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Modelling and Testing for Structural Changes in Panel Cointegration Models with Common and Idiosyncratic Stochastic Trends

by

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Modelling and Testing for Structural Changes in Panel Cointegration Models with Common and Idiosyncratic Stochastic Trends*

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Abstract

In this paper, we propose an estimation and testing framework for parameter instability in cointegrated panel regressions with common and idiosyncratic trends. We develop tests for structural change for the slope parameters under the null hypothesis of no structural break against the alternative hypothesis of (at least) one common change point which is possibly unknown. The limiting distributions of the proposed test statistics are derived. Monte Carlo simulations examine size and power of the proposed tests.

JEL classification: C32; C33; C12; C13

KEY WORDS: Panel cointegration; Common and idiosyncratic stochastic trends; testing for structural changes

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1 Introduction

Estimation and testing for structural changes is an important research topic in time series econometrics. A recent annals volume of the *Journal of Econometrics* published in 2005 entitled "Modelling structural breaks, long memory and stock market volatility" (edited by Anindya Banerjee and Giovanni Urga, 2005) and Perron (2006) offer the most recent comprehensive reviews on the topic. In contrast, scarce is the literature on the issues (estimation and testing) of structural changes in panel models, e.g., Han and Park (1989), Joseph and Wolfson (1992, 1993), Joseph et al. (1997), Hansen (1999), Chiang et al. (2002), Emerson and Kao (2001, 2002), Wachter and Tzavalis (2004) and Bai (2006). The estimation and testing for structure change in panels have many applications in economics, For example, fiscal/monetary policies may affect every unit in the economy (firms/regions), stock market crashes in the US may also cause the chain reaction in other stock markets in the world.

Despite the potential usefulness in economics, the econometric theory of the testing and estimation of structural changes in panels is still underdeveloped. This paper fills the gap in the literature by proposing an estimation and testing framework for parameter instability in cointegrated panel regression. We derive tests for structural change for the slope parameters in panel cointegration models with cross-sectional dependence that is captured by the common stochastic trends. The tests are for the null hypothesis of no structural break against the alternative hypothesis of (at least) one common change point which is possibly unknown. The framework we propose is based on a linear cointegrated panel data model where the number of cross-sectional units n and the number of time observations T are both large. The cointegrating equation we study contains unit-specific variables (idiosyncratic shocks) and a set of possibly unobservable variables that are common across all units (common shocks).

This paper makes two contributions to the existing literature. First, we develop an asymptotic theory for the estimates of the parameters in the model. We consider both the case of observed and unobserved common shocks. Ordinary large panels asymptotic theory (Phillips and Moon, 1999; Kao, 1999) cannot be applied in our framework due to the strong cross-sectional dependence introduced by the common shocks. We note that the limiting distributions of the common shocks coefficients are mixed normal, in contrast with asymptotic normality found in the literature. Second, along similar lines as Andrews (1993), we derive the limiting distribution of a Wald-type test for the null hypothesis of no structural change at an unknown point in cointegrated panels where units are cross dependent. The tests we derive are based on functionals of the Wald-type statistic.

The organization of the paper is as follows. Section 2 introduces the model. Section 3 discusses asymptotics. The limiting distribution of the OLS under the null of no structural change is established. Section 4 defines the test statistic. The limiting distributions of the proposed test are also derived. Section 5 discusses the local power. In Section 6 we report the finite sample properties, i.e., size and power, of our proposed tests. Section 7 provides concluding remarks. Some useful lemmas are given in Appendix A. In Appendix B we report the proofs of the main results in the paper.

We write the integral $\int_0^1 W(s)ds$ as $\int W$ when there is no ambiguity over limits. We define $\Omega^{1/2}$ to be any matrix such that $\Omega = \left(\Omega^{1/2}\right)\left(\Omega^{1/2}\right)'$. We use $\|\cdot\|$ to denote the Euclidean norm of a vector, $\xrightarrow{d}$ to denote convergence in distribution, $\xrightarrow{p}$ to denote convergence in probability, $[x]$ to denote the largest integer $\leq x$, $I(0)$ and $I(1)$ to signify a time-series that is integrated of order zero and one, respectively, $B = BM(\Omega)$ to denote Brownian motion with the covariance matrix Ω, and $\bar{B} = B - \int B$ to denote the demeaned version of B. We let $M < \infty$ be a generic positive number which does not depend on n or T.

2 Model and Assumptions

Consider the following panel model with common and idiosyncratic shocks

$$y_{it} = \alpha_i + \beta' F_t + \gamma' x_{it} + u_{it} \tag{1}$$

$i = 1, ..., n$ and $t = 1, ..., T$, where α_i is the individual effect. The parameters β and γ are $R \times 1$ and $p \times 1$, respectively, $F_t = (F_{1t}, ..., F_{Rt})'$ is a $R \times 1$ vector of common stochastic trends

$$F_t = F_{t-1} + \varepsilon_t \tag{2}$$

x_{it} is a $p \times 1$ vector of observable $I(1)$ individual-specific regressors,

$$x_{it} = x_{it-1} + \epsilon_{it} \tag{3}$$

and $\left(u_{it}, \varepsilon_t', \epsilon_{it}'\right)'$ are error terms. When common shocks F_t are not observable in (1), we then assume that F_t can be estimated by a set of observable exogenous variables, z_{it}, such that

$$z_{it} = \lambda_i' F_t + e_{it} \tag{4}$$

where λ_i is a vector of factor loadings and e_{it} is the error term.[1]

It is important to point out that our model in (1) is a standard common slope coefficients panel model not a factor-loading model as in Bai (2004), for example. Similar to this paper but not the same is Stock and Watson (1999, 2002, 2005). In Stock and Watson's setup, y_{it} in (1) (with $n = 1$) is the time series variable to be forecasted and $z_i = (z_{i1}, z_{i2}, ..., z_{iT})'$ is a n-dimensional multiple time series of candidate predictors.

The main aim of this paper is to develop test statistics to test the constancy over time for $\theta = \left(\beta', \gamma'\right)'$ with unknown change points. Considering the alternative hypothesis that there is only one change point k, three possible sets of alternative hypotheses can be considered as opposed to the null of no structural change

[1] Kao, Trapani and Urga (2006) provide a comprehensive asymptotic theory of the OLS estimator $\hat{\beta}$ of β when (1) does not contain idiosyncratic regressors x_{it}.

in θ: (1) only the common shocks coefficients β may change, (2) only the idiosyncratic shocks coefficients γ may change or (3) both β and γ may be affected by the break.

Denote $\theta_t = \left(\beta_t', \gamma_t'\right)'$. Given the null hypothesis

$$H_0 : \theta_t = \theta \text{ for all } t,$$

the alternative could be defined as

$$H_a : \theta_t = \begin{cases} \theta_1 & \text{for } t = 1, ..., k \\ \theta_2 & \text{for } t = k+1, ..., T \end{cases}$$

with $\theta_1 \neq \theta_2$.[2]

Note that testing for the constancy of β for the common factor, F_t, may have a different interpretation than the usual constancy of the slope parameter.[3] This is the case especially when F_t is not observed and has to be estimated e.g. using the principal component estimator (see Bai, 2003, 2004; Bai and Ng, 2002, 2004). In this case, the estimated factor matrix, $\hat{F}$, is given by T times the eigenvectors corresponding to the R largest eigenvalues of the matrix ZZ', where $Z = (z_1, z_2, ..., z_n)'$ is $T \times n$ with $z_i = (z_{i1}, z_{i2}, ..., z_{iT})'$. Since there is no guarantee that the R largest eigenvalues will have the same order for each t, the corresponding eigenvectors will have different meanings over time. For example, in the term structure literature (see e.g. Litterman and Scheinkman, 1991; Audrino et al. 2005), one usually uses a three-factor specification (level, slope and curvature) to explain the yield curves. The largest eigenvalue (and the corresponding eigenvector) for period t may not the be the same one in period s. This will make the parameter β non constant. Thus, β being non constant may indicate instability in the factor structure and not merely lack of constancy of a slope parameter. Recently, Perignon and Villa (2006) provide some discussion on the stability of the latent factor structure of interest rates over time.

We need the following assumptions.

Assumption M1: Let $\omega_{it} = (u_{it},\ \varepsilon_t', \epsilon_{it}', e_{it})'$. We assume that

(a) ω_{it} is *iid* over t and the invariance principle holds for the partial sums of ω_{it}, so that for a given i,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor T \cdot \rfloor} \omega_{it} \xrightarrow{d} B_\omega(\cdot) = \begin{bmatrix} B_u(\cdot) \\ B_\varepsilon(\cdot) \\ B_\epsilon(\cdot) \\ B_e(\cdot) \end{bmatrix}$$

[2] The formulation of the alternative hypothesis encompasses three possible cases:

$$\mathbf{H}_1^a\text{: } \boldsymbol{\theta}_t = \begin{cases} (\beta_1', \gamma') & \text{for } t = 1, ..., k \\ (\beta_2', \gamma') & \text{for } t = k+1, ..., T \end{cases}$$
$$\mathbf{H}_1^b\text{: } \boldsymbol{\theta}_t = \begin{cases} (\beta', \gamma_1') & \text{for } t = 1, ..., k \\ (\beta', \gamma_2') & \text{for } t = k+1, ..., T \end{cases}$$
$$\mathbf{H}_1^c\text{: } \boldsymbol{\theta}_t = \begin{cases} (\beta_1', \gamma_1') & \text{for } t = 1, ..., k \\ (\beta_2', \gamma_2') & \text{for } t = k+1, ..., T \end{cases}$$

where $\beta_1 \neq \beta_2$ and $\gamma_1 \neq \gamma_2$.

[3] We thank Zongwu Cai for pointing this to us.

where $B_\omega(\cdot)$ represents a multivariate Brownian motion, whose elements have covariance matrices σ_u^2, Ω_ε, Ω_ϵ and Ω_e respectively.

(b) For a given t, $\{u_{it}\}$, $\{\varepsilon_t, \epsilon_{it}\}$, and $\{e_{it}\}$ are mutually independent across i.

(c) $\{x_{it}, F_t\}$ are not cointegrated and Ω_ε and Ω_ϵ are non singular.

(d) The eigenvalues of Ω_ε and the random matrix $\int B_\varepsilon B_\varepsilon'$ are distinct with probability 1.

Assumption M2: $\|\lambda_i\| \leq M$ and $\frac{1}{n}\sum_{i=1}^{n} \lambda_i \lambda_i' \to \Sigma_\Lambda$ as $n \to \infty$, where Σ_Λ is non singular.
Assumption M3: We assume the following limits hold as in Phillips and Moon (1999):

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T} \widetilde{x}_{it}\widetilde{x}_{it}' \xrightarrow{p} \frac{1}{6}\Omega_\epsilon \tag{5}$$

and

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} \widetilde{x}_{it}u_{it} \xrightarrow{d} N\left(0, \frac{1}{6}\Omega_\epsilon \sigma_u^2\right) \tag{6}$$

as $(n, T) \longrightarrow \infty$ where $\tilde{x}_{it} = x_{it} - \frac{1}{T}\sum_{t=1}^{T} x_{it}$ and $\sigma_u^2 = Var(u_{it})$.

Assumption M1(a) considers a framework of no endogeneity of the regressors, serial dependence or contemporaneous correlation other than the one determined by the common shocks F_t are allowed for. Extensions to allow for endogeneity of the regressors, serial correlation and weak cross-sectional dependence among the regression errors are straightforward. Assumption M1(a), therefore, is considered merely for the purpose of simplification. Assumption M1(b) is a standard requirement for factor analysis and it is needed when F_t are not observable. Note here we allow non-zero covariance between ε_t and ϵ_{it}. Assumption M1(c) rules out cointegration among regressors. Assumption M1(d) is a standard requirement in large panel factor literature. Assumption M2 is also standard. Assumption M3 states that the joint limit theory developed by Phillips and Moon (1999) holds for (5) and (6).

The following proposition is important for developing the asymptotics in this paper.

Proposition 1 *Let Assumption M1 hold. As* $(n, T) \to \infty$

(a) $\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t \widetilde{x}_{it}' = O_p(1)$,

(b) $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it} \xrightarrow{d} \sigma_u \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{1/2} \times Z_1$

where $Z_1 \sim N(0, I_R)$ and $w_t = F_t - \frac{1}{T}\sum_{t=1}^{T} F_t$.

Proposition 1 states that the asymptotic magnitude of the cross term $\sum_{i=1}^{n}\sum_{t=1}^{T} w_t \widetilde{x}_{it}'$ is $O_p\left(\sqrt{n}T^2\right)$, thereby smaller than $\sum_{i=1}^{n}\sum_{t=1}^{T} \widetilde{x}_{it}\widetilde{x}_{it}'$ in (5) (and $\sum_{i=1}^{n}\sum_{t=1}^{T} w_t w_t'$ in (7) below). The asymptotic mixed normality result in part (b) is also different from the distribution limit in equation (6) where asymptotic normality holds. This result is due to the shock w_t being common to all units and $I(1)$.

We now turn to estimation of θ (under the null of no structural change).

3 Asymptotics of the Parameter Estimates Under the Null

In this section we provide asymptotics for the OLS of model (1) under the null hypothesis of no structural change. We distinguish the case of F_t observed from that where F_t needs to be estimated.

3.1 F_t is Observable

Define $W_{it} = (w_t', \tilde{x}_{it}')'$. Let $\hat{\theta}$ be the OLS of θ. Then we have

$$\begin{aligned}
\hat{\theta} - \theta &= \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix} \\
&= \left[\sum_{i=1}^{n} \sum_{t=1}^{T} W_{it} W_{it}' \right]^{-1} \left[\sum_{i=1}^{n} \sum_{t=1}^{T} W_{it} u_{it} \right] \\
&= \begin{bmatrix} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t w_t' & \sum_{i=1}^{n} \sum_{t=1}^{T} \tilde{x}_{it} w_t' \\ \sum_{i=1}^{n} \sum_{t=1}^{T} w_t \tilde{x}_{it}' & \sum_{i=1}^{n} \sum_{t=1}^{T} \tilde{x}_{it} \tilde{x}_{it}' \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t u_{it} \\ \sum_{i=1}^{n} \sum_{t=1}^{T} \tilde{x}_{it} u_{it} \end{bmatrix}. \qquad (7)
\end{aligned}$$

The following proposition characterizes the limiting distribution of $\hat{\theta}$.

Proposition 2 *Let Assumptions M1(a)-M1(d) and M3 hold. Then, as $(n, T) \longrightarrow \infty$ it holds that*

$$\sqrt{n} T \left(\hat{\theta} - \theta \right) = \sqrt{n} T \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix} \xrightarrow{d} \sigma_u \begin{pmatrix} \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon' \right)^{-1/2} \\ \sqrt{6} \Omega_\epsilon^{-1/2} \end{pmatrix} \times Z \qquad (8)$$

where

$$Z \sim N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I_R & 0 \\ 0 & I_p \end{pmatrix} \right).$$

Proposition 2 states that $\hat{\beta} - \beta$ and $\hat{\gamma} - \gamma$ are asymtotically independent. This result is a consequence of Proposition 1, i.e.,

$$\frac{1}{nT^2} \begin{bmatrix} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t w_t' & \sum_{i=1}^{n} \sum_{t=1}^{T} \tilde{x}_{it} w_t' \\ \sum_{i=1}^{n} \sum_{t=1}^{T} w_t \tilde{x}_{it}' & \sum_{i=1}^{n} \sum_{t=1}^{T} \tilde{x}_{it} \tilde{x}_{it}' \end{bmatrix} = \begin{bmatrix} O_p(1) & O_p\left(\frac{1}{\sqrt{n}}\right) \\ O_p\left(\frac{1}{\sqrt{n}}\right) & O_p(1) \end{bmatrix}. \qquad (9)$$

Note that results in Proposition 2 have $\sqrt{n}T$ convergence, as in Phillips and Moon (1999) and Kao (1999). However, the limiting distribution of $\hat{\theta}$ is different from the panel cointegration literature, where normality holds. The mixed normality found in our case is due to the shocks w_t being nonstationary and common across units, which implies $\frac{1}{nT^2} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t w_t' \xrightarrow{d} \int \bar{B}_\varepsilon \bar{B}_\varepsilon'$ being a random matrix rather than a constant as in the standard panel cointegration as in (6).

3.2 F_t is Unobservable

In order to estimate θ when F_t is unobservable, we consider a two step approach. First, we derive the estimator of the vector of common shocks, $\hat{F}_t$, using equation (4). We then plug this estimator in equation (1) to retrieve an estimate for θ.

3.2.1 Estimation of F_t

The estimator $\hat{F}_t$, can be estimated by the method of principal components, (see e.g., Bai (2004)).[4] That is, $\hat{F}_t$ can be found by minimizing

$$V_{nT}(R) = \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(z_{it} - \lambda_i' F_t\right)^2$$

subject to the normalization $\frac{1}{T^2}\sum_{t=1}^{T} F_t F_t' = I_R$, where z_{it} is given in (4). Let $F = (F_1, ..., F_T)'$ and $Z = (z_1, z_2, ..., z_n)'$ a $T \times n$ matrix with $z_i = (z_{i1}, z_{i2}, ..., z_{iT})'$. The estimator $\hat{F} = \left(\hat{F}_1, ..., \hat{F}_t\right)'$ is a $T \times R$ matrix which is found by T times the eigenvectors corresponding to the R largest eigenvalues of the $T \times T$ matrix ZZ'.

It is known that the solution to the above minimization problem is not unique, i.e., λ_i and F_t are not directly identifiable since they are identifiable only up to a transformation. Therefore, instead of estimating the factors F_t (or the loadings λ_i), what one does by employing the principal component estimator is to estimate the space spanned by them up to a $R \times R$ transformation matrix, say H, thereby finding HF_t instead of F_t. Therefore, computing the OLS of β for example, would result in estimating $H^{-1}\beta$ rather than β. However, as far as testing is concerned, knowledge of HF_t is the same as directly estimating F_t. Hence, for the purpose of notational simplicity, we assume H being a $R \times R$ identity matrix in this paper.

3.2.2 Estimation of θ

Let $\hat{w}_t = \hat{F}_t - \frac{1}{T}\sum_{t=1}^{T}\hat{F}_t$ and $\hat{W}_{it} = (\hat{w}_t', \tilde{x}_{it}')'$. The OLS estimator of θ is computed from

$$y_{it} = \alpha_i + \beta_t' \hat{F}_t + \gamma_t' x_{it} + v_{it} \tag{10}$$

where $v_{it} = u_{it} + \beta'\left(F_t - \hat{F}_t\right)$. Note

$$\begin{aligned}
\hat{\theta} - \theta &= \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix} \\
&= \left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{W}_{it}v_{it}\right] \\
&= \begin{bmatrix} \sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\hat{w}_t' & \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\hat{w}_t' \\ \sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\tilde{x}_{it}' & \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\tilde{x}_{it}' \end{bmatrix}^{-1}\begin{bmatrix} \sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t v_{it} \\ \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}v_{it} \end{bmatrix}.
\end{aligned} \tag{11}$$

[4] Throughout the paper, we assume that the number of common shocks R is known. If this is not the case, detection of R is possible using the methods derived by Bai and Ng (2002).

Let

$$\sigma_\zeta^2 = \sigma_u^2 + \sigma_\Pi^2 \tag{12}$$

where $\sigma_u^2 = Var(u_{it})$

$$\sigma_\Pi^2 = \beta' \tilde{Q}_B \left(\sigma_e^2 \Sigma_\Lambda\right) \tilde{Q}_B' \beta \tag{13}$$

$\sigma_e^2 = Var(e_{it})$ and the random variable $\tilde{Q}_B$ is defined as

$$\frac{1}{T^2} \sum_{t=1}^{T} \hat{w}_t w_t' \xrightarrow{d} \tilde{Q}_B.$$

The following theorem characterizes the limiting distribution of $\hat{\theta}$ when F_t are not observable.

Theorem 1 *Suppose Assumptions M1-M3 hold, with $n/T \to 0$ as $(n,T) \to \infty$. We get*

$$\sqrt{n}T\left(\hat{\theta} - \theta\right) = \sqrt{n}T \begin{bmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{bmatrix} \xrightarrow{d} \begin{pmatrix} \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{-1/2} \sigma_\zeta \\ \sqrt{6}\Omega_\epsilon^{-1/2} \sigma_u \end{pmatrix} \times Z \tag{14}$$

where

$$Z \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I_R & 0 \\ 0 & I_p \end{pmatrix}\right).$$

Note that $\hat{\beta}$ and $\hat{\gamma}$ are asymptotically independent due to $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T} \hat{W}_{it}\hat{W}_{it}'$ being a block diagonal matrix asymptotically similar to (9). The limiting distributions are essentially the same as those found in Proposition 2, the only difference with respect to (8), being the presence of the extra variance term σ_Π in the limiting distribution of $\hat{\beta}$. This arises from the estimation error of the common shocks, $\hat{F}_t - F_t$.

4 Test Statistics

The asymptotic theory for $\hat{\theta}$ derived in the Section 3 is used to derive the limiting distribution for the Wald-type statistic under the null hypothesis of no structural change. A variety of tests for a break, based on the Wald statistic have been discussed in the literature, e.g., Andrews (1993), Andrews and Ploberger (1994). In this section, we consider three statistics: the supremum of the Wald statistic, *Sup*W, the average Wald statistic, *Ave*W, and the logarithm of the Andrews-Ploberger exponential Wald statistic, *Exp*W.

Assumption PSE: *(Partial Sample Estimation)* $\frac{k}{T} \to r \in (0,1)$ as T and $k \to \infty$.

Assumption PSE states that the fraction of T at which the change point occurs, r, is bounded away from zero and one. Therefore, the structural break will divide the sample into two subsamples each of nontrivial size. This assumption follows an argument similar to that in Corollary 1 in Andrews (1993, p.838).

Consider the following partial sample OLS

$$\hat{\theta}_{1[Tr]} = \left(\sum_{i=1}^{n}\sum_{t=1}^{[Tr]} \hat{W}_{it}\hat{W}_{it}'\right)^{-1} \sum_{i=1}^{n}\sum_{t=1}^{[Tr]} \hat{W}_{it} y_{it}$$

and

$$\hat{\theta}_{2[Tr]} = \left(\sum_{i=1}^{n} \sum_{t=[Tr]+1}^{T} \hat{W}_{it} \hat{W}_{it}' \right)^{-1} \sum_{i=1}^{n} \sum_{t=[Tr]+1}^{T} \hat{W}_{it} y_{it}.$$

Let $\hat{\sigma}_u^2$ and $\hat{\sigma}_\zeta^2$ be consistent estimators for σ_u^2 and σ_ζ^2 respectively under H_0. Define

$$\hat{\theta}_{j[Tr]}^{*} = \begin{bmatrix} \hat{\sigma}_\zeta I_R & 0 \\ 0 & \hat{\sigma}_u I_p \end{bmatrix}^{-1} \hat{\theta}_{j[Tr]}$$

for $j = 1$, 2. Then the Wald statistic $\mathrm{W}([Tr])$ is given by

$$\mathrm{W}([Tr]) = \left(\hat{\theta}_{1[Tr]}^{*} - \hat{\theta}_{2[Tr]}^{*} \right)' \begin{bmatrix} \left(\sum_{i=1}^{n} \sum_{t=1}^{[Tr]} \hat{W}_{it} \hat{W}_{it}' \right)^{-1} \\ + \left(\sum_{i=1}^{n} \sum_{t=[Tr]+1}^{T} \hat{W}_{it} \hat{W}_{it}' \right)^{-1} \end{bmatrix}^{-1} \left(\hat{\theta}_{1[Tr]}^{*} - \hat{\theta}_{2[Tr]}^{*} \right). \tag{15}$$

Let $S_1(r) = \sigma_\zeta^{-1} \int_0^r \bar{B}_\varepsilon dB$ and $S_2(r) = \sigma_\zeta^{-1} \int_r^1 \bar{B}_\varepsilon dB$, where $B(\cdot)$ is the standard Brownian motion. Define

$$\mathbf{s}(r) = \begin{bmatrix} S_1(r) \\ S_2(r) \end{bmatrix}$$

$M_1(r) = \int_0^r \bar{B}_\varepsilon \bar{B}_\varepsilon'$, $M_2(r) = \int_r^1 \bar{B}_\varepsilon \bar{B}_\varepsilon'$, and

$$V^{-1}(r) = \begin{bmatrix} I_R & 0 \\ 0 & -I_R \end{bmatrix} \begin{bmatrix} M_1^{-1}(r) \\ M_2^{-1}(r) \end{bmatrix} \left[M_1^{-1}(r) + M_2^{-1}(r) \right]^{-1} \begin{bmatrix} M_1^{-1}(r) & M_2^{-1}(r) \end{bmatrix} \begin{bmatrix} I_R & 0 \\ 0 & -I_R \end{bmatrix}.$$

The following theorem characterizes the limiting distribution of the Wald test under the null.

Theorem 2 *Suppose Assumptions M1–M3 and PSE hold, and that $\frac{n}{T} \to 0$ as $(n, T) \to \infty$. Then, under the null H_0 of no structural change*

$$\mathrm{W}([T\cdot]) \xrightarrow{d} D(\cdot) = Q_R(\cdot) + Q_p(\cdot) \tag{16}$$

with

$$Q_R(r) = \mathbf{s}(r)' V^{-1}(r) \mathbf{s}(r), \tag{17}$$

$$Q_p(r) = \frac{\left[B((1-r)^2) - B\left(r^2\right) \right]' \left[B((1-r)^2) - B\left(r^2\right) \right]}{r^2 + (1-r)^2}, \tag{18}$$

where in this case $B(\cdot)$ is a p-dimensional standard Brownian motion. For a given r, $Q_R(r)$ and $Q_p(r)$ are independent such that

$$Q_R(r) \sim \chi_R^2$$

and

$$Q_p(r) \sim \left| \frac{(1-r)^2 - r^2}{(1-r)^2 + r^2} \right| \chi_p^2.$$

Let

$$d(r) = \left| \frac{(1-r)^2 - r^2}{(1-r)^2 + r^2} \right|.$$

Note that $B((1-r)^2) - B\left(r^2\right)$ has variance $(1-r)^2 - r^2$ if $(1-r)^2 > r^2$. Also $B((1-r)^2) - B\left(r^2\right)$ has variance $r^2 - (1-r)^2$ if $r^2 > (1-r)^2$. Then $\left\| B((1-r)^2) - B\left(r^2\right) \right\|$ is a Bessel process of order p, and

$$\frac{\left[B((1-r)^2) - B\left(r^2\right)\right]' \left[B((1-r)^2) - B\left(r^2\right)\right]}{\left|(1-r)^2 - r^2\right|}$$

is its standardized squares. Let $s = \left|(1-r)^2 - r^2\right|$, we can write

$$\frac{\left[B((1-r)^2) - B\left(r^2\right)\right]' \left[B((1-r)^2) - B\left(r^2\right)\right]}{r^2 + (1-r)^2} = \frac{s}{r^2 + (1-r)^2} \frac{BM(s)'BM(s)}{s},$$

where $BM(s)$ denotes a p-vector of independent Brownian processes on $[0, \infty]$. For a fixed r, $\left[BM(s)'BM(s)\right]/s$ has a chi-squared distribution with p degrees of freedom. However, r cannot be $1/2$ since s will be zero when $r = 1/2$.

In order to obtain a test statistic that the critical values can be taken from the literature, e.g., Andrews (1993), Andrews and Ploberger (1994), we consider the following modification to the Wald test:

$$\mathrm{W}^*([Tr]) = \left(\hat{\theta}^{**}_{1[Tr]} - \hat{\theta}^{**}_{2[Tr]}\right)' \left[\begin{array}{c} \left(\sum_{i=1}^{n} \sum_{t=1}^{[Tr]} \hat{W}_{it}\hat{W}_{it}'\right)^{-1} \\ + \left(\sum_{i=1}^{n} \sum_{t=[Tr]+1}^{T} \hat{W}_{it}\hat{W}_{it}'\right)^{-1} \end{array} \right]^{-1} \left(\hat{\theta}^{**}_{1[Tr]} - \hat{\theta}^{**}_{2[Tr]}\right)$$

where

$$\hat{\theta}^{**}_{j[Tr]} = \left[\begin{array}{cc} \hat{\sigma}_{\zeta} I_R & 0 \\ 0 & \sqrt{d(r)} \times \hat{\sigma}_u I_p \end{array} \right]^{-1} \hat{\theta}_{j[Tr]}.$$

It is clear that

$$\mathrm{W}^*([T\cdot]) \xrightarrow{d} D^*(\cdot) = Q_R(\cdot) + Q_p^*(\cdot) \tag{19}$$

where

$$Q_p^*(\cdot) = \frac{1}{d(r)} Q_p(\cdot).$$

Note that for a fixed r, $Q_R(r)$ and $Q_p^*(r)$ are independent and

$$D^*(r) \sim \chi^2_{R+p}.$$

Hence we have the following corollary:

Corollary 1 *Suppose Assumptions M1–M3 and PSE hold, and that* $\frac{n}{T} \to 0$ *as* $(n, T) \to \infty$. *Then, under the null* H_0 *of no structural change*

$$\mathrm{W}^*([T\cdot]) \xrightarrow{d} D^*(\cdot)$$

The results in Theorem 2 and the rest of the paper continue to hold if we relax some of the restrictions contained in Assumption M1. Particularly, assume that a multivariate invariance principle for ω_{it} holds, such that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[T\cdot]}\omega_{it}\xrightarrow{d}B_i\left(\Omega\right)\text{ as }T\rightarrow\infty\text{ for all }i,$$

where

$$B_i=\begin{bmatrix}B_{ui}\\B_{\varepsilon}\\B_{\epsilon i}\\B_{ei}\end{bmatrix},$$

and

$$\begin{aligned}\Omega &\equiv \sum_{j=-\infty}^{\infty}E\left(\omega_{i0}\omega_{ij}^{'}\right)=\Pi_i(1)\Sigma_{\psi}\Pi_i(1)^{'}=\Sigma+\Gamma+\Gamma^{'}\\ &= \begin{bmatrix}\Omega_u & \Omega_{u\varepsilon} & \Omega_{u\epsilon} & \Omega_{ue}\\ \Omega_{\varepsilon u} & \Omega_{\varepsilon} & \Omega_{\varepsilon\epsilon} & \Omega_{\varepsilon e}\\ \Omega_{\epsilon u} & \Omega_{\epsilon\varepsilon} & \Omega_{\epsilon} & \Omega_{\epsilon e}\\ \Omega_{eu} & \Omega_{e\varepsilon} & \Omega_{e\epsilon} & \Omega_{e}\end{bmatrix}\end{aligned}$$

where $\Gamma=\sum_{j=1}^{\infty}E\left(\omega_{i0}\omega_{ij}'\right)$ and $\Sigma=E\left(\omega_{i0}\omega_{i0}'\right)$ are partitioned conformably with ω_{it}. In this case, one can replace the OLS estimator by the fully modified (FM) estimator or dynamic OLS (DOLS), e.g., Phillips and Moon (1999) and Kao and Chiang (2000), to take account of the presence of serial correlation and exogeneity. This can be performed by replacing $\widehat{\sigma}_u^2$ by $\widehat{\Omega}_{u.\varepsilon}$ in (15) for the Wald test statistic, where $\widehat{\Omega}_{u.\varepsilon}$ is a consistent estimator for

$$\Omega_{u.b}=\Omega_u-\Omega_{ub}\Omega_b^{-1}\Omega_{bu}$$

with

$$b=\left(\varepsilon,\epsilon\right)'.$$

Further, the results in Theorem 2 are for testing the stability of θ. However, one can construct tests separately for β and γ using $Q_R(r)$ and $Q_p^*(r)$ since $Q_R(r)$ and $Q_p^*(r)$ are independent. Theorem 2 states that if one wants to test only for the constancy of β it holds that

$$\mathrm{W}([T\cdot])\xrightarrow{d}Q_R(\cdot);$$

if one is interested in testing merely for the constancy of γ it holds that

$$\mathrm{W}([T\cdot])\xrightarrow{d}Q_p^*(\cdot).$$

Finally, theorem 2 is valid for any consistent estimators of σ_u^2 and σ_ζ^2. To estimate σ_u^2, one could compute

$$\hat{\sigma}_u^2=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(y_{it}-\bar{y}_i-\hat{\theta}^{'}\hat{X}_{it}\right)^2 \tag{20}$$

which is consistent under H_0. To find a consistent estimator, $\hat{\sigma}^2_\zeta$, of σ^2_ζ, from equation (12) a possible choice is

$$\hat{\sigma}^2_\zeta = \hat{\sigma}^2_u + \hat{\sigma}^2_\Pi.$$

From equation (13), we have

$$\hat{\sigma}^2_\Pi = \hat{\beta}' \hat{\sigma}^2_\pi \hat{\beta},$$

with

$$\hat{\sigma}^2_\pi = \left(\frac{1}{T^2} \sum_{t=1}^{T} \hat{w}_t \hat{w}_t' \right) \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{T} \sum_{t=1}^{T} \hat{e}^2_{it} \right) \hat{\lambda}_i \hat{\lambda}_i' \right] \left(\frac{1}{T^2} \sum_{t=1}^{T} \hat{w}_t \hat{w}_t' \right), \tag{21}$$

where $\hat{\lambda}_i$ is a consistent estimate of λ_i and $\hat{e}_{it}$ can be computed as

$$\hat{e}_{it} = z_{it} - \hat{\lambda}_i' \hat{F}_t.$$

Therefore, we can provide an estimate for σ^2_ζ as

$$\hat{\sigma}^2_\zeta = \hat{\sigma}^2_u + \hat{\beta}' \hat{\sigma}^2_\pi \hat{\beta}. \tag{22}$$

The following proposition characterizes the consistency of $\hat{\sigma}^2_u$ and $\hat{\sigma}^2_\zeta$ under H_0.

Proposition 3 *Suppose Assumptions M1-M3 hold and that $\frac{n}{T} \to 0$ as $(n,T) \to \infty$. Then, under H_0*

$$\hat{\sigma}^2_u \xrightarrow{p} \sigma^2_u,$$

$$\hat{\sigma}^2_\zeta \xrightarrow{p} \sigma^2_\zeta.$$

The limiting distribution for the Wald test is now used to test for the presence of a structural break. Following Andrews (1993) and Andrews and Ploberger (1994), we consider three functionals of the Wald statistic $\mathrm{W}(\cdot)$:

$$Sup\mathrm{W}(k) = \underset{[Tr^*] \le k \le T-[Tr^*]}{sup} \mathrm{W}^*(k),$$

$$Ave\mathrm{W}(k) = \frac{1}{T} \sum_{k=[Tr^*]}^{T-[Tr^*]} \mathrm{W}^*(k),$$

and

$$Exp\mathrm{W}(k) = \log \left\{ \frac{1}{T} \sum_{k=[Tr^*]}^{T-[Tr^*]} \exp\left[\frac{1}{2} \mathrm{W}^*(k) \right] \right\}$$

where r^* represents the fraction of the sample trimmed away from the beginning and the end of the sample. Therefore, to carry out the test we only use data belonging to the sub-interval of the full sample $\{[Tr^*], [Tr^*]+1, ..., T-[Tr^*]-1, T-[Tr^*]\}$. Using the continuous mapping theorem (CMT) we have the following result:

Corollary 2 *Suppose Assumptions M1-M3 and PSE hold; then under H_0:*

$Sup\mathrm{W}([Tr]) \xrightarrow{d} \sup_{r^* \leq r \leq 1-r^*} D^*(r)$,

$Ave\mathrm{W}([Tr]) \xrightarrow{d} \int_{r^*}^{1-r^*} D^*(r)dr$,

$Exp\mathrm{W}([Tr]) \xrightarrow{d} \log\left\{\int_{r^*}^{1-r^*} \exp\left[\frac{1}{2}D^*(r)\right] dr\right\}$

as $(n,T) \to \infty$.

Critical values for $Sup\mathrm{W}, Ave\mathrm{W}$, and $Exp\mathrm{W}$ can be taken from Andrews (1993) and Andrews and Ploberger (1994) since $D^*(r)$ is χ^2_{R+p} for a fixed r. For example, when $r^* = 0.15$ and $R = p = 1$, the critical values of the 5% level for $Sup\mathrm{W}, Ave\mathrm{W}$, and $Exp\mathrm{W}$ are 11.79, 4.61, and 3.22 respectively.

5 Local Asymptotic Power

In this section, we evaluate the power of the Wald statistic against local alternatives. We assume the following sequence of local alternatives:

$$H_a^{(nT)} : \theta_t^{(nT)} = \theta + \frac{1}{\sqrt{nT}} g\left(\frac{t}{T}\right) \tag{23}$$

where $g(\cdot) = \left[g'_\beta(\cdot), g'_\gamma(\cdot)\right]'$ is a $(R+p) \times 1$ arbitrary function defined on the unit interval, with the sub-elements $g_\beta(\cdot)$ and $g_\gamma(\cdot)$ being $R \times 1$ and $p \times 1$ respectively.

The properties of $g\left(\frac{t}{T}\right)$ are specified in the following assumption.

Assumption LP:*(Local Power)* The function $g\left(\frac{t}{T}\right)$ belongs to the class of Riemann integrable functions and as $(n,T) \to \infty$ and for all k:

(a) $\frac{1}{T}\sum_{t=1}^{[Tr]} g\left(\frac{t}{T}\right) \to \int_0^r g(s)ds$,

(b) $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{[Tr]} W_{it}W'_{it}g\left(\frac{t}{T}\right) = O_p(1)$,

(c) $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{[Tr]} \frac{1}{\sqrt{T}}W'_{it}g\left(\frac{t}{T}\right) = O_p(1)$,

(d) $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{[Tr]} g'\left(\frac{t}{T}\right) W_{it}W'_{it}g\left(\frac{t}{T}\right) = O_p(1)$,

(e) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{[Tr]} W'_{it}g\left(\frac{t}{T}\right) u_{it} = O_p(1)$.

Possible alternative functional forms for $g(\cdot)$ include: the constant function, i.e. $g(\cdot) = c$ over the whole sample, which indicates no structural breaks; a single step function, i.e., $g(s) = 0$ if $r < r$ and $g(s) = \triangle\theta$ if $s \geq r$, which represents a one-time change on θ at $k = [Tr]$; multiple steps functions that represent multiple changes; time trending function $g(\cdot) = t/T$.

Assumptions LP(b)-(e) are technical requirements needed in order for $g(\cdot)$ to be a non-trivial local alternative, i.e., in order for $g(\cdot)$ not to vanish too quickly as $T \to \infty$.

In what follows, we derive the asymptotic behavior of the Wald statistic under the sequence of local alternatives (23). Model (1) can be rewritten as

$$y_{it}^{(nT)} = \alpha_i + X_{it}'\theta_t^{(nT)} + u_{it}.$$

Similarly, when common shocks are replaced by their estimates $\hat{X}_{it}$ we have

$$y_{it}^{(nT)} = \alpha_i + \hat{X}_{it}'\theta_t^{(nT)} + v_{it}$$

with $v_{it} = u_{it} + \left(F_t - \hat{F}_t\right)' \beta_t^{(nT)}$. Let $\hat{\theta}_{1k}^{(nT)}$ and $\hat{\theta}_{2k}^{(nT)}$ be the OLS estimators under the local alternative (23), and let $\tilde{\sigma}_u^2$ and $\tilde{\sigma}_\zeta^2$ be consistent estimators for σ_u^2 and σ_ζ^2 respectively under the local alternatives $H_a^{(nT)}$. Define

$$\hat{\theta}_{jk}^{*(nT)} = \begin{bmatrix} \tilde{\sigma}_\zeta I_R & 0 \\ 0 & \tilde{\sigma}_u I_p \end{bmatrix}^{-1} \hat{\theta}_{jk}^{(nT)},$$

for $j = 1, 2$, the Wald statistics under the local alternative can be computed as

$$W^{(nT)}(k) = \left[\hat{\theta}_{1k}^{*(nT)} - \hat{\theta}_{2k}^{*(nT)}\right]' \begin{bmatrix} \left(\sum_{i=1}^n \sum_{t=1}^k \hat{W}_{it}\hat{W}_{it}'\right)^{-1} \\ + \left(\sum_{i=1}^n \sum_{t=k+1}^T \hat{W}_{it}\hat{W}_{it}'\right)^{-1} \end{bmatrix}^{-1} \left[\hat{\theta}_{1k}^{*(nT)} - \hat{\theta}_{2k}^{*(nT)}\right]. \tag{24}$$

The local asymptotic power for the Wald statistics is given in the following theorem.

Theorem 3 *Suppose Assumptions M1-M3, PSE and LP hold. Then under the local alternative hypotheses $H_a^{(nT)}$ defined in equation (23),*

$$W^{(nT)}([T\cdot]) \xrightarrow{d} D(\cdot) + O_p(1)$$

where $D(r)$ is defined in Theorem 2.

The arguments in Theorem 3 also hold for the modified Wald test statistic. Theorem 3 indicates that the Wald statistics in (24) has nontrivial local power irrespective of the particular type of the structural change. The theorem holds for any choice of the estimators $\tilde{\sigma}_u^2$ and $\tilde{\sigma}_\zeta^2$ which is consistent under $H_a^{(nT)}$. A possible estimator for σ_u^2 is

$$\tilde{\sigma}_u^2 = \frac{1}{nT}\sum_{i=1}^n \sum_{t=1}^T \left[y_{it} - \bar{y}_i - \hat{\theta}_t^{(nT)\prime}\hat{X}_{it}\right]^2.$$

To estimate σ_ζ^2 we propose

$$\tilde{\sigma}_\zeta^2 = \tilde{\sigma}_u^2 + \hat{\beta}^{(nT)\prime}\hat{\sigma}_\pi^2\hat{\beta}^{(nT)},$$

where $\hat{\sigma}_\pi^2$ is defined in equation (21) and $\hat{\beta}^{(nT)}$ is the OLS estimator for β under $H_a^{(nT)}$. Then the following proposition establishes consistency for $\tilde{\sigma}_u^2$ and $\tilde{\sigma}_\zeta^2$ under $H_a^{(nT)}$.

Proposition 4 *Suppose Assumptions M1-M3, PSE and LP hold. Then under the local alternative hypotheses $H_a^{(nT)}$ defined in equation (23), it holds that*

$$\tilde{\sigma}_u^2 \xrightarrow{p} \sigma_u^2,$$

$$\tilde{\sigma}^2_\zeta \xrightarrow{p} \sigma^2_\zeta$$

as $(n,T) \to \infty$

6 Monte Carlo Simulations

In this section we present the simulation results that are designed to assess the null rejection probabilities and the power properties of $Sup\mathrm{W}(k)$, $Ave\mathrm{W}(k)$, and $Exp\mathrm{W}(k)$ statistics. To compare the performance of the proposed tests we conduct Monte Carlo experiments based on the following design

$$y_{it} = \alpha_i + \beta_t' F_t + \gamma_t' x_{it} \quad + \quad u_{it}$$

$$F_t = F_{t-1} + \varepsilon_t,$$

$$x_{it} = x_{it-1} + \epsilon_{it},$$

and

$$z_{it} = \lambda_i' F_t + e_{it}$$

for $i = 1, ..., n$, $t = 1, ..., T$, where the vector $[u_{it}, \varepsilon_{it}', \epsilon_{it}', e_{it}']$ is randomly drawn from a standard multivariate normal distribution.

For this experiment, we assume a single factor, i.e., $R = 1$ and λ_i is generated from i.i.d. $N(\mu_\lambda, 1)$. We set $\mu_\lambda = 2$. Under the null hypothesis of no structural change, we set the values of the parameters $\beta = 1$ and $\gamma = 1$. Also we choose $\alpha_i \sim N(0, 1)$.

We assess the power of the test considering an alternative hypothesis of structural change in both β and γ. We consider break location is assumed to take place at the 40% of the sample. To control for the break magnitude, we simulate model (1)-(4) assuming that, under H_a

$$\theta_t = \begin{cases} \theta \text{ for } t < k \\ (1+c)\,\theta \text{ for } t \geq k \end{cases}$$

where c is a scalar that defines the percentage change in the parameter values. We set $c = 0.1$. When generating the DGP, the first 1,000 observations are discarded to avoid dependence on the initial conditions. All our results are based on sample size of $n = \{20, 40, 60, 120, 240, 480\}$ and $T = \{20, 40, 60, 120, 240, 480\}$ with 10,000 iterations. The size and power are evaluated at 5% level. All programs are written by GAUSS. The the critical values of the 5% level for SupW, AveW, and ExpW are 11.79, 4.61, and 3.22 respectively. Those critical values were taken from Andrews (1993) and Andrews and Ploberger (1994).

Table 1 contains empirical rejection frequencies of the test statistics, SupW, AveW, and ExpW, under the null that β and γ are stable over time. It is clear from Table 1 that all these three test statistics are undersized if n and T are small. Overall, all three test statistics show good size when n and T are large.

Table 2 gives the power of the test statistics. All tests show very good power properties. The power gain is substantial as T increases and more moderate for increasing sizes of n. This result is consistent with the $\sqrt{n}T$ asymptotics of the three tests, as reported in the paper.

7 Conclusion

In this paper, we derive an asymptotic theory for testing for an unknown common change point in a cointegrated panel regression with common and idiosyncratic shocks. We develop the asymptotic theory for the cases of observable and unobservable common shocks and we derive the limiting distribution of the *supremum*, *average* and *exponential* Wald-type statistics under the null of no structural change. The derived limiting distributions are nuisance parameter free, depending only on the number of regressors. Monte Carlo simulations show that all three tests have good size and power properties, the power gain being substantial as T increases and more moderate for increasing sizes of n, consistent with the $\sqrt{n}T$ asymptotics of the three tests.

Appendix

Define $C_{nT} = \min\{\sqrt{n}, T\}$, $w_t = \left(F_t - \bar{F}^0\right)$, $\bar{F}^0 = T^{-1}\sum_{t=1}^T F_t$, $\widehat{w}_t = (\widehat{F}_t - \bar{F})$, and $\bar{F} = \frac{1}{T}\sum_{t=1}^T \widehat{F}_t$.

A Lemmas

Lemma A.1 *Under Assumptions M1 and M2,* as $(n,T) \to \infty$

(a) $\frac{1}{T}\sum_{t=1}^T \left\|\hat{F}_t - F_t\right\|^2 = O_p\left(C_{nT}^{-2}\right)$,

(b) $\frac{1}{T}\sum_{t=1}^T \|\hat{w}_t - w_t\|^2 = O_p\left(\frac{1}{C_{nT}^2}\right)$,

(c) $\frac{1}{T}\sum_{t=1}^T w_t'\left(F_t - \hat{F}_t\right) = O_p\left(\frac{1}{C_{nT}}\right)$.

Proof. Part (a) is taken from Lemma 1 in Bai (2004). Consider part (b).

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^T \|\hat{w}_t - w_t\|^2 &= \frac{1}{T}\sum_{t=1}^T \left\|\left(\hat{F}_t - F_t\right) + \left(\bar{F} - \bar{F}^0\right)\right\|^2 \\ &\leq \frac{2}{T}\sum_{t=1}^T \left[\left\|\hat{F}_t - F_t\right\|^2 + \left\|\bar{F} - \bar{F}^0\right\|^2\right] = I + II.\end{aligned}$$

Now, $I = T^{-1}\sum_{t=1}^T \left\|\hat{F}_t - F_t\right\|^2 = O_p\left(C_{nT}^{-2}\right)$ from part (a).. For II, it holds that

$$\left\|\bar{F} - \bar{F}^0\right\|^2 = \left\|\frac{1}{T}\sum_{t=1}^T \left(\hat{F}_t - F_t\right)\right\|^2 \leq \left(\frac{1}{T}\sum_{t=1}^T \left\|\hat{F}_t - F_t\right\|^2\right) = O_p\left(\frac{1}{C_{nT}^2}\right),$$

using the Cauchy-Schwartz inequality. Therefore $\frac{1}{T}\sum_{t=1}^T \left\|\bar{F} - \bar{F}^0\right\|^2 = O_p\left(\frac{1}{C_{nT}^2}\right)$, and consequently

$$\frac{1}{T}\sum_{t=1}^T \|\hat{w}_t - w_t\|^2 = O_p\left(\frac{1}{C_{nT}^2}\right).$$

This proves (b). Part (c) follows directly from Lemma B.4(i) in Bai (2004). ■

Lemma A.2 *Under Assumptions M1 and M2,* as $(n,T) \longrightarrow \infty$

(a)

$$\frac{1}{T^2}\sum_{t=1}^T \hat{w}_t\hat{w}_t^{'} = \frac{1}{T^2}\sum_{t=1}^T w_t w_t' + O_p\left(\frac{1}{\sqrt{T}C_{nT}}\right)$$

with

$$\frac{1}{T^2}\sum_{t=1}^T w_t w_t' = O_p(1),$$

(b)

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t u_{it} = \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it} + O_p\left(\frac{1}{C_{nT}}\right)$$

with

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it} = O_p\left(1\right),$$

(c)

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\left(F_t - \hat{F}_t\right) = \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t'\left(F_t - \hat{F}_t\right) + O_p\left(\frac{\sqrt{n}}{C_{nT}^2}\right)$$

with

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t'\left(F_t - \hat{F}_t\right) = O_p\left(\frac{\sqrt{n}}{C_{nT}}\right).$$

Proof. For part (a), note that

$$\begin{aligned}
\frac{1}{T^2}\sum_{t=1}^{T}\hat{w}_t\hat{w}_t' &= \frac{1}{T^2}\sum_{t=1}^{T}\left(w_t + \hat{w}_t - w_t\right)\left(w_t + \hat{w}_t - w_t\right)' \\
&= \frac{1}{T^2}\sum_{t=1}^{T} w_t w_t' + \frac{1}{T^2}\sum_{t=1}^{T} w_t\left(\hat{w}_t - w_t\right)' \\
&\quad + \frac{1}{T^2}\sum_{t=1}^{T}\left(\hat{w}_t - w_t\right) w_t' + \frac{1}{T^2}\sum_{t=1}^{T}\left(\hat{w}_t - w_t\right)\left(\hat{w}_t - w_t\right)' \\
&= I + II + III + IV.
\end{aligned}$$

Assumption M1 ensures that

$$I = O_p\left(1\right).$$

As far as terms II and III are concerned, application of the Cauchy-Schwartz inequality and of Lemma A.1(a) ensures that they are bounded by

$$\begin{aligned}
II &\leq \frac{1}{T^2}\left(\sum_{t=1}^{T}\left\|w_t\right\|^2\right)^{1/2}\left(\sum_{t=1}^{T}\left\|\hat{w}_t - w_t\right\|^2\right)^{1/2} \\
&= \frac{1}{T^2} O_p\left(T\right) O_p\left(\frac{\sqrt{T}}{C_{nT}}\right) = O_p\left(\frac{1}{\sqrt{T}C_{nT}}\right).
\end{aligned}$$

Use Lemma A.1(a) we have

$$\left\|\frac{1}{T^2}\sum_{t=1}^{T}\left(\hat{w}_t - w_t\right)\left(\hat{w}_t - w_t\right)'\right\| \leq \frac{1}{T^2}\sum_{t=1}^{T}\left\|\hat{w}_t - w_t\right\|^2 = O_p\left(\frac{1}{TC_{nT}^2}\right).$$

Hence,

$$\frac{1}{T^2}\sum_{t=1}^{T}\hat{w}_t\hat{w}_t' = \frac{1}{T^2}\sum_{t=1}^{T} w_t w_t' + O_p\left(\frac{1}{\sqrt{T}C_{nT}}\right) + O_p\left(\frac{1}{TC_{nT}^2}\right).$$

For part (b), note that

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t u_{it} = \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it} + \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\hat{w}_t - w_t\right) u_{it} = I + II.$$

From Proposition 1 we have

$$I = O_p(1)$$

applying Cauchy-Schwartz inequality and Lemma A.1(a) to II leads to

$$II \leq \left(\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{w}_t - w_t\right\|^2\right)^{1/2}\left(\frac{1}{T}\sum_{i=1}^{n}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n} u_{it}\right\|^2\right)^{1/2} = O_p\left(\frac{1}{C_{nT}}\right). \tag{25}$$

To prove (c) we note that

$$\frac{1}{T}\sum_{t=1}^{T}\hat{w}_t'\left(F_t - \tilde{F}_t\right) = \frac{1}{T}\sum_{t=1}^{T} w_t'\left(F_t - \tilde{F}_t\right) + \frac{1}{T}\sum_{t=1}^{T}\left(\hat{w}_t - w_t\right)'\left(F_t - \hat{F}_t\right) = I + II.$$

Lemma A.1(c) ensures that

$$I = O_p\left(\frac{1}{C_{nT}}\right).$$

For II.

$$\begin{aligned} II &\leq \left(\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{w}_t - w_t\right\|^2\right)^{1/2}\left(\frac{1}{T}\sum_{t=1}^{T}\left\|F_t - \hat{F}_t\right\|^2\right)^{1/2} \\ &= O_p\left(\frac{1}{C_{nT}}\right) O_p\left(\frac{1}{C_{nT}}\right) = O_p\left(\frac{1}{C_{nT}^2}\right). \end{aligned}$$

Hence,

$$\frac{1}{T}\sum_{t=1}^{T}\hat{w}_t\left(F_t - \hat{F}_t\right) = O_p\left(\frac{1}{C_{nT}}\right) + O_p\left(\frac{1}{C_{nT}^2}\right)$$

proving (c). ■

Let $X_{it} = \left(F_t', x_{it}'\right)'$, $\widehat{X}_{it} = \left(\widehat{F}_t', x_{it}'\right)'$, $W_{it} = X_{it} - \bar{X}_i$, and $\widehat{W}_{it} = \widehat{X}_{it} - \hat{X}_i$, with $\bar{X}_i = \frac{1}{T}\sum_{t=1}^{T} X_{it}$ and $\hat{X}_i = \frac{1}{T}\sum_{t=1}^{T}\widehat{X}_{it}$. Recall $w_t = F_t - \frac{1}{T}\sum_{t=1}^{T} F_t$, $\tilde{x}_{it} = x_{it} - \frac{1}{T}\sum_{t=1}^{T} x_{it}$, and $W_{it} = \left(w_t', \tilde{x}_{it}'\right)'$.

Lemma A.3 *Under Assumptions M1 and M3, as* $(n, T) \longrightarrow \infty$

(a)

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T} W_{it}W_{it}' \xrightarrow{d} \begin{bmatrix} \int \bar{B}_\varepsilon \bar{B}_\varepsilon' & 0 \\ 0 & \frac{1}{6}\Omega_\epsilon \end{bmatrix},$$

(b)

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it} \xrightarrow{d} \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{1/2}\sigma_u \times Z_1$$

and

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it} u_{it} \xrightarrow{d} \frac{1}{\sqrt{6}}\Omega_\epsilon^{1/2}\sigma_u \times Z_2.$$

where $Z_1 \sim N(0, I_R)$ *and* $Z_2 \sim N(0, I_p)$.

Proof. To prove (a), note

$$\begin{aligned}
\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}W_{it}W_{it}' &= \left[\begin{array}{cc} \frac{1}{T^2}\sum_{t=1}^{T}w_tw_t' & \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}w_t\tilde{x}_{it}' \\ \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}w_t' & \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\tilde{x}_{it}' \end{array}\right] \\
&= \left[\begin{array}{cc} a & b \\ b' & c \end{array}\right].
\end{aligned}$$

Assumption M1(a) ensures that

$$a = \frac{1}{T^2}\sum_{t=1}^{T}w_tw_t' \xrightarrow{d} \int \bar{B}_\varepsilon \bar{B}_\varepsilon';$$

Equation (5) in Assumption M3 states that

$$c = \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\tilde{x}_{it}' \xrightarrow{p} \frac{1}{6}\sigma_u^2\Omega_\epsilon,$$

We know from Proposition 1 that

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}w_t' = o_p(1).$$

In order to prove (b), note that

$$\begin{aligned}
\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}W_{it}u_{it} &= \left[\begin{array}{c} \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_tu_{it} \\ \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}u_{it} \end{array}\right] \\
&= \left[\begin{array}{c} c \\ d \end{array}\right].
\end{aligned}$$

From equation (6) in Assumption M3 that

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}u_{it} \xrightarrow{d} N\left(0, \frac{1}{6}\Omega_\epsilon\sigma_u^2\right) = \frac{1}{\sqrt{6}}\Omega_\epsilon^{1/2}\sigma_u \times Z_2$$

We also know from Proposition 1

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_tu_{it} \xrightarrow{d} \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{1/2}\sigma_u \times Z_1.$$

This proves part (b). ■

Lemma A.4 *Under Assumptions M1-M3 it holds that, as* $(n,T) \longrightarrow \infty$ *and* $\frac{\sqrt{n}}{T} \longrightarrow 0$

(a)

$$\frac{1}{nT^2}\sum_{t=1}^{T}\hat{W}_{it}\hat{W}_{it}' \xrightarrow{d} \left[\begin{array}{cc} \int \bar{B}_\varepsilon \bar{B}_\varepsilon' & 0 \\ 0 & \frac{1}{6}\Omega_\epsilon \end{array}\right],$$

(b)

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\left[u_{it} + \beta'(w_t - \hat{w}_t)\right] \xrightarrow{d} \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{1/2}\sigma_\zeta \times Z_1,$$

(c)

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\left(w_t-\hat{w}_t\right)'\beta=o_p\left(1\right),$$

where $Z_1\sim N\left(0,I_R\right)$, *and*

$$\sigma_\zeta=\sigma_u+\sigma_\Pi,$$

with

$$\sigma_\Pi=\sigma_e\sqrt{\beta'\tilde{Q}_B\Sigma_\Lambda\tilde{Q}_B'\beta},$$

and

$$\frac{1}{T^2}\sum_{t=1}^{T}\hat{w}_tw_t'\xrightarrow{d}\tilde{Q}_B.$$

Proof. To prove part (a), note that

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}W_{it}W_{it}'=\begin{bmatrix}\frac{1}{T^2}\sum_{t=1}^{T}\hat{w}_t\hat{w}_t' & \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\tilde{x}_{it}' \\ \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\hat{w}_t' & \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\tilde{x}_{it}'\end{bmatrix}.$$

Then Lemma A.2(a) ensures that

$$\frac{1}{T^2}\sum_{t=1}^{T}\hat{w}_t\hat{w}_t'=\frac{1}{T^2}\sum_{t=1}^{T}w_tw_t'+o_p\left(1\right),$$

so that

$$\frac{1}{T^2}\sum_{t=1}^{T}\hat{w}_t\hat{w}_t'\xrightarrow{d}\int\bar{B}_\varepsilon\bar{B}_\varepsilon'.$$

From equation (5)

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\tilde{x}_{it}'\xrightarrow{p}\frac{1}{6}\Omega_\epsilon.$$

We have

$$\begin{aligned}\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\tilde{x}_{it}' &= \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}w_t\tilde{x}_{it}'+\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\hat{w}_t-w_t\right)\tilde{x}_{it}' \\ &= I+II.\end{aligned}$$

We know from Proposition 1 that

$$I=O_p\left(\frac{1}{\sqrt{n}}\right).$$

For II, the Cauchy-Schwartz inequality and Lemma A.1(b) lead to

$$\begin{aligned}II &\leq \frac{1}{nT^2}\left(\sum_{t=1}^{T}\left\|\hat{w}_t-w_t\right\|^2\right)^{1/2}\left(n\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{x}_{it}\right\|^2\right)^{1/2} \\ &= \frac{1}{nT^2}O_p\left(\frac{\sqrt{T}}{C_{nT}}\right)O_p\left(\sqrt{n}T\right)=O_p\left(\frac{1}{\sqrt{nT}C_{nT}}\right).\end{aligned}$$

Therefore, as $(n,T)\longrightarrow\infty$

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\tilde{x}_{it}'=o_p(1).$$

To prove part (b), note that

$$\begin{aligned}\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\left[u_{it}+\beta'(w_t-\hat{w}_t)\right] &= \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_tu_{it}+\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t(w_t-\hat{w}_t)'\beta\\ &= a+b.\end{aligned}$$

As far as a is concerned, we have

$$\begin{aligned}a &= \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_tu_{it}+\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}(\hat{w}_t-w_t)u_{it}\\ &= I+II,\end{aligned}$$

and according to Lemma A.3(b) we have

$$I\xrightarrow{d}\left(\int\bar{B}_\varepsilon\bar{B}_\varepsilon'\right)^{1/2}\sigma_u\times Z_1.$$

For II, we have

$$\begin{aligned}II &\leq \frac{1}{\sqrt{n}T}\left(\sum_{t=1}^{T}\|\hat{w}_t-w_t\|^2\right)^{1/2}\left(n\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}u_{it}\right\|^2\right)^{1/2}\\ &= \frac{1}{\sqrt{n}T}O_p\left(\frac{\sqrt{T}}{C_{nT}}\right)O_p\left(\sqrt{n}T\right)=O_p\left(\frac{1}{C_{nT}}\right).\end{aligned}$$

Therefore, $II=o_p(1)$ and

$$a=\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_tu_{it}+o_p(1).$$

For b, we know from Bai (2004) that, as $(n,T)\longrightarrow\infty$ and $\frac{\sqrt{n}}{T}\longrightarrow 0$ we have

$$\sqrt{n}(\hat{w}_t-w_t)=\frac{1}{T^2}\sum_{s=1}^{T}\hat{w}_sw_s'\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_ie_{it}=O_p(1).$$

Therefore we write

$$\begin{aligned}b &= \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_t(w_t-\hat{w}_t)'\beta+\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}(w_t-\hat{w}_t)(w_t-\hat{w}_t)'\beta\\ &= \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_t(w_t-\hat{w}_t)'\beta+o_p(1).\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_t\left[u_{it}+\beta'(w_t-\hat{w}_t)\right]\\ &= \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_t\left[u_{it}+(w_t-\hat{w}_t)'\beta\right]+o_p(1).\end{aligned}$$

From Theorem 2 in Bai (2004) we know that for a given t

$$\sqrt{n}\left(w_t - \hat{w}_t\right) = \left(\frac{\widehat{w}^{'} w}{T^2}\right)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_i e_{it} + o_p\left(1\right) \xrightarrow{d} \widetilde{Q}_B N\left(0,\Gamma\right)$$

as $n \longrightarrow \infty$ where

$$\frac{\widehat{w}^{'} w}{T^2} \xrightarrow{d} \widetilde{Q}_B,$$

and

$$\begin{aligned} \Gamma &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left(\lambda_i\lambda_j^{'}e_{it}e_{jt}\right) \\ &= \sigma_e^2\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j^{'} \\ &= \sigma_e^2\Sigma_\Lambda. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T}w_t\beta^{'}\sqrt{n}\left(w_t - \hat{w}_t\right) &= \frac{1}{T}\sum_{t=1}^{T}w_t\beta^{'}\widetilde{Q}_B N\left(0,\Gamma\right) + o_p\left(1\right) \\ &= \frac{1}{T}\sum_{t=1}^{T}w_t\Pi_t + o_p\left(1\right) \end{aligned}$$

where

$$\Pi_t = \beta^{'}\widetilde{Q}_B N\left(0,\Gamma\right).$$

It is clear that

$$\frac{1}{T}\sum_{t=1}^{T}w_t\Pi_t \xrightarrow{d} \int \bar{B}_\varepsilon^{'}dB_\Pi$$

as $T \to \infty$ where B_Π is defined as

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{t}\Pi_j \xrightarrow{d} B_\Pi = \sigma_\Pi B^*. \tag{26}$$

where B^* is the standard Brownian motion. It follows that

$$\frac{1}{T}\sum_{t=1}^{T}w_t\beta^{'}\sqrt{n}\left(w_t - \widehat{w}_t\right) \xrightarrow{d} \int \bar{B}_\varepsilon^{'}dB_\Pi = \left(\int \bar{B}_\varepsilon\bar{B}_\varepsilon^{'}\right)^{1/2}\sigma_\Pi \times Z_1$$

Finally, consider the joint distribution of the elements in

$$\begin{bmatrix} \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{w}_t u_{it} \\ \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{w}_t\beta^{'}\left(w_t - \widehat{w}_t\right) \end{bmatrix}. \tag{27}$$

Any linear combination of these elements takes the form

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\frac{1}{T}\sum_{t=1}^{T}\left[\phi_1\widehat{w}_t u_{it} + \phi_2\widehat{w}_t\beta^{'}\left(w_t - \widehat{w}_t\right)\right]\right\}$$

for some ϕ_1 and ϕ_2. Let

$$\varsigma_{iT} = \frac{1}{T}\sum_{t=1}^{T}\left[\phi_1 \widehat{w}_t u_{it} + \phi_2 \widehat{w}_t \beta^{'}\left(w_t - \widehat{w}_t\right)\right].$$

For a given T, it is also clear that every element of ς_{iT} are *iid* across i conditional on C, the σ-algebra generated by $\{F_t\}$. Without loss of the generality, we assume $R = 1$. It is clear that every element of ς_{iT} are *iid* across i conditional on C which is an invariant σ-field. Thus

$$\frac{1}{n}\sum_{i=1}^{n}\varsigma_{iT}\varsigma_{iT}^{'} \xrightarrow{p} E\left(\varsigma_{iT}\varsigma_{iT}^{'}|C\right)$$

where

$$\begin{aligned}
& E\left(\varsigma_{iT}\varsigma_{iT}^{'}|C\right) \\
= & \; var\left[\frac{1}{T}\sum_{t=1}^{T}\widehat{w}_t\left[\phi_1 u_{it} + \phi_2\beta^{'}\left(w_t - \widehat{w}_t\right)\right]\right] \\
= & \; \frac{1}{T^2}\sum_{t=1}^{T} Var\left\{\widehat{w}_t\left[\phi_1 u_{it} + \phi_2\beta^{'}\left(w_t - \widehat{w}_t\right)\right]\right\} \\
= & \; \frac{1}{T^2}\sum_{t=1}^{T}\left\{\widehat{w}_t\widehat{w}_t^{'}\left[\phi_1^2 var\left(u_{it}\right) + 2\phi_1\phi_2 E\left[u_{it}\beta^{'}\left(w_t - \widehat{w}_t\right)\right] + \phi_2^2 var\beta^{'}\left(\left(w_t - \widehat{w}_t\right)\right)\right]\right\}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\left\{\widehat{w}_t\widehat{w}_t^{'}\left[\phi_1^2 var\left(u_{it}\right) + 2\phi_1\phi_2 E\left[u_{it}\beta^{'}\left(w_t - \widehat{w}_t\right)\right] + \phi_2^2 var\beta^{'}\left(w_t - \widehat{w}_t\right)\right]\right\} \\
= & \; \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}\phi_1^2 var\left(u_{it}\right) + 2\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}2\phi_1\phi_2 E\left[u_{it}\beta^{'}\left(w_t - \widehat{w}_t\right)\right] \\
& + \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}\phi_2^2 var\beta^{'}\left(w_t - \widehat{w}_t\right).
\end{aligned}$$

Notice that E and var are conditional expectation and conditional variance respectively. It follows that conditional on C,

$$\begin{aligned}
&\left\|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}2\phi_1\phi_2 E\left[u_{it}\beta^{'}\left(w_t-\widehat{w}_t\right)\right]\right\| \\
= &\left\|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}2\phi_1\phi_2 E\left[u_{it}\beta^{'}\left(w_t-\widehat{w}_t\right)\right]\right\| \\
= &\left\|\frac{1}{T^2\sqrt{n}}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}2\phi_1\phi_2 E\left[\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}u_{it}\right]\beta^{'}\left(w_t-\widehat{w}_t\right)\right]\right\| \\
= &\left\|\frac{1}{T^2\sqrt{n}}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}2\phi_1\phi_2 E\left[\overline{u}_t\beta^{'}\left(w_t-\widehat{w}_t\right)\right]\right\| \\
\leq &\frac{1}{\sqrt{n}}2\left\|\phi_1\phi_2\right\|\left(\frac{1}{T^2}\sum_{t=1}^{T}\left\|\widehat{w}_t^2\right\|^2\right)^{1/2}\left(E\frac{1}{T^2}\sum_{t=1}^{T}\left\|\overline{u}_t\beta^{'}\left(w_t-\widehat{w}_t\right)\right\|^2\right)^{1/2} \\
= &o_p\left(1\right)
\end{aligned}$$

with

$$\overline{u}_t=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}u_{it}$$

since

$$E\frac{1}{T^2}\sum_{t=1}^{T}\left\|\overline{u}_t\beta^{'}\left(w_t-\widehat{w}_t\right)\right\|^2=o_p\left(1\right)$$

and

$$\frac{1}{T^2}\sum_{t=1}^{T}\left\|\widehat{w}_t^2\right\|^2=O_p\left(1\right).$$

Also

$$\frac{1}{T^2}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}\phi_1^2 var\left(u_{it}\right)\xrightarrow{d}\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}\phi_1^2\sigma_u^2$$

and

$$\frac{1}{T^2}\sum_{t=1}^{T}\widehat{w}_t\widehat{w}_t^{'}\phi_2^2 var\beta^{'}\left(w_t-\widehat{w}_t\right)\xrightarrow{d}\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}\phi_2^2\sigma_{\Pi}^2.$$

Let I_i be the σ field generated by F_t and $(\varsigma_{1T},...,\varsigma_{iT})$. Then $\{\varsigma_{iT},I_i\}$ is a martingale difference sequence (MDS) with positive variance given by $E\left(\varsigma_{iT}\varsigma_{iT}^{'}|C\right)$ satisfying

$$\begin{aligned}
&E\left(\varsigma_{iT}\varsigma_{iT}^{'}|C\right)\xrightarrow{p}\int\overline{B}_{\varepsilon}\overline{B}_{\varepsilon}^{'}\left[\phi_1^2\sigma_u^2+\phi_2^2\sigma_{\Pi}^2\right] \\
= &\left(\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}\right)\phi^{'}\Psi\phi
\end{aligned}$$

with $\phi=\left(\phi_1,\phi_2\right)^{'}$ where

$$\Psi=\left[\begin{array}{cc}\sigma_u^2 & 0\\ 0 & \sigma_{\Pi}^2\end{array}\right].$$

Hence, we can use the MDS CLT to get

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varsigma_{iT}\xrightarrow{d}\left[E\left(\varsigma_{iT}\varsigma_{iT}^{'}|C\right)\right]^{1/2}\times Z=\left(\left(\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}\right)\phi^{'}\Psi\phi\right)^{1/2}\times Z+o_{p}\left(1\right)$$

where $Z\sim N\left(0,I_{R}\right)$ and $E\left(\xi_{i}\xi_{i}^{\prime}|C\right)$ and Z are independent. Thus, any linear combination of the two elements in the vector in (27) is asymptotically mixed normal, i.e.,

$$\left[\begin{array}{c}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{w}_{t}u_{it}\\ \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{w}_{t}\beta^{'}\left(F_{t}-\widehat{F}_{t}\right)\end{array}\right]\xrightarrow{d}\left(\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}\right)^{1/2}\Psi\times Z$$

with

$$\Psi=\left[\begin{array}{cc}\sigma_{u}^{2}&0\\0&\sigma_{\Pi}^{2}\end{array}\right]$$

and Z and $\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}$ are independent. Then

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{w}_{t}\left[u_{it}+\beta^{\prime}\left(w_{t}-\hat{w}_{t}\right)\right]\\=\;&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{w}_{t}u_{it}+\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\widehat{w}_{t}\beta^{'}\left(w_{t}-\widehat{w}_{t}\right)\\&\xrightarrow{d}\left(\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}\right)^{-1}\left(\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}\right)^{1/2}\left[\begin{array}{cc}1&1\end{array}\right]\Psi^{1/2}\times Z\\=\;&\left(\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}\right)^{-1/2}\sigma_{\zeta}\times Z\end{aligned}$$

with

$$\sigma_{\zeta}=\sigma_{u}+\sigma_{\Pi}.$$

This proves (ii).

Consider *(iii)*. Recall that we have, as $(n,T)\to\infty$ with $\frac{\sqrt{n}}{T}\to 0$

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\left(w_{t}-\hat{w}_{t}\right)^{\prime}\beta\\=\;&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\left[\frac{1}{T^{2}}\left(\sum_{s=1}^{T}\hat{w}_{s}w_{s}^{\prime}\right)\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\lambda_{j}e_{jt}\right)\right]^{\prime}\beta+o_{p}\left(1\right)\\=\;&\frac{1}{\sqrt{nT}}\left[\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{x}_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}e_{jt}\lambda_{j}^{\prime}\right)\right]\left(\frac{1}{T^{2}}\sum_{s=1}^{T}\hat{w}_{s}w_{s}^{\prime}\right)^{\prime}\beta+o_{p}\left(1\right).\end{aligned}$$

As $T\to\infty$, and for all n, we have

$$\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{x}_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}e_{jt}\lambda_{j}^{\prime}\right)=O_{p}\left(T\right);$$

therefore, for $(n,T)\to\infty$ we have

$$\frac{1}{\sqrt{n}T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\tilde{x}_{it}\right)\beta'\left(\frac{1}{T^2}\sum_{s=1}^{T}\hat{w}_s w_s'\right)\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\lambda_j e_{jt}\right)=O_p\left(\frac{1}{\sqrt{n}}\right).$$

This proves the Lemma. ■

Lemma A.5 *Let Assumptions M1-M3 and PSE hold. Then, as $(n,T)\to\infty$ with $\sqrt{n}/T\to 0$, it holds that*

(a)

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\xrightarrow{d}\left[\begin{array}{cc}\int_0^r\bar{B}_\varepsilon\bar{B}_\varepsilon' & 0\\ 0 & \frac{1}{6}r^2\Omega_\epsilon\end{array}\right],$$

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\hat{W}_{it}'\xrightarrow{d}\left[\begin{array}{cc}\int_r^1\bar{B}_\varepsilon\bar{B}_\varepsilon' & 0\\ 0 & \frac{1}{6}\left(1-r\right)^2\Omega_\epsilon\end{array}\right],$$

(b)

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\left[u_{it}+\beta'\left(\hat{w}_t-w_t\right)\right]\xrightarrow{d}\left[\begin{array}{c}\left(\int_0^r\bar{B}_\varepsilon\bar{B}_\varepsilon'\right)^{1/2}\sigma_\zeta\times Z_1\\ \frac{1}{\sqrt{6}}r\sigma_u\Omega_\epsilon^{1/2}\times Z_2\end{array}\right],$$

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\left[u_{it}+\beta'\left(\hat{w}_t-w_t\right)\right]\xrightarrow{d}\left[\begin{array}{c}\left(\int_r^1\bar{B}_\varepsilon\bar{B}_\varepsilon'\right)^{1/2}\sigma_\zeta\times Z_1\\ \frac{1}{\sqrt{6}}\left(1-r\right)\sigma_u\Omega_\epsilon^{1/2}\times Z_2\end{array}\right],$$

for all r where Z_1 and Z_1 are independent standard normals of dimensions R and p respectively.

Proof. The results are taken directly from Lemma A.4 and Chiang et al. (2002).

B PROOFS

B.1 Proof of Proposition 1

Proof. Consider (a). Note

$$\begin{aligned}&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}w_t\widetilde{x}_{it}'\\ =\;&\frac{1}{T}\sum_{t=1}^{T}w_t\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{x}_{it}'\right)\\ =\;&O_p\left(1\right).\end{aligned}$$

This is because[5]

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\widetilde{x}_{it}'\sim I(1)$$

[5]Let's assume $p=1$ and $\sigma_\epsilon^2=1$

$$x_{it}=x_{it-1}+\epsilon_{it}=\sum_{j=1}^{t}\epsilon_{ij}.$$

with

$$
\begin{aligned}
E\left(\widetilde{x}_{it}\right) &= E\left(x_{it}-\frac{1}{T}\sum_{t=1}^{T}x_{it}\right)\\
&= 0.
\end{aligned}
$$

and

$$
\frac{1}{T^2}\sum_{t=1}^{T}w_tG_t=O_p\left(1\right)
$$

for any $G_t \sim I(1)$. This proves (a).

Next we consider (b). Let C be the σ-field generated by the $\{w_t\}$ and

$$
\xi_{iT}=\frac{1}{T}\sum_{t=1}^{T}w_tu_{it}.
$$

We begin with the sequential limit. We know that

$$
\xi_{iT}\xrightarrow{d}\int \bar{B}_{\varepsilon}dB_u=\xi_i
$$

as $T\to\infty$ for a fixed n. It is clear that every element of ξ_i is *iid* across i conditional on C which is an invariant σ-field. Thus

$$
\frac{1}{n}\sum_{i=1}^{n}\xi_i\xi_i^{'}\xrightarrow{p}E\left(\xi_i\xi_i^{'}|C\right)=\sigma_u^2\int \bar{B}_{\varepsilon}\bar{B}_{\varepsilon}^{'}
$$

by an ergodic theorem. Let I_i be the σ field generated by F_t and $(\xi_1,...,\xi_i)$. Then $\{\xi_i,I_i;i\geq 1\}$ is a martingale difference sequence (MDS) because $\{\xi_i\}$ are *iid* across i conditional on C and

$$
E\left(\xi_i|I_{i-1}\right)=E\left(\xi_i|C\right)=0.
$$

A conditional Lindeberg condition holds here because for all $\delta>0$

$$
\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}E\left(\xi_i\xi_i^{'}1\left(\|\xi_i\|>\sqrt{n}\delta\right)|I_{i-1}\right) &= \lim_{n\to\infty}E\left(\xi_i\xi_i^{'}1\left(\|\xi_i\|>\sqrt{n}\delta\right)|C\right)\\
&= 0.
\end{aligned}
$$

Hence, an MDS CLT, e.g., Corollary 3.1 of Hall and Heyde (1980), implies that

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i\xrightarrow{d}\left[E\left(\xi_i\xi_i^{'}|C\right)\right]^{1/2}\times Z_1
$$

Then

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}x_{it}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{t}\epsilon_{ij}=\sum_{j=1}^{t}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\epsilon_{ij}=\sum_{j=1}^{t}N\left(0,1\right)+o_p\left(1\right)\sim I(1).
$$

This is because for a given t,

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\epsilon_{ij}\xrightarrow{d}N\left(0,1\right)
$$

as $n\to\infty$ by a CLT.

where $Z_1 \sim N(0, I_R)$ and $E\left(\xi_i \xi_i^{'}|C\right)$ are independent. Note

$$\left[E\left(\xi_i \xi_i^{'}|C\right)\right]^{1/2} = \sigma_u \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon^{'}\right)^{1/2}.$$

Denote $(n, T)_{seq} \to \infty$ as the sequential limit, i.e., $T \to \infty$ first and $n \to \infty$ later. Thus, as $(n, T)_{seq} \to \infty$, we have

$$\frac{1}{\sqrt{n}} \frac{1}{T} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t u_{it} \xrightarrow{d} \sigma_u \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon^{'}\right)^{1/2} \times Z_1.$$

We now show the limiting distribution continues to hold in the joint limit, i.e., $(n, T) \to \infty$. Given the sequential limit results derived above, establishing the joint limit results is done by verifying the conditions (i) - (iv) in Theorem 3 in Phillips and Moon (1999). Conditions (i), (ii), and (iv) are obviously satisfied. We only have to verify uniform integrability in (iii). Put in our context, the uniform integrability condition states that if $\|\xi_{iT}\| \xrightarrow{d} \|\xi_i\|$ and $E\|\xi_{iT}\| \xrightarrow{d} E\|\xi_i\|$, then $\|\xi_{iT}\|$ is uniformly integrable. We first observe that

$$\begin{aligned} E[\xi_{iT}] &= E\left[\frac{1}{T}\sum_{t=1}^{T} w_t u_{it}\right] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} E\left[\zeta_{iT}\zeta_{iT}^{'}\right] &= E\left[\left(\frac{1}{T}\sum_{t=1}^{T} w_t u_{it}\right)\left(\frac{1}{T}\sum_{t=1}^{T} w_t u_{it}\right)^{'}\right] \\ &\to E\left[\left(\int \bar{B}_\varepsilon dB_{ui}\right)\left(\int \bar{B}_\varepsilon dB_{ui}\right)^{'}\right] \\ &= \sigma_{ui}\int \bar{B}_\varepsilon \bar{B}_\varepsilon^{'} \end{aligned}$$

as $T \to \infty$. Thus ζ_{iT} is *iid* across i conditional on C with mean zero and covariance $\Omega_{ui}\int Q_i Q_i^{'}$. Now we need to show that $\|\xi_{iT}\|^2$ is uniformly integrable in T for all i. Recall

$$\int \bar{B}_\varepsilon dB_{ui} \sim \sigma_u \left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon^{'}\right)^{1/2} \times Z_1.$$

Note

$$\|\zeta_{iT}\|^2 \xrightarrow{d} \left\|\int \bar{B}_\varepsilon dB_{ui}\right\|^2$$

by a continuous mapping theorem (CMT) and

$$\begin{aligned} E\|\zeta_{iT}\|^2 &= tr\left[E\left(\zeta_{iT}\zeta_{iT}^{'}\right)\right] \\ &\to tr\left(\sigma_{ui}^2 \int \bar{B}_\varepsilon \bar{B}_\varepsilon^{'}\right) \\ &= \sigma_{ui}^2 \int \|\bar{B}_\varepsilon\|^2 \\ &= E\|\zeta_i\|^2 \end{aligned}$$

since

$$
\begin{aligned}
E\left\|\zeta_i\right\|^2 &= E\int\left\|\bar{B}_\varepsilon dB_{ui}\right\|^2 \\
&= tr\int E\left(\left[\bar{B}_\varepsilon dB_{ui}\right]\left[\bar{B}_\varepsilon dB_{ui}\right]^{'}\right) \\
&= \sigma_{ui}^2 tr\int \bar{B}_\varepsilon \bar{B}_\varepsilon^{'}.
\end{aligned}
$$

It follows that $\left\|\zeta_{iT}\right\|^2$ is uniformly integrable. We then apply Theorem 3 in Phillips and Moon (1999) to complete the proof

$$
\frac{1}{\sqrt{n}}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it} \xrightarrow{d} \sigma_u\left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon^{'}\right)^{1/2}\times Z_1.
$$

■

B.2 Proof of Proposition 2

Proof. Note

$$
\begin{aligned}
\sqrt{n}T\left(\hat{\theta}-\theta\right) &= \sqrt{n}T\begin{bmatrix}\hat{\beta}-\beta \\ \hat{\gamma}-\gamma\end{bmatrix} \\
&= \left[\frac{1}{nT^2}\begin{pmatrix}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t w_t' & \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}w_t' \\ \sum_{i=1}^{n}\sum_{t=1}^{T} w_t\tilde{x}_{it}' & \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\tilde{x}_{it}'\end{pmatrix}\right]^{-1}\frac{1}{\sqrt{n}T}\begin{bmatrix}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it} \\ \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}u_{it}\end{bmatrix}.
\end{aligned}
$$

Lemma A.3(a) ensures that, as $(n,T)\to\infty$ we have

$$
\frac{1}{nT^2}\begin{bmatrix}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t w_t' & \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}w_t' \\ \sum_{i=1}^{n}\sum_{t=1}^{T} w_t\tilde{x}_{it}' & \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}\tilde{x}_{it}'\end{bmatrix} \xrightarrow{d} \begin{bmatrix}\int \bar{B}_\varepsilon \bar{B}_\varepsilon' & 0 \\ 0 & \frac{1}{6}\Omega_\epsilon\end{bmatrix}.
$$

According to Lemma A.3(b), it is clear that conditional on conditional on C, the σ-algebra generated by $\{F_t\}$,

$$
\frac{1}{\sqrt{n}T}\begin{bmatrix}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it} \\ \sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}u_{it}\end{bmatrix} \xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)\sigma_u^2 & \Delta \\ \Delta & \frac{1}{6}\Omega_\epsilon\sigma_u^2\end{pmatrix}\right)
$$

where Δ denotes the asymptotic covariance between $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T} w_t u_{it}$ and $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{x}_{it}u_{it}$. Combining the two results, we get conditional on C,

$$
\begin{aligned}
\sqrt{n}T\left(\hat{\theta}-\theta\right) &\xrightarrow{d} N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{-1}\sigma_u^2 & 0 \\ 0 & 6\Omega_\epsilon^{-1}\sigma_u^2\end{pmatrix}\right). \\
&= \begin{pmatrix}\left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{-1/2}\sigma_u \\ \sqrt{6}\Omega_\epsilon^{-1/2}\sigma_u\end{pmatrix}\times N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}I_R & 0\\0 & I_p\end{pmatrix}\right) \\
&= \begin{pmatrix}\left(\int \bar{B}_\varepsilon \bar{B}_\varepsilon'\right)^{-1/2}\sigma_u \\ \sqrt{6}\Omega_\epsilon^{-1/2}\sigma_u\end{pmatrix}\times Z
\end{aligned}
$$

with

$$
Z\sim N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}I_R & 0\\0 & I_p\end{pmatrix}\right)
$$

Hence without conditioning C

$$\sqrt{n}T\left(\hat{\theta}-\theta\right)\xrightarrow{d}\left(\begin{array}{c}\left(\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}'\right)^{-1/2}\sigma_u\\ \sqrt{6}\Omega_{\epsilon}^{-1/2}\sigma_u\end{array}\right)\times Z.$$

This proves the proposition. ■

B.3 Proof of Theorem 1

Proof. The proof is Similar to Proposition 2. Recall

$$y_{it}=\alpha_i+\beta_t'\hat{F}_t+\gamma_t'x_{it}+v_{it},$$

where $v_{it}=u_{it}+\beta'\left(F_t-\hat{F}_t\right)$. Note

$$\begin{aligned}\sqrt{n}T\left(\hat{\theta}-\theta\right) &= \sqrt{n}T\left[\begin{array}{c}\hat{\beta}-\beta\\ \hat{\gamma}-\gamma\end{array}\right]\\ &= \left[\frac{1}{nT^2}\left(\begin{array}{cc} n\sum_{t=1}^T\hat{w}_t\hat{w}_t' & \sum_{i=1}^n\sum_{t=1}^T\hat{w}_t\tilde{x}_{it}'\\ \sum_{i=1}^n\sum_{t=1}^T\tilde{x}_{it}\hat{w}_t' & \sum_{i=1}^n\sum_{t=1}^T\tilde{x}_{it}\tilde{x}_{it}'\end{array}\right)\right]^{-1}\frac{1}{\sqrt{n}T}\left[\begin{array}{c}\sum_{i=1}^n\sum_{t=1}^T\hat{w}_tv_{it}\\ \sum_{i=1}^n\sum_{t=1}^T\tilde{x}_{it}v_{it}\end{array}\right];\end{aligned}\tag{28}$$

We know from Lemma A.4 that

$$\frac{1}{\sqrt{n}T}\left[\begin{array}{c}\sum_{i=1}^n\sum_{t=1}^T\hat{w}_tv_{it}\\ \sum_{i=1}^n\sum_{t=1}^T\tilde{x}_{it}v_{it}\end{array}\right]=\frac{1}{\sqrt{n}T}\left[\begin{array}{c}\sum_{i=1}^n\sum_{t=1}^T\hat{w}_tv_{it}\\ \sum_{i=1}^n\sum_{t=1}^T\tilde{x}_{it}u_{it}\end{array}\right]+o_p(1).$$

Hence using Lemma A.4 and the similar steps to Proposition 2 we can show that

$$\frac{1}{\sqrt{n}T}\left[\begin{array}{c}\sum_{i=1}^n\sum_{t=1}^T\hat{w}_tv_{it}\\ \sum_{i=1}^n\sum_{t=1}^T\tilde{x}_{it}v_{it}\end{array}\right]\xrightarrow{d}\left[\begin{array}{c}\left(\int\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}'\right)^{1/2}\sigma_{\zeta}\\ \frac{1}{\sqrt{6}}\Omega_{\epsilon}^{1/2}\sigma_u\end{array}\right]\times Z$$

with $Z\sim N\left(\left(\begin{array}{c}0\\0\end{array}\right),\left(\begin{array}{cc}I_R&0\\0&I_p\end{array}\right)\right)$. This proves the theorem. ■

B.4 Proof of Theorem 2

Proof. Theorem 2 states two separate results that need to be proved:

As far as equation (16) is concerned, consider the definitions of $\hat{\theta}_{1k}^*$, $\hat{\theta}_{1k}^*$, $S_1(r)$, $S_2(r)$, $M_1(r)$ and $M_2(r)$. Then, use Assumption PSE and in light of Lemma A.5 and the consistency of $\hat{\sigma}_{\zeta}^2$ and $\hat{\sigma}_u^2$, we have that, uniformly in r

$$\sqrt{n}T\left(\hat{\theta}_{1k}^*-\theta\right)\xrightarrow{d}\left[\begin{array}{c}\left[\int_0^r\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}'\right]^{-1}\sigma_{\zeta}^{-1}\int_0^r\bar{B}_{\varepsilon}dB\\ \frac{\sqrt{6}}{r}\Omega_{\epsilon}^{-1/2}N(0,I_p)\end{array}\right]=\left[\begin{array}{c}M_1(r)^{-1}S_1(r)\\ \frac{\sqrt{6}}{r^2}\Omega_{\epsilon}^{-1/2}B(r^2)\end{array}\right],\tag{29}$$

and

$$\sqrt{n}T\left(\hat{\theta}_{2k}^*-\theta\right)\xrightarrow{d}\left[\begin{array}{c}\left[\int_r^1\bar{B}_{\varepsilon}\bar{B}_{\varepsilon}'\right]^{-1}\int_r^1\bar{B}_{\varepsilon}dB\\ \frac{\sqrt{6}}{1-r}\Omega_{\epsilon}^{-1/2}N(0,I_p)\end{array}\right]=\left[\begin{array}{c}M_2(r)^{-1}S_2(r)\\ \frac{\sqrt{6}}{(1-r)^2}\Omega_{\epsilon}^{-1/2}B\left[(1-r)^2\right]\end{array}\right].\tag{30}$$

Then we note that

$$\left(\hat{\theta}_{1k}^{*}-\hat{\theta}_{2k}^{*}\right)=\left(\hat{\theta}_{1k}^{*}-\theta\right)-\left(\hat{\theta}_{2k}^{*}-\theta\right),$$

and

$$\sqrt{n}T\left(\hat{\theta}_{1k}^{*}-\hat{\theta}_{2k}^{*}\right)=\left[\begin{array}{cc} I & -I \end{array}\right]\left[\begin{array}{c} \sqrt{n}T\left(\hat{\theta}_{1k}-\theta\right) \\ \sqrt{n}T\left(\hat{\theta}_{2k}-\theta\right) \end{array}\right].$$

where I is $(R+p)\times(R+p)$ identity matrix. Therefore, use equations (29) and (30), under H_0 we have

$$\begin{aligned}
\sqrt{n}T\left(\hat{\theta}_{1k}^{*}-\hat{\theta}_{2k}^{*}\right) &\xrightarrow{d}\left[\begin{array}{cc} I & -I \end{array}\right]\left[\begin{array}{c} \left(\begin{array}{c} M_1(r)^{-1}S_1(r) \\ \frac{\sqrt{6}}{r^2}\Omega_\epsilon^{-1/2}B(r^2) \end{array}\right) \\ \left(\begin{array}{c} M_2(r)^{-1}S_2(r) \\ \frac{\sqrt{6}}{(1-r)^2}\Omega_\epsilon^{-1/2}B\left[(1-r)^2\right] \end{array}\right) \end{array}\right] \\
&= \left[\begin{array}{c} M_1(r)^{-1}S_1(r)-M_2(r)^{-1}S_2(r) \\ \sqrt{6}\Omega_\epsilon^{-1/2}\left[\frac{B(r^2)}{r^2}-\frac{B((1-r)^2)}{(1-r)^2}\right] \end{array}\right].
\end{aligned}$$

Also, use Lemma A.5(c), it follows that

$$\left[\begin{array}{c} \left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}^{\prime}\right)^{-1} \\ +\left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\hat{W}_{it}^{\prime}\right)^{-1} \end{array}\right]\xrightarrow{d}\left[\begin{array}{cc} M_1(r)^{-1}+M_2(r)^{-1} & 0 \\ 0 & \frac{6}{r^2}\Omega_\epsilon^{-1}+\frac{6}{(1-r)^2}\Omega_\epsilon^{-1} \end{array}\right]=G.$$

Therefore, by the CMT and uniformly in r we have

$$\begin{aligned}
\mathrm{W}([Tr]) &= \sqrt{n}T\left(\hat{\theta}_{1k}^{*}-\hat{\theta}_{2k}^{*}\right)^{\prime}\left[\begin{array}{c} \left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}^{\prime}\right)^{-1} \\ +\left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\hat{W}_{it}^{\prime}\right)^{-1} \end{array}\right]^{-1}\sqrt{n}T\left(\hat{\theta}_{1k}^{*}-\hat{\theta}_{2k}^{*}\right) \\
&\xrightarrow{d}\left[\begin{array}{c} M_1(r)^{-1}S_1(r)-M_2(r)^{-1}S_2(r) \\ \sqrt{6}\Omega_\epsilon^{-1/2}\left[\frac{B(r^2)}{r^2}-\frac{B((1-r)^2)}{(1-r)^2}\right] \end{array}\right]^{\prime}G^{-1}\left[\begin{array}{c} M_1(r)^{-1}S_1(r)-M_2(r)^{-1}S_2(r) \\ \sqrt{6}\Omega_\epsilon^{-1/2}\left[\frac{B(r^2)}{r^2}-\frac{B((1-r)^2)}{(1-r)^2}\right] \end{array}\right] \\
&= \left[M_1(r)^{-1}S_1(r)-M_2(r)^{-1}S_2(r)\right]^{\prime}\left[M_1(r)^{-1}+M_2(r)\right]^{-1}\times \\
&\quad \left[M_1(r)^{-1}S_1(r)-M_2(r)^{-1}S_2(r)\right] \\
&\quad +\left[\frac{1}{r^2}+\frac{1}{(1-r)^2}\right]^{-1}\times\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r(1-r)}\right]^{\prime}\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r(1-r)}\right] \\
&= I+II.
\end{aligned}$$

For I, we have, by definition of $s(r)$ and $V(r)$

$$I=\left[M_1^{-1}S_1-M_2^{-1}S_2\right]^{\prime}\left[M_1^{-1}+M_2^{-1}\right]^{-1}\left[M_1^{-1}S_1-M_2^{-1}S_2\right]=s(r)^{\prime}V^{-1}(r)\,s(r)=Q_R(r).$$

For II, we have

$$\left[\frac{1}{r^2}+\frac{1}{(1-r)^2}\right]^{-1}\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\,(1-r)}\right]'\left[\frac{B((1-r)^2)-B\left(r^2\right)}{r\,(1-r)}\right]$$

$$=\quad\frac{\left[B((1-r)^2)-B\left(r^2\right)\right]'\left[B((1-r)^2)-B\left(r^2\right)\right]}{r^2+(1-r)^2}=Q_p(r).$$

Hence

$$\mathrm{W}\left([Tr]\right)\xrightarrow{d}Q_R(r)+Q_p(r),$$

which proves equation (2). Independence of $Q_R(r)$ and $Q_p(r)$ follows from the fact that $\widehat{\beta}$ and $\widehat{\gamma}$ are asymptotically independent.

Suppose $(1-r)^2>r^2$ then $B((1-r)^2)-B\left(r^2\right)$ has variance $(1-r)^2-r^2$ so that for a fixed r

$$\frac{B((1-r)^2)-B\left(r^2\right)}{\sqrt{(1-r)^2-r^2}}\sim N\left(0,1\right).$$

Also if $r^2>(1-r)^2$ then $B((1-r)^2)-B\left(r^2\right)$ has variance $r^2-(1-r)^2$ so that for a fixed r

$$\frac{B\left(r^2\right)-B((1-r)^2)}{\sqrt{r^2-(1-r)^2}}\sim N\left(0,1\right)$$

Hence

$$Q_p(r)\sim\frac{\left|(1-r)^2-r^2\right|}{r^2+(1-r)^2}\chi^2_p.$$

As far as $Q_R(r)$ is concerned, let $\bar{W}$ be an R-dimensional demeaned standard Brownian motion, and B be a scalar standard Brownian motion. We have $\bar{B}_\varepsilon=\Omega_\epsilon^{1/2}\bar{W}$, and

$$S_1\left(r\right)=\Omega_\epsilon^{1/2}\int_0^r\bar{W}dB,$$

$$M_1\left(r\right)=\Omega_\epsilon^{1/2}\left(\int_0^r\bar{W}\bar{W}'\right)\Omega_\epsilon^{1/2},$$

and similarly for $S_2\left(r\right)$ and $M_2\left(r\right)$. Therefore we can write

$$\begin{aligned}
&\mathbf{V}^{-1}\left(r\right)\\
=\;&\begin{bmatrix} I_R & 0\\ 0 & -I_R\end{bmatrix}\begin{bmatrix} M_1^{-1}\\ M_2^{-1}\end{bmatrix}\left[M_1^{-1}+M_2^{-1}\right]^{-1}\begin{bmatrix} M_1^{-1} & M_2^{-1}\end{bmatrix}\begin{bmatrix} I_R & 0\\ 0 & -I_R\end{bmatrix}\\
=\;&\begin{bmatrix}\Omega_\epsilon^{-1/2} & 0\\ 0 & -\Omega_\epsilon^{-1/2}\end{bmatrix}\begin{bmatrix}\left(\int_0^r\bar{W}\bar{W}'\right)^{-1}\\ \left(\int_r^1\bar{W}\bar{W}'\right)^{-1}\end{bmatrix}\Omega_\epsilon^{-1/2}\Omega_\epsilon^{1/2}\left[\left(\int_0^r\bar{W}\bar{W}'\right)^{-1}+\left(\int_r^1\bar{W}\bar{W}'\right)^{-1}\right]^{-1}\\
&\Omega_\epsilon^{1/2}\Omega_\epsilon^{-1/2}\begin{bmatrix}\left(\int_0^r\bar{W}\bar{W}'\right)^{-1} & \left(\int_r^1\bar{W}\bar{W}'\right)^{-1}\end{bmatrix}\begin{bmatrix}\Omega_\epsilon^{-1/2} & 0\\ 0 & -\Omega_\epsilon^{-1/2}\end{bmatrix},
\end{aligned}$$

so that

$$
\begin{aligned}
Q_R(r) &= s(r)' \mathbf{V}^{-1}(r)\, s(r) \\
&= \left[\begin{array}{cc} \int_0^r dB\bar{W}' & \int_r^1 dB\bar{W}' \end{array}\right] \left[\begin{array}{cc} \Omega_\epsilon^{1/2} & 0 \\ 0 & \Omega_\epsilon^{1/2} \end{array}\right] \left[\begin{array}{cc} \Omega_\epsilon^{-1/2} & 0 \\ 0 & -\Omega_\epsilon^{-1/2} \end{array}\right] \\
&\quad \times \left[\begin{array}{c} \left(\int_0^r \bar{W}\bar{W}'\right)^{-1} \\ \left(\int_r^1 \bar{W}\bar{W}'\right)^{-1} \end{array}\right] \left[\left(\int_0^r \bar{W}\bar{W}'\right)^{-1} + \left(\int_r^1 \bar{W}\bar{W}'\right)^{-1}\right]^{-1} \\
&\quad \times \left[\begin{array}{cc} \left(\int_0^r \bar{W}\bar{W}'\right)^{-1} & \left(\int_r^1 \bar{W}\bar{W}'\right)^{-1} \end{array}\right] \left[\begin{array}{cc} \Omega_\epsilon^{-1/2} & 0 \\ 0 & -\Omega_\epsilon^{-1/2} \end{array}\right] \\
&\quad \times \left[\begin{array}{cc} \Omega_\epsilon^{1/2} & 0 \\ 0 & \Omega_\epsilon^{1/2} \end{array}\right] \left[\begin{array}{c} \int_0^r \bar{W} dB \\ \int_r^1 \bar{W} dB \end{array}\right] \\
&= \left[\left(\int_0^r dB\bar{W}'\right)\left(\int_0^r \bar{W}\bar{W}'\right)^{-1} - \left(\int_r^1 dB\bar{W}'\right)\left(\int_r^1 \bar{W}\bar{W}'\right)^{-1}\right] \\
&\quad \times \left[\left(\int_0^r \bar{W}\bar{W}'\right)^{-1} + \left(\int_r^1 \bar{W}\bar{W}'\right)^{-1}\right]^{-1} \\
&\quad \times \left[\left(\int_0^r \bar{W}\bar{W}'\right)^{-1}\left(\int_0^r \bar{W} dB\right) - \left(\int_r^1 \bar{W}\bar{W}'\right)^{-1}\left(\int_r^1 \bar{W} dB\right)\right].
\end{aligned}
$$

Letting C be the sigma field generated by the $\{F_t\}$, we have that, conditional on C:

$$
\left(\int_0^r \bar{W}\bar{W}'\right)^{-1}\left(\int_0^r \bar{W} dB\right)\Bigg| C = \left(\int_0^r \bar{W}\bar{W}'\right)^{-1/2} Z_1,
$$

$$
\left(\int_r^1 \bar{W}\bar{W}'\right)^{-1}\left(\int_r^1 \bar{W} dB\right)\Bigg| C = \left(\int_r^1 \bar{W}\bar{W}'\right)^{-1/2} Z_2,
$$

where Z_1 and Z_2 are two R-dimensional independent standard normals. Z_1 and Z_2 are independent since they arise from the presence of the stochastic increments $dB(r)$, which are independent across r. Therefore we also have

$$
\begin{aligned}
&\left(\int_0^r \bar{W}\bar{W}'\right)^{-1}\left(\int_0^r \bar{W} dB\right) - \left(\int_r^1 \bar{W}\bar{W}'\right)^{-1}\left(\int_r^1 \bar{W} dB\right)\Bigg| C \\
&= \left[\left(\int_0^r \bar{W}\bar{W}'\right)^{-1} + \left(\int_r^1 \bar{W}\bar{W}'\right)^{-1}\right]^{1/2} \times Z,
\end{aligned}
$$

where Z has an R-dimensional standard normal distribution. Hence we have the following passages

$$\begin{aligned}
& Q_R(r)|\, C \\
= \; & Z' \left[\left(\int_0^r \bar{W}\bar{W}' \right)^{-1} + \left(\int_r^1 \bar{W}\bar{W}' \right)^{-1} \right]^{1/2} \times \left[\left(\int_0^r \bar{W}\bar{W}' \right)^{-1} + \left(\int_r^1 \bar{W}\bar{W}' \right)^{-1} \right]^{-1} \\
& \times \left[\left(\int_0^r \bar{W}\bar{W}' \right)^{-1} + \left(\int_r^1 \bar{W}\bar{W}' \right)^{-1} \right]^{1/2} Z \\
= \; & Z'Z.
\end{aligned}$$

Therefore, conditional upon C

$$Q_R(r)|\, C = Z'Z \sim \chi^2_R.$$

Since this result does not depend on C - i.e. it holds true for all the possible elements in the sigma-field C - we have that, unconditionally on C

$$Q_R(r) \sim \chi^2_R.$$

This also proves that both $Q_p(r)$ and $Q_R(r)$ are nuisance parameters free. ■

B.5 Proof of Proposition 3

Proof. Consider $\hat{\sigma}^2_u$:

$$\hat{\sigma}^2_u = \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(y_{it} - \bar{y}_i - \hat{\theta}' \hat{X}_{it} \right)^2 .$$

Consistency of $\hat{\theta}$ under H_0, which has been proved in Theorem 1, implies that

$$\hat{\sigma}^2_u \xrightarrow{p} \sigma^2_u.$$

As far as $\hat{\sigma}^2_\zeta$ is concerned, we have

$$\hat{\sigma}^2_\zeta = \hat{\sigma}^2_u + \hat{\beta}' \hat{\sigma}^2_\pi \hat{\beta},$$

and we know that, under H_0, $\hat{\sigma}^2_u = \sigma^2_u + o_p(1)$ and likewise $\hat{\beta} = \beta + o_p(1)$. Also, it holds that

$$\hat{\sigma}^2_\pi = \left(\frac{1}{T^2} \sum_{t=1}^{T} \hat{w}_t \hat{w}_t' \right) \left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{T} \sum_{t=1}^{T} \hat{e}^2_{it} \right) \hat{\lambda}_i \hat{\lambda}_i' \right] \left(\frac{1}{T^2} \sum_{t=1}^{T} \hat{w}_t \hat{w}_t' \right),$$

and we know from Lemma A.4 (a) that

$$\frac{1}{T^2} \sum_{t=1}^{T} \hat{w}_t \hat{w}_t' = \frac{1}{T^2} \sum_{t=1}^{T} w_t w_t' + o_p(1) .$$

From Bai (2004) we know that the principal component estimator for the loadings λ_i is consistent, and therefore a LLN applies for the residuals $\hat{e}^2_{it}$ so that

$$\frac{1}{T} \sum_{t=1}^{T} \hat{e}^2_{it} \xrightarrow{p} \sigma^2_e.$$

Therefore,

$$\hat{\sigma}_{\pi}^{2} = \tilde{Q}_B \left(\sigma_e^2 \Sigma_\Lambda\right) \tilde{Q}_B' + o_p\left(1\right),$$

and hence

$$\hat{\sigma}_{\zeta}^{2} \xrightarrow{p} \sigma_{\zeta}^{2}.$$

This proves the Theorem. ■

B.6 Proof of Theorem 3

Proof. Under the local alternative hypotheses $H_a^{(nT)}$ the model can be rewritten as

$$\begin{aligned} y_{it}^{(nT)} &= \alpha_i + \hat{X}_{it}'\theta_t^{(nT)} + v_{it} \\ &= \alpha_i + \hat{X}_{it}'\theta + \frac{1}{\sqrt{nT}}\hat{X}_{it}'g\left(\frac{t}{T}\right) + v_{it}. \end{aligned}$$

The partial sample OLS estimate for θ is defined as

$$\hat{\theta}_{1k}^{(nT)} = \left[\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}y_{it}^{(nT)},$$

and we have

$$\begin{aligned} \hat{\theta}_{1k}^{(nT)} &= \left[\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\left[\alpha_i + \hat{X}_{it}'\theta + \frac{1}{\sqrt{nT}}\hat{X}_{it}'g\left(\frac{t}{T}\right) + v_{it}\right] \\ &= \theta + \left[\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\left\{\frac{1}{\sqrt{nT}}\hat{X}_{it}'g\left(\frac{t}{T}\right) + v_{it}\right\}\right] \\ &= \theta + \left[\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}v_{it} + \sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\frac{1}{\sqrt{nT}}\hat{X}_{it}'g\left(\frac{t}{T}\right)\right]. \end{aligned}$$

This leads to

$$\begin{aligned} &\sqrt{n}T\left[\hat{\theta}_{1k}^{(nT)} - \theta\right] \\ =\ &\left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}v_{it} + \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\frac{1}{\sqrt{nT}}\hat{X}_{it}'g\left(\frac{t}{T}\right)\right], \end{aligned}$$

and likewise, with respect to the partial sample OLS estimator for the second subsample $\hat{\theta}_{2k}^{(nT)}$

$$\begin{aligned} &\sqrt{n}T\left[\hat{\theta}_{2k}^{(nT)} - \theta\right] \\ =\ &\left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}v_{it} + \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\frac{1}{\sqrt{nT}}\hat{X}_{it}'g\left(\frac{t}{T}\right)\right]. \end{aligned}$$

Combining these two results, we have

$$
\begin{aligned}
& \sqrt{n}T\left[\hat{\theta}_{1k}^{(nT)}-\hat{\theta}_{2k}^{(nT)}\right] \\
= & \left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}v_{it}+\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\frac{1}{\sqrt{n}T}\hat{X}_{it}'g\left(\frac{t}{T}\right)\right] \\
& -\left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}v_{it}+\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\frac{1}{\sqrt{n}T}\hat{X}_{it}'g\left(\frac{t}{T}\right)\right]. \quad (31)
\end{aligned}
$$

We know that Lemma A.5 ensures that, as $(n,T)\to\infty$ along the path $\sqrt{n}/T\to 0$

$$
\begin{aligned}
& \left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}v_{it}\right] \\
& -\left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}v_{it}\right] \\
= & \; O_p(1),
\end{aligned}
$$

uniformly in r.

To prove that the Wald test has non trivial power against the local alternatives $H_a^{(nT)}$, we need to prove that, as far as equation (31) is concerned, it also holds that as $(n,T)\to\infty$ with $\sqrt{n}/T\to 0$

$$
\begin{aligned}
& \left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\frac{1}{\sqrt{n}T}\hat{X}_{it}'g\left(\frac{t}{T}\right)\right] \\
& -\left[\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\hat{W}_{it}'\right]^{-1}\left[\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\frac{1}{\sqrt{n}T}\hat{X}_{it}'g\left(\frac{t}{T}\right)\right] \\
= & \; O_p(1),
\end{aligned}
$$

uniformly in r. It holds that

$$
\begin{aligned}
& \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\frac{1}{\sqrt{n}T}\hat{X}_{it}'g\left(\frac{t}{T}\right) \\
= & \; \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\left[W_{it}+\left(\hat{W}_{it}-W_{it}\right)\right]\left[W_{it}+\left(\hat{W}_{it}-W_{it}\right)\right]'g\left(\frac{t}{T}\right) \\
= & \; \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}W_{it}W_{it}'g\left(\frac{t}{T}\right)+\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}W_{it}\left(\hat{W}_{it}-W_{it}\right)'g\left(\frac{t}{T}\right) \\
& +\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\left(\hat{W}_{it}-W_{it}\right)W_{it}'g\left(\frac{t}{T}\right)+\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\left(\hat{W}_{it}-W_{it}\right)\left(\hat{W}_{it}-W_{it}\right)'g\left(\frac{t}{T}\right) \\
= & \; I+II+III+IV.
\end{aligned}
$$

Assumption LP(b) states that

$$
I=\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}W_{it}W_{it}'g\left(\frac{t}{T}\right)=O_p(1).
$$

Also, for II and III, it holds that

$$\begin{aligned}\|II\| &\leq \left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\left\|\hat{W}_{it}-W_{it}\right\|^2\right)^{1/2}\left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\left\|W_{it}'g\left(\frac{t}{T}\right)\right\|^2\right)^{1/2}\\ &= \left(\frac{1}{T^2}\sum_{t=1}^{k}\|\hat{w}_t-w_t\|^2\right)^{1/2}\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{k}\left\|\frac{W_{it}'}{\sqrt{T}}g\left(\frac{t}{T}\right)\right\|^2\right)^{1/2}.\end{aligned}$$

As $(n,T)\to\infty$ with $\sqrt{n}/T\to 0$, Lemma A.1(b) ensures that

$$\left(\frac{1}{T^2}\sum_{t=1}^{k}\|\hat{w}_t-w_t\|^2\right)^{1/2}=\left[O\left(\frac{1}{T}\right)O_p\left(\frac{1}{C_{nT}^2}\right)\right]^{1/2}=o_p(1);$$

also, by Assumption LP(d)

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\left\|\frac{W_{it}'}{\sqrt{T}}g\left(\frac{t}{T}\right)\right\|^2=O_p(1).$$

Consequently, $II=III=o_p(1)$. Finally for IV, we have

$$\begin{aligned}\|IV\| &\leq \left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\left\|\hat{W}_{it}-W_{it}\right\|^2\right)\left(\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{k}\left\|g\left(\frac{t}{T}\right)\right\|^2\right)^{1/2}\\ &\leq \left(\frac{1}{T^2}\sum_{t=1}^{k}\|\hat{w}_t-w_t\|^2\right)\left(\frac{1}{T^2}\sum_{t=1}^{k}\left\|g\left(\frac{t}{T}\right)\right\|^2\right)^{1/2}\\ &= O\left(\frac{1}{T}\right)O_p\left(\frac{1}{C_{nT}^2}\right)O\left(\frac{1}{T}\right)O_p(1)=o_p(1),\end{aligned}$$

uniformly in r if $\sqrt{n}/T\to 0$. Combining these results we get

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}\frac{1}{\sqrt{n}T}\hat{X}_{it}'g\left(\frac{t}{T}\right)=O_p(1)$$

uniformly in r, and likewise

$$\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}\frac{1}{\sqrt{n}T}\hat{X}_{it}'g\left(\frac{t}{T}\right)=O_p(1).$$

Hence

$$\sqrt{n}T\left[\hat{\theta}_{1k}^{(nT)}-\hat{\theta}_{2k}^{(nT)}\right]\xrightarrow{d}\left[\begin{array}{c}M_1(r)^{-1}S_1(r)-M_2(r)^{-1}S_2(r)\\ \sqrt{6}C\frac{B((1-r)^2)-B(r^2)}{r(1-r)}\end{array}\right]+O_p(1).$$

After proving the distribution limit for the partial OLS estimates under $H_a^{(nT)}$, we can now turn to the limiting distribution of the test statistics $W^{(nT)}(k)$. This is defined as

$$\mathrm{W}^{(nT)}(k)=\left[\begin{array}{cc}\tilde{\sigma}_\zeta^2 I_R & 0\\ 0 & \tilde{\sigma}_u^2 I_p\end{array}\right]^{-1}\left[\hat{\theta}_{1k}^{(nT)}-\hat{\theta}_{2k}^{(nT)}\right]'\left[\begin{array}{c}\left(\sum_{i=1}^{n}\sum_{t=1}^{k}\hat{W}_{it}W_{it}'\right)^{-1}\\ +\left(\sum_{i=1}^{n}\sum_{t=k+1}^{T}\hat{W}_{it}W_{it}'\right)^{-1}\end{array}\right]^{-1}\left[\hat{\theta}_{1k}^{(nT)}-\hat{\theta}_{2k}^{(nT)}\right].$$

Since both $\tilde{\sigma}_\zeta$ and $\tilde{\sigma}_u$ are consistent for σ_ζ and σ_u, the Continuous Mapping Theorem ensures that, as $(n,T) \to \infty$, under the alternative $H_a^{(nT)}$

$$\mathrm{W}^{(nT)}([Tr]) \xrightarrow{d} D\left(r\right) + O_p(1).$$

uniformly in r. ■

B.7 Proof of Proposition 4

Proof. We first consider the consistency of $\tilde{\sigma}_u^2$. Let

$$\widetilde{y}_{it} = y_{it} - \bar{y}_i.$$

Then we have

$$\begin{aligned}
\tilde{\sigma}_u^2 &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\widetilde{y}_{it} - \hat{X}_{it}^{'}\hat{\theta}_t^{(nT)}\right]^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\widetilde{y}_{it} - \hat{X}_{it}^{'}\left[\hat{\theta} + \frac{1}{\sqrt{nT}}g\left(\frac{t}{T}\right)\right]\right\}^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{y}_{it} - \hat{X}_{it}^{'}\hat{\theta}\right)^2 + \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{nT^2}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}\hat{X}_{it}^{'}g\left(\frac{t}{T}\right) \\
&\quad -2\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{y}_{it} - \hat{\theta}_t^{'}\hat{X}_{it}\right)\frac{1}{\sqrt{nT}}\hat{X}_{it}^{'}g\left(\frac{t}{T}\right) \\
&= I + II + III.
\end{aligned}$$

As far as term I is concerned, we have

$$\begin{aligned}
I &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\widetilde{y}_{it} - \hat{X}_{it}^{'}\theta + \hat{X}_{it}^{'}\left(\hat{\theta} - \theta\right)\right]^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(v_{it} + \hat{X}_{it}^{'}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it}\right)^2 \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}^2 + \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}\hat{X}_{it}^{'}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it} \\
&\quad -2\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}\hat{X}_{it}^{'}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it} \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}^2 - \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}\hat{X}_{it}^{'}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it}.
\end{aligned}$$

As far as III is concerned, it holds that

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\widetilde{y}_{it}-\hat{X}_{it}^{'}\hat{\theta}\right)\frac{1}{\sqrt{n}T}\hat{X}_{it}^{'}g\left(\frac{t}{T}\right) \\
= & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[v_{it}+\hat{X}_{it}^{'}\left(\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it}\right]\frac{1}{\sqrt{n}T}\hat{X}_{it}^{'}g\left(\frac{t}{T}\right) \\
= & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{\sqrt{n}T}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}v_{it} \\
& +\frac{1}{nT}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{\sqrt{n}T}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}\hat{X}_{it}^{'}\right]\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it}\right].
\end{aligned}$$

Combing I with II and III we get

$$\begin{aligned}
\tilde{\sigma}_u^2 = & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}^2 \\
& -\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}\hat{X}_{it}^{'}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it} \\
& +\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{nT^2}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}\hat{X}_{it}^{'}g\left(\frac{t}{T}\right) \\
& -\frac{2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{\sqrt{n}T}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}v_{it} \\
& -\frac{2}{nT}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{\sqrt{n}T}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}\hat{X}_{it}^{'}\right]\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it}\right].
\end{aligned}$$

Now

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}\hat{X}_{it}^{'}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it}=O_p\left(\frac{1}{nT}\right),$$

use Lemma A.4;

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{nT^2}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}\hat{X}_{it}^{'}g\left(\frac{t}{T}\right)=O_p\left(\frac{1}{nT}\right),$$

use Assumption LP(d);

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{\sqrt{n}T}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}v_{it}=O_p\left(\frac{1}{\sqrt{n}T}\right),$$

use Assumption LP(e);

$$\frac{1}{nT}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{1}{\sqrt{n}T}g\left(\frac{t}{T}\right)^{'}\hat{X}_{it}\hat{X}_{it}^{'}\right]\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}\hat{X}_{it}^{'}\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{X}_{it}v_{it}\right]=O_p\left(\frac{1}{nT}\right),$$

due to Lemma A.4 and Assumption LP(b). Hence

$$\tilde{\sigma}_u^2=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}^2+o_p\left(1\right),$$

and it holds that

$$\begin{aligned}
\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}v_{it}^{2} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\theta^{'}\left(X_{it}-\hat{X}_{it}\right)+u_{it}\right]^{2} \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left\{\left[\theta^{'}\left(w_{t}-\hat{w}_{t}\right)\right]^{2}+u_{it}^{2}+2\theta^{'}\left(w_{t}-\hat{w}_{t}\right)u_{it}\right\}^{2} \\
&= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^{2}+\frac{1}{T}\sum_{t=1}^{T}\left[\theta^{'}\left(w_{t}-\hat{w}_{t}\right)\right]^{2}+\frac{2}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\theta^{'}\left(w_{t}-\hat{w}_{t}\right)u_{it} \\
&= I+II+III.
\end{aligned}$$

For I, use Assumption M1 and a LLN we have

$$I=\sigma_{u}^{2}+o_{p}\left(1\right).$$

As far as II and III are concerned, we have

$$\|II\|\leq\|\theta\|^{2}\frac{1}{T}\sum_{t=1}^{T}\|(w_{t}-\hat{w}_{t})\|^{2}=o_{p}\left(1\right)$$

using Lemma A.1(b), and

$$\|III\|\leq\|\theta\|^{2}\frac{1}{nT}\left(\sum_{t=1}^{T}\|(w_{t}-\hat{w}_{t})\|^{2}\right)^{1/2}\left(\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}^{2}\right)^{1/2}=o_{p}\left(1\right),$$

after equation (25). Then under the local alternatives $H_{a}^{(nT)}$ it holds that

$$\tilde{\sigma}_{u}^{2}\xrightarrow{p}\sigma_{u}^{2}.$$

We are now ready also to prove consistency of $\tilde{\sigma}_{\zeta}^{2}$. Since

$$\tilde{\sigma}_{\zeta}^{2}=\tilde{\sigma}_{u}^{2}+\hat{\beta}^{(nT)\prime}\hat{\sigma}_{\pi}^{2}\hat{\beta}^{(nT)},$$

and since

$$\hat{\sigma}_{\pi}^{2}\xrightarrow{p}\sigma_{\pi}^{2}$$

under $H_{a}^{(nT)}$ as it does not depend on $H_{a}^{(nT)}$ being true or not, in light of the consistency of $\hat{\beta}^{(nT)}$ we have

$$\tilde{\sigma}_{\zeta}^{2}\xrightarrow{p}\sigma_{u}^{2}+\beta^{(nT)\prime}\sigma_{\pi}^{2}\beta^{(nT)\prime}=\sigma_{\zeta}^{2}. \blacksquare$$

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TABLE 1:

Size at the 5% Level

Panel A: Size for SupW

$n\backslash T$	20	40	60	120	240	480
20	0.0175	0.0375	0.0372	0.0637	0.0587	0.0519
40	0.0145	0.0236	0.0248	0.0462	0.0514	0.0604
60	0.0149	0.0260	0.0340	0.0337	0.0397	0.0550
120	0.0151	0.0287	0.0346	0.0373	0.0470	0.0561
240	0.0172	0.0309	0.0306	0.0360	0.0480	0.0454
480	0.0212	0.0285	0.0351	0.0349	0.0501	0.0560

Panel B: Size for AveW

$n\backslash T$	20	40	60	120	240	480
20	0.0267	0.0375	0.0350	0.0490	0.0407	0.0350
40	0.0258	0.0267	0.0242	0.0342	0.0349	0.0403
60	0.0220	0.0273	0.0312	0.0299	0.0265	0.0339
120	0.0238	0.0298	0.0315	0.0325	0.0306	0.0354
240	0.0241	0.0333	0.0330	0.0311	0.0367	0.0314
480	0.0312	0.0300	0.0307	0.0298	0.0375	0.0349

Panel C: Size for ExpW

$n\backslash T$	20	40	60	120	240	480
20	0.0306	0.0472	0.0455	0.0653	0.0537	0.0458
40	0.0272	0.0325	0.0534	0.0475	0.0461	0.0525
60	0.0241	0.0353	0.0392	0.0337	0.0352	0.0455
120	0.0269	0.0388	0.0428	0.0396	0.0411	0.0473
240	0.0289	0.0427	0.0403	0.0362	0.0453	0.0405
480	0.0370	0.0403	0.0415	0.0377	0.0489	0.0460

TABLE 2:

Power at 5% Level

Panel A. Power for SupW

$n\backslash T$	20	40	60	120	240	480
20	0.0715	0.2085	0.4723	0.7962	0.9972	1.0000
40	0.0850	0.2332	0.5129	0.9467	1.0000	1.0000
60	0.0932	0.3281	0.6545	0.9837	1.0000	1.0000
120	0.1340	0.5551	0.8512	0.9999	1.0000	1.0000
240	0.2545	0.9640	0.9697	1.0000	1.0000	1.0000
480	0.4195	0.9327	0.9996	1.0000	1.0000	1.0000

Panel B. Power for AveW

$n\backslash T$	20	40	60	120	240	480
20	0.0859	0.2170	0.4477	0.8364	0.9983	1.0000
40	0.1055	0.2699	0.5523	0.9655	1.0000	1.0000
60	0.1172	0.3693	0.6917	0.9927	1.0000	1.0000
120	0.1686	0.6003	0.8921	1.0000	1.0000	1.0000
240	0.2991	0.8142	0.9877	1.0000	1.0000	1.0000
480	0.4700	0.9637	1.0000	1.0000	1.0000	1.0000

Panel C. Power for ExeW

$n\backslash T$	20	40	60	120	240	480
20	0.1009	0.2448	0.4993	0.8321	0.9980	1.0000
40	0.1224	0.2838	0.5628	0.9620	1.0000	1.0000
60	0.1329	0.3874	0.7006	0.9910	1.0000	1.0000
120	0.1806	0.6151	0.8899	1.0000	1.0000	1.0000
240	0.3168	0.8131	0.9850	1.0000	1.0000	1.0000
480	0.4894	0.9599	0.9999	1.0000	1.0000	1.0000