Pricing Credit Derivatives: Beyond the Market Standard Model

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Introduction

Today, the major task a financial institution has to perform is *risk management*, which we can define as evaluating and controlling exposure to risk. Risk is composed of three different components:

- market risk

- credit risk

- operational risk.

Let us start with a very short description of these risks, even though in the following we will mainly focus on market and credit risk, in particular on the last one.

Market risk is due to changes in some market variables, such as equity prices, interest and exchange rates and it is measured by looking at changes in the portfolio value, at profits and losses.

Credit risk is stricly bound to the credit merit of the debtor, in particular it reflects both its risk of default and downgrading. This is the reason for which credit risk includes *default risk, credit spread risk* and *downgrade risk*.

In order to define *operational risk*, we can look at the famous document *International Convergence of Capital Measurement and Capital Standards. A Revised Framework*, commonly known as *the Basel II Capital Accord*, issued by the Basel Committee on Banking Supervision in 2004. It describes this risk as "the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events".

Credit Risk

As we mentioned previously, *credit risk* includes three different types of risk:

- *Default risk* can be defined as the risk that the debtor will not be able to fulfill his or her obligation in full. Clearly, default can be complete in that no amount of the outstanding debt will be repaid, otherwise it may happen to have a partial default if only a portion of the original loan will be recovered. In order to have an estimation of this risk, there exist three main rating agencies, that is *Standard & Poor's Corporation*, *Moody's Investors Service* and *Fitch Ratings*, whose task is valuing default in the form of a credit rating.
  In particular, a credit rating has to be interpreted as a forward-looking assessment of the probability of default and of the relative magnitude of the eventual loss when we are looking at the long-term debt obligation.

- *Downgrade risk* is nothing else that the risk arising from a down-grade of the credit merit of an issue or issuer by a nationally recognized rating agency such as Standard & Poor's, Moody's Investors Service or Fitch Ratings. In general, a down-grade of an outstanding credit rating relies on the simultaneous evaluation of the issuer's current earning power and its capacity to fulfill its obligations as they become due.

- *Credit spread risk* is the risk that the spread associated to a specific obligation, over a benchmark rate, will increase, highlighting the financial market's reaction to a perceived credit deterioration which follows a credit review performed by an independent rating agency.

In 1988, the *Basel Accord* was issued and it established that internationally active banks had to hold a capital equal to, at least, 8% of a set of assets measured differently on the basis of their own degree of risk.
The main purpose of this accord was providing international banks with an adequate level of capital, and then, avoiding that banks were no longer able to do business without an opportune amount of capital: this could lead to an increase of the competitiveness of the overall bank system.
The 1988 accord classified assets into four different *risk-buckets*, depending on the risk-features of the debtor. In particular:

- *0% risk weight* is in general composed by claims on Organisation for Economic Coordination and Development (OECD) governments.

- *20% risk weight* is assigned to claims on banks incorporated in OECD countries.

- *50% risk weight* usually consists of residential mortgage claims.

- *100% risk weight* where claims on consumers and corporates are contained.

Once we have found these risk weights, we should multiply them by the respective exposure and in such a way we get what is called as *risk-weight assets* (RWA).
Even if the two main goals were achieved, the overall judgement about this accord could not be satisfactory, since it resulted to be in conflict with the larger

and larger level of sophistication of the internal measures of economic capital determined by banks. Furthermore, in certain cases, it was an incentive to reduce credit risk artificially, by using standard credit risk mitigation techniques. Thus, in 2004 the Basel Committee on Bank Supervision issued a new accord, *International Convergence of Capital Measurement and Capital Standards. A Revised Framework*, commonly known as *Basel II*, to improve the previous one. On the basis of this accord, banks are now allowed to choose between the so called *standardized approach* and the *internal ratings-based approach* in order to compute their capital requirements for credit risk. Here a brief description of these approaches follows:

- *Standardized approach.* In order to determine risk weights, banks can exploit assessment offered by external credit assessment institutions, if these are recognized to be eligible for capital purposes by regulators of banks.

- *Internal ratings-based approach.* Accordingly with this approach, Banks may use their internal rating systems, under the condition for which these systems have to be approved by banks' regulators.

Credit Risk Models aim to achieve mainly two different purposes. The first one consists of *measuring credit risk*. In particular, these models must give an estimation of the probability an obligor (or a set of obligors) will default before the end of a given time horizon and, when this occurs, of the magnitude of the expected loss.

Furthermore, when we deal with a portfolio of risky assets, a credit risk model should give an estimate of the credit quality correlation among all the obligors within the portfolio.

The number of problems to face when we have to build these models could be large, since they have to be consistent with both the financial theory and rational aspects reflecting the reality, such as bankruptcy laws, real credit spreads in the market, and so on.

The second goal of credit risk models is *pricing defaultable assets*. In order to do this, these models estimate credit risky cash flows using model default probabilities and expected losses. In such a way, investors may have an assessment term to know how they should be compensated if they accepted credit risk.

The method with which credit risk management is performed consists of estimating credit risk exposure and evaluating defaultable assets under several different scenarios which have to be generated.

Let us introduce the three main categories of credit risk models:

- Structural models;

- Reduced-form models;

- Credit-rating models.

All these three models are used in credit risk analysis, even though each of them displays certain pitfalls. In particular, it could be shown that *Structural*

models are not able to provide credit spreads as they are observable in the market, whereas *Reduced-form models* show difficulties in modelling dependency among defaults of different obligors within the portfolio.

1.1   Structural Models

In 1973, Black and Scholes proposed, for the first time, the widely known *Firm's Value approach* in the article *The Pricing of Options and Corporate Liabilities*. On the basis of this paper, Merton in 1974 expanded their idea and presented a model, the *Merton Model*, where default can be only triggered at maturity of the debt. In firm's value models we assume there exists a fundamental process V which is usually interpreted as the total value of the assets of the firm that has issued the bonds we are considering. Then, the value of the firm moves around it in a stochastic way and one can interpret the firm's value as a stock option, with the value of the assets as underlying. An important issue concerns the way in which a default can be triggered: these models assume that default can occur only in two different ways.

The first one, which is also the simplest case, implies that it can occur if at maturity the value of the firm V is not sufficient to pay back the outstanding bonds. Under this assumption, a default can not be triggered during the lifetime of the contract and in this case we stand in the *original Merton model framework*. The second way, reflecting a more realistic situation, allows that a default can be triggered as soon as the value of the firm falls below a given barrier value $\bar{S}$: as a clear consequence we get that, conversely with respect to the previous case, a default can occur not only at the maturity date, but also during the lifetime of the outstanding debt, just as we were considering a standard knockout barrier in equity options. These models are known as *First-passage Time models*.

In the following, we determine the variables playing a role into a structural model and then we describe their behaviour. Thus:

- The first variable is the underlying security, that is the value of the firm's assets V for which we can assume it follows a geometrical Brownian motion:

$$\frac{dV}{V} = rdt + \sigma dW \tag{1.1}$$

  where the risk-free interest rate r denotes the drift of the Brownian motion and σ its volatility.
  Note that it is possible to set the drift of the firm's value to r just because we have assumed it can be found out from traded securities: in such a way we only need the risk-neutral dynamics in order to price.

- The second variable is the amount of the claims on these assets.
  Generally, just to make the model simpler, we can assume the existence of only a single issue of debt (in particular we consider only zero-coupon bonds of total face value $\bar{D}$), even though multiple issues can be considered as well, by using a seniority structure.

The following step determines how a default is triggered and how much it depends on the capital structure of the firm. We are going to present some different alternatives:

- A default can only occur at the maturity of the outstanding debt (or at coupon dates if we consider coupon bonds) and this means that we assume the firm will continue to work until it has to pay back its debt.

- If in the issued debt contract there exist some convenants allowing the creditors to close down and liquidate the firm as soon as the value of its assets falls below a given threshold $\bar{S}$, then a default occurs when

$$V_t \leq \bar{S}.$$

- We have a variation of the previous case when $\bar{S}$ is not a constant barrier but depends on the time. This happens, for example, when there exists a covenant stating that a default can only occur when the firm's value falls below the *discounted* value of the assets outstanding, that is the case when
$$V_t \leq \bar{S}B(t,T),$$

where $B(t,T)$ denotes the value at time t of a default-free bond of the same maturity as the debt outstanding.

- Finally, we can say that within a practical implementation it could be possible to implement models based on more realistic covenants of the debt contract; however, this could be the case of a loss in terms of accuracy, since the model would be characterized by some other strong approximations.

From now on, we will consider the third case, i.e. the default is triggered when $V_t \leq \bar{S}B(t,T)$.
Another important issue we have to take into consideration is the capital structure of the firm, since the payoffs of the different securities depend upon it:

- The defaultable bonds $\bar{B}$ pay back their face value $\bar{D}$ in case no default is triggered, conversely they pay back a fraction of the value of the firm minus some bankruptcy costs c, that is:

$$\bar{B}(V, t = T) = min[\bar{D}, V]$$

where $t = T$, is the final payoff date, otherwise:

$$\bar{B}(V = \bar{S}, t) = V - c = \bar{S} - c$$

that is the case the default occurs before the final payoff date $(t < T)$.

- Differently from bonds, shares pay back

$$S(V, t = T) = (V - \bar{D})^+$$

at the final date and, of course, nothing, in case the default is occurred, that is:

$$S(V = \bar{S}, t) = 0.$$

- When we deal with a debt presenting different seniority classes, we should take into account this structure and then modify appropriately the payoffs at the default threshold.

The final step consists of incorporating uncertainty within the interest rate dynamics.

So, it could be a good assumption to consider correlation between the dynamics of the firm's value and the interest rate dynamics. By doing so, we need to add the short term risk-free interest rate r variable within our model; for example, assuming a general one-factor model we will get the following model:

$$dr = \mu_r(r,t)dt + \sigma_r(r,t)d\tilde{W}$$

denoting instantaneous correlation ρ between the Brownian motion driving the firm's value dW and the one driving the interest rates $d\tilde{W}$, that is $dW\, d\tilde{W} = \rho dt$. Conversely, under certain assumptions, it is possible modelling the firm's value process considering it independent from the risk-free interest rate dynamics, but in this case, even if we get a gain in terms of efficiency, at the same time we loose something in terms of accuracy.

1.1.1   Philosophy behind the Pricing in the Structural Models Framework

Before entering a discussion about structural models, it is necessary to specify that we are dealing within a *complete market context* and with the following assumptions:

- The *firm's value process* V is assumed to follow a geometrical Brownian motion

$$\frac{dV}{V} = rdt + \sigma dW. \tag{1.2}$$

- The *interest rate process* r is described as follows:

$$dr = \mu_r(r,t)dt + \sigma_r(r,t)d\tilde{W}. \tag{1.3}$$

  where $\tilde{W}$ denotes the Brownian motion driving the interest rate process, the drift μ_r depends on both the interest rate and time, as well as the volatility σ_r.

- ρ is the *correlation* between the driving process of the firm's value dW and that of the interest rate $d\tilde{W}$, that is:

$$dW\, d\tilde{W} = \rho dt.$$

- Finally, the *default time* τ is defined as the first time at which the firm's value V hits the barrier $\bar{S}B(t,T)$, that is:

$$\tau = min\{t \mid V_t \leq \bar{S}B(t,T)\}.$$

- The securities we are taking into consideration are a share S and a defaultable bond $\bar{B}$ with maturity T. We are assuming that the total number of shares S and bonds $\bar{D}$ and the threshhold $\bar{S}$ are all normalized to 1. Thus, if we consider the case of no default before the maturity date T, we get the following payoff

$$S(V,T) = (V-1)^+$$

for the share and

$$\bar{B}(V,T) = 1 - (1-V)^+$$

for the defaultable bond, where $(.)^+ = max(.,0)$, while, in the case a default occurs before the maturity date T we get

$$\bar{B}(V = \bar{S}B(t,r,T),t) = V - c = B(t,r,T)(\bar{S} - \tilde{c})$$

for the defaultable bond and

$$S(V = \bar{S}B(t,r,T),t) = B(t,r,T)\tilde{c}$$

for the share, where $B(t,r,T)$ is the riskless bond, while $V = S + \bar{B}$ and $c = \tilde{c}B(t,r,T)$ denotes the deviation from absolute priority in favor of the shareholders (with $\tilde{c}$ being the bankruptcy cost per unit of the riskless bond).

- Furthermore, we have a full term structure of traded default risk-free bonds $B(t,T)$.

We consider all securities like derivative instruments on the firm's value V. Therefore we have a knockout-barrier at $\bar{S}B(t,T) = B(t,T)$ at which the default payoffs are triggered, otherwhise we only get the normal payoffs specified by the contracts.

Let us consider the firm's value as a traded security: this can be done because the sum between the share and the bond leads to the firm's value as payoff in every state. Thus, since under risk-neutral valuation the drift of traded securities can be written as rdt, it is possible to re-write the (1.2) as follows:

$$dV_t = r_t V_t dt + \sigma_t V_t dW_t \tag{1.4}$$

whereas we already know the risk-neutral dynamics of the interest rate as shown in the (1.3). Now we can start to get the pricing equation.

We know that bond and share are functions of firm's value V, interest rate r and time t. Let us proceed for the bond price process, but analogously we can easily get the same result for the share price process. Recalling that:

$$dV = rV dt + \sigma V dW_t$$

and

$$dr = \mu_r dt + \sigma_r d\tilde{W}$$

their dynamics is obtained simply by applying *Itô's lemma*:

$$
\begin{aligned}
d\bar{B} &= \left(\frac{\partial \bar{B}}{\partial t} + \frac{\partial \bar{B}}{\partial V}\mu + \frac{1}{2}\sigma_v^2\frac{\partial^2 \bar{B}}{\partial^2 V} + \frac{\partial \bar{B}}{\partial r}\mu_r + \frac{1}{2}\sigma_r^2\frac{\partial^2 \bar{B}}{\partial^2 r} + \rho\sigma_v\sigma_r\frac{\partial^2 \bar{B}}{\partial V \partial r} \right) dt \\
&\quad + \sigma_v\frac{\partial \bar{B}}{\partial V}dW + \sigma_r\frac{\partial \bar{B}}{\partial r}d\tilde{W} \\
&= \left(\frac{\partial \bar{B}}{\partial t} + \frac{\partial \bar{B}}{\partial V}rV + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 \bar{B}}{\partial^2 V} + \frac{\partial \bar{B}}{\partial r}\mu_r + \frac{1}{2}\sigma_r^2\frac{\partial^2 \bar{B}}{\partial^2 r} + \rho\sigma V\sigma_r\frac{\partial^2 \bar{B}}{\partial V \partial r} \right) dt \\
&\quad + \sigma V\frac{\partial \bar{B}}{\partial V}dW + \sigma_r\frac{\partial \bar{B}}{\partial r}d\tilde{W} \\
&= \left(\frac{\partial \bar{B}}{\partial t} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 \bar{B}}{\partial^2 V} + \frac{1}{2}\sigma_r^2\frac{\partial^2 \bar{B}}{\partial^2 r} + \rho\sigma V\sigma_r\frac{\partial^2 \bar{B}}{\partial V \partial r} \right) dt \\
&\quad + \frac{\partial \bar{B}}{\partial V}rV dt + \sigma V\frac{\partial \bar{B}}{\partial V}dW + \frac{\partial \bar{B}}{\partial r}\mu_r dt + \sigma_r\frac{\partial \bar{B}}{\partial r}d\tilde{W} \\
&= \left(\frac{\partial \bar{B}}{\partial t} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 \bar{B}}{\partial^2 V} + \frac{1}{2}\sigma_r^2\frac{\partial^2 \bar{B}}{\partial^2 r} + \rho\sigma V\sigma_r\frac{\partial^2 \bar{B}}{\partial V \partial r} \right) dt \\
&\quad + \frac{\partial \bar{B}}{\partial V}\left(rV dt + \sigma V dW \right) + \frac{\partial \bar{B}}{\partial r}\left(\mu_r dt + \sigma_r d\tilde{W} \right) \\
&= \left(\frac{\partial \bar{B}}{\partial t} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 \bar{B}}{\partial^2 V} + \frac{1}{2}\sigma_r^2\frac{\partial^2 \bar{B}}{\partial^2 r} + \rho\sigma V\sigma_r\frac{\partial^2 \bar{B}}{\partial V \partial r} \right) dt \\
&\quad + \frac{\partial \bar{B}}{\partial V}dV + \frac{\partial \bar{B}}{\partial r}dr
\end{aligned}
\tag{1.5}
$$

Now, since we know that the risk-neutral drift must be equal to $rBdt$, we can write:

$$
\begin{aligned}
r\bar{B}dt &= \left(\frac{\partial \bar{B}}{\partial t} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 \bar{B}}{\partial^2 V} + \frac{1}{2}\sigma_r^2\frac{\partial^2 \bar{B}}{\partial^2 r} + \rho\sigma V\sigma_r\frac{\partial^2 \bar{B}}{\partial V \partial r} \right) dt \\
&\quad + rv\frac{\partial \bar{B}}{\partial V}dt + \mu_r\frac{\partial \bar{B}}{\partial r}dt
\end{aligned}
$$

and therefore the following partial differiantial equation has to be satisfied by $\bar{B}$:

$$
\begin{aligned}
0 &= \frac{\partial \bar{B}}{\partial t} + \frac{1}{2}\sigma^2 V^2\frac{\partial^2 \bar{B}}{\partial^2 V} + \frac{1}{2}\sigma_r^2\frac{\partial^2 \bar{B}}{\partial^2 r} + \rho\sigma V\sigma_r\frac{\partial^2 \bar{B}}{\partial V \partial r} \\
&\quad + rv\frac{\partial \bar{B}}{\partial V} + \mu_r\frac{\partial \bar{B}}{\partial r} - r\bar{B}
\end{aligned}
\tag{1.6}
$$

This last partial differential equation (1.6) is very important because every security on the firm's value, both bonds and shares, but also others, has to satisfy it. The only difference depending on the type of securities can be found in the final and boundary conditions that have to be applied. Let us continue with the

example of bonds.

Hence, *the final condition* for the bond is its payoff in the situation no default occurs:

$$\bar{B}(T, V, r) = min\{1, V\} \tag{1.7}$$

while we have four different *boundary conditions*:

$$\bar{B} = B(t, T)(1 - \tilde{c}) \qquad \text{as } V = \bar{S}B(t, T) \tag{1.8}$$
$$\bar{B} \to B(t, T) \qquad \text{as } V \to \infty \tag{1.9}$$
$$\bar{B} \to 0 \qquad \text{as } r \to \infty \tag{1.10}$$
$$\bar{B} < \infty \qquad \text{as } r = 0 \tag{1.11}$$

It is easy to understand the meaning of these conditions, but a doubt can arise with respect to the last one: the usefulness of the (1.11) can be explained with the necessity to preclude a possible singularity of the solution.

Hence, now a question could immediately arise: what happens if we consider different securities? All we need to do is applying different boundary conditions and also a different final condition.

1.1.2 Strenghts and Weaknessess of Structural Models

Structural models for defaultable bonds allow to get good results if there exist important relationships among the prices of different securities issued by the firm. Typical examples are offered by *convertible* and *callable* bonds, giving the issuer the right to convert them into shares.

Furthermore, by using these models, it is possible to obtain the prices of defaultable bonds directly from the firm's value.

Another strong advantage can be represented by the fact that they may be also used in the analysis of issues concerning corporate finance such as the definition of the optimal capital structure for the firm.

Unfortunately, this kind of orientation towards fundamentals can also represent a drawback. In fact, it is always quite difficult to define the process describing in the correct way the firm's value, or however it can be very hard to calibrate it and sometimes this process can not exist. Hence, even though this process can be found, it could become too complex to adapt it to real applications and, in certain cases, it could quickly become infeasible.

Another disadvantage is the unrealistic nature of the short-term credit spreads that can be implicitly obtained by the application of these models: in fact empirical observations show low values of them. In particular, we know they tend to zero as the maturity of the debt becomes more and more close.

All these problems lead to the conclusion that generally structural models should be more properly exploited as a rough guideline.

If we come back to the advantages, we could also mention their usefulness in providing an easy and intuitive way to encompass correlations within a portfolio framework: this is exactly what is done by the famous company *JP Morgan* in their CreditMetrics model, presented for the first time in *CreditMetrics-Technical Document* (1997).

Thus, the obvious conclusion is that, in pricing credit risk derivatives, could be more useful to have a model in which prices of defaultable bonds are taken as fundamentals, rather than be calculated. This is the main reason to introduce *Reduced-form models*.

1.2   Reduced-form Models

The origin of the name *Reduced-form* is due to Darrel Duffie who used for the first time this term to distinguish a new kind of model from the *Structural-form models.*
The great difference between a reduced-form model and a structural model is that the former does not make any consideration about the financial structure of the company, but models the default process directly.
This means that in Reduced-form models the default is an *exogeneous* event, and not *endogeneous* as it holds for Structural models. In particular, in all Reduced-form models, the modelling of the default process is done by a *Poisson process*, with a default intensity λ which can be either constant (**homogeneous Poisson process**), or time-dependent deterministic (**inhomogeneous Poisson process**) or stochastic (**Cox process**).

1.2.1   Poisson Processes

A *Poisson process* N_t is, just to get intuitively its meaning, a process which is characterized by rare value changes: in particular the probability of a change in its value gets smaller as the observable period becomes shorter, while the size of these changes in value is fixed. Let us consider $T_1, T_2, T_3...$ the *times of the jumps*, then we get

$$P[N_{t+\Delta t} - N_t = 1] = \lambda \Delta t \tag{1.12}$$

that is, we are assuming that the probability of a jump in the next time interval Δt depends proportionally on Δt and, since we are also considering *size-constant* jumps (in particular jumps by more than 1 are not allowed) we can write

$$P[N_{t+\Delta t} - N_t = 0] = 1 - \lambda \Delta t$$

and considering the interval $[t, t + 2\Delta t]$ the probabilty becomes

$$P[N_{t+2\Delta t} - N_t = 0] = P[N_{t+\Delta t} - N_t = 0]P[N_{t+\Delta t} - N_{t+\Delta t} = 0] = (1 - \lambda \Delta t)^2.$$

If we subdivide the interval $[t, s]$ into i subintervals with the same length $\Delta t = (s - t)/i$ in each of these subintervals a jump occurs with probability $\Delta t \lambda$. This leads us to say that

$$P[N_s = N_t] = (1 - \Delta t \lambda)^i = (1 - \frac{1}{i}(s - t)\lambda)^i$$

is nothing else that the probability of *no* jump over the overall time period $[t, s]$ and, since $1 + \frac{x}{i} \to e^x$ as $i \to \infty$, then

$$P[N_s = N_t] \to e^{-(s-t)\lambda}.$$

Note that the value $\frac{(s-t)}{i}$ denotes Δt, therefore $i \to \infty$ means $\Delta t \to 0$.

Let us look for the probability of exactly *one* jump within $[t, s]$. Intuitively, it is easy to understand that there are i chances to have only a single jump, therefore the total probability is given by

$$\begin{aligned}
P[N_s - N_t = 1] &= i\Delta t \lambda (1 - \Delta t \lambda)^{i-1} \\
&= i\frac{(s-t)}{i}\lambda(1 - \frac{1}{i}(s-t)\lambda)^i / (1 - \frac{1}{i}(s-t)\lambda) \\
&= \frac{(s-t)\lambda}{1 - \frac{1}{i}(s-t)\lambda}(1 - \frac{1}{i}(s-t)\lambda)^i \\
&\to (s-t)\lambda e^{-(s-t)\lambda}.
\end{aligned}$$

as $i \to \infty$.

By extending the same arguing to the case of *two* jumps, we get

$$P[N_s - N_t = 2] = \frac{1}{2}(s-t)^2\lambda^2 e^{-(s-t)\lambda}$$

and finally, for n jumps

$$P[N_s - N_t = n] = \frac{1}{n!}(s-t)^n\lambda^n e^{-(s-t)\lambda} \tag{1.13}$$

We can define what a Poisson process is, by using equation (1.13):

Definition 1 *A Poisson process with intensity λ is a non-decreasing, integer-valued process with initial value N_0 whose increments satisfy equation (1.13).*

Note that both in this approximation of the Poisson process within a discrete-time framework and in the Brownian Motion, there is a number of binomially distributed random variables we add up in order to get the process.

The only difference we can underline is in the limit behaviour of these two processes: in fact the Brownian Motion is characterized by a *decreasing jump size* (proportional to $\frac{1}{\sqrt{i}}$) and *constant probabilities*, whereas in the Poisson Process we consider exactly the opposite: a *constant jump size* (at one) and a *decreasing probability* (proportional to $\frac{1}{i}$).

Finally, by exploiting the (1.12), it becomes plausible to deal with a *large portfolio of defaultable bonds* that are all driven by *independent Poisson processes*: by doing so we will assume that Poisson events occur almost continuosly with a rate given by λdt and in such a way we could shift toward a continous rate of events.

Since defaults are rare and discrete, we can use Poisson processes in order to

model the time of default of a firm as *the time of the first jump of a Poisson process* with intensity λ. Here, some important properties of a Poisson process follow:

- It is a process for which the *Markov property holds*. In fact, Poisson processes have no memory, therefore the probability of n jumps in the interval $[t, t+s]$ is independent of N_t and thus the history of N does not affect the next occurence of the process.

- The inter-arrival times of a Poisson process $(T_{n+1} - T_n)$ are exponentially distributed with the following density:

$$P[(T_{n+1} - T_n) \in tdt] = \lambda e^{-\lambda t} dt.$$

- The probability to have more than one jump at the same point in time is zero.

Within a financial modeling framework,we should consider the following further specifications:

$$\mathbf{E}[dN] = \lambda dt \tag{1.14}$$

$$dNdN = dN$$

$$\mathbf{E}[dN^2] = \lambda dt$$

$$\mathbf{E}[dNdW] = 0$$

In addition, a modified version of Itô's lemma is required, because we need to deal with jumps in processes. Here is the reason for which we are going to consider a twice continuosly differentiable function f to decompose the process x into a continuous part x^c and a discontinuos part Δx:

$$dx = dx^c + \Delta x.$$

By doing so,the Itô's lemma is given by

$$\begin{aligned} df(t,x) = & \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx^c + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} d < x^c > \\ & + (f(t, x + \Delta x) - f(t,x)). \end{aligned} \tag{1.15}$$

What we have simply done is to add a jump term $\Delta f = f(t, x + \Delta x) - f(t,x)$ to the usual form of Itô's lemma and, [1] exactly the same could be done for a multidimensional process:

$$\begin{aligned} df(t,x) = & \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i^c + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} d < x_i^c, x_j^c > \\ & + (f(t, x + \Delta x) - f(t,x)). \end{aligned} \tag{1.16}$$

---

[1]Considering as usually $dx^c = \mu x^c + \sigma dW$, thus $d < x^c >$ is equal to $\sigma^2 (x^c)^2 dt$

So far, we have talked about so called **homogeneous Poisson processes** but now let us shift to **inhomogeneous Poisson processes**: they are characterized by an intensity λ that is a function of time $\lambda(t)$. For this kind of processes is not difficult to derive all the probabilities we have computed previously for homogeneous processes. In fact, by replacing the constant $\lambda \Delta t$ by the the integral $\int_t^{t+\Delta t} \lambda(s)ds$ we can get, quite easily the following:

- $P[N_{t+\Delta t} - N_t = 1] = \lambda(t)\Delta t$

  that is the probability of a jump in the next time interval Δt (compare it with (1.12));

- $P[N_{t+\Delta t} - N_t = 0] = e^{-\int_t^{t+\Delta t} \lambda(s)ds}$

  denoting the probability of *no* jump within the interval $[t, t+\Delta t]$ and finally

- $P[N_{t+\Delta t} - N_t = n] = \frac{1}{n!}(\int_t^{t+\Delta t} \lambda(s)ds)^n e^{-\int_t^{t+\Delta t} \lambda(s)ds}$

  that represents the probability of n jumps in the interval $[t, t+\Delta t]$.

Now, we can go on describing **compound Poisson processes**. Up to now we have talked about Poisson processes just to model the occurring of a specific event, but we still have not discussed about consequences concerning these events. If we are dealing with a credit portfolio, it is important both to forecast the time of a default and also its size. For this purpose, compound Poisson processes can help us since in these processes, at each time T_i representing the time of a default, we have a random variable Y_i which is drawn from a distribution $K(dY)$. In order to deal with these processes, we have to consider a further variable

$$X_t = \sum_{T_i \leq t} Y_i$$

which can denote, for example, the total (or cumulative) loss of the bond portfolio until the time t and an additional function $f(X)$ satisfying the following properties:

- $dX = \Delta X = YdN$

- $df = \Delta f = (f(X+Y) - f(X))dN$

- $\mathbf{E}[dX] = \int yK(dY)\lambda dt = y^e \lambda dt$

- $\mathbf{E}[df(X)] = \int (f(X+y) - f(X))K(dy)\lambda dt$

where y^e is the local expectation of Y. Note that, by these properties, we can get that the moments of the defaults and their sizes are indepedent. Furthermore, $\mathbf{E}[df(X)] \neq f(\mathbf{E}[dX])$ and Itô's lemma does not change with respect to (1.16).

If we want to perform a Monte Carlo simulation to get a Poisson process, for sure, it is better to use the equation (1.13), than the (1.12).

In particular, when we are interested in pricing defaultable securities, generally, we have to pay attention to the first jump of a Poisson process and we can correctly assume that each jump has the size equals one: therefore we have to look for the probability of at least one jump within the time interval Δt we are analyzing, that is:

$$P[N_{t+\Delta t} - N_t > 0] = 1 - e^{-\Delta t \lambda}$$

So, in order to go on with our simulation, we should draw a random number from the interval $[0, 1]$: then we assume a jump has occurred if this number is lower than $1 - e^{-\Delta t \lambda}$ (in this case we will increment the Poisson process), otherwise nothing occurs. Of course, we should also remember that this is not the best way to proceed because we need a great number of runs if we want to get an *acceptable* result.

1.2.2 Pricing: Zero Recovery and Positive Recovery Cases

In general, when we want to find the price of a defaultable bond $\bar{B}$, the simplest way to start consists in assuming that the bond has *zero recovery* in case of default, which is triggered when the first jump of the Poisson process occurs.

Once again, we need to use the Itô's lemma, but first, in order to simplify the computational task, we want to decompose the defaultable bond value process into two different parts:

$$\bar{B}(t, r, N) = \bar{B}^C(t, r, N) + \bar{B}^D(t, r, N)$$

where $\bar{B}^C(t, r, N)$ denotes the continuos part of the process and $\bar{B}^D(t, r, N)$ the discontinuos part due to jumps following defaults. By doing so we can start computing the dynamics of the bond price for the continuos part and only after this we are going to add the discontinuos part.

Thus:

$$\begin{aligned}
d\bar{B}(t, r, N)^C &= \left(\frac{\partial \bar{B}}{\partial t} + \frac{\partial \bar{B}}{\partial r}\mu_r + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2\right)dt + \sigma_r\frac{\partial \bar{B}}{\partial r}d\tilde{W} \\
&= \frac{\partial \bar{B}}{\partial t}dt + \frac{\partial \bar{B}}{\partial r}\mu_r dt + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 dt + \sigma_r\frac{\partial \bar{B}}{\partial r}d\tilde{W} \\
&= \frac{\partial \bar{B}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 dt + \frac{\partial \bar{B}}{\partial r}(\mu_r dt + \sigma_r d\tilde{W}) \\
&= \frac{\partial \bar{B}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 dt + \frac{\partial \bar{B}}{\partial r}dr
\end{aligned}$$

and now, simply by adding the discrete part up, we get

$$d\bar{B}(t, r, N) = \frac{\partial \bar{B}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 dt + \frac{\partial \bar{B}}{\partial r}dr - \bar{B}dN$$

When a jump $dN = 1$ occurs, we have $d\bar{B} = -\bar{B}$ and, simply because the recovery rate is zero, the defaultable bond value collapses to zero.

Now, it is possible to continue:

$$r\bar{B}dt = \frac{\partial \bar{B}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 dt + \frac{\partial \bar{B}}{\partial r}\mu_r dt - \bar{B}\lambda dt$$

What we have done is simply to set $\mathbf{E}[d\bar{B}] = r\bar{B}dt$ and then, by using $\mathbf{E}[dN] = \lambda dt$, we get the pricing equation

$$0 = \frac{\partial \bar{B}}{\partial t} + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 + \frac{\partial \bar{B}}{\partial r}\mu_r - \bar{B}(\lambda + r) \qquad (1.17)$$

It is very easy and intuitive to understand that, if we were pricing a default free bond, we would simply get the following

$$0 = \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma_r^2 + \frac{\partial B}{\partial r}\mu_r - Br$$

since we have not to consider the default risk interest rate λ in the last discounting term. If we come back to the (1.17), taking into consideration that B denotes the price of a riskless bond, it could be shown that

$$\bar{B}(t, r) = B(t, r)e^{-\lambda\Delta t}$$

is nothing else that the solution to the pricing equation (1.17), from which we can also derive directly the spread between the defaultable and the riskless bond

$$s(t, t + \Delta t) = \frac{1}{\Delta t}(lnB - ln\bar{B}) = \lambda$$

It should not be a surprise to see that this spread is just the intensity λ of the default process N.

Unfortunately, the reality is almost always characterized by a *positive recovery*, and not as we have done so far, by a zero recovery. This is the reason for which we are going to give a brief explanation of this situation and then we assume that the defaultable bond will pay back only a fraction of the nominal in case of default: this portion is $(1-c)$, where c denotes *bankruptcy costs*. Now, in order to get the new dynamics of the defaultable bond price, we have simply to consider a further term in the equation we computed previously. Therefore, the result becomes

$$d\bar{B}(t, r, N) = \frac{\partial \bar{B}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 dt + \frac{\partial \bar{B}}{\partial r}dr - \bar{B}dN + (1 - c)dN$$

The additional term is nothing else that the recovery term $(1 - c)dN$. Now, simply by going on as before, we can get the following equation

$$0 = \frac{\partial \bar{B}}{\partial t} + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 + \frac{\partial \bar{B}}{\partial r}\mu_r - \bar{B}(\lambda + r) + (1 - c)\lambda \qquad (1.18)$$

meaning that, in case of default, we will lose the value of the bond $(-\bar{B}dN)$ but we will get back exactly the recovery term $(1 - c)dN$.

This way to operate could seem good, but it is not, because it describes a quite unrealistic situation. In fact, in order to be more consistent with the reality, it would be better to assume that, when a bond defaults, bondholders lose a given amount q of the nominal value of their claims, but, since a reorganisation takes place, both the issuer and claims keep on continuing and thus other defaults are possible later on. The conclusion is that *multiple defaults* can happen and so, the dynamics $d\bar{B}$ has to be modified again, becoming

$$d\bar{B}(t,r) = \frac{\partial \bar{B}}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 dt + \frac{\partial \bar{B}}{\partial r}dr - q\bar{B}dN$$

The last term $q\bar{B}dN$ is simply a jump due to the default of the bond. We are working by assuming that the price of the bond is no longer dependent of the number of defaults N we had so far. Since the only factor affecting a default is now the direct loss in face value and thus, setting $\mathbf{E}[d\bar{B}] = r\bar{B}dt$, we get the new pricing equation

$$0 = \frac{\partial \bar{B}}{\partial t} + \frac{1}{2}\frac{\partial^2 \bar{B}}{\partial r^2}\sigma_r^2 + \frac{\partial \bar{B}}{\partial r}\mu_r - \bar{B}(q\lambda + r) \qquad (1.19)$$

The new pricing equation (1.19) has to be considered as a more realistic and simpler model. Its solution is

$$\bar{B}(t,r) = B(t,r)e^{-q\lambda\Delta t}$$

from which we can get the yield spread formula

$$s(t, t + \Delta t) = q\lambda$$

As we have already said, this model is simpler and more consistent with the reality and for these reasons, it should be preferred with respect to the others.

1.2.3   Poisson Processes with Stochastic Intensity: Cox Processes

We mentioned before that, for a defaultable zero coupon bond, the spread is nothing else that the constant value $q\lambda$. Unfortunately, this result is not consistent with the reality, since we can not observe constant credit spreads in the market: therefore, we should use more flexible models describing in a better way the dynamics of defaultable bond prices and credit spreads. This is the reason for which we introduce *Cox processes*. Cox processes could be defined as Poisson processes with *stocastic intensity*, thus

$$d\lambda = \mu_\lambda dt + \sigma_\lambda dW_2 \qquad (1.20)$$

Once more, we can subdivide the overall time period into i intervals Δt and performing a binomial experiment for each of them: for the probability of a jump in $[t, t + \Delta t]$ we get the following

$$P[N_{t+\Delta t} - N_t = 1] = \lambda_t \Delta t$$

This means that there is a certain kind of dependence among the jumps, given by the intensity λ. Therefore, a process N is a *Cox process* if

it is an integer-valued and non-decreasing process having the property for which, conditional on the realisation $(\lambda_t)_{t \in [0,T]}$ of λ, N is an inhomogeneous Poisson process with intensity λ_t.

We can derive some properties for Cox processes. First of all

$$E_t[dN] = \lambda_t dt$$

must hold together with the other local properties of N. Nevertheless, now we have some different properties as well, such as the probability of n jumps that we can derive from

$$P[N_T - N_t = n] = \frac{1}{n!} \left(\int_t^T \lambda(s)ds \right)^n exp\{- \int_t^T \lambda(s)ds\}$$

which holds for an inhomogeneous Poisson process. From this equation, we can obtain the following

$$P[N_T - N_t = n] = \mathbf{E}\left[P[N_T - N_t = n | \lambda] \right]$$
$$= \mathbf{E}\left[\frac{1}{n!} \left(\int_t^T \lambda(s)ds \right)^n exp\{- \int_t^T \lambda(s)ds\} \right]$$

Note that if we solve the problem for a certain λ and after this we take the expectation over all λ, then we will be able to solve the pricing problem because we are back in the continuos-time framework.

The next step consists in defining the pricing equation. Let us define the following stochastic process describing the dynamics of the risk-free interest rate

$$dr = \mu_r dt + \sigma_r dW_1$$

and for the intensity of the Cox process we consider

$$d\lambda = \mu_\lambda dt + \sigma_\lambda \left(\rho dW_1 + \sqrt{1 - \rho^2} dW_2 \right).$$

We can consider the opportunity of correlation ρ between λ and r, meaning that, tipically, the intensity of defaults increases as r gets larger (in a few words, borrowing becomes more difficult). Furthermore, both dymanics are already computed under the risk-neutral probability. Now, the price of the defaultable security will be affected also by λ and we have to take this into account when we are computing the Itô's lemma. Starting from the consideration for which

$$\bar{B}(t,T) = \bar{B}(t, r_t, \lambda_t, N_t, T)$$

we will obtain the partial differential equation in the following way

$$\begin{aligned} d\bar{B} = & \frac{\partial \bar{B}}{\partial t} dt + \frac{\partial \bar{B}}{\partial r} dr + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \bar{B}}{\partial r^2} dt \\ & + \frac{\partial \bar{B}}{\partial \lambda} d\lambda + \frac{1}{2}\sigma_\lambda^2 \frac{\partial^2 \bar{B}}{\partial \lambda^2} dt \\ & + \rho\sigma_r\sigma_\lambda \frac{\partial^2 \bar{B}}{\partial \lambda \partial r} dt - q\bar{B}dN \end{aligned}$$

In order to get the pricing equation, it is sufficient to set $\mathbf{E}[d\bar{B}] = r\bar{B}dt$. Thus, we get

$$\begin{aligned} 0 = & \frac{\partial \bar{B}}{\partial t} + \mu_r \frac{\partial \bar{B}}{\partial r} + \mu_\lambda \frac{\partial \bar{B}}{\partial \lambda} \\ & + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \bar{B}}{\partial r^2} + \rho\sigma_r\sigma_\lambda \frac{\partial^2 \bar{B}}{\partial \lambda \partial r} + \frac{1}{2}\sigma_\lambda^2 \frac{\partial^2 \bar{B}}{\partial \lambda^2} \\ & - (r + \lambda q)\bar{B} \end{aligned} \qquad (1.21)$$

Note that we used $(r + \lambda q)\bar{B}$ instead of the risk-free interest rate r. Now, we have to consider that the final condition for a defaultable zero-coupon bond with maturity T is

$$\bar{B}(T, r, \lambda) = 1$$

and the boundary conditions are $\bar{B} \to 0$ as $r, \lambda \to \infty$ and $\bar{B} < \infty$ as $r, \lambda \to 0$. From these consideration, it follows that the solution of the (1.21) depends on the given specification of the stochastic processes followed by r and λ.

Now we want to go back to a very general case, in order to show the most important properties which have to hold for the price of a defaultable bond within the modelling framework we have described so far. Let r and λ follow some stochastic processes and N be a Cox process whose intensity is λ. Then

$$B(t) = E_t \left[exp\{- \int_t^T r(s)ds\} \right] \qquad (1.22)$$

That is, the price of a default-free zero-coupon bond B with maturity T is nothing else that the expectation of the discounted value of the final payoff. The same holds for the price F of another security having a final payoff of X:

$$F(t) = E_t \left[exp\{- \int_t^T r(s)ds\}X \right] \qquad (1.23)$$

and for a defaultable zero-coupon bond

$$\bar{B}(t) = E_t \left[exp\{- \int_t^T r(s)ds\} \left(1 - q\right)^{N_T} \right] \qquad (1.24)$$

If you remember the properties of a Cox process and in particular that the expectation of the conditional expectation is the expectation itself, we can use it to expand $E_t[-]$ into

$$E_t[-] = E_t[E_t[-|(\lambda(s))_{s \leq T}]]$$

The problem is that we do not know the path $(\lambda(s))_{s \leq T}$, therefore we have to calculate the so called inner conditional expectation, that is what we would obtain if we knew it. By doing the same for different possible paths, we could consider the expectation over these several results:

$$\bar{B}(t) = E_t \left[E_t \left[exp\{-\int_t^T r(s)ds\}(1-q)^{N_T}|(\lambda(s))_{s \leq T} \right] \right] \quad (1.25)$$

which is known as the outer expectation. This is the way to proceed. Now, let us apply it to our Cox process and thus, we start computing the inner expectation:

$$E_t \left[exp\{-\int_t^T r(s)ds\}(1-q)^{N_T}|(\lambda(s))_{s \leq T} \right]$$

$$= E_t \left[exp\{-\int_t^T r(s)ds\}exp\{-\int_t^T q\lambda(s)ds\}|(\lambda(s))_{s \leq T} \right]$$

$$= E_t \left[exp\{-\int_t^T r(s) + q\lambda(s)ds\}|(\lambda(s))_{s \leq T} \right]$$

Now, we have just to plug it into the outer expectation, and thus, what we get is

$$\bar{B}(t) = E_t \left[exp\{-\int_t^T r(s) + q\lambda(s)ds\} \right]. \quad (1.26)$$

As we can note, the important conclusion is the following:

the price of a defaultable bond with maturity T can be computed, equivalently, as the price of a default-free bond with the same maturity, where we pay attention to use

$$\bar{r} = r + q\lambda \quad (1.27)$$

as the risk-free short rate. For sure, we can get a similar result for a defaultable security F'. In fact we get

$$F'(t) = E_t \left[exp\{-\int_t^T r(s)ds\}(1-q)^{N_T}X \right]$$

$$= E_t \left[exp\{-\int_t^T r(s) + q\lambda(s)ds\}X \right]. \quad (1.28)$$

Equation (1.28) is very important because it suggests a clear analogy between derivatives pricing within a default-free stochastic interest rate modelling and a multiple stochastic intensity framework. Once we have defined the defaultable short interest rate as we did in the (1.27), there is no longer difference between the (1.28) and the equivalent pricing equation where $\bar{r}$ is used as interest rate. Thus, it is possible to use one of the several standard short rate models in order to get the dynamics of $q\lambda$.

Therefore, once we have chosen a default-free interest rate model for the default-free bond prices, we have to choose a process describing the dynamics of $q\lambda$ as well. For doing this, it is better to select it such that there exist closed form solutions for the defaultable bond prices when we we deal with both (1.27) and (1.26). The next step is to fit this model to the market, by using different liquid default bonds issued by the same entity (when they are available) to calibrate the model itself. In this way, now it is possible to use the model we have built to price either bonds issued by the same issuer, or to price derivatives on defaultable bonds or to price credit risk derivatives.

1.3   Credit Rating Models

These last years have been characterized by a rapid diffusion of rating-based models in the credit risk field.

One of the possible reasons that might explain this fact is, no doubt, the straight-forwardness of the approach behind these models, but this reason is not the only one. In fact the actual popularity of these models is also due to the new Capital Accord of the Basel Committee on Banking Supervision (Basel II), since it allows banks to use both Internal and *External Rating Systems* in order to meet their capital requirements.

The natural consequence of this is that the data provided by rating agencies are becoming more and more important in credit risk management.

We have already mentioned the most known rating agencies: in general, regulators do not make distinctions among these different rating agencies, since there is a high degree of congruence between their rating systems. Nevertheless, and this may be considered as an incongruence, it might happen that different rating agencies could assign different rating for the same bond.

Generally, rating agencies publish two different kinds of ratings:

- *issue-specific credit ratings;*

- *issuer credit ratings.*

Issue-specific credit ratings are defined as *current opinions of the credit worthi-ness of an obligor with respect to a specific financial obligation, a specific class of financial obligations of a specific financial program.*

On the other hand, issuer credit ratings have to be considered as *an opinion of the obligor's overall capacity to meet its financial obligations:* it is so easy to understand that this last category of ratings denotes the fundamental credit

worthiness of a company.

Usually, ratings are kept separated with respect to long-term and short-term financial instruments. Focusing on long-term credit ratings, that is credit ratings assigned to obligations with an original maturity of more than one year are divided into several different classes. For example, Moody's credit ratings range from Aaa, denoting the best credit quality, to C, denoting the worst one.

Furthermore, the highest categories in this scale are known as *investment grades*, whereas the remaining grades are called *speculative grades* that characterize the so-called *junk bonds*.

Most of the companies requires to be rated by a rating agency prior to sale or registration of a debt issue. These requests for a rating are mainly due to the fact that, once the rating is determined, the rated company will get a gain in terms of consideration supporting it.

A general peculiarity of rating agencies analysts consists on concentrating on one or two industries only, in order to allow a specialization accumulation over these areas. In such a way, this specialization will offer a higher degree of expertise as well as better competitive information.

Here there is a presentation of the Moody's rating system criteria. Note that Moody's Bond Ratings are intended to characterize the risk of holding a bond. These ratings, or risk assessments, in part determine the interest that an issuer must pay to attract purchasers to the bonds. The ratings are expressed as a series of letters and digits. Here is how to decode those sequences.

- **Rating Aaa**
  Bonds which are rated Aaa are judjed to be of the best quality. They carry the smallest degree of investment risk and are generally referred to as "gilt edged". Interest payments are protected by a large or an exceptionally stable margin and principal is secure. While the various protective elements are likely to change, such changes, as can be visualized, are most likely to change to impair the fundamentally strong position of such issues;

- **Rating Aa**
  Bonds which are rated Aa are judjed to be of high quality by all standards. Together with the Aaa group they comprise what are generally known as high grade bonds. They are rated lower than the best bonds because margins of protection may not be as large as in Aaa securities or fluctuation of protective elements may be of greater amplitude or there may be other elements present which make the long-term risk appear somewhat larger than the Aaa securities;

- **Rating A** Bonds which are rated A possess many favorable investment attributes and are considered as upper-medium grade obligations. Factors giving security to principal and interest are considered adequate, but elements may be present which suggest a susceptibility to impairment some

time in the future;

- **Rating Baa** Bonds which are rated Baa are considered as medium-grade obligations, i.e., they are neither highly protected nor poorly secured. Interest payments and principal security appear adequate for the present but certain protective elements may be lacking or may be characteristically unreliable over any great length of time. Such bonds lack outstanding investment characteristics and in fact have speculative characteristics as well;

- **Rating Ba** Bonds which are rated Ba are judjed to have speculative elements; their future can not be considered as well-assured. Often the protection of interest and principal payments may be very moderate, and thereby not well safe-guarded during both good and bad times over the future. Uncertainty of position characterizes bonds in this class;

- **Rating B** Bonds which are rated B generally lack characteristics of the desirable investment. Assurance of interest and principal payments of maintenance of other terms of the contract over long period of time may be small;

- **Rating Caa** Bonds which are rated Caa are of poor standing. Such issues may be in default or there may be present elements of danger with respect to principal or interest;

- **Rating Ca** Bonds which are rated Ca represent obligations which are speculative in high degree. such issues are often in default or have other market shortcomings;

- **Rating C** Bonds which are rated C are the lowest rated class of bonds, and issues so rated can be regarded as having extremely poor prospects of ever attaining any real investment standing.

Furthermore, a Moody rating may have digits following the letters, for example A2 or Aa3. These digits have to be considered as sub-levels within each grade, with 1 being the highest and 3 the lowest.

1.3.1   Rating Migrations and Default

Let us suppose to consider the probability of default for a given company, say company X, which has a specific credit rating. Furthermore, let us consider the following rating criteria:

- H: high rating;

- L: low rating;

- D: default;

with a *rating migration table P* as shown below:

	H	**L**	**D**
H	P_{HH}	P_{HL}	P_{HD}
L	P_{LH}	P_{LL}	P_{LD}
D	P_{DH}	P_{DL}	P_{DD}

where P_{ij} with $i, j = H, L, D$ is the probability that an i-*th* rated company will be rated j after one year.

It should be clear that P_{Dj}, with $j = H, L$ is always equal to zero, while P_{DD} equals one. This means that state D denotes a state that, one is reached, it can never be forsaken, i.e., state D is an *absorbing state*.

Now, let us assume that the current rating of company X is H. Hence, the probability company X will default after one year is simply P_{HD} and, of course, $(1 - P_{HD})$ represents the probability it will not default over the same time horizon. So far there is nothing difficult, but, as soon as we extend our time interval from one to two years, the computation of the default probability over this new maturity becomes a little less straightforward.

In fact, at first sight, one could believe that in this case the survival probability is simply $(1 - P_{HD})^2$ and the default probability $(1 - P_{HD})P_{HD}$: but this answer is correct *only if* we do not consider the possibility of *rating migration*. Unfortunately, such an assumption does not reflect the reality, since the default state D may be reached both directly and via rating transitions:

$$
\begin{aligned}
H &\longrightarrow H \longrightarrow D & \text{with probability } P_{HH}P_{HD} \\
H &\longrightarrow L \longrightarrow D & \text{with probability } P_{HL}P_{LD} \\
H &\longrightarrow D \longrightarrow D & \text{with probability } P_{HD}P_{DD}
\end{aligned}
$$

$$(1.29)$$

and thus the total default probability is:

$$P_{HH}P_{HD} + P_{HL}P_{LD} + P_{HD}P_{DD} \neq (1 - P_{HD})P_{HD}$$

Then, by having the one-year transition matrix P, it is possible to get the two-years transition matrix simply by computing the square of the original one-year transition matrix

$$P^{(2)} = P \cdot P = P^2 \tag{1.30}$$

Several studies show that the modelling of rating transitions should be done by using *continuos-time Markov chains* via *generator matrices*:

$$
\mathbf{\Lambda} = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1K} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0
\end{bmatrix} \tag{1.31}
$$

Definition 2 *A* generator *of a time-continuos Markov chain is given by a matrix* $\Lambda = (\lambda_{ij})_{1 \leq i,j \leq K}$ *for which the following properties hold:*

1. $\sum_{j=1}^{K} = 0$ *for every* $i = 1, 2, \ldots, K$;

2. $0 \leq -\lambda_{ii} \leq +\infty$ *for every* $i = 1, 2, \ldots, K$;

3. $\lambda_{ij} \geq 0$ *for every* $i, j = 1, 2, \ldots, K$, *with* $i \neq j$;

On the basis of these properties, every diagonal element of the generator matrix Λ is negative:

$$\lambda_{ij} = -\sum_{j \neq i} \lambda_{ij} \tag{1.32}$$

where λ_{ij} denotes the *intensity* of jumping from rating i to rating j. Note that, once more, default state K is still an absorbing state.
As it was a Poisson process, the transition probability from rating i to rating j over a small time interval Δt is proportional to Δt it-self. Hence we can write:

$$P\left[R\left(t + \Delta t\right) = j | R\left(t\right) = i\right] = \lambda_{ij} \Delta t$$

where R is the *rating process* stating the credit rating of the firm at time t. Clearly, the survival probability is given by

$$P\left[R\left(t + \Delta t\right) = i | R\left(t\right) = i\right] = 1 - \sum_{j \neq i} \lambda_{ij} \Delta t$$

and thus from (1.32) we get

$$P\left[R\left(t + \Delta t\right) = j | R\left(t\right) = i\right] = 1 + \lambda_{ij} \Delta t$$

Now, by generalizing this result to the overall transition probabilities matrix P we obtain

$$P\left(t, t + \Delta t\right) = I + \Delta t \cdot \Lambda$$

where I is the unit matrix.
Then, if we consider a larger time interval $[t, s]$ and we split it into i sub-intervals of length Δt, on the basis of (1.30) it is possible to write that

$$
\begin{aligned}
P\left(t + 2\Delta t\right) &= P\left(t, t + \Delta t\right) \cdot P\left(t + \Delta t, t + 2\Delta t\right) \\
&= (I + \Delta t \Lambda) \cdot (I + \Delta t \Lambda) \\
&= (I + \Delta t \Lambda)^2
\end{aligned}
$$

and thus, more generally

$$P\left(t, t + i\Delta t\right) = (I + \Delta t \Lambda)^i = \left(1 + \frac{s - t}{i}\Lambda\right)^i. \tag{1.33}$$

This allows to reach the following (in the limit) result:

$$P(t, s) = e^{(s-t)\Lambda} \tag{1.34}$$

with

$$e^{(s-t)\Lambda} = \sum_{n=0}^{\infty} \frac{((s-t)\Lambda)^n}{n!}.$$

The next step consists in writing the transition matrix from time t to time $t + \Delta t$ as follows:

$$P(t, s + \Delta t) = P(t, s)(I + \Delta t \Lambda) = P(t, s) + \Delta t P(t, s)\Lambda$$

and then

$$\frac{1}{\Delta t}[P(t, s + \Delta t) - P(t, s)] = \frac{\partial}{\partial s}P(t, s) = P(t, s)\Lambda. \tag{1.35}$$

The differential equations represented by (1.35) are known as *Kolmogorov forward differential equations*.

For practical purposes, the problem of how to derive the generator matrix Λ may arise. In fact, rating agencies are used to publish only the transition probabilities matrix P and not the generator matrix Λ.
However, equation (1.34) shows that

$$P(t) = e^{\Lambda t}$$

whose solution would be

$$\Lambda = \frac{1}{t}lnP$$

if Λ was a scalar, otherwise, if it was a matrix (as it is), we should exploit the power series representation for the matrix logarithm

$$ln(I + X) = X - X^2 + X^3 \cdots$$

In order to avoid the computation of this infinite series, there exists an alternative method which is based on the decomposition of the original transition matrix P, for which:

$$P = MDM^{-1} \tag{1.36}$$

where M denotes a square matrix and D a diagonal one. Note that not every matrix can be decomposed by using the (1.36), but, fortunately, this decomposition usually works for the empirical rating transition matrices. Hence:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{bmatrix} =$$

$$= \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1K} \\ m_{21} & m_{22} & \cdots & m_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ m_{K1} & m_{K2} & \cdots & m_{KK} \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_K \end{bmatrix} \cdot \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1K} \\ m_{21} & m_{22} & \cdots & m_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ m_{21} & m_{22} & \cdots & m_{2K} \end{bmatrix}^{-1}$$

where the columns of the matrix M are called *eigenvectors* of P, while $d_1, d_2, \cdots, d_K$ are the *eigenvalues*.

Once this decomposition has been computed, to obtain the logarithm (exponential) of the matrix P we have just to replace the diagonal matrix D with another diagonal matrix, whose diagonal elements are the logarithm (exponential) of $d_1, d_2, \cdots, d_K$, i.e.:

$$lnP = A = M \cdot (lnD) \cdot M^{-1} = M \cdot D_A \cdot M^{-1}$$

with $lnD = D_A$. Now, considering any other time s, it is easy to calculate the transition probabilty with respect to it:

$$P(s) = e^{sA} = Me^{sD_a}M^{-1} = Me^{slnD}M^{-1} =$$
$$= MD^sM^{-1}$$

with D^s which is a diagonal matrix having the exponential of $d_1, d_2, \cdots, d_K$ as diagonal elements.

As said few pages ago, the parameters λ_{ij}, with $i, j = 1, 2, \cdots, K$, have to be considered as the intensities of different Poisson processes. This leads to the conclusion that a Markov chain may be viewed as a collection of (K) Poisson processes. In particular, the k-*th* Poisson process, with intensity λ_{nk} (if the firm we are taking into account currently stays in rating class n), models the transition to the k-*th* rating class.

Thus, if we consider the credit rating process $R(t)$, the following (infinitesimaly) properties must hold:

$$\mathbf{E}[dR] = \left(\sum_{k \neq R} (k - R)a_{Rk} \right) dt = \sum_{k=1}^{K} k \cdot a_{Rk}dt \qquad (1.37)$$

$$P(dR = k - R) = a_{Rk}dt \qquad (1.38)$$

Furthermore, if we consider a function depending on R, $f(R)$, this can be viewed as a compound Poisson process. In fact:

$$\mathbf{E}[df(R)] = \left(\sum_{l \neq R} (f(l) - f(R)) \, a_{Rl} \right) dt = \sum_{l=1}^{K} f(l) a_{Rl} dt \qquad (1.39)$$

$$P \left(df(R) = f(k) - f(R) \right) = a_{Rk} dt \qquad (1.40)$$

1.3.2 Pricing considering Rating Migrations

Within the credit-rating models setting, the price of a financial instrument must be a function of time t, of the risk-free interest rate r and, this is the novelty, of the issuer's credit rating $R(t)$. Thus, for example, the price of a defaultable zero-coupon bond $\bar{B}$ is

$$\bar{B} = \bar{B}(t, r, R)$$

In this framework, the pricing of a bond $\bar{B}$ has to be done *simultaneously* for all rating classes.

Since our model allows rating migrations, on the basis of the transition probabilities matrix, it is necessary to know the price the risky bond $\bar{B}$ will have after each transition. In a few words, the bond price $\bar{B}$ may be written as a K-*dimensional* vector (since we are considering the existence of K different rating classes):

$$\bar{\mathbf{B}}(t, r) = \begin{bmatrix} \bar{B}(t, r, R = 1) \\ \bar{B}(t, r, R = 2) \\ \vdots \\ \bar{B}(t, r, R = K) \end{bmatrix}$$

The consequence of this, in terms of final payoffs, can be expressed by the following vector:

$$\bar{\mathbf{B}}(T) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Of course, this is true because we are implicitly assuming a *zero-recovery rate*. Hence, as usual, the price of this defaultable zero-coupon bond is, given an $R(0)$ initial rating, the *risk-neutral* expectation of its discounted final payoffs:

$$\begin{aligned} \bar{B}(0) &= \mathbf{E} \left[e^{-\int_0^T r(s)ds} \bar{B}_{R(T)}(T) \right] \\ &= \mathbf{E} \left[e^{-\int_0^T r(s)ds} \right] \mathbf{E} \left[\bar{B}_{R(T)}(T) \right] \\ &= B(0, r) \mathbf{E} \left[\bar{B}_{R(T)}(T) \right] \\ &= B(0, r) \left(P(T) \bar{B}(T) \right)_{R(0)} \end{aligned}$$

Now, since the final-payoffs vector is composed for the first $K - 1$ positions by ones and only the last component is zero, then we can go on by writing:

$$\bar{B}(0) = B(0, r) \left(P(T) \bar{B}(T) \right)_{R(0)}$$
$$= B(0, r) \sum_{k=1}^{K-1} P_{R(0)k}(T)$$
$$= B(0, r) \left(1 - P_{R(0)K}(T) \right)$$

where $P(T)$ represents the transition probability matrix until time T.

Even though this model is able to include the rating migration process, it presents some drawbacks, concerning the behaviour of credit spreads, which cannot be neglected.
In fact, the implicit assumption for which the transition probability to default $P(T)$ remains constant over time leads to have *constant* credit spreads within every rating class. This may be partially due to the assumption of zero recovery rate. However, even if we made a different assumption, e.g. considering a positive recovery rate c, the result would not change, because we would get the following final pricing formula:

$$\bar{B}(0) = B(0, r) \left(1 - c P_{.K}(T) \right)$$

meaning that at default we would receive a cash-flow of $(1 - c)$.

Chapter 2

Credit Derivatives

In these last years, the credit derivative market has been characterized by a huhe growth, from a few trades in the early 1990s to the actual billions of dollars trades each year. This fact reflects the expansion of the demand from several users such as asset managers, corprations and fixed-income trading desks, buying and selling credit exposure.

Credit derivatives may be defined as financial instruments whose purpose consists in efficiently transfering the *credit risk* of an underlying asset between two or more parties, thus they allow to either increase or reduce the credit risk exposure. In such a way, credit derivatives represent an important instrument enabling asset managers to improve performances, both in terms of management of credit risk exposure and enhancement of portfolio returns.

Currently, there exist many types of credit derivatives, even though *Credit Default Swaps* (CDS) and *Collateralized Debt Obligations* (CDO) are the most popular in the market. Conversely from the other derivatives which can be both exchange-traded and over-the-counter traded (OTC), credit derivatives are only OTC products.

In particular, *Credit Default Swaps* are in general used for flow trading of single reference name credit risk, or, more extensively, for trading a basket of reference credits (*Basket Default Swaps*). Furthermore, there is another fundamental peculiarity of credit derivatives. In fact they can also be used to create debt instruments with particular structures whose payoffs derive from the credit features of a reference asset (assets), the so called *reference obligation*, and an issuer, the *reference entity*, or a basket of reference assets entities: this is the case of *Syntethic Collateralized Debt Obligations*.

Today, credit derivatives are very useful to shift credit risk from banks to non-banks investors which are willing to assume credit risk on the basis of a potential enhanced rate of return.

Recent surveys have highlighted the more and more increasing number of financial entities having positions on credit derivatives. Moreover, among those companies which do still not use these instruments, there is a large portion

planning to use them in the next future. This is one principal reason for which understanding of credit derivatives is crucial for those who does not wish to use them, as well.

Another reason is given by the fact that these derivatives are not only useful for managing credit risk, but also because their prices encompass the market expectations and thus they may provide more detailed information concerning the probability of net loss from the default of a set of borrowers.

The credit derivative market is characterized by three different entities which play a role in it. First of all, there are the so called *end-buyers of protection*, whose purpose is hedging the credit risk affecting other parts of their business. Then, there are the *end-sellers of protection* which try to diversify more efficiently their actual portfolios. Finally, *intermediaries*, such as Investment Banks, have the role to provide liquidity to end-users. In general they look for arbitrage or other opportunities by trading on their own account.

Even though credit derivatives provide an alternative instrument to manage credit risk, they are also responsible for new forms of risk arising from their use.

- *OPERATIONAL RISK*: it derives from an inprudent use of credit derivatives by traders or asset managers. In fact, credit derivatives do not appear on investors' balance sheet. Thus, it could happen that without a proper internal accounting system, the investor may not be totally aware of the total credit risk it is taking.

- *COUNTERPARTY RISK*: it is funny, but a credit protection buyer may introduce an additional amount of credit risk in the portfolio by purchasing credit derivatives. In fact, a credit protection buyer can incur in a net loss whenever a credit event on the underlying credit risky asset occurs and the credit protection seller defaults on its obligations.

- *ILLIQUIDITY RISK*: it is due to the fact that credit derivatives are traded over-the-counter. Hence, they are characterized by a strong degree of customization leading to illiquidity.

- *PRICING RISK*: this risk reflect the dependence of the pricing models on their assumptions concerning the underlying economic parameters. This means that the prices of credit derivatives are strictly sensitive to the assumptions on which the model is based on.

2.1   Credit Default Swaps

In the last decade, the trading volume of *Credit Default Swaps* (CDS) has tremendeously grown. In fact, empirical evidence shows that in 1996 the volume of CDS, in terms of outstanding credit, was *only* $20 billion, while in 2005 this amount increased up to $2.3 trillion. This huge growth is partially due also to the improvements which have been developed on these financial products, since

they are actually characterized by a much greater level of standardization than some years ago. Furthermore this allowed an increase of the liquidity within the market, as well.

The conclusion of a CDS contract involves simultaneously the specification of the following entities:

- a *credit protection buyer*;

- a *credit protection seller*;

- a *reference obligor*;

- *reference obligations.*

The *credit protection buyer* purchases a given amount of protection, the so called *notional amount* of the CDS contract, from the credit protection seller. For this reason, the former has to pay, periodically, a fee to the latter: the amount of this fee is computed by making the product between the notional amount and a proper number of basis points.

Concerning the role played by the *credit protection seller*, it consists in paying the protection buyer if a credit effect affecting the *reference obligor*, such as a default, occurs.

At this point, it is important to specify that there exist two different ways in which the payment by the credit protection seller in favour of the credit protection buyer may be made: the *physical delivery approach* and the *cash settlement approach*.

Physical delivery means that the protection buyer delivers, directly to the protection seller, a *reference obligation* of the reference obligor. Clearly, the value of the reference obligation which is delivered has to be equal to the notional amount of the CDS and thus, in exchange for it, the protection seller must pay, *in cash*, the par of the reference obligation to the protection buyer: the par amount which is transferred is nothing else that the notional amount of the CDS contract. Afterwards, the protection seller has to be considered, under every point of view, as the owner of the reference obligation and hence, it is free to take every action it believes to be appropriate in order to recover the maximum value from the reference obligation itself. Differently, from the perspective of the protection buyer, it is useful to specify its chance to choose and deliver the least expensive reference obligation to the protection seller: this is what is called *cheapest-to-deliver option*. Generally, the physical delivery approach is the method which is mainly used, since more than 80% of the outstanding CDS are based on it.

In the case of *cash settlement*, the protection seller must pay the protection buyer a cash amount corresponding to the difference between the notional amount and the market value (after the occurrence of the credit event) of the reference obligations. In particular, the market value of the reference obligation is determined by a mark-to-market auction process, by an average of the quotations which are given by qualified dealers.

A. Initial Flows

| Credit Protection Buyer | **Periodic Payments** **Bps X Notional** → | Credit Protection Seller |

B. In case of Credit Event: Physical Settlement

| Credit Protection Buyer | **Reference Obligation** → ← **Notional Amount** | Credit Protection Seller |

C. In case of Credit Event: Cash Settlement

| Credit Protection Buyer | **(100% - Market Price%)** → ← **Notional Amount** | Credit Protection Seller |

Figure 2.1: Credit Default Swap payment cash-flows

Figure 2.1 shows the different CDS payment flows.

Clearly, in the absence of a credit event, the protection buyer will keep on making its periodic (in general quarterly) payments until the expiration of the CDS.

Note that the position of the protection seller may be compared to that of a bond holder, since both receive periodically a coupon payment. The only difference is given by the fact that the buyer of a bond has to pay, immediately, its market price, while to be long a CDS it is sufficient to promise to pay eventual future losses due to the occurrence of a credit event. For what concerns the protection buyer, CDS offer the great advantage of separating the funding of a loan (or bond) from the assumption of its specific credit risk. For example, a low-cost funder might buy a bond with a relatively high coupon (and then reflecting a quite risky situation) and simulataneously, the same party could

hedge its position by purchasing protection with a CDS, getting the opportunity to achieve a gain in terms of net spread received.

It is straightforward to understand the importance of clarifying the circumstances which can be considered as acredit event for a CDS contract. In fact, only the occurence of a credit event triggers the protection payment by the protection seller in favour of the protection buyer. For this impotant reason, the *International Swaps and Derivatives Association* (ISDA) defines six possible events:

1. *Bankruptcy*;

2. *Failure to Pay*;

3. *Obligation Default*;

4. *Obligation Acceleration*;

5. *Repudiation and Moratorium*;

6. *Restructuring*.

Actually, there exist a broad consensus among the parties in excluding some of these credit events and simultaneously in specifying in a better way the definitions for the rest.
In particular, *Obligation Default* and *Obligation Acceleration* in non-emerging market corporate CDS are no longer considered credit events, since in many cases they do not lead to that amount of severity which should be necessary to trigger a protection payment. Moreover, whenever this level was reached, a Failure to Pay would follow very shortly, anyway. Furthermore, considering that a *Repudiation* or a *Moratorium* may be done only a sovereign entity like a national government, there remain only three events to specify among those which are above mentioned.
Firstly, *Bankruptcy* can be simply defined as the voluntary or involuntary filing of default. Secondly, *Failure to Pay* consists in the failure of the reference obligor to pay back the principal or to make interest payments in at least one of its outstanding obligations. Finally, for what concerns the definition of *Restructuring*, some difficulties still remain. In general, this event is considered in order to take into consideration an eventual weakened credit position of the debtor and it includes several situations such as extensions of the debt maturities, reductions of the coupon or principal amounts, and so on. In a few words, Restructuring might be seen as a less costly alternative to bankruptcy.

2.2 Collateralized Debt Obligations

Collateralized Debt Obligations (CDOs) are, for sure, the credit derivative which has been characterized by the highest growth rate in the last ten years. In 2005

there were more than \$1 trillion of outstanding CDOs, denoting their actual tremendeous popularity among investors. The reasons for such a success are several. Firstly, there exists a wide variety of different CDO structures and thus the final user has a large number of opportunities to optimize the investment choice. Secondly, CDOs enable investors to get particular gain exposures which could not be obtained via other financial instruments. In fact, for example, they offer the chance to combine an investment-grade risk with speculative-grade assets or, conversely, a speculative-grade risk with investment-grade assets. Finally, CDOs should be considered as instruments allowing *normal* investors to acquire certain assets that they could not otherwise invest on. Moreover, CDOs represent a tool with which an investor may improve the return profile of its portfolio, since their returns show low correlations to those of the underlying assets.

Briefly, *a CDO issues both debt and equity and exploits the money it raises by investing in portfolios of corporate loans or mortgage-backed securities. Then, on the basis of the relative seniority of the outstanding liabilities, it offers the cash flow originated by the collateral portfolio to the debt and equity holders.*

However, even though there exists a lot of different types of CDOs, any of them is characterized by the following three attributes.

- *CDO assets.* CDOs own different kinds of assets, in general corporate loans or mortgage-backed securities, but not only. In order to understand this, maybe it is simpler to illustrate a little of history of CDOs. In 1987, the first CDO was created even if it was not named CDO but *Collateralized Bond Obligation* (CBO), since it owned a portfolio of high-yield bonds. In the following years, CDOs owning corporate and real estate loans were introduced and thus the name *Collateralized Loan Obligation* (CLO) was invented. With the beginning of the 1990s, CDOs with loans and bonds issued by emerging markets and sovereign governments as collateral were created and then the the term *Emerging Market CDO* (EM CDO) was conied. In 1995, it was the turn of *Residential Mortgage-Backed Securities CDOs* (RMBS CDOs) and, up to actual days, a lot of different *Structured Finance CDOs* (SF CDOs) has been introduced. Figure 2.2 shows the typical collateral portfolio of CDOs, with respect to 2005.

- *Liabilities.* The liabilities of a CDO are characterized by a detailed and strict ranking of *seniority.* Hence, the CDO's capital structure comprises *equity or preferred shares*, *subordinated debt*, *mezzanine debt* and *seniority debt*, that in general are denoted by usual names as *Class A*, *Class B* and so on. The following table shows how these tranches of notes and equities range from the most secured AAA rated, with the greatest degree of subordination under it, to the most levered equity tranche.
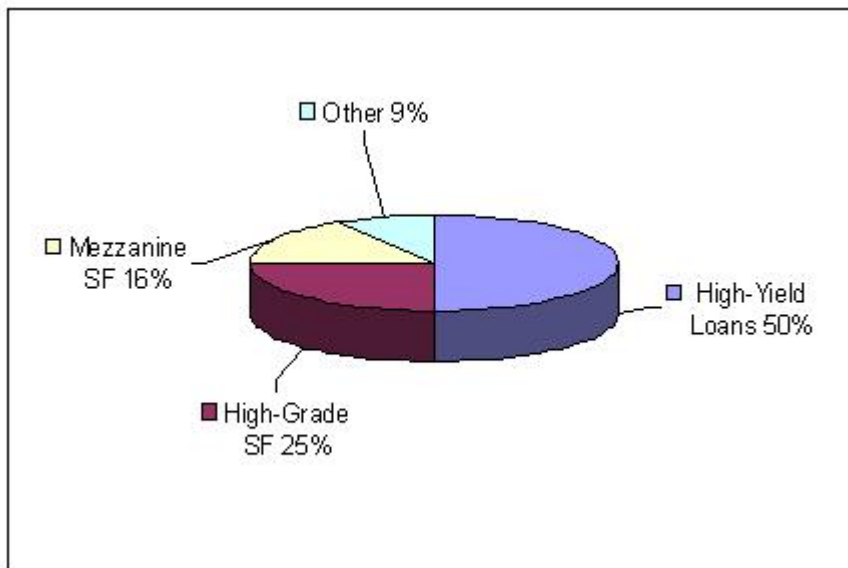
Figure 2.2: Collateral backing cash CDOs in 2005

Tranche	% of Capital Structure	Rating	Coupon
Class A	77.5	AAA	$LIBOR + 26$
Class B	9	A	$LIBOR + 75$
Class C	2.75	BBB	$LIBOR + 180$
Class D	2.75	BB	$LIBOR + 475$
Equity	8	NR	*Residual*

The reason for this severe seniority is allowing the CDO structures to raise funds at the lowest possible cost.

- *Purposes.* Generally speaking, there is more than one reason for which CDOs have been created. First of all, holding a CDO affects the company's balance sheet. Thus, by means of them, it is possible to shrink the balance sheet itself and, in such a way, both the required regulatory and economical capital may be reduced. In fact, for example, a bank might removing a portion of its loans from the balance sheet simply by selling them to a CDO, with the consequence of lower capital requirements to meet.
  Obviously, CDOs can also be seen as an alternative way with which a financial entity provides its services to customers. By doing so, investors do not get their returns proportionally to the investment done, but on the basis of the CDO tranche the buy. Hence, purchasing a CDO allows to separate and distribute the implicit risk of its assets among buyers with different risk attitudes, with the further advantage of getting an investment in a diversified assets portfolio.

CDO Cash Flows

A CDO might be seen as a joint set of financial instruments which are splitted into different tranches to be negotiated separately from the collateral. In particular, each tranche presents an *upper* and a *lower boundary* determining the fraction of the loss coming from the collateral which has to be attributed to the tranche itself. In general, the so called *waterfall* scheme is given by the following tranches:

- *Super-Senior* tranche

- *Senior* tranche

- *Mezzanine* tranche

- *Equity* tranche

which are periodically rated by a *Special Purpose Vehicle* (SPV). Furthermore, the SPV performs also the task to allocate the different tranches on the market. Probably, in order to understand how a CDO works, a simple example may be very helpful.

Example.
Let us consider a CDO characterized by a collateral of EUR 1.0 billion, paying a periodic premium of $Euribor + 100bps$. Moreover, let us suppose the Euribor rate equals 3.00% while the administrative costs are equal to 2 bps.

As shown in Figure 2.3, the SPV splits it into four different tranches. Thus, there are two situations which have to be evaluated:

1. No default affecting the collateral within the maturity of the CDO;

2. Occurrence of defaults affecting the collateral within the maturity of the CDO (e.g. *Total loss of EUR 30 million*).

In both cases, the principle on which the CDO payoff is based on is that investors are paid in a manner such that the more senior tranches are paid off first, followed by the subordinated tranches and finally by the equity class. On the contrary, eventual losses are absorbed by the equity tranche first, then by the mezzanine tranche and so on. However, any cash flow is subordinated to the payment of the admistrative costs.

1. *No default affecting the collateral within the maturity of the CDO.*

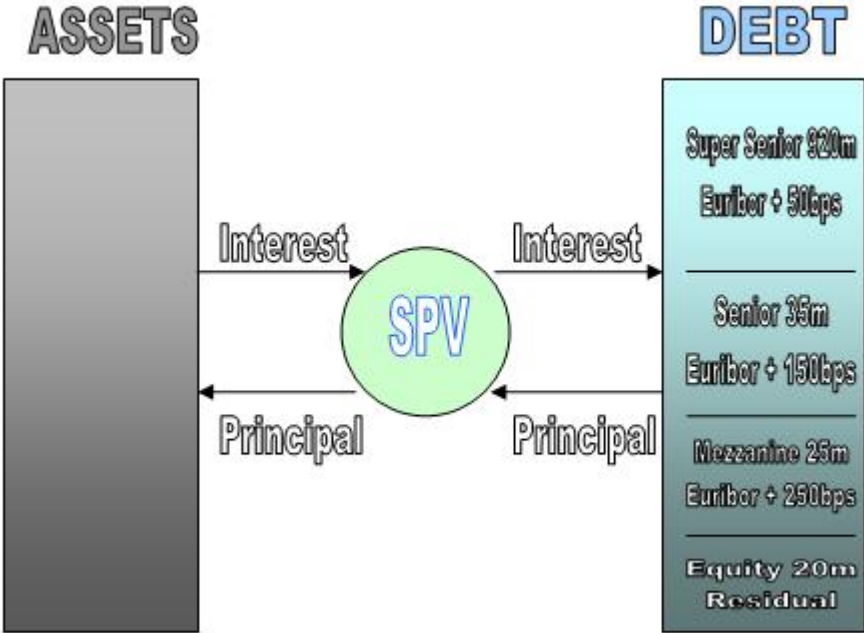| Total Portfolio	$(3.00 + 100bps) \times 1b) = 40m$
Administr. Costs	$(2bps \times 1b) = 0.2m$
Super-Senior	$(3.00\% + 50bps) \times 920m = 32.2m$
Senior	$(3.00\% + 150bps) \times 35m = 1.6m$
Mezzanine	$(3.00\% + 250bps) \times 25m = 1.4m$
Equity	$(40 - 0.2 - 32.2 - 1.6 - 1.4)m = 4.6m$

Figure 2.3: CDO cash flows

Table 1: Interest payments per year, in Euros.

Total Portfolio	$1b$
Super-Senior	$920m$
Senior	$35m$
Mezzanine	$25m$
Equity	$1b - 920m - 35m - 25m = 20m$

Table 2: Principal redemption at maturity date, in Euros.

2. *Defaults determining a total net loss of EUR 30 million.*

Total Portfolio	$(3.00 + 100bps) \times 970m) = 38.8m$
Administr. Costs	$(2bps \times 970m) = 0.1952m$
Super-Senior	$(3.00\% + 50bps) \times 920m = 32.2m$
Senior	$(3.00\% + 150bps) \times 35m = 1.6m$
Mezzanine	$(3.00\% + 250bps) \times 15m = 0.825m$
Equity	$(38.8 - 0.1952 - 32.2 - 1.6 - 0.825)m = 3.98m$

Table 3: Interest payments per year, in Euros.

Total Portfolio	$970m$
Super-Senior	$920m$
Senior	$35m$
Mezzanine	$15m$
Equity	0

Table 4: Principal redemption at maturity date, in Euros.

By looking at these tables, it is possible to note that, in case of defaults, the overall portfolio value decreases from the initial amount of EUR 1 billion to EUR 970 million. Thus, the total loss is absorbed, for its total value, partially by the mezzanine tranche and by the equity tranche, for what concerns the interest payments. Nevertheless, with regard to the principal redemption, the portfolio loss makes the cash flow of the equity tranche zero, while it absorbes only partially the one of the mezzanine tranche, which passes to EUR 15 million from the original 25 in case of no defaults.

Finally, it is important to emphasize the existence of two main categories of CDO: *cash-flow CDOs* and *synthetic CDOs*. The difference between them stands in the rights on the collateral. In fact, in cash-flow CDOs the original issuer gives up its assets portfolio to the SPV, while the same does not happen with synthetic CDOs. In this last case, the original issuer transfers to the SPV *only* the portfolio risk, while the portfolio value keeps on appearing on the issuer's balance sheet. Obviously, in exchange for this, the *originator* will periodically pay a premium to the SPV: it a few words it purchases *credit protection* from the Special Purpose Vehicle.
Actually, synthetic CDOs represent the largest slice of the credite derivatives market.

2.3 Pricing of Credit Derivatives

The first basic step for pricing credit derivatives, consists in deriving the *implied default probabilities* under which their fair prices can be found out. In general, this may be done by exploiting the knowledge of CDS market prices. The first reason for such a thing is given by the fact that these instruments are characterized by high levels of liquidity and standardization, leading to a relative simplicity and convenience in using their market quotations. Furthermore, CDS premia include market expectations concerning the credit merit of a given obligor and thus, it is possible to use these *risk-neutral probabilities* for pricing purposes.

Hence, let us consider a probabilty space given by $(\Omega, F, \{\Im_t\}_{t \geq 0}, \mathbb{P}^*)$ where $\mathbb{P}^*$ denotes the equivalent martingale measure which makes the price processes (of any tradeable security), discounted by the *risk-free interest rate*, $\mathbb{P}^*$-martingales with respect to the filtration $\{\Im_t\}_{t \geq 0}$. In addition, it is a common practise to make the following assumptions:

1. Independence between the interest rate and the default processes, under the martingale measure $\mathbb{P}^*$;

2. Defaults can only occur at discrete dates and, at their occurrence, the protection payments are immediately settled.

Under these hypothesis, it is now possible to illustrate the pricing analysis for some more and more widely used credit risk derivatives: *Basket Default Swaps* and *Collateralized Debt Obligations*.

2.3.1 Pricing of Basket Default Swaps

Among the several credit risk derivatives, actually *Basket Default Swaps* (BDS) show a great success. The structure of these instruments derives directly from that of CDS, where the only difference stands in the definition of the credit event which triggers the protection payment from the protection seller to the protection buyer. In fact, for a BDS, the credit event is represented by the defaults of a given number of reference entities within the the reference basket. Thus, in a *k-th to Default Swap*, the protection buyer pays a periodic premium to the protection seller either until the natural expiration of the contract or until the occurrence of the k-*th* default within the reference portfolio: in this last case, the protection seller is forced to make the protection payment, on the basis of the loss given default resulting from the k-*th* default. In a few words, a BDS may be seen as a credit derivative on a portfolio of reference entities which offers credit protection until the k-*th* default in the reference basket.

Obviously, a BDS provides a protection over a set of underlying assets at a lower price than that which should be paid in the case of joint single protections. Furthermore, intuitively, it is clear that the price of a BDS will be affected by both the default risks characterizing each obligors and especially by the dependence among their defaults. For this reason the choice of the right model

describing the default dependence will result to be crucial for the pricing of these instruments.

Within this framework, the risk-neutral price of a BDS is simply defined as the fraction s of the contract nominal value M, expressed in yearly basis points, such that the present value of the contingent payment in case of default, i.e. the *Default Leg* (DL), and that of the so called *Premium Leg* (PL), i.e. the flow of the periodic payments paid by the protection buyer, are equal. Hence, before showing how to price a BDS, first it is necessary to determine the notation which will be used.

- N denotes the number of entities composing the reference portfolio;

- k is the seniority level of the BDS structure, thus the protection payment is triggered as soon as the k-*th* default occurs;

- t_i with $i = 1, \ldots, n$ are the discrete dates at which the protection buyer has to pay its periodic premia, with $t_n = T$ denoting the natural expiration date of the contract;

- τ_i denotes the default time of the i-*th* reference entity;

- M is the notional amount of the contract. For simplicity we will consider a homogeneous portfolio, such that M also coincides with the notional amount of each reference entity;

- s denotes the fair price of the BDS contract, in yearly basis points, to be paid $1/\Delta$ times per year as long as the BDS contract lasts;

- $B(0,t)$ represents the risk-free discount factor from time 0 to time t;

- R_j is the recovery rate associated to the j-*th* default;

- AP denotes the accrued payment, i.e. the portion of interest which has to be attributed to the period starting from the last payment date to the k-*th* default date.

In such a way, the Premium Leg PL is the present value of the periodic premia paid by the protection buyer:

$$PL = \mathbb{E}^* \left[\sum_{i=1}^{n} (sM\Delta)\, B(0, t_i) \mathbf{I}_{\{t_k > t_i\}} \right]$$

$$= \sum_{i=1}^{n} (sM\Delta)\, B(0, t_i)\, (1 - \mathbb{P}^*(\tau_k \leq t_i)) \tag{2.1}$$

while the Default Leg, that is the protection payment which is triggered by the occurrence of the k-*th* default is computed as the difference between the default Payment DP and the Accrued Payment AP

$$DL = DP - AP$$

In details, the Default Payment DP may be written as

$$DP = \mathbb{E}^* \left[M \sum_{j=1}^{N} (1 - R_j) B(0, \tau_k) \mathbf{I}_{\{t_k \leq T\}} \right]$$

$$= M \sum_{j=1}^{N} (1 - R_j) \int_0^T B(0, t) \cdot \mathbb{P}^*(\tau_k = t) dt \qquad (2.2)$$

while the Accrued Payment AP is

$$AP = \mathbb{E}^* \left[\sum_{i=1}^{n} M \left(s \cdot \frac{\tau_k - t_{i-1}}{t_i - t_{i-1}} \cdot \Delta \right) B(0, \tau_k) \mathbf{I}_{\{t_{i-1} < t_k \leq T\}} \right]$$

$$= sM \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{u - t_{i-1}}{t_i - t_{i-1}} \cdot \Delta \cdot B(0, u) \cdot \mathbb{P}^*(\tau_k = u) du \qquad (2.3)$$

Now, assuming to know exogenously the recovery rates (e.g. by looking at data published by rating agencies), the fair price s^* of the k-*th* BDS can be direcly obtained from the equation

$$PL(s^*) = DL(s^*)$$

that is

$$s^* = \frac{\sum_{j=1}^{N} (1 - R_j) \int_0^T B(0, t) \cdot \mathbb{P}^*(\tau_k = t) dt}{\sum_{i=1}^{n} \Delta B(0, t_i) \left(1 - \mathbb{P}^*(\tau_k \leq t_i) \right) + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{u - t_{i-1}}{t_i - t_{i-1}} \Delta B(0, u) \mathbb{P}^*(\tau_k = u) du}$$

$$(2.4)$$

2.3.2 Pricing of Collateralized Debt Obligations

In pricing a CDO, the Accrued Payment AP does not appear into the pricing equation and so only two cash flows have to be kept into account, i.e. the Default Leg DL and the Premium Leg PL. It is important understanding that both must be computed with regard to a specific tranche of the CDO contract. Once more, before entering the details of the pricing analysis, it is better to show the notation which will be used later on.

- N denotes the number of reference entities composing the collateral portfolio;

- $T = t_n$ is the naturalexpiration date of the CDO contract;

- τ_i represents the default time of the i-*th* reference obligation;

- A_i stands for the nominal amount of the i-*th* reference obligation;

- $s_{\alpha,\beta}$ denotes the fair price of the CDO tranche with α and β respectively as the lower and the upper boundary of the tranche itself;

- $B(0,t)$ is the risk-free discount factor;

- R_i denotes the recovery rate with respect to the i-*th* reference obligation;

- Δ represents the frequency (during the year) of the interest payments;

- L_i is the loss given default associated to the i-*th* reference obligor;

- Q_i stands for the default indicator of the i-*th* reference obligor;

- L denotes the cumulative loss of the collateral portfolio.

On the basis of this notation, the loss given default L_i due to the i-*th* reference obligation can be written as

$$L_i = (1 - R_i) \cdot A_i$$

then, the cumulative loss L of the collateral portfolio at time t simply results to be the sum of the single losses L_i:

$$L(t) = \sum_{i=1}^{N} L_i(t) \cdot Q_i(t) \tag{2.5}$$

It is clear that the amount of the loss suffered by the debt holders strictly depends on the *seniority* of the CDO tranches. This may be found out by looking at the values assumed by the lower and the upper boundaries α and β of the single tranches. In particular

- $\alpha = 0 \implies$ EQUITY TRANCHE;

- $\alpha > 0$ and $\beta < \sum_{i=1}^{N} A_i \implies$ MEZZANINE TRANCHE;

- $\beta = \sum_{i=1}^{N} Ai \implies$ SENIOR TRANCHE.

As a consequence, the loss $L^{\alpha,\beta}$ absorbed by the tranche (α,β) is related to the total loss L of the collateral portfolio. In fact

$$L^{\alpha,\beta} = \begin{cases} 0, & \text{if } L(t) < \alpha \\ L(t) - \alpha, & \text{if } \alpha \leq L(t) < \beta \\ \beta - \alpha, & \text{if } L(t) \geq \beta \end{cases}$$

Thus, $L^{\alpha,\beta}$ can be written as

$$L^{\alpha,\beta} = (L(t) - \alpha) \cdot \mathbf{I}_{\{\alpha \leq L(t) < \beta\}} + (\beta - \alpha) \cdot \mathbf{I}_{\{\beta \leq L(t) \leq \sum_{i=1}^{N} A_i\}} \tag{2.6}$$

Equivalently, the previous formula (2.6) may also be expressed as

$$L^{\alpha,\beta} = min\left[max\left(0; L(t) - \alpha\right); \beta - \alpha\right] \tag{2.7}$$

Again, the fair price $s_{\alpha,\beta}$ for the CDO tranche (α, β) is obtained by imposing the equivalence between the Default Leg DL and the Premium Leg PL.

The Default Leg DL is nothing else that the present value of the contingent payments in terms of the expected tranche loss:

$$DL = \mathbb{E}^*\left[\int_0^T B(0,t)dL^{\alpha,\beta}(t)\right] \tag{2.8}$$

while the premium leg PL is given by the expected value of the discounted premium payments computed on the outstanding capital (i.e. nominal tranche minus the tranche loss)

$$PL = \mathbb{E}^*\left[\sum_{i=1}^N s_{\alpha,\beta} \cdot \Delta \cdot B(0,t_i) \cdot min\left[max\left(\beta - L_i(t); 0\right); \beta - \alpha\right]\right] \tag{2.9}$$

and finally, the fair spread $s_{\alpha,\beta}^*$ can be easily obtained from the equivalence

$$PL(s_{\alpha,\beta}^*) = DL(s_{\alpha,\beta}^*),$$

that is

$$s_{\alpha,\beta}^* = \frac{\mathbb{E}^*\left[\int_0^T B(0,t)dL^{\alpha,\beta}(t)\right]}{\mathbb{E}^*\left[\sum_{i=1}^N \Delta \cdot B(0,t_i) \cdot min\left[max\left(\beta - L_i(t); 0\right); \beta - \alpha\right]\right]} \tag{2.10}$$

Chapter 3

Copula Functions

In general, the dependence among a set of random variables is represented by their *joint probability function*. Unfortunately, the computation of these functions is in many cases hard, thus *copula functions* may be interpreted as an important tool to model the dependence more easily.

3.1 Copula Functions: Definitions and Properties.

The analysis of copula functions starts from their basic definition suggested *An Introduction to Copulas*, by Nelsen in [38].

Definition 3 *An n-dimensional copula is a function of $C : [0,1]^n \to [0,1]$ with the following properties:*

1. *$C(\boldsymbol{u})$ is increasing for each component u_k, with $k = 1, \ldots, n$;*

2. *for every vector $\boldsymbol{u} \in [0,1]^n$, $C(\boldsymbol{u}) = 0$ if at least one component of $\boldsymbol{u}$ equals zero, while $C(\boldsymbol{u}) = u_k$ if every coordinate of $\boldsymbol{u}$ is one, except the k-th one;*

3. *for every $\boldsymbol{a}, \boldsymbol{b} \in [0,1]^n$, with $\boldsymbol{a} \leq \boldsymbol{b}$, given a hypercube $\boldsymbol{B} = [\boldsymbol{a}, \boldsymbol{b}] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ whose vertices lie in the domain of C, its volume $V_C(B) \geq 0$.*

Alternatively,as well as equivalently, a copula function can be also defined as follows:

Definition 4 *A copula is the distribution function of a random vector in $\mathbb{R}^n$ with uniform margins in $[0,1]$.*

No doubt, the most important theorem concerning copulas is the well known *Sklar's theorem*. Its importance results into a wide use of it in several practical applications.

Theorem 1 . ***Sklar's theorem**. Let F be an n-dimensional distribution function with continuos margins $F_1, \ldots, F_n$. Then it has the following* unique *copula representation:*

$$F(x_1, ..., x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \tag{3.1}$$

From *Sklar's theorem* we can observe that the uniqueness of the copula function C is guaranteed only if the margins $F_1, \ldots, F_n$ are all continuous. Conversely, when these are not all continuous, the copula function still exists, even if it is no longer unique.

The fundamental idea offered by *Sklar's theorem* is that the use of copula functions for dependence modeling allows to split the multivariate distribution function of n random variables into two different parts:

- the distribution functions of each random variable F_i, with $i = 1, ..., n$;

- the *copula function* which completely describes the dependence structure of the random variables X_i, with $i = 1, \ldots, n$.

In a few words, any multivariate probability distribution function $F_{X_1, \ldots, X_n}$ can be written by means of a copula function as follows:

$$
\begin{aligned}
F_{X_1, \ldots, X_n}(x_1, \ldots, x_n) &= P(X_1 \le x_1, \ldots, X_n \le x_n) \\
&= C\left(P(X_1 \le x_1), \ldots, P(X_n \le x_n)\right) \\
&= C\left(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\right) \tag{3.2}
\end{aligned}
$$

where $X = (X_1, \ldots, X_n)$ denotes any n-dimensional random vector and F_{X_i} represents the marginal distribution functions of the random variables X_i, with $i = 1, \ldots, n$.

Such a conclusion must be considered as a milestone in the credit risk modeling: in fact it would be translated into the possibility to determine and calibrate the processes describing the individual defaults in a completely independent way with respect to their joint behaviour.

Furthermore, the following corollary can be attained from Sklar's theorem (*theorem 1*).

Corollary 1 *Let F be an n-dimensional distribution function with continuous margins $F_1, \ldots, F_n$ and copula C which satisfies (3.1). Then, for any vector $\boldsymbol{u} = (u_1, \ldots, u_n)$ in $[0,1]^n$, the following holds*

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)),$$

where F_i^{-1} is the generalized inverse of F_i.

The importance of the previuos corollary will be essentially evident in the procedures used for simulating random numbers generated by a specific copula function.

Finally, the expression for the *copula density* $c(F_{X_1}(x_1), \ldots, F_{X_n}(x_n))$ associated to a copula function $C(F_{X_1}(x_1), \ldots, F_{X_n}(x_n))$ can be derived. In general, this will be very useful in order to calibrate the copula parameters to real market data.

Hence, starting from the *joint density function* $f(x_1, \ldots, x_n)$ of an n-*dimensional* random variable $(X_1, \ldots, X_n)$, obtained by (3.2), we can define the *multivariate copula density* $c(F_{X_1}(x_1), \ldots, F_{X_n}(x_n))$:

$$f(X_1, \ldots, X_n) = \frac{\partial^n \left[C(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)) \right]}{\partial F_{X_1}(x_1) \cdots \partial F_{X_n}(x_n)} \prod_{i=1}^{n} f_{X_i}(x_i)$$

$$= c(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)) \prod_{i=1}^{n} f_{X_i}(x_i)$$

and thus, simply by re-arranging the terms, we get

$$c(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)) = \frac{f(x_1, \ldots, x_n)}{\prod_{i=1}^{n} f_{X_i}(x_i)} \tag{3.3}$$

3.2 Examples of Copula Functions

The aim of this section is to show some of the most exploited copula functions for credit risk modeling purposes. Hence, two functions included in the class of *elliptical copulas*, i.e. the *Gaussian* and *Student's t copula* are going to be presented first, while, after these, another important class of copula such as *Archimedean copulas* will be briefly described.

3.2.1 The Gaussian Copula

The *Normal* or *Gaussian copula* is the copula function of the multivariate normal distribution.

If we consider a symmetric, positive definite matrix R, with $diag(R) = \mathbf{1}$, then Φ_R denotes the standardized multivariate Gaussian distribution with correlation matrix R. Hence, on the basis of this notation, the *multivariate Gaussian copula* C_R^{Ga} is given by

$$C_R^{Ga}(u_1, \ldots, u_n) = \Phi_R(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_n)) \tag{3.4}$$

where $\phi^{-1}(u)$ is the inverse of the normal cumulative distribution function. Now, simply by applying the equation (3.3), it is possible to get also the expression for the *normal copula density function* $c_R^{Ga}(u_1, \ldots, u_n)$:

$$c_R^{Ga}(u_1,\ldots,u_n) = \frac{f^{Ga}(x_1,\ldots,x_n)}{\prod_{i=1}^n f_{X_i(x_i)}}$$

$$= \frac{\frac{1}{(2\pi)^{n/2}|R|^{1/2}}exp(-\frac{1}{2}X'R^{-1}X)}{\prod_{i=1}^n \frac{1}{\sqrt{2\pi}}exp(-\frac{1}{2}x_i^2)} \qquad (3.5)$$

where f^{Ga} is the joint normal density function. So, if we set $u_i = \phi(x_i)$ and $\zeta = (\phi^{-1}(u_1),\ldots,\phi^{-1}(u_n))'$, then the equation (3.5) can be written in the following way:

$$c_R^{Ga}(u_1,\ldots,u_n) = \frac{1}{|R|^{1/2}}exp\left[-\frac{1}{2}\zeta'(R^{-1}-I)\zeta\right] \qquad (3.6)$$

with I denoting the identity matrix.

3.2.2   The Student's t Copula

As the Gaussian copula, also the *Student's t copula* belongs to the family of the elliptical copulas and it is nothing else that the copula of the multivariate Student's t distribution.
When we consider a symmetric, positive definite matrix R, with $diag(R) = \mathbf{1}$, then $T_{R,\nu}$ represents the standardized multivariate Student's t distribution with correlation matrix R and ν degrees of freedom. Thus, we can define the multivariate Student's t copula as

$$C_{R,\nu}^t(u_1,\ldots,u_n) = T_{R,\nu}(t_\nu^{-1}(u_1),\ldots,t_\nu^{-1}(u_n)) \qquad (3.7)$$

where t_ν^{-1} stands for the inverse of the Student's t cumulative distribution function.
For sure, analogously as in the Gaussian case, the relative *Student's t copula density function* may be obtained from the equation (3.3). Thus, what we get is

$$c_{R,\nu}^t(u_1,\ldots,u_n) = \frac{f^{stud}(x_1,\ldots,x_n)}{\prod_{i=1}^n f_i^{stud}(x_i)}$$

$$= |R|^{-1/2}\frac{\Gamma(\frac{\nu+n}{2})}{\Gamma(\frac{\nu}{2})}\left[\frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})}\right]^n \frac{(1+\frac{\zeta'R^{-1}\zeta}{\nu})^{-\frac{\nu+n}{2}}}{\prod_{i=1}^n(1+\frac{\zeta_i^2}{\nu})^{-\frac{\nu+1}{2}}} \qquad (3.8)$$

where f^{stud} denotes the joint Student's t density function and $\zeta = (t_\nu^{-1}(u_1),\ldots,t_\nu^{-1}(u_n))$.

3.2.3   Archimedean Copulas

Archimedean copulas are another family of copula functions. The importance of this class derives directly from their analytical tractability, since most of them

are provided with a closed formula, but it is also reflected into their specific capability to describe a large set of different dependence structures.

However, before presenting some of the most common Archimedean copulas, it is necessary to make a short introduction. Let us define a continuous function $\varphi : [0,1] \rightarrow [0, \infty]$, for which the following two propertis hold:

- $\varphi'(u) < 0$ for every $u \in [0, 1]$;

- $\varphi(1) = 0$.

Now, let us consider another function, the *pseudo-inverse* of φ, $\varphi^{[-1]}$: $[0, \infty] \rightarrow [0, 1]$ such that:

$$\varphi^{[-1]} = \begin{cases} \varphi^{-1}(t), & \text{for } 0 \leq t \leq \varphi(0); \\ 0, & \text{for } \varphi(0) \leq t \leq \infty. \end{cases}$$

If φ is a convex function, then the function $C : [0,1]^2 \rightarrow [0,1]$ given by

$$C(u,v) = \varphi^{[-1]} \left[\varphi(u) + \varphi(v) \right] \tag{3.9}$$

is an *Archimedean copula*, while φ is the *generator* of the copula.

The Gumbel Copula

A *Gumbel copula* C_θ^{Gumbel} is obtained by starting form the generator function φ

$$\varphi(t) = (-lnt)^\theta$$

with $\theta \geq 1$. Now, simply by applying equation (3.9), we get

$$C_\theta^{Gumbel}(u,v) = \varphi^{-1} \left[\varphi(u) + \varphi(v) \right] = exp \left[-[(-lnu)^\theta + (-lnv)^\theta]^{1/\theta} \right] \tag{3.10}$$

The Clayton Copula

A *Clayton copula* $C_\theta^{Clayton}$ is given by the generator function φ

$$\varphi(t) = \frac{t^{-\theta} - 1}{\theta}$$

with $\theta \in [-1, \infty]/\{0\}$. Thus, applying equation (3.9) and having to take into account the properties of the generator function, we obtain

$$C_\theta^{Clayton}(u,v) = max \left[(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, 0 \right] \tag{3.11}$$

Nevertheless, when $\theta > 0$, then $\varphi(0) = \infty$, the closed formula for the same copula results easier:

$$C_\theta^{Clayton}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1 \right)^{-1/\theta} \tag{3.12}$$

The Frank Copula

The generator function φ for the *Frank copula* C_θ^{Frank} is

$$C_\theta^{Frank}(u,v) = -ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$$

with $\theta \in \mathbb{R}/\{0\}$. As in the previous cases, we get the equation for the Frank copula directly from the application of the equation (3.9):

$$C_\theta^{Frank}(u,v) = -\frac{1}{\theta}ln\left[1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{(e^{-\theta}-1)}\right] \qquad (3.13)$$

Finally, it is important to highlight that, in order to extend this setting to the multivariate case, we need to take into consideration the so called *Kimberling theorem*, as suggested in Embrechts, Lindskog and McNeil. This theorem states the following:

Theorem 2 *(Kimberling). Let $\varphi : [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $\varphi(0) = \infty$ and $\varphi(1) = 0$, and let φ^{-1} be the inverse of φ. Then, for $n \geq 2$, the function $C : [0,1]^n \to [0,1]$ defined as*

$$C(u_1, u_2, \ldots, u_n) = \varphi^{-1}\left[\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_n)\right]$$

is a n-dimensional Archimedean copula if and only if φ^{-1} is completely monotone on $[0,\infty]$.

3.3   Tail Dependence

One of the fundamental reasons making copulas so important in finance is due to their ability in modelling *tail dependence*. This concept is particularly relevant for risk managers, and thus let us explain it.

If we consider a vector $(X,Y)^T$ of continuos random variables, with marginal distribution functions F and G, we can imagine tail dependence as the probability of getting a high (low) extreme value of Y, given a high (low) extreme value of X. Of course, this concept is very important and thus it is one of the major problems risk managers have to face in their daily activities. In fact, if we consider a credit portfolio, you can easily understand that underestimating the probability of simultaneous joint defaults of several counterparties could lead to catastrophic consequences in terms of losses. Here copulas represent important tools, since they are able to model such extreme events effectively.

Hence, when we are considering a bivariate distribution, the concept of tail dependence expresses either the amount of dependence in the upper-right quadrant tail or lower-left quadrant tail. In particular, tail dependence between two continuos random variables is determined directly by means of copulas and then, as we saw with last proposition, the amount of tail dependence does not change under strictly increasing trasformations of X and Y. There exist two coefficients to measure the degree of tail dependence between two continuos random

variables X and Y with distribution functions F and G: the *coefficient of upper tail dependence* λ_U and the *coefficient of lower tail dependence* λ_L.

Definition 5 *The coefficient of upper tail dependence of X and Y is*

$$\lim_{u \nearrow 1} P\left(Y > G^{-1}(u) | X > F^{-1}(u)\right) = \lambda_U$$

given that the limit $\lambda_U \in [0,1]$ exists. In the case $\lambda_U \in (0,1]$, we say that X and Y are asymptotically dependent in the upper tail, *while, if $\lambda_U = 0$, X and Y are said to be* asymptotically independent in the upper tail.

It is possible to give an alternative, but equivalent, definition in order to show that the concept of tail dependence is a copula property. In fact, the probability $P\left(Y > G^{-1}(u) | X > F^{-1}(u)\right)$ could be written, equivalently, as

$$\frac{1 - P\left(X \leq F^{-1}(u)\right) - P\left(X \leq G^{-1}(u)\right) + P\left(X \leq F^{-1}(u), Y \leq G^{-1}(u)\right)}{1 - P\left(X \leq F^{-1}(u)\right)}$$

Now, we can write the following definition.

Definition 6 *If we have a bivariate copula C, such that the following limit*

$$\lim_{u \nearrow 1} \frac{1 - 2u + C(u,u)}{1 - u} = \lambda_U$$

exists, then C has upper tail dependence *if $\lambda_U \in (0,1]$, otherwise, when $\lambda_U = 0$, it is characterized by* upper tail independence.

We can show this concept by looking at a simple example.
Let us consider the bivariate Gumbel family of copulas

$$C_\theta(u,v) = exp\left(-[(-lnu)^\theta + (-lnv)^\theta]^{\frac{1}{\theta}}\right)$$

where $\theta \geq 1$. Hence,

$$\frac{1 - 2u + C(u,u)}{1 - u} = \frac{1 - 2u + exp\left(2^{\frac{1}{\theta}}lnu\right)}{1 - u} = \frac{1 - 2u + u^{2/\theta}}{1 - u}$$

and finally we get

$$\lim_{u \nearrow 1} \frac{1 - 2u + C(u,u)}{1 - u} = 2 - \lim_{u \nearrow 1} 2^{1/\theta}u^{1/\theta - 1} = 2 - 2^{1/\theta}$$

Since $\theta > 1$, it means that $\lambda_U \in (0,1]$, that is C_θ has upper tail dependence.

When a simple closed formula for the copula function does not exist, then it is possible to use an alternative and more useful formula for getting the value of λ_U. In fact, let us consider two uniformly distributed $U(0,1)$ random variables U and V, having a copula C. The first thing to do is considering that

$$P\left(V \le v|U = u\right) = \frac{\partial C(u, v)}{\partial u}$$

and then

$$P\left(V > v|U = u\right) = 1 - \frac{\partial C(u, v)}{\partial u}.$$

Of course, the same could be done when conditioning on V.
Hence:

$$\begin{aligned}
\lambda_U &= \lim_{u \nearrow 1} \frac{\bar{C}(u, u)}{1 - u} \\
&= -\lim_{u \nearrow 1} \frac{d\bar{C}(u, u)}{du} \\
&= -\lim_{u \nearrow 1} \left(-2 + \frac{\partial C(s, t)}{\partial s}\big|_{s=t=u} + \frac{\partial C(s, t)}{\partial t}\big|_{s=t=u}\right) \\
&= \lim_{u \nearrow 1} [P\left(v > u|U = u\right) + P\left(U > u|V = u\right)]
\end{aligned}$$

This result shows that $C(u, v) = C(v, u)$, meaning that C is an exchangeable copula. This leads to a simple expression for λ_U:

$$\lambda_U = 2 \lim_{u \nearrow 1} P\left(V > u|U = u\right)$$

Analogously, we can define the coefficient of lower tail dependence. Provided that the following limit

$$\lim_{u \searrow 0} C(u, u) = \lambda_L$$

exists, we will say that C has lower tail dependence if $\lambda_L \in (0, 1]$, while, if $\lambda_L = 0$, thus C is characterized by having tail independence.
In the case copulas have not a simple closed formula, as previously, an alternative formula for λ_L results to be more helpful. Here, let us consider a random vector $(U, V)^T$, with copula C. Thus, we can get the following:

$$\begin{aligned}
\lambda_L &= \lim_{u \searrow 0} \frac{C(u, u)}{u} \\
&= \lim_{u \searrow 0} \frac{dC(u, u)}{du} \\
&= \lim_{u \searrow 0} \left(\frac{\partial C(s, t)}{\partial s}\big|_{s=t=u} + \frac{\partial C(s, t)}{\partial t}\big|_{s=t=u}\right) \\
&= \lim_{u \searrow 0} [P\left(v < u|U = u\right) + P\left(U < u|V = u\right)]
\end{aligned}$$

Thus, once again, $C(u,v) = C(v,u)$, that is we get another time that C is an exchangeable copula. This leads to a simpler formula for λ_L:

$$\lambda_L = 2 \lim_{u \searrow 0} P\left(V < u | U = u\right)$$

Now, let us take the last step.
The *survival copula* $\hat{C}$ of two random variables having copula C can be defined as

$$\hat{C}(u,v) = u + v - 1 + C(1 - u, 1 - v)$$

while the joint survival function for two uniformly distributed $U(0,1)$ random variables with joint distribution function C is given by

$$\bar{C}(u,v) = 1 - u - v + C(u,v) = \hat{C}(1 - u, 1 - v).$$

From this partial result, it is possible to write

$$\lim_{u \nearrow 1} \frac{\bar{C}(u,u)}{1-u} = \lim_{u \nearrow 1} \frac{\hat{C}(1-u,1-u)}{1-u} = \lim_{u \searrow 0} \frac{\hat{C}(u,u)}{u}$$

This final result suggests the non-trivial conclusion for which the coefficient of upper tail dependence of C equals the coefficient of lower tail dependence of $\hat{C}$. Furthermore, the viceversa holds, as well: the coefficient of lower tail dependence of C equals the coefficient of upper tail dependence of $\hat{C}$.

3.4 One-Factor Copula Models for Pricing Credit Derivatives

The major difficulty in pricing credit derivatives is given by the estimation of the *default correlation* between the underlying assets, in order to measure the tendency of two companies to default about simultaneously. As widely explained in chapter one, researchers and practioners have suggested two main default correlation models: reduced form models and structural models. However, the major problem implied in these models arises from the fact that they are, in general, very time consuming when are used for the evaluation of credit derivatives. This is one of the main reasons for which alternative models such as *one-factor copula models* have been devoloped: in such a way, the joint probability distribution for a set of companies may be derived from the marginal distributions.
Let us consider a portfolio of N companies and define

- τ_i as the time to default of the i-*th* company, with $i = 1, \ldots, N$;

- $Q_i(t)$ as the cumualtive *risk-neutral* probability that the i-*th* company will default within time t. In other words, $Q_i(t) = Prob(\tau_i \leq t)$, with $i = 1, \ldots, N$;

- $S_i(t)$ as the *risk-neutral* probability that the i-*th* company will survive beyond time t, i.e. $S_i(t) = 1 - Q_i(t)$, with $i = 1, \ldots, N$;

The first assumption behind one-factor copula models consists in creating a correspondence between the default time τ_i for the i-*th* company and a random variable X_i such that, any given time t correspond to a specific value of x:

$$Prob(X_i < x) = Prob(\tau_i < t) \tag{3.14}$$

with $i = 1, \ldots, N$.

On the basis of this assumption, the one-factor copula model is generated by the sum of two components:

$$X_i = a_i M + \sqrt{1 - a_i^2} Z_i \tag{3.15}$$

for every $i = 1, \ldots, N$.

In this model, M represents a common factor affecting the dynamics of each asset and thus, a good approximation for it may be a well diversified market index (e.g. S&P 500). On the contrary, Z_i is an idiosyncratic factor, specic for each company included in the portfolio. Both M and the Z_i's are characterised by zero mean and unit variance. Concerning a_i, which has to satisfy the condition $-1 \leq a_i \leq 1$, it may be set equal to the correlation between the equity returns of the i-*th* company and those of the market index M. Furthermore, mutually independence among M and the Z_i's is assumed and thus, as it can be simply proved, the default correlation between company i and company j, with $i, j = 1, \ldots, N$ results to be $a_i a_j$.
This is due to the fact that is reasonable to assume that the default correlation between company i and company j is the same as the correlation between X_i and X_j. In fact, the covariance is defined as

$$cov(X_i, X_j) = E\left[(X_i - E(X_i))(X_j - E(X_j))\right]$$

while

$$X_i = a_i M + \sqrt{1 - a_i^2} Z_i \tag{3.16}$$

$$X_j = a_j M + \sqrt{1 - a_j^2} Z_j \tag{3.17}$$

It is clear that the covariance between X_i and X_j may be write simply as

$$cov(X_i, X_j) = E\left[X_i X_j\right]$$

since by definition the first moments of all the Z_i's are equal to zero. Now, by replacing X_i and X_j respectively with equations (3.16) and (3.17), and taking into consideration that M and the Z_i's are assumed to be independent, after

developing the product between X_i and X_j, the only term which results to be different from zero is $a_i a_j E[M^2]$. Thus

$$cov(X_i, X_j) = a_i a_j E[M^2] = a_i a_j.$$

Finally, since the correlation coefficient ρ between X_i and X_j is nothing else that the ratio between the covariance and the standard deviations of X_i and X_j, then

$$\rho = \frac{cov(X_i, X_j)}{\sigma_i \sigma_j} = cov(X_i, X_j) = a_i a_j.$$

Let F_i denote the cumulative distribution of X_i under the one-factor copula model, the correspondence between the X_i's and the t_i's are obtained by setting every general point $X_i = x$ to $t_i = t$ where $t = Q_i^{-1}[F_i(x)]$.
Now, by denoting the cumulative distribution of the Z_i's as H_i, after looking at (3.15), it is possible to write

$$Prob(X_i < x|M) = H_i \left[\frac{x - a_i M}{\sqrt{1 - a_{i^2}}} \right],$$

and since

$$x = F_i^{-1}[Q_i(t)] \implies Prob(\tau_i < t) = Prob(X_i < x)$$

Thus

$$Prob(\tau_i < t|M) = H_i \left[\frac{F_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1 - a_{i^2}}} \right]. \tag{3.18}$$

It is clear that the conditional survival probability $S_i(t|M)$ of the i-*th* company, with $i = 1, \dots, N$, is nothing else that

$$S_i(t|M) = 1 - H_i \left[\frac{F_i^{-1}[Q_i(t)] - a_i M}{\sqrt{1 - a_{i^2}}} \right]. \tag{3.19}$$

The one-factor copula model offers the great advantage of creating a tractable multivariate joint distribution by exploiting the known marginal distributions of the single variables. Since the major property of copula functions consists in splitting the marginal distributions from the dependence structure, in this case the nature of the default dependence is governed by the choice of the copula.
The use of such a model in credit risk modeling, for the estimation of the correlation default, was indicated for the first time by Li (1999, 2000). In particular, he suggested the use of the *Gaussian copula*, i.e. he supposed that both the common market factor M and the idiosyncratic components Z_i's were standard normally distributed. Obviously, since the sum of two *independent* normal distributed random variables is still normal, the final result is that the X_i's in the equation (3.15) are standard normally distributed, as well.

The aim of the next chapter is to study the effects of removing and generalizing the usual assumptions which make the one factor copula model as the *standard market model*. This will be done and analyzed on the prices of an empirical *Basket Default Swaps*.

Chapter 4

Beyond the Market Standard Model

During the recent years, with the large increase in trading volume of basket credit derivatives, dependency modelling for such instruments has evolved enormously. The first attempts for this purpuse were represented by binomial extension techniques but, after other approaches, copula based models have certainly become more widely studied and used over the last three years. However, the standard Gaussian copula of Li (2001) may be considered as the historical starting point. This approach has been implemented in *CreditMetrics CDO Manager* and today it is one of the most used models for the management and monitoring of correlated products. Since then, several modifications and extensions have been proposed, especially for trying to remove and replacing some of the too restrictive assumptions on which this model is based on.

In this chapter, we are going to use a one-factor copula model in order to price an empirical Basket Default Swap composed by N reference entities.
In particular, we will perform an analysis under the standard hypothesis framework. A natural extensions of the original *Li model* will be proposed, in order to make it more complying with the reality conditions.
First of all, we show the fundamental hypothesis which our work relies on. Given a usual standard filtered probability space $(\Omega, F, \mathbb{P}^*, \{\Im_t\}_{t \geq 0})$, it assumes the absence of arbitrage condition, in order to guarantee the existence of a unique risk-neutral probability measure $\mathbb{P}^*$. On the basis of such assumptions, non dividend paying assets (default free) are martingales when they are discounted using the risk-free interest rate.
To evaluate the probability of joint defaults, the one-factor copula model presented in the last section of the previous chapter will be exploited. As it may be inferred from equation (3.15), the main concept the one-factor copula model relies on is that all the reference entities are influenced by the same source of uncertainty, i.e. there is only one common factor M that affects the dynamics

of all the names, while the other factors Z_i, with $i = 1, \ldots, N$ are idiosyncratic components.

Beyond the fact that the one-factor copula model allows for fast calculations which do not imply the use of Monte Carlo simulations, the huge success of this kind of model may be essentially explained by their endogeneous features. First of all, they certainly represent an intuitive framework, but not only. In fact, the use of these models does not require the modelling of the full correlation matrix. This peculiarity is a particular point of strength, otherwise given a portfolio of N reference entities, $\frac{N(N-1)}{2}$ different pairwise correlation should be estimated. The hard part of this task is not given only by the large number of estimations to be performed, but also by the fact that default correlations are in general very difficult to estimate, since joint defaults are particularly uncommon events. For this reason, due to the lack of suitable data, a reliable statistical analysis cannot be done and thus it is acceptable to take CDS spreads correlations or equity correlations as they were default correlations, even though it should always kept in mind that they represent only a proxy, since these values certainly incorporates other external conditions, such as liquidity factors.

The existence of the risk-neutral probability measure $\mathbb{P}^*$ is fundamental for pricing correctly our credit derivative instrument. In order to price credit derivatives, it is necesssary to model both the risk neutral probability of default of each reference asset in the portfolio and the probability of joint defaults. Because CDS contracts are actually characterized by an extremely high degree of liquidity and standardization, it is a common practice to calibrate the risk-neutral default probabilities directly from their quotations. In general, it is assumed that the default event may be well described by the first jump of a Poisson process, assuming that the risk-free interest rate curve and the credit spreads structures are independent. Given the standard probability space defined above, we may consider the $\{\Im_t\}$-stopping time τ_i for modelling the default time of the *i-th* name in the portfolio. Let $\lambda_i(t)$ be the so called *hazard rate* function or simply the *intensity* of the Poisson process. We define it as

$$\lambda_i(t) = \frac{f_i(t)}{1 - F_i(t)} \tag{4.1}$$

where $F_i(t) = \mathbb{P}^* \{\tau_i < t\}$ and $f_i(t)$ respectively denote the default probability distribution and the probability density function with respect to the *i-th* obligor.

Not that $\lambda_i(t)$ may be interpreted as the value of the *conditional probability density function* of the time until default τ_i at the t-*th* year, given its survival until this time.

Defining $S_i(t) = 1 - F_i(t)$ the survival probability until time t of the *i-th* name, an alternative way to write $\lambda_i(t)$ is the following:

$$\lambda_i(t) = \frac{f_i(t)}{1 - F_i(t)} = -\frac{S_i'(t)}{S_i(t)} \tag{4.2}$$

In fact $f_i(t)$ is nothing else that $F_i'(t) = -S_i(t)$ and $1 - F_i(t)$ is just equal

to $S_i(t)$.

Hence, considering the integral of $\lambda_i(t)$, we get

$$\int \lambda_i(t)dt = -\int \frac{S_i'(t)}{S_i(t)}dt = -ln|S_i(t)| = -lnS_i(t)$$

being $S_i(t)$ the survival function that assumes only non-negative values. Thus, we can express the survival function S_i in terms of the hazard rate function λ_i, obtaining

$$S_i(t) = e^{-\int_0^t \lambda_i(s)ds}$$

and obviously

$$F_i(t) = 1 - S_i(t) = 1 - e^{-\int_0^t \lambda_i(s)ds} \tag{4.3}$$

Thus, once the hazard rate $\lambda_i(t)$ has been defined, it is not hard to bootstrap the default probabilities.

In this section, we will suppose to price a basket default swaps composed by $N = 10$ names and considering a time horizon $T = 10$ years. The following table shows all the reference entities considered[1], and their relative ratings[2].

COMPANY	RATING
American Express	A
Chevron	AA
Citigroup	A
Coca Cola	AA
Coventry Health	BBB
Exxon Mobil	AAA
Ford	CCC
General Motors	CCC
Pfizer	AAA
Time Warner	BBB

This credit derivative instrument will be first priced under the usual hypothesis for which the one-factor copula model is known , i.e.the *standard market model*. Since most of the assumptions characterizing this framework are essentially too strong, the same model will be generalized by using more general, as well as more realistic conditions and, after this, the effects of this model extensions will be analyzed.

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[1]All these companies are traded on the New York Stocks Exchange.
[2]Ratings assigned by the rating agency Moody's.

4.1   One-Factor Copula Model: the Standard Framework

The use of the Gaussian copula in credit risk modelling, for the estimation of the default correlation between two obligors, was suggested for the first time by Li in 2000. Considering the one-factor copula model showed by equation (3.15), Li assumed that both the common market factor M and the idiosyncratic component Z_i, with $i = 1, \ldots, N$ were standard normal distributed. In such a way, the one-factor copula model became the well known *one-factor Gaussian copula model*. As a clear consequence of this assumption, the random variables X_i's in equation (3.15) can simply be written in a closed formula, since the sum of two independent standard normal distributions is still a standard normal distribution.

The one-factor Gaussian copula model represents the basis of the *market standard model*, when implemented taking into account some given assumptions. First of all, it supposes to deal with a homogeneous portfolio. This means that every reference asset composing the basket is treated as it was the same, and thus all the names present essentially equal features in terms of conditional default probabilities (as shown by equation (3.18)). This is a direct consequence of the assumption for which all the names in the reference portfolio are characterized by the same CDS premia, implying that the value of the bootstrapped hazard rates do not change when considering different names in the portfolio. Furthermore, the market standard model supposes to deal with constant default intensities, independently from the time horizon considered. In addition, the pairwise correlation between the obligors is assumed to be the same for all the reference entities, as well as the recovery rate R: a typical value for it in empirical applications is $R = 40\%$. Finally, the standard market model exploits the hypothesis of a flat risk-free interest rate curve and in many empirical applications this value is tipically taken equal to 5%.

4.2   One-Factor Copula Model: Beyond the Standard Hypothesis

The main issue in pricing correlated-based products is offering a good estimation of the default dependence among the underlying assets. As said above, copula functions represent a popular method for getting this dependence structure. Li (2000) [33] can be considered the father of one-factor copula models, even though, from that date, several extensions to the one-factor copula model have been proposed, see [25] and [29]. These works tried to improve the basic standard Li model, by using probability distributions characterized by fatter tails than those of the Gaussian one (Student t and Normal Inverse Gaussian distributions). These extensions are essentially motivated by the fact that most of the assumptions underlying the one-factor copula model seem to be too unrealistic and this opinion is confirmed by a wide literature where the insufficiency

of this model to match market quotes of CDO products is shown.

We will show the impact of using alternative distibutions with fatter tails than the Gaussian one. These distributions are the *Generalized Tempered Stable* (GTS) and *CGMY* (Carr-Geman-Madan-Yor) distributions and we will evaluate the use of them on Basket Default Swap pricing with one-factor copula model.

In addition, every model parameter has been estimated by using real market data, differently from the common technique of many practitioners which deal with parameters values considered to be constant and unaffected from the market conditions. Thus, the pairwise correlations among the names have been estimated from historical data, and not set to be constant on the basis of an *a priori* assumption. The same can be said for the risk-neutral interest rate whose dynamics, conversely from ordinary assumptions, has been taken from the market. With respect to the calibration of the marginal default probabilities, as already mentioned, CDS quotes may be used. In fact, the prices of these derivative instruments reflect the market expectations about the credit merit of a company and thus it is correct to use them as the first step in the pricing process of a more structured instrument such as a basket default swap. Generally, it is assumed that the default process is described by the first jump of a Poisson process and the usual assumption, as confirmed in [25], consists in supposing a constant default intensity over the entire time horizon. If the same CDS spreads are taken into account for this calibration procedure, and this is another largely common assumption, equal marginal default probabilities will result for every name in the collateral. It is clear that such a framework may easily considered as too restrictive and not compliant with the reality. Dealing with such assumptions would mean considering that the default probability of a given company is not affected by its own rating and in addition the time horizon would not play any role: in a few words many of the previous works assume that an AAA rated company has the same probability to default of a CCC rated company. Moreover, with such assumptions the default intensity is constant over a time period, indefferently from its length.

In our modelling, this behaviour has overcome. First of all, each marginal default probability has been calibrated from the relative CDS market quotes: by doing so, a different credit merit reflects an appropriate probability of default and, furthermore, the default intensities have been supposed to be time-varying and not constant. On the basis of these conditions, and considering the risk-neutral interest rate and the spreads dynamics to be independent, the fair value of a CDS may be computed under the *non-arbitrage condition*, simply by setting the CDS *premium leg* equal to its *default leg*.

In a CDS contract, the protection buyer (premium leg) periodically pays a fixed premium S to the protection seller (default leg), which has to carry out the payment of the bond nominal value in case of default. The premium, expressed in yearly basis points with respect to the notional, is usually paid quarterly, either until the natural expiration date of the contract or until the default date. Hence the premium leg (PL) may be written as

$$PL(0,T) = M \sum_{i=1}^{n} \Delta S_T B(0,t_i) S(0,t_i) \qquad (4.4)$$

where $(t_1, t_2, \ldots, t_n)$, with $t_i = t_{i-1} + \Delta$ denote the periodic (Δ) payment dates, t_n represents the expiration date and M is the notional value. Furthermore, $B(0,t_i)$ and $S(0,t_i)$ respectively denote the risk-neutral discount factor and the survival probability at time t_i.
The value of the default leg DL may be written as the present value of the loss given default at the moment in which the default of the reference entity occurs. Thus:

$$DL(0,T) = M(1-R) \sum_{j=1}^{m} \left(S(0,t_{j-1}) - S(0,t_j) \right) B(0,t_j) \qquad (4.5)$$

where R denotes the recovery rate, $S(0,t_{j-1}) - S(0,t_j)$ the probability that the default happens in the time interval $[t_{j-1}, t_j]$ and $(t_1, t_2, \ldots, t_m)$ the discrete observation dates. Now, simply by setting the premium leg PL equal to the default leg DL, the fair CDS spread may be found as

$$S_T = \frac{(1-R) \sum_{j=1}^{m} \left(S(0,t_{j-1}) - S(0,t_j) \right) B(0,t_j)}{\sum_{i=1}^{n} \Delta S_T B(0,t_i) S(0,t_i)} \qquad (4.6)$$

Note that in general the premium leg PL may include an accrued component in order to take into consideration the premium amount correspondent to the period going from the last payment date to the default date. However, as proven by Arvanitis-Gregory (2002), this value has only a little impact on the CDS premium, especially when entities with a high rating are considered and thus this component is very often negleted.
Once the recovery rate R[3] is set to a given value, equation (4.6) can be exploited to extrapolate the marginal survival probability S with respect to the i-th company. The standard hyphotesis, as already mentioned above, consists in considering the default intensity λ constant over the overall time horizon of the CDS contract. Conversely from this unrealistic assumption, our modeling shows an alternative approach to obtain more realistic default intensities, which will result to be time-varying.
On the basis of the different maturity dates relative the CDS market prices available, the default intensity for the i-th name, λ_i, may be considered constant only over the time period laying between two expiration dates t_{i-1} and t_i. Hence:

$$\lambda(u) = \alpha_i$$

for $i \in [t_{i-1}, t_i]$ and thus

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[3]The recovery rate R should be fixed on the basis of the historical recovery values. In general, the observed values for such parameters range from 20% to 50%, and so in this chapter it is used a constant value of 40%, as a good approximation of the recovery rates R of all the names in the reference portfolio.

$$S(0, t_j) = exp\left(-\sum_{j=1}^{k} \alpha_i(t_j - t_{j-1})\right)$$

where $j = 1, \ldots, k$ represent the maturity dates of the CDS contracts considered. In such a way, all the values α_i can be simply obtained, minimizing the difference between the observed market prices and those of the model.
For this work, a basket default swap composed by $N = 10$ different CDS contracts has been considered. The default intensities with respect to each of the N names of the basket have been estimated considering five different maturity dates: $T_1 = 1$ year, $T_2 = 3$ years, $T_3 = 5$ years, $T_4 = 7$ years, $T_5 = 10$ years. Furthermore, the U.S. Treasury rate has been taken as the risk-free interest rate and quarterly premia payments have been assumed. The example below shows the CDS premia which have been used in order to find out the default intensities for two of the N names[4] inside the collateral.

Expiration (years)	CDS Spreads (bps)
1	7.8
3	15.7
5	33.5
7	46.7
10	62.80

Table 4.1: Time Warner CDS premia for different expiration dates

Expiration (years)	CDS Spreads (bps)
1	7.7
3	8.9
5	16.5
7	23.10
10	29.40

Table 4.2: Coca Cola CDS premia for different expiration dates

On the basis of these assumptions and exploiting the CDS premia showed by table 4.1 and table 4.2, the *time-varying* default intensities with respect to the relative companies have been computed, as showed by figure 4.1 and figure 4.2.

Looking at figure 4.3 as the Time Warner CDS premia always dominate those relative to Coca Cola: this is an obvious consequence of the better credit quality characterizing the AA rated company. However, it is very interesting to

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[4]In the example CDS market premia for Time Warner (BBB rated) and Coca Cola (AA rated) are showed.
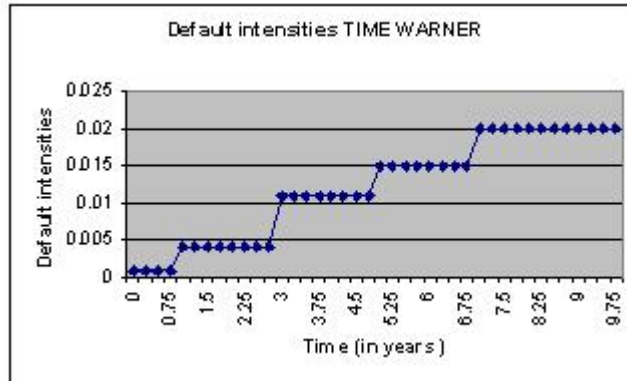
Figure 4.1: default intensities calibrated for a BBB rated company (Time Warner).
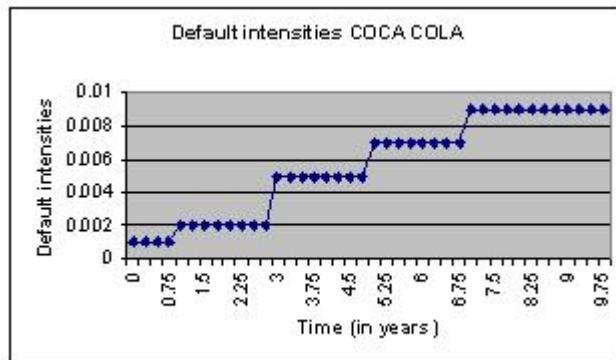
Figure 4.2: default intensities calibrated for an AA rated company (Coca Cola)
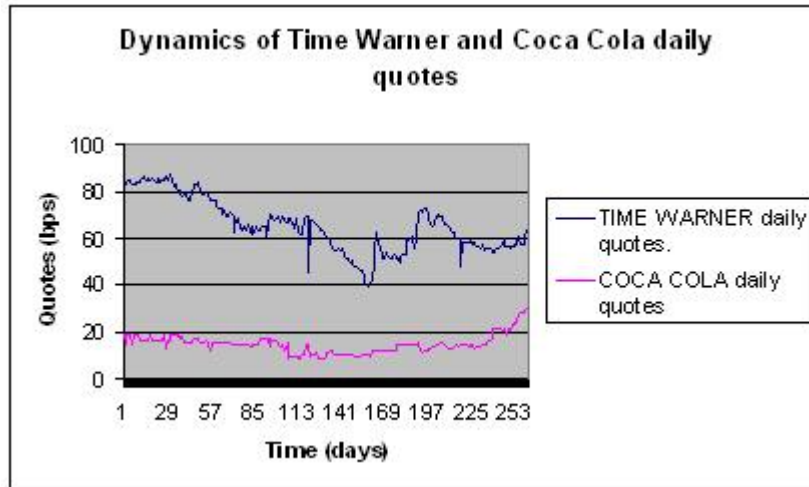
Figure 4.3: Comparison between one year of daily quotes for Time Warner (BB rated) and Coca Cola (AA rated), with expiration date of ten years.
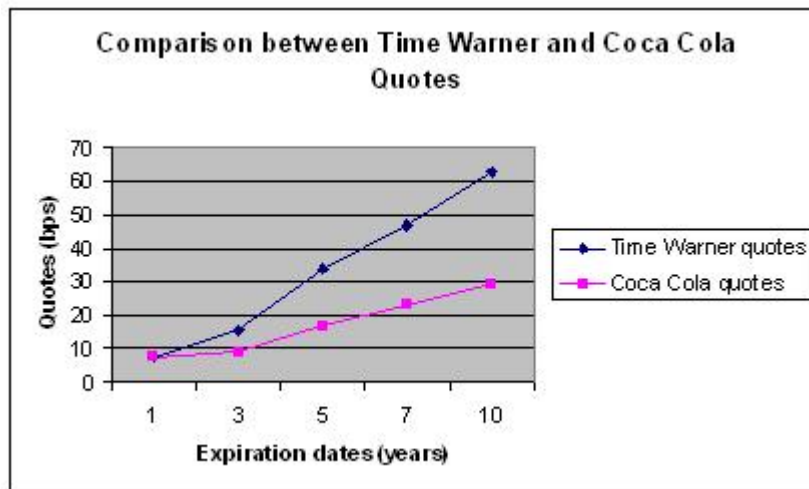

Figure 4.4: Comparison among CDS quotes with respect to different maturities, for Time Warner (BB rated) and Coca Cola (AA rated).

find out that the difference between the CDS premia requested by the market for this companies change daily: in a few words, the market expectations about the probability of default is affected every day by several factors, and this leads to those spread fluctuations which are so evident in figure 4.3. Figure 4.4 confirms what has been said so far about the credit qualities of the two companies taken into consideration: the higher default risk associated to a BBB rated company with respect to an AA rated company appears by looking at the CDS premia for different expiration dates: the market will require an higher premium as soon as the default probability of a given company becomes larger.

Hence, so far, the usual *standard market model* has been extended on the basis of the following considerations:

- pairwise correlations have not been considered constant for all the names in the collateral, but they have been estimated on real market data;

- a real market risk-neutral interest rate curve has been used instead of supposing a flat curve;

- default intensities have been estimated from the market, and not assumed to be the same for each name of the collateral;

- default intensities have been computed in order to be time-varying and not constant.

These extension certainly represent a first improvement of the market standard model. However, the main drawback of such a model derives from the nature of the dependence structure driving the default correlation. In fact, the insufficiency of the single factor Gaussian model to model the default dependence has already been underlined by a lot of practitioners and academics.
For a good comprehension of the problem it is fundamental to compare the joint default probabilities resulting from the Gaussian copula with those implied by the market: a consistent higher probability is allocated by the market to high default scenarios than the probability which the Gaussian copula is able to assign. This means that, in general, the market expects a low probability of having no (or few) defaults, while in the Gaussian framework the same probability is certainly higher. This is the reason for which in this chapter we decided to furtherly extend the market standard model by using two alternative distributions: the CGMY and the Generalized Tempered Stable (GTS) distributions.

4.3   Alternative Distributions for One-Factor Copula Models

The Black and Scholes model, describing the dynamics of the stock price evolution with geometric Brownian motion, has become a standard model especially for option pricing and hedging purposes. However, from an empirical point

of view, this widely celebrated model presents different shortcomings. First of all, it is well known that real measured probability distributions are clearly leptokurtotic, because of their much heavier tails than those of the Gaussian distribution. In addition, real assets are characterized by prices exhibiting jumps in which prices move too quickly such that a dynamic hedging cannot be carried out. Finally, the continuous trading framework does not comply with the reality and options prices show to deviate from those computed with the Black-Scholes formula.

These problems leaded many authors and practitioners to develop alternative models, incorporating both Gaussian processes and jumps. Hence, *jump-diffusion* models are able to considering simultaneously frequent small movements with the diffusion part and rare large movements with jumps. These kinds of models consider heavy tails and market jumps, as well. An other alternative is given by stochastic volatility models, in which a second random process must be introduced in order to describe the instantaneous volatility of the underlying. These family of models offer the advantage of taking into account the typical volatility clustering which is observed in most financial time series, but, from the other side, they do not explain heavy tails.

In the 60's, Mandelbrot (1963) and Fama (1965) proposed the use of *stable Lévy processes* to model stock prices, but the main difficult in this approach was the infinity of the second moment of such processes.

It was only in the 90's that several families of Lévy processes with probability distributions of semi-heavy tails, i.e. exponential decaying, started to be used, especially for modelling stock returns and pricing options. The first remarkable step was taken by Madan and Seneta who studied *Variance Gamma processes* in 1990. Only few years ago (2002), Carr, Geman, Madan and Yor (*CGMY processes*) generalized the VG process and reported on the goodness of fit on stocks and indices. A further step forward is represented by *Generalized Tempered Stable processes* introduced by Cont and Tankov (2004). If its deterministic drift parameter is not considered, the GTS process is a model with six parameters which allows the jump component to have either infinite or finite variation. The GTS process may be considered as a better alternative than the CGMY process, since it allows for greater modelling freedom, thanks to two additional parameters which are not present into the CGMY processes.

The one-factor copula model showed by equation (3.15) can be set up by using alternative distributions for the common factor M and the idiosyncratic components Z_i, with $i = 1, \dots, N$. Regarding this possible extension, there exists a large literature which is partcularly focused on the use of some famous copulas such as Student t, Archimedean and Marshall-Olkin copulas.

The aim of this section is to analyze the impact of two new copula frameworks, given by the use of the CGMY and GTS distributions. These distributions reveal to be very versatile and useful, especially for their ability to reproduce heavy-tailed processes.

4.4   Introduction of Lévy Processes

Before starting with the description of the distributions on which the two alternative copula models proposed in the following are based on, it is necessary to briefly introduce the topic of *Lévy processes*. As already mentioned, the widely celebrated Black and Scholes model presents the great problems deriving from the fact the empirical log-returns of stocks do not follow a Normal law. The direct consequence is that it would be better to deal with more flexible processes with independent and stationary increments, such that the classical Brownian Motion can be generalized. Hence, Lévy processes may be considered for such a purpose.

Let $\phi(u)$ denote the characteristic function of a distribution. Then, the distribution will be said to be *infinitely divisible* if, for every integer number n, $\phi(u)$ is also the n-th power of the characteristic function.

Having an infinitely divisible distribution allows for the definition of a stochastic process $X = \{X_t, t \geq 0\}$ starting at zero and with independent and stationary increments, such that the increment of the process $X_{t+s} - X_s$, with $s, t \geq 0$, has $(\phi(u))^t$ as its characteristic function. A process with these features is called a *Lévy process*. Every sample paths of a Lévy process is almost surely continuous from the right and has limits from the left, this because every Lévy process has an RCLL (Right Continuos and Left Limited) modification which is still a Lévy process.

The *cumulant characteristic function $\psi(u)$*, also known as *characteristic exponent*

$$\psi(u) = log\phi(u)$$

satisfies the following formula (*Lévy-Khintchine formula*):

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} \left(exp(iux) - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx) \qquad (4.7)$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}/\{0\}$ with

$$\int_{-\infty}^{+\infty} inf\{1, x^2\}\nu(dx) = \int_{-\infty}^{+\infty} (1 \wedge x^2)\nu(dx) < \infty.$$

If these properties hold, it is possible to say that the infinitely distribution has a *Lévy triplet* $[\gamma, \sigma^2, \nu(dx)]$, where the measure ν represents the *Lévy measure* of the process X.

Furthermore, if the Lévy measure ν can be written as

$$\nu(dx) = u(x)dx$$

then $u(x)$ denotes the *Lévy density*, having the same mathematical characteristics as a usual probabilty density, except that it does not require to be integrable

and must have zero mass at the origin.

In general, a Lévy process is characterized by three different independent components: a linear term given by the first term of the (4.7), a Brownian part and a pure jump component respectively represented by the second and the third term of the (4.7). The Lévy measure $\nu(dx)$ governs the way in which the jumps may occur. Jumps with size laying in the set A happen on the basis of a Poisson process with intensity parameter given by $\int_A \nu(dx)$.

Finally, let us explain the concept of subordinator. A subordinator can be defined as a non-negative and non-decreasing Lévy process. It has no Brownian component, i.e. $\sigma^2 = 0$, a non-negative drift and a Lévy measure having only positive increments.

When $\sigma^2 = 0$ (there is no Brownian part) and $\int_{-1}^{+1} |x|\nu(dx) < \infty$ the process is said to be of finite variation and the characteristic exponent may be written as

$$\psi(u) = i\nu'u + \int_{-\infty}^{+\infty} \left(exp(iux) - 1 \right) \nu(dx)$$

for some ν' which denotes the drift coefficient. In this case the process can be decomposed into the difference of two increasing processes. Note that a subordinator is always of finite variation.

In the situation in which $\sigma^2 = 0$ and $\int_{-1}^{+1} \nu(dx) < \infty$, the process will be characterized by finitely many jumps in any finite interval, meaning that the process is of finite activity.

Since the Brownian motion is of infinite variation, every Lévy process with Brownian part is of infinite variation, as well. When a Lévy process does not include a Brownian component, and thus is said a pure jump Lévy process, it is of infinite variation if and only if $\int_{-1}^{+1} |x|\nu(dx) = \infty$.

Now, let

$$\Delta X_t = X_t - X_{t-}$$

be the jump that a process $X = \{X_t, t \geq 0\}$ makes at time t. Under some given weak moment assumptions, it may be proved that a Lévy process $X = \{X_t, 0 \leq t \leq T\}$ has a version of the predictable representation property (PRT), i.e. every square integral random variable F has a representation of the form

$$F = E[F] + \sum_{i=1}^{\infty} \int_0^T a_s^{(i)} d \left(H_s^{(i)} - E[H_s^{(i)}] \right)$$

where $a^{(i)} = \{a_s^{(i)}, 0 \leq s \leq T\}$ is predictable and $H^{(i)} = \{H_s^{(i)}, 0 \leq s \leq T\}$ is the power jump process of order i, i.e. $H_s^{(1)} = X_s$ and

$$H_s^{(i)} = \sum_{0 < u \leq s} (\Delta X_u)^i$$

with $i = 2, 3, \ldots$.

In the next sections an introduction of CGMY processes and Generalized Tempered Stable processes follows. These distributions will be used in order to generalize the one-factor copula model, trying to overcome the restrictive and unrealistic assumptions which the standard market model relies on.

4.4.1   The CGMY Distribution

CGMY processes belong to the family of Lévy Processes. In particular a Lévy process is said to be a CGMY process with parameters (C, G, M, Y) if its Lévy triplet $(\gamma, \sigma^2, \nu(dx))$ is given by

$$\sigma = 0,$$

$$\nu(dx) = C \left(\frac{e^{-Mx}}{x^{Y+1}} \mathbb{I}_{\{x>0\}} + \frac{e^{-G|x|}}{|x|^{Y+1}} \mathbb{I}_{\{x<0\}} \right),$$

$$\nu(0) = 0,$$

$$\gamma = \begin{cases} m + \int_{|x| \leq 1} x\nu(dx), & Y < 1, \\ m - \int_{|x| > 1} x\nu(dx) - CY\Gamma(-Y)(M^{Y-1} - G^{Y-1}) & Y \in (1,2), \end{cases}$$

with $C, M, G > 0, Y \in (-\infty, 2) / \{0, 1\}$ and $m \in \mathbb{R}$.

CGMY processes can be defined as processes

- starting at zero;

- having independent and stationary increments;

- where the increment over a time interval t is described by a CGMY distribution with parameters (tC,G,M,Y),

characterized by an infinitely divisible distribution.

Considering $(X_t)_{t \geq 0}$ as a CGMY process, characterized by parameters (C, G, M, Y, m), then its characteristic function ϕ_{X_t} may be expressed as follows:

$$\phi_{X_t}(u; C, G, M, Y, m) =$$
$$= \exp \left(iumt + tC\Gamma(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y) \right)$$

with $u \in \mathbb{R}$.

The CGMY process is a pure jump process, meaning that it does not include any Brownian component.

The Y parameter is fundamental for the description of the path behaviour of the process: when $Y < 0$ the process will be characterized by a finite number of jumps on a finite time interval. On the contrary, if $Y \geq 0$, the process is said to have infinite activity, i.e. it shows infinitely many jumps in any finite time interval.

In addition, when the parameter Y stands in the interval $[1, 2)$, the CGMY process is of infinite variation.

Note that the Variance Gamma process may be considered as a special case of the CGMY process. In fact, when $Y = 0$, the CGMY process becomes a VG process, that is:

$$CGMY(C, G, M, 0) = VG(C, G, M).$$

4.4.2 The Generalized Tempered Stable Distribution

Generalized Tempered Stable (GTS) processes have been introduced by Cont and Tankov (2004). A GTS process is a Lévy process on $\mathbb{R}$ with no Gaussian component and a specific Lévy density. These processes are characterized by probability densities with an exponentially decaying in the far tails after which the small jumps maintain their initial stable behaviour and the large jumps start to be exponentially tempered.

The Lévy triplet (σ^2, ν, γ) of a GTS process is given by

$$\sigma = 0,$$

$$\nu(x) = \left(C_+ \frac{e^{-Mx}}{x^{Y_+ + 1}} \mathbb{I}_{\{x > 0\}} + C_- \frac{e^{-G|x|}}{|x|^{Y_- + 1}} \mathbb{I}_{\{x < 0\}} \right) dx,$$

$$\gamma(0) = 0$$

and since Y_+ / Y_- are considered to be greater than one

$$\gamma = m - \int_{|x| > 1} x\nu(dx) - C_+ Y_+ \Gamma(-Y_+) M^{Y_+ - 1} - C_- Y_- \Gamma(-Y_-) G^{Y_- - 1}$$

with C_+, C_-, M, $G > 0$, and Y_+, $Y_- \in (1, 2)$ and $m \in \mathbb{R}$.

Each of the parameters of a GTS process control a different aspect of the stochastic process. In particular C_+ / C_- respectively control the overall and the relative frequencies of upward and downward jumps and more specifically they provide information regarding how often we should expect jumps larger than a given value. If we look at M and G, they rule the tail behaviour of the Lévy measure, telling how far the process may present a jump. Furtermore, if

they are not equal, the resulting distribution will be skewed. Finally, Y_+ and Y_- govern the local dynamics of the process, determining if it will be characterized by finite or infinite activity or variation.

Considering $(X_t)_{t\geq0}$ as a GTS process with parameters $(C_+, C_-, G, M, Y_+, Y_-, m)$, then its characteristic function can be written as

$$\phi(u; C_+, C_-, G, M, Y_+, Y_-, m) =$$
$$= exp[iumt + tC_+\Gamma(-Y_+)[(M - iu)^{Y_+} - M^{Y_+}]+$$
$$+ tC_-\Gamma(-Y_-)[(G + iu)^{Y_-} - G^{Y_-}]]$$

with $u \in \mathbb{R}$.

GTS processes are characterized by having infinite activity and finite variation if $Y_+/Y_- < 1$, but at least one of them is non-negative.

Finally, the CGMY can be obtained from a GTS process simply setting the following conditions:

$$C_- = C_+$$

and

$$Y_- = Y_+$$

while the same can be said for a Variance Gamma process which derives from a GTS process under the following assumptions:

$$C_- = C_+$$

and

$$Y_- = Y_+ = 0.$$

4.5   Computational Results

In this section, the results relative to a ten years empirical Basket Default Swap will be presented. This correlated-based product is supposed to be composed by the following names:

1. AMERICAN EXPRESS with rating (Moody's): A

2. CHEVRON CORPORATION with rating (Moody's): AA

3. CITIGROUP with rating (Moody's): A

4. COCACOLA with rating (Moody's): AA

5. COVENTRY HEALTH with rating (Moody's): BBB

6. EXXON MOBIL with rating (Moody's): AAA

7. FORD with rating (Moody's): CCC

8. GENERAL MOTORS with rating (Moody's): CCC

9. PFIZER with rating (Moody's): AA

10. TIME WARNER with rating (Moody's): BBB

With respect to the correlation parameters of the one-factor copula model given by equation (3.15), they have been estimated on the basis of the last ten years daily returns (until July 19, 2007) of the respective stocks, and as common component the *Standard and Poor's 500 index* of the New York Stock Exchange was considered. The parameters of the distributions on which the common market component M and the idiosyncratic terms Z_i's of equation (3.15) are modelled have been analogously estimated by using the relative daily stock prices of the last ten years, on the basis of the classical Maximum Likelihood Estimation Approach, using data until July 19, 2007, while the time-varying default intensities have been estimated by using the CDS quotes of July 19, 2007 of the relative names with maturities respectively of one year, three years, five years, seven years and ten years. Furthermore, the U.S. Treasury yield curve has been taken as the risk-free interest rate, extrapolating those values corresponding to time maturities for which there was no correspondence.
Finally, the Discrete Fourier Transform was used in order to invert the characteristic function, both in the case of CGMY copula model and for GTS copula model.

Change of Measure for CGMY Processes

Before starting with the pricing of the Basket Default Swaps, a problem arises: all the parameters of the CGMY distributions are calibrated on the basis of ten years daily stock prices, meaning that their values are obtained under the market measure $\mathbb{P}$. Clearly, for pricing reasons, it is strictly necessary to return to the risk-neutral measure $\mathbb{Q}$ and thus a change of measure has to be carried out.
Hence, first it is necessary to show the general result on the equivalence of measures for Lévy processes on the basis of the following theorem by Cont and Tankov and applying it to CGMY processes.

Theorem 3 (Cont and Tankov 2004b, p. 308). *Let $(X_t, \mathbb{P})$ and $(X_t, \mathbb{Q})$ be Lévy processes on $\mathbb{R}$, with Lévy triplets (σ^2, ν, γ) and $(\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})$ respectively. Then $\mathbb{P}|_{\Im_t}$ and $\mathbb{Q}|_{\Im_t}$ are equivalent for all $t > 0$ if and only if the Lévy triplets satisfy*

$$\sigma^2 = \tilde{\sigma}^2, \tag{4.8}$$

$$\int_{-\infty}^{+\infty} (e^{\psi(x)/2} - 1)^2 \nu(dx) < \infty, \tag{4.9}$$

$$\tilde{\gamma} - \gamma = \int_{|x| \leq 1} x(\tilde{\nu} - \nu)(dx). \tag{4.10}$$

When $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, the Radon-Nikodym derivative can be written as follows

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \mid_{\Im_t} = e^{U_t} \tag{4.11}$$

where $(U_t, \mathbb{P})$ denotes a Lévy process whose relative Lévy triplet $(\sigma_U^2, \nu_U, \gamma_U)$ of $(U_t)_{t \in [0,t]}$ is given by

$$\sigma_U^2 = \sigma^2 \eta^2, \tag{4.12}$$

$$\nu_U = \nu \circ \psi^{-1}, \tag{4.13}$$

$$\gamma_U = -\frac{\sigma^2 \eta^2}{2} - \int_{-\infty}^{+\infty} (e^y - 1 - y\mathbb{I}_{\{|y| \leq 1\}})\nu_U(dy), \tag{4.14}$$

such that ν respects the following condition

$$\tilde{\gamma} - \gamma - \int_{|x| \leq 1} x(\tilde{\nu} - \nu)(dx) = \sigma^2 \eta$$

if $\sigma > 0$ and *zero* if $\sigma = 0$.

As [9] suggests, theorem (3) can be applied to CGMY processes in order to set two equivalent measures for these family of processes:

Corollary 2 *Let $(X_t, \mathbb{P})$ and $(X_t, \mathbb{Q})$ be CGMY processes on $\mathbb{R}$ with parameters (C, G, M, Y, m) and $(\tilde{C}, \tilde{G}, \tilde{M}, \tilde{Y}, \tilde{m})$. Then $\mathbb{P}|_{\Im_t}$ and $\mathbb{Q}|_{\Im_t}$ are equivalent for all $t > 0$ if and only if $C = \tilde{C}$, $Y = \tilde{Y}$, and $m = \tilde{m}$.*

Proof. The example 9.1 contained in Cont and Tankov (Cont and Tankov, 2004b, p. 309) shows that condition (4.9) holds if and only if $C = \tilde{C}$ and $Y = \tilde{Y}$. Under these conditions, $\int_{|x| \leq 1} |x|\tilde{\nu}(dx) < \infty$ and $\int_{|x| \leq 1} |x|\nu(dx) < \infty$ if $Y < 1$, and thus it can be showed that $\int_{-\infty}^{+\infty} x(\tilde{\nu} - \nu)(dx) = CY\Gamma(-Y)(\tilde{M}^{Y-1} - \tilde{G}^{Y-1} - M^{Y-1} + G^{Y-1})$ when $Y \in (1, 2)$. Consequently, equation (4.10) holds when $\tilde{m} = m$, meaning that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $C = \tilde{C}$, $Y = \tilde{Y}$ and $m = \tilde{m}$.

Denoting with $T > 0$ a time horizon and with $r > 0$ the risk-free rate, the probability space $(\Omega, \Im_T, (\Im_t)_{t \in [0,T]}, \mathbb{P})$ can be modeled, where the filtration $(\Im_t)_{t \in [0,T]}$ satisfies the conditions $\Im_0 = \{\varnothing, \Omega\}$ and $\Im_S \subset \Im_t$ if $s \leq t$. With regard to the stock price process, it is modeled by an adapted cadlag and strictly positive Lévy process $S = (S_t)_{t \in [0,T]}$, while the process $\tilde{S} = (e^{-rt}S_t)_{t \in [0,T]}$

simply represents the *discounted price process* of S. Finally, let $\mathbb{P}$ be the *market measure*: a probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and thus it is called as an *Equivalent Martingale Measure* (EMM) of $\mathbb{P}$ if the discounted price process $\tilde{S}$ is a $\mathbb{Q}$-martingale with respect to the filtration $(\Im_t)_{t\in[0,T]}$.

Definition 7 *For $C > 0$, $G > 0$, $M > 1$, $Y \in (0,2) - \{1\}$ and $\mu > 0$, let $(X_t)_{t\in[0,T]}$ be a CGMY process with parameters (C, G, M, Y, 0). Then the process given by $(S_t)_{t\in[0,T]} = (S_0 e^{\mu t - t\psi_0(-i;C,G,M,Y)+X_t})_{t\in[0,T]}$ is called the CGMY stock price process with parameters (C,G,M,Y,μ).*

Simply by applying the corollary (2) to the above defined CGMY stock price process, the following lemma is obtained.

Lemma 1 *Let $(S_t)_{t\in[0,T]}$ be the CGMY stock price process respectively with parameters (C,G,M,Y,μ) under the market measure $\mathbb{P}$ and with parameters $(\tilde{C},\tilde{G},\tilde{M},\tilde{Y},r)$ under the measure $\mathbb{Q}$. Then $\mathbb{Q}$ is an EMM of $\mathbb{P}$ if and only if $\tilde{C} = C$, $\tilde{Y} = Y$ and*

$$r = \mu - \psi_0(-i;C,G,M,Y) + \psi_0(-i;C,\tilde{G},\tilde{M},Y). \tag{4.15}$$

Proof. For any $0 \le u < t < T$ it holds

$$E_{\mathbb{Q}}[e^{-rt}S_t|\Im_u] = e^{-ru}S_u e^{-(t-u)\psi_0(-i;\tilde{C},\tilde{G},\tilde{M},\tilde{Y})} E_{\mathbb{Q}}[e^{X_t-X_u}|\Im_u],$$

with $(X_t,\mathbb{Q})$ denoting a CGMY process with parameters $(\tilde{C},\tilde{G},\tilde{M},\tilde{Y},0)$. Since

$$E_{\mathbb{Q}}[e^{X_t-X_u}|\Im_u] = E_{\mathbb{Q}}[e^{X_{t-u}}] = e^{(t-u)\psi_0(-i;\tilde{C},\tilde{G},\tilde{M},\tilde{Y})},$$

then $E_{\mathbb{Q}}[e^{-rt}S_t|\Im_u] = e^{-ru}S_u$. This means that the discounted price process $(e^{-rt}S_t)_{t\in[0,T]}$ is a martingale under the measure $\mathbb{Q}$. In addition, since corollary (2) states that two measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\tilde{C} = C$, $\tilde{Y} = Y$ and

$$r - \psi_0(-i;\tilde{C},\tilde{G},\tilde{M},\tilde{Y}) = \mu - \psi_0(-i;C,G,M,Y)$$

then equation (4.15) is obtained.

The next step consists in defining a sub-family of EMMs of $\mathbb{P}$ in the following way:

$EMM^{CGMY}(\mathbb{P}) \equiv$
$\equiv \{\mathbb{Q}$ is an EMM of $\mathbb{P}|(S_t)_{t\in[0,T]}$ is a CGMY stock price process under $\mathbb{Q}\}$.

If $(S_t)_{t\in[0,T]}$ represents a CGMY stock price process with parameters (C,G,M,Y,μ) under the measure $\mathbb{P}$, then the following equation can be inferred from lemma (1):

$EMM^{CGMY}(\mathbb{P})=$
$= \{\mathbb{Q}|\mathbb{Q}$ is a measure induced by a $(C, \tilde{G}, \tilde{M}, Y, \mu - \psi_0(-i; C, G, M, Y))$ CGMY process where $(\tilde{G}, \tilde{M})$ respects equation (4.15)$\}$.

Since equation (4.15) is characterized by two degrees of freedom, then the set $EMM^{CGMY}(\mathbb{P})$ contains more than two elements: thus it is necessary to find an approach enabling us to select one EMM among them.
This method is offered by the *Esscher transform*.

Let us consider a pure jump Lévy process $(X_t)_{t \in [0,T]}$ with Lévy measure $\nu(dx)$ and drift γ under a measure $\mathbb{P}$. Supposing the existence of a real number θ such that $\int_{|x| \geq 1} e^{\theta x} \nu(dx) < \infty$ and by using the measure transform with the function $\psi(x) = \theta x$ presented in theorem (3), then it is possible to get an equivalent martingale measure $\mathbb{Q}^\theta$ making the process $(X_t)_{t \in [0,T]}$ a pure jump Lévy process characterized by Lévy measure $\tilde{\nu}(dx) = e^{\theta x} \nu(dx)$ and drift $\tilde{\nu} = \nu + \int_{-1}^{+1} x(e^{\theta x} - 1)\nu(dx)$. This may be done by carrying out the so called *Esscher transform*, which has been applied to the CGMY stock price process in order to switch from the market measure $\mathbb{P}$, under which the distribution parameters were estimated, to the risk-neutral measure $\mathbb{Q}$ on which we want to base the pricing of the Basket Default Swap.
Let $(X_t)_{t \in [0,T]}$ be the CGMY process with parameters $(C, G, M, Y, 0)$ and consequently $(S_t)_{t \in [0,T]} = (S_0 e^{\mu t - t\psi_0(-i; C,G,M,Y) + X_t})_{t \in [0,T]}$ denote the CGMY stock price process with respect to the market measure $\mathbb{P}$. When it is verified that $-G < \theta < M$ then $\int_{|x| \geq 1} e^{\theta x}(dx) < \infty$ and thus the process $(X_t)_{t \in [0,T]}$ is a Lévy process with Lévy measure $\tilde{\nu}(dx)$ under the measure $\mathbb{Q}^\theta$ which is equal to

$$\tilde{\nu}(dx) = C\left(\frac{e^{-(M-\theta)x}}{x^{Y+1}}\mathbb{I}_{\{x>0\}} + \frac{e^{-(G+\theta)|x|}}{|x|^{Y+1}}\mathbb{I}_{\{x<0\}}\right)(dx).$$

On the basis of the definition of a CGMY process with parameters (C, G, M, Y, m) and Lévy triplet (σ^2, ν, γ), the drift $\tilde{\gamma}$ under the equivalent measure $\mathbb{Q}^\theta$ is given by

$$\tilde{\gamma} = \gamma + \int_{-1}^{+1} x(e^{\theta x} - 1)\nu(dx)$$

which it is equal to

$$\tilde{\gamma} = \begin{cases} \int_{-1}^{+1} x\tilde{\nu}(dx), & \text{if } Y < 1 \\ \int_{|x|>1} x\tilde{\nu}(dx) + \int_{-\infty}^{+\infty} x(\tilde{\nu} - \nu)(dx) \\ -CY\Gamma(-Y)(M^{Y-1} - G^{Y-1}), & \text{if } Y \in (1,2) \end{cases}$$

As it could be shown that

$$\int_{-\infty}^{+\infty} x(\tilde{\nu} - \nu)(dx) = -CY\Gamma(-Y)((M-\theta)^{Y-1} - (G+\theta)^{Y-1} - M^{Y-1} + G^{Y-1}),$$

simply by substitution, the drift $\tilde{\gamma}$ under the equivalent measure $\mathbb{Q}^\theta$ becomes

$$\tilde{\gamma} = \begin{cases} \int_{-1}^{+1} x\tilde{\nu}(dx), & \text{if } Y < 1 \\ -\int_{|x|>1} x\tilde{\nu}(dx) - CY\Gamma(-Y)((M-\theta)^{Y-1} - (G+\theta)^{Y-1}) & \text{if } Y \in (1,2) \end{cases}$$

The meaning of this is that now $(X_t)_{t \in [0,T]}$ represents a CGMY process characterized by parameters $(C, G+\theta, M-\theta, Y, 0)$ under the measure $\mathbb{Q}^\theta$. Hence, what has to be done consists in finding the unique value of θ, θ^*, such that $-G < \theta^* < M$ and

$$r - \psi_0(-i; C, G + \theta^*, M - \theta^*, Y) = \mu - \psi_0(-i; C, G, M, Y). \tag{4.16}$$

In such a way, it is possible to get

$$E_{\mathbb{Q}^{\theta^*}}[e^{-rt}S_t|\mathfrak{S}_u] = e^{-ru}S_u$$

for every $0 \le u < t < T$: thus the *Esscher martingale measure* $\mathbb{Q}^{\theta^*} \in EMM^{CGMY}(\mathbb{P})$.

Change of Measure for GTS Processes

Analogously to what occurred for CGMY processes, the same mismatching problem between the two measures involved in the estimation process exists for GTS processes, as well. In fact, the parameters of the distributions have been estimated from the market, and thus it is necessary a change of measure from the market measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$ in order to correctly proceed with the pricing of the Basket Default Swap.
On the basis of Cont and Tankov (2004), the general result on the equivalence of measures for Lévy processes, reported by *theorem 3* keeps on holding also for GTS processes, with the consequence that the *corollary 2* is valid for this family processes, as well, becoming:

Corollary 3 *Let $(X_t, \mathbb{P})$ and $(X_t, \mathbb{Q})$ be GTS processes on $\mathbb{R}$ with parameters $(C_+, C_-, G, M, Y_+, Y_-, m)$ and $(\tilde{C}_+, \tilde{C}_-, \tilde{G}, \tilde{M}, \tilde{Y}_+, \tilde{Y}_-, \tilde{m})$. Then $\mathbb{P}|_{\mathfrak{S}_t}$ and $\mathbb{Q}|_{\mathfrak{S}_t}$ are equivalent for all $t > 0$ if and only if $C_+ = \tilde{C}_+$, $C_- = \tilde{C}_-$ $Y_+ = \tilde{Y}_+$, $Y_- = \tilde{Y}_-$ and $m = \tilde{m}$.*

Under the same conditions holding for CGMY processes, the measure $\mathbb{P}$ still identifies the *market measure*, while the measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, can be again defined as an Equivalent Martingale Measure of $\mathbb{P}$ if the discounted price process $\tilde{S}$ results to be a $\mathbb{Q}$-martingale with respect to the filtration $(\mathfrak{S}_t)_{t \in [0,T]}$. Furthermore, as a natural step forward, the following definition may be given:

Definition 8 *For $C_+ > 0$, $C_- > 0$, $G > 0$, $M > 0$, Y_+ and $Y_- \in (1,2)$ and $\mu > 0$, let $(X_t)_{t \in [0,T]}$ be a GTS process with parameters $(C_+, C_-, G, M, Y_+, Y_-, 0)$. Then the process given by $(S_t)_{t \in [0,T]} = (S_0 e^{\mu t - t\psi_0(-i; C_+, C_-, G, M, Y_+, Y_-) + X_t})_{t \in [0,T]}$ is called the GTS stock price process with parameters $(C_+, C_-, G, M, Y_+, Y_-, \mu)$.*

As it was done for CGMY processes, it is possible to apply the corollary 3, in order to get the following lemma, by which an EMM $\mathbb{Q}$ of the market measure $\mathbb{P}$ can be obtained.

Lemma 2 *Let $(S_t)_{t\in[0,T]}$ be the GTS stock price process respectively with parameters $(C_+, C_-, G, M, Y_+, Y_-, \mu)$ under the market measure $\mathbb{P}$ and with parameters $(\tilde{C}_+, \tilde{C}_-, \tilde{G}, \tilde{M}, \tilde{Y}_+, \tilde{Y}_-, r)$ under the measure $\mathbb{Q}$. Then $\mathbb{Q}$ is an EMM of $\mathbb{P}$ if and only if $\tilde{C}_+ = C_+$, $\tilde{C}_- = C_-$, $\tilde{Y}_+ = Y_+$, $\tilde{Y}_- = Y_-$ and*

$$r = \mu - \psi_0(-i; C_+, C_-, G, M, Y_+, Y_-) + \psi_0(-i; C_+, C_-, \tilde{G}, \tilde{M}, Y_+, Y_-). \quad (4.17)$$

Once again, now it is possible to define a sub-family of EMMs of $\mathbb{P}$ in the following way:

$EMM^{GTS}(\mathbb{P}) \equiv$
$\equiv \{\mathbb{Q}$ is an EMM of $\mathbb{P}|(S_t)_{t\in[0,T]}$ is a GTS stock price process under $\mathbb{Q}\}.$

If $(S_t)_{t\in[0,T]}$ represents a GTS stock price process with parameters $(C_+, C_-, G, M, Y_+, Y_-, \mu)$ under the measure $\mathbb{P}$, then the following equation can be obtained from lemma 2:

$EMM^{GTS}(\mathbb{P})=$
$= \{\mathbb{Q}|\mathbb{Q}$ is a measure induced by a $(C_+, C_-, \tilde{G}, \tilde{M}, Y_+, Y_-, \mu - \psi_0(-i; C_+, C_-, G, M, Y_+, Y_-))$
GTS process where $(\tilde{G}, \tilde{M})$ respects equation (4.17)$\}.$

Analogously to what said for CGMY, the set $EMM^{GTS}(\mathbb{P})$ still contains more than two elements: thus the *Esscher transform* was the mean through which only one EMM of $\mathbb{P}$ among the set $EMM^{GTS}(\mathbb{P})$ was selected.

Number of Defaults Distribution

As soon as the change of measure has been performed, there is no more mismatching between the measure under which the marginal default probabilities were estimated and the measure under which the distribution parameters were calibrated. Hence, the last issue to solve is given by the computation of the number of defaults probabilities, i.e. those probabilities that have to be plugged into the equation (2.4) in order to get the fair price of the Basket Default Swap. Coming back to the one-factor copula model, represented in equation (3.15) and to the conditional default probability computed as shown by equation (3.18) shows, it is necessary to find a method allowing us to obtain the number of defaults distribution.
Denoting with

$$N(t) = \sum_{i=1}^{N} \mathbb{I}_{\{\tau_i \leq t\}} \quad (4.18)$$

the counting process of the number of defaults occurred in the reference portfolio until time t, fixing the time t, the characteristic function of the random variable $N(t)$ may be simply computed as

$$\varphi_{N(t)}(u) = E[e^{iuN(t)}].$$

Looking at equation (3.15), let f represent the density function of the random variable M. Thus, considering the independence between the idiosyncratic factors Z_i, with $i = 1, \ldots, N$ and the definition of the counting process $N(t)$ given by equation (4.18), it is possible to write what follows:

$$
\begin{aligned}
E[e^{iuN(t)}] &= E[e^{iu\sum_{j=1}^{N}\mathbb{I}_{\{\tau_j \leq t\}}}] \\
&= \int_{\mathbb{R}} E[e^{iu\sum_{j=1}^{N}\mathbb{I}_{\{\tau_j \leq t | M=m\}}}] f(m)dm \\
&= \int_{\mathbb{R}} E[e^{iu\mathbb{I}_{\{\tau_1 \leq t | M=m\}}}] \cdots E[e^{iu\mathbb{I}_{\{\tau_N \leq t | M=m\}}}] f(m)dm \\
&= \int_{\mathbb{R}} \prod_{j=1}^{N} \left(S_{j|M=m}(t) + e^{iu}Q_{j|M=m}(t)\right) f(m)dm
\end{aligned}
$$

where the conditional survival probability $S_{j|M=m}(t)$ and the conditional default probability $Q_{j|M=m}(t)$ may be numerically obtained by means of the *Fast Fourier Transform* (FFT) algorithm. So, denoting with τ^k the k-th default within the reference portfolio, the survival distribution $S^k(t)$ of the k-th default time is given by

$$S^k(t) = Prob(\tau^k > t) = Prob(N(t) < k) = \sum_{i=0}^{k-1} Prob(N(t) = i)$$

while the default probability $Q^k(t)$ is simply

$$Q^k(t) = 1 - S^k(t).$$

The probabilities $Prob(N(t) = i)$ have been computed in a simple way by inverting the characteristic function, since only previously estimated data, $Q_i(t)$ and a_i, were involved. Thus, $P(N(t) = k)$ has been computed as the Inverse Fourier Transform for a discrete distribution:

$$P(N(t) = k) = \frac{1}{N+1} \sum_{n=0}^{N} \varphi_{N(t)}\left(\frac{2\pi n}{N+1}\right) e^{\frac{-2\pi ink}{N+1}}$$

Here, all the estimation steps have been now illustrated and thus the impact of different pricing assumptions on the quotes of the empirical Basket Default Swap analyzed are ready to be presented[5]. We will discuss the effects of correlations and default intensities. Furthermore, it will be analysed how the different

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[5]All the results are showed in Appendix I.

dependence structures among defaults play a fundamental role in the determination of the fair BDS spreads. Finally, the results obtained considering the usual standard market hypothesis will be compared with those obtained by the introduction of the different dependence structures introduced so far.

4.5.1   Effects of Correlation

As it is possible to see from table 4.3, table 4.4 and table 4.5, the first to default premia show a monotonically decreasing tendency, independently from the the copula function chosen to govern the dependence structure among the defaults of the obligors. This dynamics may be justified by considering a no-arbitrage concept: if the pairwise correlations were all null, and the term structure of the CDS premia was flat, then it would be possible to create a perfect hedging strategy for a short postion on a first to default BDS simply by taking a long position on each of the CDS composing the basket. This holds since, as soon as a default happens, the payment required would find a compensation from the positive cash flow offered by the corresponding CDS. Furthermore, the absence of correlation will not affect the credit spreads of the other reference entities and thus, under these conditions, the first to default BDS premium has to be equal to the sum of the the underlying CDS premia.

On the other hand, when the correlation gets higher, the probability of joint defaults becomes larger as well, and this should be viewed as a lower degree of protection offered by the first to default contract which is reflected into a lower premia required.

However, correlations affect BDS prices in a less predictable way for the other orders of seniority[6]. Empirical evidence shows that, for a small number of reference entities (e.g. $N = 2$), second to default premia generally increase with correlation but, as soon as the number of obligors gets higher (e.g. $N = 10$), this tendency cannot be confirmed, as shown by table 4.3, table 4.4 and table 4.5. This opposite dynamics finds a justification into two separate effects which should be taken into account every time a correlation sensitivity analysis is carried out:

1. variation in the correlation among defaulters needed to trigger the default payment;

2. variation in the correlation between the remaining credits.

The first effect plays a role increasing the probability of joint defaults and thus the premium requested, while the second effect affects second to default swaps in the same way as for first to default swaps. Thus, for small portfolios, the first effect dominates the second one, while the opposite holds for larger portfolios.

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[6]See the complete results in Appendix I.

Gaussian Copula	1st BDS	2nd BDS
ρ=0.3	0.041196	0.015829
ρ=0.5	0.031097	0.014632
ρ=0.7	0.021782	0.012331

Table 4.3: 1*st and* 2*nd BDS spreads under the Gaussian copula framework, obtained for different values of the pairwise correlation ρ and default intensity $\lambda = 0.01$.*

CGMY Copula	1st BDS	2nd BDS
ρ=0.3	0.041342	0.014862
ρ=0.5	0.030756	0.0133
ρ=0.7	0.020972	0.010925

Table 4.4: 1*st and* 2*nd BDS spreads under the CGMY copula framework, obtained for different values of the pairwise correlation ρ and default intensity $\lambda = 0.01$.*

GTS Copula	1st BDS	2nd BDS
ρ=0.3	0.040428	0.015624
ρ=0.5	0.030539	0.014324
ρ=0.7	0.021503	0.012088

Table 4.5: 1*st and* 2*nd BDS spreads under the GTS copula framework, obtained for different values of the pairwise correlation ρ and default intensity $\lambda = 0.01$.*

Gaussian Copula	1st BDS	2nd BDS
λ=0.01	0.041196	0.015829
λ=0.02	0.076681	0.03498
λ=0.03	0.11082	0.054407

Table 4.6: 1*st and* 2*nd BDS spreads under the Gaussian copula framework, obtained for different values of the default intensity* λ *and constant pairwise correlation* $\rho = 0.3$.

CGMY Copula	1st BDS	2nd BDS
λ=0.01	0.041342	0.014862
λ=0.02	0.076968	0.033614
λ=0.03	0.11087	0.052692

Table 4.7: 1*st and* 2*nd BDS spreads under the CGMY copula framework, obtained for different values of the default intensity* λ *and constant pairwise correlation* $\rho = 0.3$.

4.5.2 Effect of Default Intensities

This section analyzes the effects of different default intensities determining the marginal default probabilities of the single names on the BDS prices.

It is clear, as table 4.6, table 4.7 and table 4.8 show, that the default intensity is positively correlated to the BDS spreads: the higher the default intensity is, the larger the premium gets. This may be explained by the relation for which,under the hypothesis of constant spreads for each maturity date, the link between the spread s and the default intensity λ could be written as

$$s = \lambda(1 - R)$$

where R is the (constant) recovery rate assumed.

Moreover, the most interesting issue in the analysis of default intensities effects stands in discovering that, under the same conditions, the impact of an increase of the default intensity becomes much larger as soon as the seniority of the BDS gets higher.

Table 4.9, table 4.10 and table 4.11 confirm this statement: the same variation of the default intensity has clearly a more evident effect on a second to

GTS Copula	1st BDS	2nd BDS
λ=0.01	0.040428	0.015624
λ=0.02	0.074747	0.034581
λ=0.03	0.10737	0.053689

Table 4.8: 1*st and* 2*nd BDS spreads under the GTS copula framework, obtained for different values of the default intensity* λ *and constant pairwise correlation* $\rho = 0.3$.

Seniority	$\lambda = 0.01$	$\lambda = 0.02$	Δ	$\lambda = 0.03$	Δ
1	411.96	766.81	86%	1108.2	169%
2	158.29	349.8	120%	544.07	244%
3	72.56	189.5	161%	319.03	340%
4	34.49	106.12	208%	194.24	463%
5	16.15	58.46	262%	116.83	623%

Table 4.9: *First five BDS spreads (in basis points) under the Gaussian copula framework, obtained for different values of the default intensity λ and constant pairwise correlation $\rho = 0.3$, and the relative spread increments with respect to the first case in which $\lambda = 0.01$*

Seniority	$\lambda = 0.01$	$\lambda = 0.02$	Δ	$\lambda = 0.03$	Δ
1	413.42	769.68	86%	1108.7	168%
2	148.62	336.14	126%	526.42	254%
3	67.56	179.36	165%	305.52	352%
4	33.85	101.31	199%	186.03	449%
5	17.81	57.93	225%	113.45	537%

Table 4.10: *First five BDS spreads (in basis points) under the CGMY copula framework, obtained for different values of the default intensity λ and constant pairwise correlation $\rho = 0.3$, and the relative spread increments with respect to the first case in which $\lambda = 0.01$*

Seniority	$\lambda = 0.01$	$\lambda = 0.02$	Δ	$\lambda = 0.03$	Δ
1	404.28	747.47	85%	1073.7	165%
2	156.24	345.81	121%	536.89	243%
3	72.79	190.01	161%	319.9	339%
4	35.57	107.82	203%	197.13	454%
5	17.41	60.24	246%	119.6	587%

Table 4.11: *First five BDS spreads (in basis points) under the GTS copula framework, obtained for different values of the default intensity λ and constant pairwise correlation $\rho = 0.3$, and the relative spread increments with respect to the first case in which $\lambda = 0.01$*

Seniority	Normal	CGMY	GTS
1	0.041196	0.041342	0.040428
2	0.015829	0.014862	0.015624
3	0.0072563	0.006756	0.0072794

Table 4.12: *First, second and third to default basket spreads respectively computed under the one-factor copula model, one-factor CGMY copula model and one-factor GTS copula model. The portfolio is supposed to be homogeneous, characterized by a constant pairwise correllation $\rho = 0.3$ and constant default intensity $\lambda = 0.01$.*

Seniority	Normal	CGMY	GTS
1	0.031097	0.030756	0.030539
2	0.014632	0.0133	0.014324
3	0.0083263	0.0074906	0.0082257

Table 4.13: *First, second and third to default basket spreads respectively computed under the one-factor copula model, one-factor CGMY copula model and one-factor GTS copula model. The portfolio is supposed to be homogeneous, characterized by a constant pairwise correllation $\rho = 0.5$ and constant default intensity $\lambda = 0.01$.*

default BDS than on a first to default BDS. And the same may be said for BDSs with a greater level of seniority.

4.5.3   Effects of Distributional Assumptions

In this section, we analyse the effects of different distributional assumptions on Basket Default spreads are presented.

In particular, table (4.12), table (4.13) and table (4.14) contain the fair prices obtained for a first, second and third to default BDS, under three different default dependence frameworks: the Normal, the CGMY and the GTS copula. Furthermore, this analysis considers three alternative scenarios concerning the value assumed by the constant pairwise correlation, respectively set equal to

Seniority	Normal	CGMY	GTS
1	0.021782	0.020972	0.021503
2	0.012331	0.010925	0.012088
3	0.0082255	0.0072184	0.0081175

Table 4.14: *First, second and third to default basket spreads respectively computed under the one-factor copula model, one-factor CGMY copula model and one-factor GTS copula model. The portfolio is supposed to be homogeneous, characterized by a constant pairwise correllation $\rho = 0.7$ and constant default intensity $\lambda = 0.01$.*

0.3, 0.5 and 0.7. Finally, the default intensity λ is kept constant, to 0.01 for every reference asset.
By looking at table (4.12), table (4.13) and table (4.14), it is possible to note the impact of these different distributional assumptions on the BDS spreads: the result is that the prices obtained under the Gaussian framework clearly tend to dominate those resulting from the other distributional hypothesis.

The reason for such an evidence, as figure (4.5) shows, has to be certainly attributed to the tail peculiarities of the relative distribution considered. Once more, distributions as CGMY and GTS are characterized by fatter tails than those which can be observed in a standard Normal distribution. Thus, this implies that within the CGMY and GTS model, the probabilities of joint defaults are clearly higher than in the normal setting, meaning that the level of protection offered by the CGMY and GTS models is surely lower and hence this fact is reflected into lower prices requested for the same correlated-based product.
Note that the domain of the Normal framework becomes more and more clear when the value of the pairwise correlation gets higher: in a few words, by increasing the correlation among the obligors, the risk of joint defaults increases as well, and thus the peculiarity of fat tails of the distribution plays a more and more important role in the price determination.

4.5.4   Computational Results with Market Data

This section offers a comparison among the results obtained with the usual hypothesis of the *standard market model* described by [33], and those coming from a better estimation from the market of the parameters involved in the BDS pricing. In particular, the main differences between the two approaches include a costant pairwise correlation for every name against the use of market correlations, constant default intensities for all the reference assets instead of time-varying default intensities estimated from the relative CDS market quote. The same comparison has been done for each of the copula models considered: Normal copula (standard hypothesis), CGMY and GTS copula.

As table (4.15), table (4.16) and table (4.17) show, all the copula frameworks implemented with a constant pairwise correlation ρ equal to 0.3 and with constant default intensity λ equal to 0.01 and 0.02 tend to be totally mismatched with respect to the results obtained under the inhomogeneous portfolio assumption. When the homogeneous portfolio framework with correlation $\rho = 0.3$ and default intensity $\lambda = 0.03$ is considered, acceptable results are found in terms of comparison among the spreads obtained under the two different hypothesis. Thus, from a first rough analysis, this could mean that the condition given by a pairwise correlation $\rho = 0.3$ and a default intensity $\lambda = 0.3$, constant for each of the reference asset in the empirical portfolio, may be considered satisfactory for its ability to replicate quite well the results obtained with a more careful and precise estimation of every model parameter (*inhomogeneous portfolio* approach).
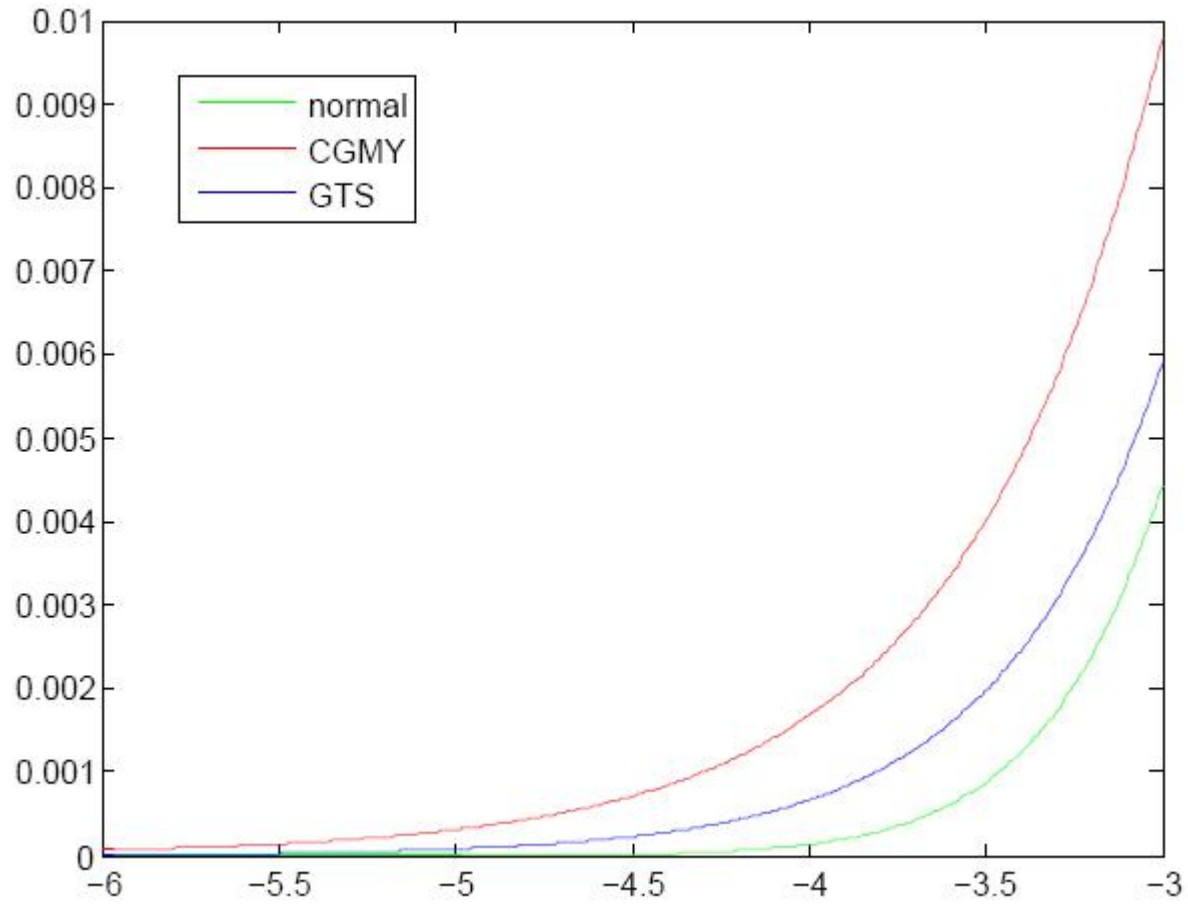However, this statement holds only for the first and second to default BDS

Figure 4.5: Focus on left tails of Standard Normal, CGMY and GTS distributions

Seniority	Market Data	$\lambda = 0.01$	$\lambda = 0.02$	$\lambda = 0.03$
1	0.10614	0.041196	0.076681	0.11082
2	0.047447	0.015829	0.03498	0.054407
3	0.015679	0.0072563	0.01895	0.031903
4	0.006076	0.0034487	0.010612	0.019424
5	0.0028229	0.0016147	0.0058465	0.011683
6	0.0013548	0.0007186	0.0030614	0.0067076
7	0.00060531	0.0002919	0.0014643	0.0035337
8	0.00022808	0.000101	0.0005974	0.0015948
9	0.0000594	0.0000248	0.0001725	0.0005109
10	0	0	0	0

Table 4.15: *Comparison among the spreads computed under the hypothesis of inhomogeneous portfolio (Market Data) and those obtained assuming homogeneous portfolios with constant pairwise correlation $\rho = 0.3$ for each name in the basket and default intensity λ respectively equal to 0.01, 0.02 and 0.03. The model used is the one-factor Normal copula model.*

Seniority	Market Data	$\lambda = 0.01$	$\lambda = 0.02$	$\lambda = 0.03$
1	0.1071	0.041342	0.076968	0.11087
2	0.04745	0.014862	0.033614	0.052692
3	0.014977	0.006756	0.017936	0.030552
4	0.0055573	0.0033856	0.010131	0.018603
5	0.00267	0.0017806	0.0057935	0.011345
6	0.0014224	0.0009492	0.0032617	0.0067316
7	0.0007446	0.0004933	0.0017503	0.0037619
8	0.0003523	0.0002343	0.0008432	0.0018656
9	0.0001281	0.0000849	0.0003056	0.0006882
10	0	0	0	0

Table 4.16: *Comparison among the spreads computed under the hypothesis of inhomogeneous portfolio (Market Data) and those obtained assuming homogeneous portfolios with constant pairwise correlation $\rho = 0.3$ for each name in the basket and default intensity λ respectively equal to 0.01, 0.02 and 0.03. The model used is the one-factor CGMY copula model.*

Seniority	Market Data	$\lambda = 0.01$	$\lambda = 0.02$	$\lambda = 0.03$
1	0.10708	0.040428	0.074747	0.10737
2	0.047469	0.015624	0.034581	0.053689
3	0.014949	0.0072794	0.019001	0.03199
4	0.0055382	0.0035576	0.010782	0.019713
5	0.0026658	0.0017406	0.0060241	0.01196
6	0.0014254	0.0008271	0.0032113	0.0069092
7	0.0007487	0.0003687	0.0015759	0.0036594
8	0.0003551	0.0001449	0.0006683	0.0016636
9	0.0001296	0.000042	0.0002046	0.0005403
10	0	0	0	0

Table 4.17: *Comparison among the spreads computed under the hypothesis of inhomogeneous portfolio (Market Data) and those obtained assuming homogeneous portfolios with constant pairwise correlation $\rho = 0.3$ for each name in the basket and default intensity λ respectively equal to 0.01, 0.02 and 0.03. The model used is the one-factor GTS copula model.*

spreads since, as soon as the spreads of the BDS with other seniorities are analyzed, it can be discovered how the homogeneous portfolio approach clearly tends to overestimate the requested premium: this happens under each of the copula models considered. In particular, the discrepancies among the spreads gets more and more larger as the order of the seniority increases. The reason of such a result has to be attributed to the misleading assumptions implicitely considered in the Li standard market model: constant pairwise correlations and constant default intensities for each name of the basket cannot be considered as a good proxy of the real conditions characterizing the market. Moreover, the overestimation of the fair BDS spreads becomes more evident under the Gaussian copula framework: once more another standard hypothesis of the Li model leads to results far to be considered as acceptable if compared with the other approaches.

Hence, this work suggests the interesting concept for which dealing with the assumptions of the standard market model may be dangerous from the point of view of the correct estimation of the multiple risk of default. In fact, under this framework, the financial product sold by the protection seller seems to protect the protection buyer more than what it effectively does: but this is not true. In fact, its ability to protect from the risk of joint defaults essentially derives from the wrong underestimation of the risk itself: the reality is that the market standard model does not estimate the real risk and thus seems to offer an higher protection. The result is that, for the same reason, the price requested to the protection buyer is greater than that it should be paid, since the model seems to offer an high degree of protection.

The natural step of such an uncorrectly process is clearly the underestimation of the risk, both for the protection seller and the protection buyer, which may lead to terribly dangerous results as pointed out by the recent *subprime crises*

in which the financial markets has been involved, with huge losses for most of the investment banking divisions of the largest banks of the world.

The obvious questions are the following: when will the effects of this crises be transferred from the financial world to the real economy? And how long will they last? The correct answer will be evident very soon.

4.6 Conclusions

The thesis starts with an introduction of credit risk and the different approaches proposed in the literature to model it. The main differences between structural models and reduced-form models were discussed. Even though the formers may seem more attractive from a pure intuitive point of view, they are generally characterized by poorer performances in terms of fitting real credit spreads. I personally share the opinion for which intensity models probabably represent the best choice, thanks to their direct modelling of the default event.

The second chapter contains a detailed description of two credit derivatives, CDSs and CDOs, that in these last years became so popular, especially with the recent subprime crises and actually are the object of the efforts of many researchers.

The third chapter introduces copula functions, as a fundamental and innovative concept to model the dependence between relevant credit events such as defaults. For pricing correlated-based products, the estimation of the marginal default probabilities represents the first step to take, but certainly it is not the hardest one. In fact, modelling the dependence among defaults of a set of obligors may be not so trivial: here copulas play a fundamental role, thanks to their peculiar ability to split the marginal probabilities from the dependence structure among them. This is the main reason for which copula models have become so popular in credit risk modelling: their use has to be considered a valid and effective alternative to the time consuming Monte Carlo simulations, but not only: copula models allows for computing default probability with closed formulae.

Chapter four discusses the famous Li one-factor copula model and in particular of the standard market model, with its simplicity but also with its too restrictive and sometimes unrealistic assumptions which it relies on.

The thesis proposes an improvement of the market standard model, based on the consideration of more realistic assumptions which try to consider the characteristcs of the market (instead of neglecting them), by taking standard and arbitrary hyphotesis as it happens with the standard market model. The aim of this work is to evaluate the impacts of some more realistic hyphotesis, on the prices of an empirical BDS. Thus, the improvements proposed relies on the consideration of a non-homogeneous portfolio, the modelling of the marginal default probabilities on the basis of a Poisson process with time-varying instead of constant default intensities, and by trying to replace the common assumption of a Gaussian dependence among the defaults of the set of obligors considered with other more flexible distributions, such as CGMY and GTS ones. In such a

way, the default event results to be better modelled because of the fatter tails characterizing these distributions if compared with the Normal case.

Finally a comment of the obtained results is contained in the last section, where we show the interesting effects of these different assumptions on the prices of an empirical BDS. This last part allows to prove that the prices are strongly affected by the hypothesis underlying the model, highlighting the importance of a suitable choice for the description of default event, both from the side of the marginal probabilities and also from the point of view of the dependence structure.

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