

# On the impact of association measures in portfolio theory

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## Abstract

This paper discusses the use of different association measures in portfolio problems. From among the association measures we highlight those that are consistent with the choices of risk-averse investors and we characterize semidefinite positive association measures. Additionally, we propose new portfolio selection problems that optimize the association between the portfolio and one or two market benchmarks. Finally, we discuss when, and how, we can use association measures to reduce the dimensionality of portfolio problems. An empirical analysis shows the impact of different association measures in portfolio selection problems and in portfolio reduction problems. It is documented that although the proper usage of both a risk measure and an association measure can increase the performance of the portfolio, the impact of the latter is higher.

*Keywords:* Concordance measure, reward measure, semidefinite positive association measure, dimensionality reduction, large scale portfolio selection.

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## 1. Introduction

The dependency structure among sources of randomness plays a crucial role in the portfolio theory and in several pricing and risk management problems. In particular, the classic Pearson linear correlation measure is regularly used to measure and optimize the dispersion of portfolio return and to reduce the dimensionality of large scale portfolio problems. However, it is not clear why this measure of linear correlation is still so much popular, despite its drawbacks.

For example, it is well known that Pearson linear correlation works well only with elliptically distributed vectors (so that they admit finite variance-covariance matrix). Unfortunately, the behavior of financial returns is more complex and at least the Gaussian distributional assumption has to be usually rejected, see e.g. Mandelbrot (1963a,b) and Fama (1965), or Rachev and Mittnik (2000) and Bulla and Bulla (2006) and the references therein. Moreover, the empirical evidence (see, among others, Rachev et al. (2008) and Biglova et al. (2009)) suggests that the dependence model has to account for dependence of the tail events (“huge losses go together”). Many other measures have been proposed in literature to deal and summarize the dependence among random variables (see, among others, Scarsini (1984), Cherubini et al. (2004), Nelsen (2006) and the references therein). However, most of these measures cannot be used directly to order investors’ choices, since they are not consistent with investors’ preferences.

In searching for an acceptable model to describe the dependence structure of financial returns, we first identify the most desirable and useful characteristics of the Pearson linear correlation. In particular, we characterize the class of semidefinite positive association measures and we distinguish the measures consistent with preferences of risk-averse investors. Moreover, we show that other linear correlation measures can be used in portfolio selection problems as an alternative to the Pearson linear correlation.

By a practical point of view, we propose to use the association measures for two distinct portfolio problems: 1) to identify portfolio strategies that optimize the association between the portfolio and one or two market benchmarks; 2) to reduce the dimensionality of large scale portfolio selection problems. For both problems we perform an empirical analysis on the US stock market.

With respect to the first problem we propose new portfolio optimization models that account two logical investors' behavior: a) investors want to maximize the concordance and/or the association with the upper stochastic bound of the market; b) investors want to minimize the concordance and/or the association with the lower stochastic bound of the market. Therefore, we compare ex-post sample paths of wealth obtained using portfolio optimization strategies based on different association measures.

With respect to the second problem, we suggest to use different linear correlation measures to perform a principal components analysis (PCA) that identifies the main portfolio factors whose dispersion is significantly different from zero. These factors are then used to approximate the portfolio returns in large scale portfolio selection problems. Therefore, using more than 1300 assets of the US stock market, we compare the results obtained by the principal components analysis applied to different linear correlation matrixes. Then the performance of some large scale portfolio selection strategies applied to the approximated returns obtained by the portfolio dimensionality reduction is compared ex-post.

We proceed as follows. Section 2 summarizes some of the basic characteristics of concordance/association measures and characterizes the semidefinite positive association measures. In Section 3 we discuss when, and how, we should use association measures for portfolio problems. Section 4 proposes an empirical comparison among portfolio strategies based on the use of different association measures. We summarize our principal findings in Section 5.

## 2. Concordance and semidefinite positive association measures

One of the most essential tasks of financial decision-making is the measurement of the dependency among the realizations of particular random variables. Specifically, let us consider  $n$  risky assets with gross returns<sup>1</sup>  $z = [z_1, z_2, \dots, z_n]'$ . As a consequence of the Sklar theorem (Sklar, 1959) the joint distribution function is given by:

$$F_{\mathbf{z}}(\mathbf{x}) = \mathcal{C}(F_{z_1}(x_1), F_{z_2}(x_2), \dots, F_{z_n}(x_n)), \quad (1)$$

where  $F_{z_i}(x_i) = \Pr(z_i \leq x_i)$  are the marginal distribution functions and  $\mathcal{C}: [0, 1]^n \rightarrow [0, 1]$  is the copula function. The copula function can therefore be defined by inverting (1):

$$\mathcal{C}(\mathbf{u}) = F_{\mathbf{z}}(F_{z_1}^{-1}(u_1), F_{z_2}^{-1}(u_2), \dots, F_{z_n}^{-1}(u_n)). \quad (2)$$

Therefore, the dependency among particular variables is fully described by suitable copula function  $\mathcal{C}$ . Furthermore, the copula function can be regarded as the joint distribution function of the marginal distribution functions.

In several financial contexts it is convenient to express the dependency among random variables by a single number (more generally, for  $n$  random variables we get an  $n$ -dimensional matrix). The most widespread is *the Pearson coefficient of correlation* defined as follows:

$$\text{cor}(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}}, \quad (3)$$

where  $\text{var}(X)$  states the variance of  $X$  and  $\text{cov}(X, Y)$  the covariance of  $X$  and  $Y$ . This measure is the inner product of standardized random variables in the Hilbert  $L^2 = \{X | E(|X|^2) < \infty\}$  space and it derives most of its properties from

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<sup>1</sup> Generally, we assume the standard definition of gross return between time  $t$  and time  $t+1$  of asset  $i$ , as  $z_{i,t+1} = \frac{S_{i,t+1} + d_{i,[t,t+1]}}{S_{i,t}}$ , where  $S_{i,t}$  is the price of the  $i$ -th asset at time  $t$  and  $d_{i,[t,t+1]}$  is the total amount of cash dividends paid by the asset between  $t$  and  $t+1$ .

this characteristic. However, the Pearson *coefficient of correlation* is only one among possible measures of dependency between two random variables.

Generally, a concordance measure is used to measure the concordance/dependence/association between random variables. In the following example (Nelsen, 2006), two random variables  $(X, Y)$  with independent replications,  $(X_1, Y_1)$  and  $(X_2, Y_2)$ , are concordant if  $X_1 < X_2$  ( $X_1 > X_2$ ) implies  $Y_1 < Y_2$  ( $Y_1 > Y_2$ ). Similarly, the two variables are discordant if  $X_1 < X_2$  ( $X_1 > X_2$ ) implies  $Y_1 > Y_2$  ( $Y_1 < Y_2$ ). The concordance measures are easily definable by copula functions, since they rely only on the "joint" features, having no relation to the marginal characteristics. Formally a concordance measure  $\rho$  is any functional that satisfies the following seven properties:

- 1)  $\rho : H \times H \rightarrow [-1, 1]$  where  $H$  is a given class of random variables;
- 2) for any random variable  $X \in H$  :  $\rho(X, X) = 1$ ;  $\rho(X, -X) = -1$ ;
- 3)  $\rho(X, Y) = \rho(Y, X)$ ;
- 4)  $\rho(-X, Y) = \rho(X, -Y) = -\rho(X, Y)$ ;
- 5) if  $X$  and  $Y$  are independent random variables, then  $\rho(X, Y) = 0$ ;
- 6) if we consider two bivariate random vectors  $\mathbf{X} = (X_1, X_2)$ ,  $\mathbf{Y} = (Y_1, Y_2)$ , with the same marginal distributions  $(F_1, F_2)$  such that  $F_{\mathbf{X}}(\mathbf{x}) = \Pr(X_1 \leq x_1, X_2 \leq x_2) \leq F_{\mathbf{Y}}(\mathbf{x})$  for any  $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$  (i.e.  $X$  dominates  $Y$  with respect to concordance ordering<sup>2</sup>) then  $\rho(X_1, X_2) \leq \rho(Y_1, Y_2)$  (or  $\rho_{\mathcal{C}_1} \leq \rho_{\mathcal{C}_2}$  where  $\mathcal{C}_1, \mathcal{C}_2$  are the copulas associated with  $\mathbf{X}, \mathbf{Y}$ );
- 7) given a sequence of continuous bivariate random vectors  $\{(X_n, Y_n)\}_{n \geq 1}$  with copulas  $\mathcal{C}_n$  that converge pointwise to the copula  $\mathcal{C}$ , then  $\rho_{\mathcal{C}_n}$  converge to  $\rho_{\mathcal{C}}$ .

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<sup>2</sup>Analogously, we say that  $X$  dominates  $Y$  with respect to concordance ordering if and only if the copulas  $\mathcal{C}_1, \mathcal{C}_2$  associated to  $\mathbf{X}, \mathbf{Y}$  are ordered i.e.  $\mathcal{C}_1 \leq \mathcal{C}_2$ . This definition is also equivalent to saying that  $\mathbf{cov}(h_1(X_1), h_2(X_2)) \leq \mathbf{cov}(h_1(Y_1), h_2(Y_2))$  for any increasing function  $h_1, h_2$  such that covariance exists).

Observe that  $\rho(X_1, X_2) = \rho(h_1(X_1), h_2(X_2))$  for any concordance measure  $\rho$ , for any couple of continuous random variables  $(X_1, X_2)$  and for any two strictly monotone functions  $h_1, h_2$ . The Pearson correlation coefficient is not a concordance measure, since it does not satisfy Property 7 of concordance measures. For further details on all properties of concordance measures and their proofs see Cherubini et al. (2004) and Nelsen (2006).

*Examples.* The most popular measures of concordance are: Kendall's tau, Spearman's rho, Gini's gamma, and Blomqvist's beta.

The *Kendall's tau*,  $\tau_K$  (also called Kendall correlation), is defined as the probability of concordance reduced by the probability of discordance:

$$\tau_K(X, Y) = \Pr((X_1 - X_2)(Y_1 - Y_2) > 0) - \Pr((X_1 - X_2)(Y_1 - Y_2) < 0), \quad (4)$$

where  $(X_1, Y_1)$  and  $(X_2, Y_2)$  are independent replications of  $(X, Y)$ . Therefore,

$$\begin{aligned} \tau_K(X, Y) &= \mathbb{E}(\text{sign}((X_1 - X_2)(Y_1 - Y_2))) \\ &= \text{cor}(\text{sign}(X_1 - X_2), \text{sign}(Y_1 - Y_2)), \end{aligned}$$

where  $\text{sign}(x) = 1$  if  $x > 0$ ,  $\text{sign}(x) = 0$  if  $x = 0$  and  $\text{sign}(x) = -1$  if  $x < 0$ .

Clearly, Kendall's tau can be defined in terms of the copula function:

$$\tau_K(\mathcal{C}) = 4 \int_0^1 \int_0^1 \mathcal{C}(u, v) d\mathcal{C}(u, v) - 1, \quad (5)$$

where  $\mathcal{C}$  is the copula associated to the bivariate vector  $(X, Y)$ .

The second most popular measure of concordance, *Spearman's rho*,  $\rho_S$ , is given by:

$$\begin{aligned} \rho_S &= 3 (\Pr((X_1 - X_2)(Y_1 - Y_3) > 0) - \Pr((X_1 - X_2)(Y_1 - Y_3) < 0)) = \\ &= 3\mathbb{E}(\text{sign}((X_1 - X_2)(Y_1 - Y_3))) = 3 \text{cor}(\text{sign}(X_1 - X_2), \text{sign}(Y_1 - Y_3)). \end{aligned} \quad (6)$$

where  $(X_1, Y_1)$ ,  $(X_2, Y_2)$  and  $(X_3, Y_3)$  are independent replications of  $(X, Y)$ .

This measure is very similar to the linear correlation coefficient, except for the fact that it measures the dependency among marginal distribution functions.

$$\rho_S = \text{cor}(F_X(X), F_Y(Y)) = \frac{\text{cov}(F_X(X), F_Y(Y))}{\sqrt{\text{var}(F_X(X)), \text{var}(F_Y(Y))}}. \quad (7)$$

It follows, that it can be regarded as the correlation of copula functions:

$$\begin{aligned} \rho_S(X, Y) &= 12 \int_0^1 \int_0^1 uv d\mathcal{C}(u, v) - 3 = \\ &= 12 \int_0^1 \int_0^1 \mathcal{C}(u, v) dudv - 3, \end{aligned} \quad (8)$$

where  $\mathcal{C}$  is the copula associated to the bivariate vector  $(X, Y)$ .

Another measure used to quantify the concordance among random variables is *Gini's gamma*,  $\gamma_G$ . It can be defined in terms of copula functions as follows:

$$\gamma_G(\mathcal{C}) = 4 \left[ \int_0^1 \mathcal{C}(u, 1-u) du - \int_0^1 [u - \mathcal{C}(u, u)] du \right], \quad (9)$$

where  $\mathcal{C}$  is the copula associated to the bivariate vector  $(X, Y)$ . Its sample estimation is given by ranks  $p_i$  and  $q_i$  of random variables  $X$  and  $Y$ , respectively:

$$\gamma_G(X, Y) = \frac{1}{[n^2/2]} \left[ \sum_{i=1}^n |p_i + q_i - n - 1| - \sum_{i=1}^n |p_i - q_i| \right]. \quad (10)$$

Finally, we should mention *Blomqvist beta*,  $\beta_B$ , defined as follows:

$$\begin{aligned} \beta_B(X, Y) &= \Pr[(X - \tilde{x})(Y - \tilde{y}) > 0] - \Pr[(X - \tilde{x})(Y - \tilde{y}) < 0] = \\ &= \mathbb{E}(\text{sign}((X - \tilde{x})(Y - \tilde{y}))), \end{aligned} \quad (11)$$

where  $\tilde{x}$  and  $\tilde{y}$  are the medians of some given continuous random variables  $X$  and  $Y$ , respectively. With certain simplifications, this measure may also be rewritten in terms of copula functions:

$$\beta_B(\mathcal{C}) = 4\mathcal{C}\left(\frac{1}{2}, \frac{1}{2}\right) - 1. \quad (12)$$

The proof that all these measures are really measures of concordance can be found, for example, in Nelsen (2006).

In order to consider a larger class of "dependence" measures, rather than concordance measures, we next introduce the class of association measures.

**Definition 1.** The association measure defined on a given class of random variables  $H$  is any functional  $\rho : H \times H \rightarrow [-1, 1]$  that is law invariant (i.e.  $\rho(X, Y)$  is uniquely determined by the joint distribution of  $(X, Y)$ ) and satisfies the first five properties of concordance measures as given above. We say that an association measure  $\rho$  is semidefinite positive if for any vector  $\mathbf{X} = (X_1, X_2, \dots, X_N)'$  with  $X_i \in H$  the association matrix  $Q = [\rho_{i,j}]$ , where  $\rho_{i,j} = \rho(X_i, X_j)$ , is semidefinite positive. We call  $\varphi$ -association measure any association measure that satisfies the following additional property:

6bis)  $|\rho(X, Y)| = 1$  if and only if  $Y = \varphi(X)$  almost surely (a.s.) for a given class of real monotone functions  $\varphi$ .

Clearly, the concordance measures and the Pearson correlation coefficient are association measures. In particular, the Pearson correlation coefficient satisfies the property  $|\rho(X, Y)| = 1$  if and only if  $Y = aX + b$  a.s. for certain real  $a$  and  $b$ . Similarly, for a pair of monotone real functions  $h_1, h_2$  we can define a  $\varphi$ -association measure  $\tilde{\rho}(X, Y) = \rho(h_1(X), h_2(Y))$  where  $\rho$  is the Pearson correlation coefficient. In this case  $|\tilde{\rho}(X, Y)| = 1$  if and only if  $Y = h_2^{-1}(ah_1(X) + b)$  a.s. for certain real  $a$  and  $b$ .

Any association measure can be used to value the dependence between random variables, but only some particular semidefinite positive association measures can be also used to reduce the dimensionality of statistical problems and to value the dispersion of portfolios. For any couple of random variables  $X, Y$  and for any association measure  $\rho(X, Y)$  the association matrix

$$Q = \begin{bmatrix} 1 & \rho(X, Y) \\ \rho(X, Y) & 1 \end{bmatrix}$$

is semidefinite positive, since

$$0 \leq (|x_1| - |x_2|)^2 \leq x'Q_{ij}x = x_1^2 + x_2^2 + 2x_1x_2\rho_{i,j} \leq (|x_1| + |x_2|)^2$$

for any  $x = [x_1, x_2]' \in \mathbb{R}^2$ . However, this property is not sufficient to guarantee that an association measure is semidefinite positive. Think, for example, of a



(3 × 3) matrix

$$Q = \begin{bmatrix} 1 & 0.71 & -0.01 \\ 0.71 & 1 & 0.71 \\ -0.01 & 0.71 & 1 \end{bmatrix},$$

where every principal submatrix is semidefinite positive but the determinant of the matrix  $|Q| = -0.018$  is negative. A sufficient condition for obtaining a semidefinite positive association measure is represented by the following lemma.

**Lemma 1.** *If the following two properties are satisfied, then  $\rho$  is a semidefinite positive association measure:*

1. *there exists an inner vectorial product  $\langle \cdot, \cdot \rangle: V \times V \rightarrow R$  and a function  $g : H \times H \rightarrow V \times V$  such that  $g(X, Y) = (v_X, v_Y)$ ;  $\langle v_X, v_Y \rangle = \langle v_Y, v_X \rangle$ ;  $\langle v_{-X}, v_Y \rangle = \langle v_X, v_{-Y} \rangle = -\langle v_X, v_Y \rangle$  and*

$$\rho(X, Y) = \frac{\langle g(X, Y) \rangle}{\sqrt{\langle g(X, X) \rangle \langle g(Y, Y) \rangle}} = \frac{\langle v_X, v_Y \rangle}{\sqrt{\langle v_X, v_X \rangle \langle v_Y, v_Y \rangle}}; \quad (13)$$

2. *if  $X$  and  $Y$  are independent random variables, then  $v_X, v_Y$  are orthogonal vectors with respect to the inner product, i.e.  $\langle v_X, v_Y \rangle = 0$ .*

We can easily proof the converse of the previous lemma when the number of random variables is finite. Thus semidefinite association measures are characterized by the following Theorem.

**Theorem 1.** *An association measure  $\rho$  defined on a space of real random variables  $H$ , is semidefinite positive if and only if for any finite subspace of random variables  $H_1 \subseteq H$  the following two properties are satisfied:*

1. *there exists an inner vectorial product  $\langle \cdot, \cdot \rangle: V \times V \rightarrow R$  and a function  $g : H_1 \times H_1 \rightarrow V \times V$  such that  $g(X, Y) = (v_X, v_Y)$ ;  $\langle v_X, v_Y \rangle = \langle v_Y, v_X \rangle$ ;  $\langle v_{-X}, v_Y \rangle = \langle v_X, v_{-Y} \rangle = -\langle v_X, v_Y \rangle$  and*

$$\rho(X, Y) = \frac{\langle g(X, Y) \rangle}{\sqrt{\langle g(X, X) \rangle \langle g(Y, Y) \rangle}} = \frac{\langle v_X, v_Y \rangle}{\sqrt{\langle v_X, v_X \rangle \langle v_Y, v_Y \rangle}};$$

2. *if  $X$  and  $Y$  are independent random variables, then  $v_X, v_Y$  are orthogonal vectors with respect to the inner product, i.e.  $\langle v_X, v_Y \rangle = 0$ .*

Moreover, as a consequence of Cauchy–Schwarz inequality, we get that  $|\rho(X, Y)| = 1$  if and only if  $v_X = av_Y$  for a given real  $a$  (where  $v_X, v_Y$  are defined as in Theorem 1).

*Examples.* From the above results we easily deduce that Pearson, Kendall, Spearman, and Blomqvist measures are semidefinite association measures since they satisfy the Properties 1 and 2 of Lemma 1. Moreover, for any  $L^p = \{X | \mathbb{E}(|X|^p) < \infty\}$  space of random variables defined in a probability space  $(\Omega, \mathfrak{F}, \Pr)$  we can introduce the following classes of semidefinite positive association measures.

**Proposition 1.** *For any  $p > 0$  the following functionals defined on  $L^p$  space are semidefinite association measures*

$$\begin{aligned}
\text{M1 } \rho_{P,p}(X, Y) &= \frac{\mathbb{E}\left((X - V_{p/2}(X))^{<p/2>} (Y - V_{p/2}(Y))^{<p/2>} \right)^{<\min(2/p, 2)>}}{\|X - V_{p/2}(X)\|_p \|Y - V_{p/2}(Y)\|_p}, \text{ where } (x)^{<q>} = \\
&\quad \text{sign}(x) |x|^q, V_q(X) \text{ is the unique real value such that } \mathbb{E}\left((X - V_q(X))^{<q>}\right) = \\
&\quad 0 \text{ and } \|X\|_p = \mathbb{E}(|X|^p)^{\min(1, 1/p)} \text{ is the classic metric in } L^p. \text{ Moreover } \\
&\quad |\rho_{P,p}(X, Y)| = 1 \text{ if and only if } Y = aX + b \text{ a.s. for some real } a \text{ and } b. \\
\text{M2 } \tau_{K,p}(X, Y) &= \frac{\mathbb{E}\left((X - X_1)^{<p/2>} (Y - Y_1)^{<p/2>} \right)^{<\min(2/p, 2)>}}{\|(X - X_1)\|_p \|(Y - Y_1)\|_p} \text{ where } (X_1, Y_1) \text{ is an in-} \\
&\quad \text{dependent identically distributed (i.i.d.) copy of } (X, Y). \\
&\quad O_{p, \mathfrak{F}_1}(X, Y) = \text{cor}\left((X^{<p/2>} - \mathbb{E}(X^{<p/2>} | \mathfrak{F}_1)), (Y^{<p/2>} - \mathbb{E}(Y^{<p/2>} | \mathfrak{F}_1))\right)^{<\min(2/p, 2)>} \\
\text{M3 } &= \frac{\mathbb{E}\left((X^{<p/2>} - \mathbb{E}(X^{<p/2>} | \mathfrak{F}_1))(Y^{<p/2>} - \mathbb{E}(Y^{<p/2>} | \mathfrak{F}_1))\right)^{<\min(2/p, 2)>}}{\left(\|X^{<p/2>} - \mathbb{E}(X^{<p/2>} | \mathfrak{F}_1)\|_2 \|Y^{<p/2>} - \mathbb{E}(Y^{<p/2>} | \mathfrak{F}_1)\|_2\right)^{<\min(2/p, 2)>}} \\
&\quad \text{where } \mathfrak{F}_1 \text{ is a sub-sigma algebra of } \mathfrak{F} \text{ (i.e. } \mathfrak{F}_1 \subset \mathfrak{F}) \text{ and } X \text{ and } Y \text{ are not } \\
&\quad \mathfrak{F}_1 \text{ measurable.}
\end{aligned}$$

All these measures are a logical extension of the Pearson correlation measure. We obtain the Pearson correlation measure with measures of type M1 and M2 when  $p = 2$ . We obtain the Pearson correlation measure with measures of type M3 when  $p = 2$  and  $\mathfrak{F}_1 = \{\emptyset; \Omega\}$ . In addition, if  $X$  and  $Y$  are continuous random variables, then  $V_0(X)$  and  $V_0(Y)$  is the median of  $X$  and  $Y$ , respectively. Thus  $\lim_{p \rightarrow 0} \rho_{P,p}(X, Y) = \beta_B(X, Y)$  and measures of the type M1 are an extension of the Blomqvist measure (that we obtain when  $p = 0$ ). Similarly, measures of the type M2 are a logical extension of the Kendall correlation (that we get for  $p = 0$ ). About measures of the type M3 we suggest to use a sigma algebra that is not too rich of events, in order to obtain an association measure that can be easily used. For example, we can use the sigma algebra  $\mathfrak{F}_1$  generated by a finite partition of  $\Omega$ , that is,  $\mathfrak{F}_1 = \langle \{A_i; i = 1, \dots, n\} \rangle$  where  $A_i \in \mathfrak{F}$ ;

$A_i \cap A_j = \emptyset, \forall i \neq j$ ; and  $\cup_{i=1}^n A_i = \Omega$ . In portfolio problems we can think that  $A_1 = \{z_b \leq F_{z_b}^{-1}(\alpha_1)\}$ ,  $A_i = \{F_{z_b}^{-1}(\alpha_{i-1}) < z_b \leq F_{z_b}^{-1}(\alpha_i)\}$  for  $i=2, \dots, n-1$ ;  $A_n = \{z_b > F_{z_b}^{-1}(\alpha_{n-1})\}$  where  $0 < \alpha_1 < \dots < \alpha_{n-1} < 1$ ;  $F_{z_b}^{-1}(\beta) = \inf\{u \mid \Pr(z_b \leq u) \geq \beta\}$  and  $z_b$  is a benchmark of the market. Under these assumptions, the conditional expectation can be easily estimated, since it is given by the simple function:

$$\mathbb{E}(X/\mathfrak{S}_1)(w) = \sum_{i=1}^n I_{[X(w) \in A_i]} \frac{1}{\Pr(A_i)} \int_{A_i} X d\Pr \quad \forall w \in \Omega$$

where

$$I_{[X(w) \in A]} = \begin{cases} 1 & \text{if } X(w) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Given a sample of  $n$  i.i.d. copies  $(X_i, Y_i)$  of the bivariate vector  $(X, Y)$  and, assuming a suitable sigma algebra  $\mathfrak{S}_1$  as above, then:

1. a consistent estimator of  $V_q(X)$  is simply obtained by solving the estimating equation  $\sum_{i=1}^n \text{sign}(X_i - \bar{V}_q(X)) |X_i - \bar{V}_q(X)|^q = 0$ .
2.  $\frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j>i} (Y_i - Y_j)^{<p/2>} (X_i - X_j)^{<p/2>}$  is a consistent unbiased estimator of  $\mathbb{E} \left( (X - X_1)^{<p/2>} (Y - Y_1)^{<p/2>} \right)$  and thus we can estimate  $\tau_{K,p}(X, Y)$ .
3.  $\frac{1}{\#(X_i \in A)} \sum_{X_i \in A} X_i^{<p/2>}$  (where  $\#(X_i \in A)$  is the number of observations  $X_i$  belonging to  $A$ ) is a consistent estimator of  $\frac{1}{\Pr(A)} \int_A X^{<p/2>} d\Pr$  and thus we can estimate  $O_{p,\mathfrak{S}_1}(X, Y)$ .

Working with semidefinite association matrixes is fundamental in several statistical problems. However, the estimator of semidefinite association matrixes could not be semidefinite positive (see Rousseuw and Molenberghs, 1993).

### 3. Possible use of association measures in portfolio problems

One of the most popular measures proposed to order admissible portfolios according to their risk is standard deviation. Several papers in recent literature

discuss the possibility of using other measures of risk and uncertainty to optimize investor's choices (see, for a review, Rachev et al. (2008)). Measures of uncertainty can be introduced axiomatically (Ortobelli, 2001). Typically, uncertainty measure is defined as any increasing function of a positive functional  $D$  that is law invariant (i.e.  $D(X) = D(Y)$  for any  $X$  and  $Y$  with the same distribution) and that satisfies the following properties:

**P1**  $D(X + C) \leq D(X)$  for all  $X$  and constant  $C > 0$ ;

**P2**  $D(0) = 0$ , and  $D(aX) = aD(X)$  for all  $X$  and  $a > 0$ ;

**P3**  $D(X) \geq 0$  for all  $X$ , with  $D(X) = 0$  if and only if  $X$  is constant.

Moreover, in a certain sense, semidefinite positive association matrixes represent a multivariate measure of dispersion and generally cannot be used to value the dispersion of a given portfolio (except for special cases). In practical terms, let us consider  $n$  assets with gross returns  $z = [z_1, z_2, \dots, z_n]'$  and the vector of portfolio weights  $x = (x_1, x_2, \dots, x_n)'$ . Given a semidefinite positive association matrix  $Q_\rho = [\rho_{i,j}]$  of the gross returns, then we could consider the following measure of the portfolio dispersion (Tichý and Ortobelli, 2009)):

$$d_\rho(x'z) = \sqrt{x'Q_{\rho,\sigma}x} \quad (14)$$

where  $Q_{\rho,\sigma} = [\sigma_{z_j}\sigma_{z_i}\rho_{i,j}]$ ,  $\rho$  is a semidefinite positive association measure and  $\sigma_z$  is an uncertainty measure. Observe that if there is a riskless return among the asset returns (say, the first component), then  $\rho_{1,j} = \rho_{j,1} = 0$  for any  $j$ , since a constant is independent of any random variable. Therefore the riskless asset does not make any contribution to the measure  $d_\rho$ .

Even if the measure  $d_\rho(x'z)$  appears to be a logical extension of portfolio variance, it does not satisfy the law invariance property. For example, let us assume there are three assets with gross returns  $z = (z_1, z_2, z_3)'$  and suppose the third gross return has the same distribution of the portfolio  $xz_1 + yz_2$ , i.e.

the portfolios  $[x, y, 0]z$  and  $[0, 0, 1]z$  have the same distributions. Since any uncertainty measure  $\sigma_z$  is law invariant then  $\sigma_{z_3}^2 = \sigma_{xz_1 + yz_2}^2$ . However, unless  $Q_{\rho, \sigma}$  is the variance covariance matrix, we obtain the following inequality:

$$d_\rho([0, 0, 1]z)^2 = \sigma_{xz_1 + yz_2}^2 \neq x^2\sigma_{z_1}^2 + y^2\sigma_{z_2}^2 + 2xy\sigma_{z_1}\sigma_{z_2}\rho(z_1, z_2) = d_\rho([x, y, 0]z)^2.$$

A sufficient condition that guarantees that measure (14) is invariant in law is given by the following proposition.

**Proposition 2.** *Proposition 2. Suppose  $\rho$  is semidefinite positive association measure defined on all possible portfolios of gross returns  $x'z$ . Suppose the functional  $\rho$  can be represented for all portfolios as in Theorem 1, i.e.  $\rho : H \times H \rightarrow [-1, 1]$  and*

$$\rho(X, Y) = \frac{\langle g(X, Y) \rangle}{\sqrt{\langle g(X, X) \rangle \langle g(Y, Y) \rangle}} = \frac{\langle v_X, v_Y \rangle}{\sqrt{\langle v_X, v_X \rangle \langle v_Y, v_Y \rangle}},$$

where  $H$  is the class of all admissible portfolios  $x'z$  and  $\langle \cdot, \cdot \rangle : V \times V \rightarrow R$  is a vectorial inner product. Let us assume the function  $g : H \times H \rightarrow V \times V$  such that  $g(X, Y) = (v_X, v_Y)$ , is bilinear, i.e.  $g(aX + bZ, Y) = (av_X + bv_Z, v_Y)$  and  $g(X, aY + bZ) = (v_X, av_Y + bv_Z)$ . If  $\sigma_X = \sqrt{\langle v_X, v_X \rangle}$  is an uncertainty measure, then  $d_\rho(x'z) = \sqrt{x'Q_{\rho, \sigma}x}$  is invariant in law.

More generally we can define the semidefinite positive association measures that satisfy the properties of Proposition 2 as follows.

**Definition 2.** We say that  $\rho$  is a linear correlation measure in the class of the random variables  $H$  if it satisfies the following properties:

1.  $\rho$  is a semidefinite positive association measure defined on the class of random variables  $H$ ;
2. for all random  $X, Y$  belonging to  $H$ ,  $\rho(X, Y) = \frac{\langle g(X, Y) \rangle}{\sqrt{\langle g(X, X) \rangle \langle g(Y, Y) \rangle}} = \frac{\langle v_X, v_Y \rangle}{\sqrt{\langle v_X, v_X \rangle \langle v_Y, v_Y \rangle}}$ , where  $\langle \cdot, \cdot \rangle : V \times V \rightarrow R$  is a vectorial inner product and  $g : H \times H \rightarrow V \times V$  such that  $g(X, Y) = (v_X, v_Y)$ , is a bilinear function;
3. the functional  $\sigma_X = \sqrt{\langle v_X, v_X \rangle}$  is an uncertainty measure (i.e. it is invariant in law and satisfies properties P1, P2 and P3).

Examples of linear correlation measures are all those measures that can be seen as an inner product of an Hilbert space of random variables. Thus, the Pearson correlation measure, the measure  $O_{2,\mathfrak{S}_1}(X,Y)$  defined in Proposition 1 and some combinations of linear correlation measures (as described in the following corollary) are linear correlation measures.

**Corollary 1.** *The convex combination of concordance measures (association measures, semidefinite association measures) is still a concordance measure (association measure, semidefinite association measure). Moreover, let  $\rho_i$  ( $i = 1, \dots, m$ ) be  $m$  linear correlation measures defined for all random variables  $X, Y$  belonging to  $H$  (thus we suppose it contains also its centered random variables) as  $\rho_i(X, Y) = \frac{\langle g_i(X, Y) \rangle_i}{\sqrt{\langle g_i(X, X) \rangle_i \langle g_i(Y, Y) \rangle_i}} = \frac{\langle v_X^{(i)}, v_Y^{(i)} \rangle_i}{\sqrt{\langle v_X^{(i)}, v_X^{(i)} \rangle_i \langle v_Y^{(i)}, v_Y^{(i)} \rangle_i}}$ , where  $\langle \cdot, \cdot \rangle_i: V_{(i)} \times V_{(i)} \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$ , are vectorial inner products. Then  $\langle X, Y \rangle = \sum_{i=1}^m a_i \langle v_X^{(i)}, v_Y^{(i)} \rangle_i$  ( $a_i \geq 0$ ;  $\sum_{i=1}^m a_i = 1$ ) is an inner product in the class of centered random variables belonging to  $H$  and thus  $\rho(X, Y) = \frac{\langle X, Y \rangle}{\sqrt{\langle X, X \rangle \langle Y, Y \rangle}}$  is a linear correlation measure.*

The lack of invariance in law does not permit the usage of measures of the type  $d_\rho(x'z)$  within portfolio selection problem. Generally, when we use concordance measures such as Kendall, Spearman, and Blomqvist measures, the law invariance property of  $d_\rho(x'z)$  is not satisfied since these concordance measures are not linear correlation measures. When  $d_\rho(x'z)$  is invariant in law, we get the following proposition.

**Proposition 3.** *Suppose the matrix  $Q_{\rho,\sigma}$  does not depend on the portfolio weights  $x$  and all random variables are defined in a finite probability space where the probability is uniform. If  $d_\rho(x'z)$  is invariant in law, it is consistent with preferences of risk-averse investors.*

Clearly, the assumption, that we are in a finite probability space  $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$  with probability  $\Pr(\{\omega_i\}) = \frac{1}{n}$ , is not very realistic. However, several consistent estimators  $\tilde{Q}_{\rho,\sigma}$  of  $Q_{\rho,\sigma}$  are computed as if the gross returns were defined in a finite probability space with uniform probability and, also for this reason, the estimator  $\tilde{Q}_{\rho,\sigma}$  is still semidefinite positive. So, for example, if  $\sigma_{z_i} = \mathbb{E}(f(z_i))$  and  $\rho_{i,j} = \mathbb{E}(g(z_i, z_j))$  for some functions  $f$  and  $g$ , then

$\tilde{\sigma}_{z_j} = \frac{1}{n} \sum_{k=1}^n f(z_j^{(k)})$ ,  $\tilde{\rho}_{i,j} = \frac{1}{n} \sum_{k=1}^n g(z_i^{(k)}, z_j^{(k)})$  (where  $z_j^{(k)}$  is the  $k$ -th observation of  $z_j$ ) are consistent estimators of  $\sigma_{z_i}$  and  $\rho_{i,j}$  and  $\tilde{Q}_{\rho,\sigma} = [\tilde{\sigma}_{z_j} \tilde{\sigma}_{z_i} \tilde{\rho}_{i,j}]$  is a consistent estimator of  $Q_{\rho,\sigma}$ . Therefore when  $d_\rho(x'z)$  is invariant in law (according to Bauerle and Müller (2006)) and the estimated distribution of  $w'z$  is dominated in the convex order by the estimated distribution of  $y'z$ , then  $w'\tilde{Q}_{\rho,\sigma}w \leq y'\tilde{Q}_{\rho,\sigma}y$ . Moreover, as it was pointed out by Bauerle and Müller (2006), when the probability space is non-atomic we can guarantee that a measure  $D$  is consistent with the choices of risk-averse investors if  $D$  is an invariant in law, convex measure that satisfies the Fatou property (that is, for any sequence of integrable random variables  $\{X_n\}_{n \in \mathbb{N}}$  such that  $\mathbb{E}(|X_n - X|) \rightarrow 0$ , implies  $D(X) \leq \liminf D(X_n)$ ). Thus the following corollary holds.

**Corollary 2.** *Suppose the matrix  $Q_{\rho,\sigma}$  does not depend on the portfolio weights  $x$ . If  $d_\rho(x'z)$  is invariant in law and satisfies the Fatou property, then it is consistent with the choices of risk-averse investors.*

As suggested from the following definition the linear correlation measures are not the unique association measures related to functionals consistent with risk averse preferences.

**Definition 3.** Let  $\rho$  be a semidefinite positive association measure defined for all random variables  $X, Y$  belonging to a Polish space of random variables  $H$  as  $\rho(X, Y) = \frac{\langle v_X, v_Y \rangle}{\sqrt{\langle v_X, v_X \rangle \langle v_Y, v_Y \rangle}}$ . We say that the functional  $\sigma_X = \sqrt{\langle v_X, v_X \rangle}$  is an uncertainty measure in  $H$  derived from the association measure  $\rho$  if it is a convex uncertainty measure that satisfies the Fatou property.

Clearly any uncertainty measure  $\sigma_X = \sqrt{\langle v_X, v_X \rangle}$  derived from an association measure  $\rho$  is consistent with risk averse choices in  $H$ . Typical examples of uncertainty measures derived from association measures in  $L^p$  spaces are the functionals  $\sigma_X = \|(X - X_1)\|_p$  of Proposition 1 for any  $p \geq 1$ . Moreover, from this definition we also deduce that concordance measures cannot be used as uncertainty measures.

**Corollary 3.** *An uncertainty measure  $\sigma_X = \sqrt{\langle v_X, v_X \rangle}$  in  $H$  cannot be derived from a semidefinite positive concordance measure.*

Next we distinguish two possible uses of association measures in portfolio theory. In particular, we propose to use them either in optimization problems or in order to reduce the dimensionality of the problem. Let us briefly discuss both problems.

### 3.1. Portfolio selection problems

The classic portfolio selection problem among  $n$  assets is a portfolio that minimizes a given risk measure  $q$  provided that the reward measure  $v$  is constrained by some minimal value  $m$ ; that is,

$$\begin{aligned} \min_x & q(x'z - z_b) \\ \sum_{i=1}^n x_i &= 1; \quad x_i \geq 0; \\ v(x'z - z_b) &\geq m, \end{aligned} \tag{15}$$

where  $z_b$  denotes the gross return of a given benchmark. The portfolio that provides the maximum reward per unit of risk is called the market portfolio. In particular, when the reward and risk are both positive measures, the market portfolio is the solution for the optimization problem:

$$\begin{aligned} \max_x & \frac{v(x'z - z_b)}{q(x'z - z_b)} \\ \sum_{i=1}^n x_i &= 1; \quad x_i \geq 0. \end{aligned} \tag{16}$$

Generally, we can distinguish two different types of benchmarks: *artificial benchmarks* and *traded benchmarks*. Traded benchmarks are some indexes traded on the market that represent some sectors and/or markets. For these benchmarks we can obtain historical observations. Artificial benchmarks are not traded on the market and they are artificially created by portfolio managers to represent the best/worst indicators of the assets used. Typical examples are the upper and lower stochastic bounds (see, among others, Ortobelli and Rachev (2001) or Ortobelli and Pellerey (2007, 2008)).



The most simple upper and lower stochastic bounds are respectively given by  $\max_i z_i$  and  $\min_i z_i$  that satisfy the relation  $\max_i z_i \geq x'z \geq \min_i z_i$  for all vectors of portfolio weights  $x$  belonging to the simplex  $S = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1; x_i \geq 0\}$ . Thus investors would like to maximize the concordance and/or the association with the upper bound benchmark  $\max_i z_i$  and to minimize the concordance and/or the association with the lower bound benchmark  $\min_i z_i$ . Alternatively, with traded benchmarks investors would like to:

- a) maximize the association between the portfolio  $x'z$  and the benchmark  $z_b$  when the traded benchmark is on the right tail;
- b) minimize the association between the portfolio  $x'z$  and the traded benchmark  $z_b$  when the benchmark is on the left tail.

From this brief discussion we deduce that investors would like to maximize utility functionals of the type:

- 1.  $f_{x'z, z_b}(\alpha, \beta) = \rho_1(x'z, \max_i z_i) v(x'z) - \rho_2(x'z, \min_i z_i) q(x'z)$  when we use the upper and lower stochastic bounds ( $\max_i z_i$  and  $\min_i z_i$ );
- 2.  $f_{x'z, z_b}(\alpha, \beta) = \rho_1(x'z, z_b | z_b \geq F_{z_b}^{-1}(\beta)) v(x'z) - \rho_2(x'z, z_b | z_b \leq F_{z_b}^{-1}(\alpha)) q(x'z)$  when we use a traded benchmark gross return  $z_b$ ;

where  $F_{z_b}^{-1}(\beta) = \inf \{u \mid \Pr(z_b \leq u) \geq \beta\}$ ,  $\rho_1$  and  $\rho_2$  are two association measures, and  $v(x'z)$ ,  $q(x'z)$  are given measures of reward and risk, respectively. We still call market portfolio the portfolio that provides the maximum reward per unit of risk, optimizing the differences between association measures

$$f_{x'z, z_b}(\alpha, \beta) = \rho_1(x'z, \max_i z_i) - \rho_2(x'z, \min_i z_i)$$

or

$$(\rho_1(x'z, z_b | z_b \geq F_{z_b}^{-1}(\beta)) - \rho_2(x'z, z_b | z_b \leq F_{z_b}^{-1}(\alpha))).$$

In particular, when the reward and risk are both positive measures, and there exists a portfolio  $x'z$  such that the difference between association measures is

positive (i.e.  $\rho_1(x'z, \max_i z_i) - \rho_2(x'z, \min_i z_i) > 0$  or  $\rho_1(x'z, z_b | z_b \geq F_{z_b}^{-1}(\beta)) - \rho_2(x'z, z_b | z_b \leq F_{z_b}^{-1}(\alpha)) > 0$ ), then the market portfolio is the solution for the optimization problems:

$$\begin{aligned} \max_x \frac{v(x'z)}{q(x'z)} (\rho_1(x'z, \max_i z_i) - \rho_2(x'z, \min_i z_i)) \\ \sum_{i=1}^n x_i = 1; \quad x_i \geq 0; \end{aligned} \quad (17)$$

or

$$\begin{aligned} \max_x \frac{v(x'z)}{q(x'z)} [\rho_1(x'z, z_b | z_b \geq F_{z_b}^{-1}(\beta)) - \rho_2(x'z, z_b | z_b \leq F_{z_b}^{-1}(\alpha))] \\ \sum_{i=1}^n x_i = 1; \quad x_i \geq 0. \end{aligned} \quad (18)$$

Clearly the optimization problems (17, 18) generally admit more local optima, and thus, we have to use heuristics for global optimization. Moreover, as dispersion a measure  $q(x'z)$  we can use a measure of type  $d_\rho(x'z)$  when the conditions of Proposition 2 are satisfied.

### 3.2. The portfolio dimensional problem

Papp et al. (2005) and Kondor et al. (2007) have shown that the number of observations should increase proportionally with the number of assets in order to get a good approximation of the portfolio risk-reward measures. Therefore, it is necessary to find the right trade-off between a statistical approximation of the historical series depending only on a few parameters and the number of historical observations. In practice, portfolio managers reduce the dimensionality of the problem approximating the return series with a  $k$ -fund separation model (or other regression-type models) that depends on an adequate number (not too large) of parameters.

Thus we can perform a PCA of the gross returns of the stocks used in order to identify the few factors (portfolios) with the highest return variability (see Biglova et al. (2009)). Therefore, we replace the original  $n$  correlated time series  $z_i$  with  $n$  uncorrelated time series  $R_i$  assuming that each  $z_i$  is a linear combination of the series  $R_i$ . This is always possible when we use a linear correlation

measure  $\rho$ . Then we implement a dimensionality reduction by choosing only those factors whose uncertainty measure  $\langle R_i, R_i \rangle$  is significantly different from zero. We call portfolio factors  $f_i$  the  $s$  time series  $R_i$  with a significant dispersion measure, while the remaining  $n - s$  series with very small dispersion measure are summarized by an error. Thus, each series  $z_i$  is a linear combination of the factors plus a small uncorrelated noise:

$$z_i = \sum_{j=1}^s a_{ij} f_j + \sum_{j=s+1}^n a_{ij} R_j = \sum_{j=1}^s a_{ij} f_j + \varepsilon_i. \quad (19)$$

We can apply the PCA either to the Pearson correlation matrix or to any other linear correlation measure, for example  $Q = [\rho_{i,j}]$  where  $\rho_{i,j} = O_{2,\mathfrak{S}_1}(z_i, z_j)$  for a suitable sigma algebra  $\mathfrak{S}_1$ .<sup>3</sup> Once identified the  $s$  factors  $f_j = \sum_{i=1}^n x_i z_i$  ( $j = 1, \dots, s$ ; such that  $\sum_{k=1}^n x_k^2 = 1$ ) that account for most of the variability of the gross returns, we further reduce the variability of the error by regressing the series on the factors  $f_j$  so that we get:

$$z_i = b_{i,0} + \sum_{j=1}^s b_{i,j} f_j + \varepsilon_i. \quad (20)$$

Once reduced the dimensionality of the problem we can apply portfolio selection optimization problems (17, 18) to the approximated portfolio gross returns:  $x'z \simeq x'\widehat{\mathbf{b}}_0 + \sum_{j=1}^s x'\widehat{\mathbf{b}}_j f_j$ , where  $\widehat{\mathbf{b}}_j = [\widehat{b}_{1,j}, \dots, \widehat{b}_{n,j}]'$  is the vector of estimated coefficients  $\widehat{b}_{i,j}$  ( $j = 0, 1, \dots, s$ ). This procedure is computational efficient and can be applied using any linear correlation measure.

#### 4. Some empirical applications

In this section we employ various uncertainty measures and association measures, as defined in Sections 2 and 3, in the portfolio selection problem. We evaluate two distinct tasks: (i) *portfolio performance optimization* and (ii) *portfolio dimensionality reduction and large scale portfolio selection*.

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<sup>3</sup>For some other alternatives and application issues see e.g. Hubert et al. (2009).

In the portfolio performance optimization problem we use 34 equities<sup>4</sup> and upper and lower market bounds  $\max_i z_i$  and  $\min_i z_i$  as artificial benchmarks. We use daily data for the period Januray, 1994 to November, 2009 (a total of 3999 observations).

For the portfolio dimensionality reduction problem and the large scale portfolio selection we consider 1304 equities on the US stock Market (450 equities from Nasdaq and 854 equities from NYSE). We use daily data for the period Januray, 1997 to December, 2009 (a total of 3258 observations). All data sets are taken from DataStream.

Table 1: Number of assets and observations

<i>Problem</i>	# of assets	# of observations
portfolio performance optimization	34	3999
portfolio dimension reduction	1304	3258

#### 4.1. Portfolio performance optimization

In this context we propose an ex-post comparison among several versions of optimization problem (17) based on different risk and association measures. The objective function that we maximize is regularly evaluated on the basis of daily observations of market prices over the preceding 10 years (2600 daily observations).

We use as a reward measure  $v(x'z)$  in (17), the expected return of a portfolio,

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<sup>4</sup>The 34 assets are: Home de Pot, 3M, Alcoa, Boeing, Caterpillar, Coca Cola, The Du Pont E I De Nem, Exxon Mobil, Gen Electric, Hewlett Packard, IBM, Johnson and Johnson, McDonalds, Merck, Procter Gamble, United tech, Wal Mart Stores, Walt Disney-Disney, American Express, AT&T, Intel, Microsoft, Pfizer, Travelers, Verizon, Chevron, Adobe, Amgen, Apple, Applied Materials, CA, Costco whole Sale Corporation, Ross Stores, Sun Microsystem.

given that it belongs to the right tail as bounded by 95th percentile, i.e.:

$$v(x'z) = \mathbb{E}[x'z | x'z \geq F_{x'z}^{-1}(0.95)]. \quad (21)$$

In contrast to a unique reward measure, we consider as measures of risk either the standard deviation of the portfolio of gross returns or the average value at risk of the centered portfolio  $AVaR_{0.05}(x'z - \mathbb{E}(x'z))$  – i.e. either  $q(x'z) = std(x'z) = \sqrt{x'Qx}$  where  $Q$  is the variance covariance matrix of  $z$  or  $q(x'z) = AVaR_{0.05}(x'z - \mathbb{E}(x'z)) = \frac{-1}{0.05} \int_0^{0.05} F_{x'z - \mathbb{E}(x'z)}^{-1}(u) du$ . Moreover, we consider three different factors of type:

$$\left( \rho_1 \left( x'z, \max_i z_i \right) - \rho_2 \left( x'z, \min_i z_i \right) \right)$$

measuring differently the association of the return portfolio with the upper and lower bounds. In particular, we set  $\rho_1$  and  $\rho_2$  as follows:

1.  $\rho_1 = \rho_2 = \gamma_G$  (i.e.  $\rho_i$  ( $i = 1, 2$ ) is the Gini concordance measure);
2.  $\rho_1 = \rho_2 = \text{cor}$  (i.e.  $\rho_i$  ( $i = 1, 2$ ) is the Pearson correlation measure);
3.  $\rho_1 = \rho_S$  and  $\rho_2 = \tau_K$  (i.e. as  $\rho_1$  we use the Spearman concordance measure and as  $\rho_2$  we use the Kendall concordance measure).

We compare ex-post sample paths of wealth considering an initial wealth  $W_0 = 1$ . In this empirical analysis we recalibrate the portfolios every 6 months (125 working days). At  $k$ -th recalibration ( $k = 0, 1, 2, \dots$ ), three main steps are performed to compute the ex-post final wealth.

Step 1 Determine the "market" portfolio  $x_M^{(k)}$  – a solution to:

$$\max_x \frac{v(x'z)}{q(x'z)} \left[ \rho_1 \left( x'z, \max_i z_i \right) - \rho_2 \left( x'z, \min_i z_i \right) \right]$$

s.t.  $\sum_{i=1}^n x_i^{(k)} = 1$  and  $x_i^{(k)} \geq 0$ . To solve these problems we use the heuristic for global optimization proposed by Angelelli and Ortobelli (2009).

Step 2 The ex-post final wealth is given by:

$$W_{k+1} = W_k((x_M^{(k)})' z^{(ex\ post)}),$$

where  $z^{(ex\ post)}$  is the vector of observed gross returns between time  $k$  and  $k + 1$ .

Step 3 The new starting point for the  $(k + 1)$ -th optimization problem is portfolio  $x_M^{(k)}$ .

Steps 1, 2 and 3 are repeated for different risk and association measures until the observations are available.

We illustrate the results in Figures 1 and 2. In particular, Figure 1 reports the wealth sample paths of four strategies: two that use the standard deviation as risk measure  $q(x'z)$  (Pearvar, and SpeaKendvar) and the analogous that use the average value at risk (AVaR) of the centered portfolio as risk measure  $q(x'z)$  (PearAVaR, and SpeaKendAVaR). With Pearvar and PearAVaR strategies we use as association measures the Pearson linear correlation (i.e.  $\rho_1 = \rho_2 = \text{cor}$ ). While with SpeaKendvar and SpeaKendAVaR strategies we use the Spearman and the Kendall concordance measures (i.e.  $\rho_1 = \rho_S$  and  $\rho_2 = \tau_K$ ). Observe that the strategies based on AVaR risk measure present an higher performance than those based on the standard deviation. Similarly, the strategies that use the Spearman and the Kendall concordance measures present a higher performance than those based on the Pearson linear correlation. Moreover, Figure 1 suggests that the use of suitable association measures  $\rho_i$  could have a higher impact than the use of a suitable risk measure  $q(x'z)$ .

Figure 2 compares the wealth sample paths when we use the average value at risk of the centered portfolio as risk measure  $q(x'z)$  the Spearman and the Kendall concordance. Even Figure 2 reports the wealth sample paths of four strategies: one that use the standard deviation as risk measure (Ginivar) and three strategies that use the average value at risk (AVaR) of the centered portfolio as risk measures (GiniAVaR, PearAVaR, and SpeaKendAVaR). While the two

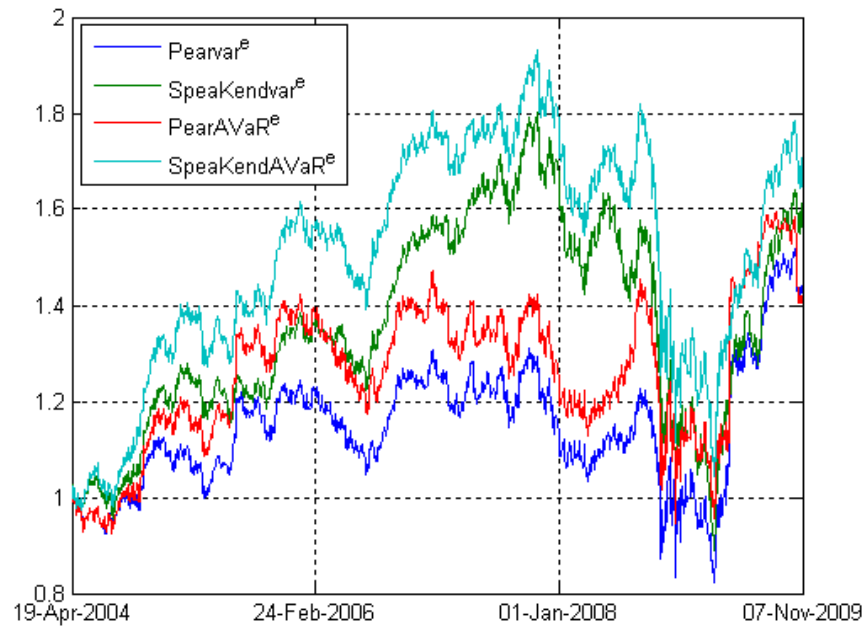


Figure 1: Ex-post final wealth and different risk measures I. We compare ex-post final wealth sample paths when the risk measure is either the standard deviation or AVaR and the association measures are either Spearman combined with Kendall, or the Pearson linear correlation.

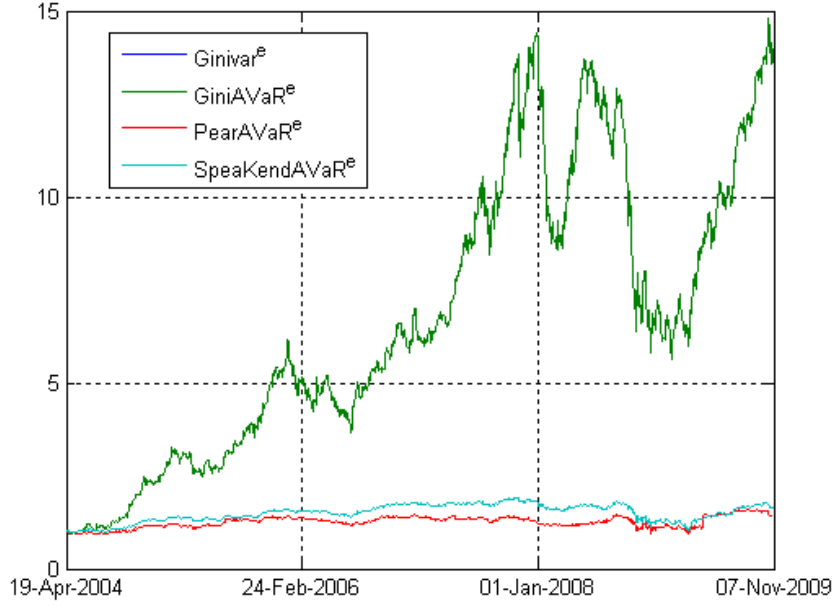


Figure 2: Ex-post final wealth and different risk measures II. Ex-post final wealth sample paths when the association measures are either Gini, or Spearman combined with Kendall, or the Pearson linear correlation.

Gini type strategies with different risk measures give exactly the same choices and the best performance, the other strategies present significant differences. Thus this comparison essentially confirms the results of Figure 1. Gini type strategies give an ex-post final wealth with more than 50% return for year, that is not comparable with all the other strategies. However, Gini type strategies do not present a diversification of the portfolio choices, since the wealth is invested in only one asset (Apple) at any recalibration step. On the other hand, it seems clear that the use of concordance type measures give better performance than Pearson correlation measure.



#### 4.2. Portfolio dimensionality reduction and large scale portfolio selection

Our next task is to reduce the dimensionality of the portfolio problem in order to compare portfolio strategies in a large scale framework. Generally, we can consider two possible criteria for selecting the "best" principal components:

- a) take the first principal components that together explain at least 50% of the dispersion;
- b) take only those principal components that explain not less than  $100/N\%$  of the dispersion measure (*Kaiser type rule*) (where  $N = 1304$  is the number of all assets in our empirical analysis).

However, in several large portfolio problems we should choose only few principal components in order to guarantee a sufficiently good approximation to the optimal portfolio problem (see, among others, Papp et al. (2005), Kondor et al. (2007)). The right number of principal components should account the limited number of historical observation of the dataset. For this reason in the portfolio dimensionality reduction analysis we also consider the criterium:

- c) take the first 35 principal components and show how much dispersion is explained.

As linear correlation measures we consider:

1. the Pearson correlation measure  $\rho_{i,j} = \text{cor}(z_i, z_j)$ , and its conditional version  $\rho_{i,j} = \text{cor}(z_i, z_j | x'z \leq F_{x'z}^{-1}(0.05))$ , where  $x'z = \frac{1}{N} \sum_{i=1}^N z_i$  is the equidiversified portfolio;
2. the linear correlation measure  $\rho_{i,j} = O_{2,\mathfrak{S}_1}(z_i, z_j)$  defined in Proposition 1 and its conditional version  $\rho_{i,j} = O_{2,\mathfrak{S}_1}(z_i, z_j | x'z \leq F_{x'z}^{-1}(0.1))$ , where  $x'z = \frac{1}{N} \sum_{i=1}^N z_i$  and  $\mathfrak{S}_1 = \langle \{A_i; i = 1, \dots, 10\} \rangle$  where  $A_1 = \{\max_i z_i \leq F_{\max_i z_i}^{-1}(0.1)\}$ ,  $A_i = \{F_{\max_k z_k}^{-1}(0.1(i-1)) < \max_k z_k \leq F_{\max_k z_k}^{-1}(0.1i)\}$  for  $i=2, \dots, 9$ ; and  $A_{10} = \{\max_k z_k > F_{\max_k z_k}^{-1}(0.9)\}$ ;

3. the linear correlation measure  $\rho_{i,j} = O_{2,\mathfrak{S}_2}(z_i, z_j)$  defined in Proposition 1 and its conditional version  $\rho_{i,j} = O_{2,\mathfrak{S}_2}(z_i, z_j | x'z \leq F_{x'z}^{-1}(0.05))$ , where  $x'z = \frac{1}{N} \sum_{i=1}^N z_i$  and  $\mathfrak{S}_1 = \{A_i; i = 1, \dots, 40\}$  where  $A_1 = \{\max_i z_i \leq F_{\max_i z_i}^{-1}(0.025)\}$ ,  $A_i = \{F_{\max_k z_k}^{-1}(0.025(i-1)) < \max_k z_k \leq F_{\max_k z_k}^{-1}(0.025i)\}$  for  $i = 2, \dots, 39$ ; and  $A_{40} = \{\max_k z_k > F_{\max_k z_k}^{-1}(0.975)\}$ .

The results are reviewed in Table 2. From Table 2 we observe very little differences between  $O_{2,\mathfrak{S}_1}$  and  $O_{2,\mathfrak{S}_2}$  conditional and unconditional measures. Moreover, we need more than 12% (22%) of the principal components to explain most of the variability using the Kaiser rule applied either to conditional (unconditional) measures. Therefore, the number of principal components selected with the Kaiser rule is still too big to apply a portfolio selection of type (17, 18) to the approximated portfolio of gross returns. In addition, less than 20 components are sufficient to explain more than 50% of variability when we use conditional correlation measures. While we can explain more than 40% (61%) of variability using only 35 principal components for unconditional (conditional) correlation measures.

Table 2: Portfolio dimensionality reduction

Measures	cor	$O_{2,\mathfrak{S}_1}$	$O_{2,\mathfrak{S}_2}$	Cond. cor	Cond. $O_{2,\mathfrak{S}_1}$	Cond. $O_{2,\mathfrak{S}_2}$
# PCs Kaiser rule	294	300	300	158	157	157
% explained Kaiser r.	69.48%	69.55%	69.6%	99.8%	99.73%	99.73%
# PCs to explain 50%	97	102	102	19	18	18
% explained 35 PCs	40.92%	40.15%	40.13%	61.98%	63.03%	62.95%

Given these results we want to compare portfolio strategies considering two different approximations of gross returns: one based on 35 factors derived from conditional and unconditional Pearson correlation measure; the second based on 35 factors derived from conditional and unconditional  $O_{2,\mathfrak{S}_1}$  correlation measure. In both cases we regress the gross return series on 35 factors  $f_j$  (i.e.  $z_i =$

$b_{i,0} + \sum_{j=1}^{35} b_{i,j} f_j + \varepsilon_i$ ) and we approximate the vector of gross returns  $\hat{\mathbf{z}} \simeq \hat{\mathbf{b}}_0 + \sum_{j=1}^{35} \hat{\mathbf{b}}_j f_j$  using OLS estimates of parameters  $\mathbf{b}_j$ . In particular, in the first case the 35 factors  $f_j$  are shared as follows:

- a) the first 19 principal components obtained with conditional Pearson correlation measure (that explain at least 50% of conditional dispersion) and the first 16 principal components obtained with unconditional Pearson correlation measure.

Similarly, in the second case the 35 factors  $f_j$  are shared as follows:

- b) the first 18 principal components obtained with conditional  $O_{2,\mathfrak{S}_1}$  correlation measure (that explain at least 50% of conditional dispersion) and the first 17 principal components obtained with unconditional  $O_{2,\mathfrak{S}_1}$  correlation measure.

Using the approximation of gross returns  $\hat{\mathbf{b}}_0 + \sum_{j=1}^{35} \hat{\mathbf{b}}_j f_j$  the randomness of the choices is uniquely determined by the 35 factors  $f_j$ . As in Section 4.1 we propose an ex-post portfolio comparison where the portfolio decision is taken using the approximated returns and the valuation of future wealth is taken with real ex-post gross returns. In particular, we recalibrate the portfolio every 6 month (125 working days) using the last 10 years daily approximated observations (2600 working days). The objective function is (17) where the reward measure  $v(x'\hat{\mathbf{z}})$  is given by formula (21) and the risk measure  $q(x'\hat{\mathbf{z}}) = AVaR_{0.05}(x'\hat{\mathbf{z}} - \mathbb{E}(x'\hat{\mathbf{z}}))$  is the average value at risk of the centered approximated portfolio. We consider three possible strategies based on different association measures. The first strategy (SpeaAVaR) is based on the Spearman concordance measure (i.e.  $\rho_1 = \rho_2 = \rho_S$ ); the second strategy (PearAVaR) is based on the Pearson correlation measure (i.e.  $\rho_1 = \rho_2 = \text{cor}$ ); the third strategy (GiniSpeaAVaR) uses as  $\rho_1$  the Gini concordance measure and as  $\rho_2$  the Spearman concordance measure (i.e.  $\rho_1 = \gamma_G$ , and  $\rho_2 = \rho_S$ ). Then we value the effects in portfolio selection of the two different dimensional reductions. In particular, we compare the ex-

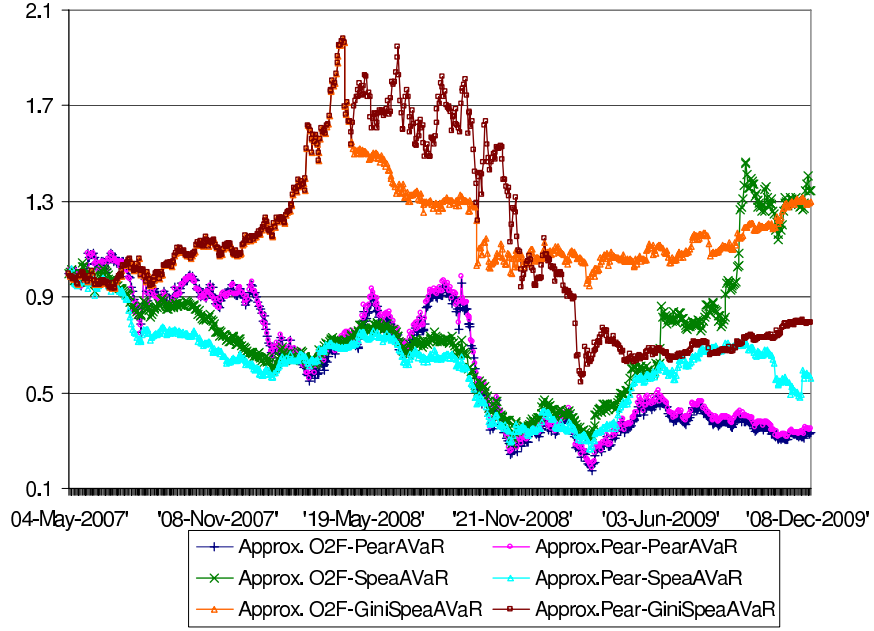


Figure 3: Ex-post final wealth after dimensionality reduction. Ex-post final wealth obtained with three portfolio strategies (PearAVaR, SpeaAVaR, GiniSpeaAVaR) considering returns approximated either with dimensional reduction derived from Pearson correlation measure (namely, Approx. Pear) or with the reduction derived from  $O_{2,\mathfrak{S}_1}$  correlation measures (namely, Approx. O2F).

post wealth sample paths of the three strategies obtained considering either the dimensional reduction derived from (conditional and unconditional) Pearson correlation measure or the reduction derived from (conditional and unconditional)  $O_{2,\mathfrak{S}_1}$  correlation measure. The results of this comparison are reported in Figure 3.

Figure 3 shows that the unique two strategies that present a wealth greater than one at the end of the last three years of crisis are those based on concordance measures (GiniSpeaAVaR and SpeaAVaR) obtained considering the reduction derived from (conditional and unconditional)  $O_{2,\mathfrak{S}_1}$  correlation measures. Thus, these results partially confirms the results of Section 4.1. Moreover,

we have proved that the dimensional reduction of large scale portfolio problems could have a substantial impact in portfolio choices, and future researches should account of different linear correlation measures to reduce the dimensionality of the portfolio problems.

## 5. Conclusion

In this paper we have discussed when, and how, we can use association measures in portfolio problems. First of all, we have characterized the semidefinite positive association measures distinguishing those that can be used as uncertainty measures in portfolio problems. Then we have discussed the use of association measures with respect to the portfolio performance optimization and the portfolio dimensionality reduction. In particular, we have formulated new portfolio selection problems that use the association between portfolios and the stochastic bounds of the market. To deal with large scale portfolio problems, we have discussed the possibility of using different linear correlation measures to reduce the dimensionality of the problems. Finally, we have proposed an empirical analysis of the problems discussed. All the empirical experiments have shown that the use of different association measures have an important impact on portfolio performance and on portfolio dimensionality reduction.

## Appendix

*Proof of Lemma 1.* Cauchy–Schwarz inequality guarantees property one and two of association measures, i.e.  $\rho : H \times H \rightarrow [-1, 1]$  and  $\rho(X, X) = 1$ ;  $\rho(X, -X) = -1$ . Since  $\rho(X, Y)$  can be seen as an inner product of vectors then from the Gram representation theorem we deduce that any association matrix is semidefinite positive. Properties 3 and 4 of association measure are a logical consequence of  $\langle v_X, v_Y \rangle = \langle v_Y, v_X \rangle$  (and it follows the symmetry) and  $\langle v_{-X}, v_Y \rangle = \langle v_X, v_{-Y} \rangle = -\langle v_X, v_Y \rangle$ .

*Proof of Theorem 1.* Suppose  $\rho$  to be a semidefinite association measure. Then any association matrix  $Q = [\rho_{i,j}]$  should be semidefinite positive. Consider the association matrix defined on all the random variables belonging to  $H_1$  that we suppose are  $n$ ). From the Gram representation theorem we know that an  $n$ -dimensional association matrix  $Q = [\rho_{i,j}]$  is semidefinite positive if, and only if, there exists  $n$  vectors  $v_1, v_2, \dots, v_n$ , in a vector space  $V$  such that  $\rho_{i,j} = \langle v_i, v_j \rangle$ . Since  $\rho : H_1 \times H_1 \rightarrow [-1, 1]$  and  $\rho(X, X) = 1$  then  $\langle v_i, v_i \rangle = 1$  and it is equivalent to say  $\rho_{i,j} = \frac{\langle v_i, v_j \rangle}{\sqrt{\langle v_i, v_i \rangle \langle v_j, v_j \rangle}}$ . Moreover property 4 of association measures implies  $\langle v_{-X}, v_Y \rangle = \langle v_X, v_{-Y} \rangle = -\langle v_X, v_Y \rangle$ . If  $X$  and  $Y$  are independent random variables  $\rho(X, Y) = 0$  and thus  $\langle v_X, v_Y \rangle = 0$ .

*Proof of Proposition 1.* We first prove Case 1. Observe that for any constant  $a, b$   $V_{p/2}(aX + b) = aV_{p/2}(X) + b$  and the functional

$$cv(X, Y) = \mathbb{E} \left( (X - V_{p/2}(X))^{\langle p/2 \rangle} (Y - V_{p/2}(Y))^{\langle p/2 \rangle} \right)^{\langle \min(2/p, 2) \rangle}$$

is an inner product in the  $L^p$  space that satisfies the properties of Lemma 1. In addition,  $cv(X, Y) = cv(Y, X)$  and  $cv(aX + b, Y) = cv(X, Y)a$ . Moreover, if  $X$  is independent of  $Y$ , then

$$cv(X, Y) = \mathbb{E} \left( (X - V_{p/2}(X))^{\langle p/2 \rangle} \right)^{\langle \min(2/p, 2) \rangle} \mathbb{E} \left( (Y - V_{p/2}(Y))^{\langle p/2 \rangle} \right)^{\langle \min(2/p, 2) \rangle} = 0.$$

If  $|\rho_p(X, Y)| = 1$ , then  $|cv(X, Y)| = \|X - V_{p/2}(X)\|_p \|Y - V_{p/2}(Y)\|_p$ , and then there exists a real  $c$  such that  $(X - V_{p/2}(X))^{\langle p/2 \rangle} = c(Y - V_{p/2}(Y))^{\langle p/2 \rangle}$  and the thesis follows. Regarding Case 2, observe that when  $Y = X$  a.s. then  $X_1 = Y_1$  a.s. and consequently  $\tau_{K,p}(X, X) = 1$ . As a matter of fact, for any  $x_n \in \mathbb{Q}$ ;  $\Pr(X_1 \leq x_n, Y_1 > x_n) + \Pr(X_1 > x_n, Y_1 \leq x_n) = 0$ . Consider  $\tilde{\Omega} = (\cup_n \Omega_n)^C$  (where  $\Omega_n = \{w | X_1(w) \leq x_n, Y_1(w) > x_n \text{ or } X_1(w) > x_n, Y_1(w) \leq x_n\}$ ). Then  $\Pr(\tilde{\Omega}) = 1$  and  $X_1(w) = Y_1(w)$  for any  $w \in \tilde{\Omega}$  (otherwise if  $X_1(w) > Y_1(w)$  – or  $X_1(w) < Y_1(w)$  – there exists  $x_n \in \mathbb{Q}$  such that  $X_1(w) > x_n \geq Y_1(w)$  – or  $X_1(w) \leq x_n < Y_1(w)$  – against the assumption that  $w \in \tilde{\Omega}$ ). Moreover since  $(X - X_1)$  and  $(Y - Y_1)$  are symmetric random variables  $\mathbb{E} \left( (X - X_1)^{\langle p/2 \rangle} \right) =$

$\mathbb{E}\left((Y - Y_1)^{<p/2>}\right) = 0$  and when  $X$  and  $Y$  are independent random variables  $\tau_{K,p}(X, Y) = 0$ . All the other properties of Lemma 1 are satisfied by the measures  $\tau_{K,p}(X, Y)$ . Regarding Case 3, the proof is similar to the cases 1 and 2. Observe only that

$$\begin{aligned} \mathbb{E}\left((X^{<p/2>} - \mathbb{E}(X^{<p/2>}|\mathfrak{S}_1))(Y^{<p/2>} - \mathbb{E}(Y^{<p/2>}|\mathfrak{S}_1))\right) = \\ = \mathbb{E}(X^{<p/2>}Y^{<p/2>}) - \mathbb{E}(\mathbb{E}(X^{<p/2>}|\mathfrak{S}_1)\mathbb{E}(Y^{<p/2>}|\mathfrak{S}_1)). \end{aligned}$$

Thus, if  $X$  is independent by  $Y$  for any  $A \in \mathfrak{S}_1$ ;

$$\begin{aligned} \int_A X^{<p/2>}Y^{<p/2>}d\Pr = \int_A X^{<p/2>}d\Pr \int_A Y^{<p/2>}d\Pr = \\ = \int_A \mathbb{E}(X^{<p/2>}|\mathfrak{S}_1)d\Pr \int_A \mathbb{E}(Y^{<p/2>}|\mathfrak{S}_1)d\Pr \end{aligned}$$

then  $O_p(X, Y) = \mathbb{E}\left((X^{<p/2>} - \mathbb{E}(X^{<p/2>}|\mathfrak{S}_1))(Y^{<p/2>} - \mathbb{E}(Y^{<p/2>}|\mathfrak{S}_1))\right) = 0$ .

*Proof of Proposition 2.* Under these assumptions the matrix  $Q_{\rho, \sigma} = [\sigma_{z_j} \sigma_{z_i} \rho_{i,j}] = [ < v_{z_i}, v_{z_j} > ]$ . Moreover, from bilinearity we deduce  $x'Q_{\rho, \sigma}x = < v_{x'z}, v_{x'z} > .$  If we have two portfolios  $x'z$  and  $y'z$  with the same distribution

$$y'Q_{\rho, \sigma}y = < v_{y'z}, v_{y'z} > = \sigma_{y'z}^2 = \sigma_{x'z}^2 = < v_{x'z}, v_{x'z} > = x'Q_{\rho, \sigma}x,$$

where we used the invariance in law of the uncertainty measure  $\sigma_X = \sqrt{< v_X, v_X >}$ .

Thus  $d_\rho(x'z)$  is invariant in law.

*Proof of Corollary 1.* Let us consider  $\rho(X, Y) = \sum_{i=1}^m a_i \rho_i(X, Y)$  such that  $a_i \geq 0$ ;  $\sum_{i=1}^m a_i = 1$ . Clearly  $\rho(X, Y)$  is a concordance (association) measure if all  $\rho_i(X, Y)$  satisfy the seven (five) properties of concordance (association) measures. Similarly, if  $\rho_i$  for  $i = 1, \dots, m$  are semidefinite positive association measures, then also  $\rho$  is semidefinite positive because any association matrix  $Q = \sum_{i=1}^m a_i Q_i$  of  $\rho$  is the convex combination of the association matrixes  $Q_i$  of the measures  $\rho_i$ . Observe that if  $\rho_i(X, Y)$   $i = 1, \dots, m$  are linear correlation measures  $< v_X^{(i)}, v_X^{(i)} >_i \geq 0$  and  $< v_X^{(i)}, v_X^{(i)} >_i = 0$  if and only if  $X$  is a constant. Thus  $< X - \mathbb{E}(X), X - \mathbb{E}(X) > \geq 0$  and it is equal to

zero if and only if  $X - \mathbb{E}(X) = 0$ . Since  $\langle v_X^{(i)}, v_Y^{(i)} \rangle_i = \langle v_Y^{(i)}, v_X^{(i)} \rangle_i$  then  $\langle X - \mathbb{E}(X), Y - \mathbb{E}(Y) \rangle = \langle Y - \mathbb{E}(Y), X - \mathbb{E}(X) \rangle$ . In addition, for any  $a, b \in \mathbb{R}$  and  $X, Y, Z \in H$ ,  $\langle v_{aX+bZ}^{(i)}, v_Y^{(i)} \rangle_i = a \langle v_X^{(i)}, v_Y^{(i)} \rangle_i + b \langle v_Z^{(i)}, v_Y^{(i)} \rangle_i$ , thus  $\langle aX + bZ, Y \rangle = a \langle X, Y \rangle + b \langle Z, Y \rangle$  and  $\langle X, Y \rangle$  is an inner product in the class of centered random variables belonging to  $H$ .

*Proof of Proposition 3.* The matrix  $Q_{\rho, \sigma}$  is still semidefinite positive since  $x'Q_{\rho, \sigma}x = y'Q_{\rho}y \geq 0$  where  $y = (x_1\sigma_{z_1}, x_2\sigma_{z_2}, \dots, x_n\sigma_{z_n})'$ . Moreover, Bauerle and Müller (2006) have proven that in a finite probability space where the probability  $\Pr$  is uniform, any invariant in law, convex measure  $D$  (i.e.  $D(aX + (1-a)Y) \leq aD(X) + (1-a)D(Y)$  for any  $a \in [0, 1]$ ) is consistent with choices of risk-averse investors. The measure  $x'Q_{\rho, \sigma}x$  (or its estimator  $x'\tilde{Q}_{\rho, \sigma}x$  with semidefinite positive matrix  $\tilde{Q}_{\rho, \sigma}$ ) is convex in the class of portfolio gross returns  $x'z$  since the function  $f(x) = x'Q_{\rho, \sigma}x$  ( $f(x) = x'\tilde{Q}_{\rho, \sigma}x$ ) is a convex function. Then, for any  $a \in [0, 1]$ :

$$(ax + (1-a)y)'Q_{\rho, \sigma}(ax + (1-a)y) \leq ax'Q_{\rho, \sigma}x + (1-a)y'Q_{\rho, \sigma}y.$$

The measure  $x'Q_{\rho, \sigma}x$  (or its estimator  $x'\tilde{Q}_{\rho, \sigma}x$ ) is strictly convex when matrix  $Q_{\rho, \sigma}$  ( $\tilde{Q}_{\rho, \sigma}$ ) is definite positive. According to Bauerle and Müller (2006), if  $w'z$  is dominated in the sense of the convex order by  $y'z$ , then  $w'Q_{\rho, \sigma}w \leq y'Q_{\rho, \sigma}y$ .

*Proof of Corollary 3.* Any uncertainty measure  $\sigma_X = \sqrt{\langle v_X, v_X \rangle}$  in  $H$  derived from a semidefinite positive association measure  $\rho$  should satisfy  $\sigma_{aX} = a\sigma_X$  for any  $a > 0$  (P2). Since for any semidefinite positive concordance measure  $\rho_C$  and for any increasing function  $h$   $\rho_C(X, h(X)) = \frac{\langle v_X, v_{h(X)} \rangle}{\sqrt{\langle v_X, v_X \rangle \langle v_{h(X)}, v_{h(X)} \rangle}} = 1$ , then for Cauchy-Schwarz inequality  $v_{h(X)} = bv_X$  for a given  $b > 0$ . On the other hand, if  $\sigma_X$  is derived from  $\rho_C$  for any  $a > 0$ ,  $\langle v_X, v_{aX} \rangle = a\sigma_X^2$  and thus  $\langle av_X - v_{aX}, av_X - v_{aX} \rangle = 0$  that implies  $av_X = v_{aX}$ . Now for any  $p > 1$  and for any increasing function  $h$  that is homogeneous of degree  $p$  i.e.  $h(cX) = c^ph(X)$ , let us consider  $b > 0$  such that  $v_{h(X)} = bv_X$ . Then



$v_{h(bX)} = b^p v_{h(X)} = b^{p+1} v_X$  but we also get  $v_{h(bX)} = b v_{bX} = b^2 v_X$ . Thus it must be  $b = 1$  and  $v_{h(X)} = v_X$ . Consider  $a, p > 1$  and let  $X$  be a bounded random variable. Given  $h_n(X) = (aX)^{<1+\frac{p}{n}>}$  then  $\mathbb{E}(|h_n(X) - aX|) \rightarrow 0$ . Therefore from the Fatou property  $a^2 < v_X, v_X > \leq \liminf < v_{h_n(X)}, v_{h_n(X)} > = < v_X, v_X >$  against the hypothesis  $a > 1$ . We deduce the thesis and  $\sigma_X$  cannot be derived from a semidefinite positive concordance measure  $\rho_C$ .

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