzy sets and rough approximations), Fuzzy Sets and Systems, 108, (1999) 201–222.
- [CL, 98] G. Cattaneo, F. Lombardo, Independent axiomatization of MValgebras, Tatra Mountains Mathematical Publication, 15, (1998) 227– 232.
- [CN, 89] G. Cattaneo, G. Nisticò, Brouwer-Zadeh posets and three-valued Lukasiewicz posets, Fuzzy Sets and Systems, 33, (1989) 165–190.
- [Ch, 58] C.C. Chang, Algebraic analysis of many valued logics, Trans. Americ. Math. Soc., 88, (1958) 467–490.
- [Ch, 59] C.C. Chang, A new proof of the completeness of Łukasiewicz axioms, Trans. Americ. Math. Soc., 93, (1959) 74–80.
- [Ch, 88] B. F. Chellas, Modal logic, An introduction, Cambridge University Press, Cambridge (1988).
- [Cl, 62] R.E. Clay, A simple proof of functional completeness in manyvalued logics based on Łukasiewicz's C and N, Notre Dame J. of Formal Logic, 3, (1962) 114–117.
- [FT, 82] E. Fredkin, T. Toffoli, Conservative Logic, Intern J. Theor. Phys., 21, (1982) 219–253.
- [Gr, 77] R. Grigolia, Algebraic analysis of Łukasiewicz-Tarski's n-valued logical systems, in "Selected papers on Łukasiewicz sentential calculus", R. Wojcicki, G. Malinowski (eds.), Polish Acad. of Sc. (1977) 81–92.

- [Ki, 76] K. Kinoshita et al., On Magnetic Bubble Circuits, IEEE Trans. Computers C-25 (1976) 247-253.
- [La, 61] R. Landauer, Irreversibility and heat generation in the computing process, IBM J. Res. Dev., 3, (1961) 183–191.
- [LR, 90] H. S. Leff, A. F. Rex (eds.), Maxwell's Demon: Entropy, Information, Computing, Princeton University Press (1990).
- [Lu, 20] J. Łukasiewicz, O logice trójwartościowjei, Ruch Filozoficzny, 5, (1920) 170–171, Engl. Ver.: On three–valued logic, in [Lu, 70], page 87.
- [Lu, 30] J. Lukasiewicz, Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküs, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, 23, (1930) 51–77, Engl. Ver.: Philosophical remarks on many-valued systems of propositional logic, in [Lu, 70], page 153.
- [Lu, 70] J. Lukasiewicz, Selected works, L. Borkowski (ed.), North-Holland, Amsterdam (1970).
- [Ma, 73] P. Mangani, On certain algebras related to many valued logics, Boll. Un. Mat. Ital., 8, (1973) 68–78.
- [Mc, 51] R. McNaughton, A theorem about infinite-valued sentential logic, J. Symb. Logic, 16, (1951) 1–13.
- [RT, 52] J.B. Rosser A.R. Turquette, *Many–valued logics*, North Holland, Amsterdam (1952).
- [Su, 67] S. Surma, Logical Works. Polish Academy of Sciences, Wroclaw (1977).
- [To, 80] T. Toffoli, *Reversible Computing*, MIT/LCS Technical Report 151, February 1980.
- [Ur, 86] A. Urquhart, Many valued logic, in G. Gabbay, F. Guenter (eds.), "Handbook of philosophical logic", vol. III, D. Reidel Publishing Company, 1986.
- [vN, 56] J. von Neumann, Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components, Automata Studies (edited by C.E. Shannon and J. McCarthy), Princeton University Press, (1956) 43–98.
- [Wa, 31] M. Wajsberg, Aksjomatyzacja trówartościowego rachunkuzdań [Axiomatization of the three-valued propositional calculus]. Comptes Rendus des Séances de la Societé des Sciences et des Lettres de Varsovie 24, (1931) 126–148 (English Translation in [Su, 67]).

- [Wa, 35] M. Wajsberg, Beiträge zum Metaaussagenkalkül I. Monashefte fur Mathematik un Physik 42, (1935) 221–242 (English Translation in [Su, 67]).
- [Za, 34] Z. Zawirski, Relation of many-valued logic to probability calculus (in Polish, original title: Stosunek logiki wielowartosciowej do rachunku prawdopodobienstwa), *Poznanskie Towarzystwo Przyjaciol Nauk*, (1934).