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The analysis of the perpetual option markets: Theory and evidence.

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To Antonella

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Part I

Introduction

In literature it is generally acknowledged the importance of the Black & Scholes framework in the derivative pricing analysis [Black and Scholes, 1973]. From then on the derivatives instruments have recorded an exponential interest trend in the global economy. Likewise the derivative analysis literature has experienced a fruitful creation of numerous methods and instruments proposed to "explain" options market price, especially to explain the biases between the theoretical price provided by the Black & Scholes model and the market price

The explanations proposed in literature to model the theoretical price, as attempts to interpret these biases, are various but there is general consensus on the importance to model the time varying volatility (Engle [1982], Bollerslev [1986]) and the leverage effect (Black [1976]). Also in the empirical studies these financial return aspects have showed to be suitable to solve the biases (see Bakshi, Cao and Chen [1997], Engle and Mustafa [1992], and Heston and Nandi [2000]).

Because there is wide consensus that the variance of the financial asset returns is time variant, a great amount of efforts are directing to realize mathematical models which, by choosing the variance dynamics as the model corner-stone, should be effectively able to explain the option prices. Surely the GARCH model is a reference instrument to study the volatility dynamics, and among its advantages there is its highly flexibility to be suitable to capture the most important features of the financial variables. It is to note that the passage from the GARCH parametric characterization of financial asset series to the computation of the theoretical option price is not immediate.

The volatility is neither time constant nor time homogenous therefore the researchers have developed wide class of GARCH models and Stochastic Volatility (SV) models to describe the stochastic time evolution of the volatility.

The Black & Scholes model has yield, without doubt, a great interest to the research to improve the option price techniques and it has end by influence the literature but so much that the pricing model development seems to be confined to a set of assumptions often cited as cause of the poor pricing and hedging performance (Christoffersen and Jacobs (2004)). We refer to some assumptions (i.e., parametric risk premium, normal error) which implies a simplified passage to the risk neutral measure (as in Black & Scholes framework) but their realism remain discussible.

In a time varying volatility model, as in the GARCH model, the Black & Scholes argument to price the derivative becomes useless, it is no more possible the replication of the derivative with a portfolio composed by the underlying asset and the risk-free asset. For this reason the market is said to be incomplete. It is necessary to exploit other theoretical arguments to formulate a correct option pricing framework. It is known in the financial theory that the existence of a sufficient number of derivatives in the market yields the market completeness, and a well-defined option pricing measure can be stated. In this sense the market incompleteness represents the key point to define the option pricing method. In this direction a new GARCH option pricing model in incomplete markets condition is proposed by G. Barone-Adesi, R. Engle and L. Mancini (2007) through their article "A GARCH Option Pricing Model in Incomplete Markets" (BAEM in the following). This article endows this work of the underlying hypotheses framework and the pricing theory used in performing the empirical analysis and in developing the GARCH option pricing models.

The analysis of the Nikkei Put Warrants (NPWs) market as long-term options and proxy to the perpetual options is the empirical object of this study. There is not yet an official market which offers perpetual options, therefore we study an option prices database of derivatives with long maturity. The analysis of this market area allows us to study the long-term financial phenomenons behavior and it allows to highlight crucially the different performances of the models in comparison with particular reference to the option price dynamics.

This work is organized in three parts: introduction, theoretical part and empirical part. It follows the conclusion and some critical comments.

The introduction presents the study object that is the long-term American options market and gives the reasons of the theoretical instruments choice to face the research.

The theoretical part presents the BAEM approach which is the main theoretical reference point of the entire work. The empirical part shows the analysis results performed on the Nikkei 225 index historical series and the Nikkei Put Warrants data collected.

Both parts (i.e., the theoretical part and the empirical part) are again divided in two sections.

First sections refer to three asymmetric GARCH models, widely known in literature. We name these models "Simple GARCH models". These models use four parameters to describe the volatility dynamics, one more than the Standard GARCH. This additional parameter has the end to model the asymmetric response of the volatility to the market news (i.e., leverage effect).

Second sections deal with extended versions of the same GARCH models, known in literature as "Components GARCH models". They decompose the volatility dynamics in a long-run component and a short-run component. This extension was developed by Engle, R.F. and G. Lee (1999) and used in option pricing by Christoffersen, P., K. Jacobs and Y. Wang (2004).

Unfortunately there does not exist an analytical close-form formula to compute the price of an American derivative when the variance has a GARCH dynamics. To this end we suggest two solutions. The first one is used in the empirical analysis and it is an approximation based on the Monte Carlo method simulation to price the American option.

The second one is proposed in Duan, J.C. and J.G. Simonato (2001) and here it is revisited in light of the assumptions and of the models used in this work. A particular arrangement is required in the Component GARCH models case. Even if the sparsity feature of the transition matrix were exploited, as the authors (Duan, Simonato) suggest, this approach is more computational demanding than the previous one, particularly in long-term derivatives pricing. In the last part of this work it is presented a suggestion to reduce further the computational effort by improving both memory usage and computation speed.

0.1 Presentation of the problem and instruments choice.

The empirical analysis of the long-term options constitutes the object of this study. The analysis is performed on the Nikkei Put Warrants (NPWs) market. The volatility dynamics is studied by a GARCH model characterization. The reason to choose the GARCH model is relative to its potential ability in describing the volatility dynamics, its flexibility in capturing many financial variables features and its aptitude to extensions in managing both long-term and short-term financial phenomenons.

The NPWs have two main features: They are American-style Put Option and they are offered in a wide range of maturities.

Unluckily there is not a close-form pricing formula to compute an American Option price in a GARCH framework. From an empirical point of view this makes indispensable to use some approximation methods. The derivative price will be computed by an approximation techniques based on Monte Carlo simulation method. These price approximations will be furnished with some theoretical justifications and empirically validated with regard to the characteristics of the long-term derivatives.

Secondly we present the instruments development able to compute the American GARCH option price by means of Duan and Simonato's approximation which describes the GARCH asset price dynamics through a multi-states Markov Chain. This is only a theoretical presentation but this method is not used in the empirical analysis because it needs some adjustment in order to improve the computational efficiency. The last part of the work is dedicated to propose one possible improvement for this method.

The long-term options study implies to model the underlying financial variables dynamics for a long lapse of time. Therefore a comparison between the simple GARCH models (4 parameters) and the Component GARCH models (7 parameters), which model richer volatility dynamics, is well regarded.

Nevertheless even if the volatility dynamics is enriched by the "time-varying volatility", the "volatility-clustering" and the "leverage effect" features the log-returns show a non-normal behavior.

Many assumptions of non-normal shocks distribution to model the innovation has been proposed to cope with this inconsistency. In the parametric ambit it is the case to cite the stable distributions and the related Levy processes. While in the non-parametric ambit the Filtering Historical Simulation (Barone-Adesi at al. [1998]) allows to use the empirical shocks distribution to model the future innovation with respect to the theoretical assumption about the financial variable dynamics.

We verify through the empirical analysis the GARCH framework performance in the long-term American Option Pricing. To ascertain the analysis results, as in the BAEM framework, we show both statistical measures of the model mispricing and economic check of decreasing monotonicity of the SPD with respect to the underlying asset price.

It is important to underline that the analysis is conjointly result of the approximation method used to price the options, the option pricing underlying assumptions (which allow to pass from the historical probability measure to the probability risk-adjusted) and the error distribution assumption.

The assumptions framework, the financial theory and the analytic instruments of this work are based on "A GARCH Option Pricing Model in Incomplete Markets", G. Barone-Adesi ,R. Engle, L. Mancini [2007].

0.2 Perpetual options and Long-term options: The official market.

An official market trading perpetual options does not exist until today, albeit some exchange organization seems to promise an open soon. We can consider the long-term options as a proxy for such derivatives. If we would list the Options with long maturities traded on organized exchanges we have to number certainly:

- The so called LEAPS (Long-term Equity AnticiPation Securities) with maturities up to 5 years.
- The Incentive stock options whose maturity can be up to 15 years.
- Variable annuity contracts (in the U.S.) and segregated mutual funds (in Canada) have embedded equity put options with maturities of up to 30 years.

Even if the long-maturities options have received increasing attention from the market, at the moment it remains some perplexity about these derivatives. The main problem is that as the maturity increases as the difficulties to model and to verify the derivative market price become greater. This is mainly due to the real difficulty to describe efficaciously a future event far in the time.

There are also derivatives traded by some Financial Institutions with maturity very long (up to 40 years), for example the Nikkei Put Warrants (NPWs) are American put options on the NIKKEI 225 index and they are available for different strike prices and maturities. The following table shows some Financial Institutions which offer NPWs:

Currency	Country	Financial Institution	Option Type	Underlying Type	City
Hong Kong Dollar	Hong Kong	DEUTSCHE BANK AG	Put	Index	Hong Kong
Hong Kong Dollar	Hong Kong	SGA SOCIETE GENERALE ACCEPTANCE NV	Put	Index	Hong Kong
Singaporean Dollar	Singapore	BNP PARIBAS ARBITRAGE ISSUANCE BV	Put	Index	Singapore
Singaporean Dollar	Singapore	DEUTSCHE BANK AG LONDON	Put	Index	Singapore
Swiss Franc	Switzerland	ABN AMRO BANK NV	Put	Index	Zurich
Swiss Franc	Switzerland	ABN AMRO BANK NV LONDON BRANCH	Put	Index	Zurich
Swiss Franc	Switzerland	CREDIT SUISSE	Put	Index	Zurich
Swiss Franc	Switzerland	GOLDMAN SACHS INTERNATIONAL	Put	Index	Zurich
Swiss Franc	Switzerland	VONTOBEL HOLDING AG	Put	Index	Zurich
United Kingdom Pound	United Kingdom	MERRILL LYNCH & COMPANY INCORPORATED	Put	Index	London

Figure 0.1: International Financial Institutions offering Nikkei Put Warrants

The long-term option market in Italy is at the beginning but the market tendency to offer derivatives with long maturities is recorded. Since 23rd October 2006 the Italian Exchange(Borsa Italiana) has introduced on the IDEM market (Italian Derivatives Market) the long stock options. The maturity is 3 years for such derivatives. There are two types of long options traded on IDEM:

-The European-style derivatives written on the S&P/MIB index

-The American-style derivatives written on single stocks traded in the market.

The empirical analysis of this work will be based on the Nikkei Put Warrants databases and the historical time series of the Nikkei 225 index. All data was collected from DataStream.

Chapter 1

Theoretical part

1.1 Section I - Simple GARCH models

In this part we present the basic version of the models used in the Section I: the Simple GARCH models. On the basis of these models we present the option pricing theory underlying the empirical work. Subsequently it is showed the approximative Monte Carlo method to compute the theoretical American option price. Moreover it is presented the alternative approximation (due to Duan and Simonato) to price this type of contingent claim in the GARCH models. Some notes are dedicated to explain how the innovations has been modeled both parametrically and non-parametrically.

Finally it follows the summarizing schemes of the entire procedure used in the Empirical analyses.

1.1.1 Simple GARCH volatility dynamics models

We consider a discrete-time economy. Let S_t be the price of the Nikkei 225 index at day t and y_t the daily log-return, $y_{t+1} := \ln \frac{S_{t+1}}{S_t}$.

Suppose that under the objective or historical measure P the daily log-return is described by the following relation:

$$y_{t+1} = \ln \frac{S_{t+1}}{S_t} = \mu + \sigma_{t+1} z_{t+1} \tag{1.1}$$

where $z_{t+1}|\phi_t \sim (0,1)$ under P.

The Simple GARCH models are an extension of the standard GARCH(1,1) and they could be represented as:

$$\sigma_{t+1} = f(\rho, \sigma_t, z_t) \tag{1.2}$$

where the relation expresses that the conditional variance at time $t+1$ is function of the lagged value of the variance (σ_t), the lagged shock (z_t) and a set of parameters (ρ).

We consider three different models for the variance dynamics in the first section of the work:

Model I: N-GARCH (*N-G*):

$$\sigma_{t+1}^2 = \omega + \beta \sigma_t^2 + \alpha \sigma_t^2 (z_t - \gamma)^2 \tag{1.3}$$

where $\omega, \alpha > 0$ and $0 < \beta < 1$.

Model II: GJR-GARCH (*GJR-G*):

$$\sigma_{t+1}^2 = \omega + \beta \sigma_t^2 + \alpha \sigma_t^2 z_t^2 - \gamma \sigma_t^2 z_t^2 I_t \tag{1.4}$$

where $\omega, \alpha > 0$,$0 < \beta < 1$ and $I_t = \begin{cases} 1; & z_t < 0 \\ 0; & \text{otherwise} \end{cases}$.

Model III: E-GARCH (E-G):

$$\ln\left(\sigma_{t+1}^2\right) = \omega + \beta \ln\left(\sigma_t^2\right) + \alpha\left(|z_t| - \gamma z_t\right) \tag{1.5}$$

where $\alpha > 0$ and $0 < \beta < 1$.

The parameters of the models are $\theta = (\mu, \rho)$ where μ is the constant drift term and $\rho = (\omega, \beta, \alpha, \gamma)$ is the parameter vector related to the variance dynamics.

The parameter γ allows to model the asymmetric behaviour of the variance, sometime called Black's effect. It consists of a greater response of the variance when the news arrived in the market are negative ($z_t < 0$) than when the news are positive ($z_t > 0$).

All conditions on the parameters ω, α are used to avoid theoretical inconsistence on the value for σ_t^2 (i.e., $\omega < 0$ should mean a potential negative value for the variance, while $\alpha < 0$ should mean that greater shock movements induce a decreasing variance), while the conditions on β allow the variance process to be covariance-stationary. Let consider the value of the stationary variance level for each model, supposing the weakly stationarity on σ_t^2 (i.e.,$E\left(\sigma_t^2\right) = h^* \; \forall t$ (or $E^P\left(\ln\left(\sigma_t^2\right)\right) = \ln\left(h^*\right)$ for the E-G)):

in the N-G

$$h^* = \frac{\omega}{1 - \beta - \alpha E^P\left(z_t - \gamma\right)^2} \tag{1.6}$$

in the GJR-G

$$h^* = \frac{\omega}{1 - \beta - \alpha E^P\left(z_t^2\right) - \gamma E^P\left(z_t^2 I_t\right)} \tag{1.7}$$

in the *E-G*

$$h^* = \exp\left(\frac{\omega - \alpha E^P\left(|z_t| - \gamma z_t\right)}{1-\beta}\right) \tag{1.8}$$

We have used the GARCH property that: $E\left(f\left(\sigma_t\right)g\left(z_t\right)\right) = E\left(f\left(\sigma_t\right)E\left(g\left(z_t\right)|\phi_{t-1}\right)\right)$ where f and g are some real function, since σ_t, and thus each its function, is measurable with respect to the sigma-algebra at time $t-1$.

Since we have to avoid the asymptotic divergence and the negativity of the variance process we need the following additional conditions:

$\beta + \alpha E^P\left(z_t - \gamma\right)^2 < 1$ in the *N-G*

$\beta + \alpha E^P\left(z_t^2\right) + \gamma E^P\left(z_t^2 I_t\right) < 1$ in the *GJR-G*

(i.e. the denominators in 1.6 and 1.7 must be greater than 0).

Note that in the normal innovation case (i.e., $z_{t+1}|\phi_t \overset{P}{\sim} N(0,1)$)

$h^* = \dfrac{\omega}{1-\beta-\alpha\left(1+\gamma^2\right)}$ in *N-G*, $h^* = \dfrac{\omega}{1-\beta-\alpha-\gamma/2}$ in *GJR-G*,

$h^* = \exp\left(\dfrac{\omega - \alpha\sqrt{2/\pi}}{1-\beta}\right)$ in *E-G*.

and the conditions are:

$\beta + \alpha\left(1+\gamma^2\right) < 1$ in the *N-G*

$\alpha + \beta + \gamma/2 < 1$ in the *GJR-G*

The main characteristics of the simple GARCH models can be summarized in:

- Parsimonious model, only 4 parameters to model the variance dynamics.

- Two state variables: price and variance.

- Time varying variance: Simple GARCH models drive the variance process.

- All model are potentially able to explain the well-known stylised-facts as the "Leverage effect" (γ parameter) and the "Clustering effect" in the stochastic volatility (β parameter).

1.2 Theoretical approach to the option pricing

In the theory of asset pricing it is well known that in a dynamic equilibrium (such as Rubinstein 1976 or Lucas 1978) the price of any financial asset can be represented as a discounted expected value of its future payoffs. If r is the riskless rate, S_T represents the state variable of the economy, ϕ_t is the information to time t available to the agent and Q is the risk neutral measure, then the price at time t (i.e., ξ_t) of a single future payoff at time T is

$$\xi_t = E^Q[\xi_T \cdot e^{-r(T-t)}|\phi_t] = e^{-r(T-t)} \int_0^\infty \xi_T \cdot q_{t,T}\left(S_T\right) dS_T \qquad (1.9)$$

The right-hand side highlights the role of $q_{t,T}\left(s\right)$ which represents the probability density function of the payoffs under the risk neutral measure Q, and it is called State Price Density (SPD) or Pricing Density (or Risk neutral Probability Density Function (Cox and Ross 1976)) or Equivalent Martingale Measure (Harrison and Kreps (1979))).

The SPD completely holds the implicit characteristics of the pricing model used. In a continuum state setting the SPD allows to define for each possible state of the economy the price of a security (Arrow-Debreu security price) paying one dollar at time T if the state variable S_T at time T falls into that state (i.e., $S_T \in (s,\, s+ds)$).

Using the notion of State Price Density per unit probability (SPD per unit probability), $M_{t,T}(s) := e^{-r(T-t)}\frac{q_{t,T}(s)}{p_{t,T}(s)}$ allows us to rewrite the pricing formula 1.9 as:

$$\xi_t = \int_0^\infty \xi_T \cdot \left(e^{-r(T-t)}\frac{q_{t,T}(S_T)}{p_{t,T}(S_T)}\right) p_{t,T}(S_T)\, dS_T = E^P[\xi_T \cdot M_{t,T}|\phi_t] \tag{1.10}$$

where P is the historical measure and $p_{t,T}(s)$ represents the historical probability density function at time t for payoff of time T.

The SPD per unit probability can interpreted for each state of the economy as the price of a pure state contingent claim which pays $1/p_{t,T}(s)$ dollars at time T if and only if the state variable S_T at time T falls into that state.

Let consider now the SPD approach in the American-style contingent claim case. In the discrete-case time setting the American option can be exercised only at integer time instant. The option price can be formulated as a recursive procedure from the end in backward.

Let ξ_T be the final payoff at time T of the American option then the value of the American option (i.e., V) at time T corresponds to the payoff:

$V_T = \xi_T$.

In general the American price is the maximum between the value of the option if immediately exercised and the discounted expected value of the option if left "live" for the next time, to say with a formula:

$V_t = \max\left\{\xi_t, E^P\left[V_{t+1}M_{t,t+1}|\phi_t\right]\right\}$ for $t = 0, 1, ..., T-1$.

The American option price can be written in compact way by using $\vee$ as Max operator in the following way:

$V_0 = \xi_0 \vee E^P\{\{\xi_1 \vee E^P[[...\xi_{T-2} \vee E^P((\xi_{T-1} \vee E^P(\xi_T M_{T-1,T}|\phi_{T-1}))\, M_{T-2,T-1}|\phi_{T-2})...$

$...]M_{1,2}|\phi_1]\}M_{0,1}|\phi_0\}$, and because $M_{t-1,t}$ is ϕ_t-measurable we can rewrite the option price as:

$$V_0 = \xi_0 \vee E^P\{\{\xi_1 N_{0,1} \vee E^P[[...\xi_{T-2} N_{0,T-2} \vee E^P((\xi_{T-1} N_{0,T-1} \vee E^P(\xi_T N_{0,T}|\phi_{T-1}))|\phi_{T-2})...$$
$$...]|\phi_1]\}|\phi_0\}$$

where $N_{0,t} = \prod_{i=1}^{t} M_{i-1,i}$, having use the property that $(a \vee b)c = (ca \vee cb)$ where a, b, c have positive value.

This formulation highlights that the American option price can be defined as function of the sequence of the State Price Densities per unit probability from the current time up to maturity.

We return on the SPD approach applied on the long-term American option in a next paragraph after we have introduced other problem specifications.

1.3 GARCH option price in incomplete markets (Barone Adesi-Engle-Mancini approach)

The equation (1.10) can be used to determine the equilibrium asset prices given the historical price dynamics and the SPD per unit probability.

The BAEM approach supposes that two different GARCH models can approximate the historical and the risk-neutral asset price dynamics. Therefore two different sets of GARCH parameters allow to determine the volatility pricing process and the historical volatility of the asset process respectively. Distinguishing the two processes is very important. The first one is related to the market asset price dynamics and the first set of GARCH parameters can be historically estimated. The second set of the GARCH parameters related to the pricing process is estimated by inferring the investors aggregate preferences (i.e., how they set state prices) through a calibration procedure.

The BAEM framework offers a novel option pricing approach in literature. It exploits the market incompleteness to determine the pricing distribution distinctly by the historical distribution. The two distinct distributions allow to characterize the SPD per unit probability

The BAEM approach develops in the following three points:

1) Historical GARCH parameters estimation by using an asset return time series to determine the historical asset price dynamics (Asset-based setting estimation)

2) "Risk-neutral" or Pricing GARCH parameters calibration by using option prices cross sections to determine the pricing process dynamics (Option-based setting estimation)

3) The option pricing process identification: the SPD per unit probability is obtained by discounting the ratio of the historical and pricing densities derived from the two GARCH models

The next paragraph shows the detailed description for each point.

It is important to highlight some characteristics of the BAEM approach:

a) This approach does not attempt to specify the SPD per unit probability directly through a specification of the change of measure from P to Q, but it requires to approximate the change of measure through a calibration procedure directly on the market option prices and infers the risk-neutral GARCH parameters.

b) The assumption of the two different processes (i.e., the historical and the pricing process) allows to treat the risk premium in nonparametric manner (the premium is inferred by the aggregate investors' behaviour summarized in the SPD).

c) The valuation results can be validated both statistically and economically, by using statistical measures to evaluate mispricing error of the model and economic measures to verify a minimum economic criteria on the SPD per unit probability.

1.3.1 Asset-based setting estimation: the historical GARCH parameter estimation

In the BAEM approach the historical GARCH parameters estimation is performed by the maximum likelihood (ML) method. We present a briefly a description of this procedure:

Let consider the price equation 1.1 as previously stated, we can rewrite it as:

$$y_t = \mu + \varepsilon_t, \tag{1.11}$$

where $y_t = \ln \frac{S_t}{S_{t-1}}$, $\varepsilon_t = \sigma_t z_t$ and $\varepsilon_t|\phi_{t-1} \overset{P}{\sim} (0, \sigma_t)$.

Note that σ_t is a measurable function w.r.t ϕ_{t-1} in each GARCH model presented before.

In order to obtain maximum likelihood parameters estimation assume that the error term is conditionally normal distributed (i.e., $\varepsilon_t|\phi_{t-1} \overset{i.i.d}{\sim} N(0, \sigma_t)$), we can thus write the conditional probability density of Y_t as:

$$f_{Y_t|\phi_{t-1}(y_t|\phi_{t-1};\theta)} = \frac{1}{\sqrt{2\pi}\sigma_t(\rho)} \exp\left(-\frac{1}{2}\left(\frac{\varepsilon_t - \mu}{\sigma_t(\rho)}\right)^2\right) \tag{1.12}$$

where $\theta = (\mu, \rho)$ where $\rho = (\omega, \beta, \alpha, \gamma)$

Let $y_{-(n-1)}, ..., y_{-1}, y_0$ be an observed sample of asset log returns of n historical observations.

The maximum likelihood principle view at the probability density $f_{Y_{-n+1},\dots,Y_{-1},Y_0}(y_{-n+1},\dots,y_{-1},y_0;\theta)$ as the probability of having observed this particular sample. The Maximum Likelihood Estimate of θ is the value for which this probability density is maximized.

The i.i.d. assumption allow us to write the joint density as:

$$f_{Y_{-n+1},\dots,Y_{-1},Y_0}\left(y_{-n+1},\dots,y_{-1},y_0;\theta\right) =$$

$$= f_{Y-n+1,\dots,Y_{-2},Y_{-1}}\left(y_{-n+1},\dots,y_{-2},y_{-1};\theta\right) f_{Y_0|Y_{-1}}\left(y_0|y_{-1};\theta\right) =$$
$$= f_{Y-n+1}\left(y_{-n+1};\theta\right) \prod_{t=0}^{-n+2} f_{Y_t|Y_{t-1}}\left(y_t|y_{t-1};\theta\right)$$

The log-likelihood function can be calculated as:

$$l\left(\theta\right) = \log f_{Y_{-n+1}}\left(y_{-n+1};\theta\right) + \sum_{t=0}^{-n+2} \log f_{Y_t|Y_{t-1}}\left(y_t|y_{t-1};\theta\right) \text{ and using 1.12 we obtain:}$$
$$l\left(\theta\right) = \log f_{Y_{-n+1}}\left(y_{-n+1};\theta\right) - \tfrac{n-1}{2}\log\left(2\pi\right) - \tfrac{1}{2}\sum_{t=0}^{-n+2}\log\left(\sigma_t^2\left(\rho\right)\right) - \tfrac{1}{2}\sum_{t=0}^{-n+2}\left(\tfrac{\varepsilon_t-\mu}{\sigma_t(\rho)}\right)^2$$

Note that as long as the sample path becomes as the contribution of the first term becomes negligible.

In conclusion the maximum likelihood estimate is:

$$\widehat{\theta} = \left(\widehat{\mu},\widehat{\rho}\right) = \arg\max_{\mu,\rho} l\left(\theta\right) =$$
$$= \arg\max_{\mu,\rho} \log f_{Y_{-n+1}}\left(y_{-n+1};\theta\right) - \tfrac{n-1}{2}\log\left(2\pi\right) - \tfrac{1}{2}\sum_{t=0}^{-n+2}\log\left(\sigma_t^2\left(\rho\right)\right) - \tfrac{1}{2}\sum_{t=0}^{-n+2}\left(\tfrac{\varepsilon_t-\mu}{\sigma_t(\rho)}\right)^2$$

where $f_{Y_{-n+1}}\left(y_{-n+1};\theta\right) = \frac{1}{\sqrt{2\pi}\sigma_{-n+1}}\exp\left(-\frac{\left(\sigma_{-n+1}z_{-n+1}-\mu\right)^2}{2\sigma_{-n+1}^2}\right)$

However in the GARCH case to compute the likelihood function we need the initial values for the variance and the shock term (i.e., σ_{-n+1} and z_{-n+1}). There exist different approaches to perform this initialization. A simple approach is to pose the initial variance to its unconditional value, while Bollerslev suggested (1986) to set the initial variance as:

$\sigma_{-n+1} = \sqrt{(n-1)^{-1}\sum_{t=0}^{-n+2}\left(y_t-\mu\right)^2}$. The shock term z_{-n+1} can be deduced from the price equation by knowing the sample value y_{-n+1}. Anyway both σ_{-n+1} and z_{-n+1} are function of the unknown parameters.

In our empirical work we set the variance to its unconditional expected value to perform the ML estimation. Because we use a very long historical sample we consider unlikely that the initial parameters setting can strongly affect the estimate. In the BAEM approach it is to note that the final values for z_0 and σ_0 will be used as starting values for the calibration procedure.

Although this procedure could appear to restrict the estimation at the normal error assumption this is no the case. Bollerslev and Wooldridge in 1992 suggested the way to adjust the standard errors to consider a possible misspecification of the error density and prove that under some regularity condition we can rely on the consistency of the estimate. So when the error is non-normal the estimate is called Pseudo Maximum Likelihood(PML) estimate. Let $\widehat{\theta}_T$ be the estimate that maximizes the Gaussian log likelihood and let θ be the true value that characterizes the GARCH model, then even if it is non-Gaussian under certain regularity conditions:

$$\sqrt{T}\left(\widehat{\theta}_T - \theta\right) \xrightarrow{d} N\left(0, D^{-1} S D^{\cdot 1}\right) \tag{1.13}$$

where $S = \underset{T\to\infty}{p\lim} T^{-1} \sum_{t=1}^{T} \left[s_t(\theta)\right]\left[s_t(\theta)\right]'$,where $s_t(\theta) = \frac{\partial \log f\left(y_t|\phi_{t-1};\theta\right)}{\partial\theta}$

and $D = \underset{T\to\infty}{p\lim} T^{-1} \sum_{t=1}^{T} -E\left\{\frac{\partial s_t(\theta)}{\partial\theta'}|\phi_{t-1}\right\}$

The standard errors robust to misspecification of the family of densities can be consistently obtained from the square root of diagonal elements of $T^{-1}\widehat{D}_T^{-1}\widehat{S}_T\widehat{D}_T^{-1}$ where $\widehat{D}$ and $\widehat{S}$ are computed with $\widehat{\theta}$ in place of the true parameter value.

1.3.2 Option-based setting calibration: the “risk-neutral” or pricing GARCH parameters

In order to approximate the risk neutral GARCH price dynamics it is used a calibration procedure in the BAEM approach.

The aim of this procedure is to determine the pricing GARCH parameters $\rho^* = (\omega^*, \alpha^*, \beta^*, \gamma^*)$ of the risk-neutral dynamics under the probability measure Q:

$y_{t+1} = \ln \frac{S_{t+1}}{S_t} = \mu^* + \sigma_{t+1}(\rho^*) z_{t+1}$ where $z_{t+1}|\phi_t \sim (0,1)$ under Q, where μ^* is chosen such that $E^Q\left(\frac{S_t}{S_{t-1}}|\phi_{t-1}\right) = e^{r-\delta}$ and the variance is described for each model as:

$\sigma^2_{t+1} = \omega^* + \beta^*\sigma^2_t + \alpha^*\sigma^2_t (z_t - \gamma^*)^2$ (*N-G*)

$\sigma^2_{t+1} = \omega^* + \beta^*\sigma^2_t + \alpha^*\sigma^2_t z^2_t - \gamma^*\sigma^2_t z^2_t I_t$ (*GJR-G*)

$\ln\left(\sigma^2_{t+1}\right) = \omega^* + \beta^* \ln\left(\sigma^2_t\right) + \alpha^*\left(|z_t| - \gamma^* z_t\right)$ (*E-G*)

The calibration procedure returns the pricing parameters (ρ^*) which minimize some error criteria computed on the difference between the theoretical price and the market option price of the cross-section, in particular the pricing GARCH parameters are deducted by the following optimization problem:

$\rho^* = \arg\min_{\rho} \sum_{i=1}^{N_t} \left(P^{GARCH}(K_i, T_i; \theta) - P^{MKT}(K_i, T_i)\right)^2$ where $\theta = (\mu^*, \rho)$ and $P^{MKT}(K_i, T_i)$ is the market price of the i-th option with strike price K_i, maturity T_i and N_t is the number of the options present in the cross-section at time t. $P^{GARCH}(K_i, T_i; \theta)$ is the theoretical GARCH price with parameters θ, strike price K_i and maturity T_i.

It is important to note that in the BAEM approach the change of measure from P to Q relies only on the GARCH parameters but not on the risk neutral distribution of the scaled innovation z. Indeed the distribution of z_t under the Q measure is left the same as the historical distribution, it is assured that the new parameters already provides an appropriate change of measure.

In order to perform the calibration and solve the optimization problem it is used the *fminsearch* Matlab function which implements the Nelder-Mead simplex direct search method and it does not require the computation of the gradients.

Also in the Calibration procedure, as in the Historical estimation, we need of initial guess for the conditional variance to perform the optimization. The starting value for the current risk neutral conditional volatility σ_0^* is the same as the historical conditional variance σ_0 computed as explained in the previous paragraph. This choice has its theoretical justification because when $\tau \to 0$ the SPD per unit probability $(M_{t,t+\tau}(s))$ tend to one and so the historical and the pricing estimate coincide.

A measure of the calibration quality is offered by computing:

$$APE = \frac{\sum_{i=1}^{N_t} \left| P^{GARCH}(K_i, T_i; \theta) - P^{MKT}(K_i, T_i) \right|}{\sum_{i=1}^{N_t} \left| P^{MKT}(K_i, T_i) \right|} \tag{1.14}$$

In our empirical application the theoretical American GARCH option price $P^{GARCH}(K_i, T_i; \theta)$ is computed by an approximation based on Monte Carlo simulation method described in next paragraphs.

It exists the theoretical explanation which excludes the possibility of use normal innovations to drive the GARCH dynamics in the BAEM approach. In other words the assumption of the existence of two different GARCH dynamics (the historical and the neutral) both driven by normal innovation to derive the SPD per unit probability are in conflicts[1].

But in the BAEM approach it is proposed to model the error by the Filtering Historical Simulation (FHS) method (introduced by Barone-Adesi, Bourgin, and Giannopoulos (1998)). However the Gaussian innovations could be used to give an interesting comparison.

To use FHS in modeling the GARCH innovations first it needs the historical scaled GARCH innovations estimate:

Let $\widehat{\theta} = (\widehat{\mu}, \widehat{\rho})$ the historical GARCH parameters estimated previously (i.e., PML estimation) on the historical log-returns series $y_{-n+1}, y_{-n+2}, ..., y_0$, then the scaled innovations are given by $\widehat{z}_t = \dfrac{y_t - \widehat{\mu}}{\sigma_t(\widehat{\rho})}$, for $t = -n+1, ..., 0$.

The filtered historical scaled innovation $\widehat{z}_t$ can be used to drive the GARCH simulation during the calibration procedure and this provides a non-parametric way to model the error.

1.3.3 Option pricing process identification.

The State Price Density per unit probability (i.e.,$M_{t,T}(s) := e^{-r(T-t)} \frac{q_{t,T}(s)}{p_{t,T}(s)}$) is derived in non-parametric manner by a simulation technique. The simulation is used because it is known that the distribution of temporally aggregated asset returns is not a simple task under GARCH dynamics.

To understand the procedure it is useful to consider the asset return behavior both under the historical measure and under the risk neutral measure.

Under the historical probability measure (P) the asset return dynamics of the

[1]In Barone-Adesi, G., R.F. Engle, and L. Mancini L., (2007, p.10) with reference to Duan (1995, Lemma A.1)

models presented before are described by:

$\ln\left(\frac{S_t}{S_{t-1}}\right) = \mu + \sigma_t z_t$ where $z_t|\phi_{t-1} \overset{P}{\sim} (0,1)$ and $\sigma_t = f(\sigma_{t-1}, z_{t-1}, \rho)$ where f depends on the model chosen.

While under the risk neutral probability measure (Q) the asset return dynamics of the models are:

$\ln\left(\frac{S_t}{S_{t-1}}\right) = \mu^* + \sigma_t z_t$ where $z_t|\phi_{t-1} \overset{Q}{\sim} (0,1)$ and $\sigma_t = f(\sigma_{t-1}, z_{t-1}, \rho^*)$.

The SPD per unit probability $M_{t,T}(s)$ is estimated by discounting the ratio of the historical and pricing densities. The historical density is estimated assuming that $E^P\left(S_t/S_{t-1}|\phi_{t-1}\right) = \exp\left(r + \frac{\lambda}{365}\right)$ (a risk premium of λ per year), this implies under the Gaussian innovation case that $\mu = r + \frac{\lambda}{365} - \frac{\sigma_t^2}{2}$. In the empirical work λ is fixed at 8% per year (i.e., $\lambda = 0.08$).

In the risk neutral density case it is assumed, as usual, that $E^Q\left(S_t/S_{t-1}|\phi_{t-1}\right) = e^{r-\delta}$, this implies that $\mu^* = r - \delta - \frac{\sigma_t^2}{2}$.

When the Filtering Historical Simulation is used to drive the innovation then we have to fix the drift parameters as:

$\mu = \left(r + \frac{\lambda}{365} - \zeta_t\right)$ $\quad \mu^* = (r - \delta - \zeta_t)$ where ζ_t is computed by the upgrade of an historical estimate ζ_0 computed as $\hat{\zeta}_0 = \log\left(\frac{1}{T}\sum_1^T \exp(\sigma_t z_t)\right)$ on the basis of the following relation:

$$E^Q\left(\frac{S_t}{S_{t-1}}|\phi_{t-1}\right) = e^{\mu^*} E^Q\left(\exp(\sigma_t z_t)|\phi_{t-1}\right) = e^{\mu^* + \zeta_t}.$$

The estimates for the risk-free rate and the dividend yield for the Japanese market on daily basis used in the empirical work are $r = 0,01931/365$ and $\delta = 0,005/365$;

The historical and the risk-neutral densities are estimated in non parametric manner by the Matlab function *ksdensity* with a Gaussian Kernel[2] and the optimal default bandwidth for estimating Gaussian Density.

The SPD per unity probability is numerically obtained by discounting the two densities as previously obtained.

1.4 Theoretical GARCH option prices methodologies:

In this section we present the Option pricing techniques underlying the work. First we present the solution to the problem of the perpetual option price in the Gaussian case as known in literature. Secondly we derive some justifications on the use of our "Monte Carlo American GARCH option price approximations" when the variance process is driven by a GARCH model. It is the case to anticipate that since the object of the empirical study are the long-term options then the perpetual option price could represent a first degree of approximation. We could remit to the empirical study to judge how much this approximation can be considered good or from the other hand side how much it affects the State Price Density estimation. In the second part of the section we present the Duan and Simonato's approximation method to price American GARCH option as a possible alternative approximation to compute the option price. Because this method is computational expensive especially in the case of the Components variance process models, where the state variables becomes three, we suggest some algorithmic shrewdness to increase the speed. In the last part we summarize the entire procedure Estimation-Calibration-Identification.

[2]see B.W. Silverman (1986), Density estimate for Statistics and Data Analysis

1.4.1 The Perpetual option price in the Gaussian case

The problem of the Perpetual Call option, under the classical assumption of a Brownian motion which drives the price dynamics, was solved by McKean H.P. in the 1965.

From then on the solution of this problem was rewritten in various ways, now we briefly draw one of this procedure useful to explain in the next paragraph the approximation solution in the GARCH setting case, what we call "Monte Carlo American GARCH option price approximations".

Let the asset price S_t be driven by the standard Gaussian dynamics, then under the risk neutral setting the price dynamics is given by the following stochastic differential equation:

$dS_t = (r - \delta) S_t dt + \sigma S_t dB_t$ where B_t is a standard Brownian motion, r is the risk-free rate, δ is the dividend-yield and σ is the standard deviation.

Using the Ito's lemma it is possible to derive the solution given the starting value for S_t at time $t = 0$:

The solution of this equation is

$$S_t = S_0 \exp\left(\sigma\left(\alpha t + B_t\right)\right) \quad \text{where} \quad \alpha = \frac{r - \delta - \frac{\sigma^2}{2}}{\sigma} \tag{1.15}$$

The Perpetual Call option price at time t is given by the solution to the following optimal stopping problem:

$$C\left(S_t\right) = \sup_{\tau} E^Q\left(\left(S_\tau - K\right)^+ \exp\left(-r\left(\tau - t\right)\right) | \phi_t\right) \tag{1.16}$$

This formulation expresses the call price as the expected value of the discounted payoff of the option exercised at time τ, where τ is a stopping time. Let T_H be an optimal stopping time to exercise the option (i.e., $T_H = \inf\{t \geq 0; S_t \geq H\}$) the problem can be reformulated as:

$$C(S_t) = \sup_H E^Q\left((S_{T_H} - K)^+ \exp(-r(T_H - t))\,|\phi_t\right) \text{ where } S_{T_H} = H.$$

In conclusion the call price at time $t = 0$ is expressed with:

$$C(S_0) = \sup_H (H - K)\, E^Q(\exp(-rT_H)) \tag{1.17}$$

In this form the problem of a Perpetual Call pricing can be interpreted as the so-called free boundary problem. In order words the maximization now is respect an optimal price level H while before it was with respect to an optimal time. In the Gaussian framework the perpetual option optimal barrier level is time-constant and this constitutes the advantage of this reformulation.

This kind of problem has an intrinsic characteristic: the solution (the call price) is function of the barrier (H) which has an unknown value. Although the value of H is unknown, the problem admits a close-form solution in the Gaussian case setting.

The problem is now to compute the expected value $E^Q(\exp(-rT_H))$ where $T_H = \inf\{t \geq 0; S_t \geq H\}$.

First note that at time $t = T_H$, by using 1.15 it results

$$S_{T_H} = S_0 \exp\left(\sigma\left(\alpha T_H + B_{T_H}\right)\right) = H \tag{1.18}$$

Or equivalently

$$\alpha T_H + B_{T_H} = \frac{1}{\sigma} \ln\left(\frac{H}{S_0}\right) \tag{1.19}$$

where the left hand side is sometime called Brownian motion with drift.

Now let consider the martingale property that $E^Q\left(\exp\left(-\frac{\lambda^2}{2}t+\lambda B_t\right)\right) = 1$, it can be rewritten as

$$E^Q\left(\exp\left(-rt+\sqrt{2r}B_t\right)\right) = 1 \text{ with } r = \frac{\lambda^2}{2} \tag{1.20}$$

The relation is valid for each value of t and also when t is a stochastic time then

$E^Q\left(\exp\left(-rT_H+\sqrt{2r}B_{T_H}\right)\right) = 1$, and by using 1.19 we can write:

$E^Q\left(\exp\left(-\left(r+\alpha\sqrt{2r}\right)T_H\right)\right) = \left(\frac{S_0}{H}\right)^{\frac{\sqrt{2r}}{\sigma}}$

When $\alpha = 0$ we immediately verify that

$$E^Q\left(\exp\left(-rT_H\right)\right) = \left(\frac{S_0}{H}\right)^{\frac{\sqrt{2r}}{\sigma}} \tag{1.21}$$

but because $\alpha \neq 0$ the Girsanov Theorem can be used to change the probability measure, toggle the drift to zero and compute the optimal stopping time expectation as we need. At this end we formulate a Radon-Nykodin derivative to change the probability measure: $\frac{dQ'}{dQ}|\phi_t = \exp\left(-\frac{\alpha^2}{2}t-\alpha B_t\right)$

$E^Q\left(\exp\left(-rT_H\right)\right) = E^{Q'}\left(\exp\left(-rT_H+\frac{\alpha^2}{2}T_h+\alpha B_{T_H}\right)\right) =$

$= E^{Q'}\left(\exp\left(-\left(r+\frac{\alpha^2}{2}\right)T_H\right)\right)\left(\frac{S_0}{H}\right)^{-\frac{\alpha}{\sigma}}$

By 1.21 we obtain expected value of the optimal stopping time in the perpetual call problem:

$$E^Q\left(\exp\left(-rT_H\right)\right) = \left(\frac{S_0}{H}\right)^{\theta} \quad \text{where } \theta = \frac{-\alpha+\sqrt{2r+\alpha^2}}{\sigma} \tag{1.22}$$

The perpetual call option problem is now given by:

$$C\left(S_0\right)=\sup_H\left(H-K\right)\left(\frac{S_0}{H}\right)^{\theta} \tag{1.23}$$

The first order conditions applied to 1.23 allow us to deduce the optimal exercise level H as:$H=K\dfrac{\theta}{\theta-1}$

In conclusion the call price is:

$C\left(S_0,K,r,\delta,\sigma\right)=S_0\left(\frac{S_0}{H}\right)^{\theta-1}-K\left(\frac{S_0}{H}\right)^{\theta}$ where $\alpha=\dfrac{r-\delta-\frac{\sigma^2}{2}}{\sigma}$,

$\theta=\dfrac{-\alpha+\sqrt{2r+\alpha^2}}{\sigma}$ and $\alpha=\dfrac{r-\delta-\frac{\sigma^2}{2}}{\sigma}$ as previously written.

In the Perpetual Put Option problem:

$$P\left(S_0,K,r,\delta,\sigma\right)=\sup_L\left(K-L\right)E^Q\left(-rT_L\right) \tag{1.24}$$

one obtains the following relations:

$P\left(S_0,K,r,\delta,\sigma\right)=K\left(\frac{S_0}{L}\right)^{\theta^P}-S_0\left(\frac{S_0}{L}\right)^{\theta^P-1}$ where $L=\frac{\theta^P}{\theta^P-1}K$ and $\theta^P=\frac{-\alpha-\sqrt{2r+\alpha^2}}{\sigma}$.

1.4.2 Monte Carlo American GARCH option price approximations

In this paragraph we describe the approximations used in order to compute the American GARCH price in our empirical work.

Let consider the asset return dynamics under the risk-neutral measure and when the innovation are conditionally Gaussian, i.e.,

$\ln\left(\frac{S_t}{S_{t-1}}\right)=\mu^*+\sigma_t z_t \qquad z_t|\phi_{t-1}\overset{Q}{\sim}N\left(0,1\right)$

In a GARCH framework we can write:

$S_t=S_0\exp\left(t\mu^*+\sum_{i=1}^{t}\sigma_i\left(\rho^*\right)z_i\right)$ where σ_i is a ϕ_{i-1}-measurable function dependent on the GARCH model chosen, and ρ^* is the GARCH risk neutral parameters set.

The problem of a Perpetual Put under a GARCH variance process remain unchanged as in eq 1.24.

The solution is not simple to compute but we can expect that changing volatility involves a non-horizontal optimal barrier to exercise the perpetual option.

So we propose some approximations to obtain the theoretical GARCH option price.

Let assume that it exists a constant horizontal barrier (L) which delimits the early exercise area from the "waiting to exercise" area, then the American Put option problem can be rewritten as:

$$P\left(S_0, K, T\right) = \sup_t E^Q\left(\xi_t e^{-rt}|\phi_0\right) = E_0^Q\left(e^{-r\tau}\xi_\tau|A\right) + E_0^Q\left(e^{-rT}\xi_T|\overline{A}\right)$$

where $\tau = \inf\{t \geq 0; S_t \leq L\}$, $\xi_t = (K - S_t)^+$, A is the set of the price trajectories which cross the horizontal barrier.

Note that in term of SPD approach the American Put price can be written as:

$$P\left(S_0, K, T\right) = \xi^* E_0^P\left(M_{0,\tau}|A\right) + E_0^P\left(\xi_T M_{0,T}|\overline{A}\right) \tag{1.25}$$

having used $\xi^* = \xi_\tau = (K - S_\tau)^+ = (K - L)$.

In order to obtain an horizontal barrier in the case of GARCH perpetual derivatives we first consider the Gaussian optimal barrier to exercise the option as computed in the previous paragraph.

This first approximation is derived by substituting the stationary variance level h^*(see 1.6, 1.7 and 1.8) in place of the variance σ^2 in the Gaussian optimal exercise level L.

This is probably a rude approximation but there are some reasons that invite us to start by this one:

1) This approximation will be used only to compute the barrier while the trajectories simulated with the Monte Carlo method are computed in the original GARCH framework.

2) The stationary variance level used in place of the conditional variance could be reasonable when we work with out-of-the-money derivatives and when they have a long maturity as it is in this case. In such cases the probability of an early exercise is very low in the first period of the life of the derivative, just when it is more probable that the Gaussian barrier is inaccurate. After that we could expect that the conditional variance tends to be more near the asymptotic variance and the approximation could results not too rude.

First approximation: $H_1 = K\dfrac{\theta^{Gauss}}{\theta^{Gauss}-1}$ where $\theta^{Gauss} = \dfrac{-\alpha^* - \sqrt{2r+h^*}}{\sqrt{h^*}}$ and $\alpha = \dfrac{r-\delta-\frac{h^*}{2}}{\sqrt{h^*}}$

The second approximation presented, in order to derive an horizontal barrier more accurate that the previous, is obtained by a parameter added to the barrier in this form:

$$H_2 = K\left(\frac{\left(\theta^{Gauss}\right)}{\left(\theta^{Gauss}\right)-1} + \eta\right) \tag{1.26}$$

where η is a real number such that the total value of the derivatives under calibration are maximized.

The Nikkei Put Warrants prices are computed as the sum of the corresponding European option price conditional on the non-optimal early exercise cases, plus the early exercise value for the remaining cases. The barrier has the end to estimate this early exercise premium.

In conclusion the American option price is computed following the next procedure:

Let $t = 0$ be the current time, S_0 be the current asset price, σ_0 and z_0 the initial values for the variance and the shock under the risk-neutral measure (which coincide to the values under the historical measure as explained previously)

- Asset price paths simulation under the pricing measure Q from current time 0 to T^{max} (the max time to maturity of the options in the cross section).

  - Start for $l = 1$
  - Start for $t = 1$
  - draw the innovation z_t from the historical distribution hypothesized (for example Gaussian distribution or FHS distribution)
  - update the conditional variance $\sigma_t^2 = f\left(\rho^*, \sigma_{t-1}, z_{t-1}\right)$ according to the GARCH model chosen, remember that the value of z_0 and σ_0 are given.
  - repeat for next t until $t = T^{max}$.
  - The l-th simulated price path at time t where $0 \leq t \leq T^{max}$ is $S_t^{(l)} = S_0 \exp\left(t\mu^* + \sum_{i=1}^{t} \sigma_i z_i\right)$
  - repeat the procedure for the next path l until $l = 10.000$.

- A Monte Carlo American Call price approximation can be estimated as $P\left(K,T\right) = \frac{1}{10000} \sum_{l=1}^{10000} \exp\left(-r\tau\left(l\right)\right) \left[\left(K - S_{\tau(l)}^{(l)}\right)^+\right]$ where $\tau\left(l\right) = \begin{cases} T & \text{if } T^H \text{ does not exist} \\ T^H & \text{otherwise} \end{cases}$,

  $T^H = \min\left\{t \geq 0; S_t^{(l)} \geq H\right\}$ and H is the optimal horizontal barrier to exercise.

If we define A as the set of the trajectories which cross the exercise barrier, the American Put price can be rewritten as:

$$P\left(K,T\right) = \frac{\#(A)}{10000} \left(\frac{1}{\#(A)} \sum_{l \in A} \left[\exp\left(-r\tau\left(l\right)\right) \left(K - S_{\tau(l)}^{(l)}\right)^+\right]\right) +$$

$$+\tfrac{10000-\#(A)}{10000}\left(\tfrac{\exp(-rT)}{10000-\#(A)}\sum_{l\notin A}\left[\left(K-S_T^{(l)}\right)^+\right]\right)$$

or equivalently

$$P\left(K,T\right)=p(K,T)+eep\left(K,T\right) \tag{1.27}$$

where $p(K,T)=\frac{\exp(-rT)}{10000}\sum_{l}\left[\left(S_T^{(l)}-K\right)^+\right]$ is the corresponding European put price while eep represents the early exercise premium and is defined by:

$$eep\left(K,T\right)=\tfrac{\#(A)}{10000}\left(\tfrac{1}{\#(A)}\sum_{l\in A}\left[\exp\left(-r\tau\left(l\right)\right)\left(K-S_{\tau(l)}^{(l)}\right)^+\right]+\right.$$
$$\left.-\tfrac{\exp(-rT)}{\#(A)}\sum_{l\in A}\left[\left(K-S_T^{(l)}\right)^+\right]\right).$$

As in the BAEM framework to reduce the variance of the Monte Carlo estimates we use the empirical martingale simulation method proposed by Duan and Simonato (1998). This method rescales the simulated price path $S_t^{(l)}$ such that the expected value of the underlying asset is equal to the forward price under the risk neutral measure, so done the simulated prices will have this theoretical asset pricing property reducing the estimation error.

1.4.3 American GARCH Option price by Duan and Simonato's Markov Chain approximation

The Duan and Simonato's GARCH approximation is based on the Markovian property of the GARCH process. The GARCH models used in this work and in general the GARCH(1,1) models can be represented as a bivariate Markovian system (i.e., the state of the process is uniquely represented by (S_t,σ_{t+1}^2), so the process is Markovian of the first order). This feature allows to approximate GARCH models by a discrete Markov chain.

Now we present the Markov chain approximation of a GARCH (1,1) process (Duan, Simonato 2001) adapted to the models of Section I, included when the process is driven by FHS innovations.

Let consider an underlying asset log-return modelled by the usual equation 1.1 or equivalently let consider $\ln S_t = \ln S_{t-1} + \mu + \sigma_t z_t$ where S_t denote the asset closing price at day t. Let Q be some probability measure (Objective or risk-neutral or other) and σ_t be the variance modelled by 1.3, 1.4 or 1.5. Let z_t a standardized random variable independently distributed with respect to the information up to time $t-1$, i.e., $z_t|\phi_{t-1} \overset{Q}{\sim} (0,1)$.

As previously stated when Q is the pricing measure (or risk neutral measure) μ is such that $E\left(S_t/S_{t-1}|\phi_{t-1}\right) = e^{r-\delta}$, in particular in the Gaussian innovation cases, when the z_t is distributed normally, by Ito's lemma we can conclude that $\mu = r - \delta - \frac{1}{2}\sigma_t^2$, where as usually r is the risk-free rate and δ is the dividend-yield, or in the FHS innovation cases $\mu = r - \delta - \zeta_t$ [3].

Following Duan and Simonato's suggestions, we form the partition of the states by using the log of adjusted prices and log variances for the two state variables considered. The adjusted prices are used to reduce the dimension of the transition matrix by a price conversion. The log values used are justified mainly for its better convergence behavior.

The adjusted price is computed by $S_t^* = e^{-\widetilde{\mu}t}S_t$ where $\widetilde{\mu} = r - \delta - h^*/2$ and h^* is the stationary variance, the pre-adjusted price can be easily recover later. The unconditional variance can be computed in the *N-G*, *GJR-G* and in the *E-G* with 1.6, 1.7 and 1.8 respectively.

Note that in term of log-adjusted prices the price dynamics becomes: $\ln\left(\frac{S_t^*}{S_{t-1}^*}\right) =$

[3] See 1.3.3

$\ln\left(\frac{S_t}{S_{t-1}}\right) - \widetilde{\mu} = \frac{1}{2}\left(h^* - \sigma_t^2\right) + \sigma_t z_t$.

The unconditional expectation of the continuously compounded return on the adjusted price is zero, since $E^Q\left(\sigma_t^2\right) = h^*$ and $E\left(\sigma_t z_t\right) = E\left(\sigma_t E\left(z_t|\phi_{t-1}\right)\right) = 0$.

Let p_t and q_t be the log of the adjusted price and the log of the variance respectively (i.e., $p_t = \ln\left(S_t^*\right)$ and $q_t = \ln\left(\sigma_t^2\right)$) then the models can be rewritten with:

$$p_t = p_{t-1} + \frac{1}{2}\left(h^* - e^{q_t}\right) + \sqrt{e^{q_t}} z_t \tag{1.28}$$

$$q_{t+1} = \ln\left(\omega + \beta e^{q_t} + \alpha e^{q_t}\left(z_t - \gamma\right)^2\right) \tag{1.29}$$

in the *N-G* case

$$q_{t+1} = \ln\left(\omega + \beta e^{q_t} + \alpha e^{q_t} z_t^2 - \gamma e^{q_t} z_t^2 I_t\right) \tag{1.30}$$

in the *GJR-G* case or

$$q_{t+1} = \omega + \beta q_t + \alpha\left(|z_t| - \gamma z_t\right) \tag{1.31}$$

in the *E-G* case.

In order to find a states partition to approximate the GARCH process to the option pricing end, Duan and Simonato propose:

- Log price partition:$[p_0 - I_p, p_0 + I_p]$, where I_p is determined by studying the conditional behavior of the logarithm of the adjusted asset price over the life of the option contract (i.e., $I_p = \delta_p\left(m\right)\sqrt{\sum_{t=1}^{T} E^Q\left(\sigma_t^2|\phi_0\right)}$). Let consider the log price at the end of the option life: $p_T = p_0 + \frac{1}{2}\sum_{t=1}^{T}\left(h^* - e^{q_t}\right) + \sum_{t=1}^{T}\sqrt{e^{q_t}} z_t$, it follows that: $E^Q\left(p_T|\phi_0\right) =$

p_0 and $Var^Q(p_T|\phi_0) = Var^Q\left(\sum_{t=1}^{T}\sigma_t z_t|\phi_0\right) = \sum_{t=1}^{T} E^Q\left(\sigma_t^2|\phi_0\right)$. An analytical formula of the conditional variance of the log price can be derived for many GARCH processes.

- Log variance partition: To form the partition we would study the conditional behavior of the logarithm of the variance $q_T = \ln\left(\sigma_T^2\right)$. From the GARCH process characteristics we know that there are two notable values of the variance: the initial variance, which the process starts from, and the unconditional variance, $h^* = E^Q\left(\sigma_t^2\right)$, to whom the process asymptotically is attracted. Both these values have to be considered in the variance partition, but the second has increasing importance in dependence with the maturity of the option. The partition center can be computed as: $q_1^* = \ln\left(\frac{\tau-\min(\tau,T)}{\tau}\sigma_1^2 + \frac{\min(t,T)}{\tau}h^*\right)$. The value of τ is a temporal index used to form the weights. As it increases as the relative weight of the unconditional variance respect to the initial variance increases. Then in the study of long-term derivatives τ has to be small. Anyway it is important to ensure that q_1 belongs to the partition. The log variance partition is $[q_1^* - I_q, q_1^* + I_q]$. In order to compute the width I_q of the partition it should be enough to study $Var^Q(q_T|\phi_0)$, but in *N-G* and *GJR-G* it could result analytically complex. We know by the Jensen inequality that $Var^Q(q_T|\phi_0) \leq \ln\left(Var^Q\left(\sigma_T^2|\phi_0\right)\right)$ so Duan proposes to use a width: $I_q = \ln\left(e^{q_1} + \delta_q(n)\sqrt{Var^Q\left(\sigma_T^2|\phi_0\right)}\right) - q_1$.

  Only in the *E-G* case we have to note that the log variance partition can be constructed directly by the E-GARCH equation, because it expresses the variance in logarithmic terms: $[q_1^* - I_q, q_1^* + I_q]$ where $q_1^* = \frac{\tau-\min(\tau,T)}{\tau}\ln\left(\sigma_1^2\right) + \frac{\min(t,T)}{\tau}\ln(h)^*$ and $I_q =$

$\delta_q(n)\sqrt{\sum_{t=1}^{T} Var^Q(q_t|\phi_0)} - q_1.$

Note that $Var^Q(q_t|\phi_{t-2}) = E^Q\left(\left(q_t - E^Q(q_t|\phi_{t-2})\right)^2|\phi_{t-2}\right) =$

$= \alpha^2 E^Q\left(|z_{t-1}| - \gamma z_{t-1} - \sqrt{2/\pi}\right)^2 = \alpha^2\left(1+\gamma^2-\frac{2}{\pi}\right).$

In conclusion in the *E-G* the sum of the conditional variance up to maturity is given by:

$$\sum_{t=1}^{T} Var^Q(q_t|\phi_0) = T\alpha^2\left(1+\gamma^2-\frac{2}{\pi}\right). \tag{1.32}$$

Duan and Simonato showed that $\delta_p(m) \overset{m\to\infty}{\to} \infty$ and $\frac{\delta_p(m)}{m} \overset{m\to\infty}{\to} 0$ are sufficient partition conditions for the approximating Markov chain to converge to its target GARCH process.

The logarithmic adjusted price partition and the logarithmic variance partition are equally divided in m and n odd parts respectively in order to determine the state of the bivariate process:

$$\overline{p}(i) = p_0 + \frac{2i-1-m}{m-1} I_p \tag{1.33}$$

and the corresponding cells are

$$C(i) = [c(i), c(i+1)) \tag{1.34}$$

for $i = 1, ..., m$, where $c(1) = -\infty$,

$$c(i) = \frac{\overline{p}(i-1) + \overline{p}(i)}{2} \tag{1.35}$$

for $i = 2, ..., m$ and $c(m+1) = +\infty$

$\overline{q}(j) = q_1^* + \frac{2j-1-n}{n-1} I_q$ and the corresponding cells are $D(j) = [d(j), d(j+1))$ for $j = 1, ..., n$, where $d(1) = -\infty$, $d(j) = \frac{\overline{q}(j-1)+\overline{q}(j)}{2}$ for $j = 2, ..., n$ and $d(n+1) = +\infty$.

The Markov transition probability from state (i, j) at time t to state (k, l) at time $t+1$ is defined as

$$\pi(i, j; k, l) = \Pr^P \{p_{t+1} \in C(k), q_{t+2} \in D(l) | p_t = \overline{p}(i), q_{t+1} = \overline{q}(j)\}$$

for $t = 0, ..., T-1$

It is typical in the GARCH(1,1) models that the variance at time $t+2$ is a deterministic function of the information set at time $t+1$. In particular in the models investigated we can write the variance as function of its lagged value, and two lagged prices, i.e., :$q_{t+2} = \Phi(q_{t+1}, p_{t+1}, p_t)$

First we recover z_{t+1} from the log price equation 1.28 written one time forward:

$z_{t+1} = \frac{p_{t+1}-p_t+\frac{1}{2}(e^{q_{t+1}}-h^*)}{\sqrt{e^{q_{t+1}}}}$ and substituting in the log variance equation we obtain:

$$q_{t+2} = \Phi^{N-G}(q_{t+1}, p_{t+1}, p_t) = \ln\left(\omega + \beta e^{q_{t+1}} + \alpha\left(p_{t+1} - p_t + \frac{1}{2}\left(e^{q_{t+1}} - h^*\right) - \gamma\sqrt{e^{q_{t+1}}}\right)^2\right)$$

(N-G)

$$q_{t+2} = \Phi^{GJR-G}(q_{t+1}, p_{t+1}, p_t) = \ln\left(\omega + \beta e^{q_{t+1}} + (\alpha + \gamma I_t)\left(p_{t+1} - p_t + \frac{1}{2}\left(e^{q_{t+1}} - h^*\right)\right)^2\right)$$

(GJR-G)

$$q_{t+2} = \Phi^{E-G}(q_{t+1}, p_{t+1}, p_t) = \omega + \beta e^{q_{t+1}} + \alpha\left(\left|\frac{p_{t+1}-p_t+\frac{1}{2}(e^{q_{t+1}}-h^*)}{\sqrt{e^{q_{t+1}}}}\right| - \gamma\frac{p_{t+1}-p_t+\frac{1}{2}(e^{q_{t+1}}-h^*)}{\sqrt{e^{q_{t+1}}}}\right)$$

(E-G)

This implies a first source of sparsity in the Markovian transition matrix: for each combination of (i, j, k) it exits only an index l where the transition probability can be non zero. Thus we can rewrite the Markov transition probability as:

$$\pi(i,j;k,l) = \begin{cases} \Pr^Q\{p_{t+1} \in C(k) \,|p_t = \overline{p}(i), q_{t+1} = \overline{q}(j)\} \\ \qquad\qquad\qquad \text{if } \Phi(\overline{q}(j), \overline{p}(k), \overline{p}(i)) \in D(l) \\ 0, \text{ otherwise} \end{cases}$$

The conditional probability can be computed with:

$$\pi(i,j;k,l) = \Pr^Q\{p_{t+1} \in C(k) \,|p_t = \overline{p}(i), q_{t+1} = \overline{q}(j)\} =$$

$$= \Pr^Q\{(c(k) \leq p_{t+1} < c(k+1)) \,|p_t = \overline{p}(i), q_{t+1} = \overline{q}(j)\} =$$

$$= \Pr^Q\left\{\left(c(k) \leq \overline{p}(i) + \tfrac{1}{2}\left(h^* - e^{\overline{q}(j)}\right) + \sqrt{e^{\overline{q}(j)}} z_{t+1} < c(k+1)\right)\right\}.$$

$$\pi(i,j;k,l) = \Pr^Q\left\{\frac{c(k) - \overline{p}(i) + \frac{1}{2}\left(e^{\overline{q}(j)} - h^*\right)}{\sqrt{e^{\overline{q}(j)}}} \leq z_{t+1} < \frac{c(k+1) - \overline{p}(i) + \frac{1}{2}\left(e^{\overline{q}(j)} - h^*\right)}{\sqrt{e^{\overline{q}(j)}}}\right\} \tag{1.36}$$

1.4.4 American GARCH option price via Markov Chain approximation

The American GARCH option price problem can be formulated via dynamic programming. Let $f(X,K)$ be the payoff function of a European-style option. Let $V(S_t, h_{t+1}, t)$ be the value of an American option at time t and let the underlying be determined by a bivariate Markovian system of order one by the state variables S_t and h_t, then the American option price can be expressed as a dynamic programming formulation:

$$V(S_t, h_{t+1}, t) = \max\left\{f(S_t, K), e^{-r} E^Q(V(S_{t+1}, h_{t+2}, t+1)\,|\phi_t)\right\} \tag{1.37}$$

where $V(S_T, h_{T+1}, T) = f(S_T, K)$.

From the fact that the GARCH model is a time-homogenous Markov Chain the problem can be restate by a Markov chain structure.

First we define Π the Markov transition probability matrix for the bivariate system with dimensions $mn \times mn$ where m and n are the total number of the states of the log price and of the log variance respectively.

The structure of the transition matrix proposed by Duan and Simonato is the following:

$$
\begin{array}{ccc}
 & & \begin{array}{ccccccccc} k & 1 & 2 & \cdots & m & \cdots & 1 & 2 & \cdots & m \\ l & 1 & 1 & \cdots & 1 & \cdots & n & n & \cdots & n \end{array} \\
\begin{array}{cc} i & j \\ 1 & 1 \\ 2 & 1 \\ \vdots & \vdots \\ m & 1 \\ \vdots & \vdots \\ 1 & n \\ 2 & n \\ \vdots & \vdots \\ m & n \end{array} & \Pi= &
\begin{pmatrix}
\pi(1,1;1,1) & \pi(1,1;2,1) & \cdots & \pi(1,1;m,1) & \cdots & \pi(1,1;1,n) & \pi(1,1;1,n) & \cdots & \pi(1,1;m,n) \\
\pi(2,1;1,1) & \pi(2,1;2,1) & \cdots & \pi(2,1;m,1) & \cdots & \pi(2,1;1,n) & \pi(1,1;1,n) & \cdots & \pi(2,1;m,n) \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
\pi(m,1;1,1) & \pi(m,1;2,1) & \cdots & \pi(m,1;m,1) & \cdots & \pi(m,1;1,n) & \pi(1,1;1,n) & \cdots & \pi(m,1;m,n) \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
\pi(1,n;1,1) & \pi(1,n;2,1) & \cdots & \pi(1,n;m,1) & \cdots & \pi(1,n;1,n) & \pi(1,1;1,n) & \cdots & \pi(1,n;m,n) \\
\pi(2,n;1,1) & \pi(2,n;2,1) & \cdots & \pi(2,n;m,1) & \cdots & \pi(2,n;1,n) & \pi(1,1;1,n) & \cdots & \pi(2,n;m,n) \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
\pi(m,n;1,1) & \pi(m,n;2,1) & \cdots & \pi(m,n;m,1) & \cdots & \pi(m,n;1,n) & \pi(1,1;1,n) & \cdots & \pi(m,n;m,n)
\end{pmatrix}
\end{array}
$$

where i and j are the characteristic indexes of the provenance state while l and k represent the achievable state.

Let $\bar{S}$ be the vector of the possible asset prices corresponding to the structure of the transition matrix (i.e., a $mn \times 1$ vector $\bar{S} = [\bar{s}(1), \bar{s}(2), ..., \bar{s}(m), ..., \bar{s}(1), \bar{s}(2), ..., \bar{s}(m)]$). Note that $\overline{S}$ contains m discretized price values repeated for n times (as many as the variance states number). Although the repetition of the asset price could appear in some sense redundant, note that the transition matrix is bi-dimensional even if the system is bivariate.

The option pricing problem can be rewritten by the Markov chain structure as: $\bar{V}(t) = \max[g(\bar{S}, K), e^{-r}\Pi\bar{V}(t+1)]$ with $\bar{V}(T) = g(\bar{S}, K)$ where the max operator works element-by-element and the $g(\cdot,\cdot)$ function represents the option payoff. In American or European Put option case $g(\bar{S}, K) = \max\left\{\left(K\mathbf{1} - \bar{S}\right), \mathbf{0}\right\}$ where $\mathbf{1}$ and $\mathbf{0}$ are $mn \times 1$ vectors of ones and zeros respectively.

Now the problem has to be rewritten to allow us to work with the log adjusted price in place of the simpler price.

If the transition probability matrix Π is computed with respect to the log price states, and in the payoff function is added a correction term to allow us to work with log prices but to recover the correct asset price, (i.e., $g\left(\bar{P}, K\right) = \max\{[K\mathbf{1} - e^{\left(r-\frac{h^*}{2}\right)t} \exp\left(\bar{P}\right)], \mathbf{0}\}$ then the option value can computed by:

$$\bar{V}\left(t\right) = \max[g\left(\bar{P}, K\right), e^{-r}\Pi\bar{V}\left(t+1\right)] \text{ with } \bar{V}\left(T\right) = g\left(\bar{P}, K\right)$$

The last step is to computed the approximated option price knowing that by solving the maximization problem we obtain an $mn \times 1$ dimensional vector $\bar{V}\left(0\right)$. The option prices computed are function of the current state situated at the center of the price partition. In order to obtain the correct option price the formula is:

$$C\left(S_0, h_1\right) = \frac{d\left(j+1\right) - \ln\left(h_1\right)}{d\left(j+1\right) - d\left(j\right)} v\left(j\right) + \frac{\ln\left(h_1\right) - d\left(j\right)}{d\left(j+1\right) - d\left(j\right)} v\left(j+1\right)$$

where j and $j+1$ locates two adjacent discretized logarithmic volatility values (i.e., $d\left(j\right)$ and $d\left(j+1\right)$) such that $d\left(j\right) \leq \ln\left(h_1\right) \leq d\left(j+1\right)$.

1.4.5 Parametric and non parametric Innovations in the transition matrix

The probability 1.36 can be computed when it is chosen a theoretical distribution of under Q (Gaussian or non-Gaussian), or if it is used other distributional specification, such as the Filtering Historical Simulation (FHS), to infer the transition probability.

In the Gaussian-case one obtains:

$$\pi\left(i, j; k, l\right) = N\left(\frac{c(k+1) - \overline{p}(i) + \frac{1}{2}\left(e^{\overline{q}(j)} - h^*\right)}{\sqrt{e^{\overline{q}(j)}}}\right) - N\left(\frac{c(k) - \overline{p}(i) + \frac{1}{2}\left(e^{\overline{q}(j)} - h^*\right)}{\sqrt{e^{\overline{q}(j)}}}\right)$$

where $N\left(\cdot\right)$ is the cumulative distribution function of a standard normal random variable.

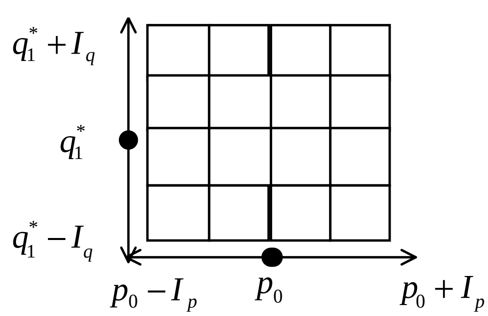
In the Filtering Historical Simulation (FHS) approach, by letting $\hat{z}_{-T+1}, ..., \hat{z}_0$ to be the filtered historical innovation series estimated to time $t = 0$ then the transition probability can be estimated by:

$$\pi\left(i,j;k,l\right)=\frac{1}{T}\left(\#\left(\widehat{z}_t|\widehat{z}_t<\frac{c(k+1)-\overline{p}(i)+\frac{1}{2}\left(e^{\overline{q}(j)}-h^*\right)}{\sqrt{e^{\overline{q}(j)}}}\right)+\right.$$
$$\left.-\#\left(\widehat{z}_t|\widehat{z}_t\leq\frac{c(k)-\overline{p}(i)+\frac{1}{2}\left(e^{\overline{q}(j)}-h^*\right)}{\sqrt{e^{\overline{q}(j)}}}\right)\right)$$

where $\#\left(A\right)$ indicates the cardinality of the set A.

1.4.6 Appendix: Computation of $E^Q\left(\sigma_t^2|\phi_0\right)$ and $Var^Q\left(\sigma_T^2|\phi_0\right)$ in the GARCH models:

The GARCH Markov Chain approximation of Duan and Simonato (1999) uses a first state variable partition for the log price and a second one for the log variance. In order to construct this two fixed partitions it is necessary to compute $E^Q\left(\sigma_t^2|\phi_0\right)$ and $Var^Q\left(\sigma_T^2|\phi_0\right)$, in particular we have to be able to compute: $I_p=\delta_p\left(m\right)\sqrt{\sum_{t=1}^{T}E^Q\left(\sigma_t^2|\phi_0\right)}$ and $I_q=\ln\left(e^{q_1}+\delta_q\left(n\right)\sqrt{Var\left(\sigma_T^2|\phi_0\right)}\right)-q_1$ which indicate the widths of the partitions form their centre.



Markovian states representation of the bi-variate system

We present the procedure suggested by Duan (1995) to determine the conditional expectation of the variance, and the conditional variance of the variance.

Let consider the volatility dynamics of the GARCH models as described in the equations 1.3, 1.4 or 1.5 each equation shows the relation between future volatility and its lagged value and one lagged shock.

We can rewrite these equations to give the variance at time t as function of all and only the lagged shocks up to time 0, given an initial value of the variance($\sigma_1 \in \phi_0$):

N-G:$\sigma_t^2 = \omega\left(1+\sum_{i=1}^{t-2}\prod_{j=1}^{i}\left(\beta+\alpha\left(z_{t-j}^2-\gamma\right)^2\right)\right)+\sigma_1^2\prod_{j=1}^{t-1}\left(\beta+\alpha\left(z_{t-j}^2-\gamma\right)^2\right)$

GJR-G:$\sigma_t^2 = \omega\left(1+\sum_{i=1}^{t-2}\prod_{j=1}^{i}\left(\beta+\alpha z_{t-j}^2+\gamma z_{t-j}^2 I_{t-j}\right)\right)+\sigma_1^2\prod_{j=1}^{t-1}\left(\beta+\alpha z_{t-j}^2+\gamma z_{t-j}^2 I_{t-j}\right)$

E-G:$\ln\left(\sigma_t^2\right) = \omega\left(1+\sum_{i=1}^{t-2}\beta^i\right)+\beta^t\ln\left(\sigma_1^2\right)+\alpha\sum_{i=1}^{t-1}\beta^{i-1}\left(|z_{t-i}|-\gamma z_{t-i}\right)$

Or equivalently:

N-G: $\sigma_t^2 = \omega\left(1+\sum_{i=1}^{t-2}G_i\right)+\sigma_1^2 G_{t-1}$ where $G_i = \prod_{j=1}^{i}\left(\beta+\alpha\left(z_{t-j}-\gamma\right)^2\right)$

GJR-G:$\sigma_t^2 = \omega\left(1+\sum_{i=1}^{t-2}H_i\right)+\sigma_1^2 H_{t-1}$ where $H_i = \prod_{j=1}^{i}\left(\beta+\alpha z_{t-j}^2+\gamma z_{t-j}^2 I_{t-j}\right)$

E-G: $\ln\left(\sigma_t^2\right) = \omega\left(\frac{\beta^{t-1}-1}{\beta-1}\right)+\beta^t\ln\left(\sigma_1^2\right)+\alpha\sum_{i=1}^{t-1}L_i$ where $L_i = \beta^{i-1}\left(|z_{t-i}|-\gamma z_{t-i}\right)$

Before we compute the conditional expected value of the variance at time t, we compute the conditional expected value of G_i,H_i and L_i and we consider that they are independent on z_{t-i+1}:

$$E^Q\left(G_i|\phi_0\right) = E^Q\left(G_{i-1}\left(\beta+\alpha\left(z_{t-i}-\gamma\right)^2\right)|\phi_0\right) =$$

$$= E^Q\left(G_{i-1}|\phi_0\right)E^Q\left(\left(\beta+\alpha\left(z_{t-i}-\gamma\right)^2\right)|\phi_0\right) = \left(\beta+\alpha E\left(z_t-\gamma\right)^2\right)^i = \left(v_G\right)^i$$

$$E^Q\left(H_i|\phi_0\right) = E^Q\left(H_{i-1}\left(\beta+\alpha z_{t-i}^2+\gamma z_{t-i}^2 I_{t-1}\right)|\phi_0\right) =$$

$$= E^Q\left(H_{i-1}|\phi_0\right)E^Q\left(\beta+\alpha E\left(z_{t-i}^2\right)+\gamma E\left(z_{t-i}^2 I_{t-i}\right)\right) =$$

$$= \left(\beta+\alpha E\left(z_t^2\right)+\gamma E\left(z_t^2 I_t\right)\right)^i = \left(v_H\right)^i$$

$$E^Q\left(L_i|\phi_0\right) = E^Q\left(\beta^{i-1}\left(|z_{t-i}|-\gamma z_{t-i}\right)|\phi_0\right) =$$

$$= \beta^{i-1}\left(E^Q\left(|z_t|\right)-\gamma E^Q\left(z_t\right)\right)$$

Note that in the Gaussian innovation case:

$v_G = \beta + \alpha\left(1+\gamma^2\right)$, $v_H = \beta + \alpha + \gamma/2$ and $\left(E^Q\left(|z_t|\right) - \gamma E^Q\left(z_t\right)\right) = 2/\sqrt{2\pi}$.

By these calculations we can write the conditional expected value of the variance, $E^Q\left(\sigma_t^2|\phi_0\right)$, in each model:

N-G $\quad :E^Q\left(\sigma_t^2|\phi_0\right) = \omega\left(\frac{1-v_G^{t-1}}{1-v_G}\right) + \sigma_1^2 v_G^{t-1}$

GJR-G $\quad :E^Q\left(\sigma_t^2|\phi_0\right) = \omega\left(\frac{1-v_H^{t-1}}{1-vH}\right) + \sigma_1^2 v_H^{t-1}$

In the E-G case we find immediately that:

$E^Q\left(\ln\left(\sigma_t^2|\phi_0\right)\right) = \left(\omega + \alpha\left(E^Q|z_t| - \gamma E^Q(z_t)\right)\right)\left(\frac{1-\beta^{t-1}}{1-\beta}\right) + \beta^t \ln\left(\sigma_1^2\right)$

But we need $E^Q\left(\sigma_t^2|\phi_0\right)$ to find an appropriate partition width, moreover by the Jensen inequality we know that:$E\left[\ln\sigma_t^2|\phi_0\right] \leq \ln E\left[\sigma_t^2|\phi_0\right]$

In the E-GARCH case in order to compute $E\left(\sigma_t^2|\phi_0\right)$ it is enough to note that:

$$\sigma_t^2 = \exp\left(\omega\right)\left(\sigma_{t-1}^2\right)^\beta \exp\left(\alpha\left(|z_{t-1}| - \gamma z_{t-1}\right)\right) \tag{1.38}$$

and this implies that

$\sigma_t^2 = \left[\exp\left(\omega\right)\right]^{\sum\limits_{i=0}^{t-2}\beta^i} \left(\sigma_1^2\right)^{\beta^t} \prod\limits_{i=1}^{t-1}\left\{\exp\left[\alpha\left(|z_{t-i}| - \gamma z_{t-i}\right)\right]\right\}^{\beta^{i-1}}$

$\sigma_t^2 = \exp\left(\omega\frac{1-\beta^{t-1}}{1-\beta}\right)\left(\sigma_1^2\right)^{\beta^t} \exp\sum\limits_{i=1}^{t-1}\left\{\alpha\beta^{i-1}\left(|z_{t-i}| - \gamma z_{t-i}\right)\right\}$

$E\left(\sigma_t^2|\phi_0\right) = \exp\left(\omega\frac{1-\beta^{t-1}}{1-\beta}\right)\left(\sigma_1^2\right)^{\beta^t} \prod\limits_{i=1}^{t-1} E\left(\exp\left\{\alpha\beta^{i-1}\left(|z_{t-i}| - \gamma z_{t-i}\right)\right\}|\phi_0\right)$

The expected value could be computed as function of cumulative normal distribution. However this procedure can be used but it could results computational burden considering that it require to call the cumulative normal distribution a number of times equal to the length of the sample and it computes only the width of the price state partition. While by applying the Jensen inequality to 1.38 we can be satisfied by using the following limitation in which the cumulative distribution is computed only twice:

$E\left[\sigma_t^2(\beta)|\phi_0\right] \leq E\left[\sigma_t^2(1)|\phi_0\right]$ with $0 < \beta < 1$

For $\beta = 1$ from 1.38 we get

$\sigma_t^2 = (\exp(\omega))^{t-1}\sigma_1^2 \prod_{i=1}^{t-1} \exp(\alpha(|z_{t-1}| - \gamma z_{t-1}))$

$E\left[\sigma_t^2(1)|\phi_0\right] = \sigma_1^2 \left\{\sqrt{2\pi}\exp\left(\omega + \frac{a^2}{2}\right)\left(\exp\left(\frac{b^2-a^2}{2}\right)(N(b)-1) - (N(a)-1)\right)\right\}^{t-1}$

where $a = \alpha(\gamma - 1)$, $b = \alpha(\gamma + 1)$ and $N(x)$ is the cumulative normal distribution (i.e., $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{x^2}{2}\right)dx$.)

However it is the case to anticipate that β represents in the models the coefficient of the variance persistence from a day to next day, and its empirical value is close to 1.

In order to determine the width of the price state partition (i.e., I_p)

It is now useful to compute $\sum_{i=1}^{T} E\left(\sigma_i^2|\phi_0\right)$

N-G:

$$\sum_{i=1}^{T} E\left(\sigma_i^2|\phi_0\right) = \omega\left(\frac{T}{1-v_G} - \frac{1-v_G^T}{(1-v_G)^2}\right) + \sigma_1^2\frac{1-v_G^T}{1-v_G} \tag{1.39}$$

GJR-G:

$$\sum_{i=1}^{T} E\left(\sigma_i^2|\phi_0\right) = \omega\left(\frac{T}{1-v_H} - \frac{1-v_H^T}{(1-v_H)^2}\right) + \sigma_1^2\frac{1-v_H^T}{1-v_H} \tag{1.40}$$

E-G:

$$\sum_{i=1}^{T} E\left(\sigma_i^2(\beta)|\phi_0\right) \leq \sum_{i=1}^{T} E\left(\sigma_i^2(1)|\phi_0\right) = \sigma_1^2\frac{1-v_L^T}{1-v_L} \tag{1.41}$$

with $v_L = \sqrt{2\pi}\exp\left(\omega + \frac{a^2}{2}\right)\left(\exp\left(\frac{b^2-a^2}{2}\right)(N(b)-1) - (N(a)-1)\right)$ and a, b and $N(\cdot)$ as previously stated.

The computation of $Var^Q\left(\sigma_T^2|\phi_0\right)$ is based on the general formula: $Var^Q\left(\sigma_T^2|\phi_0\right) = E^Q\left(\sigma_T^4|\phi_0\right) - \left[E^Q\left(\sigma_T^2|\phi_0\right)\right]^2$

Therefore we have to compute $E\left(\sigma_T^4|\phi_0\right)$ in each model, but first note that:

in N-G: $E\left(G_k^2|\phi_0\right) = E\left(G_{k-1}^2|\phi_0\right) E\left(\left(\beta + \alpha\left(z_{t-k} - \gamma\right)^2\right)^2 |\phi_0\right) =$

$= E\left(G_{k-1}^2|\phi_0\right) E\left(\beta^2 + 2\beta\alpha\left(1+\gamma^2\right) + \alpha^2\left(3 + 6\gamma^2 + \gamma^4\right)\right) =$

$= \left[\beta^2 + 2\beta\alpha\left(1+\gamma^2\right) + \alpha^2\left(3 + 6\gamma^2 + \gamma^4\right)\right]^k = u_G^k$ where $u_G = \beta^2 + 2\beta\alpha\left(1+\gamma^2\right) + \alpha^2\left(3 + 6\gamma^2 + \gamma^4\right)$

and for $k > j$ $E\left(G_k G_j|\phi_0\right) = E\left[G_j^2 \prod_{i=j}^{k}\left(\beta + \alpha\left(z_{t-2k+1} + \gamma\right)^2\right)|\phi_0\right] =$

$= E\left(G_j^2|\phi_0\right) E\left(\prod_{i=j}^{k}\left(\beta + \alpha\left(z_{t-2k+1} + \gamma\right)^2\right)|\phi_0\right) = u_G^i\left(v_G^i\right)$

We obtain then: $E\left[\left(\sigma_t^2\right)^2|\phi_0\right] = \omega^2 E\left\{\left(\sum_{i=0}^{t-2} G_i\right)^2 \middle| \phi_0\right\} +$

$+2\omega\sigma_1^2\left[\sum_{i=0}^{t-2} E\left(G_{t-1}G_i|\phi_0\right)\right] + \left(\sigma_1^2\right)^2 E\left(G_{t-1}^2|\phi_0\right) = ...$

$= \beta_0^2\left[\frac{u_G^{t-1}-1}{u_G - 1} + 2\frac{v_G}{u_G - v_G}\left(\frac{u_G^{t-1}-1}{u_G-1} - \frac{v_G^{t-1}-1}{v_G-1}\right)\right] +$

$+2\omega\sigma_1^2 v_G \frac{u_G^{t-1} - v_G^{t-1}}{u_G - v_G} + \sigma_1^4 u_G^{t-1}$

similarly in the GJR-G one obtains:

$E\left(H_k^2|\phi_0\right) = u_H^k$ where $u_H = \left(\beta^2 + 3\alpha^2 + \frac{3}{2}\gamma^2 + 2\alpha\beta + \alpha\gamma + 3\beta\gamma\right)$

and for $k > j$ $E\left(H_k H_j|\phi_0\right) = u_H^j\left(v_H^{k-j}\right)$

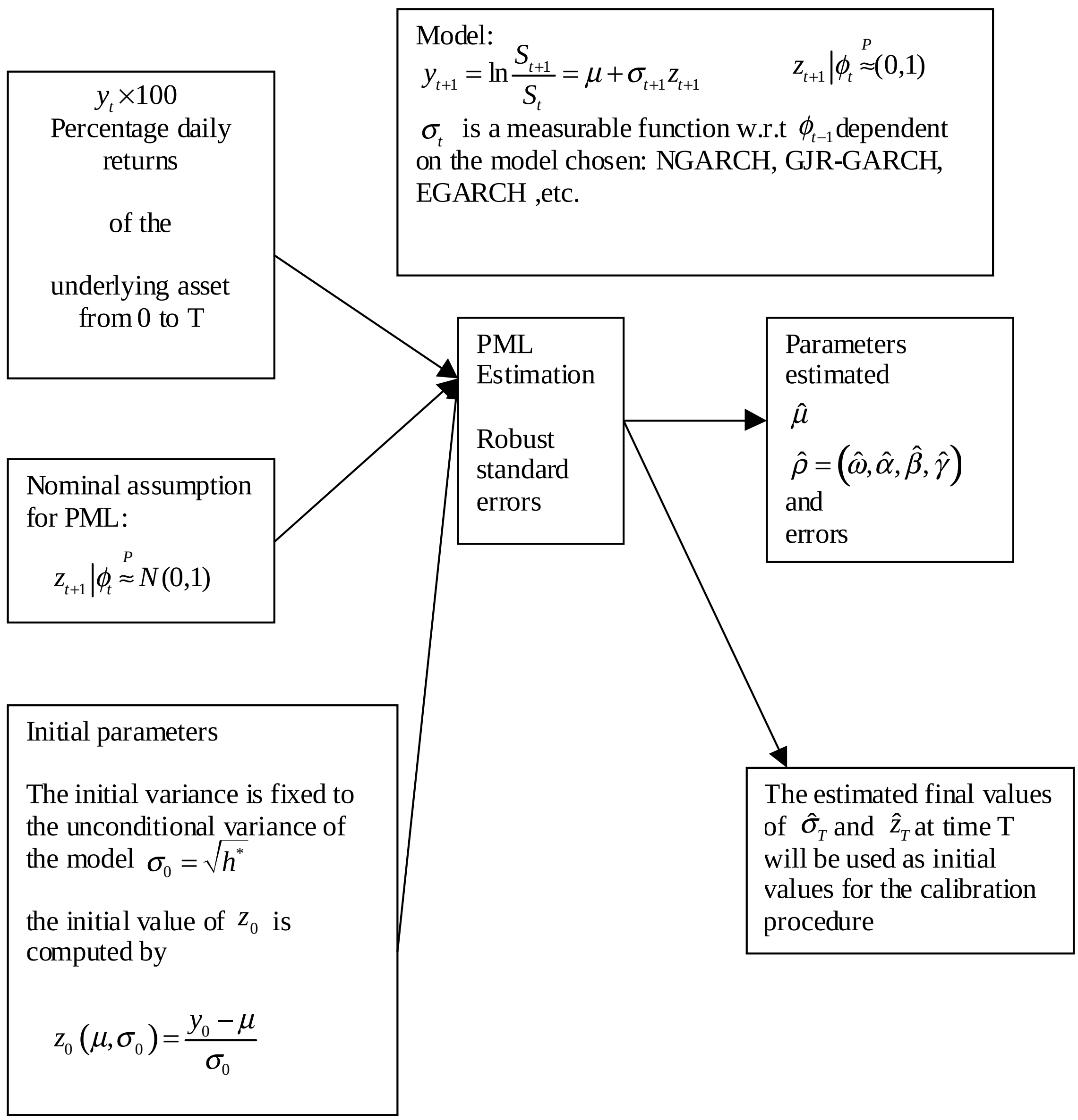
so $E\left[\left(\sigma_t^2\right)^2|\phi_0\right] = \beta_0^2\left[\frac{u_H^{t-1}-1}{u_H-1} + 2\frac{v_H}{u_H - v_H}\left(\frac{u_H^{t-1}-1}{u_H-1} - \frac{v_H^{t-1}-1}{v_H-1}\right)\right] +$

$+2\omega\sigma_1^2 v_H \frac{u_H^{t-1} - v_H^{t-1}}{u_H - v_H} + \sigma_1^4 u_H^{t-1}$

In the E-G case it is simpler to form the log variance partition than in the other GARCH models presented because the variance appears in the logarithmic form. The formula to compute the log variance partition was presented in 1.32

1.4.7 Summarizing scheme Estimation-Calibration:

Step 1: Parameters Estimation under historical measure P



Estimation procedure scheme

Step 2: GARCH Calibration ($P \Rightarrow Q$)

Initial values of σ_0^*, z_0^* are the final values estimated with the PML previously, used to simulate the risk neutral GARCH dynamics during the Calibration phase

$$\sigma_0^* = \hat{\sigma}_T,\ z_0^* = \hat{z}_T$$

Cross section of closing prices of an enough number of options on the same underlying asset at time T (excluding illiquid options to avoid microstructure effects misguide the results)
Cross section series: $K_i, T_i, P^{mkt}(K_i, T_i)$

Initial parameter values for the numerical minimization procedure are the parameters estimated previously, $\rho = \hat{\rho}$ but the drift μ (continuously compounded) is set to r - δ where r and δ are the market risk free rate and the dividend yield on daily base respectively.

Models to calibrate
In Gaussian innovation case

$$y_{t+1} = r - \delta - \frac{1}{2}\sigma_{t+1}^2 + \sigma_{t+1} z_{t+1}$$

In FHS innovation case

$$y_{t+1} = r - \delta - \zeta + \sigma_{t+1} z_{t+1}$$

σ_t is a measurable function w.r.t ϕ_{t-1} dependent on the model chosen: NGARCH, GJR-GARCH, EGARCH ,etc.

Minimization problem
The calibration is obtained minimizing the mean squared error between theoretical option prices and the market prices

$$\rho^* := \arg\min_{\rho} \sum_i \left(P^{theor}(K_i, T_i; \rho) - P^{mkt}(K_i, T_i) \right)^2$$

Risk neutral parameters resulting by the calibration:

$$\rho^* = (\omega^*, \alpha^*, \beta^*, \gamma^*)$$

Average absolute pricing error over the mean price used as measure of the quality of the calibration:

$$ape := \frac{\sum_i \left| P^{theor}(K_i, T_i; \rho^*) - P^{mkt}(K_i, T_i) \right|}{\sum_i P^{mkt}(K_i, T_i)}$$

Calibration procedure scheme

The theoretical price of the American option is computed by the approximations for long-term American GARCH option presented previously.

In a first analysis the innovation are assumed to be Gaussian, in a second analysis the innovation are modelled by the Filtering Historical Simulation approach (Barone-Adesi, Bourgin and Giannopoulos (1998))

1.5 Section II - The Component volatility dynamics GARCH model

1.5.1 Construction of a Component volatility model: the Long-run and Short-run components

In this section we construct a component version model for each model considered in Section I, as in Christoffersen P. Jacobs and Wang (2004), following the work of Engle and Lee (1999). The end of this extension is to model the variance dynamics in a characteristic way.

Let us reconsider the three GARCH models analyzed in the Section I:

N-G : (1.3) $\sigma_{t+1}^2 = \omega + \beta\sigma_t^2 + \alpha\sigma_t^2 (z_t - \gamma)^2$

GJR-G:(1.4) $\sigma_{t+1}^2 = \omega + \beta\sigma_t^2 + \alpha\sigma_t^2 z_t^2 - \gamma\sigma_t^2 z_t^2 I_t$ where $I_t = \begin{cases} 1; & z_t < 0 \\ 0; & \text{otherwise} \end{cases}$.

E-G :(1.5) $\ln\left(\sigma_{t+1}^2\right) = \omega + \beta\ln\left(\sigma_t^2\right) + \alpha\left(|z_t| - \gamma z_t\right)$

We rewrite the models by subtracting the unconditional expected value from both sides, we obtain the zero unconditional mean innovation form:

N-G: $\sigma_{t+1}^2 - h^* = \beta\left(\sigma_t^2 - h^*\right) + \alpha\left(\sigma_t^2 (z_t - \gamma)^2 - h^*\left(E(z_t - \gamma)^2\right)\right)$

GJR-G: $\sigma_{t+1}^2 - h^* = \beta\left(\sigma_t^2 - h^*\right) + \alpha\left(\sigma_t^2 z_t^2 - h^* E\left(z_t^2\right)\right) + \gamma\left(\sigma_t^2 z_t^2 I_t - h^* E\left(z_t^2 I_t\right)\right)$

E-G: $\ln\left(\sigma_{t+1}^2\right) - \ln(h^*) = \beta\left(\ln\sigma_t^2 - \ln h^*\right) + \alpha\left(|z_t| - \gamma z_t - E|z_t| + \gamma E(z_t)\right)$

In this form the variance process has the following scheme:

$$\sigma_{t+1}^2 = \underset{\text{constant term}}{h^*} + \underset{\text{speed of reversion}}{\beta} \left(\sigma_t^2 - \underset{\text{long-run mean}}{h^*}\right) + \underset{\text{zero-mean "variance-shock" term}}{\alpha v_t \left(\sigma_t, z_t, \grave{\gamma}\right)} \tag{1.42}$$

(N-G and GJR-G)

$$\ln \sigma_{t+1}^2 = \underset{\text{constant term}}{\ln h^*} + \underset{\text{speed of reversion}}{\beta} \left(\ln \sigma_t^2 - \underset{\text{long-run mean}}{\ln h^*}\right) + \underset{\text{zero-mean "variance-shock" term}}{\alpha v_t \left(z_t, \grave{\gamma}\right)} \tag{1.43}$$

(E-G)

The zero-mean "variance-shock" term is the variance innovation term which has zero unconditional mean.

In order to build the Component model version we rewrite the volatility dynamics for each model by substituting the constant unconditional mean of the conditional variance process with a time-varying component, denoted by Σ_t, and by renaming the parameters we obtain the short-run variance equation:

(N-G):$\sigma_{t+1}^2 = \Sigma_{t+1}^2 + \beta_1 \left(\sigma_t^2 - \Sigma_t^2\right) + \alpha_1 \left(\sigma_t^2 \left(z_t - \gamma_1\right)^2 - \Sigma_t^2 E \left(z_t - \gamma_1\right)^2\right)$

(GJR-G):$\sigma_{t+1}^2 = \Sigma_{t+1}^2 + \beta_1 \left(\sigma_t^2 - \Sigma_t^2\right) + \alpha_1 \left(\sigma_t^2 z_t^2 - \Sigma_t^2 E \left(z_t^2\right)\right) + \gamma_1 \left(\sigma_t^2 z_t^2 I_t - \Sigma_t^2 E \left(z_t^2 I_t\right)\right)$

(E-G):$\ln\left(\sigma_{t+1}^2\right) = \ln\left(\Sigma_t^2\right) + \beta_1 \left(\ln \sigma_t^2 - \ln \Sigma_t^2\right) + \alpha_1 \left(|z_t| - \gamma_1 z_t - E|z_t| + \gamma_1 E\left(z_t\right)\right)$

The short-run variance equation can be rearranged in the zero conditional innovation form, useful to compute the conditional expected value, by managing the expressions as indicate in the following.

In the N-G model, adding and subtracting $\alpha_1 \sigma_t^2 \left(E\left(z_t^2\right) - 2\gamma_1 E\left(z_t\right)\right)$ and grouping the factors of the term $\sigma_t^2 - \Sigma_t^2$ gives:

$$\sigma_{t+1}^2 = \Sigma_{t+1}^2 + \left(\beta_1 + \alpha_1 E\left(z_t - \gamma_1\right)^2\right)\left(\sigma_t^2 - \Sigma_t^2\right) + \alpha_1 \sigma_t^2 \left(z_t^2 - E\left(z_t^2\right) - 2\gamma_1\left(z_t - E\left(z_t\right)\right)\right) \tag{1.44}$$

similarly in *GJR-G* model, adding and subtracting: $\alpha_1 \sigma_t^2 E\left(z_t^2\right) + \gamma_1 \sigma_t^2 E\left(z_t^2 I_t\right)$ gives:

$$\sigma_{t+1}^2 = \Sigma_{t+1}^2 + \left(\beta_1 + \alpha_1 E\left(z_t^2\right) + \gamma_1 E\left(z_t^2 I_t\right)\right)\left(\sigma_t^2 - \Sigma_t^2\right) + \sigma_t^2\left(\alpha_1\left(z_t^2 - E\left(z_t^2\right)\right) + \gamma_1\left(z_t^2 I_t - E\left(z_t^2 I_t\right)\right)\right) \tag{1.45}$$

Or equivalently we can rewrite the short-run variance component equation for the N-GARCH model or for the GJR-GARCH model as:

Short-run variance equation (Component N-GARCH model (*cN-G*) or Component GJR-GARCH model (*cGJR-G*)):

$$\sigma_{t+1}^2 = \Sigma_{t+1}^2 + \tilde{\beta}_1\left(\sigma_t^2 - \Sigma_t^2\right) + \alpha_1 \sigma_t^2 v_{1,t} \tag{1.46}$$

where:

a) in the *cN-G*: $\tilde{\beta}_1 = \beta_1 + \alpha_1 E\left(z_t - \gamma_1\right)^2$ and $v_{1,t} = z_t^2 - E\left(z_t^2\right) - 2\gamma_2\left(z_t - E\left(z_t\right)\right)$

b) in the *cGJR-G*: $\tilde{\beta}_1 = \beta_1 + \alpha_1 E\left(z_t^2\right) + \gamma_1 E\left(z_t^2 I_t\right)$ and $v_{1,t} = \left(z_t^2 - E\left(z_t^2\right)\right) + \dfrac{\gamma_1}{\alpha_1}\left(z_t^2 I_t - E\left(z_t^2 I_t\right)\right)$.

The Short-run variance equation for the component E-GARCH model(*cE-G*) is expressed with:

$$\ln\left(\sigma_{t+1}^2\right) = \ln\left(\Sigma_{t+1}^2\right) + \tilde{\beta}_1 \ln\left(\sigma_t^2 / \Sigma_t^2\right) + \alpha_1 v_{1,t} \tag{1.47}$$

where $\tilde{\beta}_1 = \beta_1$ and $v_{1,t} = |z_t| - \gamma_1 z_t - E|z_t| + \gamma_1 E\left(z_t\right)$.

The expressions for the conditional variance can be considered as a generalization of the variance process, and following Engle and Lee (1999), the term Σ_{t+1}^2denotes the long-run component while $\sigma_t^2 - \Sigma_t^2$ (or $\ln \sigma_t^2 - \ln \Sigma_t^2$ in the *cE-G*) is the short-run component. The term $v_{1,t}$ represents the zero conditional mean innovation term in each model.

The component GARCH model is completed by specifying the long-run volatility component dynamics. Its functional form can be assumed as:

(*cN-G* and *cGJR-G*)

$$\Sigma_{t+1}^2 = \omega + \beta_2 \Sigma_t^2 + \alpha_2 \sigma_t^2 v_{2,t} \tag{1.48}$$

(*cE-G*)

$$\ln\left(\Sigma_{t+1}^2\right) = \omega + \beta_2 \ln\left(\Sigma_t^2\right) + \alpha_2 v_{2,t} \tag{1.49}$$

where

cN-G: $\quad v_{2,t} = z_t^2 - E\left(z_t^2\right) - 2\gamma_2\left(z_t - E\left(z_t\right)\right)$

cGJR-G: $\quad v_{2,t} = \left(z_t^2 - E\left(z_t^2\right)\right) + \dfrac{\gamma_2}{\alpha_2}\left(z_t^2 I_t - E\left(z_t^2 I_t\right)\right)$

cE-G : $\quad v_{2,t} = |z_t| - \gamma_2 z_t - E|z_t| + \gamma_2 E\left(z_t\right)$

The parameters of the models are (μ, ρ_1, ρ_2) where is $\rho_1 = (\alpha_1, \beta_1, \gamma_1)$ the parameter vector related to the short-run volatility component, and $\rho_2 = (\omega, \alpha_2, \beta_2, \gamma_2)$ is the parameter vector related to the long-run volatility component.

Note that the stationarity of the variance process in the Component models implies that the unconditional variances are:

$$H^* = E^Q\left(\Sigma_{t+1}^2\right) = \omega + \beta_2 E^Q\left(\Sigma_t^2\right) + \alpha_2 E^Q\left(\sigma_t^2 v_{2,t}\right) = \frac{\omega}{1-\beta_2} \tag{1.50}$$

moreover the unconditional expectation of the short-run component is:

$E^Q\left(\sigma_{t+1}^2 - \Sigma_{t+1}^2\right) = \tilde{\beta}_1 E^Q\left(\sigma_t^2 - \Sigma_t^2\right) + \alpha_1 E^Q\left(\sigma_t^2 v_{1,t}\right) = 0$, and this implies that also the stationary level of the variance is:

$$h^* = E^Q\left(\sigma_{t+1}^2\right) = E^Q\left(\sigma_{t+1}^2 - \Sigma_{t+1}^2\right) + E^Q\left(\Sigma_{t+1}^2\right) = H^* = \frac{\omega}{1-\beta_2} \tag{1.51}$$

To derive the expectation we have used that:

$E\left(\sigma_t^2 v_{i,t}\right) = E\left(\sigma_t^2 E\left(v_{i,t}|\phi_{t-1}\right)\right) = 0$ with $i = 1, 2$ since σ_t^2 is ϕ_{t-1}-measurable in any GARCH models.

The Component Models presented have the following characteristics:

- Parsimonious models, only 7 parameters to model the component volatility dynamics.
- Three state variables: price, short-run volatility component, long-run volatility component.
- The constant parameter ω represents the constant long-run variance component.
- Two shock parameter (α_1 and α_2) to model differently the shock in the short-run and in the long-run respectively.
- Two persistence parameters ($\widetilde{\beta}_1$ and β_2) to model the persistence of the short-run component and the long-run component respectively.

- Two asymmetric parameters (γ_1 and γ_2) to model the asymmetric variance response to the news in the short-run and in the long-run respectively.

One can note that Component model and its corresponding Simple version are nested. The Component model is reduces to the Simple version when α_2 and β_2 approaches to zero, or in words when the long-term variance component becomes constant.

The following scheme summarizes the three component models analyzed in the Section II:

Price equation	$y_{t+1} = \ln \frac{S_{t+1}}{S_t} = \mu + \sigma_{t+1} z_{t+1}$	$z_{t+1} \vert \phi_t \overset{P}{\sim} (0,1)$
	Model IV, Model V	
	cN-G, *cGJR-G*	
Short-run variance component equation	$\sigma_{t+1}^2 = \Sigma_{t+1}^2 + \tilde{\beta}_1 \left(\sigma_t^2 - \Sigma_t^2\right) + \alpha_1 \sigma_t^2 v_{1,t}$	1.46
Long-run variance component equation	$\Sigma_{t+1}^2 = \omega + \beta_2 \Sigma_t^2 + \alpha_2 \sigma_t^2 v_{2,t}$	1.48
	Model VI	
	cE-G	
Short-run variance component equation	$\ln\left(\sigma_{t+1}^2\right) = \ln\left(\Sigma_{t+1}^2\right) + \tilde{\beta}_1 \ln\left(\sigma_t^2 / \Sigma_t^2\right) + \alpha_1 v_{1,t}$	1.47
Long-run variance component equation	$\ln\left(\Sigma_{t+1}^2\right) = \omega + \beta_2 \ln\left(\Sigma_t^2\right) + \alpha_2 v_{2,t}$	1.49

(1.52)

where

cN-G	$v_{i,t} = z_t^2 - E\left(z_t^2\right) - 2\gamma_i\left(z_t - E\left(z_t\right)\right)$	$\tilde{\beta}_1 = \beta_1 + \alpha_1 E\left(z_t - \gamma_1\right)^2$
cGJR-G	$v_{i,t} = \left(z_t^2 - E\left(z_t^2\right)\right) + \frac{\gamma_i}{\alpha_i}\left(z_t^2 I_t - E\left(z_t^2 I_t\right)\right)$	$\tilde{\beta}_1 = \beta_1 + \alpha_1 E\left(z_t^2\right) + \gamma_1 E\left(z_t^2 I_t\right)$
cE-G	$v_{i,t} = \lvert z_t\rvert - \gamma_1 z_t - E\lvert z_t\rvert + \gamma_i E\left(z_t\right)$	$\tilde{\beta}_1 = \beta_1$

in particular in the Gaussian innovations case:

cN-G	$v_{i,t} = z_t^2 - 1 - 2\gamma_1 z_t$	$\tilde{\beta}_1 = \beta_1 + \alpha_1\left(1 + \gamma_1^2\right)$
cGJR-G	$v_{i,t} = \left(z_t^2 - 1\right) + \frac{\gamma_1}{\alpha_1}\left(z_t^2 I_t - 1/2\right)$	$\tilde{\beta}_1 = \beta_1 + \alpha_1 + \gamma_1/2$
cE-G	$v_{i,t} = \lvert z_t\rvert - \gamma_1 z_t - \sqrt{2/\pi}$	$\tilde{\beta}_1 = \beta_1$

and $i = 1, 2$.

1.5.2 Option pricing with the Component GARCH models

To price the long-term American options in the Component GARCH models of Section II we use the same approaches of the Section I.

Also in this section the Monte Carlo American GARCH option price approximation is effectively used in the empirical study while the Duan and Simonato's approximation is theoretical studied and adapted to operate with the Component Models.

Note on the Monte Carlo American GARCH option price approximation in the Component Models

The Monte Carlo American GARCH option price approximations, proposed in the Section I, remain unchanged in the option pricing procedure applied to the Component Models of the Section II. The trajectories are simulated correctly as indicate in the model dynamics equation (1.52). The optimal horizontal barriers are computed by the same procedure as described in 1.4.2

The Monte Carlo pricing approximation has a great advantage: it can be applied to a wide class of models without effort to adaptation. This is no true for the Duan and Simonato's approximation which requires some theoretical determinations as we see in the following paragraphs.

Duan and Simonato's American price approximation in the Component models

The construction of the Component model partitions The GARCH models used in PART II use three state variables to explain the asset price dynamics (the stock price, the short-run variance component and the long-run variance component) and uses only one lagged value for each of these state variables. This allows us to represent the GARCH models of PART II as a trivariate Markovian system (i.e., the state of the process is uniquely represented by $\left(S_t, \sigma_{t+1}^2 - \Sigma_{t+1}^2, \Sigma_{t+1}^2\right)$ (or $\left(S_t, \ln\left(\sigma_{t+1}^2/\Sigma_{t+1}^2\right), \ln\Sigma_{t+1}^2\right)$ in the *cE-G* case). Therefore the process is Markovian of the first order. Moreover conditional on the information to time t we can write the state of the price-variance-system of each component model to time $t+1$ as:

(cN-G,cGJR-G):

$S_t = f_1\left(S_{t-1}, \sigma_t^2 - \Sigma_t^2, \Sigma_t^2, z_t\right) = S_{t-1}\exp\left\{\mu + \left[\left(\sigma_t^2 - \Sigma_t^2\right) + \Sigma_t^2\right] z_t\right\}$

$\sigma_{t+1}^2 - \Sigma_{t+1}^2 = f_2\left(\sigma_t^2 - \Sigma_t^2, \Sigma_t^2, z_t\right) = \beta_1\left(\sigma_t^2 - \Sigma_t^2\right) + \alpha_1\left[\left(\sigma_t^2 - \Sigma_t^2\right) + \Sigma_t^2\right] v_{1,t}$

$\Sigma_{t+1}^2 = f_3\left(\sigma_t^2 - \Sigma_t^2, \Sigma_t^2, z_t\right) = \omega + \beta_2\Sigma_t^2 + \alpha_2\left[\left(\sigma_t^2 - \Sigma_t^2\right) + \Sigma_t^2\right] v_{2,t}$

(cE-G):

$S_t = f_1\left(S_{t-1}, \ln\left(\sigma_t^2/\Sigma_t^2\right), \ln\left(\Sigma_t^2\right), z_t\right) = S_{t-1}\exp\left\{\mu + \exp\left[\ln\left(\sigma_t^2/\Sigma_t^2\right) + \ln\left(\Sigma_t^2\right)\right] z_t\right\}$

$\ln\left(\sigma_{t+1}^2/\Sigma_{t+1}^2\right) = f_2\left(\ln\left(\sigma_t^2/\Sigma_t^2\right), z_t\right) = \beta_1\ln\left(\sigma_t^2/\Sigma_t^2\right) + \alpha_1 v_{1,t}$

$\ln\left(\Sigma_{t+1}^2\right) = f_3\left(\ln\left(\Sigma_t^2\right), z_t\right) = \omega + \beta_2\ln\left(\Sigma_t^2\right) + \alpha_2 v_{2,t}$

where $v_{i,t}$ are function of z_t and theirs functional form are reported in (1.52) for each model.

These GARCH models can be approximated by a discrete Markov chain following the Duan and Simonato's approach, but it needs some adjustments, and to simplify the exposition we consider only the *cN-G* and the *cGJR-G* cases, excluding the *cE-G* case.

In order to build the state partitions to approximate the GARCH process we study the conditional behavior of the logarithm of the adjusted asset price over the life of the option contract in the trivariate Markovian system $\left(S_t, \sigma_{t+1}^2 - \Sigma_{t+1}^2, \Sigma_{t+1}^2\right)$.

We consider as done for the bivariate case the log price partition: $[p_0 - I_p, p_0 + I_p]$ where $I_p = \delta\left(m\right)\sqrt{\sum_{t=1}^{T} E^Q\left(\sigma_t^2|\phi_0\right)}$

The sum of the conditional short-term variances at each time up to maturity (i.e.,$\sum_{t=1}^{T} E^Q\left(\sigma_t^2|\phi_0\right)$) can be determined by the following procedure:

First we compute $E\left(\Sigma_t^2|\phi_0\right)$. From the equation 1.48 we derive:

$$E\left(\Sigma_t^2|\phi_{t-2}\right) = \omega + \beta_2\Sigma_{t-1}^2 \text{ and so}$$
$$E\left(\Sigma_t^2|\phi_0\right) = \omega\left(\sum_{j=0}^{t-2}\beta_2^j\right) + \beta_2^{t-1}\Sigma_1^2 = \omega\left(\tfrac{1-\beta_2^{t-1}}{1-\beta_2}\right) + \beta_2^{t-1}\Sigma_1^2$$

Secondly we note that $E\left(\sigma_t^2 - \Sigma_t^2|\phi_{t-2}\right) = \tilde{\beta}_1\left(\sigma_{t-1}^2 - \Sigma_{t-1}^2\right)$, this implies that:

$$E^Q\left(\sigma_t^2 - \Sigma_t^2|\phi_0\right) = \widetilde{\beta}_1^{t-1}\left(\sigma_1^2 - \Sigma_1^2\right)$$

Now we can write the expected value of the short-run variance as:

(*cN-G*,*cGJR-G*)

$$E^Q\left(\sigma_t^2|\phi_0\right) = E^Q\left(\Sigma_t^2|\phi_0\right) + \widetilde{\beta}_1^{t-1}\left(\sigma_1^2 - \Sigma_1^2\right) = \omega\left(\frac{1-\beta_2^{t-1}}{1-\beta_2}\right) + \left(\beta_2^{t-1} - \widetilde{\beta}_1^{t-1}\right)\Sigma_1^2 + \widetilde{\beta}_1^{t-1}\sigma_1^2. \tag{1.53}$$

In conclusion:

$$\sum_{t=1}^{T} E^{Q}\left(\sigma_t^2|\phi_0\right) = \omega\left(\frac{T}{1-\beta_2} - \frac{1-\beta_2^T}{\left(1-\beta_2\right)^2}\right) + \left(\frac{1-\beta_2^T}{1-\beta_2} - \frac{1-\beta_1^T}{1-\beta_1}\right)\Sigma_1^2 + \frac{1-\beta_1^T}{1-\beta_1}\sigma_1^2. \quad (1.54)$$

It can be interesting to compare this equation 1.54 to the equations 1.39 or 1.40 obtained for the Simple GARCH models.

Note that the stationarity level of the variance process in the Component models are reported in 1.50, 1.51.

In order to find a centered value for the short-run component partition we compute a weighted value of the initial variances difference $q_1^* = \ln\left(\frac{\tau-\min(\tau,T)}{\tau}\left(\sigma_1^2 - \Sigma_1^2\right)\right)$. In the Component models the short-run component partition will be $[q_1^* - I_q, q_1^* + I_q]$. I_q can be computed (as in Duan and Simonato) by the formula:

$$I_q = \ln\left(\delta_q\left(n\right)\sqrt{Var^Q\left(\sigma_T^2 - \Sigma_T^2|\phi_0\right)}\right)$$

Similarly the long-run log variance partition is composed on the basis of the interval $[Q_1^* - I_Q, Q_1^* + I_Q]$,where $Q_1^* = \ln(\frac{\tau-\min(\tau,T)}{\tau}\Sigma_1^2 + \frac{\min(\tau,T)}{\tau}H^*)$,

$$I_Q = \ln(e^{Q_1} + \delta_Q\left(l\right)\sqrt{Var^Q\left(\Sigma_T^2|\phi_0\right)}) - Q_1 \text{ and } Q_1 = \ln\left(\Sigma_1^2\right).$$

$Var\left(\Sigma_t^2|\phi_{t-2}\right) = E\left[\left(\Sigma_t^2 - E\left(\Sigma_t^2|\phi_{t-2}\right)\right)^2|\phi_{t-2}\right] = \alpha_2^2\sigma_{t-1}^4\sigma_{v_2}$ where $\sigma_{v_2} = Var\left(v_{2,t}\right)$.

$Var\left(\sigma_t^2 - \Sigma_t^2|\phi_{t-2}\right) = Var\left(\alpha_1\sigma_{t-1}^2 v_{1,t-1}|\phi_{t-2}\right) = \alpha_1^2\sigma_{t-1}^4\sigma_{v_1}$ where $\sigma_{v_1} = Var\left(v_{1,t}\right)$.

Let for simplicity $\kappa_\sigma = E\left(\sigma_{t-1}^4|\phi_0\right)$ then

$Var\left(\Sigma_t^2|\phi_0\right) = \alpha_2^2\kappa_\sigma\sigma_{v_2}$ and $Var\left(\sigma_t^2 - \Sigma_t^2|\phi_0\right) = \alpha_1^2\kappa_\sigma\sigma_{v_1}$.

Appendix: computation of $\kappa_\sigma = E\left(\sigma_{t-1}^4|\phi_0\right)$ Now we show a possible path to compute κ_σ similarly at the procedure suggested in Duan (1995) and presented in paragraph 1.4.6.

First we write Σ_{t+1}^2 as function of the lagged values of σ_t^2 and $v_{2,t}$ by using the equation (1.48) but first subtracting the unconditional expected value:

$$\Sigma_{t+1}^2 - H^* = \beta_2^t \left(\Sigma_1^2 - H^*\right) + \alpha_2 \sum_{i=0}^{t-1} \beta_2^i \sigma_{t-i}^2 v_{2,t-i} \tag{1.55}$$

The same type of relation is obtained for $\sigma_{t+1}^2 - \Sigma_{t+1}^2$ by using the relation (1.46) :

$$\sigma_{t+1}^2 - \Sigma_{t+1}^2 = \widetilde{\beta}_1^t \left(\sigma_1^2 - \Sigma_1^2\right) + \alpha_1 \sum_{i=0}^{t-1} \beta_1^i \sigma_{t-i}^2 v_{1,t-i} \tag{1.56}$$

Now we substitute the relation 1.55 in 1.56 we obtain a relation which expresses σ_{t+1}^2 as function of lagged value of σ_t^2 and the lagged shocks $v_{1,t}, v_{2,t}$ up to time 1 given the initial variances values (Σ_1^2, σ_1^2)

$\sigma_{t+1}^2 = H^* + \widetilde{\beta}_1^t \left(\sigma_1^2 - \Sigma_1^2\right) + \beta_2^t \left(\Sigma_1^2 - H^*\right) + \sum_{i=0}^{t-1} \left(\alpha_1 \beta_1^i v_{1,t-i} + \alpha_2 \beta_2^i v_{2,t-i}\right) \sigma_{t-i}^2$

The last step consists in expressing σ_t^2 from the last recursive relation as function of all and only the lagged shocks given the initial variance values:

$\sigma_t^2 = H^* + \widetilde{\beta}_1^{t-1} \left(\sigma_1^2 - \Sigma_1^2\right) + \beta_2^{t-1} \left(\Sigma_1^2 - H^*\right) + \sum_{i=0}^{t-2} \left(\alpha_1 \beta_1^i v_{1,t-i} + \alpha_2 \beta_2^i v_{2,t-i}\right) \sigma_{t-i}^2$

Let for convenience $\widetilde{\omega} = H^* + \widetilde{\beta}_1^{t-1} \left(\sigma_1^2 - \Sigma_1^2\right) + \beta_2^{t-1} \left(\Sigma_1^2 - H^*\right)$ then we can rewrite the variance as:

$\sigma_t^2 = \widetilde{\omega} + \sum_{i=0}^{t-2} \left(\alpha_1 \beta_1^i v_{1,t-i} + \alpha_2 \beta_2^i v_{2,t-i}\right) \sigma_{t-i}^2$ or by cumulating the recursive computation we write:

$\sigma_t^2 = \widetilde{\omega} \left(1 + \sum_{i=1}^{t-2} \prod_{j=1}^{i} \left(\alpha_1 \beta_1^i v_{1,t-i} + \alpha_2 \beta_2^i v_{2,t-i}\right)\right) + \sigma_1^2 \prod_{j=0}^{t-1} \left(\alpha_1 \beta_1^i v_{1,t-i} + \alpha_2 \beta_2^i v_{2,t-i}\right)$

or equivalently

$\sigma_t^2 = \widetilde{\omega} \left(1 + \sum_{i=1}^{t-2} V_i\right) + \sigma_1^2 V_{t-1}$ where $V_i = \prod_{j=1}^{i} \left(\alpha_1 \beta_1^i v_{1,t-i} + \alpha_2 \beta_2^i v_{2,t-i}\right)$

With this last expression we have found the same formulation of the variance as in Duan (1995) (see paragraph 1.4.6), we can thus follow the same procedure to compute the variance of the variance as we would.

The transition probability matrix in the component GARCH model We derive in this paragraph the transition probabilities in the component GARCH model case. Also here to remove the trend in the price process, or in other words to limit the transition matrix dimension, we work with the adjusted price following the Duan and Simonato's construction.

The adjusted price is computed by $S_t^* = e^{-\widetilde{\mu}t} S_t$ where $\widetilde{\mu} = r - \delta - h^*/2$ and h^* is the variance stationary level as computed in 1.51.

Let p_t,q_t and Q_t be the log adjusted price, the log short-run variance component and the log long-run variance component respectively (i.e., $p_t = \ln\left(S_t^*\right)$, $q_t = \ln\left(\sigma_t^2 - \Sigma_t^2\right)$ and $Q_t = \ln\left(\Sigma_t^2\right)$) then the component GARCH models can be rewritten with:

$$p_t = p_{t-1} + \frac{1}{2}\left(h^* - e^{q_t} - e^{Q_t}\right) + \sqrt{e^{q_t} + e^{Q_t}} z_t \tag{1.57}$$

in the c*N-G, cGJR-G* cases

$$q_{t+1} = \ln\left(\widetilde{\beta}_1 e^{q_t} + \alpha_1\left(e^{q_t} + e^{Q_t}\right) v_{1,t}\right) \tag{1.58}$$

$$Q_{t+1} = \ln\left(\omega + \beta_2 e^{Q_t} + \alpha_2\left(e^{q_t} + e^{Q_t}\right) v_{2,t}\right) \tag{1.59}$$

in the c*E-G* case:

$$q_{t+1} = \widetilde{\beta}_1 q_t + \alpha_1 v_{1,t} \tag{1.60}$$

$$Q_{t+1} = \omega + \beta_2 Q_t + \alpha_2 v_{2,t} \tag{1.61}$$

Similarly to the bivariate system we assign the partition as it follows:

$\overline{p}(i) = p_0 + \frac{2i-1-m}{m-1} I_p$ and the corresponding cells are $C(i) = [c(i), c(i+1))$ for $i = 1, ..., m$, where $c(1) = -\infty$, $c(i) = \frac{\overline{p}(i-1) - \overline{p}(i)}{2}$ for $i = 2, ..., m$ and $c(m+1) = +\infty$

$\overline{q}(j) = q_1^* + \frac{2j-1-n}{n-1} I_q$ and the corresponding cells are $D(j) = [d(j), d(j+1))$ for $j = 1, ..., n$, where $d(1) = -\infty$, $d(j) = \frac{\overline{q}(j-1) - \overline{q}(j)}{2}$ for $j = 2, ..., n$ and $d(n+1) = +\infty$.

$\overline{Q}(r) = Q_1^* + \frac{2r-1-o}{o-1} I_Q$ and the corresponding cells are $E(r) = [e(r), e(r+1))$ for $r = 1, ..., o$, where $e(1) = -\infty$, $e(r) = \frac{\overline{Q}(r-1) - \overline{Q}(r)}{2}$ for $r = 2, ..., o$ and $e(o+1) = +\infty$.

The Markov transition probability from state (i, j, r) at time t to state (k, l, s) at time $t+1$ is defined as

$$\pi(i, j, r; k, l, s) = \Pr{}^P \{ p_{t+1} \in C(k), q_{t+2} \in D(l), Q_{t+2} \in E(s) | p_t = \overline{p}(i), q_{t+1} = \overline{q}(j), Q_{t+1} = \overline{Q}(r) \}$$

for $t = 0, ..., T-1$

In the Component GARCH models the variance components at time $t+2$ are deterministic functions of the information set at time $t+1$. In particular in the models investigated we can write the variance components as function of their lagged values, and two lagged prices, i.e.:

$q_{t+2} = \Phi(Q_{t+1}, q_{t+1}, p_{t+1}, p_t)$

$Q_{t+2} = \Theta(Q_{t+1,} q_{t+1}, p_{t+1}, p_t)$

First we recover z_{t+1} from the log price equation 1.57 written one time forward:

$z_{t+1} = \frac{p_{t+1} - p_t + \frac{1}{2}\left(e^{q_t} + e^{Q_t} - h^*\right)}{\sqrt{e^{q_{t+1}} + e^{Q_{t+1}}}}$ and substituting in the log variance equation in each model we obtain:

in *cN-G*:

$$q_{t+2} = \Phi^{cN-G}\left(Q_{t+1,}q_{t+1}, p_{t+1}, p_t\right) =$$

$$= \ln\{\widetilde{\beta}_1 e^{q_{t+1}} + \alpha_1\left(e^{q_{t+1}} + e^{Q_{t+1}}\right)[\frac{(p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right))^2}{e^{q_{t+1}}+e^{Q_{t+1}}}+$$

$$-1-2\gamma_1\frac{p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right)}{\sqrt{e^{q_{t+1}}+e^{Q_{t+1}}}}]\}$$

$$Q_{t+2} = \Theta^{cN-G}\left(Q_{t+1,}q_{t+1}, p_{t+1}, p_t\right) =$$

$$= \ln\{\omega + \beta_2 e^{Q_{t+1}} + \alpha_2\left(e^{q_{t+1}} + e^{Q_{t+1}}\right)[\frac{\left(p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right)\right)^2}{e^{q_{t+1}}+e^{Q_{t+1}}}+$$

$$-1-2\gamma_2\frac{p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right)}{\sqrt{e^{q_{t+1}}+e^{Q_{t+1}}}}]\}$$

in *cGJR-G*

$$q_{t+2} = \Phi^{cGJR-G}\left(Q_{t+1,}q_{t+1}, p_{t+1}, p_t\right) =$$

$$= \ln\{\widetilde{\beta}_1 e^{q_{t+1}} + \alpha_1\left(e^{q_{t+1}} + e^{Q_{t+1}}\right)\{[\frac{(p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right))^2}{e^{q_{t+1}}+e^{Q_{t+1}}}-1]+$$

$$+\frac{\gamma_1}{\alpha_1}[\frac{(p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right))^2}{e^{q_{t+1}+Q_{t+1}}}I_t - 1/2]\}$$

$$Q_{t+2} = \Theta^{cGJR-G}\left(Q_{t+1,}q_{t+1}, p_{t+1}, p_t\right) =$$

$$= \ln\{\omega + \beta_2 e^{Q_{t+1}} + \alpha_2\left(e^{q_{t+1}} + e^{Q_{t+1}}\right)\{[\frac{(p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right))^2}{e^{q_{t+1}}+e^{Q_{t+1}}}-1]+$$

$$-\frac{\gamma_2}{\alpha_2}[\frac{(p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right))^2}{e^{q_{t+1}}+e^{Q_{t+1}}}I_t - 1/2]\}$$

where

$$I_t = \begin{cases} 1; & p_{t+1} < p_t \\ 0; & \text{otherwise} \end{cases}.$$

In *cE-G* case

$$q_{t+2} = \Phi^{cE-G}\left(Q_{t+1,}q_{t+1}, p_{t+1}, p_t\right) =$$

$$= \widetilde{\beta}_1 q_{t+1} + \alpha_1\left[\left|\frac{p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right)}{\sqrt{\left(e^{q_{t+1}}+e^{Q_{t+1}}\right)}}\right| - \gamma_1\left(\frac{p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right)}{\sqrt{\left(e^{q_{t+1}}+e^{Q_{t+1}}\right)}}\right)\right]$$

$$Q_{t+2} = \Theta^{cE-G}\left(Q_{t+1,}q_{t+1}, p_{t+1}, p_t\right) =$$

$$= \omega + \beta_2 Q_{t+1} + \alpha_2\left[\left|\frac{p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right)}{\sqrt{\left(e^{q_{t+1}}+e^{Q_{t+1}}\right)}}\right| - \gamma_2\left(\frac{p_{t+1}-p_t+\frac{1}{2}\left(e^{q_{t+1}}+e^{Q_{t+1}}-h^*\right)}{\sqrt{\left(e^{q_{t+1}}+e^{Q_{t+1}}\right)}}\right)\right]$$

The measurability of the both components (q_t and Q_t) one time before implies an highly sparse Markovian transition matrix, because for each combination of (i,j,k,r) there exits only an index l and an index s where the transition probability can be non zero. Thus we can rewrite the Markov transition probability as:

$$\pi(i,j,r;k,l,s) = \begin{cases} \Pr^Q\left(p_{t+1} \in C(k) \,|p_t = \overline{p}(i), q_{t+1} = \overline{q}(j), Q_{t+1} = \overline{Q}(r)\right), \\ \qquad \text{if } \Phi\left(\overline{Q}(r), \overline{q}(j), \overline{p}(k), \overline{p}(i)\right) \in D(l) \\ \qquad \text{and } \Theta\left(\overline{Q}(r), \overline{q}(j), \overline{p}(k), \overline{p}(i)\right) \in E(s) \\ 0, \text{ otherwise} \end{cases}$$

The conditional probability can be computed with:

$$\Pr^Q\left\{p_{t+1} \in C(k) \,|p_t = \overline{p}(i), q_{t+1} = \overline{q}(j), Q_{t+1} = \overline{Q}(r)\right\} =$$

$$= \Pr^Q\left\{(c(k) \le p_{t+1} < c(k+1)) \,|p_t = \overline{p}(i), q_{t+1} = \overline{q}(j), Q_{t+1} = \overline{Q}(r)\right\} =$$

$$= \Pr^Q\left\{\left(c(k) \le \overline{p}(i) + \tfrac{1}{2}\left(h^* - e^{\overline{q}(j)} - e^{\overline{Q}(r)}\right) + \sqrt{e^{\overline{q}(j)} + e^{\overline{Q}(r)}}\, z_{t+1} < c(k+1)\right)\right\} =$$

$$= \Pr^Q\left\{\frac{c(k) - \overline{p}(i) + \frac{1}{2}\left(e^{\overline{q}(j)} + e^{\overline{Q}(r)} - h^*\right)}{\sqrt{e^{\overline{q}(j)} + e^{\overline{Q}(r)}}} \le z_{t+1} < \frac{c(k+1) - \overline{p}(i) + \frac{1}{2}\left(e^{\overline{q}(j)} + e^{\overline{Q}(r)} - h^*\right)}{\sqrt{e^{\overline{q}(j)} + e^{\overline{Q}(r)}}}\right\}$$

A computational improvement for the Duan and Simonato's Markovian approximation In this paragraph we present briefly a computational improvement usable in the Markovian approximation proposed by Duan and Simonato.

To do this we start speaking about the transition matrix features in the GARCH approximation useful to understand the improvement.

The sparsity feature of the transition matrix, when we exploit the Markovian property in describing a GARCH dynamics, means two things:

1. The sparsity feature can be exploit to accelerate the matrix multiplication in the option pricing procedure, as the authors suggest in their article.

2. The Markovian structure is a super-structure in comparison to the GARCH structure.

The fact that the GARCH model is much more than a Markovian system reflects not only in the depopulated transition matrix but also in the real information contained.

Let look to the transition probability formula (1.36) for the bivariate system, which we report here for convenience:

$$\pi\left(i,j;k,l\right)=\Pr^{Q}\left\{\frac{c(k)-\overline{p}(i)+\frac{1}{2}\left(e^{\overline{q}(j)}-h^{*}\right)}{\sqrt{e^{\overline{q}(j)}}}\leq z_{t+1}<\frac{c(k+1)-\overline{p}(i)+\frac{1}{2}\left(e^{\overline{q}(j)}-h^{*}\right)}{\sqrt{e^{\overline{q}(j)}}}\right\}$$

The sparsity feature, as explained by the authors, is manly related to the observation that the index l can be determined if i,j,k are known. In the GARCH models this comes from the measurability of the variance at time $t+1$ with respect to sigma algebra at time t. If we know the current price, the lagged price and the current variance (i.e.,i,j,k is known) the next-step variance is measurable (i.e., the index l is computable) This implies that for each triple of indexes i,j,k can exist only one probability not null in the matrix position indicated by the indexes i,j,k,l. The secondary sparsity is due to the typical distributional phenomenon that often events in the distribution tails have probability numerically null, since they are rare events.

Apart from the sparsity it is important to note that what real matters in the transition probability computation is the distance between the log price states determined by indexes i and k, not the individual value in the state, in other word the log-returns matter

This is equivalent to say that if we equally divide the log price state partition we obtain equal probabilities for each j when $i-k=\Delta$ with $-(m-1)<\Delta<m-1$ and $(i\notin\{1,m\})\wedge(k\notin\{1,m\})$(the operator $\wedge$ is the logical operator AND).

Let us consider the transition probabilities for a given transaction $(i,j;k,l)$, where we exclude the cases when i or k take values in $\{1,m\}$, as we show $\forall\Delta\colon\ (i+\Delta\notin\{1,m\})\vee(k+\Delta\notin\{1,m\})$ it results that

$$\pi\left(i+\Delta,j;k+\Delta,l\right)=\pi\left(i,j;k,l\right) \tag{1.62}$$

from the fact that :

$$\Pr{}^{Q}\left\{\frac{c(k+\Delta)-\overline{p}(i+\Delta)+\frac{1}{2}\left(e^{\overline{q}(j)}-h^{*}\right)}{\sqrt{e^{\overline{q}(j)}}}\leq z_{t+1}<\frac{c(k+\Delta+1)-\overline{p}(i+\Delta)+\frac{1}{2}\left(e^{\overline{q}(j)}-h^{*}\right)}{\sqrt{e^{\overline{q}(j)}}}\right\}=$$
$$=\Pr{}^{Q}\left\{\frac{c(k)-\overline{p}(i)+\frac{1}{2}\left(e^{\overline{q}(j)}-h^{*}\right)}{\sqrt{e^{\overline{q}(j)}}}\leq z_{t+1}<\frac{c(k+1)-\overline{p}(i)+\frac{1}{2}\left(e^{\overline{q}(j)}-h^{*}\right)}{\sqrt{e^{\overline{q}(j)}}}\right\}$$

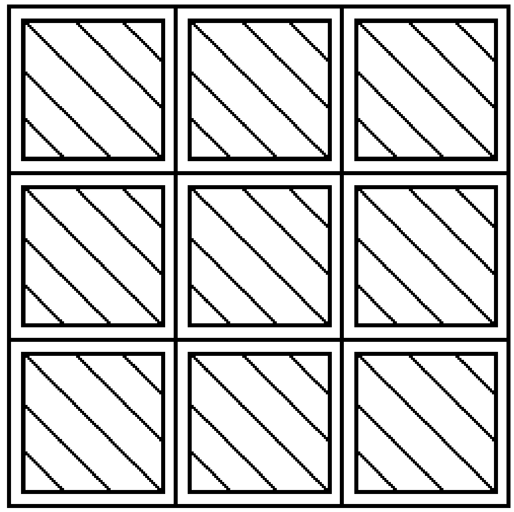
where we have used that:

$\overline{p}\left(i+\Delta\right)=\overline{p}\left(i\right)+2\frac{\Delta}{m-1}I_p$ that follows from 1.33

and $c\left(k+\Delta\right)=\frac{\overline{p}(k+\Delta+1)+\overline{p}(k+\Delta)}{2}=\frac{\overline{p}(k+1)+\overline{p}(k)}{2}+2\frac{\Delta}{m-1}I_p$ taken from 1.35.

Therefore $c\left(k+\Delta\right)-\overline{p}\left(i+\Delta\right)=c\left(k\right)-\overline{p}\left(i\right)$

In deed there are redundant probabilities in the matrix, and this can be represented using the Duan and Simonato's matrix by the following figure:



Redundance in the Duan and

Simonato's Matrix .

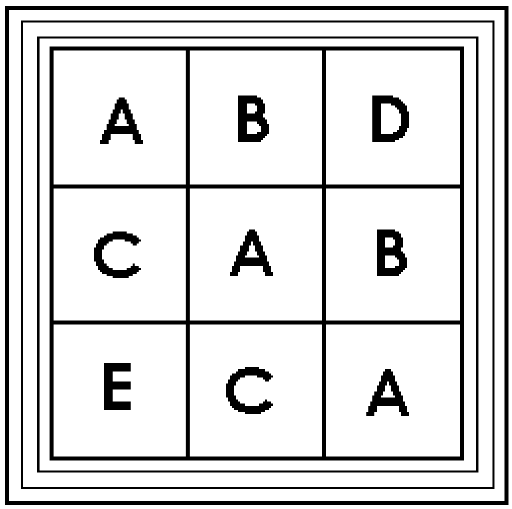
The figure represents a stylized transition matrix for a bivariate system composed by a nine block sub-matrixes . Each sub-matrix is composed by an external frame and an inner matrix. The external frames of each block represents the probabilities identified by the condition $(i \in \{1, m\}) \vee (k \in \{1, m\})$ ($\vee$ is the logical operator OR). These probabilities cannot considered redundant because are computed by approximating the distribution in the tail and so by using $-\infty$ or $+\infty$ in localizing the log price state, differently from what happen for the internal states.

The diagonal lines represents the redundant probabilities, in other word along the diagonal there are equal probabilities in each block.

To better appreciate the potential application of this redundancy in order to improve the computational method let rewrite the matrix in a different way.

We can note that Duan and Simonato's matrix structure is primarily ordered with respect to the variance state indexes (i.e., j, l) which identify the sub-matrix. After each sub-matrix is internally ordered with respect to prices states indexes (i.e.,i, k)

We invert the order (i.e., ordering primarily w.r.t. i and k and each block internally on j and l) and obtain the following result:



Markovian Transition matrix rearranged

The transition matrix rearranged has the following features:

1. The great frame containing all and only the probabilities in the tails which we know to be non-redundant.

2. The internal block matrixes are exactly repeated in diagonal, for the same reason given in.1.62.

Perhaps it is important to repeat that the matrix rearrangement has only the value to well-explain the structured matrix redundancy, all the suggestions could be applied direct to the Duan and Simonato's structure. The main difference between the two structures is that the Duan and Simonato's structure involves redundant elements while our rearrangement treats with redundant sub-matrixes.

On the base of these considerations we can draw the first obvious computational improvement.

We need compute only:

a) the probabilities where $(i \in \{1, m\}) \vee (k \in \{1, m\})$ which form the great frame.

b) the probabilities of non-redundant block (in the example D and E)

c) the probabilities of redundant blocks only one time (in the example A,B and C)

It is clear that by rearranging the matrix to correctly compute the discounted payoff we have to consider the vector of the possible asset prices corresponding to the new structure of the transition matrix as $\bar{S} = [\bar{s}(1), \bar{s}(1), ..., \bar{s}(1), \bar{s}(2), \bar{s}(2), ...\bar{s}(2), ...$

$..., \bar{s}(m), \bar{s}(m) ..., \bar{s}(m)]$ where each price is repeated by n time, in place of the old $\bar{S} = [\bar{s}(1), \bar{s}(2), ..., \bar{s}(m), ..., \bar{s}(1), \bar{s}(2), ..., \bar{s}(m)]$.

Precisely we have to consider the log adjusted prices $\overline{p}(i) = p_0 + \frac{2i-1-m}{m-1} I_p$. The partition is then:

$\overline{P} = [\overline{p}(1), \overline{p}(1), ..., \overline{p}(1), \overline{p}(2), \overline{p}(2), ..., \overline{p}(2), ..., \overline{p}(m), \overline{p}(m) ..., \overline{p}(m)]$ where each log adjusted price is repeated by n time.

It is at least the case to say that although we refer to a simple bivariate system all remains valid for a general multivariate Markovian GARCH approximation.

But this could not complete the computational improvement, because a potential improvement can be obtained by considering the matrix multiplication procedure involved during the option pricing procedure.

This could be possible by exploiting conjointly two features of the GARCH option in the Markovian approximation:

1) The well-structured redundancy in the Transition matrix

2) The linear representation of the payoff for successive states

(i.e.,if $g(i,K,t) = \left(K1_n - e^{\left(r-\frac{h^*}{2}\right)t} \exp\left(p_0 + \frac{2i-1-m}{m-1} I_p\right)\right)$ then

$g(i+1,K,t) = g(i,K,t) * \exp\left(\frac{2}{m-1} I_p\right) + K * (1 - \exp\left(\frac{2}{m-1} I_p\right)))$

We left the development of this further potential improvement to a future research. But we terminate this part by underlying the great importance to find an efficient and accurate procedure to price American GARCH Options.

Chapter 2

Empirical analysis

2.1 Study objective and methodology

In the next paragraph we present the data used in the estimation procedure and some graphical representations of the Nikkei 225 index. The PML estimation is applied on about fifty years of historical returns to better estimate the GARCH parameters.

Also in this Chapter there is a first section regarding the Simple GARCH models and a second section related to the Component GARCH models.

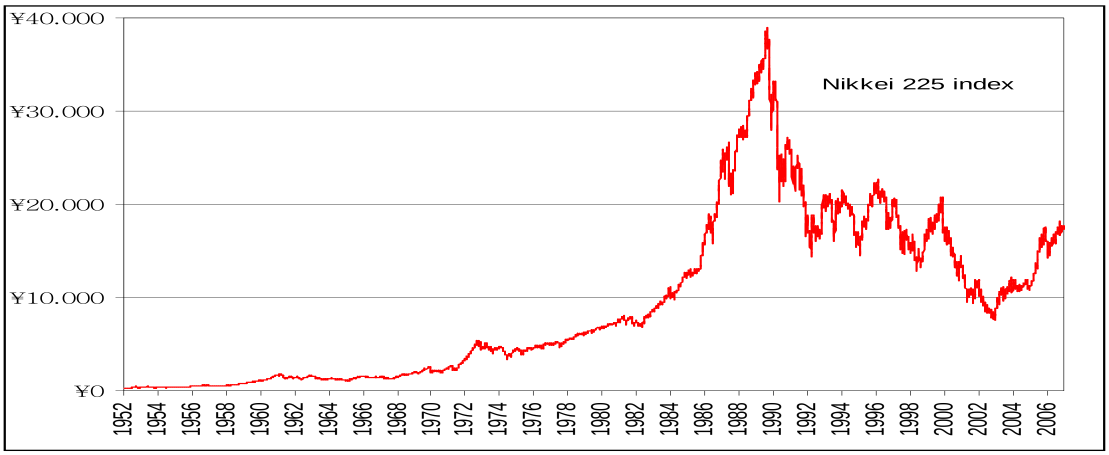
About the calibration results we give a preliminary comparison of all models analyzed in a random data (February 7, 2007). This analysis is performed by using both Gaussian innovations and FHS innovations.

As test of a solution stability we continue the analysis in the following days (February 8, 9 and so on).

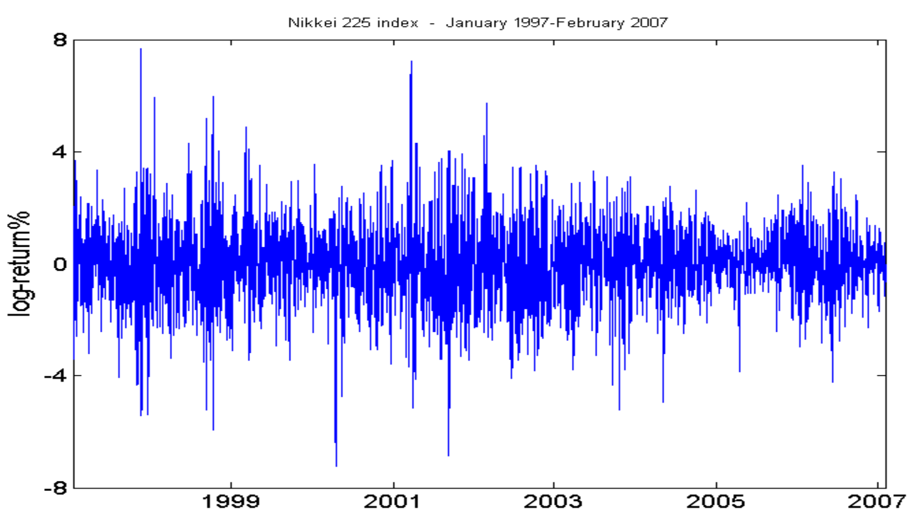
For each analysis we show the calibration results and the pricing error with respect strike price and maturity.

2.2 Data: Nikkei 225 index and Nikkei Put Warrants

In the next figures we give a graphical representation of the Nikkei 225 index historical series from 1952 and its log-returns during about the last ten years: from January, 15 1997 to February, 7 2007.



Nikkei 225 index from 1952 to 2007.



Daily log-return in percentage of Nikkei 225 index from January 15, 1997 to February 7, 2007.

The test of the models presented in this work is based on a Nikkei Put Warrants database provided by DataStream, all prices are expressed in Japanese Yen. This derivatives are offered in a large range of maturities and strike prices from different financial institutions, as the table 2.1 shows:

Nikkei Put Warrants		Time to Maturity(days)					
Moneyness		< 60	60-180	180-360	360-720	720-3000	>3000
	Price (¥)	1,63	15,51	62,87	192,65	384,67	1826,64
<0,85	Stand.Dev.	0,01	21,58	71,02	153,82	249,80	1914,67
	Observations	47	397	1064	1957	345	68
	Price (¥)	89,90	291,15	542,98	915,07	1351,84	4309,13
0,85-1	Stand.Dev.	101,34	197,64	248,52	308,75	366,71	15482,10
	Observations	34	272	693	1216	244	662
	Price (¥)	987,76	1243,17	1679,66	2119,96	2588,68	4089,18
1-1,15	Stand.Dev.	688,60	518,73	499,36	486,92	639,80	8013,08
	Observations	35	157	459	1017	161	1132
	Price (¥)	3826,28	3702,88	4005,52	4281,99	4701,23	11749,90
>1,15	Stand.Dev.	546,62	521,11	899,18	1004,71	908,07	15411,19
	Observations	9	33	215	697	209	567

Figure 2.1: The Table shows mean, standard deviation and number of observations for Nikkei Put Warrants grouped by moneyness/maturity category each Wednesday from July 6, 2005 to May 16, 2007. Time to Maturity range 15-16094 days. Strike price range 9000¥-25000¥. Asset price range 11603,53¥-17913,21¥.

The parameters calibration is avoided to be driven by microstructure effects in illiquid options by the following contrivances:

- Only Out-of-the-money derivatives with moneyness less than 0,98 are considered in the calibration.(i.e., moneyness is computed as current asset price over option strike price)

- Only options, whose prices are greater than ¥10, are included in the analysis.

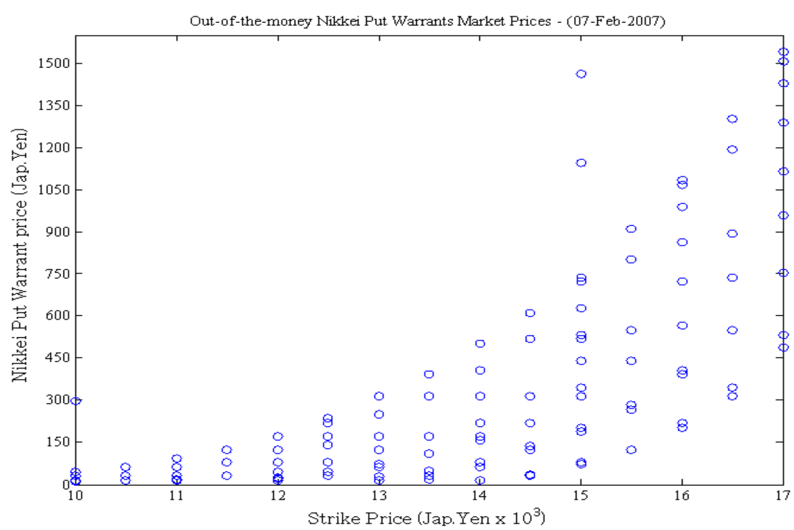
To orient the analysis on the long-term options we discard the NPWs with maturity less than 60 days. Moreover to reduce the computational burden we consider only the long-term options with maturity less than 3000 days. The thick rectangle in the figure 2.1

indicates approximately the area of the Nikkei Put Warrants subjected to the analysis.

The next table summarizes an example of a cross-section effectively used in a calibration exercise on February 7, 2007.

Nikkei Put Warrants		Time to Maturity(days)				
Moneyness		60-180	180-360	360-720	720-3000	Tot
	Price (¥)	22,93	74,52	177,25	298,40	
<0,85	Stand.Dev.	9,59	59,87	157,75		
	Observations	5	20	37	1	63
	Price (¥)	250,65	486,86	947,30	1303,52	
0,85-0,98	Stand.Dev.	164,91	219,94	331,86	222,10	
	Observations	10	17	22	2	51
						114

The Table shows mean, standard deviation and number of observations for the 114 Out-of-the-money Nikkei Put Warrants grouped by moneyness/maturity category used in the calibration procedure for Wednesday Febraury 7, 2007. Time to Maturity 143, 235, 326, 417, 509, 601, 692, 1057, 1422 days.Strike prices range: from ¥10.000 to ¥17.000 with increments of ¥500. ($S_0 =$ ¥$17.292, 32$)



Out-of-the-money Nikkei Put Warrants used in the calibration procedure for Wednesday Febraury 7, 2007. Time to Maturity 143, 235, 326, 417, 509, 601, 692, 1057, 1422 days.($S_0 =$ ¥$17.292, 32$)

2.3 Section I - Simple volatility dynamics models

2.3.1 GARCH Model PML Estimation results

The historical daily log-returns y_t of the Nikkei 225 index we consider start from December 31, 1957 to February 7, 2007. Model I,II,III (1.3, 1.4, 1.5) are estimated using Pseudo Maximum Likelihood (PML) estimator. The PML estimation is repeated for the next days. The PML estimation results for the models are reported in the following table:

Time series	Nikkei 225											
PML Estimation results												
Range data	<31-Dec-1957..07-Feb-2007>						<31-Dec-1957..08-Feb-2007>					
	NGarch		$Garch_{GJR}$		EGarch		NGarch		$Garch_{GJR}$		EGarch	
parameter	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error
μx10^2	3,99E-02	2,62E-04	4,60E-02	1,91E-04	4,50E-02	1,23E-06	3,99E-02	5,57E-04	4,60E-02	3,77E-02	4,47E-02	2,64E-04
ωx10^4	2,46E-02	2,69E-04	2,22E-02	1,99E-04	-4,28E-01	9,79E-06	2,46E-02	1,78E-04	2,22E-02	1,01E-01	-4,47E-01	1,27E-02
α	1,29E-01	2,08E-03	6,03E-02	2,00E-04	2,16E-01	5,66E-04	1,29E-01	9,93E-03	6,03E-02	1,05E-01	2,20E-01	2,18E-01
β	8,26E-01	2,61E-03	8,61E-01	6,00E-04	9,71E-01	1,71E-05	8,26E-01	9,20E-03	8,61E-01	2,81E-01	9,70E-01	2,76E-03
γ	5,25E-01	1,17E-06	1,37E-01	1,02E-05	4,09E-01	5,42E-04	5,25E-01	6,47E-06	1,37E-01	3,52E-04	4,11E-01	2,15E-01
Persistence	0,9899		0,9896		0,9713		0,9899		0,9896		0,9696	
Final values estimated												
h_0x10^4	0,6179		0,6269		0,6731		0,5611		0,5620		0,5920	
z_0	-0,8907		-0,8919		-0,8595		-0,0521		-0,0601		-0,0570	
LogLKLHD	41735		41687		41698		41739		41691		41702	
Range data	<31-Dec-1957..09-Feb-2007>						<31-Dec-1957..12-Feb-2007>					
	NGarch		$Garch_{GJR}$		EGarch		NGarch		$Garch_{GJR}$		EGarch	
parameter	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error
μx10^2	4,01E-02	1,01E-03	4,61E-02	2,45E-05	4,53E-02	1,87E-09	4,00E-02	1,59E-02	4,61E-02	2,36E-03	4,51E-02	7,21E-05
ωx10^4	2,46E-02	7,31E-04	2,22E-02	4,89E-05	-4,25E-01	7,49E-04	2,47E-02	2,11E-01	2,22E-02	1,18E-02	-4,23E-01	4,90E-05
α	1,29E-01	1,00E-02	6,03E-02	2,87E-04	2,16E-01	1,38E-05	1,29E-01	1,00E-03	6,03E-02	3,34E-04	2,15E-01	1,49E-06
β	8,26E-01	9,75E-03	8,61E-01	3,17E-04	9,72E-01	3,08E-06	8,26E-01	6,34E-04	8,61E-01	4,74E-04	9,72E-01	3,97E-06
γ	5,25E-01	1,95E-06	1,37E-01	5,21E-06	4,08E-01	1,17E-05	5,25E-01	9,56E-07	1,37E-01	2,43E-06	4,08E-01	4,26E-06
Persistence	0,9899		0,9896		0,9716		0,9899		0,9896		0,9718	
Final values estimated												
h_0x10^4	0,5672		0,5889		0,6155		0,5175		0,5293		0,5392	
z_0	1,5636		1,5267		1,4943		-0,0557		-0,0634		-0,0614	
LogLKLHD	41742		41694		41705		41745		41697		41709	

Daily returns from December 31, 1957 to February 7, 2007 on Nikkei 225 index are used to estimate the three simple GARCH models using Psuedo Maximum Likelihood. Robust standard errors are computed using numerical derivative at the optimum parameter value. h_0 and z_0 are the current estimates of the variance and the shock. LogLKLHD is the logarithm of the likelihood at the optimal parameter values. Persistence refers to the persistence of the conditional variance in each model. Note: in the E-G case ω is not multiplied by 10^4.

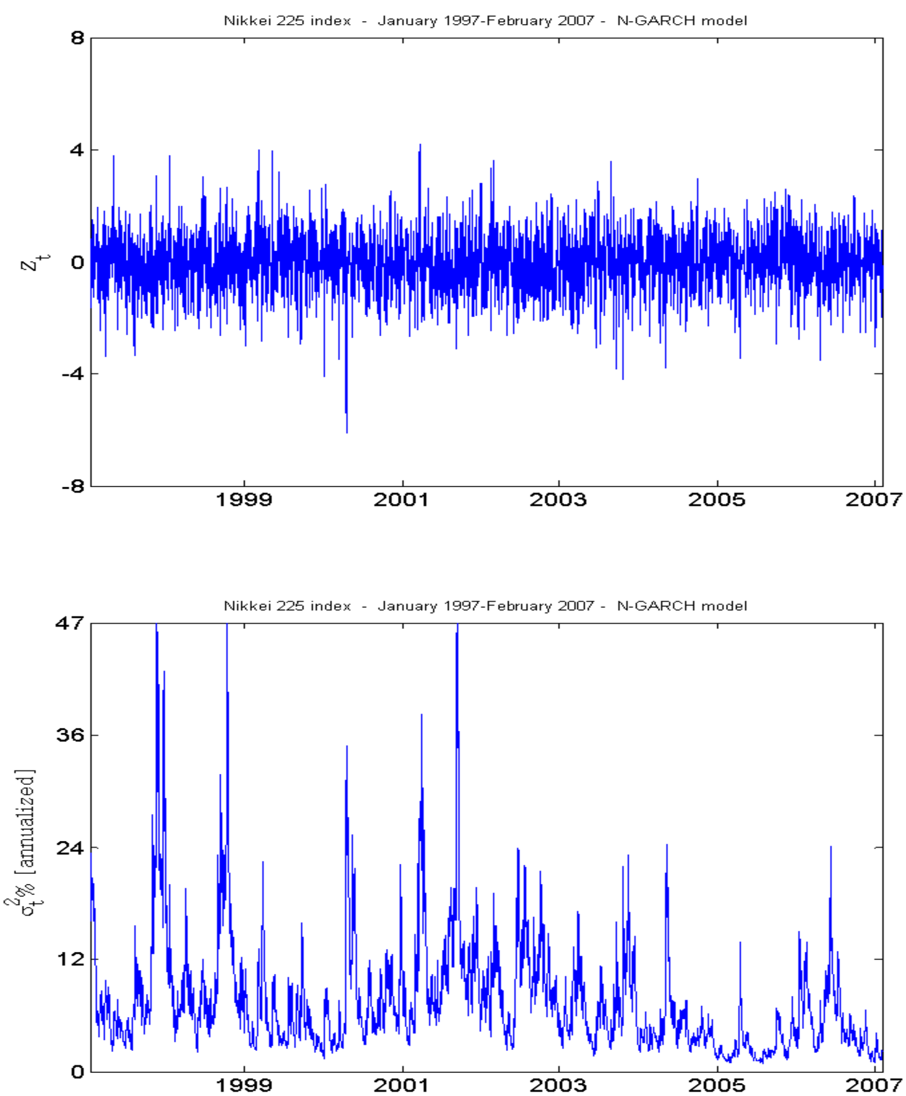
Almost all parameters are estimated significantly different from zero at conventional significance level. Some problems is presented by the E-GARCH estimated up to February 8, 2007. The log likelihood values indicate that the N-GARCH fits better than E-GARCH and E-GARCH better than GJR-GARCH. The models behave only in slightly different way in estimating the current volatility (h_0) and the current shock (z_0). In this terms N-GARCH and GJR-GARCH are much alike. The persistence are high in all the models. The more persistent model seems to be the GJR-GARCH.

The estimated parameters are quite stable during the period analyzed. The expected return parameter estimate(μ) is very different between the models although is more stable across the time for each model. It is notoriously known the difficulty to estimate μ. We remember that the drift (μ) is not used in the BAEM approach. It no needs its value in the calibration procedure, where the drift is adjusted such that the riskfree rate is the correct rate to use in the pricing procedure While in the historical density estimation it is used a constant positive risk premia (following a Merton argument (1980))

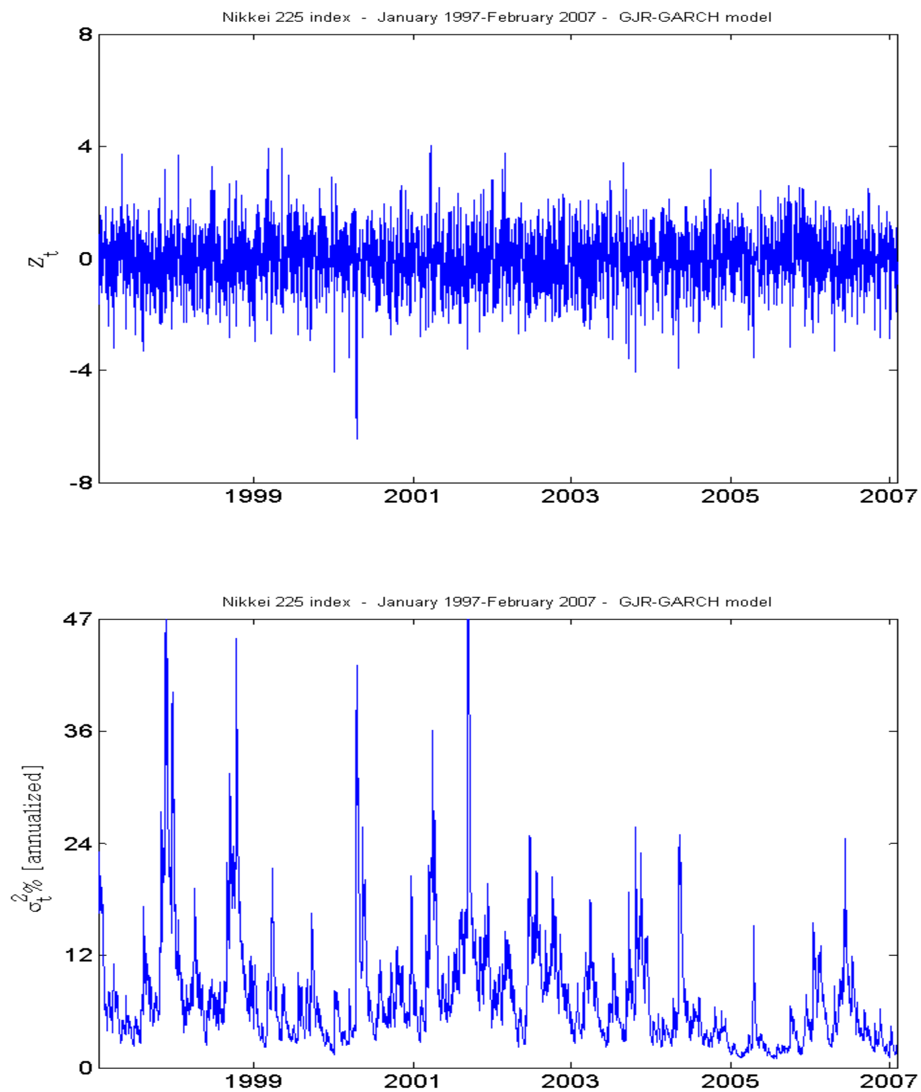
It is important to underline that the E-GARCH model has a functional form very different among the models analyzed, because it expresses the variance in logarithmic terms. Moreover this implies a no immediate meaning of the parameter values in comparison with the other models. In particular note that the parameter ω is typically negative in E-GARCH while it is positive for the other models. This is a coherent result among the models because in the E-GARCH case to do a sort of comparison it needs convert ω by the exponential function.

The estimated current volatility (h_0) and current shock (z_0) reported in the table will be used as starting values in the calibration procedure.

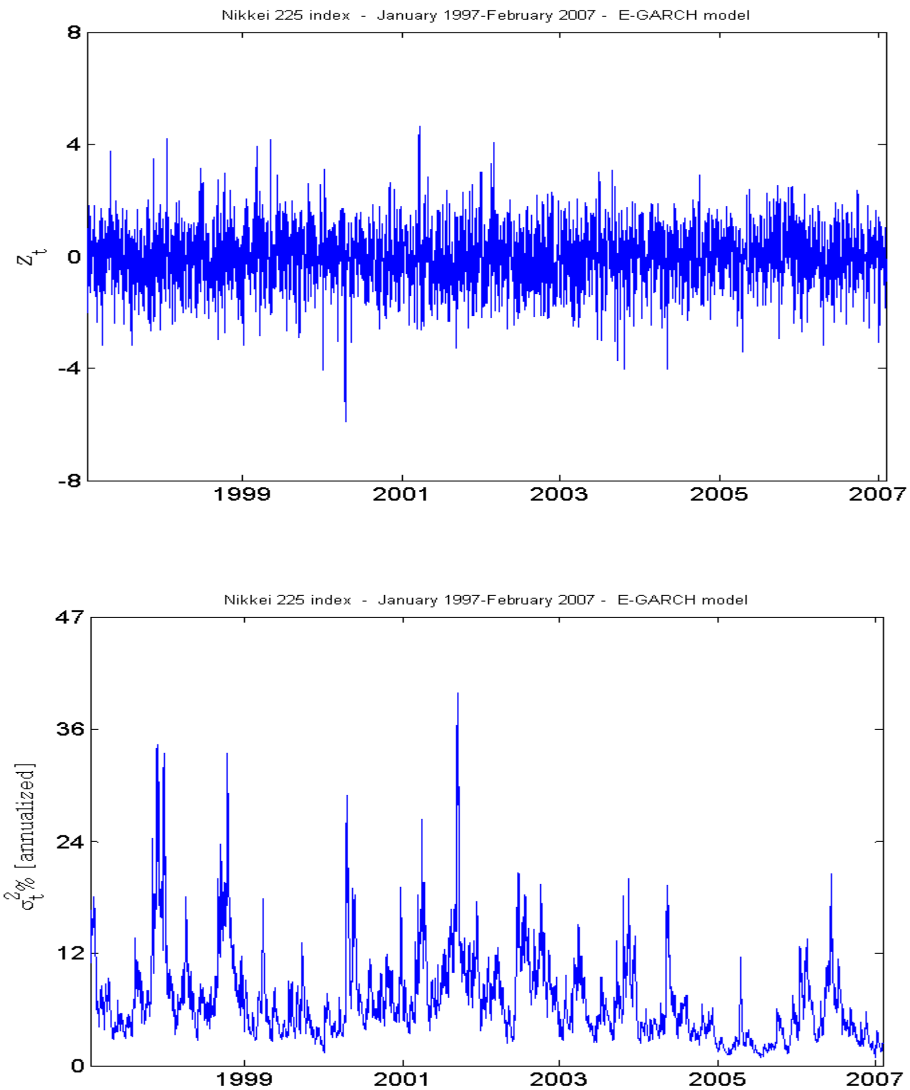
To graphically appreciate the model differences we show in this paragraph the estimated scaled innovations and the historical variance dynamics based on the PML results as interpreted by each simple GARCH model during about the last 10 years.



N-GARCH model: Estimated scaled innovation of Nikkei 225 index from January 15, 1997 to February 7, 2007(first panel) and estimated conditional variances (second panel) obtained from PML estimation.



GJR-GARCH model: Estimated scaled innovation of Nikkei 225 index from January 15, 1997 to February 7, 2007(first panel) and estimated conditional variances (second panel) obtained from PML estimation.



E-GARCH model: Estimated scaled innovation of Nikkei 225 index from January 15, 1997 to February 7, 2007(first panel) and estimated conditional variances (second panel) obtained from PML estimation.

2.3.2 Calibration results

We calibrate the simple GARCH models to the market prices of the Out-of-the-money Nikkei Put Warrants observed on Wednesday February 7, 2007. Starting values for the risk neutral parameters $\rho^* = (\omega^*, \alpha^*, \beta^*, \gamma^*)$ are the parameters obtained by the PML estimation procedure and given in the table showed in the previous paragraph 2.3.1.

We report two preliminary tables to begin with the empirical analysis of the calibration results starting from February 7, 2007 to February 12, 2007.

The next table is obtained by computing the American option price by the first Monte Carlo American option approximation which uses the Gaussian barrier to approximate the optimal exercise barrier.

Time series	NK225											
Range data	<31-Dec-1957..07-Feb-2007>			<31-Dec-1957..08-Feb-2007>			<31-Dec-1957..09-Feb-2007>			<31-Dec-1957..12-Feb-2007>		
Cross-Section Nikkei Put Warrants :	07-Feb-2007			08-Feb-2007			09-Feb-2007			12-Feb-2007		
Gaussian Calibration results	NGarch	$\text{Garch}_{\text{GJR}}$	EGarch	NGarch	$\text{Garch}_{\text{GJR}}$	EGarch	NGarch	$\text{Garch}_{\text{GJR}}$	EGarch	NGarch	$\text{Garch}_{\text{GJR}}$	EGarch
ω*x10^4	2,79E-02	4,22E-02	-3,76E-02	3,63E-02	6,70E-02	-1,74E-01	3,06E-02	2,21E-02	-1,60E+00	7,23E-03	2,20E-02	-1,09E-01
α*	5,74E-04	6,61E-02	3,02E-07	6,58E-02	2,57E-02	3,35E-02	2,55E-05	5,90E-02	9,09E-07	7,14E-03	5,93E-02	4,38E-02
β*	9,76E-01	8,54E-01	9,96E-01	8,58E-01	9,97E-01	9,83E-01	9,74E-01	8,45E-01	8,24E-01	9,81E-01	8,46E-01	9,91E-01
γ*	5,61E-01	6,92E-02	-1,42E+00	7,59E-01	-1,55E-01	1,49E+00	6,10E-01	1,34E-01	5,29E-01	9,51E-01	1,35E-01	5,21E-01
Persistence	0,9765	0,9544	0,9957	0,9614	0,9455	0,9835	0,9738	0,9044	0,8238	0,9948	0,9726	0,9914
APE	7,3%	9,3%	4,2%	10,2%	7,7%	3,8%	9,5%	15,0%	11,2%	4,5%	12,7%	3,8%
RMSE	42,99	48,82	21,43	52,26	43,43	19,27	51,17	65,65	59,85	24,45	56,29	17,60
Max ABS(E)	219,24	242,22	86,63	271,45	197,61	63,50	290,26	332,76	328,04	117,52	299,95	61,54
Cross-Section Nikkei Put Warrants :	07-Feb-2007			08-Feb-2007			09-Feb-2007			12-Feb-2007		
FHS Calibration												
ω*x10^4	3,95E-02	3,36E-02	-6,79E-01	4,32E-02	1,43E-02	-6,13E-02	3,62E-02	2,22E-02	-1,43E-01	7,40E-03	1,04E-02	-6,85E-02
α*	7,89E-02	1,49E-01	1,64E-01	6,00E-02	1,30E-01	9,42E-07	7,55E-02	5,92E-02	4,47E-02	1,63E-02	7,50E-02	1,25E-05
β*	8,83E-01	8,67E-01	9,38E-01	8,97E-01	8,65E-01	9,93E-01	8,88E-01	8,55E-01	9,87E-01	9,73E-01	9,41E-01	9,92E-01
γ*	2,58E-01	-7,68E-02	5,28E-01	3,66E-01	-3,73E-03	4,27E-01	3,01E-01	1,33E-01	-1,65E+00	6,33E-01	-4,08E-02	5,97E-01
Persistence	0,9667	0,9777	0,9378	0,9653	0,9929	0,9930	0,9708	0,9141	0,9870	0,9954	0,9961	0,9922
APE	10,4%	7,5%	4,8%	8,3%	10,5%	5,7%	11,1%	16,1%	4,8%	6,2%	5,0%	5,5%
RMSE	55,46	45,13	26,54	49,42	47,37	30,89	55,83	64,59	20,55	28,54	25,59	28,60
Max ABS(E)	306,62	266,24	112,21	315,33	187,33	156,10	344,71	309,64	69,15	131,00	127,71	157,14

	mean	NGarch	$\text{Garch}_{\text{GJR}}$	EGarch
Gaussian	APE	7,9%	11,1%	5,8%
	RMSE	42,72	53,55	29,54
	Max ABS(E	224,62	268,14	134,93
FHS	APE	9,0%	9,8%	5,2%
	RMSE	47,31	45,67	26,64
	Max ABS(E	274,41	222,73	123,65

Calibration results of Simple GARCH Models I,II,II using Gaussian innovations (Panel 1) and FHS Innovations (Panel 2) on February 7, 2007 out of the money Nikkei Put Warrants (114), up to February 12, 2007. American price approximation is computed by Monte Carlo simulation and

Gaussian Barrier (H_1). Time to maturities start from: 143, 235, 326, 417, 509, 601, 692, 1057, 1422 days. The root mean squared error (RMSE) is in ¥, the APE measure is defined in .Note: in the E-G case ω^* is not multiplied by 10^4.

The results presented in the next table obtained by using the second Monte Carlo American option approximation as explained in paragraph 1.4.2.

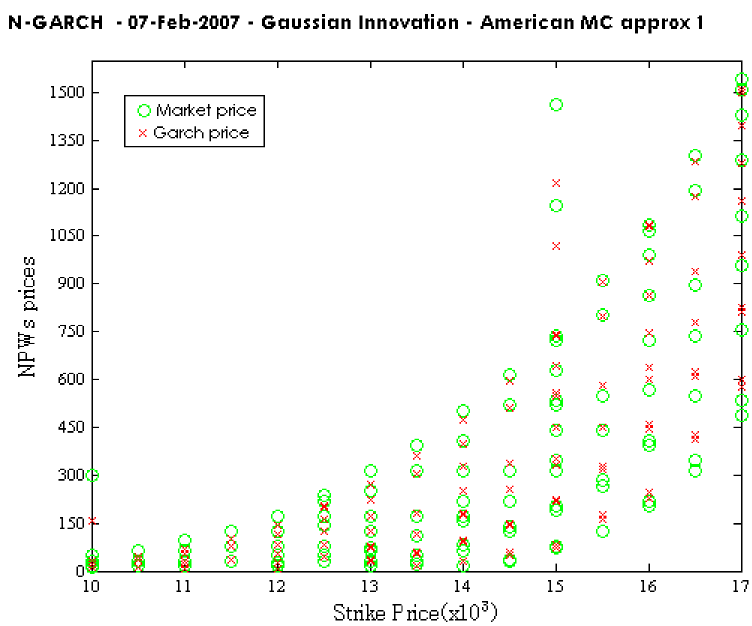
Time series Range data	NK225 <31-Dec-1957..07-Feb-2007>			<31-Dec-1957..08-Feb-2007>			<31-Dec-1957..09-Feb-2007>			<31-Dec-1957..12-Feb-2007>		
Cross-Section NPWs	07-Feb-2007			08-Feb-2007			09-Feb-2007			12-Feb-2007		
Gaussian Calibration	NGarch	Garch$_{GJR}$	EGarch	NGarch	Garch$_{GJR}$	EGarch	NGarch	Garch$_{GJR}$	EGarch	NGarch	Garch$_{GJR}$	EGarch
ω*x10^4	2,61E-02	2,75E-02	-3,49E-01	4,04E-02	2,21E-02	-4,94E-01	4,02E-03	2,16E-02	-9,95E-01	4,77E-02	1,26E-02	-4,21E-01
β*	9,57E-01	8,81E-01	9,70E-01	8,91E-01	8,58E-01	9,57E-01	9,74E-01	9,03E-01	8,91E-01	8,19E-01	8,89E-01	9,71E-01
α*	1,45E-02	8,22E-02	1,04E-01	5,38E-02	5,94E-02	1,33E-01	1,40E-02	9,86E-02	3,94E-05	1,39E-01	1,20E-01	2,12E-01
γ*	6,30E-01	2,75E-02	-4,31E-02	6,34E-01	1,32E-01	1,29E+00	8,49E-01	-3,83E-02	5,62E-01	1,11E-01	-3,42E-02	3,94E-01
Persistence	0,9776	0,9772	0,9702	0,9663	0,9836	0,9574	0,9977	0,9823	0,8906	0,9594	0,9920	0,9712
APE	7,3%	7,1%	8,1%	8,7%	11,8%	4,7%	5,8%	7,7%	10,9%	7,3%	4,1%	8,8%
RMSE	42,64	38,87	41,73	46,83	51,92	23,28	24,95	44,36	58,64	42,20	21,26	39,74
Max ABS(E)	232,78	203,25	155,21	250,56	211,38	61,20	79,49	264,96	328,41	266,93	97,03	117,55
η	0,30	0,30	0,30	0,30	0,40	0,40	0,40	0,30	0,40	0,30	0,30	0,50
Mean Feb, 7-12	NGarch	Garch$_{GJR}$	EGarch									
APE	7,3%	7,7%	8,1%									
RMSE	39,15	39,10	40,85									
Max ABS(E)	207,44	194,16	165,59									

Calibration results of Simple GARCH Models I,II,II using Gaussian innovations on February 7, 2007 out of the money Nikkei Put Warrants (114), up to February 12, 2007. American price approximation is computed by Monte Carlo simulation and Optimal Horizontal Barrier (H_2). Time to maturities start from: 143, 235, 326, 417, 509, 601, 692, 1057, 1422 days. The root mean squared error (RMSE) is in ¥, the APE measure is defined in 1.14. the parameter η is defined in 1.26. Note: in the E-G case ω^* is not multiplied by 10^4.

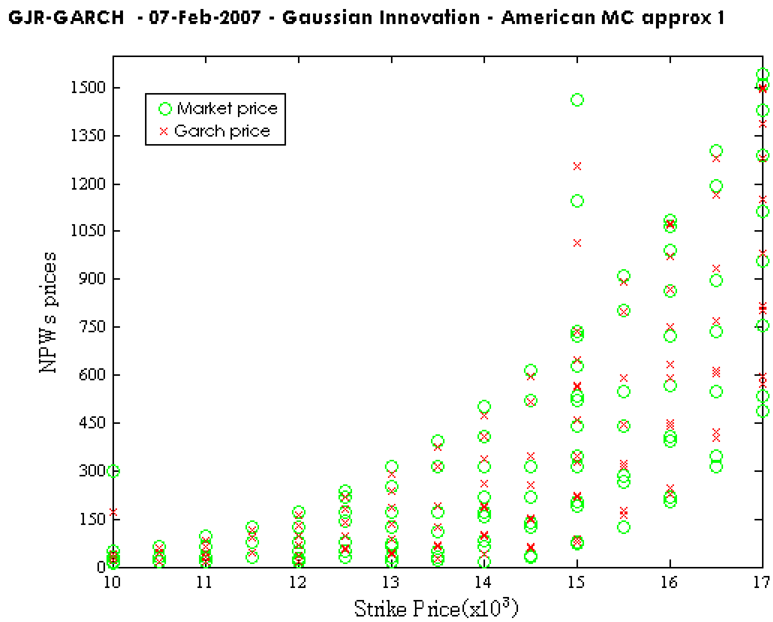
A preliminary look to the parameter estimations induces to remain unsatisfied from different point of views. First of all the RMSE and APE is high for all models. This implies that the Simple GARCH models don't achieve a good quality in the calibration fitting exercise using our American option approximations for the Strike price-Maturity space of the derivative cross-section analyzed.

Moreover in the calibration procedure left free to interpret the cross section generate some negative leverage effect parameter in particular for the GJR-GARCH and the E-GARCH. In order to obtain a pricing process parameter with a positive leverage effect as obtained by the PML estimation we impose the positivity constraint for the next calibration procedure. This is equivalent to find a GARCH pricing process by minimizing the pricing error in the class of the GARCH process which admits positive leverage effect.

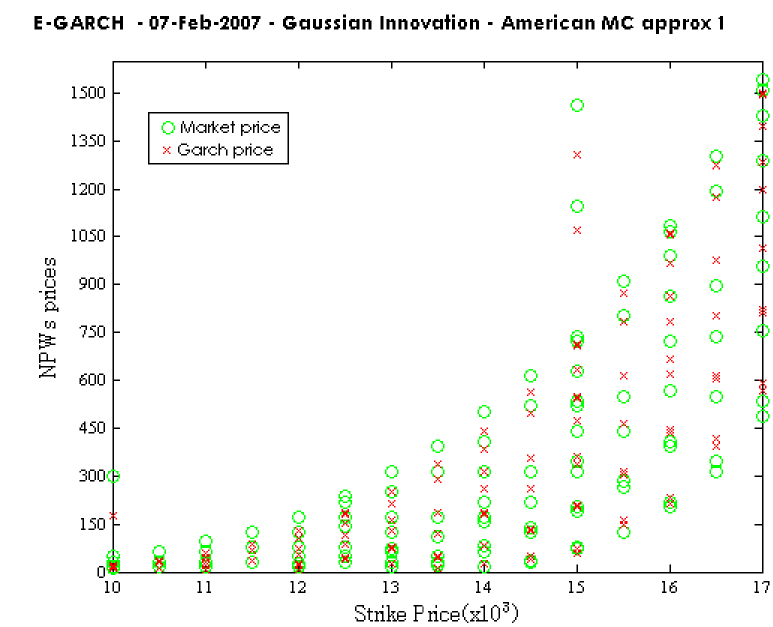
While at the moment we postpone additional comments we presents some graphical calibration results to visualize other empirical problems of the model investigated:



Monte Carlo (Approx 1) calibration results of the N-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 7, 2007.

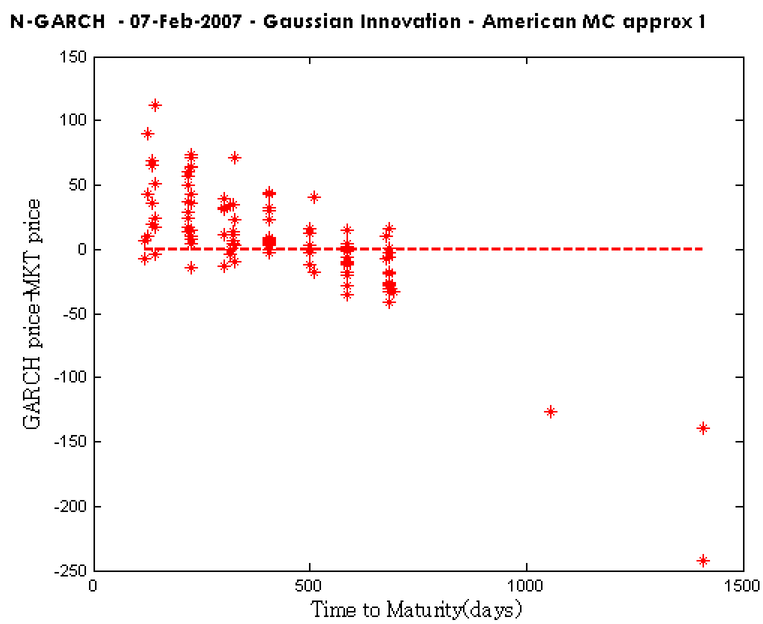


Monte Carlo (Aprox 1) calibration results of the GJR-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 7, 2007.



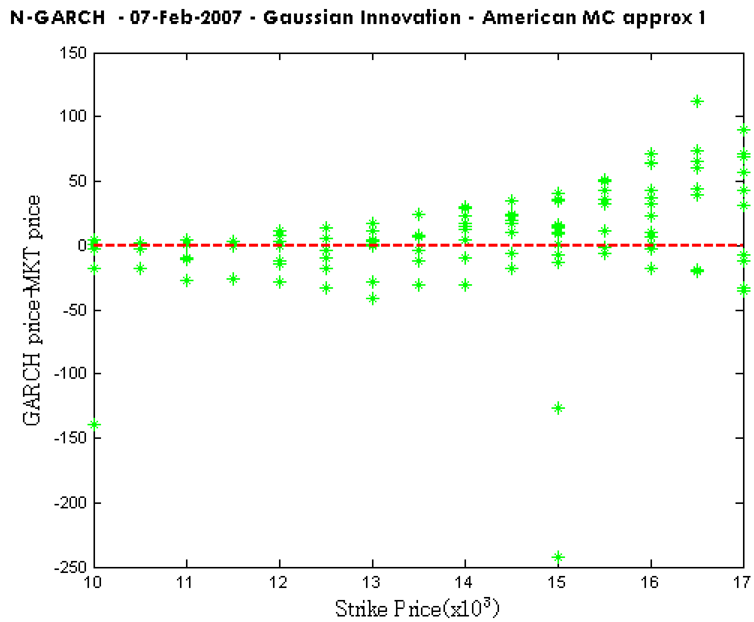
Monte Carlo (Approx 1) calibration results of the E-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 7, 2007.

If we refer to the last three calibration graphics (related to the three simple GARCH models) we can observe similar mispricing errors. The short maturity NPWs are generally overpriced and the long maturity NPWs are underpriced. The same result is obtained if we use the second approximation in order to compute the theoretical option price.



Pricing errors with respect to the time to Maturity of the N-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 7, 2007.

The same observation is confirmed if we take a look to the last figure and although this shows only the N-GARCH error pricing with respect to the time to maturity, the empirical evidence of the other models is much alike.



Pricing errors with respect to strike price of the N-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 7, 2007.

A moderate mispricing is present with respect to the strike price. Exactly as the options are near the money as such error tend to be larger. Also the pricing errors with respect to the strike price behave in the same way for all models therefore we have showed only the figure related to the N-GARCH model.

We limit our consideration to the fact that the simple GARCH models present a significant mispricing with respect to the time maturity. Because our study is oriented on the long-term maturity derivatives we pass directly to the empirical study of the Component models.

We only add an obligatory consideration:

These first conclusions, we have drawn, are conjointly result of the approximation method used to price the options, the option pricing underlying assumptions (which mainly allow to pass from the historical probability measure to the probability risk-adjusted) and the error distribution assumption.

While the option pricing assumptions are very weak, due to the numerous strong hypotheses relaxed in the BAEM-framework, and while the error distribution assumption is also treated in non parametric manner, some right doubts are to be moved with regard to the option pricing approximation accuracy. We cannot exclude that these same models give better performances by exploiting other more accurate pricing techniques.

After all these same models have known better performances in European derivatives analysis, as in the BAEM article.

But the next section induces to moderate our criticism to the option price approximation, while highlights that the Simple GARCH models tend to have much difficulties to model both short-term options and long-term options, as confirmed in Christoffersen, P., K. Jacobs and Y. Wang (2004).

2.4 Section II - Component volatility dynamics models

2.4.1 GARCH Models PML Estimation results

As done for the Simple GARCH model, the Component GARCH models (i.e., Model IV,V,VI (1.52)) are estimated using Pseudo Maximum Likelihood (PML) estimator.

The PML estimation results for the three Component models are reported in the next table:

Time series	Nikkei 225 index											
PML Estimation results												
Range data	<31-Dec-1957..07-Feb-2007>						<31-Dec-1957..08-Feb-2007>					
	NGarch		Garch$_{GJR}$		EGarch		NGarch		Garch$_{GJR}$		EGarch	
parameter	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error
μx10^2	4,03E-02	1,58E-02	4,46E-02	2,06E-04	4,64E-02	9,62E-07	4,04E-02	1,34E-03	4,48E-02	1,42E-02	4,60E-02	7,68E-03
β_1	7,12E-01	5,85E-02	7,80E-01	5,22E-06	8,18E-01	1,13E-04	7,12E-01	1,30E-06	7,80E-01	5,35E-01	8,17E-01	6,18E-02
α_1	9,28E-02	5,19E-03	1,10E-05	1,34E-08	1,13E-01	9,26E-04	9,30E-02	2,07E-04	1,47E-05	1,28E-04	1,13E-01	**4,02E-02**
γ_1	1,05E+00	2,40E-06	2,17E-01	1,56E-04	1,39E+00	1,77E-05	1,05E+00	2,67E-07	2,16E-01	4,87E-02	1,40E+00	1,67E-02
ωx10^4	3,34E-03	3,28E-05	3,21E-03	3,98E-06	-6,97E-02	1,36E-07	3,30E-03	8,59E-07	3,26E-03	4,10E-02	-7,00E-02	1,30E-03
β_2	9,98E-01	4,18E-03	9,98E-01	1,12E-04	9,92E-01	1,03E-06	9,98E-01	4,64E-06	9,98E-01	4,80E-05	9,92E-01	2,81E-04
α_2	3,59E-02	3,83E-03	3,32E-02	2,85E-05	1,06E-01	1,89E-05	3,53E-02	1,14E-03	3,35E-02	3,47E-03	1,06E-01	**1,91E-01**
γ_2	1,18E-01	5,15E-02	6,16E-03	3,43E-05	1,32E-01	1,03E-03	1,15E-01	4,16E-04	6,28E-03	2,89E-02	1,34E-01	**2,56E-01**
Short-run persistence	0,907186		0,887995		0,817804		0,906565		0,887591		0,817295	
Long-run persistence	0,997941		0,998089		0,991880		0,997865		0,998022		0,991847	
Final values estimated												
H_0x10^4	0,8621		0,7570		0,6216		0,8355		0,7273		0,5793	
h_0x10^4	0,7498		0,7168		0,7628		0,6724		0,6175		0,6345	
z_0	-0,8090		-0,8325		-0,8090		-0,0482		-0,0558		-0,0566	
LogLKLHD	41862		41812		41836		41866		41816		41840	
Range data	<31-Dec-1957..09-Feb-2007>						<31-Dec-1957..12-Feb-2007>					
	NGarch		Garch$_{GJR}$		EGarch		NGarch		Garch$_{GJR}$		EGarch	
parameter	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error	Estimate	Std.Error
μx10^2	4,06E-02	3,00E-04	4,48E-02	9,00E-05	4,64E-02	1,64E-06	4,05E-02	1,72E-04	4,47E-02	3,49E-02	4,65E-02	5,93E-05
β_1	7,14E-01	9,06E-05	7,80E-01	1,52E-05	8,30E-01	3,37E-05	7,14E-01	6,88E-05	7,79E-01	1,86E-04	8,34E-01	3,61E-04
α_1	9,56E-02	5,72E-04	1,24E-06	7,76E-11	1,25E-01	3,78E-04	9,58E-02	2,00E-04	9,03E-06	1,03E-07	1,28E-01	2,56E-04
γ_1	1,02E+00	2,06E-06	2,17E-01	1,54E-05	1,26E+00	1,63E-05	1,01E+00	3,45E-05	2,17E-01	2,85E-04	1,23E+00	1,17E-04
ωx10^4	2,86E-03	7,84E-07	3,06E-03	1,69E-05	-5,60E-02	1,85E-07	2,79E-03	1,23E-05	3,20E-03	3,05E-06	-5,22E-02	2,12E-06
β_2	9,98E-01	2,20E-05	9,98E-01	1,83E-06	9,93E-01	4,46E-07	9,98E-01	1,91E-05	9,98E-01	3,25E-07	9,94E-01	4,02E-06
α_2	3,30E-02	4,75E-04	3,27E-02	1,08E-04	9,60E-02	2,14E-04	3,28E-02	5,21E-05	3,34E-02	2,69E-04	9,31E-02	2,99E-04
γ_2	1,09E-01	4,65E-04	5,65E-03	3,76E-05	1,18E-01	1,47E-04	1,08E-01	1,22E-04	6,04E-03	3,57E-05	1,13E-01	2,84E-04
Short-run persistence	0,907684		0,888277		0,830099		0,907790		0,887869		0,833660	
Long-run persistence	0,998151		0,998176		0,993415		0,998205		0,998126		0,993843	
Final values estimated												
H_0x10^4	0,8779		0,7618		0,6299		0,8630		0,7379		0,5976	
h_0x10^4	0,5934		0,5921		0,5772		0,5521		0,5259		0,5096	
z_0	1,5281		1,5243		1,5416		-0,0546		-0,0617		-0,0651	
LogLKLHD	41869		41819		41845		41873		41823		41849	

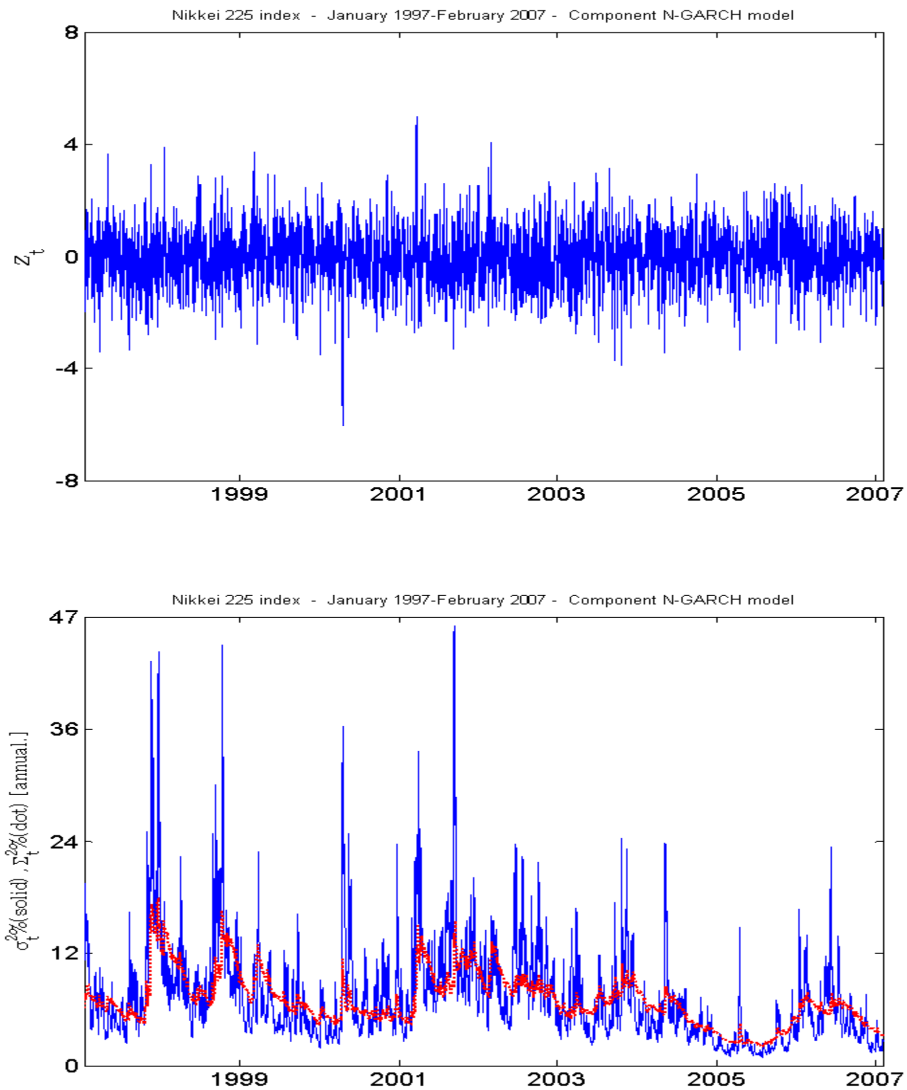
Daily returns from December 31, 1957 to February 7, 2007 on Nikkei 225 index are used to estimate the three Component GARCH models using Psuedo Maximum Likelihood. Robust standard errors are computed using numerical derivative at the optimum parameter value. $H_{0,}$ h_0 and z_0 are the current estimates of the log-term variance, the short-term variance and the shock. LogLKLHD is the logarithm of the likelihood at the optimal parameter values. Short-run Persistence refers to the persistence of the conditional short-term variance component $(\widetilde{\beta}_1)$. Long-run Persistence is the persistence of the conditional long-term variance component (β_2). Note: in the E-G case ω is not multiplied by 10^4.

The Component model PML estimation results appear to be similar to the Simple models PML estimation results for many aspects. Also here some problems is presented by the E-GARCH estimated up to February 8, 2007. In terms of fit the log likelihood values indicate that the cN-GARCH fits better than cE-GARCH and cE-GARCH better than cGJR-GARCH. In estimating the current volatility (h_0) and the current shock (z_0) the models behave in similar fashion, while the current long-term variance component (H_0) are differently estimated in each model. The long-run persistence are very high, proxy to 1, in all the models in particular in cN-GARCH and cGJR-GARCH. At this end we cite the work of Christoffersen et.al. (2004). Fixing the long-run persistence to 1, as possible alternative model, they have shown that the resulting "persistent" model is dominated by the component model related also in out-the-sample comparison. Although the log-run persistence is proxy to 1, modelling this difference appears to be very important.

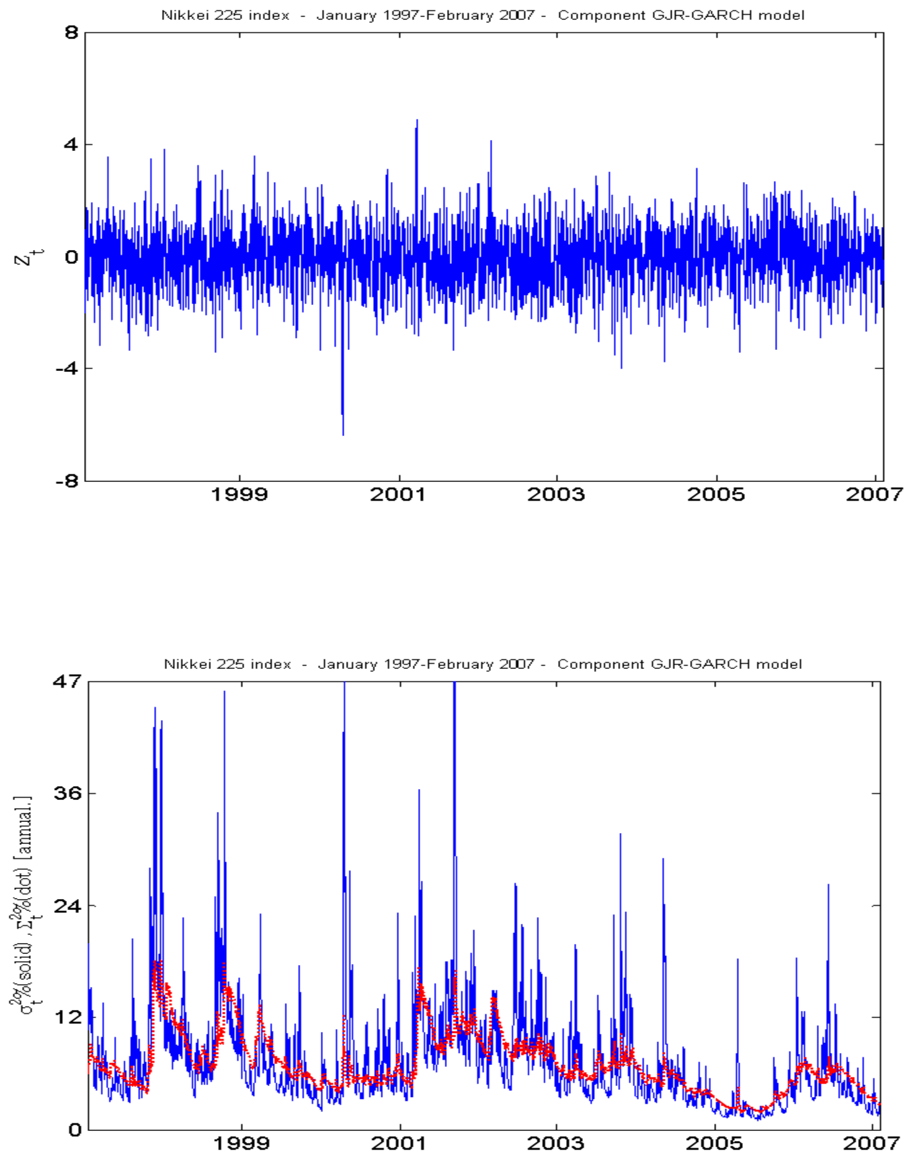
The estimated parameters are quite stable during the period analyzed. The expected return parameter estimate(μ) is very different between the models although is more stable across the time for each model.

The estimated current long-term variance component (H_0), current variance(h_0) and current shock (z_0) reported in the table will be used as starting values in the calibration procedure.

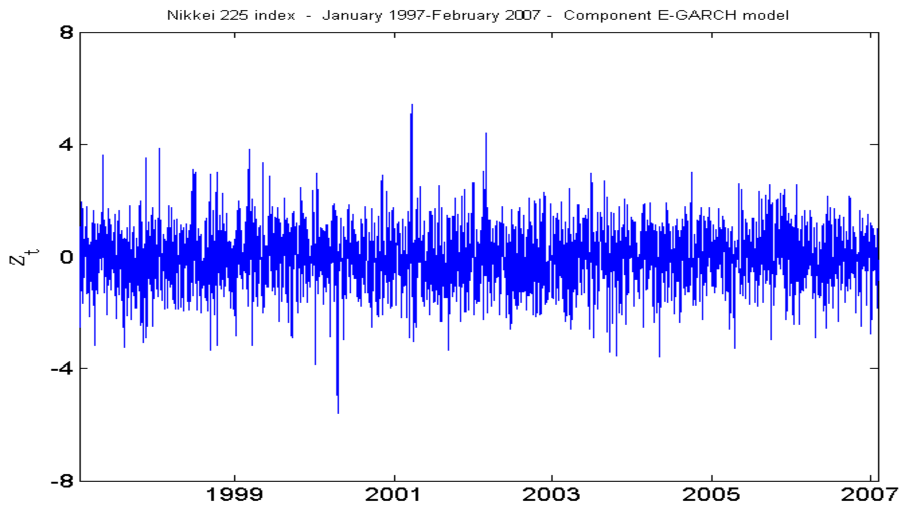
To graphically appreciate the model differences we present in the next figures the estimated scaled innovations and the historical variance component dynamics based on the PML results as interpreted by each component GARCH model during about the last 10 years.

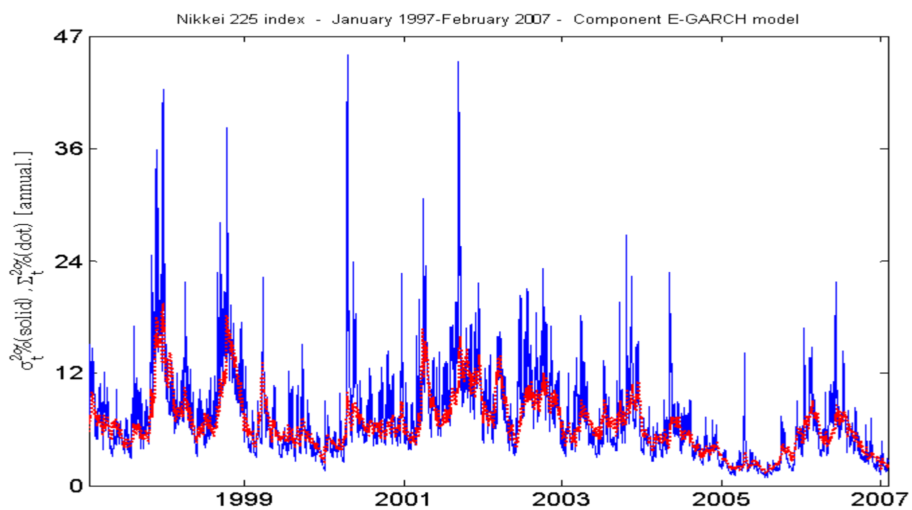


Component N-GARCH model: Estimated scaled innovation (z_t) of Nikkei 225 index from January 15, 1997 to February 7, 2007(first panel) and estimated conditional variances (σ_t^2,Σ_t^2) (second panel) obtained from PML estimation.



Component GJR-GARCH model: Estimated scaled innovation (z_t) of Nikkei 225 index from January 15, 1997 to February 7, 2007(first panel) and estimated conditional variances (σ_t^2,Σ_t^2) (second panel) obtained from PML estimation.





Component E-GARCH model: Estimated scaled innovation (z_t) of Nikkei 225 index from January 15, 1997 to February 7, 2007(first panel) and estimated conditional variances (σ_t^2,Σ_t^2) (second panel) obtained from PML estimation.

2.4.2 Calibration results

We repeat the calibration exercise as done for the Simple GARCH model. We calibrate the Component GARCH models to the market prices of the Out-of-the-money

Nikkei Put starting from Wednesday February 7, 2007. Starting values for the risk neutral parameters $\rho^* = (\beta_1^*, \alpha_1^*, \gamma_1^*, \omega^*, \beta_2^*, \alpha_2^*, \gamma_2^*)$ are the parameters obtained by the PML estimation procedure and given in figure 2.4.1.

We report a summarizing table to begin with the empirical analysis of the calibration results starting from February 7, 2007 to February 12, 2007.

The next table is obtained by computing the American option price by the second Monte Carlo American option approximation which uses the optimal horizontal barrier as explained in paragraph 1.4.2.

Time series	Nikkei 225 index											
Historical Range data	<31-Dec-1957..07-Feb-2007>			<31-Dec-1957..08-Feb-2007>			<31-Dec-1957..09-Feb-2007>			<31-Dec-1957..12-Feb-2007>		
Gaussian Calibration	cNGarch	cGarchGJR	cEGarch	cNGarch	cGarchGJR	cEGarch	cNGarch	cGarchGJR	cEGarch	cNGarch	cGarchGJR	cEGarch
Cross-Section NPWs	07-Feb-2007			08-Feb-2007			09-Feb-2007			12-Feb-2007		
β'_1	4,36E-01	6,32E-01	4,28E-01	3,27E-01	4,51E-01	7,92E-01	7,63E-01	7,37E-01	7,61E-01	3,65E-02	4,58E-01	5,60E-01
α'_1	6,73E-02	1,23E-05	2,09E-01	3,92E-02	1,62E-05	1,10E-01	1,05E-01	1,28E-06	1,09E-01	9,12E-02	6,92E-06	1,28E-01
γ'_1	2,10E+00	2,00E-01	8,80E-01	1,67E+00	2,26E-01	1,41E+00	1,07E+00	2,11E-01	1,68E+00	1,01E+00	1,32E-01	1,38E+00
ω'	2,37E-07	3,88E-07	-6,35E-02	2,85E-07	4,83E-07	-7,07E-02	2,67E-07	3,08E-07	-3,46E-03	5,78E-07	4,39E-07	-7,86E-02
β'_2	1,00E+00	9,98E-01	9,93E-01	9,99E-01	9,97E-01	9,92E-01	9,99E-01	9,98E-01	1,00E+00	9,97E-01	9,97E-01	9,91E-01
α'_2	4,64E-03	2,64E-02	2,96E-02	8,54E-06	1,92E-02	1,03E-01	2,80E-02	3,23E-02	1,21E-02	1,41E-02	2,82E-02	6,27E-02
γ'_2	1,50E-01	6,47E-03	1,39E-01	3,28E-01	5,57E-03	1,34E-01	1,00E-01	5,61E-03	2,21E-01	1,01E-01	4,26E-03	1,24E-01
Short-run Persistence	0,8004	0,7322	0,4279	0,4751	0,5639	0,7916	0,9892	0,8421	0,7611	0,2215	0,5244	0,5600
Long-run persistence	0,9998	0,9978	0,9927	0,9986	0,9972	0,9921	0,9987	0,9983	0,9995	0,9973	0,9973	0,9909
APE	6,5%	3,6%	4,3%	4,3%	3,6%	12,2%	10,1%	6,5%	5,3%	4,4%	3,5%	4,2%
RMSE	52,23	17,27	22,12	21,69	17,99	63,67	43,60	26,78	23,00	19,93	16,86	19,66
Max ABS(E)	423,83	62,93	77,14	79,27	76,72	302,87	107,84	92,54	78,62	67,84	50,88	69,90
η	0,40	0,30	0,30	0,30	0,30	0,40	0,20	0,30	0,30	0,30	0,10	0,40
Cross-Section NPWs	07-Feb-2007			08-Feb-2007			09-Feb-2007			12-Feb-2007		
FHS Calibration												
β'_1	5,96E-01	6,24E-01	3,73E-01	7,01E-01	4,87E-01	8,03E-01	5,28E-03	3,74E-01	1,13E-02	4,34E-01	4,23E-03	1,03E-02
α'_1	8,43E-02	2,41E-05	1,37E-01	9,33E-02	1,21E-05	1,17E-01	1,68E-01	9,63E-07	1,82E-01	1,63E-01	1,34E-05	1,98E-01
γ'_1	3,69E-02	1,43E-01	2,00E+00	1,01E+00	3,93E-01	1,42E+00	7,28E-01	6,43E-01	2,69E+00	1,09E+00	7,03E-02	9,17E-01
ω'	3,29E-07	1,82E-07	-1,25E-01	3,28E-07	3,46E-07	-6,79E-02	7,91E-08	4,67E-07	-5,88E-02	4,07E-07	2,84E-07	-5,71E-02
β'_2	1,00E+00	9,99E-01	9,86E-01	9,91E-01	9,98E-01	9,91E-01	1,00E+00	9,97E-01	9,93E-01	9,98E-01	9,99E-01	9,93E-01
α'_2	1,41E-02	4,12E-03	3,01E-02	3,51E-02	2,59E-03	1,03E-01	1,13E-06	9,18E-04	6,44E-02	4,43E-03	1,73E-02	3,62E-04
γ'_2	4,91E-01	4,29E-03	1,86E-01	1,18E-01	4,47E-03	1,34E-01	1,98E-02	4,66E-03	8,83E-02	7,05E-02	1,14E-02	1,48E-01
Short-run Persistence	0,6807	0,6951	0,3731	0,8887	0,6834	0,8028	0,2630	0,6952	0,0113	0,7894	0,0394	0,0103
Long-run persistence	1,0000	0,9993	0,9858	0,9913	0,9980	0,9915	1,0000	0,9966	0,9931	0,9977	0,9990	0,9934
APE	7,0%	4,1%	9,3%	21,0%	7,1%	12,0%	6,8%	9,3%	12,7%	10,0%	3,8%	6,8%
RMSE	120,93	21,78	40,95	89,50	33,19	187,74	34,16	44,35	69,73	51,69	19,28	31,88
Max ABS(E)	345,18	96,71	114,22	123,21	84,87	98,78	186,98	269,10	167,80	129,14	68,74	131,81
η	0,30	0,40	0,40	0,10	0,40	0,50	0,80	0,30	0,40	0,40	0,30	0,50

Gaussian	Mean Feb, 7-12	cNGarch	cGarchGJR	cEGarch	FHS	Mean Feb, 7-12	cNGarch	cGarchGJR	cEGarch
	APE	6,3%	4,3%	6,5%		APE	11,2%	6,1%	10,2%
	RMSE	34,36	19,73	32,11		RMSE	74,07	29,65	82,57
	Max ABS(E)	169,69	70,77	132,13		Max ABS(E)	196,13	129,85	128,15

Calibration results of Component GARCH Models IV,V,VI using Gaussian/FHS innovations on February 7, 2007 out of the money Nikkei Put Warrants (114), up to February 12, 2007. American price approximation is computed by Monte Carlo simulation and Optimal Horizontal Barrier (H_2). Time to maturities start from: 143, 235, 326, 417, 509, 601, 692, 1057, 1422 days. The root mean squared error (RMSE) is in ¥, the APE measure is defined in 1.14. the parameter η is defined in 1.26. Note: in the cE-G case ω^* is not multiplied by 10^4.

The Gaussian calibration results for the Component models are satisfactory with particular regard to the RMSE and APE, which present very low values in all models if we consider the wide Strike price-Maturity space of the derivative cross-section analyzed. However Component GJR-GARCH is conspicuous for its good performance, especially because it shows to have a good extrapolation ability to interpret long-term option price if measured with Max ABS(E).

Before we pass to show other analyses on the calibration results we have interest to compare the risk neutral parameters to the historical parameters.

If we compare the calibration results table with the historical estimates table we can observe some interesting empirical evidence.

First note that our results derived through the BAEM approach allows us to rely on the market coherence of the risk neutral parameters. Instead of make hypotheses on the risk neutral pricing process, such as many other methods in literature, we can study its features directly from the market.

In the Gaussian innovation case the risk neutral process remains highly long-run persistence like as the historical process while the short-run persistence is generally reduced by passing from the historical to the risk-neutral measure.

This evidence is valid also in the FHS innovation case but in a more marked way: some days suggest a risk neutral pricing process without mean-reversion features ($\widetilde{\beta}_1 \simeq 0$). This can be revised with our intuition by highlighting a possible exploration path for future research. If the historical price process is modelled by a component GARCH model, an interesting risk-neutral pricing hypothesis seems to be to model the pricing process through a component GARCH model with zero-short-run-persistence. If the investor have a long-run and a short-run volatility expectation in order to describe the market volatility, under the risk-neutral setting these two expectations tend to coincide.

We conclude the calibration section by a series of graphic figures which allow us to highlight the model performance along some critical dimensions.

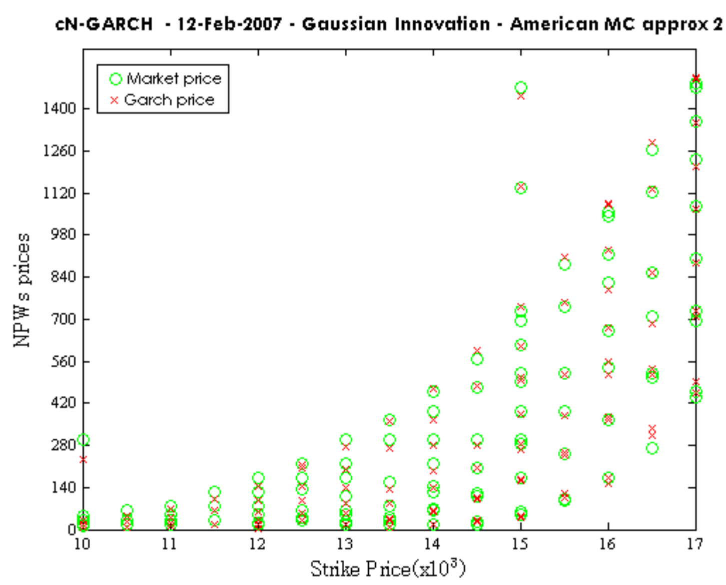


Figure 2.2: Monte Carlo (approx 2) calibration results of the cN-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007 (Gaussian Innovation)

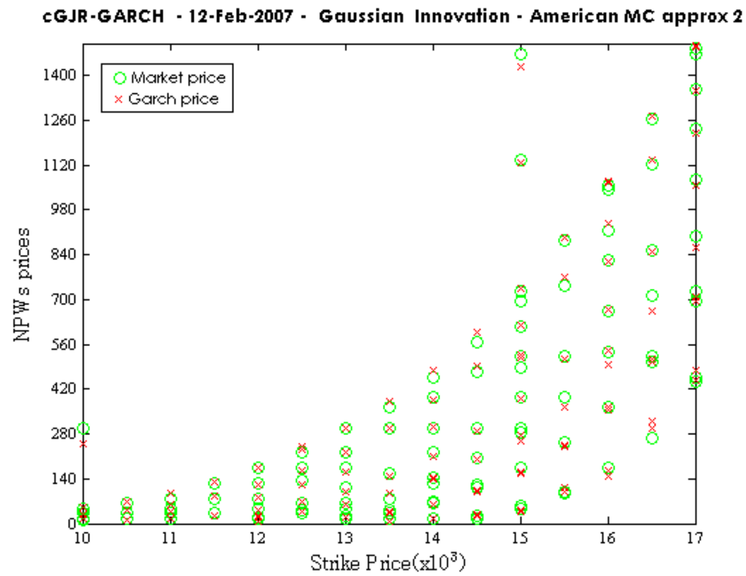


Figure 2.3: Monte Carlo (approx 2) calibration results of the cGJR-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007 (Gaussian Innovation)

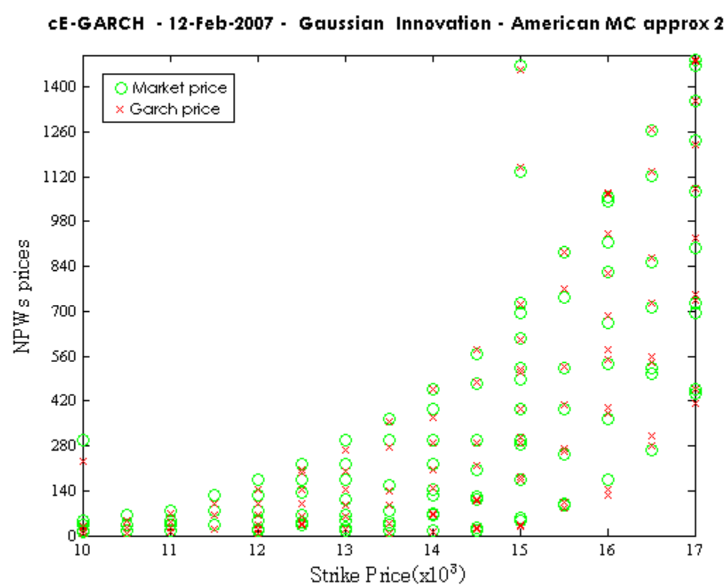


Figure 2.4: Monte Carlo (approx 2) calibration results of the cE-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007 (Gaussian Innovation)

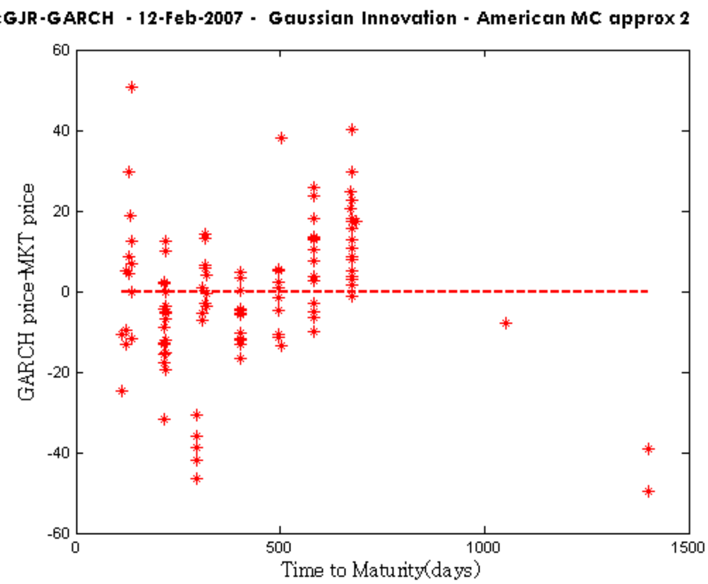


Figure 2.5: Pricing errors with respect to the time to Maturity of the cGJR-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007 (Gaussian Innovation)

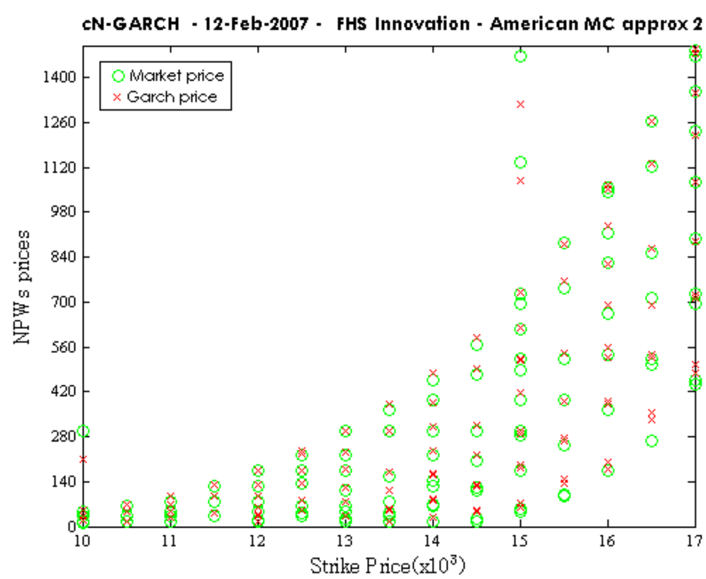


Figure 2.6: Monte Carlo (approx 2) calibration results of the cN-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007 (FHS Innovation)

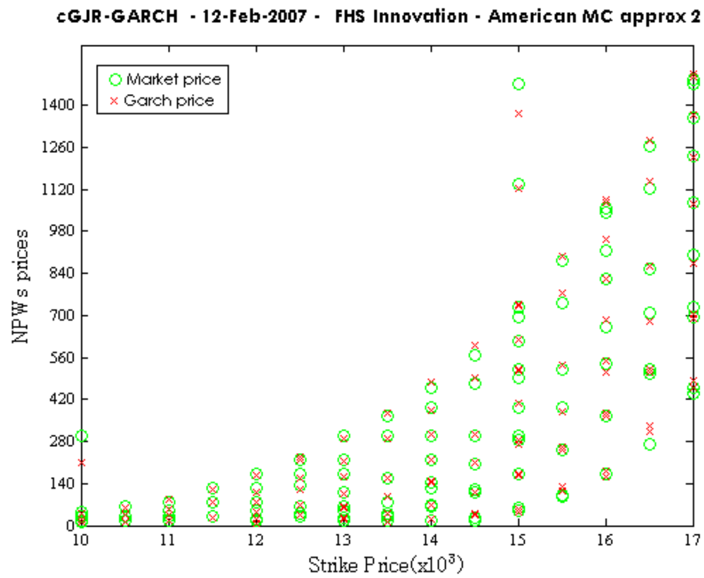


Figure 2.7: Monte Carlo (approx 2) calibration results of the cGJR-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007 (FHS Innovation)

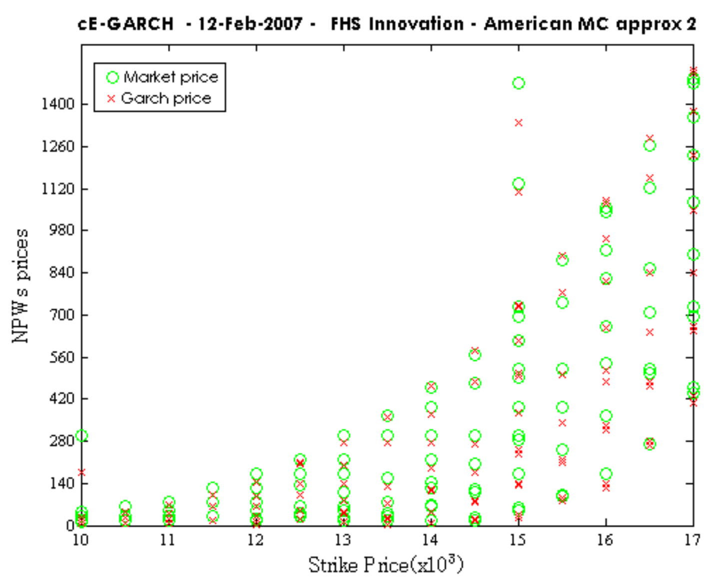


Figure 2.8: Monte Carlo (approx 2) calibration results of the cE-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007 (FHS innovation)

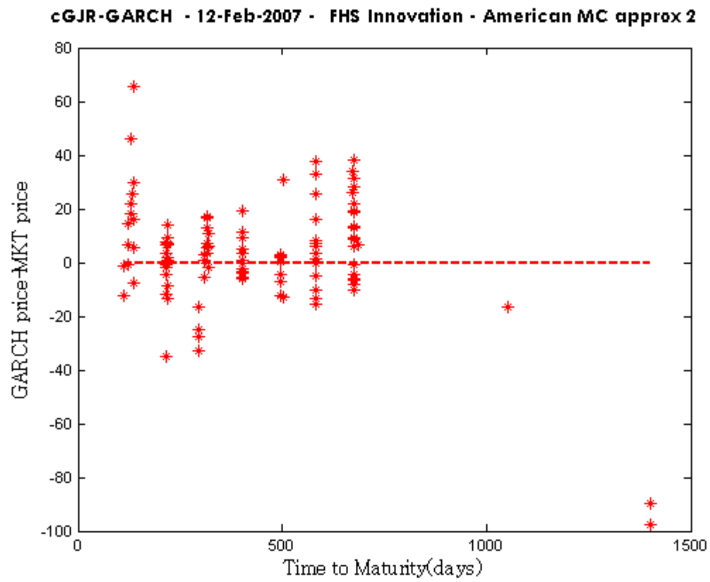


Figure 2.9: Pricing errors with respect to the time to Maturity of the cGJR-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007 (FHS Innovation)

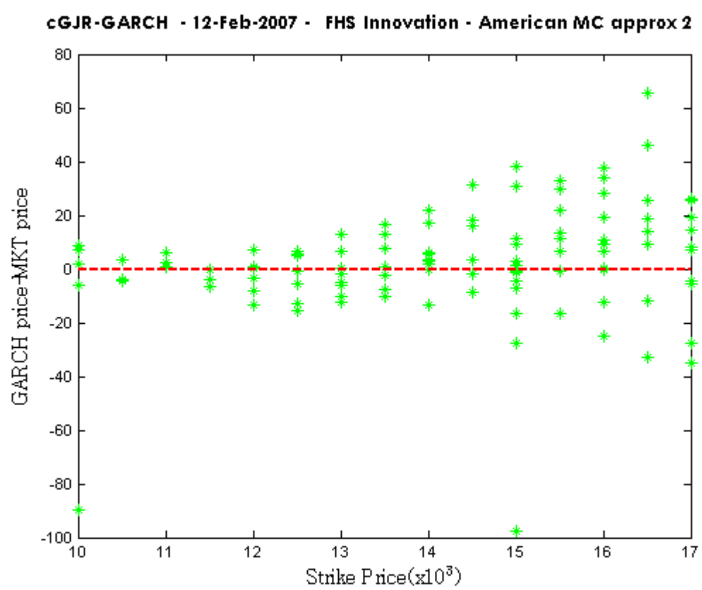


Figure 2.10: Pricing errors with respect to strike price of the cGJR-GARCH model to 114 out of the money Nikkei Put Warrants prices observed on February 12, 2007

The calibration exercise results using Gaussian Innovations on February 12, 2007 is showed in figure 2.2, 2.3 and 2.4. The figure 2.5 shows the pricing error of the cGJR-GARCH model with respect to the time to maturity. As we can see the increasing bias trend present in the Simple GARCH model are completely disappeared in the Component GARCH models.

The Component models appear to be suitable in short-term and long-term option prices calibration more that the simple models. Some problem on the long-term maturity options appear in the FHS innovation calibration exercise. As showed in figure 2.6, 2.7 and 2.8 the mispricing for the long-term options is evident, and it is also verifiable in figure 2.9.

Here it is the case to suspect that the horizontal barrier, used to compute the option price in our approximation and deduced from the Gaussian dynamics of the asset price process, could be not reasonable when we use FHS innovation to lead the price process This observation will be discussed in the conclusion.

Also for the Component models a moderate mispricing is present with respect to the time to maturity, but it has a smaller error. This evidence is valid all models both in Gaussian innovation case and FHS innovation case, therefore we present only the figure related to the cGJR-GARCH model (figure 2.10).

After we have shown some statistical characteristic of the NPWs analyzed by the BAEM approach using our Monte Carlo American option approximations we conclude our analysis by adding the economical analysis of the State Price Density per unit probability.

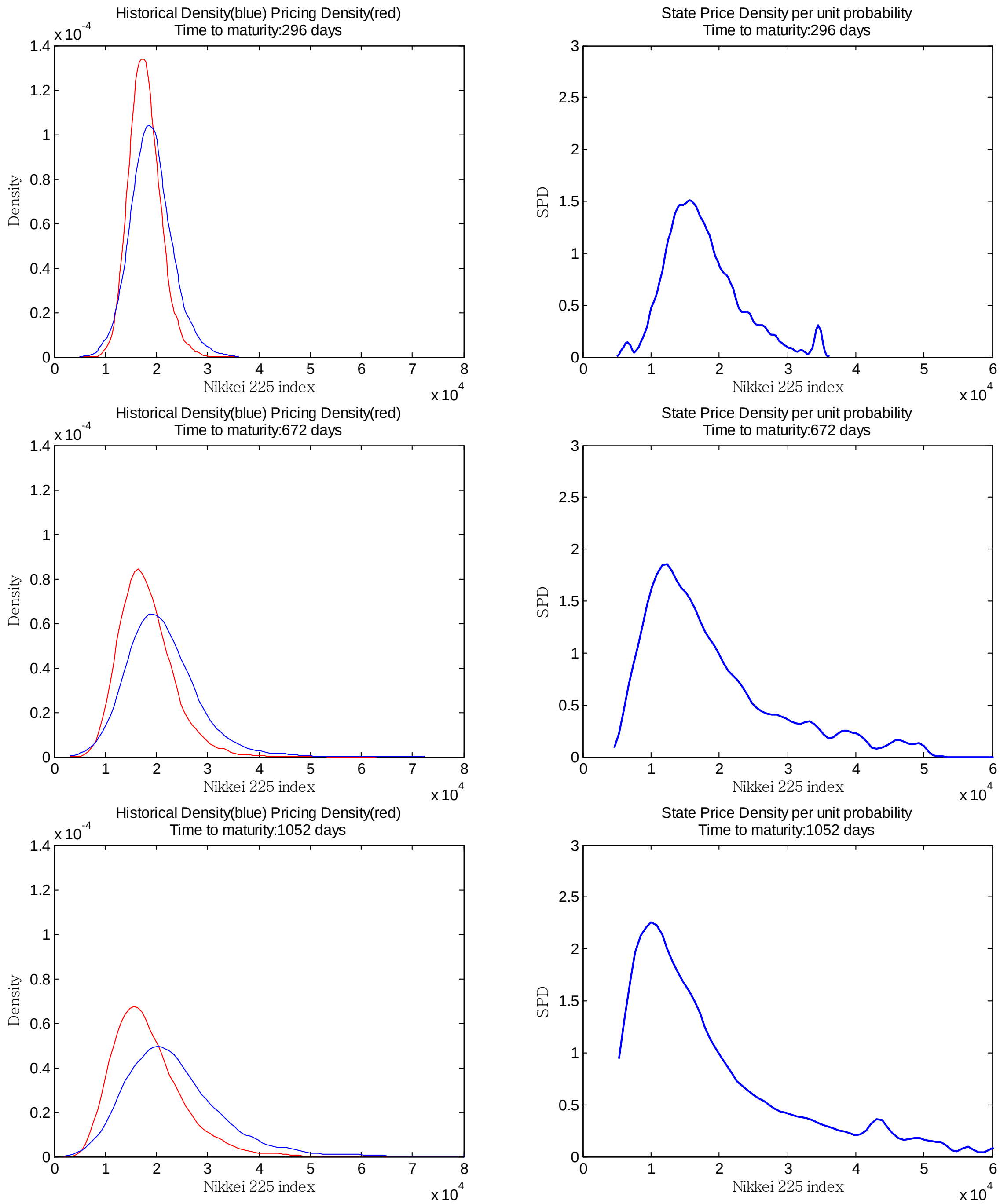
2.4.3 State price Density Estimations

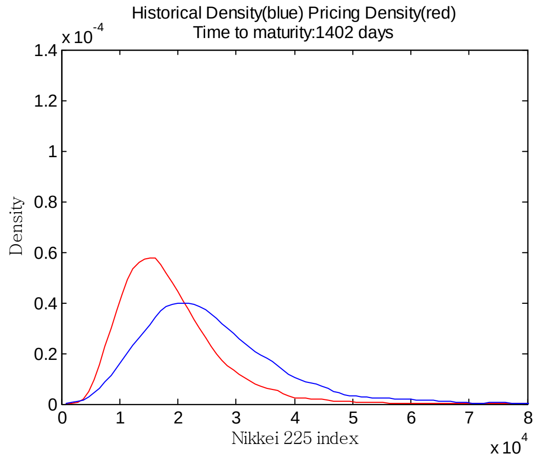
The calibration exercise could appear as a simple fitting exercise involving the absence of arbitrage and some pricing process stability over time and across maturities. The State Price Density investigation can endow the option pricing work of the economic meaning underlying. The SPD summarizes the investor preferences for aggregate wealth in different states of economy. From the restrictions on the investor preferences imposed by Economic Theory we expect a decreasing SPD with respect to the economy state. In order to realize an economic validation of the model we perform some empirical verification on the SPD in this direction.

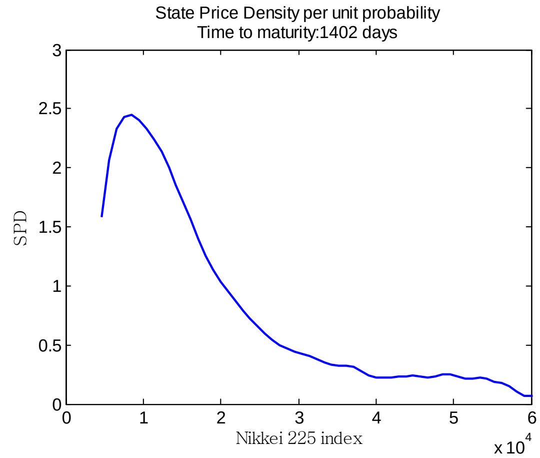
One possibility is to refer to the Put option price formulation in 1.27: we can estimate State Price density per unit probability ($M_{0,T}$) as discounted ratio of the pricing and the historical densities derived from the two related GARCH models.

In order to assess the economical meaning of the SPD we study the historical density and the pricing (risk-neutral) density as derived through the Monte Carlo simulation of the two GARCH processes previously estimated.

We present in the next figures the Historical distributions and the Pricing distributions and the related SPD for different time to maturity of the cGJR-GARCH on February12, 2007.


Historical Density(blue) Pricing Density(red)
Time to maturity:296 days
State Price Density per unit probability
Time to maturity:296 days
Historical Density(blue) Pricing Density(red)
Time to maturity:672 days
State Price Density per unit probability
Time to maturity:672 days
Historical Density(blue) Pricing Density(red)
Time to maturity:1052 days
State Price Density per unit probability
Time to maturity:1052 days
Density
SPD
Nikkei 225 index
x 10^-4
x 10^4





These figures confirm some results obtained before and allow to redirect some question for future researches.

The SPD presents an initial part which appear to be inconsistent with fundamental assumptions about investor behavior. If we interpret the SPD as the investor's marginal utility as the economic theory suggests, in this initial state range investors behave as if they are risk seeking (and not risk averse).

We also note that as the maturity increase as the SPD tends to become monotonically decreasing and the empirical evidence reconcile with the economic theory.

Therefore although the economic investigation on our results highlights some inconsistencies in the short-term modelling, in the long-term the SPD validation appear reasonable considering also that in the lowest states and lowest maturities we have very few observations to rely on the SPD estimation goodness.

Also here we can suspect that our American option approximation doesn't work very well for the short-term options, but as maturity becomes longer as the performance becomes reasonable.

This suggests our setting suitable in treating American derivatives more long in terms of maturity respect to our database.

Chapter 3

Conclusions and critical comments

In this work we have shown a study on long-term American options and the empirical evidences emerging from the Nikkei Put Warrants analysis. The main problems encountered in this work are related to the difficulty in forecasting long-term phenomenon with accuracy, the American option pricing closed-formula inexistence, the strong assumptions of the currently used option pricing theory. In our study we can relax some strong assumptions by using the BAEM approach (Barone-Adesi, Engle, Mancini (2007)). In the BAEM work the authors studied European options on S&P500 index while we use their approach to analyze American options on Nikkei 225 index.

We study three basic GARCH models widely known in literature: N-GARCH (Engle and Ng (1993)) GJR-GARCH (Glosten et.al.(1993)) and E-GARCH (Nelson (1991)). These models have one additional parameter respect to the standard GARCH(1,1) devoted to model the asymmetric response of the conditional variance to the market news arrival.

The empirical analysis shows that these basic GARCH models don't achieve a performance acceptable level in explain the cross sections of the Nikkei Put Warrants investigated.

If a first explanation for the bad performance seems to be impute to our option price approximation accuracy, a succeeding analysis incline us to moderate our criticism while suggests that the basic models have to be extended in order to manage with long-term options.

It is known in empirical analysis studies that as the maturity range increases as the option pricing model performance generally deteriorate. This is mainly due to the increasing difficult to perform forecast as the horizon become long or, in other words, to the difficulty in correctly model a stochastic financial phenomenon for a long-term period.

In order to cope with this problem we have extended our three basic GARCH models by constructing the related component GARCH model versions as suggested by Engle and Lee (1999). Christoffersen et al. (2004) built the component version of the Heston and Nandi's model and studied European derivatives rely on a European option price close-form solution. We study American contingent claims and we cannot rely on a closed form solution. In order to price the American options we propose some approximations. The approximations used in the empirical analysis are derived starting from the perpetual option framework under Gaussian dynamics. We use an horizontal barrier and a Monte Carlo simulation to compute the theoretical American option price.

The Component GARCH models show to have an appreciable ability to explain the option cross-section. Among the models investigated the Component GJR-GARCH model often dominates the other models. We have used both Gaussian innovation and FHS innovation to drive the simulation in order to obtain both a parametric estimation and a non-parametric estimation of the error.

Unluckily the FHS based estimations don't improve the Gaussian results as we expect. But we have to underline that our American option prices are approximations derived on theoretical option price based on Gaussian innovation therefore we cannot exclude that other option price approximation can realize better results. We observe that while the horizontal barrier used to localize the early exercise area seems to work well in the Gaussian innovation case, in the FHS innovation case the same procedure makes worse the results if measured by the RMSE (Root Mean Squared Error) and by the APE (Absolute Pricing Error).

The State Price Density investigation have allowed to obtain two important results:

1) Our setting is confirmed to be suitable in treating American derivatives very long in terms of maturity

2) In order to treat both short-term and long-term American derivatives it needs a more accurate American option price approximation.

The importance of an efficient American option price under GARCH dynamics is more times underlined in this work. As alternative option price approximation we propose to use the Duan and Simonato's method. We theoretically develop their approximation for each model investigated in our work. Finally we present some suggestions to improve the computational efficiency of this approximation method and show some possible future development.

We terminate this work by underling the importance of the research on American

Options especially in a flexible framework such as the GARCH one. The BAEM approach has allowed to consider conjointly the statistical properties and the economic meaning of the pricing process. We left to future researches a deeper analysis in and out-the sample based on an American GARCH Option price formula more general than our Monte Carlo approximation which should allow us to compute both short term than long-term options accurately.

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