UNIVERSITÀ DEGLI STUDI DI BERGAMO

Facoltà di Economia

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On The Reformulation of a Particular Class of Bilevel Problems

Relatore:

Chiar.mo Prof./Chiar.ma M.T. Vespucci

Correlatore:

Chiar.mo Prof./Chiar.ma A. A. Gaivoronski

Tesi di

Paolo PISCIELLA

Matricola n. 700190

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Chapter 1

Introduction

This thesis is devoted to the analysis of bilevel programming problems in the view of obtaining a nonlinear reformulation which does not encompass complementarity constraints, which are known for rendering nonlinear programs in which they are featured very hard to handle.

The use of optimization has become a prominent part of the design and analysis process of most industrial and socio-economic systems. Great developments have been made in solution methods tailored to many different problem structures and allowing for solution large scale problems arising from many different areas such as revenue management, congestion management, airline scheduling and network design problems amongst others [5].

Nevertheless, it can be argued that most of the managerial decisions are of a bilevel nature, as they impact and influence systems very often with conflicting goals.

Bilevel programming addresses the problem in which two decision makers one referred to as the leader and the second referred to as the follower, each with their own individual objectives, act and react in a interdependant, sequencial manner. The actions taken by the leader affect the reactions that the follower will take, as a response of the leader's decisions [4].

Namely, the decision nature of the follower actor is defined as an optimization problem which is parametrized on the decisions taken by the leader. In turn, when taking her decisions has to take into account which reaction the follower will have as a consequence. The actions of one affect the choices and payoffs available to the other [2]. The hierarchical relationship results from the fact that the mathematical program related to the follower's behaviour is part of the leader constraints. This is the major feature of bilevel programs: they include two mathematical programs into a single instance, one of these problems being part of the constraints of the other ones.

A very important aspect of bilevel programs is linked on the expectation on the extent in which the follower is available to engage in a collaborative interaction. This means that if a multiple number of optimal solutions for the follower problem are available for a given solution of the leader's problem, the follower can choose whether to select the optimal point that delivers the highest objective value to the leader or the other way around. If the leader assumes that the follower will be collaborative and will pick the solution most favourable to the leader then we are considering the so called *optimistic solution* of the bilevel program. Most of the contributions to bilevel programming treat this kind of problem, and one of the reasons can be that this problem has an optimal solution under quite reasonable assumptions [3].

In fact, if a regularity condition is satisfied for the lower level problem, then the Karush-Kuhn-Tucker conditions can be used to reformulate the problem as an ordinariy mathematical program. KKT conditions allow to express the bilevel program as a so-called Mathematical Program with Equilibrium Constraints, which features a feasible set defined by a set of complementarity constraints, besides the ordinary constraints. This implies that is possible to solve the optimistic bilevel programming problem via an MPEC, but only if the lower level problem has a unique optimal solution for all values of the parameter. The solution of a pessimistic bilevel programming problem via MPEC is not possible.

Although the optimistic view allows for the use of a established toolkit of mathematical programming to seek for solutions, such position cannot be applied at least in cases when cooperation is not allowed, not possible or when the follower's seriousness of keeping the agreement is not granted. This leads to the so called *pessimistic bilevel programming problem*.

The method developed in the thesis allows in some cases to model a pessimistic bilevel position or, at least, to obtain a solution in which the follower chooses the output that the leader has forecasted for a particular choice of upper level variables.

A prominent example of bilevel relationships are Stackelberg games. Other applications can be found, for example, in game theory, investigations of oligopolies, network design problems or traffic management. Another field where bilevel programming finds a wide application is ICT and telcommunication services modelling. ICT service provision is, in fact, characterized by collaboration and competition between a multitude of actors. As a mention, Mobile Network Operators (MNO) offer heterogeneous communications services, and compete on the same markets not only with other MNOs, but also with Mobile Virtual Network Operators (MVNO). However, MVNOs do not possess the physical network to be used in order to deliver services to customers, and have to buy large amounts of traffic-minutes from the MNO. Thus on one hand the MNO competes with the MVNO on market share and on the other hand MNO receives money from lending capacity to the MVNO. The interplay between such two actors and their decisions frameworks are well described by bilevel models.

The thesis is heavily inspired on the work of [1], and the ideas introduced in the thesis were thought as a solution method for the model introduced by such authors.

In such model, the owner of a service platform, i.e. a set of tools for building complex services stemming from the composition of a set of basic services, wants to attract a group of service providers in order to enable the provision of a group of service bundles. These service bundles, normally referred to as service portfolios, are composite services built up by composing together basic services supplied, more or less independantly, by service providers. The service portfolio is feasible only if each service provider supplies a minimum amount of her service. A scheme for revenue sharing between such actors must then be devised by the platform operator in order to get each provider 'on board'. The model that the authors propose is a stochastic programming model with bilevel structure. Each service provider solve a portfolio optimization model to decide which service portfolios to provide their service to in order to maximize their returns over the investment done while, at the same time, controlling the losses they can incur. The platform operator balances the basic service provision by offering a proper revenue share to each service provider so that the result of their portfolio selection fits with the group of service portfolios that the platform operator wants to provide to the customers. More details about the model and the reformulation suggested will be provided in the last chapter.

The rest of the thesis is organized as follows. After this first chapter devoted to the introduction of the problem an overview of the main ideas and solution approaches of bilevel programming is treated in the second chapter. The third chapter introduces the ideas underlying the reformulation of the KKT conditions for the solution of bilevel programs with convex lower levels and the fourth chapter is dedicated to an application of such reformulations.

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Chapter 2

Overview on Bilevel Programming

2.1 introduction

From a historical point of view, multilevel optimization is closely related to the economic problem of Stackelberg [7] in the field of game theory, which we briefly describe now. To this end, we consider an economic planning process involving interacting agents at two distinct levels: some of the individuals Ñ collectively called the leader issue directives to the remaining agents called the followers. In the particular framework of Stackelberg games, the leader is assumed to anticipate the reactions of the followers; this allows him to choose his best or optimal strategy accordingly. More precisely, the leader chooses a strategy x in a set $X \subseteq \Re^n$, and every follower i has a strategy set $Y_i(x) \subseteq \Re^m{}_i$ corresponding to each $x \in X$. The sets $Y_i(x)$ are assumed to be closed and convex. Any follower i also has a cost function depending on both the leader's and all followers' strategies and which may be expressed as

$$\theta_i(x,.) : \prod_{j=1}^{M} \Re^{m_j} \to \Re,$$

where M is the number of followers. It is further assumed that for fixed values of $x \in X$ and $y_j(j \neq i)$ the function θ_i is convex and continuously differentiable in $y_i \in Y_i(x)$. The followers behave collectively according to the noncooperative principle of Nash [12] which means that, for each $x \in X$, they will choose a joint response vector

$$y^{opt} = (y^{opt})_{j=1}^{M} \quad \in C(x),$$

where $C(x) = \prod_{i=1}^{M} Y_i(x)$, such that, for every $i = 1, ..., M$, there holds

$$y_i^{opt} \in \arg\min \left\{ \theta_i \left(x, y, y_{j \neq i}^{opt} \right) : y_i \in Y_i(x) \right\}.$$

In the above setting, considered by Sherali et al. [13] in an oligopolistic situation, Stackelberg problems possess a hierarchical structure similar to that of BLPP, although the lower level program is an equilibrium rather than an optimization problem.

Bilevel programs were initially considered by Bracken and McGill [9] in a series of papers that dealt with applications in the military field as well as in production and marketing decision making. By that time, such problems were called *mathematical programs with optimization problems in the constraints*, the terms bilevel and multilevel programming being introduced later by Candler and Norton [14].

Basically, all real-world problems involving a hierarchical relationship between two decision levels may be modeled by bilevel programs. These are encountered in fields as diverse as management (facility location, environmental regulation, credit allocation, energy policy, hazardous materials), economic planning (social and agricultural policies, electric power pricing, oil production), engineering (optimal design, structures and shape), chemistry, environmental sciences, optimal control, etc.

For one, it can be argued that most managerial decisions are of a bilevel nature, in the sense that they impact systems with some degree of autonomy and conflicting objectives, few real-life studies have adopted this paradigm. In the following, we provide a small selection of actual or potential fields of application considered in the literature.

Revenue Management
Revenue management is a generic term that covers a set of optimization procedures aimed at maximizing the profitability of firms characterized by high investment costs, low operating costs, and perishable inventories. It was initially implemented in the airline industry, under the name 'yield management', and involved four issues: ticket pricing, seat allocation, demand forecasting, and overbooking. Notwithstanding the third issue, which is of a tactical nature, the first three lend themselves to a bilevel formulation that extends the toll setting problem described in the introduction of this survey. Such model, that involves the pricing and seat allocation policies, is described in Côté et al. [15].

Congestion Management
In urban areas, marginal tolls can be used to minimize overall congestion. If only a subset of the arcs are subject to tolls, the latter scheme is not applicable, and one faces a second best problem of true bilevel nature. See Hearn and Ramana

[16] and Larsson and Patriksson [17] for more details on this topic.

Network Design Problems
Design problems involving autonomous agents are a rich source of bilevel models. One such example is concerned with capacity improvement of a road network, where one must bal- ance investment costs against congestion reduction, in a network where traffic flows achieve an equilibrium compatible with the design parameters, i.e., optimize their own objective. Introduced by LeBlanc [18], this model was further analyzed by Marcotte [19], who provided worst-case bounds on the performance of easily implementable heuristic procedures.

Energy Sector
The energy sector, in particular the power sector, has been the topic of some interesting bilevel modelizations. Hobbs and Nelson [20] consider an electric utility that seeks to minimize costs or maximize benefits while controlling electric rates and subsidizing energy conservation programs. In the model of Haurie et al. [21], the interaction between a power utility and cogenerators is set within the framework of a leader-follower game, where the demand side is modelled as a large-scale techno-economic model. In a joint energy-agriculture setting, Bard et al. [22] address the problem faced by a government (the leader) that wishes to induce, through minimal subsidies, the conversion of food to biofuel crops. The model, involving bilinear objectives at both levels of decision-making, is reminiscent of the toll-setting model discussed above.

A Stackelberg-Nash Game
Sherali et al. analyze an oligopoly where one firm acts as the leader, and the remaining ones achieve a Cournot-Nash equilibrium parameterized by the production level of the leader firm.

2.2 General Formulation

The general formulation of a bilevel programming problem (BLPP) is given by the following

$$\begin{array}{lll} \min\limits_{x \in X} & F(x,y) & \\ \text{subject to} & G(x,y) \leq 0 & \\ & \min\limits_{y \in Y} & f(x,y) \\ & \text{subject to} & g(x,y) \leq 0 \end{array}$$

where $x \in X \subseteq R^n$ are called *upper level variables* and $y \in Y \subseteq R^m$ are called *lower level variables*. The functions $F : R^{n+m} \to R$, $f : R^{n+m} \to R$ are the *upper level objective function* and the *lower level objective function*, while the vector valued functions $G : R^{n+m} \to R^p$, $g : R^{n+m} \to R^q$ are called the *upper level constraints* and *lower level constraints* respectively. The upper level problem is generally termed as leader problem, meanwhile the lower level problem is also called follower problem.

To analyze such type of problem let us start by introducing some considerations about the following sets:

- Constraint region of the BLPP
  $S = \{(x, y) \in X \times Y : G(x, y) \leq 0 \text{ and } g(x, y) \leq 0\}$
- Feasible set for the follower for fixed x
  $S(x) = \{y \in Y : g(x, y) \leq 0\}$
- Rational reaction set
  $P(x) = \{y \in Y : y \in \arg\min \left(f(x, \tilde{y}),\ \tilde{y} \in S(x)\right)\}$
- Inducible region
  $IR = \{(x, y) : (x, y) \in S : y \in P(x)\}$

It is then possible to reformulate the BLPP as

$$\min_{(x,y) \in IR} \quad F(x, y).$$

The first main issue linked to BLP problems is that even in case in which all of the constraints of the BLP are linear it is possible that $P(x)$ might consist of some nontrivial subset of an hyperplane. In this case the follower would be indifferent to any point on that hyperplane; however the leader might have a specific preference. To cope with this problem two modelling approaches to bilevel programming are used. In the case of *optimistic bilevel programming*, it is assumed that, whenever

the reaction set $P(x)$ is not a singleton, the leader is allowed to select the element in $P(x)$ that suits him best.
In this case a point (x^*, y^*) is called *local optimistic solution* for the BLPP if

$$\begin{aligned} & x^* \in X \\ & y^* \in P(x^*) \\ & G(x^*, y^*) \leq 0 \\ & F(x^*, y^*) \leq F(x^*, y), \ \forall y \in P(x^*) \end{aligned}$$

and x^* is a local minimizer for the function

$$\phi_o(x) = \min_y (F(x, y) : y \in P(x))$$

When cooperation of the leader and the follower is not allowed, or if the leader is risk-averse and wishes to limit the damage resulting from an undesirable selection of the follower, then a point (x^*, y^*) is said to be a *local pessimistic solution* for the BLPP if

$$\begin{aligned} & x^* \in X \\ & y^* \in P(x^*) \\ & G(x^*, y^*) \leq 0 \\ & F(x^*, y^*) \geq F(x^*, y), \ \forall y \in P(x^*) \end{aligned}$$

and x^* is a local minimizer for the function

$$\phi_p(x) = \max_y (F(x, y) : y \in P(x))$$

the optimistic solution results from a friendly or cooperative behaviour while an aggressive follower produces a pessimistic solution.

Although early work on bilevel programming dates back to the nineteen seventies, it was not until the early nineteen eighties that the usefulness of these mathematical programs in modelling hierarchical decision processes and engineering design problems prompted researchers to pay close attention to bilevel programs, thus it is not surprising that most algorithmic research to date has focused on the simplest cases of bilevel programs, that is problems having nice properties such as linear, quadratic or convex objective and/or constraint functions. In particular, the most studied instance of bilevel programming problems has been for a long time the linear BLPP. Over the years, more complex bilevel programs were studied. In particular the research has focused on the convex BLPP and linear BLPP with discrete variables. This work is mainly focused on the practical/algorithmic viewpoint of the bilevel programming programs. We will consider the main results along with the most used algorithms for the following types of BLPP:

- Linear BLPP with continuous variables
- Linear BLPP with discrete variables
- Convex BLPP
- General BLPP

2.3 Linear BLPP with continuous variables

The vast majority of research on bilevel programming has been centered on the linear version of the problem. The formulation is the following.

$$\begin{array}{ll} \min\limits_{x \in X} & c_1 x + d_1 y \\ \text{subject to} & A_1 x + B_1 y \leq b_1 \\ & \begin{array}{ll} \min\limits_{y \in Y} & c_2 x + d_2 y \\ \text{subject to} & A_2 x + B_2 y \leq b_2 \end{array} \end{array}$$

where $c_1, c_2 \in R^n$, $d_1, d_2 \in R^m$, $b_1 \in R^p$, $b_2 \in R^q$, $A_1 \in R^{p \times n}$, $B_1 \in R^{p \times m}$, $A_2 \in R^{q \times n}$, $B_2 \in R^{q \times m}$.

As previously done when we introduced the general formulation of a BLPP, let us define the following sets

- Constraint region of the BLPP

  $S = \{(x, y) \in X \times Y : A_1 x + B_1 y \leq b_1 \text{ and } A_2 x + B_2 y \leq b_2\}$

- Feasible set for the follower for fixed x

  $S(x) = \{y \in Y : A_2 x + B_2 y \leq b_2\}$

- Projection of S onto the leader's decision space

  $S(X) = \{x \in X : \exists y \in Y : A_1 x + B_1 y \leq b_1, A_2 x + B_2 y \leq b_2\}$

- Rational reaction set

  $P(x) = \{y \in Y : y \in \arg\min (c_2 x + d_2 \tilde{y}, \tilde{y} \in S(x))\}$

- Inducible region
  $IR = \{(x, y) : (x, y) \in S : y \in P(x)\}$

For sake of simplicity it will be assumed, throughout the analysis, that $P(x)$ is a point-to-point map.

2.3.1 Theoretical properties

Before introducing some of the prominent algorithms used for solving linear BLPPs, we need to introduce the following results

Theorem 1 *IR can be written equivalently as a piecewise linear equality constraint comprised of supporting hyperplanes of S.*

Solving the linear BLLP is equivalent to minimizing the upper level objective function over a piecewise linear equality constraint In other words we can reformulate the inducible region as

$$IR = \{(x, y) \in S : Q(x) = d_2 y\}$$

where

$$Q(x) = \min\{d_2 y : B_2 y \leq b_2 - A_2 x,\ y \geq 0\}$$

and piecewise linearity comes from duality on the linear program defined in $Q(x)$. A solution of the linear BLPP occurs at a vertex of IR.

Theorem 2 *The solution of the linear BLPP occurs at a vertex of S.*

If x is an extreme point of IR, then it is an extreme point of S. One more interesting property is that when the set of optimal solutions to the linear BLPP is not single valued, than IR is not necessarily convex, In fact the inducible region is not generally convex, which means that if we take two points lying on the function $Q(x)$ and we consider a linear combination of these two points, the resulting points are not in the function. Linearity of the constraints leads to qualification conditions on the lower level constraint functions, which in turn allow us to obtain an explicit representation of IR by replacing the lower level problem with the equivalent Karush-Kuhn-Tucker (KKT) conditions. Under the assumption that $X = R^n_+$ and $Y = R^m_+$ we have the following

Proposition 1 *A necessary condition that (x^*, y^*) solves the linear BLPP is that there exist vectors u^* and v^* such that (x^*, y^*, u^*, v^*) solves (KKTR)*

$$\begin{aligned}
\min \quad & c_1x + d_1y \\
\text{subject to} \quad & A_1x + B_1y \leq b_1 \\
& A_2x + B_2y \leq b_2 \\
& uB_2 - v = -d_2 \\
& u(b_2 - A_2x - b_2y) + vy = 0 \\
& x, y, u, v \geq 0.
\end{aligned} \tag{2.1}$$

This formulation plays a key role in the development of the algorithms. From a conceptual point of view KKTR is a standard mathematical program and should be relatively easy to solve because all but one constraint is linear. Nevertheless, virtually all commercial nonlinear codes find the complementary terms in KKTR notoriously difficult to handle.

2.3.2 Algorithms for the linear BLPP

In general, there are three different approaches for solving the linear BLPP that can be considered workable

- Vertex enumeration based method. This method relies on the fact that the optimal solution has to be on a vertex of the constraint region of the BLPP. Such a method systematically explores basic solutions.

- KKT conditions based method. The BLPP is converted into problem KKTR, then a branch and bound strategy is used to deal with the complementarity constraint.

- Penalty approach based method. The follower's problem is converted into an unconstrained minimization problem using a barrier method. Alternatively a penalty on the duality gap for the lower level problem can be introduced on the upper level objective function.

K-th Best Algorithm

This algotithm proceeds enumerating basic solutions that are in the IR. It has been introduced by Bialas and Karwan [23]. The assumptions are that IR is bounded and the follower's reaction set $P(x)$ is a singleton for all $x \in S(X)$. Let $(x_{[1]}, y_{[1]}), (x_{[2]}, y_{[2]}), ..., (x_{[N]}, y_{[N]})$ denote the N ordered basic feasible solutions for the LP (called LPR)

$$\begin{aligned} \min \quad & c_1x + d_1y \\ \text{subject to} \quad & A_1x + B_1y \leq b_1 \\ & A_2x + B_2y \leq b_2 \\ & x, y \geq 0 \end{aligned} \tag{2.2}$$

and suppose that these solutions are such that

$$c_1x_{[i]} + d_1y_{[i]} \leq c_1x_{[i+1]} + d_1y_{[i+1]}, \quad i = 1, ..., N-1$$

then solving the linear BLPP is equivalent to finding the index $K^* = \min\{i \in \{1, ..., N\} : (x_{[i]}, y_{[i]}) \in IR\}$ Let us also define the lower level problem for a given value $\tilde{x}$, $LL(\tilde{x})$ as

$$\begin{aligned} \min_y \quad & c_2\tilde{x} + d_2y \\ \text{subject to} \quad & A_2\tilde{x} + B_2y \leq b_2 \\ & y \geq 0 \end{aligned}$$

Let W be the set of basic solutions to be investigated, $W_{[i]}$ the set of the basic solutions adjacent to the incumbent, T is the set of the basic solutions which have already been tested.
The algorithm is the following

Algorithm

1. i=1.
   solve LPR $\rightarrow (x_{[i]}, y_{[i]})$.
   set $W = \{(x_{[i]}, y_{[i]})\}$.
   set $T = \emptyset$
2. solve $LL(x_{[1]}) \rightarrow (x_{[i]}, \tilde{y})$
   if $\tilde{y} = y_{[i]}$
       STOP: $(x_{[i]}, y_{[i]})$ is the optimal solution of the linear BLPP.
   else
       NEXT
3. set $T = T \cup \{(x_{[i]}, y_{[i]})\}$
   set $W = (W \cup W_{[i]}) \setminus T$
4. set k=k+1.
   choose $(x_{[i]}, y_{[i]})$ s.t.
   $c_1x_{[i]} + d_1y_{[i]} = \min\{c_1x + d_1y : (x, y) \in W\}$
   go to step (2).

KKT Approach

The algorithm [2] is based on the KKT reformulation and attempts to solve problem KKTR with the complementarity constraints suppressed. A branch and bound method is then used to alternatively set the i-th lagrangean multiplier or the i-th constraint to zero.
Let W be the index set of the complementarity constraints in KKTR.
Let $\bar{F}$ be the incumbent upper bound on the leader's objective function.
At the k-th iteration we define a subset of indices $W_k \subseteq W$ and a path P_k corresponding to a vector whose elements are the indexes in W_k: i if the multiplier of the i-th constraint is zero, $-i$ if the i-th constraint is binding for $i \in W_k$.
Let $u_i = 0$ when the multiplier associated to the i-th constraint is zero.
Let $g_i = 0$ when the i-th inequality constraint is binding.
Let

$$\begin{aligned} S_k^+ &= \{i : i \in W_k \text{ and } u_i = 0\} \\ S_k^- &= \{i : i \in W_k \text{ and } g_i = 0\} \\ S_k^0 &= \{i : i \notin W_k\} \end{aligned}$$

when $i \in S_k^0$ the i-th complementarity constraint can be violated.

Algorithm

1. Set $k = 0$,
   Set $S_k^+ = \emptyset$.
   Set $S_k^- = \emptyset$.
   Set $S_k^0 = \{1, ..., q + m\}$.
   Set $\bar{F} = \infty$

2. Set $u_i = 0$ for $i \in S_k^+$
   Set $g_i = 0$ for $i \in S_k^-$.
   Solve KKTR without complementarity constraints.
   if KKTR is infeasible
       go to (6) (Backtracking)
   else
       $k = k + 1$ and label solution (x^k, y^k, u^k)

3. if $F(x^k, y^k) \geq \bar{F}$
       go to (6) (Backtracking)
   else
       NEXT

4. if $u_i g_i(x^k, y^k) = 0 \quad i = 1, ..., m+q$
   go to (5) (Updating)
   else
   select i s.t. $u_i g_i(x^k, y^k) = 0$ is largest and label it i_1
   Set $S_k^+ = S_k^+ \cup \{i_1\}$
   Set $S_k^0 = S_k^0 \setminus \{i_1\}$
   Set $S_k^- = S_k^-$
   Append i_1 to P_k
   go to (2)

5. $\bar{F} = F(x^k, y^k)$

6. if no live node exists
   go to (7) (Termination)
   else
   branch to the newest live vertex and update S_k^+, S_k^-, S_k^0 and P_k
   go to (2).

7. if $\bar{F} = \infty$
   BLPP is infeasible
   else
   $\bar{F}$ is the optimal solution of the linear BLPP.

Penalty Function Approach

For sake of completeness we will sketch the main ideas of the Penalty Function Approach [24] even though we will not treat the algorithm. The Penalty Function Approach exploits the fact that if $(x, y) \in IR$, the duality gap for the lower level problem is zero. Let us consider the problem $LL(x)$ and, ignoring the constant term $c_2 x$, let us write the dual as

$$\begin{aligned} \max_{u} \quad & u\,(A_2 x - b_2) \\ \text{subject to} \quad & u B_2 \geq -d_2 \\ & u \geq 0 \end{aligned}$$

The linear BLPP is reformulated as

$$\begin{aligned} P(K) = \min \hat{F} \ &= \ c_1 x + d_1 y + K\,[d_2 y - u\,(A_2 x - b_2)] \\ \text{subject to} \quad & A_1 x + B_1 y \leq b_1 \\ & A_2 x + B_2 y \leq b_2 \\ & u B_2 \geq -d_2 \\ & x, y, u \geq 0. \end{aligned}$$

A well known result in linear programming states that when the duality gap is zero the basic solution is optimal, then either the slack variable in the primal or the related dual variable in the dual is zero for every constraint of the LP. (Strong Duality Theorem and Complementary Slackness Theorem).
Let S be the feasible region for (x, y) and $u \in U = \{u : uB_2 \geq -d_2, u \geq 0\}$. If S and U are nonempty bounded polyhedra, their extreme points are denoted by the sets S^E and U^E, respectively. The following theorems hold.

Theorem 3 *For a given value of $u \in U$ and fixed $K \geq 0$ define*

$$\Theta(u, K) = \min_{x,y}\{\hat{F}(x, y, u, K) : (x, y) \in S\}$$

then $\Theta(., K)$ is concave on R^q and a solution to the problem

$$\min_{u}\{\Theta(u, K) : u \in U\}$$

will occur at some $u^* \in U^E$

Theorem 4 *For fixed $K \geq 0$, an optimal solution to problem $P(K)$ is achievable in $S^E \times U^E$, and $S^E \times U^E = (S \times U)^E$*

Theorem 5 *Let (x^*, y^*) solve the linear BLPP and assume that the rational reaction set $P(x^*)$ is unique. Then there exists a finite value $K^* \geq 0$ for which an optimal solution to the penalty function problem $P(K)$ yields an optimal solution to the linear BLPP for all $K \geq K^*$.*

Theorem 6 *If $(x(K), y(K), u(K))$ solves $P(K)$ as a function of K, the leader's objective function is monotonically nondecreasing and the duality gap of the follower's problem is monotonically nonincreasing in K.*

Those theorems provide the foundations for an algorithm that could be used to derive a quasi-local optimum for the linear BLPP. For a given K , the first step is to begin with an arbitrary (x^0, y^0) and solve the LP $\min_u\{\hat{F}(x^0, y^0, u, K) : u \in U^E\}$ to get an optimal u^0. Then with $u = u^0$ find $(x^1, y^1) \in \arg\min_{x,y}\{\hat{F}(x, y, u^0, K) : (x, y) \in S^E\}$ and continue iteratively the procedure.

2.4 Linear BLPP with discrete variables

Let $x_1 \in R_+^{n_1}$, $x_2 \in Z_+^{n_2}$, $y_1 \in R_+^{m_1}$, $y_2 \in Z_+^{m_2}$, where R_+ and Z_+ are respectively the set of nonnegative real numbers and the set of nonnegative integer numbers. We define the mixed integer linear BLPP as

$$\begin{array}{lll} \min & c_{11}x_1 + c_{12}x_2 + d_{11}y_1 + d_{12}y_2 & \\ \text{subject to} & A_{11}x_1 + A_{12}x_2 + B_{11}y_1 + B_{12}y_2 \le b_1 & \\ & x_1 \ge 0, x_2 \in Z_+^{n_2} & \\ & \min & d_{21}y_1 + d_{22}y_2 \\ & \text{subject to} & A_{21}x_1 + A_{22}x_2 + B_{21}y_1 + B_{22}y_2 \le b_2 \\ & & y_1 \ge 0, y_2 \in Z_+^{m_2} \end{array} \tag{2.3}$$

2.4.1 Properties of the zero-one linear BLPP

Let us consider the problem

$$\begin{array}{lll} \min\limits_{x \in X} & c_1x + d_1y & \\ \text{subject to} & A_1x + B_1y \le b_1 & \\ & \min\limits_{y \in Y} & d_2y \\ & \text{subject to} & A_2x + B_2y \le b_2 \end{array}$$

where $c_1, c_2 \in R^n$, $d_1, d_2 \in R^m$, $b_1 \in R^p$, $b_2 \in R^q$, $A_1 \in R^{p\times n}$, $B_1 \in R^{p\times m}$, $A_2 \in R^{q\times n}$, $B_2 \in R^{q\times m}$, $X \subseteq R^n$, $Y \subseteq R^m$.
In addition to the definitions in the previous section, let

$$S_L(y) = \{x \in X : A_2x + B_2y \le b_2\}$$

for all values of y, and

$$S_U = \{(x, y) : A_1x + B_1y \le b_1\}$$

A previously done we assume that the optimal solution of the lower level problem is unique. Along with the linear BLPP (L-BLPP) we consider the following models,

- discrete linear BLPP (DL-BLPP), where $X = B^n$ and $Y = B^m$
- discrete-continuous linear BLPP (DCL-BLPP), where $X = B^n$ and $Y = R^m$
- continuous-discrete linear BLPP (CDL-BLPP), where $X = R^n$ and $Y = B^m$

The existence of optimal solutions for these problems depends on the presence or absence of upper-level constraints. The following property holds for L-BLPP, DL-BLPP, DCL-BLPP and CDL-BLPP.

Property A: if $S_U = R^{n+m}$ then IR is nonempty if $S \neq \emptyset$. If $S_U \neq R^{n+m}$ then IR is nonempty if there exists an $\bar{x} \in X$ such that $(\bar{x}, \bar{y}) \in S_U$.

Property B: The inducible regions of DCL-BLPP and DL-BLPP are respectively included in the inducible regions of L-BLPP and CDL-BLPP.

Moreover for L-BLPP, DL-BLPP and DCL-BLPP we have the following property regarding the existence of optimal solutions.

Property C: Let S be a bounded set. If $S_U = R^{n+m}$ then L-BLPP, DL-BLPP and CDL-BLPP have an optimal solution if $S \neq \emptyset$. If $S_U \neq R^{n+m}$ then L-BLPP, DL-BLPP and CDL-BLPP have an optimal solution if there exists an $\bar{x} \in X$ such that $(\bar{x}, \bar{y}) \in S_U$.

In other words, if there exists a feasible solution, then there also exists an optimal solution provided that the feasible region is bounded.

For what concerns the investigation of the inducible region and the characterization of the optimal solution of CDL-BLPP the following holds

Proposition 2 *Consider the CDL-BLPP where $S_U = R^{n+m}$ and $S \neq \emptyset$. Then IR is composed of a finite union of quasi polyhedral sets, i.e. sets whose closure is a polyhedral set.*

Theorem 7 *Let $S_U = R^{n+m}$ and $S \neq \emptyset$ and suppose that there exists an optimal solution (x^*, y^*) to CDL-BLPP. Then (x^*, y^*) is a boundary point of S*

2.4.2 Reductions to Linear Three-Level Programs

Let $X = \{0,1\}^n$, $Y = \{0,1\}^m$ and e_x and e_y be the vectors of ones with dimensions n and m respectively and consider the standard optimization problem:

$$\begin{array}{ll} \min\limits_{x \in X} & c_1 x + d_1 y \\ \text{subject to} & (x,y) \in \mathcal{P} \end{array} \tag{2.4}$$

where $\mathcal{P} = \cup_{i=1}^{l} P_i$ such that P_i is a polyhedral set for $i = 1, ..., l$. Denote the solution of the problem by (x^*, y^*) and let $\theta : R^n \to R$ be a continuous function such that $\theta(x) \geq 0$ for all $0 \leq x \leq e_x$ and $\theta(x) = 0$ iff $x \in X$. The following theorem is a standard result for integer linear programs

Theorem 8 *If θ is a concave function, there exists a positive real number M such that (2.4) and the following problem*

$$\begin{array}{ll} \min\limits_{x \in X} & c_1 x + d_1 y + M\theta(x) \\ \text{subject to} & 0 \leq x \leq e_x \\ & (x,y) \in \mathcal{P} \end{array}$$

have the same solutions.

DCL-BLPP and DL-BLPP

The discrete-continuous bilevel programming problem can be stated as

$$\begin{array}{ll} \min\limits_{x \in X} & c_1 x + d_1 y \\ \text{subject to} & A_1 x + B_1 y \leq b_1 \\ & \begin{array}{ll} \min\limits_{y} & d_2 y \\ \text{subject to} & A_2 x + B_2 y \leq b_2 \end{array} \end{array}$$

The following theorem shows that this problem is equivalent to a linear bilevel programming problem

Theorem 9 *There exists a positive real M such that DCL-BLPP and the linear bilevel problem, L-BLPP(M)*

$$\begin{array}{lll}
\min\limits_{x} & c_1x + d_1y + Me_xu & \\
\text{subject to} & A_1x + B_1y \leq b_1 & \\
& 0 \leq x \leq e_x & \\
& \min\limits_{y,u} & d_2y - e_xu \\
& \text{subject to} & A_2x + B_2y \leq b_2 \\
& & u \leq x \\
& & u \leq e_x - x
\end{array}$$

have the same optimal solutions.

With the same approach we can reformulate the DL-BLPP. Consider the sets $X = \{0,1\}^n$ and $Y = \{0,1\}^m$ along with the problem

$$\begin{array}{lll}
\min\limits_{x \in X} & c_1x + d_1y & \\
\text{subject to} & A_1x + B_1y \leq b_1 & \\
& \min\limits_{y \in Y} & d_2y \\
& \text{subject to} & A_2x + B_2y \leq b_2
\end{array}$$

the following theorem holds

Theorem 10 *There exist two positive real numbers M_x and M_y such that DL-BLPP and the linear three-level problem:*

$$\begin{array}{llll}
\min\limits_{x} & c_1x + d_1y + M_xe_xu & & \\
\text{subject to} & A_1x + B_1y \leq b_1 & & \\
& 0 \leq x \leq e_x & & \\
& \min\limits_{y,u} & d_2y - e_xu + M_ye_yv & \\
& \text{subject to} & A_2x + B_2y \leq b_2 & \\
& & 0 \leq y \leq e_y & \\
& & u \leq x & \\
& & u \leq e_x - x & \\
& & \min\limits_{v} & -e_yv \\
& & \text{subject to} & v \leq y \\
& & & v \leq e_y - y
\end{array}$$

have the same optimal solutions.

Even though it is possible to convert some types of BLPP with discrete variables into continuous three-level programs, solving the resulting three-level program is hard. This is due to the fact that no efficient algorithms exist for the three-level problem.

2.4.3 Properties of the Mixed Integer Linear BLPP

Apart from the reduction into three-level programs it is possible to apply a branch and bound procedure to solve the MIBLPP. For regular MIP, algorithms generally rely on some form of separation, relaxation and fathoming to construct tighter bounds on the solution. This approach is directly applicable to the MIBLPP. The natural relaxation derives from removing the integrality requirements on the variables. Fathoming, however, presents several difficulties. In regular MIP we have three general fathoming rules:

1. The relaxed subproblem has no feasible solution
2. The solution of the relaxed subproblem is no less than the value of the incumbent
3. The solution of the relaxed subproblem is feasible for the original problem

Only rule 1 in its original form still holds for MIBLPP. in particular the solution of the relaxed BLPP does not provide a valid bound on the solution of the mixed integer BLPP and solutions to the relaxed BLPP that are in the inducible region cannot, in general, be fathomed.

2.4.4 Moore-Bard Algorithm for the Mixed Integer Linear BLPP

Let us define the following
$N = 1, ..., n+m$ the set of the decision variables
$N^1 = 1, ..., n_2$ set of the integer variables x_2 controlled by the leader
$N^2 = 1, ..., m_2$ set of integer variables y_2 controlled by the follower
u^1 upper bound for x_2
u^2 upper bound for y_2
For subproblem (node) k we denote

$$H_k^1 = \{(\alpha^{1k}, \beta^{1k}) : 0 \leq \alpha_j^{1k} \leq x_{2j} \leq \beta_j^{1k} \leq u_j^1; \quad j \in N^1\}$$

$$H_k^2 = \{(\alpha^{2k}, \beta^{2k}) : 0 \leq \alpha_j^{2k} \leq y_{2j} \leq \beta_j^{2k} \leq u_j^2; \quad j \in N^2\}$$

as the set of upper and lower bounds for the integer variables controlled respectively by the leader and the follower at node k. If a node k is in the path to node l we have $H_k^1 \subseteq H_l^1$ and $H_k^2 \subseteq H_l^2$
Furthermore, for subproblem k, the notation $H_k^2(0,\infty)$ is user to indicate that no bounds other than the original bounds specified in problem (2.3) and placed on the integer variables controlled by the follower.
The set of the variables that have been restricted at node k is defined by

$$S_k^1 = \{j : \alpha^{1k} > 0 \text{ or } \beta^{1k} < u_j^1; \quad j \in N^1\}$$

$$S_k^2 = \{j : \alpha^{2k} > 0 \text{ or } \beta^{2k} < u_j^2; \quad j \in N^2\}.$$

Let us also define F_k^C as the optimal solution of the relaxed BLPP for subproblem k. It consists of problem (2.3) without integrality requirements and augmented by the bound constraints $x_2 \in H_k^1$ and $y_2 \in H_k^2$. If the follower's objective function is removed from this formulation we obtain the *high point solution*. We denote this value by F_k^H for subproblem k.

Bounding Theorems

Sufficient conditions are derived that indicate when a solution of a relaxed subproblem may be used as an upper bound for the mixed integer BLPP.

Theorem 11 *Given H_k^1 and $H_k^2(0,\infty)$, let (x^k, y^k) be the high point solution to the corresponding relaxed BLPP. Then $F_k^H = F(x^k, y^k)$ is a lower bound on the solution of the mixed integer BLPP at node k*

The high point solution computed at node k can be used as a bound to determine if the subproblem can be fathomed. This bound is, unfortunately, not applicable when some restrictions have been made on the variables controlled by the follower.

The following theorem indicates when F_k^H provides a valid lower bound for the case where the lower level integer variables have been bounded at iteration k with H_k^2

Theorem 12 *Given H_k^1 and H_k^2, let (x^k, y^k) be the high point solution to the corresponding relaxed BLPP. Then $F_k^H = F(x^k, y^k)$ is a lower bound on the solution of the mixed integer BLPP at node k if none of the y_{2j}^k are either at α^{2k} or β^{2k}.*

Given H_k^1 and H_k^2, let (x^k, y^k) be the high point solution to the corresponding relaxed BLPP with the restrictions in H_k^2 relaxed. Then $F_k^H = F(x^k, y^k)$ is a lower bound on the solution of the mixed integer BLPP at node k.

Algorithm

1. Set $k = 0$,
   Set H_k^1 and H_k^2 at their original bounds.
   Set $S_k^1 = \emptyset$, $S_k^2 = \emptyset$,
   Set $\bar{F} = \infty$

2. Attempt to solve the high point solution of MIBLPP with constraints H_k^1 and H_k^2 and without integrality requirements. $\rightarrow F_k^H$
   if infeasible or $F_k^H \geq \bar{F}$
       go to (7)(backtracking)
   else
       NEXT

3. Attempt to solve the relaxation of MIBLPP
   if infeasible
       go to (7) (backtracking)
   else
       $\rightarrow (x^k, y^k)$
   $F_k^C = F(x^k, y^k)$

4. if integrality requirements are satisfied by (x^k, y^k)
       NEXT
   else
       choose x_{2j}^k, $j \in N^1$ or y_{2j}^k, $j \in N^2$ which is fractional valued
       Place a new bound on the variable
       $k = k + 1$
       update H_k^1, H_k^2, S_k^1 and S_k^2
       go to (2)

5. Set $x = x^k$ and solve $LL(x^k) \rightarrow (x^k, \hat{y}^k)$
   Compute $F(x^k, \hat{y}^k)$
   Set $\bar{F} = \min\{\bar{F}, F(x^k, \hat{y}^k)\}$

6. if $\alpha_j^{1k} = \beta_j^{1k}$ for each $j \in N^1$, and $\alpha_j^{2k} = \beta_j^{2k}$ for each $j \in N^2$
       go to (7) (backtracking)
   else
       select an integer variable such that $\alpha_j^{1k} \neq \beta_j^{1k}$n $j \in N^1$ or $\alpha_j^{2k} \neq \beta_j^{2k}$, $j \in N^2$
       Place new bound
       k=k+1
       update H_k^1, H_k^2, S_k^1 and S_k^2
       go to (2)

7. if no live node exists
   go to (8) (termination)

   else

   branch to the newest live node
   k=k+1
   update H_k^1, H_k^2, S_k^1 and S_k^2
   go to (2)

8. if $\bar{F} = \infty$
   the problem is infeasible

   else

   the current solution is optimal.

2.5 Convex BLPP

Some of the algorithms presented for the linear BLPP can be modified to be used to solve particular instances of Convex BLPP. In particular it is very easy to extend most of the approaches to solve the linear-quadratic case, where the functions F, G and g are linear and the function f is quadratic and strictly convex. The convex version of the BLPP (CBLPP) is given by

$$\begin{array}{lll} \min\limits_{x\geq 0} & F(x,y) & \\ \text{subject to} & G(x,y)\leq 0 & \\ & \min\limits_{y\geq 0} & f(x,y) \\ & \text{subject to} & g(x,y)\leq 0 \end{array}$$

where $F : R^{n+m} \rightarrow R$, $f : R^{n+m} \rightarrow R$ and $G : R^{n+m} \rightarrow R^p$, $g : R^{n+m} \rightarrow R^q$ are continuous, twice differentiable convex functions. As usual, we assume that f is strictly convex, for fixed x. This assures, along with a constraint qualification, that the solution of the follower's problem is unique. This implies that the rational reaction set $P(x)$ is single valued. Thus the inducible region, IR, can be represented by the KKT conditions. Accordingly, CBLPP can be rewritten as the following single level mathematical program

$$\begin{array}{ll}\min\limits_{x,y,u\geq 0} & F(x,y)\\ \text{subject to} & G(x,y)\leq 0\\ & \nabla_y f(x,y)+u^T\nabla_y g(x,y)\leq 0\\ & g(x,y)\leq 0\\ & u^T g(x,y)=0\\ & y^T\left(\nabla_y f(x,y)+u^T\nabla_y g(x,y)\right)=0\\ & x,y,u\geq 0.\end{array}\tag{2.5}$$

Proposition 3 *If $f(x,y)$ is quadratic in (x,y) and the constraint region S is polyhedral, then IR is piecewise linear.*

In fact the solution of the lower level problem occurs either on a face of S or in its interior as x is varied. In the latter case we have that $u=0$ and the stationarity condition becomes $\nabla_y f(x,y)=0$ with $\nabla_y f(x,y)$ linear in x.

Proposition 4 *Let $F(x,y)$ be strictly convex in (x,y), $f(x,y)$ be quadratic in (x,y) and the constraint region S be polyhedral, then if z^1 and z^2 are two distinct local solutions of (2.5) and both lie on the same face of S, then that face cannot be in IR.*

2.5.1 Descent Approaches for the Quadratic BLPP

Two descent algorithms for solving the BLPP in which the upper level objective function F is quadratic, the lower level objective function f and the lower level constraint set is polyhedral are presented here. The first one is based on movements along the inducible region by complementary pivoting in a simplex framework. This procedure converges to a global optimum when F is concave. The second approach is a modification of the steepest descent approach.

Problem Definition and Properties

The quadratic BLPP (Q-BLPP) under consideration has the following form

$$\min_{x\geq 0}\ \frac{1}{2}[x^T,y^T]\begin{bmatrix} C_1 & C_3\\ C_3^T & C_2\end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix}+c_1x+d_1y$$

$$\min_y\ \frac{1}{2}y^TQy+y^TDx+d_2y$$

$$\text{subject to } Ax+By\leq b$$

where $C = \begin{bmatrix} C_1 & C_3 \\ C_3^T & C_2 \end{bmatrix}$ and Q are respectively positive semidefinite and positive definite matrices. This implies that both the leader relaxation (the BLPP without the follower's objective function) and the lower level problem are convex quadratic programs. We also assume, as usual, that the set $P(x)$ is a point-to-point mapping. It is furthermore possible to express the inducible region with the following linear complementarity conditions:

$$Qy + Dx + d_2 + B^T\gamma - \beta = 0 \tag{2.6}$$

$$Ax + By + \alpha = b \tag{2.7}$$

$$x, y, \alpha, \beta, \gamma \geq 0 \tag{2.8}$$

$$\alpha^T\gamma = \beta^T y = 0 \tag{2.9}$$

The point $u = (x, y)$ is said to be in the extreme inducible region (EIR) if there exist α, β and γ such that $(x, y, \alpha, \beta, \gamma)$ is an extreme point of the polyhedral set defined by the previously introduced complementarity system. An EIR point is said to be nondegenerate if the values of the basic variables are positive.
From here on, it is assumed that all points in EIR are nondegenerate. Adjacent EIR points differ in exactly one column. Thus, movement along adjacent EIR points can be achieved with a pivot step that maintains complementarity. The vector $d \in R^{n+m}$ is an EIR direction if it lies along the edge connecting two adjacent points in EIR.

Theorem 13 *Let u be a point in EIR. If u is not a local minimum of Q-BLPP, then there is at least one descent direction at u.*

An EIR Point Descent Algorithm If $(\bar{x}, \bar{y})$ is a nondegenerate EIR point then one of the following situations hold:

- $(\bar{x}, \bar{y})$ is a local minimum of Q-BLPP
- $(\bar{x}, \bar{y})$ is not a local minimum of Q-BLPP and there exist an adjacent EIR point $(\hat{x}, \hat{y})$ and corresponding EIR direction satisfying:

$$F(\hat{x}, \hat{y}) < F(\bar{x}, \bar{y})$$

- $(\bar{x}, \bar{y})$ is not a local minimum of Q-BLPP and

$$F(\hat{x}, \hat{y}) \geq F(\bar{x}, \bar{y})$$

for all adjacent EIR points $(\hat{x}, \hat{y})$. We call $(\bar{x}, \bar{y})$ a *local star inducible region* (LSIR) point. In other words, not all the EIR descent directions decrease the value of the leader's objective function. It is possible to design an algorithm that finds at least a LSIR point for Q-BLPP. The first step is to find an initial point in EIR. At each iteration the current EIR point is either an LSIR point or a local minimizer and the algorithm terminates, or an adjacent EIR point with a lower value of F.

A Modified Steepest Descent Approach

- Direction Search

The aim of this part of the algorithm is to find a descent direction for the leader's objective function which, at the same, time does not violate the active constraints for the lower level (to preserve complementarity conditions) and keeps the stationarity condition at zero. The direction is found as the solution of the problem

$$\begin{array}{lll} \min\limits_{z} & (C_1x^k + C_3y^k + c_1)^T z + (C_3^T x^k + C_2y^k + d_1)^T w & \\ \text{subject to} & -1 \leq z_i \leq 1, \quad i = 1, ..., n & \\ & \min\limits_{w\geq 0} & w^T Qw + 2w^T Dz \\ & \text{subject to} & A'z + B'w \leq 0 \\ & & -\phi^k A'z + (Qy^k + Dx^k + d_2)^T w = 0 \end{array}$$

and then update the solution

$$(x^{k+1}, y^{k+1}) = (x^k, y^k) + \sigma_k(z^k, w^k)$$

with σ_k denoting the appropriate stepsize at iteration k.

At each iteration we look for a direction d which decreases the value of the leader's objective function. This implies that we do not necessarily need to solve the previous problem to optimality: we can stop when we get a descent direction. To accomplish this we need to replace the lower level of the problem with the equivalent KKT conditions and then solve it using a sequential Linear Complementarity Problem method. - Line Search

if the number of binding constraints at a given solution is zero, all the slack variables are positive and all the related lagrangean multipliers are at zero. The constraints of the QBLPP becomes

$$\begin{aligned} &Q(y^k + \sigma_k w^k) + D(x^k + \sigma_k z^k) + d_2 = 0 \\ &A(x^k + \sigma_k z^k) + B(y^k + \sigma_k w^k) \leq b \\ &x^k + \sigma_k z^k \geq 0, y^k + \sigma_k w^k \geq 0 \end{aligned}$$

and σ_k can be computed using the minimum ratio rule.

if the number of binding constraints at a given solution is positive the stepsize has to minimize the leader's objective function holding feasibility, both in the lower level active constraints and in the stationarity condition. This is accomplished by including the stationarity constraints into the lower level constraints and expressing these constraints as an overall system with stepsize and lagrangean multipliers as unknowns.

2.6 General BLPP

The General BLPP is the most challenging of the mathematical programs with two level structure. It has already been mentioned about the issues linked to the fact that the rational reaction set is not necessarily single valued. We describe another property of bilevel programs that is taken for granted when dealing with standard optimization problems. For single-level programs, an optimal solution remains optimal when an inactive constraint (irrelevant) is added to the formulation. Denote the problem

$$\begin{aligned} \min_{x \in X} \quad & F(x,y) \\ \text{subject to} \quad & G(x,y) \le 0 \\ & \min_{y \in Y} \quad f(x,y) \\ & \text{subject to} \quad g(x,y) \le 0 \end{aligned}$$

by $BLP(g)$ and let (x^*, y^*) be its optimal solution. Define the following sets

$$\Delta_g = \{(x,y) : G(x,y) \le 0, g(x,y) \le 0\}$$
$$\Delta_s = \{(x,y) : G(x,y) \le 0, s(x,y) \le 0\}$$

and let us use the symbol $g \cap s$ to refer to $\Delta_g \cap \Delta_s$ A bilevel program $BLP(g)$ is independent of irrelevant constraints (IIC) if its solution (x^*, y^*) is also a solution to the bilevel program $BLP(g \cap s)$ for every set Δ_s containing (x^*, y^*).

Let (x^*, y^*) be a solution of the $BLP(g)$ and suppose that $(x^*, y^*) \in \Delta_s$. Denote the optimal objective function values of $BLP(g)$ and $BLP(g \cap s)$ by F_g^* and F_{gs}^* respectively. Then $(x^*, y^*) \in \Delta_g \cap \Delta_s \subseteq \Delta_g$ so that $F_{gs}^* \le F_g^*$. The point now is that, differently from what happens for a single level feasibility set, the inducible region of the problem for $BLP(g \cap s)$ is not necessarily contained into the inducible region for $BLP(g)$. So it does not happen that $F_{gs}^* \ge F_g^*$.

A bilevel program $BLP(g)$ is degenerate if a solution of its associate single-level problem is feasible to the unconstrained level two problem.

Theorem 14 *A nondegenerate bilevel program is independent of any constraint $s(x,y) \le 0$ iff there is an optimal solution to the associated single level program that is in the inducible region of BLP(g).*

2.6.1 Preview of the Algorithms

The most used algorithms for general BLPPs are based on penalty approach and KKT conditions.

Branch and Bound Algorithm

The following properties are assumed

- f and g are twice continuously differentiable in $y \in S(x)$
- f is strictly convex in $y \in S(x)$
- $S(x)$ is a compact convex set
- F is continuous and convex in x and y

Under these assumptions the rational reaction set is a continuous point-to-point map and it is possible to represent the BLPP through the KKT reformulation for the lower level.

$$\begin{aligned} \min_{x,y,u} \quad & F(x, y) \\ \text{subject to} \quad & G(x, y) \leq 0 \\ & \nabla_y f(x, y) + u\nabla_y g(x, y) = 0 \\ & g(x, y) \leq 0 \\ & ug(x, y) = 0 \\ & u \geq 0. \end{aligned} \tag{2.10}$$

The general idea for the algorithm is to relax the complementarity requirements and then trying to reintroduce them via some sort of enumeration scheme. If the solution violates one or more constraints a combination of depth-first and breadth-first branch and bound is used. With the breadth-first search, one or more of the violated complementarity slackness conditions is selected and two or more subproblems are set up and solved Let n_e be the number of slackness complementarity conditions to satisfy at a given iteration, then 2^{n_e} subproblems will be set up and solved.

Steepest Descent Direction This approach is similar to the one introduced for the case of CBLPP. We assume unicity of the lower level solution for a given upper level decision, linear independence of the gradients of the constraints for the lower level and that the Hessian matrix of the Lagrangian function related to the lower level problem is positive definite. Under these assumptions we have

Theorem 15 *Let $(x^*, y(x^*))$ be an optimal solution for the general BLPP. Then for any upper level direction $d \in R^n$ at x^* the directional derivative of the objective* *function of the upper level problem satisfies*

$$F'(x^*, y(x^*); d) = \nabla_x F(x^*, y(x^*))d + \nabla_y F(x^*, y(x^*))w(x^*; d) \geq 0 \tag{2.11}$$

where $w(x, d)$ *is the optimal solution for* $x = x^*$ *of the quadratic program (QP)*

$$\begin{array}{lll} \min\limits_{x \geq 0} & [d^T, w^T]\nabla^2_{xy}\mathcal{L}(x, y(x), u(x)) \begin{bmatrix} d \\ w \end{bmatrix} & \\ \text{subject to} & \nabla_y g_i(x, y(x))w \leq -\nabla_x g_i(x, y(x))d & i \in I(x) \\ & \nabla_y g_i(x, y(x))w \leq -\nabla_x g_i(x, y(x))d & i \in J \\ & \nabla_y f_i(x, y(x))w \leq -\nabla_x f_i(x, y(x))d + \nabla_x \mathcal{L}(x, y(x), u(x))d & \end{array}$$

2.7 Conclusions

As evidenced in this survey, bilevel programming is the subject of important research efforts from the mathematical programming and operations research communities. Many classes of bilevel programs now have dedicated solution algorithms, and researchers have started to study more complicated instances, like bilevel programs with integer variables or without derivatives, which to our view is an indication that some maturity has been reached in the field. It is nevertheless the case that challenges remain to be tackled, in particular concerning nonlinear bilevel problems. Besides the improvement of existing methods and derivation of proper convergence results, our feeling is that a promising approach would be to develop tools similar to those by Scholtes (2002) allowing to take advantage of the inherent combinatorial structure of bilevel problems. These ideas, combined with well-tried tools from nonlinear programming like sequential quadratic programming, should allow the development of a new generation of solution methods.

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Chapter 3

On the reformulation of a class of bilevel programs

3.1 introduction

One tool often used to reformulate the optimistic bilevel programming problem as an one-level are the Karush-Kuhn-Tucker conditions. If a regularity condition is satisfied for the lower level problem, then the Karush-Kuhn-Tucker conditions are necessary optimality conditions. They are also sufficient in the case when the lower level is a convex optimization problem in the x variables for fixed parameters y. This suggests to replace problem

$$\begin{array}{lll} \min\limits_{x\geq 0} & F(x,y) & \\ \text{subject to} & G(x,y)\leq b_1 & \\ & \min\limits_{y\geq 0} & f(x,y) \\ & \text{subject to} & g(x,y)\leq b_2 \end{array} \tag{3.1}$$

by the problem

$$\begin{array}{ll} \min\limits_{x,y,u} & F(x,y) \\ \text{subject to} & G(x,y)\leq b_1 \\ & \nabla_y f(x,y) + \left[\nabla_y g(x,y)\right]^T u \geq 0 \\ & g(x,y)\leq b_2 \\ & u^T g(x,y) = 0 \\ & y^T\left(\nabla_y f(x,y) + u^T \nabla_y g(x,y)\right) = 0 \\ & x,y,u\geq 0. \end{array} \tag{3.2}$$

Even under suitable convexity assumptions on the functions F, G and the constraints set, the above mathematical program is not easy to solve, due mainly to the nonconvexities that occur in the complementarity and Lagrangean constraints. While the Lagrangean constraint is linear in certain important cases (linear or convex quadratic functions), the complementarity constraint is intrinsically combinatorial, and is best addressed by enumeration algorithms, such as branch-and-bound, possibly performed in a smart way, without introducing a new set of binary variables, as in the Bard-Moore algorithm introduced in the previous chapter. We introduce a reformulation of a class of bilevel programs having convex lower level with linear or quadratic constraints Under the assumption of *optimistic approach*, and assuming that, for each fixed x, the lower level problem is a convex optimization problem, each local optimal solution for the problem (3.1) corresponds to a local optimal solution for problem (3.2).

3.2 Linear Lower Level

The basic reformulation case is provided by the case in which the lower level problem is linear. As we will see, the method does not necessarily overperform a branching scheme on the complementarity constraints done directly on the KKT reformulation, but it lays the foundations on the way to eliminate the complementarity constraints through the use of an equivalent formulation for the lower level optimality conditions. Incidentally, the method will deploy the inner link between optimality conditions and duality theory in linear programming. Let us consider the following linear bilevel optimization problem

$$\begin{array}{lll} \min\limits_{x\geq 0} & F(x,y) & \\ \text{subject to} & G(x,y) \leq b_1 & \\ & \min\limits_{y\geq 0} & c_2x + d_2y \\ & \text{subject to} & A_2x + B_2y \leq b_2 \end{array} \tag{3.3}$$

where $c_1, c_2 \in R^n$, $d_1, d_2 \in R^m$, $b_1 \in R^p$, $b_2 \in R^q$, $A_1 \in R^{p\times n}$, $B_1 \in R^{p\times m}$, $A_2 \in R^{q\times n}$, $B_2 \in R^{q\times m}$, and, in particular, let us consider the lower level problem for a fixed value of the upper level decision variables, denoted by

$$\begin{array}{ll} \min\limits_{y\geq 0} & c_2x + d_2y \\ \text{subject to} & B_2y \leq b_2 - A_2x \end{array}$$

The KKT conditions attached to such problem are defined by the system

$$
\begin{aligned}
& d_2 + B_2^T \lambda \geq 0 \\
& B_2 y \leq b_2 - A_2 x \\
& \lambda^T \left(B_2 y - b_2 + A_2 x \right) = 0 \\
& y^T \left(d_2 + B_2^T y \right) = 0 \\
& y, \lambda \geq 0
\end{aligned}
$$

and the first inequality is equivalently expressed by $d_2^T y \geq -\lambda^T B_2 y$ given the fact that $y \geq 0$. Moreover, considering as well the second constraint we have the double inequality

$$
d_2^T y \geq -\lambda^T B_2 y \geq -\lambda^T b_2 + \lambda^T A_2 x. \tag{3.4}
$$

It is then possible to exclude the complementarity constraints from the KKT system by forcing the equality

$$
d_2^T y = -\lambda^T b_2 + \lambda^T A_2 x
$$

which, as a consequence of 3.4 will imply the satisfaction of both the complementarity constraints found in the original KKT system.

We can now reformulate the optimality conditions of the lower level, parametrized by the upper level variables as the nonlinear system

$$
\begin{aligned}
& d_2^T y + \lambda^T b_2 - \lambda^T A_2 x = 0 \\
& d_2 + B_2^T \lambda \geq 0 \\
& B_2 y \leq b_2 - A_2 x \\
& y, \lambda \geq 0
\end{aligned}
$$

And, consequently, we can restate the problem (3.3) into the ordinary mathematical problem

$$
\begin{array}{ll}
\min\limits_{x,y,\lambda \geq 0} & F(x,y) \\
\text{subject to} & d_2^T y + \lambda^T b_2 - \lambda^T A_2 x = 0 \\
& G(x,y) \leq b_1 \\
& A_2 x + B_2 y \leq b_2 \\
& B_2^T \lambda \geq -d_2
\end{array} \tag{3.5}
$$

which is equivalent to (3.3) since it satisfies the KKT conditions for the lower level problem, but it does not involve complementarity constraints.

In general, if we assume to have a linear programming problem of the form

$$\begin{aligned}\max_{x\geq 0} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b\end{aligned}$$

and considering its dual problem, given by

$$\begin{aligned}\min_{y\geq 0} \quad & b^T y \\ \text{subject to} \quad & A^T y \geq c\end{aligned}$$

we can restate the optimization problem as the following system of optimality conditions

$$\begin{aligned}& c^T x - b^T y = 0 \\ & Ax \leq b \\ & A^T y \geq c \\ & x, y \geq 0\end{aligned}$$

with y denoting the dual variables of the LP. Notice that in this case, the reformulation of the KKT conditions are a well known fact in mathematical programming: the LP has an optimal solution when both the primal and the dual constraints are satisfied and, at the same time, the duality gap is zero. Thus, when the lower level of a bilevel program is linear for a given choice of the upper level decision variables, it can be reformulated by bundling the primal and the dual constraints of the lower level linear problem and requiring, at the same time, the duality gap to be zero.

Observe that, in the case of bilevel linear problems as defined in (3.3) with the additional assumption that $F(x, y) = c_1x + d_1y$ and $G(x, y) = A_1x + B_1y$ it is possible to cast the bilevel problem onto an ordinary mixed integer linear problem by converting each complementarity constraint $\lambda_i g_i = 0$ with $\lambda \geq 0$ and $g_i \geq 0$ into

$$\begin{aligned}& \lambda_i \leq M z_i \\ & g_i \leq M(1 - z_i) \\ & \lambda \geq 0 \\ & g_i \geq 0 \\ & z_i \in \{0, 1\}\end{aligned}$$

to obtain a single level equivalent problem

$$
\begin{aligned}
\min \quad & c_1 x + d_1 y \\
\text{subject to} \quad & A_1 x + B_1 y \leq b_1 \\
& A_2 x + B_2 y \leq b_2 \\
& B_2^T \lambda \geq -d_2 \\
& \lambda \leq M_1 z_1 \\
& A_2 x + B_2 y - b_2 \leq M_1 (e - z_1) \\
& y \leq M_2 z_2 \\
& B_2^T + d_2 \leq M_2 (e - z_2) \\
& x, y, \lambda \geq 0 \\
& z_1, z_2 \text{ binary}
\end{aligned}
\tag{3.6}
$$

with $e^T = \{1, 1, ..., 1\}$ and M_1, M_2 are large scalars. Reformulation (3.6) can be solved to global optimality, meanwhile, using reformulation (3.5) one can only seek for local optima.

Let us introduce an example to show how a linear bilevel problem is reformulated as an ordinary mathematical program substituting complementarity constraints with the requirement that duality gap is zero.

The test problem for the linear bilevel case is taken from Bard and it is reported as follows

$$
\begin{aligned}
\min_{x \geq 0} \quad & -8x_1 - 4x_2 + 4y_1 - 40y_2 - 4y_3 \\
\text{subject to} \quad & \min_{y \geq 0} \quad x_1 + 2x_2 + y_1 + y_2 + 2y_3 \\
& \text{subject to} \quad -y_1 + y_2 + y_3 \leq 1 \\
& \qquad 2x_1 - y_1 + 2y_2 - 0.5y_3 \leq 1 \\
& \qquad 2x_2 + 2y_1 - y_2 - 0.5y_3 \leq 1
\end{aligned}
$$

MPEC reformulation

$$\begin{aligned}
\min \quad & -8x_1 - 4x_2 + 4y_1 - 40y_2 - 4y_3 \\
\text{subject to} \quad & y_1 - y_2 - y_3 \geq -1 \\
& -2x_1 + y_1 - 2y_2 + 0.5y_3 \geq -1 \\
& -2x_2 - 2y_1 + y_2 + 0.5y_3 \geq -1 \\
& 1 - \lambda_1 - \lambda_2 + 2\lambda_3 \geq 0 \\
& 1 + \lambda_1 + 2\lambda_2 - \lambda_3 \geq 0 \\
& 2 + \lambda_1 - 0.5\lambda_2 - 0.5\lambda_3 \geq 0 \\
& \lambda_1 (y_1 - y_2 - y_3 + 1) = 0 \\
& \lambda_2 (-2x_1 + y_1 - 2y_2 + 0.5y_3 + 1) = 0 \\
& \lambda_3 (-2x_2 - 2y_1 + y_2 + 0.5y_3 + 1) = 0 \\
& y_1 (1 - \lambda_1 - \lambda_2 + 2\lambda_3) = 0 \\
& y_2 (1 + \lambda_1 + 2\lambda_2 - \lambda_3) = 0 \\
& y_3 (2 + \lambda_1 - 0.5\lambda_2 - 0.5\lambda_3) = 0 \\
& x_1, x_2, y_1, y_2, y_3, \lambda_1, \lambda_2, \lambda_3 \geq 0
\end{aligned}$$

The duality gap reformulation is given by the following mathematical program

$$\begin{aligned}
\min \quad & -8x_1 - 4x_2 + 4y_1 - 40y_2 - 4y_3 \\
\text{subject to} \quad & y_1 + y_2 + 2y_3 - \lambda_1 - (1 - 2x_1)\lambda_2 - (1 - 2x_2)\lambda_3 = 0 \\
& y_1 - y_2 - y_3 \geq -1 \\
& -2x_1 + y_1 - 2y_2 + 0.5y_3 \geq -1 \\
& -2x_2 - 2y_1 + y_2 + 0.5y_3 \geq -1 \\
& 1 - \lambda_1 - \lambda_2 + 2\lambda_3 \geq 0 \\
& 1 + \lambda_1 + 2\lambda_2 - \lambda_3 \geq 0 \\
& 2 + \lambda_1 - 0.5\lambda_2 - 0.5\lambda_3 \geq 0 \\
& x_1, x_2, y_1, y_2, y_3, \lambda_1, \lambda_2, \lambda_3 \geq 0
\end{aligned}$$

In this case, solving the problem by means of nonlinear optimization techniques deliver the same result, whether using the MPEC formulation or the duality gap formulation. Namely, we have that $x^* = (0.5, 0.5)$, $y^* = (0, 0, 0)$ and $F^* = -6$. When the MPEC formulation is solved using specialized algorithms, such as Bard-Moore, or recasting the MPEC into a mixed integer linear program, global optimum is reached. The optimal solution is $x^* = (0, 0.9)$, $y^* = (0, 0.6, 0.4)$ and $F^* = -29$.

Nevertheless, besides pure linear bilevel case, it is not normally possible to reformulate the problem obtaining treatable structures such as the mixed integer linear programming formulation. Normally, bilevel problems feature a nonconvex feasible set even if the lower level is a convex problem and the upper level only entails convex functions. Let us now look at a case with linear lower level, but with a quite different structure than the pure linear bilevel problem. We consider the case with the upper level variables are the coefficients of the objective function for the lower level problem. Such problems are referred to as bilinear bilevel problems. Such problems are introduced and studied by Marcotte, Savard, Labbé, and their use is naturally required to model a wide range of problems, such as network models with toll setting problems.

As an example of such problem, here we propose a game between a regulator setting a tax x on a particular component y_1 used together with a second component y_2 to bundle a product to be distributed to end users. We assume that part of the tax is payed by the producer and part of the tax is payed by the end users. We assume revenues for the firm bundling the product are fixed and stemming from a public auction, to have the rights to distribute the particular product. The producer seeks to minimize total costs from the production and the delivery of the product

The producer problem is thus given by

$$\begin{array}{ll} \min & (c_1 + \alpha x)^T y_1 + c_2 y_2 \\ \text{subject to} & A_1 y_1 + A_2 y_2 \geq b - Bx \\ & y_1, y_2 \geq 0 \end{array}$$

where the term $b - Bx$ could be considered as a sort of demand function depending on the part of tax level x to be payed by the end user, while α is the proportion of tax to be payed by the producer.

The problem for the regulator is thus given by the following bilevel mathematical program

$$\begin{array}{lll} \max\limits_{x \geq 0} & x^T y_1 & \\ \text{subject to} & \min & (c_1 + \alpha x)^T y_1 + c_2 y_2 \\ & \text{subject to} & A_1 y_1 + A_2 y_2 \geq b - Bx \\ & & y_1, y_2 \geq 0 \end{array} \tag{3.7}$$

which corresponds to a Stackelberg-type game where the regulator takes into account the optimal response of the producer in terms of quantity produced as the tax level changes. Labbé et al. consider a similar problem, but without upper level decision variables into the lower level constraints. This allows them to reformulate the problem as a mixed integer linear problem using duality theory to find a linear equivalent for the bilinear objective functions. Here the presence of upper level problem variables in the constraints of the lower level problem hinders the possibility to use a mixed integer linear program equivalent to (3.7) and dedicated algorithms must be used to cope with nonlinearities provided by the complementarity constraints stemming by the reformulation of the lower level problem using KKT conditions. It is anyway possible to simplify the problem avoiding the complementarity constraints linked to KKT conditions using the same approach used for the pure linear case (3.3). Let us first write the KKT conditions related to the producer problem

$$\begin{aligned}
&c_1 + \alpha x - A_1^T \lambda \geq 0 \\
&c_2 - A_2^T \lambda \geq 0 \\
&A_1 y_1 + A_2 y_2 \geq b - Bx \\
&\lambda^T \left(A_1 y_1 + A_2 y_2 + Bx - b\right) = 0 \\
&y_1^T \left(c_1 + \alpha x - A_1^T \lambda\right) = 0 \\
&y_2^T \left(c_2 - A_2^T \lambda\right) = 0 \\
&y_1, y_2, \lambda \geq 0
\end{aligned}$$

and use the first three inequalities so to obtain

$$(c_1 + \alpha x)^T y_1 + c_2 y_2 \geq \lambda^T \left(A_1 y_1 + A_2 y_2\right) \geq (b - Bx)^T \lambda$$

Then we have that imposing the equality

$$(c_1 + \alpha x)^T y_1 + c_2 y_2 = (b - Bx)^T \lambda$$

we automatically satisfy the complementarity constraints related to the problem. Thus solving problem (3.7) is equivalent to finding the solution of the following:

$$\begin{aligned}
\max_{x, y_1, y_2, \lambda} \quad & x^T y_1 \\
\text{subject to} \quad & (c_1 + \alpha x)^T y_1 + c_2 y_2 - (b - Bx)^T \lambda = 0 \\
& c_1 + \alpha x - A_1^T \lambda \geq 0 \\
& c_2 - A_2^T \lambda \geq 0 \\
& A_1 y_1 + A_2 y_2 + Bx \geq b \\
& x, y_1, y_2, \lambda \geq 0
\end{aligned}$$

which can be solved using commercial software for nonlinear problems without having to take care of complementarities.

3.3 Quadratically constrained linear lower level

The method introduced can be used also when we have a bilevel problem whose lower level contains quadratic constraints. In particular, the reformulation works in all cases in which the lower level constraints have the property $kg(x,y) = \nabla_y g(x,y)y$, with $k \in \Re$. We consider a bilevel program having in the lower level the following quadratic constraint

$$\begin{array}{lll} \min\limits_{x\geq 0} & F(x,y) & \\ \text{subject to} & G(x,y) \leq b_1 & \\ & \min\limits_{y\geq 0} & c_2x + d_2y \\ & \text{subject to} & A_2x + B_2y \leq b_2 \\ & & \frac{1}{2}[x^T, y^T]\begin{bmatrix} C_1 & C_3 \\ C_3^T & C_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + c_3x + d_3y \leq b_3 \end{array} \tag{3.8}$$

where the quadratic constraint on the lower level problem can be rewritten as $\frac{1}{2}\left(x^TC_1x + 2x^TC_3y + y^TC_2y\right) + c_3x + d_3y \leq b_3$ and the KKT conditions for the lower level are given by

$$\begin{aligned} & d_2 + B_2^T\lambda_1 + C_3^Tx\lambda_2 + C_2y\lambda_2 + d_3\lambda_2 \geq 0 \\ & A_2x + B_2y \leq b_2 \\ & \frac{1}{2}\left(x^TC_1x + 2x^TC_3y + y^TC_2y\right) + c_3x + d_3y \leq b_3 \\ & \lambda_1^T\left(A_2x + B_2y - b_2\right) = 0 \\ & \lambda_2\left[\frac{1}{2}\left(x^TC_1x + 2x^TC_3y + y^TC_2y\right) + c_3x + d_3y - b_3\right] = 0 \\ & y^T\left(d_2 + B_2^T\lambda_1 + C_3^Tx\lambda_2 + C_2y\lambda_2 + d_3\lambda_2\right) = 0 \\ & y, \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

As we did in the previous case, we can consider the first inequality of the system defining the KKT conditions for the lower level problem to obtain,

$$y^Td_2 \geq -y^TB_2^T\lambda_1 - y^TC_3^Tx\lambda_2 - y^TC_2y\lambda_2 - y^Td_3\lambda_2 \tag{3.9}$$

and from the second and third inequality we have

$$\begin{aligned} & -y^TB_2^T\lambda_1 - y^TC_3^Tx\lambda_2 - y^TC_2y\lambda_2 - y^Td_3\lambda_2 \geq \\ & -b_2^T\lambda_1 + x^TA^T\lambda_1 - 2b_3\lambda_2 + 2c_3^Tx\lambda_2 + x^TC_1x\lambda_2 + y^TC_3^Tx\lambda_2 + y^Td_3\lambda_2. \end{aligned} \tag{3.10}$$

Now, by imposing the equality

$$y^T d_2 = -b_2^T \lambda_1 + x^T A^T \lambda_1 - 2b_3\lambda_2 + 2c_3^T x\lambda_2 + x^T C_1 x\lambda_2 + y^T C_3^T x\lambda_2 + y^T d_3\lambda_2$$

we automatically require (3.9) and (3.10) to be satisfied by equalities. In particular, by satisfying (3.9) by equality we force the last complementarity constraint of the previous system of KKT conditions to hold, meanwhile by satisfying 3.10 with equality we have that

$$\lambda_2 \left[\left(x^T C_1 x + 2x^T C_3 y + y^T C_2 y \right) + 2c_3 x + 2d_3 y - 2b_3 \right] = \lambda_1^T \left(-A_2 x - B_2 y + b_2 \right)$$

with the first term being nonpositive and the second term being nonnegative. Such equality is therefore satisfied only when both terms equal zero, i.e. when we have

$$\begin{aligned} &\lambda_1^T \left(A_2 x + B_2 y - b_2 \right) = 0 \\ &\lambda_2 \left[\frac{1}{2} \left(x^T C_1 x + 2x^T C_3 y + y^T C_2 y \right) + c_3 x + d_3 y - b_3 \right] = 0 \end{aligned}$$

Thus the optimal solution of the nonlinear problem

$$\begin{aligned} \min_{x \geq 0} \quad & F(x,y) \\ \text{subject to} \quad & d_2^T y + b_2^T \lambda_1 - Ax\lambda_1 + 2b_3\lambda_2 - 2c_3^T x\lambda_2 - x^T C_1 x\lambda_2 - y^T C_3^T x\lambda_2 - d_3^T y\lambda_2 = 0 \\ & G(x,y) \leq b_1 \\ & A_2 x + B_2 y \leq b_2 \\ & d_2 + B_2^T \lambda_1 + C_3^T x\lambda_2 + C_2 y\lambda_2 + d_3\lambda_2 \geq 0 \\ & \frac{1}{2} \left(x^T C_1 x + 2x^T C_3 y + y^T C_2 y \right) + c_3 x + d_3 y \leq b_3 \end{aligned}$$

is the same as the optimal solution of problem (3.8). More generally, consider the bilevel program with quadratic constraints in the lower level, and with the lower level depending, in some way by the upper level decisions as the following one

$$\begin{aligned} \min_{x \geq 0} \quad & F(x,y) \\ \text{subject to} \quad & G(x,y) \leq b_1 \\ & \min_{y \geq 0} \quad d_2(x)y \\ & \text{subject to} \quad B_2(x)y \leq b_2 \\ & \qquad\qquad\quad\; y^T Q(x)y \leq b_3 \end{aligned} \tag{3.11}$$

where $d_2(x)$, $B_2(x)$ are respectively a vector and a matrix of proper dimensions parametrized by the upper level variables, while $Q(x)$ is a symmetric semi-positive definite matrix parametrized by the upper level variables.
We can follow the reasoning done for the previous case to formulate a problem equivalent to (3.12), in the sense that they have the same (global) optimal solution. The equivalent problem is as follows:

$$\begin{aligned}
\min_{x,y,\lambda_1,\lambda_2} \quad & F(x,y) \\
\text{subject to} \quad & d_2(x)y + b_1^T\lambda_1 + 2b_2\lambda_2 = 0 \\
& G(x,y) \leq b_1 \\
& B_2(x)y \leq b_2 \\
& y^T Q(x) y \leq b_3 \\
& d_2(x) + B_2(x)^T\lambda_1 + 2Q(x)y\lambda_2 \geq 0 \\
& x, y, \lambda_1, \lambda_2 \geq 0
\end{aligned}$$

3.4 Quadratic Lower Level

Let us now consider the case when the lower level is a quadratic problem with a set linear constraints. Using the same arguments seen in the previous sections of the chapter the results can easily be extended to the case of a quadratic lower level problem with quadratic constraints. The problem reads as follows:

$$\begin{aligned}
\min_{x\geq 0} \quad & F(x,y) \\
\text{subject to} \quad & G(x,y) \leq b_1 \\
& \min_{y\geq 0} \quad \frac{1}{2}[x^T, y^T] \begin{bmatrix} C_1 & C_3 \\ C_3^T & C_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + c_3 x + d_3 y \\
& \text{subject to} \quad A_2 x + B_2 y \leq b_2
\end{aligned} \tag{3.12}$$

The KKT conditions for the lower level problem are formulated through the system

$$\begin{aligned}
& C_3^T x + C_2 y + d_3 + B_2^T \lambda \geq 0 \\
& A_2 x + B_2 y \leq b_2 \\
& \lambda^T (A_2 x + B_2 y - b) = 0 \\
& y^T \left(C_3^T x + C_2 y + d_3 + B_2^T \lambda \right) = 0 \\
& y, \lambda \geq 0
\end{aligned}$$

and the first and second inequalities yield

$$y^T C_3^T x + y^T C_2 y + y^T d_3 \geq -y^T B_2^T \lambda \geq -b_2^T \lambda + x^T A_2^T \lambda$$

Thus, by imposing the equality

$$y^T C_3^T x + y^T C_2 y + y^T d_3 = -b_2^T \lambda + x^T A_2^T \lambda$$

the complementarity constraints for the KKT conditions referring to the lower level of the bilevel problem (3.12) are satisfied. Therefore problem (3.12) and the following problem

$$\begin{array}{ll} \min\limits_{x,y,\lambda} & F(x,y) \\ \text{subject to} & G(x,y) \leq b_1 \\ & y^T C_3^T x + y^T C_2 y + y^T d_3 + b_2^T \lambda - x^T A_2^T \lambda = 0 \\ & C_3^T x + C_2 y + d_3 + B_2^T \lambda \geq 0 \\ & A_2 x + B_2 y \leq b_2 \\ & x, y, \lambda \geq 0 \end{array} \tag{3.13}$$

share the same global optimal solution.

For what concerns the quadratic case, the test problem is given by the following bilevel program with convex quadratic objective function at the lower level problem.

$$\begin{array}{lll} \min\limits_{x \geq 0} & (x-5)^2 + (2y-1)^2 & \\ \text{subject to} & \min\limits_{y \geq 0} & (y-1)^2 + 1.5xy \\ & \text{subject to} & -3x + y \leq -3 \\ & & x - 0.5y \leq 4 \\ & & x + y \leq 7 \end{array}$$

The MPEC reformulation for such a bilevel problem is given by

$$\begin{aligned}
\min \quad & (x-5)^2 + (2y-1)^2 \\
\text{subject to} \quad & 2y - 1.5x + \lambda_1 - 0.5\lambda_2 + \lambda_3 \geq 2 \\
& -3x + y \leq -3 \\
& x - 0.5y \leq 4 \\
& x + y \leq 7 \\
& \lambda_1 (-3x + y + 3) = 0 \\
& \lambda_2 (x - 0.5y - 4) = 0 \\
& \lambda_3 (x + y - 7) = 0 \\
& y (2y - 1.5x + \lambda_1 - 0.5\lambda_2 + \lambda_3 - 2) = 0 \\
& x_1, x_2, y_1, y_2, y_3, \lambda_1, \lambda_2, \lambda_3 \geq 0
\end{aligned}$$

while, with the duality gap reformulation we get

$$\begin{aligned}
\min \quad & (x-5)^2 + (2y-1)^2 \\
\text{subject to} \quad & -2y^2 + 1.5xy + 2y + 3\lambda_1 - 4\lambda_2 - 7\lambda_3 - 3x\lambda_1 + x\lambda_2 + x\lambda_3 = 0 \\
& 2y - 1.5x + \lambda_1 - 0.5\lambda_2 + \lambda_3 \geq 2 \\
& -3x + y \leq -3 \\
& x - 0.5y \leq 4 \\
& x + y \leq 7 \\
& x_1, x_2, y_1, y_2, y_3, \lambda_1, \lambda_2, \lambda_3 \geq 0
\end{aligned}$$

Similarly to what happened in the linear case, the results delivered under this case are the same, corresponding to $x = 1.27027$, $y = 0.810811$ and $F = 14.2972973$.

3.5 Generalization

We turn our attention to the case of a bilevel program of type (3.1) with linear or quadratic constraints and generalize our result through the following

Theorem 16 *Let $f(x,y) : \Re^{n\times m} \to \Re$ be and $g(x,y) : \Re^{n\times m} \to \Re^q$ be convex functions on y. Assume also that $kg(x,y) = \nabla_y g(x,y)y$, with $k \in \Re$. Then (3.1)* *and problem*

$$\begin{aligned}
\min_{x,y,\lambda} \quad & F(x,y) \\
\text{subject to} \quad & G(x,y) \leq b_1 \\
& g(x,y) \leq b_2 \\
& \nabla_y f(x,y) + \left[\nabla_y g(x,y)\right]^T \lambda \geq 0 \\
& \left[\nabla_y f(x,y)\right]^T y + k b_2^T \lambda = 0 \\
& x, y, \lambda \geq 0
\end{aligned} \tag{3.14}$$

have the same global optimum.

Proof If lower level is convex KKT conditions are necessary and sufficient for follower's reaction to be optimal (rational). Thus we can restate (3.1) by appending to the upper level problem the KKT conditions related to the lower level problem. These conditions are as follows

$$\begin{aligned}
\nabla_y f(x,y) + \left[\nabla_y g(x,y)\right]^T \lambda \geq 0 \\
g(x,y) \leq b_2 \\
\lambda^T \left[g(x,y) - b_2\right] = 0 \\
y^T \left(\nabla_y f(x,y) + \left[\nabla_y g(x,y)\right]^T \lambda\right) = 0 \\
x, y, \lambda \geq 0.
\end{aligned} \tag{3.15}$$

From the first and the second inequalities results

$$\nabla_y f(x,y)^T y \geq -\lambda^T \nabla_y g(x,y) y$$

and from the assumption $kg(x,y) = \nabla_y g(x,y) y$ we further have

$$\nabla_y f(x,y)^T y \geq -\lambda^T \nabla_y g(x,y) y = -kg(x,y) \geq -k b_2^T \lambda.$$

Now, by forcing the equality

$$\nabla_y f(x,y)^T y + k b_2^T \lambda = 0$$

we automatically satisfy the complementarities

$$\lambda^T \left[g(x,y) - b_2\right] = 0$$

and

$$y^T \left(\nabla_y f(x,y) + \left[\nabla_y g(x,y)\right]^T \lambda\right) = 0$$

Q.E.D.

Chapter 4

A Stochastic Coordination Model Implementation

4.1 Introduction

The model [1] is composed of two levels interacting in a hierarchical way. Service Provider Problem (lower level) is an allocation model: it defines the amount of Service i to be supplied to Service Portfolio, or to External Service j for a given deal on the revenue sharing scheme between Service Providers. Platform Operator Problem (upper level) is a coordination model: How to define the optimal sharing scheme pushing each Service Provider to supply the necessary amount of their service to each Platform Service Portfolio and delivering, at the same time, the highest possible return on investment on the actor taking the further role of Platform Operator. The core of the work is the reformulation of the Lower Level Problem for the model in Gaivoronski et al. and the formulation of the overall bilevel problem with multiple followers as a regular, single level nonlinear problem.

4.2 Model formulation

The following notation is used throughout the model:

SETS

- $\mathcal{P}$ set of Platform Service Portfolios
- $\mathcal{E}$ set of External Services
- $\mathcal{I}_j$ set of Service Providers involved in collaborative provision of Service Portfolio j

PARAMETERS

- v_j Expected unit revenue for Platform Service Portfolio j

- v_{ij} Expected unit revenue for External Service j for Service Provider i
- λ_{ij} Amount of service i needed to deliver one unit of Service Portfolio j
- c_i Operating cost for providing one unit of Service i
- σ_{hk} Covariance between Platform Portfolio services h and k
- σ_{ihk} Covariance between Services h and k (Platform/External or External/External) for Service Provider i
- $\bar{R}_i$ Maximum risk level that Service Provider i is willing to accept
- $\bar{r}_i$ Minimum return level that Service Provider i is willing to accept
- B_j Target Supply for Service Portfolio j
- W_i Provision capability for Service Provider i

VARIABLES

- x_{ij} Fraction of capability W_i that Service Provider i supplies to Service Portfolio j
- γ_{ij} Share of revenues v_j from Platform Service Portfolio j accorded to Service Provider i

Service Provider

$$
\begin{aligned}
\underset{x_{ij}}{\text{maximize}} \quad & \sum_{j\in\mathcal{P}} \left(\frac{\gamma_{ij}v_j}{\lambda_{ij}c_i} - 1\right) x_{ij} + \sum_{j\in\mathcal{E}} \left(\frac{v_{ij}}{c_i} - 1\right) x_{ij} \\
\text{subject to} \quad & \frac{1}{c_i^2}\left(\sum_{h\in\mathcal{P}}\sum_{k\in\mathcal{P}} \frac{\gamma_{ih}}{\lambda_{ih}}\frac{\gamma_{ik}}{\lambda_{ik}}\sigma_{hk}x_{ih}x_{ik} + 2\sum_{h\in\mathcal{P}}\sum_{k\in\mathcal{E}} \frac{\gamma_{ih}}{\lambda_{ih}}\sigma_{ihk}x_{ih}x_{ik} + \sum_{h\in\mathcal{E}}\sum_{k\in\mathcal{E}} \sigma_{ihk}x_{ih}x_{ik}\right) \le \bar{R}_i \\
& \sum_{j\in\mathcal{P}} \left(\frac{\gamma_{ij}v_j}{\lambda_{ij}c_i} - 1\right) x_{ij} + \sum_{j\in\mathcal{E}} \left(\frac{v_{ij}}{c_i} - 1\right) x_{ij} \ge \bar{r}_i \\
& \sum_{j\in\mathcal{P}\cup\mathcal{E}} x_{ij} = 1 \\
& x_{ij} \ge 0 \qquad j \in \mathcal{P}\cup\mathcal{E}
\end{aligned}
$$

Platform Operator

$$
\begin{aligned}
\underset{\gamma_{ij}}{\text{maximize}} \quad & \sum_{j \in \mathcal{P}} \left(\frac{\gamma_{1j} v_j}{\lambda_{1j} c_1} - 1 \right) x_{1j} + \sum_{j \in \mathcal{E}} \left(\frac{v_{1j}}{c_1} - 1 \right) x_{1j} \\
\text{subject to} \quad & x_{ij} \geq \frac{B_j \lambda_{ij}}{W_i} \qquad i \in \mathcal{I}_j \quad j \in \mathcal{P} \\
& \sum_{i \in \mathcal{I}_j} \gamma_{ij} = 1 \quad j \in \mathcal{P} \\
& \gamma_{ij} \geq 0 \qquad i \in \mathcal{I}_j \quad j \in \mathcal{P}
\end{aligned}
$$

4.3 Model Properties

The optimal solution of the mathematical program is obtained by first reformulating the Service Provider Problem into its equivalent Karush-Kuhn-Tucker conditions. Solutions to KKT conditions deliver a globally optimal solution to the mathematical problem provided that such problem is convex. So we first show that the each Service Provider Problem is, in fact, convex for every choice of revenue sharing scheme vector. Let us first suppose that no sharing scheme are involved in the portfolio allocation [2], i.e., the i-th provider is the sole supplier of all the platform and external services. The risk constraint would look like the following:

$$\mathbf{x_i}^{\mathbf{T}} \mathbf{Q_i} \mathbf{x_i} \leq \bar{R}_i$$

where the matrix $\mathbf{Q_i}$ is given by the following blocks

$$\mathbf{Q_i} = \begin{pmatrix} \mathbf{Q_p} & \mathbf{Q_{pe}} \\ \mathbf{Q_{pe}^T} & \mathbf{Q_e} \end{pmatrix}$$

Moreover, since $\mathbf{Q_i}$ is a covariance matrix, it is built in the following way

$$\mathbf{Q_i} = \begin{pmatrix} \mathbf{A_p^T} \\ \mathbf{A_e^T} \end{pmatrix} \begin{pmatrix} \mathbf{A_p} & \mathbf{A_e} \end{pmatrix} = \mathbf{A^T A}$$

which implies that $\mathbf{x^T Q_i x} = \mathbf{x^T A^T A x} = \|\mathbf{Ax}\|^2 \geq 0$. Thus the covariance matrix is positive semidefinite [**?**]. Let us now consider the introduction of a vector of revenue sharing scheme $\lambda_\mathbf{i}$ for the set of the platform services. The covariance matrix becomes

$$\mathbf{Q_{\lambda i}} = \begin{pmatrix} \mathbf{D Q_p D} & \mathbf{D Q_{pe}} \\ \mathbf{Q_{pe}^T D} & \mathbf{Q_e} \end{pmatrix}$$

with $\mathbf{D} = diag(\lambda_\mathbf{i})$. Now, from the definition of $\mathbf{Q_i}$ we have that

$$\mathbf{Q}_{\lambda\mathbf{i}} = \begin{pmatrix} \mathbf{DA_p^T} \\ \mathbf{A_e^T} \end{pmatrix} \begin{pmatrix} \mathbf{A_pD} & \mathbf{A_e} \end{pmatrix} = \mathbf{A}_{\lambda\mathbf{i}}^\mathbf{T}\mathbf{A}_{\lambda\mathbf{i}}$$

which indeed proves positive semidefiniteness of the covariance matrix parametrized by the revenue sharing scheme vector. Return vector is, as well, parametrized by the revenue sharing scheme vector, but since it is a linear function of the portfolio weights x_{ij} for every value of revenue shares, the portfolio return is indeed a convex function of the portfolio weights.

Thus we claim that both, objective function and constraints for the component provider problem are convex functions for each choice of the revenue sharing scheme. The overall lower level problem is convex.

4.4 Primal-Dual reformulation of the lower level problem

In this section we show how the bilevel programming problem with multiple followers can be restated as a regular single level optimization problem by using arguments from duality theory. This is accompished by reformulating the component providers' problem into an equivalent system of nonlinear inequalities. We want to stress out that such a reformulation does not entail any complementarity condition as opposite to what happens when the lower level of a bilevel program is reformulated using KKT conditions. Let us first restate, in matrix form, the general component provider problem, without accounting for the upper level decision variables about the revenue sharing scheme.

$$\begin{aligned} \underset{\mathbf{x_i}}{\text{maximize}} \quad & \mathbf{r}_{\lambda\mathbf{i}}^\mathbf{T}\mathbf{x_i} \\ \text{subject to} \quad & \mathbf{x_i}^\mathbf{T}\mathbf{Q}_{\lambda\mathbf{i}}\mathbf{x_i} \leq \bar{R}_i \\ & \mathbf{r}_{\lambda\mathbf{i}}^\mathbf{T}\mathbf{x_i} \geq \bar{r}_i \\ & \mathbf{e^T}\mathbf{x_i} = 1 \\ & \mathbf{x_i} \geq 0 \end{aligned}$$

We tackle the problem by stating the related KKT conditions, and then we reformulate this conditions in order to get rid of the complementarity constraint. Our aim is to define a system which is equivalent to KKT conditions, but at the same time easier to treat. As a matter of example we can state that such conditions, in case of linear optimization problem, correspond to bundling primal and dual constraints and require the duality gap to be zero. A similar approach is taken in this case.

The KKT conditions of the component provider problem are given by the system

$$
\begin{aligned}
&\mathbf{r_{\lambda i}} - \theta_{1i} 2\mathbf{Q_{\lambda i}}\mathbf{x} + \theta_{2i}\mathbf{r_{\lambda i}} + \theta_{3i}\mathbf{e} + \mu_i = \mathbf{0}\\
&\mathbf{x_i}^{\mathbf{T}}\mathbf{Q_{\lambda i}}\mathbf{x_i} \le \bar{R}_i\\
&\mathbf{r_{\lambda i}^T}\mathbf{x_i} \ge \bar{r}_i\\
&\mathbf{e^T}\mathbf{x_i} = 1\\
&\theta_{1i}(\mathbf{x_i}^{\mathbf{T}}\mathbf{Q_{\lambda i}}\mathbf{x_i} - \bar{R}_i) = 0\\
&\theta_{2i}(\mathbf{r_{\lambda i}^T}\mathbf{x_i} - \bar{r}_i) = 0\\
&\mu_{\mathbf{i}}^{\mathbf{T}}\mathbf{x_i} = 0\\
&\mathbf{x_i} \ge \mathbf{0}, \qquad \theta_{1i} \ge 0, \qquad \theta_{2i} \ge 0, \qquad \mu_i \ge \mathbf{0}
\end{aligned}
$$

or equivalently, reformulated as

$$
\begin{aligned}
&\mathbf{r_{\lambda i}} - \theta_{1i} 2\mathbf{Q_{\lambda i}}\mathbf{x} + \theta_{2i}\mathbf{r_{\lambda i}} + \theta_{3i}\mathbf{e} \le \mathbf{0}\\
&\mathbf{x_i}^{\mathbf{T}}\mathbf{Q_{\lambda i}}\mathbf{x_i} \le \bar{R}_i\\
&\mathbf{r_{\lambda i}^T}\mathbf{x_i} \ge \bar{r}_i\\
&\mathbf{e^T}\mathbf{x_i} = 1\\
&\theta_{1i}(\mathbf{x_i}^{\mathbf{T}}\mathbf{Q_{\lambda i}}\mathbf{x_i} - \bar{R}_i) = 0\\
&\theta_{2i}(\mathbf{r_{\lambda i}^T}\mathbf{x_i} - \bar{r}_i) = 0\\
&\mathbf{r_{\lambda i}^T}\mathbf{x_i} = \theta_{1i} 2\mathbf{Q_{\lambda i}}\mathbf{x} - \theta_{2i}\mathbf{r_{\lambda i}^T}\mathbf{x_i} - \theta_{3i}\mathbf{e^T}\mathbf{x_i}\\
&\mathbf{x_i} \ge \mathbf{0}, \qquad \theta_{1i} \ge 0, \qquad \theta_{2i} \ge 0
\end{aligned}
\tag{4.1}
$$

by substituting for vector μ.

Let us now make some considerations about the system. First and foremost, from the first inequality, we can obtain $\mathbf{r_{\lambda i}^T}\mathbf{x} \le \theta_{1i} 2\mathbf{x^T}\mathbf{Q_{\lambda i}}\mathbf{x} - \theta_{2i}\mathbf{r_{\lambda i}^T}\mathbf{x} - \theta_{3i}\mathbf{e^T}\mathbf{x}$, meanwhile from the second, third and fourh inequalities we have that $2\theta_{1i}\bar{R}_i - \theta_{2i}\bar{r}_i - \theta_{3i} \ge \theta_{1i} 2\mathbf{x^T}\mathbf{Q_{\lambda i}}\mathbf{x} - \theta_{2i}\mathbf{r_{\lambda i}^T}\mathbf{x} - \theta_{3i}\mathbf{e^T}\mathbf{x}$. Thus, by imposing the equality

$$\mathbf{r_{\lambda i}^T}\mathbf{x} = 2\theta_{1i}\bar{R}_i - \theta_{2i}\bar{r}_i - \theta_{3i} \tag{4.2}$$

we automatically satisfy the equality $\mathbf{r_{\lambda i}^T}\mathbf{x_i} = \theta_{1i} 2\mathbf{Q_{\lambda i}}\mathbf{x} - \theta_{2i}\mathbf{r_{\lambda i}^T}\mathbf{x_i} - \theta_{3i}\mathbf{e^T}\mathbf{x_i}$ and more importantly we have that $2\theta_{1i}\bar{R}_i - \theta_{2i}\bar{r}_i = \theta_{1i} 2\mathbf{x^T}\mathbf{Q_{\lambda i}}\mathbf{x} - \theta_{2i}\mathbf{r_{\lambda i}^T}\mathbf{x}$.

In particular, the last equality corresponds to

$$2\theta_{1i}(\bar{R}_i - \mathbf{x^T}\mathbf{Q_{\lambda i}}\mathbf{x}) = \theta_{2i}(\bar{r}_i - \mathbf{r_{\lambda i}^T}\mathbf{x})$$

where the first term is always non-negative, while the second term is always non-positive. This means that the equality is satisfied only when

$$2\theta_{1i}(\bar{R}_i - \mathbf{x^T Q_{\lambda i} x}) = 0$$

and

$$\theta_{2i}(\bar{r}_i - \mathbf{r^T_{\lambda i} x}) = 0$$

which means that the first four inequalities of system (1), together with (4.2) and the related restrictions on sign for the variables are enough to describe system (1). Formally, we claim that KKT conditions are equivalent to the system

$$\begin{aligned}
&\mathbf{r^T_{\lambda i} x} - 2\theta_{1i}\bar{R}_i + \theta_{2i}\bar{r}_i + \theta_{3i} = 0\\
&\mathbf{r_{\lambda i}} - \theta_{1i}2\mathbf{Q_{\lambda i}x} + \theta_{2i}\mathbf{r_{\lambda i}} + \theta_{3i}\mathbf{e} \leq \mathbf{0}\\
&\mathbf{x_i}^\mathbf{T}\mathbf{Q_{\lambda i}x_i} \leq \bar{R}_i\\
&\mathbf{r^T_{\lambda i} x_i} \geq \bar{r}_i\\
&\mathbf{e^T x_i} = 1\\
&\mathbf{x_i} \geq \mathbf{0}, \quad \theta_{1i} \geq 0, \quad \theta_{2i} \geq 0, \quad \theta_{3i} \in \Re
\end{aligned} \tag{4.3}$$

Let us restate such conditions to fit the formulation of the bilevel problem we have introduced

$$\begin{aligned}
&\sum_{j\in\mathcal{P}}\left(\frac{\gamma_{ij}v_j}{\lambda_{ij}c_i} - 1\right)x_{ij} + \sum_{j\in\mathcal{E}}\left(\frac{v_{ij}}{c_i} - 1\right)x_{ij} + 2\bar{R}_i\theta_{2i} - \bar{r}_i\theta_{1i} - \theta_{3i} = 0\\
&(1-\theta_{2i})\left(\frac{\gamma_{ij}v_j}{\lambda_{ij}c_i} - 1\right) + \frac{2}{c_i^2}\left(\sum_{h\in\mathcal{P}}\frac{\gamma_{ih}\gamma_{ij}}{\lambda_{ih}\lambda_{ij}}\sigma_{hj}x_{ih}\sum_{h\in\mathcal{E}}\frac{\gamma_{ij}}{\lambda_{ij}}\sigma_{ihj}x_{ih}+\right)\theta_{i1} - \theta_{3i} \leq 0 \qquad j\in\mathcal{P}\\
&(1-\theta_{2i})\left(\frac{v_{ij}}{c_i} - 1\right) + \frac{2}{c_i^2}\left(\sum_{h\in\mathcal{P}}\frac{\gamma_{ih}}{\lambda_{ih}}\sigma_{ihj}x_{ih}\sum_{h\in\mathcal{E}}\sigma_{ihj}x_{ih}+\right)\theta_{i1} - \theta_{3i} \leq 0 \qquad j\in\mathcal{E}\\
&\frac{1}{c_i^2}\left(\sum_{h\in\mathcal{P}}\sum_{k\in\mathcal{P}}\frac{\gamma_{ih}}{\lambda_{ih}}\frac{\gamma_{ik}}{\lambda_{ik}}\sigma_{hk}x_{ih}x_{ik} + 2\sum_{h\in\mathcal{P}}\sum_{k\in\mathcal{E}}\frac{\gamma_{ih}}{\lambda_{ih}}\sigma_{ihk}x_{ih}x_{ik} + \sum_{h\in\mathcal{E}}\sum_{k\in\mathcal{E}}\sigma_{ihk}x_{ih}x_{ik}\right) \leq \bar{R}_i\\
&\sum_{j\in\mathcal{P}}\left(\frac{\gamma_{ij}v_j}{\lambda_{ij}c_i} - 1\right)x_{ij} + \sum_{j\in\mathcal{E}}\left(\frac{v_{ij}}{c_i} - 1\right)x_{ij} \geq \bar{r}_i\\
&\sum_{j\in\mathcal{P}\cup\mathcal{E}} x_{ij} = 1\\
&x_{ij} \geq 0 \quad j\in\mathcal{P}\cup\mathcal{E} \qquad \theta_{1i},\theta_{2i} \geq 0, \quad \theta_{3i}\in\Re
\end{aligned}$$

We can now restate our bilevel programming problem through its single level equivalent, substituting each component provider problem with its equivalent Primal-Dual formulation. Altogether the problem is as follows

$$
\begin{aligned}
\max \quad & \sum_{j\in\mathcal{P}} \left(\frac{\gamma_{1j} v_j}{\lambda_{1j} c_1} - 1\right) x_{1j} + \sum_{j\in\mathcal{E}} \left(\frac{v_{1j}}{c_1} - 1\right) x_{1j} \\
\text{s.t.} \quad & x_{ij} \geq \frac{B_j \lambda_{ij}}{W_i} \qquad j \in \mathcal{P} \quad i \in \mathcal{I}_j \\
& \sum_{j\in\mathcal{P}} \left(\frac{\gamma_{ij} v_j}{\lambda_{ij} c_i} - 1\right) x_{ij} + \sum_{j\in\mathcal{E}} \left(\frac{v_{ij}}{c_i} - 1\right) x_{ij} + 2\bar{R}_i \theta_{2i} - \bar{r}_i \theta_{1i} - \theta_{3i} = 0 \qquad i \in \bigcup_{j\in\mathcal{P}} \mathcal{I}_j \\
& (1-\theta_{2i}) \left(\frac{\gamma_{ij} v_j}{\lambda_{ij} c_i} - 1\right) + \frac{2}{c_i^2} \left(\sum_{h\in\mathcal{P}} \frac{\gamma_{ih}\gamma_{ij}}{\lambda_{ih}\lambda_{ij}} \sigma_{hj} x_{ih} \sum_{h\in\mathcal{E}} \frac{\gamma_{ij}}{\lambda_{ij}} \sigma_{ihj} x_{ih} + \right) \theta_{i1} - \theta_{3i} \leq 0 \qquad j \in \mathcal{P} \quad i \in \mathcal{I}_j \\
& (1-\theta_{2i}) \left(\frac{v_{ij}}{c_i} - 1\right) + \frac{2}{c_i^2} \left(\sum_{h\in\mathcal{P}} \frac{\gamma_{ih}}{\lambda_{ih}} \sigma_{ihj} x_{ih} \sum_{h\in\mathcal{E}} \sigma_{ihj} x_{ih} + \right) \theta_{i1} - \theta_{3i} \leq 0 \qquad j \in \mathcal{E} \quad i \in \bigcup_{j\in\mathcal{P}} \mathcal{I}_j \\
& \frac{1}{c_i^2} \left(\sum_{h\in\mathcal{P}} \sum_{k\in\mathcal{P}} \frac{\gamma_{ih}}{\lambda_{ih}} \frac{\gamma_{ik}}{\lambda_{ik}} \sigma_{hk} x_{ih} x_{ik} + 2 \sum_{h\in\mathcal{P}} \sum_{k\in\mathcal{E}} \frac{\gamma_{ih}}{\lambda_{ih}} \sigma_{ihk} x_{ih} x_{ik} + \sum_{h\in\mathcal{E}} \sum_{k\in\mathcal{E}} \sigma_{ihk} x_{ih} x_{ik} \right) \leq \bar{R}_i \qquad i \in \bigcup_{j\in\mathcal{P}} \mathcal{I}_j \\
& \sum_{j\in\mathcal{P}} \left(\frac{\gamma_{ij} v_j}{\lambda_{ij} c_i} - 1\right) x_{ij} + \sum_{j\in\mathcal{E}} \left(\frac{v_{ij}}{c_i} - 1\right) x_{ij} \geq \bar{r}_i \qquad i \in \bigcup_{j\in\mathcal{P}} \mathcal{I}_j \\
& \sum_{j\in\mathcal{P}\cup\mathcal{E}} x_{ij} = 1 \qquad i \in \bigcup_{j\in\mathcal{P}} \mathcal{I}_j \\
& \sum_{i\in\mathcal{I}_j} \gamma_{ij} = 1 \qquad j \in \mathcal{P} \\
& x_{ij} \geq 0 \quad j \in \mathcal{P}\cup\mathcal{E} \quad \gamma_{ij}, \theta_{1i}, \theta_{2i} \geq 0, \quad \theta_{3i} \in \Re \quad i \in \mathcal{I}_j \quad j \in \mathcal{P}
\end{aligned}
$$

4.5 Model Implementation

We implemented the model through commercial software AMPL/MINOS using simulated data, as real data collected and placed in databases held by Telenor NO is covered by industrial secret. Data is the following:

- Number of Service Providers involved in provision of Service Portfolio j, ($|\ \mathcal{I}_j\ |$)=3
- Number of Plaftorm Services ($|\ \mathcal{P}\ |$)=3
- Number of External Services for each Service Provider ($|\ \mathcal{E}\ |$)=1

	Service Porfolios		
	Expected Unit Revenue	Estimated Demand	
Service Portfolio 1	60	10	
Service Portfolio 2	35	30	
Service Portfolio 3	45	60	
	External Service		
	Expected Unit Revenue		
Service Provider 1	4.8		
Service Provider 2	1.6		
Service Provider 3	6.4		
	Service Amounts per Service Portfolio Unit		
	ES Provider 1	ES Provider 2	ES Provider 3
Service Portfolio 1	10	5	2
Service Portfolio 2	8	5	1
Service Portfolio 3	12	6	0.5
	Service Providers Profiles		
	Service Provider 1	Service Provider 2	Service Provider 3
Capacity	1991	1800	1125
Return Bound	0.05	0.05	0.05
Risk Bound	0.170765	0.2	0.1582
Operating Costs	1	1	20
	Platform Services Covariance Matrix		
	Service Portfolio 1	Service Portfolio 2	Service Portfolio 3
Service Portfolio 1	20	20.7846	3.1623
Service Portfolio 2	20.7846	60	-27.3861
Service Portfolio 3	3.1623	-27.3861	50
	Platform Services / External Service Covariance		
	ES Provider 1	ES Provider 2	ES Provider 3
Service Portfolio 1	2.1466	0.3578	4.2933
Service Portfolio 2	-3.7181	-1.8590	-2.4787
Service Portfolio 3	0	-0.5657	5.5255
	External Services Variance		
	Service Provider 1	Service Provider 2	Service Provider 3
	5.760	0.640	10.240

Primal-Dual reformulation, solved using data introduced in the previous table delivers the following optimal sharing scheme for revenues. We also report the service allocation chosen by each Service Provider for all Service Portfolios provided by the Platform and for the External Services.

	Revenue Shares		
	Service Portfolio 1	Service Portfolio 2	Service Portfolio 3
Service Provider 1	0.458782	0.621327	0.0907092
Service Provider 2	0.200522	0.21354	0.0332471
Service Provider 3	0.340696	0.165133	0.876044
	Portfolio Services: Bilevel solution		
	Service Portfolio 1	Service Portfolio 2	Service Portfolio 3
Service Provider 1	0.523727	0.328179	0.0635147
Service Provider 2	0.0444444	0.266704	0.102809
Service Provider 3	0.106667	0.24923	0.644103
	External Services: Bilevel solution		
	External Service		
Service Provider 1	0.0845791		
Service Provider 2	0.586042		
Service Provider 3	0		
	Portfolio Services: Lower level solution		
	Service Portfolio 1	Service Portfolio 2	Service Portfolio 3
Service Provider 1	0.523728	0.328179	0.0635135
Service Provider 2	0.0426559	0.267571	0.103585
Service Provider 3	0.106603	0.249293	0.644104
	External Services: Lower level solution		
	External Service		
Service Provider 1	0.084579		
Service Provider 2	0.586188		
Service Provider 3	0		

It is easy to see that the solution of the Service Provider Problem (lower level), once fixed the decision variables of the Platform Operator Problem (upper level), are the same as the ones obtained by solving the Primal-Dual reformulation of the stochastic coordination model with bilevel structure, up to approximation errors, whereas by using KKT reformulation, we get

	Portfolio Services: Bilevel solution		
	Service Portfolio 1	Service Portfolio 2	Service Portfolio 3
Service Provider 1	0.8494016	0.0870901	0.0635082
Service Provider 2	0.3100078	0.0833333	0.0362612
Service Provider 3	0.1066667	0.1600000	0.7333333
	External Services: Bilevel solution		
	External Service		
Service Provider 1	0		
Service Provider 2	0.570398		
Service Provider 3	0		
	Portfolio Services: Lower level solution		
	Service Portfolio 1	Service Portfolio 2	Service Portfolio 3
Service Provider 1	0.849402	0.0870902	0.0635082
Service Provider 2	0	0.989544	0.0104564
Service Provider 3	0	0	1
	External Services: Lower level solution		
	External Service		
Service Provider 1	0.0605649		
Service Provider 2	0		
Service Provider 3	0		

Where it is shown that the choice picked up by the followers, does not correspond to the one 'forecasted' by the leader.

Bibliography

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Conclusions

We have shown a reformulation of Bilevel Programs which can be used for modelling problems with linear and quadratic convex lower levels. The reformulation has led, in some instances to force the lower level to pick the exact values forcasted by the leader. This makes such reformulation suitable for modelling pessimistic bilevel positions. Furthermore some authors, such as Anandalingam and White have used a gap reduction approach for linear bilevel programs solved using penalty methods. The use of duality gap into penalty functions has proved as an improvement over the use of complementarities. This suggests that these reformulations could be used in further research to devise algorithms based on penalty functions on the duality gap.