Università degli Studi di Bergamo

Dottorato di Ricerca

XXIII Ciclo

Metodi Computazionali per le Previsioni

e Decisioni Economiche e Finanziarie

S.S.D. SECS-S/06

# Maximal Monotone Operators, Convex Representations and Duality

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Anno Accademico 2009/2010

# Contents

# Introduction

The thesis proposes some advances in the research area of convex representations of monotone operators.

The mathematical concept of an operator, intended, roughly speaking, as an extension of the notion of a function to the case in which the domain and the range are infinite-dimensional (or at least multidimensional) spaces, has been a central notion in the development of modern Functional Analysis. Implicitly present in eighteenth century studies on partial differential equations and variational calculus, like those of Laplace and Fourier, this concept acquired more and more importance during the first three decades of the twentieth century, due to the work of Hilbert, von Neumann and Banach, whose book [6], published in 1932, is the first book on modern Operator Theory. These studies mainly focused on linear operators, while, in the second half of the century, nonlinear, possibly multivalued, operators were extensively investigated as well, under the impulse of several fields of application.

In this connection, particular attention has been paid to monotone operators, i.e. multifunctions $T : X \rightrightarrows X^*$ defined on a real Banach space $X$ and taking values in its topological dual $X^*$, such that

$$\langle x - y, x^* - y^* \rangle \geq 0,$$

for all $x, y \in X$, $x^* \in T(x)$, $y^* \in T(y)$, where $\langle x, x^* \rangle = x^*(x)$. This notion of monotonicity constitutes a straightforward but important generalization of the one-dimensional definition of a monotonically nondecreasing function and it is a very useful notion in different contexts. For instance, in Microeconomics it is studied in connection with demand correspondences.

A relevant feature of monotone operators, to which increasing importance has been credited, is the strong resemblance and the connections they have with notions from Convex Analysis. We will review these links in detail in the first two chapters. Now it suffices to recall that a key tool when dealing with convexity, namely the subdifferential, provides a solid bridge between the two realms of Convex and Functional Analysis. Indeed, as is well-known after the work of Rockafellar

[84], the subdifferential of a proper lower semicontinuous convex function is a maximal monotone operator. Subdifferentials thus act as a paradigm for the study of properties of general maximal monotone operators, while, on the other hand, research on monotone operators in an abstract setting can shed new light on our knowledge of convex functions. This going back and forth from convexity to monotonicity has become standard in the last few years, after acquiring a new method of investigation of properties of maximal monotone operators via convex representations, following the example of [27, 34, 63]. Once again, the subdifferential plays an archetypal role in this connection. Given indeed a proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ on a real Banach space $X$, it is known that the subdifferential of $f$ at a point $x$ in its domain is the set of elements $x^* \in X^*$ such that

$$f(x) + f^*(x^*) = \langle x, x^* \rangle.$$

Thus, the proper lower semicontinuous convex function $h : X \to \mathbb{R} \cup \{+\infty\}$ defined as $h(y, y^*) = f(y) + f^*(y^*)$ for all $(y, y^*) \in X \times X^*$ completely characterizes the graph of the subdifferential of $f$, or, as we will say, *represents* the operator $\partial f$. The references we cited above extend this methodology to a general maximal monotone operator $T : X \rightrightarrows X^*$, attaching to it a whole family $\mathcal{H}_T$ of proper lower semicontinuous convex functions that represent the operator, in the sense that, for all $h \in \mathcal{H}_T$, one has $h(x, x^*) = \langle x, x^* \rangle$ if and only if $x^* \in T(x)$ [27]. The set $\mathcal{H}_T$ is now known in the literature as the Fitzpatrick family associated with $T$ and the use of its members for the study of maximal monotone operators has led to considerable simplifications in the proofs of some classical properties, as well as to the discovery of new results.

The present thesis contributes to this research area considering convex representations both as an instrument to gain new insight on monotonicity and as an independent object of study, of intrinsic interest.

The material can be ideally split into two parts, discerning the review chapters from the original part.

The first part (Chapters 1 and 2) collects notions and already known results that are required in the remaining of the thesis. The first chapter sets notation and recalls basic definitions and theorems from Convex Analysis and from the theory of monotone operators, while the second one reviews some important contributions from the literature on convex representations (this survey is confined to those results that are of immediate reference for the following chapters). In particular, Section 1.3 is specular to Section 2.2, the former presenting the main features of maximal monotone operators obtained via standard functional analytical techniques, while the latter goes through the same topics, now revisited under the lens of convex representations.

The second part (Chapters 3 to 7) is the original contribution of the thesis and is based on [22, 69, 78, 79, 80].

Chapter 3 provides conditions for the coincidence of two maximal monotone operators. It is well-known that, as a consequence of [84, Theorem B], the difference of two proper lower semicontinuous convex functions is constant if and only if their subdifferentials coincide. It can be proved that, if the common domain of the two functions is either convex [46] or open, the equality of the two subdifferentials can be replaced by a weaker condition, namely, that they have nonempty intersection at each point. We prove that, analogously, under some weakened convexity requirement, two maximal monotone operators coincide if they map each point of their common domain to non disjoint images. The proof is purely algebraic, but a similar result involving enlargements of the operators is provided (the notion of an enlargement is strictly related to that of a convex representation, as we will recall in Section 2.3 below).

Chapter 4, instead, is explicitly focused on the study of the Fitzpatrick family. More precisely, the interest is directed towards autoconjugate elements of that family, providing a necessary and sufficient condition for the minimal element of the Fitzpatrick family (the so-called Fitzpatrick function) to be autoconjugate. This condition is then applied to the case of the subdifferential of a proper lower semicontinuous convex function.

Chapter 5, refining and generalizing [59] to the case of nonreflexive Banach spaces, studies surjectivity-type properties of extensions to the bidual of maximal monotone operators of type (D), i.e., roughly speaking, maximal monotone operators that are particularly well-behaved, like operators defined on reflexive Banach spaces[1] (see Section 1.3.3). In this chapter, special attention is paid to the relations between the Fitzpatrick family and the sum of the graphs $\mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$ of the extensions to the bidual of two maximal monotone operators $S$ and $T$, by means of techniques from convex duality theory.

The last two chapters can be considered as further extensions of this approach. In Chapter 6 a new family of representable extensions of monotone operators to the bidual is introduced for operators that are not of type (D), yielding as a consequence also a new characterization of operators of type (D). On the other hand, subfamilies of the family of operators of type (D) are also defined and studied in the second part of the chapter.

Finally, Chapter 7 presents a generalization of a main surjectivity result of Chapter 5 to the setting of abstract convexity, to which the notions of monotonicity and convex representations can be suitably extended.

[1] Actually, every maximal monotone operator defined on a reflexive Banach space is of type (D).

# Chapter 1

# Preliminary Notions of Convex and Nonlinear Analysis

In this chapter, after setting some basic notation that we will use extensively in the present thesis (Section 1.1), we collect the main notions from Convex Analysis (Section 1.2) and from the theory of monotone operators (Section 1.3) that we will need in the following.

## 1.1 Notation

Given sets $X_i$, for $i = 1, \cdots, n$, $A \subseteq \Pi_{i=1,\cdots,n} X_i$ and $k \in \{1, \cdots, n\}$, we will denote by $\mathrm{Pr}_{X_k} A$ the projection of $A$ on $X_k$. Moreover, adopting a notation introduced in [72], we define, for any given set $B \subseteq X_1 \times X_2$ and function $f : X_1 \times X_2 \to \overline{\mathbb{R}}$,

$$B^\top := \{(y, x) \in X_2 \times X_1 : \quad (x, y) \in B\},$$

$$\forall (y, x) \in X_1 \times X_2 : \quad f^\top(y, x) := f(x, y).$$

In our presentation we will always work in real Banach spaces, unless otherwise specified. Given a Banach space $X$, its topological dual and bidual will be denoted by $X^*$ and $X^{**}$, respectively[1]. For ease of notation, we will identify $X$ with its image in $X^{**}$ through the canonical inclusion $\iota : X \to X^{**}$ and we will use the same notation $\pi(\cdot, \cdot)$ or $\langle \cdot, \cdot \rangle$ for the duality product both on $X \times X^*$ and $X^{**} \times X^*$. Given a subset $A$ of a Banach space $X$, we will use standard notation to denote the interior and the closure of $A$ (int $A$ and cl $A$, respectively), while adding the short-hand notation $\mathrm{cl}^* B$ for the closure of $B \subseteq X^*$ in the weak* topology.

[1] The same notation will be used for general topological vector spaces as well

Furthermore, we will need to consider elementary isometries in Banach spaces. Translations of vector $y \in X$ will be denoted by $\tau_y$, i.e., $\tau_y : X \to X$ will be the function defined by $\tau_y(x) = x+y$, for all $x \in X$, while reflections in the first or in the second component of an ordered pair will be written

$$\begin{aligned} \varrho_1 : X \times Y \to X \times Y, \qquad & (x,y) \mapsto (-x,y) \\ \varrho_2 : X \times Y \to X \times Y, \qquad & (x,y) \mapsto (x,-y), \end{aligned}$$

where $Y$ is another Banach space.

Finally, for any $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$, we will also define

$$f \oplus g : X \times Y \to \overline{\mathbb{R}}, \quad (x,y) \mapsto (f \oplus g)(x,y) = f(x) + g(y).$$

Therefore, $\mathrm{dom}\,(f \oplus g) = \mathrm{dom}\, f \times \mathrm{dom}\, g$.

## 1.2 Convex Functions, Fenchel Conjugation and Subdifferentials

Several textbooks on Convex Analysis are nowadays available, such as [14, 42, 83], just to mention a few of them. We will mainly make reference to [108], which focuses on an infinite-dimensional framework, well suited to our perspective.

Let $X$ be a real vector space. Given a subset $C \subseteq X$, we say that $C$ is convex if, for any $x, y \in C$ and $\lambda \in \,]0,1[$ (or, equivalently, $\lambda \in [0,1]$), the point $\lambda x + (1-\lambda)y$, a *convex combination* of $x$ and $y$, belongs to $C$ as well. For any $D \subseteq X$, the convex hull of $D$ is the intersection of all convex subsets of $X$ that contain $D$ and is denoted by $\mathrm{conv}\, D$.

Given a function $f : X \to \overline{\mathbb{R}}$, we define its *domain* (or *effective domain*) and its *epigraph*, respectively, as

$$\mathrm{dom}\, f := \{x \in X : \ f(x) < +\infty\}, \quad \mathrm{epi}\, f := \{\lambda \in \mathbb{R} : \ f(x) \leq \lambda\}.$$

We say that the function $f$ is:

(a) *proper*, if $\mathrm{dom}\, f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$;

(b) *convex*, if $\mathrm{dom}\, f$ is convex and, for all $x, y \in \mathrm{dom}\, f$ and $\lambda \in \,]0,1[$, one has

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y); \tag{1.1}$$

(c) *strictly convex*, if all the conditions of (*b*) hold, with the inequality in (1.1) replaced by strict inequality whenever $x \neq y$.

When $f$ is proper, we will usually write $f : X \to \mathbb{R} \cup \{+\infty\}$, to stress the fact that it cannot take the value $-\infty$.

Concerning items (*b*) and (*c*), the related notion of (strict) concavity is easily defined, saying that $f$ is *(strictly) concave* if the function $-f$ is (strictly) convex. A very well-known characterization of the convexity of $f$ (connecting convexity of functions and convexity of sets) is the property of epi $f$ being a convex subset of $X \times \mathbb{R}$.

When $X$ is a topological vector space, we can consider a particularly important subfamily of convex functions, i.e. that of proper lower semicontinuous convex functions. Recall that a function is lower semicontinuous at $x \in X$ if

$$f(x) \leq \liminf_{y \to x} f(y) \tag{1.2}$$

and, simply, *lower semicontinuous* if (1.2) holds at every $x \in \operatorname{dom} f$. The following characterization of lower semicontinuous convex functions is well-known [108, Theorem 2.2.1] (indeed, under even less restrictive assumptions on $X$).

**Theorem 1.2.1** *Let $X$ be a Banach space and let $f : X \to \overline{\mathbb{R}}$. The following conditions are equivalent:*

*(i) $f$ is convex and lower semicontinuous;*

*(ii) $f$ is convex and weak-lower semicontinuous;*

*(iii)* epi $f$ *is convex and closed;*

*(iv)* epi $f$ *is convex and weak-closed.*

Given a function $f : X \to \overline{\mathbb{R}}$ defined on a Banach space $X$, one can define cl $f$ (conv $f$) as the biggest lower semicontinuous (convex, respectively) function majorized by $f$. Joining these two operations, we can obtain a lower semicontinuous convex function, cl conv $f$, from an arbitrary function $f$.

Another simple transformation that, given $f$, generates a lower semicontinuous convex function from it, is Fenchel conjugation. The *(Fenchel) conjugate* of $f$ is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined as

$$f^*(x^*) := \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\},$$

for all $x^* \in X^*$, where $\langle \cdot, \cdot \rangle$ denotes the duality product between $X$ and $X^*$. It can be proved [108, Theorem 2.3.1] that $f^*$ is convex and weak*-lower semicontinuous, that conjugation is inequality inverting, i.e.

$$f \leq g \quad \Longrightarrow \quad g^* \leq f^*,$$

and that the very important *Fenchel inequality* (or *Young-Fenchel inequality*) holds

$$\forall (x, x^*) \in X \times X^* : \quad f(x) + f^*(x^*) \geq \langle x, x^* \rangle,$$

with the convention $(-\infty) + (+\infty) = +\infty$. It is interesting to notice that, in general, when iterating Fenchel conjugation, we don't end up with the initial function $f$, but with the function $f^{**} := (f^*)^*$, such that $(f^{**})|_X \leq f$. Anyway, when $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function, then $f^* : X^* \to \overline{\mathbb{R}}$ is proper as well and $(f^{**})|_X = f$ (*Fenchel-Moreau theorem*).

As we have just pointed out, convex functions of interest are usually not even continuous, so that we cannot expect them to be differentiable. For convex functions, though, the key notion of subdifferential provides a very useful instrument to tackle this problem.

Let $X$ be a Banach space and $f : X \to \overline{\mathbb{R}}$. The *(Fenchel) subdifferential* of $f$ is the multivalued mapping[2] $\partial f : X \rightrightarrows X^*$ defined as

$$\partial f(x) := \begin{cases} \{x^* \in X^* : \quad f(y) \geq f(x) + \langle y - x, x^* \rangle, \ \forall y \in X\}, & \text{if} \quad f(x) \in \mathbb{R} \\ \emptyset, & \text{if} \quad f(x) \notin \mathbb{R}. \end{cases}$$

In practice, when numerically computing the subdifferential of $f$ at a given point $x \in X$, we are actually able to determine only an approximation of it. In this connection, the following relaxation of the notion of subdifferential is useful. Given $\varepsilon \geq 0$, we call *$\varepsilon$-subdifferential* (or *approximate subdifferential*) of $f$ the multivalued mapping $\partial_\varepsilon f : X \rightrightarrows X^*$ defined as

$$\partial_\varepsilon f(x) := \begin{cases} \{x^* \in X^* : \quad f(y) \geq f(x) + \langle y - x, x^* \rangle - \varepsilon, \ \forall y \in X\}, & \text{if} \quad f(x) \in \mathbb{R} \\ \emptyset, & \text{if} \quad f(x) \notin \mathbb{R}. \end{cases}$$

Obviously, $\partial_0 f = \partial f$. It can be proved that, for all $x \in X$ and $\varepsilon \geq 0$, the set $\partial_\varepsilon f(x)$ is convex and weak*-closed (possibly empty). The following properties of approximate subdifferentials are also very useful:

$$\forall x \in X, \forall \varepsilon_1, \varepsilon_2 \geq 0 : \quad \varepsilon_1 \leq \varepsilon_2 \implies \partial_{\varepsilon_1} f(x) \subseteq \partial_{\varepsilon_2} f(x)$$

[2] See Section 1.3 below for a formal definition of this notion and for an explanation of the corresponding notation.

and

$$\forall x \in X, \forall \varepsilon \geq 0 : \quad \partial_\varepsilon f(x) = \bigcap_{\eta > \varepsilon} \partial_\eta f(x). \tag{1.3}$$

Moreover, one can prove the following characterization of approximate subdifferentials [108, Theorem 2.4.2].

**Theorem 1.2.2** *Let $X$ be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$. Then*

$$x^* \in \partial_\varepsilon f(\overline{x}) \quad \Longleftrightarrow \quad f(\overline{x}) + f^*(x^*) \leq \langle \overline{x}, x^* \rangle + \varepsilon.$$

In the particular case of subdifferentials, taking Fenchel inequality into account, the previous characterization can be rewritten as

$$x^* \in \partial f(\overline{x}) \quad \Longleftrightarrow \quad f(\overline{x}) + f^*(x^*) = \langle \overline{x}, x^* \rangle.$$

Finally, we mention the famous Brønsted-Rockafellar property [17], which further clarifies how $\varepsilon$-subdifferentials work as approximations of ordinary subdifferentials.

**Theorem 1.2.3 (Brønsted-Rockafellar)** *Let $X$ be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$. If $\overline{x}^* \in \partial_\varepsilon f(\overline{x})$, then there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that*

$$x_\varepsilon^* \in \partial_\varepsilon f(x_\varepsilon), \quad \|x_\varepsilon - \overline{x}\| \leq \sqrt{\varepsilon}, \quad \|x_\varepsilon^* - \overline{x}^*\| \leq \sqrt{\varepsilon}.$$

## 1.3 Monotone Operators

Many textbooks and survey papers on Nonlinear Analysis, and on maximal monotone operators in particular, have been published during the last two decades, such as [4, 5, 20, 43, 74, 75, 93, 96, 110].

### 1.3.1 Definitions and Examples

Let $X, Y$ be nonempty sets. A *multivalued mapping* (or *multifunction*, or *point-to-set operator*, or, simply, *operator*) is a binary relation between $X$ and $Y$, i.e. a map from $X$ to $2^Y$, that is to say, a map which associates to any point of $X$ a (possibly empty) subset of $Y$. In general, we will denote operators with capital letters of the Latin alphabet. Moreover, to recall that we are considering point-to-set mappings, given an operator $T$, we will write $T : X \rightrightarrows Y$. According to this set-theoretical definition an operator coincides with its graph. Anyway, we will stick to the more intuitive notion which distinguishes the operator from its graph and introduce a specific notation for the graph

$$\mathcal{G}(T) := \{(x, y) \in X \times Y : \ \ y \in T(x)\}.$$

It remains nonetheless true that the operator is univocally determined by its graph. The *domain* and *range* of the operator are defined respectively as

$$\mathcal{D}(T) := \{x \in X : \ \ T(x) \neq \emptyset\}, \qquad \mathcal{R}(T) := \{y \in Y : \ \ \exists x \in X \ (y \in T(x))\}.$$

Thus, $\mathcal{D}(T) = \mathrm{Pr}_X \mathcal{G}(T)$ and $\mathcal{R}(T) = \mathrm{Pr}_Y \mathcal{G}(T)$.

Notice that a single-valued operator $S : X \to Y$ can be regarded as a multivalued one, identifying $S(x)$ with $\{S(x)\}$ for all $x \in X$. Similarly, we will also consider functions $f : \mathbb{R} \to \mathbb{R}$, or $f : X \to \overline{\mathbb{R}}$ as operators [3].

An immediate example of how a multifunction may arise and, at the same time, one possible justification for the introduction of this kind of mappings, is the possibility to consider in a unified framework any function $f$ and its inverse $f^{-1}$ (which is not a single-valued function, unless $f$ is injective). In general, given an operator $T : X \rightrightarrows Y$, the *inverse* operator $T^{-1} : Y \rightrightarrows X$ can be defined by means of its graph, setting

$$\mathcal{G}(T^{-1}) := \{(y, x) \in Y \times X : \ \ y \in T(x)\}.$$

Equivalently, we can write $\mathcal{G}(T^{-1}) := (\mathcal{G}(T))^{\top}$.

In the broad class of (multivalued) operators, we will restrict our attention to those which display the monotonicity property that we now define. Taking as a reference point a monotonically nondecreasing function $f : \mathbb{R} \to \mathbb{R}$, the property

$$\forall x, y \in \mathrm{dom}\, f : \quad x \leq y \implies f(x) \leq f(y)$$

can be rewritten as

$$\forall x, y \in \mathrm{dom}\, f : \quad (x - y)(f(x) - f(y)) \geq 0.$$

The previous formulation immediately extends to $\mathbb{R}^n$ and, in general, to inner product spaces, simply replacing the product in $\mathbb{R}$ by the inner product. More generally, the property can be

[3] In the latter case, though, an inconsistency may arise concerning the domain of $f$, since the inclusion $\mathrm{dom}\, f \subseteq \mathcal{D}(f)$ may be strict. In this case we will be mainly interested in the effective domain, so that the word domain will make reference to $\mathrm{dom}\, f$, unless otherwise specified.

restated for operators from a Banach space[4] to its dual, where the role of the inner product is now played by the duality product[5].

**Definition 1.3.1** *Let $X, Y$ be Banach spaces such that $Y \subseteq X^*$ and let $T : X \rightrightarrows Y$ be an operator. We say that $T$ is:*

*(a)* monotone*, if, for all $(x_1, y_1), (x_2, y_2) \in \mathcal{G}(T)$,*

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0; \tag{1.4}$$

*(b)* maximal monotone*, if $T$ does not admit proper monotone extensions, i.e., for any monotone operator $S : X \rightrightarrows Y$,*

$$\mathcal{G}(T) \subseteq \mathcal{G}(S) \implies S = T;$$

*(c)* premaximal monotone[6]*, if $T$ admits a unique maximal monotone extension, i.e. there exists a unique maximal monotone operator $S : X \rightrightarrows Y$ such that $\mathcal{G}(T) \subseteq \mathcal{G}(S)$.*

The previous definition only deals with the graph of the operator. As a consequence, given the symmetric role played by $X$ and $Y$, we have that $T$ is (maximal) monotone if and only if $T^{-1}$ is. Moreover, we can define the notion of monotonicity for an arbitrary subset $A$ of $X \times Y$, asking that relation (1.4) be satisfied for all $(x_1, y_1), (x_2, y_2) \in A$ (an analogous adaptation holds for the notions of maximal monotonicity and premaximal monotonicity). On some occasions we will also need to consider monotonicity of a point with respect to a given set. Specifically, given $A \subseteq X \times Y$ and $(x_1, y_1) \in X \times Y$ we will say that $(x_1, y_1)$ is *monotonically related* to $A$ if (1.4) holds for all $(x_2, y_2) \in A$. Thus, $A$ is monotone if and only if any point of $A$ is monotonically related to $A$, while a monotone set $A$ is maximal monotone if and only if any point $(x, y) \in X \times Y$ which is monotonically related to $A$ actually belongs to $A$. Analogous considerations hold for

[4]Actually, the definition could be given for more general spaces. Analogously, some of the results of the present thesis immediately extend to locally convex spaces. Anyway, we will not pursue this level of generality and rather stick to a Banach space setting, both for the sake of a uniform treatment of the material and given the wider diffusion of Banach spaces in the applications.

[5]The definition of a monotone operator that we provide next is not standard, in the sense that, usually, only the case $Y = X^*$ is considered. The slight, obvious change that we make here allows us to keep notation and terminology as simple as possible when dealing with extensions of monotone operators to the bidual, i.e. operators of the form $S : X^{**} \rightrightarrows X^*$, where $X^*$ can be identified with its image in $X^{***}$ via the canonical injection.

[6]This notion was introduced in [61].

operators as well. In this connection, a useful instrument of analysis that we will employ in Chapter 3 is the polar of an operator, which was introduced in [61].

**Definition 1.3.2** *Let $X, Y$ be Banach spaces such that $Y \subseteq X^*$ and $T : X \rightrightarrows Y$ be an operator. The* polar *of $T$ is the operator $T^\mu : X \rightrightarrows Y$, the graph of which consists of all points of $X \times Y$ that are monotonically related to $\mathcal{G}(T)$.*

Notice that $T^\mu$ is not monotone itself, unless $T$ is premaximal monotone, in which case $T^\mu$ is the unique maximal monotone extension of $T$.

We end this section of preliminary definitions by considering the most important example of a maximal monotone operator, i.e. the subdifferential of any proper lower semicontinuous convex function.

It is easy to prove that the subdifferential of any function is a monotone operator.

**Proposition 1.3.3** *Let $X$ be a Banach space. Given the function $f : X \to \overline{\mathbb{R}}$, its subdifferential $\partial f : X \rightrightarrows X^*$ is a monotone operator.*

*Proof.* For all $(x, x^*), (y, y^*) \in \mathcal{G}(\partial f)$, by definition we have

$$f(y) \geq f(x) + \langle y - x, x^* \rangle \quad \text{and} \quad f(x) \geq f(y) + \langle x - y, y^* \rangle,$$

with $f(x), f(y) \in \mathbb{R}$. Adding up these two inequalities, we obtain

$$f(y) + f(x) \geq f(x) + \langle y - x, x^* \rangle + f(y) + \langle x - y, y^* \rangle,$$

from which $\langle x - y, x^* - y^* \rangle \geq 0$. □

In the case when $f$ is proper lower semicontinuous and convex, then $\partial f$ is maximal monotone. The proof is not trivial and was provided by Rockafellar [84]. Much easier proofs were recently obtained [52, 97].

**Theorem 1.3.4 ([84, Theorem A])** *Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then $\partial f : X \rightrightarrows X^*$, the subdifferential of $f$, is a maximal monotone operator.*

Actually, one can prove more, i.e. that subdifferentials of proper lower semicontinuous convex function are cyclically monotone and that they are the only maximal monotone operators which are cyclically monotone.

**Definition 1.3.5** *Let $X, Y$ be Banach spaces such that $Y \subseteq X^*$. The operator $T : X \rightrightarrows Y$ is:*

*(a)* $n$-cyclically monotone*, for some $n \geq 2$, if, for all $(x_0, y_0), \cdots, (x_n, y_n) \in \mathcal{G}(T)$ with $x_0 = x_n$,*

$$\sum_{1 \leq k \leq n} \langle x_k - x_{k-1}, y_k \rangle \geq 0;$$

*(b)* cyclically monotone*, if $T$ is $n$-cyclically monotone for all $n \geq 2$.*

Obviously, the notions of 2-cyclical monotonicity and monotonicity coincide. Moreover, notice that the subdifferential of a function is cyclically monotone (the proof is an easy generalization of that of Proposition 1.3.3).

**Example [75, Example 2.23 (a)]** As a consequence, we can immediately establish that the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(x^{(1)}, x^{(2)}) = (x^{(2)}, -x^{(1)})$, though being maximal monotone (as a linear and positive single-valued operator [75, Example 1.5 (b)]), is not the subdifferential of a proper lower semicontinuous convex function. Indeed, it is not 3-cyclically monotone (hence neither cyclically monotone), since, choosing $x_0 = x_3 = (1, 1)$, $x_1 = (0, 1)$ and $x_2 = (1, 0)$, we obtain $y_1 = (1, 0)$, $y_2 = (0, -1)$ and $y_3 = (1, -1)$, from which

$$(x_1 - x_0) \cdot y_1 + (x_2 - x_1) \cdot y_2 + (x_3 - x_2) \cdot y_3 = -1 < 0.$$

A converse statement holds for proper lower semicontinuous convex functions, but, again, the proof is not immediate and it was provided by Rockafellar. As a consequence, the following important characterization holds.

**Theorem 1.3.6 ([84, Theorem B])** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a multifunction. In order that there exist a proper lower semicontinuous convex function $f$ on $X$ such that $T = \partial f$, it is necessary and sufficient that $T$ be a maximal cyclically monotone operator. Moreover, in this case, $T$ determines $f$ uniquely up to an additive constant.*

As a consequence of this result, subdifferentials of proper lower semicontinuous convex functions, which can be used, to some extent, as a paradigm for the study of maximal monotone operators, are completely characterized. In this prototypical class of maximal monotone operators, there is one special example that we have to point out, since it will be crucial in the results that we will recall in the next section.

**Example** Let $(X, \|\cdot\|)$ be a Banach space and consider the function $f : X \to \mathbb{R}$, defined as

$$f(x) := \frac{1}{2}\|x\|^2,$$

for all $x \in X$. Since $f$ is finite-valued, continuous and convex, its subdifferential is a maximal monotone operator and is called the *duality mapping* of $X$. We will denote it by $J_X^{\|\cdot\|}$, or simply $J_X$, or $J$, when no confusion may arise. Explicitly, we can write $J : X \rightrightarrows X^*$ as the operator that associates to each $x \in X$ the set

$$\begin{aligned} J(x) &= \{x^* \in X^* : \quad 1/2\|x\|^2 + 1/2\|x^*\|^2 = \langle x, x^* \rangle\} \\ &= \{x^* \in X^* : \quad \|x\|^2 = \|x^*\|^2 = \langle x, x^* \rangle\}. \end{aligned}$$

Another useful example is represented by the case when $f = \delta_K$, for some nonempty closed convex set $K \subseteq X$. The maximal monotone operator $N_K := \partial\delta_K$ is called the *normal cone operator* to $K$ and is given by

$$N_K(x) := \begin{cases} \{x^* \in X^* : \langle y - x, x^* \rangle \leq 0, \quad \forall y \in K\}, & x \in K \\ \emptyset, & x \notin K. \end{cases}$$

Obviously, for any $x \in K$ its image $N_K(x)$ is a convex cone in $X^*$. Notice that, in the following, we will always consider cones as containing the origin, according to the following definition.

**Definition 1.3.7** *Let $Y$ be a normed space and $K \subseteq Y$. We say that $K$ is a* cone *if*

$$\forall k \in K, \forall \lambda \geq 0 : \qquad \lambda k \in K.$$

### 1.3.2 Main Theorems Concerning Maximal Monotone Operators

In this section we will recall some fundamental results about maximal monotone operators, mainly focusing on three aspects: convexity of the interior/closure of the domain and the range of a maximal monotone operator; the problem of the maximality of the sum of two maximal monotone operators and the related Attouch-Brézis conditions; the famous surjectivity theorem that characterizes the maximality of a monotone operator.

**Properties of the Domain and the Range**

In principle, one could conjecture that the domain of the subdifferential $\partial f$ of a proper lower semicontinuous convex function $f : X \to \overline{\mathbb{R}}$ be convex, though not necessarily coincident with $\operatorname{dom} f$. Anyway, this is not true in general (see [96, Chapter 27] for an example). Therefore, one cannot expect the domain of a maximal monotone operator to be necessarily convex, though Rockafellar proved that, under a suitable hypothesis, it is near to be convex, in the sense that its interior and its closure are both convex (see also [75, Theorem 1.9]).

**Theorem 1.3.8 ([82, Theorem 1])** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a maximal monotone operator and suppose that* $\operatorname{int}\operatorname{conv}\mathcal{D}(T) \neq \emptyset$*. Then:*

*(a)* $\operatorname{int}\mathcal{D}(T) = \operatorname{int}\operatorname{conv}\mathcal{D}(T)$*;*

*(b)* $\operatorname{cl}\mathcal{D}(T) = \operatorname{cl}\operatorname{int}\mathcal{D}(T)$*;*

*(c) $T$ is locally bounded at every point of* $\operatorname{int}\mathcal{D}(T)$*.*

Two remarks are in order in connection with the previous theorem. The first one is that, when $X$ is reflexive, one can apply the same theorem to $T^{-1} : X^* \rightrightarrows X$, obtaining the convexity of $\operatorname{int}\mathcal{R}(T)$ and (when this set is nonempty) of $\operatorname{cl}\mathcal{R}(T)$ [75, Corollary 1.10].

**Corollary 1.3.9** *Let $X$ be a reflexive Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then:*

*(a)* $\operatorname{int}\mathcal{R}(T)$ *is convex;*

*(b) if* $\operatorname{int}\mathcal{R}(T) \neq \emptyset$*, the set* $\operatorname{cl}\mathcal{R}(T)$ *is convex as well.*

Notice that, in the previous corollary, the reflexivity assumption is necessary, as proven by a counterexample provided by Fitzpatrick (see [75, Example 2.21]).

The second remark concerns the local boundedness property. It was proved by Borwein and Fitzpatrick [13] that this property actually holds on a set that is bigger (at least in principle) than the interior of the domain of the operator. Recall that a point $a \in A \subseteq X$ is an *absorbing point* of $A$ if the set $A - a$ is absorbing, i.e. if, for all $x \in X$, there exists $\lambda > 0$ such that $\lambda x \in A - a$ (the condition $0 \in A - a$ is satisfied by definition in this case).

**Theorem 1.3.10** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a maximal monotone operator and $x \in \mathcal{D}(T)$. If $x$ is an absorbing point of $\mathcal{D}(T)$, then $T$ is locally bounded at $x$.*

Though $\operatorname{int} \mathcal{D}(T)$ is contained in the set of absorbing points of $\mathcal{D}(T)$, the inclusion is actually an equality when $\operatorname{cl} \mathcal{D}(T)$ is convex. Indeed, Veselý proved that, in this case, the local boundedness of $T$ at a point $x \in \operatorname{cl} \mathcal{D}(T)$ implies $x \in \operatorname{int} \mathcal{D}(T)$ (see [75, Theorem 1.14]).

Finally, still concerning boundedness properties, it can be proved that a maximal monotone operator $T : X \rightrightarrows X^*$ such that its range, $\mathcal{R}(T)$, is bounded, has full domain, i.e. $\mathcal{D}(T) = X$ (see [96, Chapter 25] for a proof using convex representations and for references to other, more traditional proofs).

### Maximality of the Sum

An important problem to which much attention has been paid is that of the maximal monotonicity of the sum of two maximal monotone operators $S, T : X \rightrightarrows X^*$, defined as the operator $S + T : X \rightrightarrows X^*$ such that

$$x^* \in (S+T)(x) \quad \Longleftrightarrow \quad \exists y^* \in X: \;\; y^* \in S(x), \;\; x^* - y^* \in T(x).$$

The sum is indeed the simplest and most common combination of two operators, by means of which one could hope to obtain a new element in the family of maximal monotone operators, given two of them. Anyway, it turns out that, while the family of monotone operators is closed with respect to addition, this is not the case for the subfamily of maximal monotone ones. A sufficient condition in the reflexive setting was provided by Rockafellar [85].

**Theorem 1.3.11 ([85, Theorem 1])** *Let $X$ be a reflexive Banach space and $S, T : X \rightrightarrows X^*$ be two maximal monotone operators. If*

$$\mathcal{D}(S) \cap \operatorname{int} \mathcal{D}(T) \neq \emptyset, \tag{1.5}$$

*then the sum $S + T : X \rightrightarrows X^*$ is a maximal monotone operator.*

The problem whether (1.5) is a sufficient condition also for arbitrary maximal monotone operators defined on nonreflexive Banach spaces is still open and is known as Rockafellar's conjecture. At least for some particular cases we know that condition (1.5) is indeed sufficient. This is true, for instance, when both $S$ and $T$ are subdifferentials of proper lower semicontinuous convex functions, or when $S$ is of this kind, while $T$ is an arbitrary maximal monotone operator such that $\mathcal{D}(T) = X$ [104]. In the general case, intense research has been conducted in order to find less restrictive conditions one can add to (1.5) to ensure the maximality of the sum. The most famous one, apart from that of Rockafellar, is probably Attouch-Brézis condition, which reads

$$\bigcup_{\lambda>0} \lambda[\mathcal{D}(S) - \mathcal{D}(T)] \quad \text{is a closed subspace of } X. \tag{1.6}$$

The search for new conditions has been fostered by the new approach to maximal monotonicity via convex representations, so that we will have to come back to this issue in the next chapter.

We only point out here that another way to tackle the problem has been pursued. Namely, instead of adding conditions on the domains of the operators in order to have the graph of the sum big enough to be maximal monotone, new generalized notions of sum have been introduced. In particular, the so called *extended sum* [77], which is based on the concept of enlargement (see Section 2.3 below), and the *variational sum* [2, 76], which is based instead on the Yosida regularization of a maximal monotone operator. For both these generalized sums we have that the subdifferential of the sum of two proper lower semicontinuous convex functions is equal to the generalized sum of the subdifferentials of the two functions, without imposing any further condition (except that the domains of the two functions have to intersect at some point). Moreover, it has been proved in [36] that the graph of the extended sum of two maximal monotone operators is contained in the graph of the variational sum.

#### The Surjectivity Theorem

A special case, which is of particular interest when considering the sum of maximal monotone operators, arises if the Banach space $X$ is reflexive and one chooses the duality mapping to be one of the two operators to be added. Obviously, in this case the sum is a maximal monotone operator, since $\mathcal{D}(J) = X$, so that (1.5) is satisfied, but the interest in this specific instance is mainly due to the following two-fold reason:

(a) from a theoretical point of view, when $X$ is reflexive, surjectivity of the sum of a monotone operator $T$ with the duality mapping is equivalent to the maximal monotonicity of $T$;

(b) from a practical perspective, the previous property allows to solve inclusions involving the operator $T$ via a perturbation method.

Concerning $(a)$, the characterization we mentioned is the well-known surjectivity theorem proved by Minty [66] for monotone operators defined on Hilbert spaces and extended by Browder [18] and Rockafellar [85] to the more general case of a reflexive Banach space $X$ such that the norm of $X$ and the norm of $X^*$ are everywhere Gâteaux differentiable, except at the origin. This assumption is not excessively restrictive, since Asplund [1] proved that any reflexive Banach space admits an equivalent norm for which the previous property holds. Thus, it is always possible to renorm a given space in such a way that the differentiability assumption on the norm be fulfilled, while this operation does not affect the maximal monotonicity of the operators defined on that space.

**Theorem 1.3.12 ([85, Corollary p. 78])** *Let $X$ be a reflexive Banach space such that the norm of $X$ and the norm of $X^*$ are everywhere Gâteaux differentiable, except at the origin, and let $T : X \rightrightarrows X^*$ be a monotone operator. Then the following are equivalent:*

*(a) $T$ is maximal monotone;*

*(b) $\mathcal{R}(T + J) = X^*$.*

In the last decades several generalizations of this theorem appeared in the literature, mainly along three paths: first, the dismissing of the renorming assumption; second, the possibility to extend in some way the result to nonreflexive Banach spaces; finally, the possibility to replace the duality mapping by more general maximal monotone operators satisfying appropriate conditions. The latter generalization was very recently proposed in [59], using convex representations (see Section 2.2 below), while the second one was already considered by J.-P. Gossez [37] in the 1970s and was recently rediscovered and enriched in [58] (again, by means of convex representations of maximal monotone operators). On the other hand, the first generalization we mentioned found a very nice formulation in [93, Theorem 10.6], in terms of the sum of the graphs of the operator $T$ and the negative of the duality mapping.

**Theorem 1.3.13 ([93, Theorem 10.6])** *Let $X$ be a reflexive Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. Then the following are equivalent:*

*(a) $T$ is maximal monotone;*

*(b) $\mathcal{G}(T) + \mathcal{G}(-J) = X \times X^*$.*

Though, properly speaking, item $(b)$ is not a surjectivity property, we will call it this way in Chapter 5, motivated by the fact that, in the case of the duality mapping, it is equivalent to surjectivity, according to the following easy corollary.

**Corollary 1.3.14** *Let $X$ be a reflexive Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. Then the following are equivalent:*

*(a) $T$ is maximal monotone;*

*(b) $\mathcal{R}(T(\cdot + w) + J(\cdot)) = X^*$ for all $w \in X$.*

*Proof.* The result follows from Theorem 1.3.13, since, taking into account the symmetry of the duality mapping, we obtain, for all $(w, w^*) \in X \times X^*$,

$$\begin{aligned}
(w, w^*) \in \mathcal{G}(T) + \mathcal{G}(-J) \quad &\Longleftrightarrow \quad \exists (x, x^*) \in X \times X^* : \ (x + w, x^* + w^*) \in \mathcal{G}(T), \ (-x, x^*) \in \mathcal{G}(J) \\
&\Longleftrightarrow \quad \exists (x, x^*) \in X \times X^* : \ (x + w, x^* + w^*) \in \mathcal{G}(T), \ (x, -x^*) \in \mathcal{G}(J) \\
&\Longleftrightarrow \quad w^* \in \mathcal{R}(T(\cdot + w) + J(\cdot)).
\end{aligned}$$

□

Notice that, in the literature, the name of Rockafellar's surjectivity theorem is employed to make reference both to the characterization contained in Theorem 1.3.12 and to implication $(a) \Longrightarrow (b)$ of the same theorem alone. As the latter implication is concerned, the same paper of Rockafellar [85] provides the result on which a perturbation method for finding solutions of operator inclusions can be based.

**Theorem 1.3.15 ([85, Proposition 1])** *Let $X$ be a reflexive Banach space such that the norm of $X$ and the norm of $X^*$ are everywhere Gâteaux differentiable, except at the origin, and let $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then, for all $\lambda > 0$, $\mathcal{R}(T + \lambda J) = X^*$ and $(T + \lambda J)^{-1} : X^* \rightrightarrows X$ is a single-valued maximal monotone operator, which is demicontinuous, i.e. continuous from the strong topology to the weak topology.*

### 1.3.3 Operators of Type (D) and Extensions to the Bidual

In the previous section we have seen that several nice properties of maximal monotone operators, like convexity-type properties of the range, maximality of the sum under condition (1.5) or (1.6) and Rockafellar's surjectivity theorem, hold in the case where $X$ is a reflexive Banach space. To recover these properties in nonreflexive spaces, Gossez introduced (in [39], modifying a previous definition given in [37]) a special class of monotone operators, called of type (D), with D standing for dense.

In the following, we will denote by $\sigma(X, X^*)$ and $\sigma(X^*, X)$ the weak topology of a Banach space $X$ and the weak* topology of its dual space $X^*$, respectively.

**Definition 1.3.16** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. We say that $T$ is of* type (D) *if, for all $(x^{**}, x^*) \in X^{**} \times X^*$, there exists a bounded net $(x_\alpha, x^*_\alpha)$ in $\mathcal{G}(T)$ that converges to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*)\otimes$ norm topology of $X^{**} \times X^*$.*

Another way to state the same definition implies reasoning on extensions of $T$ to the bidual, meant as operators $T' : X^{**} \rightrightarrows X^*$ such that $\mathcal{G}(T) \subseteq \mathcal{G}(T')$ (here and throughout in the following we will identify $X$ with its canonical embedding in $X^{**}$). In Chapters 5 and 6 we will use extensively the two extensions that we now define.

**Definition 1.3.17** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator.*

*(a) Let $\overline{T} : X^{**} \rightrightarrows X^*$ be the operator such that $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$ if and only if there exists a bounded net $(x_\alpha, x^*_\alpha)$ in $\mathcal{G}(T)$ that converges to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*)\otimes$ norm topology of $X^{**} \times X^*$.*

*(b) Let $\widetilde{T} : X^{**} \rightrightarrows X^*$ be the operator such that $(x^{**}, x^*) \in \mathcal{G}(\widetilde{T})$ if and only if $(x^{**}, x^*)$ is monotonically related to $\mathcal{G}(T)$.*

Comparing the previous definitions, we immediately conclude that $T$ is of type (D) if and only if $\overline{T} = \widetilde{T}$. While, in general, $\widetilde{T}$ is not monotone, if $T$ is monotone of type (D), then $\widetilde{T}$ is maximal monotone [75] and it can be proved that, in fact, it is the only maximal monotone extension of $T$ to the bidual.

The consideration of maximal monotone operators of type (D) is the most natural generalization of the theory of maximal monotone operators on reflexive spaces that one could conceive for the nonreflexive setting, according to the following examples.

**Example**

(a) When $X$ is a reflexive Banach space, any maximal monotone operator is of type (D), given that $T = \overline{T} = \widetilde{T}$.

(b) Even if $X$ is nonreflexive, if $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function, then its subdifferential, $\partial f : X \rightrightarrows X^*$, is a maximal monotone operator of type (D) and

$$\widetilde{\partial f} = (\partial f^*)^{-1}, \tag{1.7}$$

(see for instance [37], or Lemma 5.1.2 below).

Gossez proved that, as we anticipated above, some nice features of maximal monotone operators defined on reflexive Banach spaces also hold in the nonreflexive setting, provided that the operator under consideration is of type (D). In the following, for all $\varepsilon > 0$ we denote by $J_\varepsilon$ the $\varepsilon$-subdifferential of the function

$$f : X \to \mathbb{R}, \qquad x \mapsto \frac{1}{2}\|x\|^2.$$

**Theorem 1.3.18 ([37, Théorèmes 7.1, 4.1, Corollaire 5.1])** *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator of type (D). Then:*

*(a)* $\operatorname{cl} \mathcal{R}(T)$ *is a convex set;*

*(b)* $\mathcal{R}(T + \lambda(J_\varepsilon)) = X^*$*, for all* $\lambda, \varepsilon > 0$

*(c)* $\mathcal{R}(\overline{T} + \lambda\overline{J}) = X^*$*, for all* $\lambda > 0$.

Item $(a)$ extends to the nonreflexive framework item $(b)$ of Corollary 1.3.9 and, at the same time, shows that, even in the reflexive case, the condition that $\mathcal{R}(T)$ have nonempty interior is not necessary (applying the previous theorem to $T^{-1}$, in the reflexive case one also concludes that $\operatorname{cl} \mathcal{D}(T)$ is convex).

Item $(b)$, on the other hand, generalizes Theorem 1.3.15 substituting the duality mapping by its approximate version, while item $(c)$ generalizes the same theorem replacing the operators $T$ and $J$ by their maximal monotone extensions to the bidual.

# Chapter 2

# The Interplay between Convexity and Monotonicity

In the previous chapter we reviewed basic facts and definitions concerning both Convex Analysis and the theory of monotone operators. We now turn to the relationships between these two domains. Important similarities between convexity and maximal monotonicity appear at different levels, considering e.g. the almost convexity of the domains of maximal monotone operators, the local boundedness on the interior of the domains, or the qualification conditions for the sum. Exploitation of this close affinity has been conducted by several authors, in different ways, since the seventies, but a substantial increasing in the popularity of convex analytical tools for dealing with maximal monotone operators and the emergence of a predominant approach can be clearly recognized in the last decade. This new wave was fostered by the independent rediscovery and generalization of the approach proposed by S. Fitzpatrick [34] in 1988, operated by Martínez-Legaz and Théra [63] and Burachik and Svaiter [27] almost ten years ago.

The present chapter is organized as follows. The first section briefly reviews the approach to maximal monotone operators via skew-symmetric saddle functions developed by Krauss [47, 48, 49] in the mid-eighties. To our purposes, this section is meant as an introduction to motivate and historically locate the paper by Fitzpatrick [34], which proposes itself as an improvement upon the approach of Krauss. The second section describes Fitzpatrick's article and records some pioneering results obtained in the literature which refers to this approach. The criterion we adopted for the choice of the results to be included in this section is essentially their relevance for the understanding of the remaining of the thesis, so that our account of the existing literature will necessarily be incomplete[1]. According to the same criterion, in the third section we will mention a topic that is closely related to convex representations of maximal monotone operators, i.e. the notion of an enlargement of a maximal monotone operator, which we will need in the following.

Notice that several results that we will review were originally stated in, or could be extended to, locally convex spaces. Anyway, we will restrict ourselves to Banach spaces, the most common case considered both in the applications and in the literature.

## 2.1 Skew-Symmetric Saddle Functions and Monotone Operators

In a series of papers E. Krauss [47, 48, 49] showed the possibility to associate monotone operators to saddle functions and vice versa, while demonstrating the profitability of this procedure for the study of variational inequalities and differential equations.

We present here only some basic features of this approach, since we will not make reference to it in the following. Recall that, given two Banach spaces $X$ and $Y$, a saddle function $L : X \times Y \to \overline{\mathbb{R}}$ is a function that is concave in the first argument and convex in the second one. $L : X \times X \to \overline{\mathbb{R}}$ is skew-symmetric if

$$\mathrm{cl}_2 L(x, y) = -\mathrm{cl}_1 L(y, x), \quad \forall x, y \in X,$$

where $\mathrm{cl}_2 L$ is the closure of the convex function $L(x, \cdot)$ for each $x \in X$, while $\mathrm{cl}_1 L$ is the closure of the concave function $L(\cdot, y)$ for each $y \in X$. For skew-symmetric saddle functions the two sets $\mathrm{dom}_1 L := \{x \in X : \ \mathrm{cl}_2 L(x, y) > -\infty, \ \forall y \in X\}$ and $\mathrm{dom}_2 L := \{y \in X : \ \mathrm{cl}_1 L(x, y) < +\infty, \ \forall x \in X\}$ coincide and one can define $\mathrm{Dom}\, L := \mathrm{dom}_1 L = \mathrm{dom}_2 L$. In addition, verifying the lower (upper) closure of $L$ amounts to check the equality $\mathrm{cl}_2 L = L$ ($\mathrm{cl}_1 L = L$, respectively).

Given an arbitrary skew-symmetric saddle function $L : X \times X \to \overline{\mathbb{R}}$, Krauss [47] associates to it the operator $T_L : X \rightrightarrows X^*$ such that, for all $x \in X$,

$$x^* \in T(x) \quad \Longleftrightarrow \quad (-x^*, x^*) \in \partial L(x, x),$$

where $\partial L(x, y) := \partial_1 L(x, y) \times \partial_2 L(x, y)$ ($\partial_i L$ is the subdifferential of $L$ as a function of its $i$th argument, considering the other argument as fixed), and proves the following results.

[1] In particular, we will not survey here the new interesting abstract framework for the study of monotonicity introduced by S. Simons [95] with the notions of SSD space and $q$-positive set. For a detailed introduction, we refer the reader to [60, 95, 96].

**Theorem 2.1.1 ([47, Theorems 1,2])** *Let $X$ be a Banach space.*

*(a) $(x, x^*) \in \mathcal{G}(T_L)$ if and only if $\langle y - x, x^* \rangle \leq L(x, y)$, for all $y \in X$.*

*(b) The operator $T_L$ is monotone. If, additionally, $L$ is lower closed and $X$ is reflexive, then $T_L$ is maximal monotone.*

Notice that, when $L$ is defined as

$$L(x,y) := \begin{cases} f(y) - f(x), & x, y \in \operatorname{dom} f \\ +\infty, & x \in \operatorname{dom} f, y \notin \operatorname{dom} f \\ -\infty, & x \notin \operatorname{dom} f, \end{cases}$$

for some proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, then $L_T$ coincides with the subdifferential of $f$. Thus, as shown in the previous theorem, $T_L$ is obtained from $L$ similarly to the subdifferential of a convex function $f$ from $f$ itself.

The argument also works the other way around, giving the possibility to associate to a monotone operator $T$ a skew-symmetric saddle function $L_T$. In the case when $T$ is maximal monotone, this function represents $T$, in the sense that $T_{L_T} = T$, and is called a *Lagrangian saddle function* to $T$.

**Theorem 2.1.2 ([47, Theorem 4])** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator with $\mathcal{D}(T) \neq \emptyset$. Then there exists a lower closed skew-symmetric saddle function $L_T : X \times X \to \overline{\mathbb{R}}$ with*

$$\operatorname{conv} \mathcal{D}(T) \subseteq \operatorname{Dom} L_T \subseteq \operatorname{cl} \operatorname{conv} \mathcal{D}(T),$$

*such that $T_{L_T}$ is a monotone extension of $T$ with $\mathcal{D}(T_{L_T}) \subseteq \operatorname{cl} \operatorname{conv} \mathcal{D}(T)$. If $X$ is reflexive, then $T_{L_T}$ is a maximal monotone extension of $T$.*

Notice that, in general, the saddle function mentioned in the previous theorem is not unique. The previous result is useful in conjunction with the following variational formulation for inclusions with monotone operators introduced by Krauss.

**Theorem 2.1.3 ([47, Theorem 3])** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $L : X \times X \to \overline{\mathbb{R}}$ be a skew-symmetric saddle function such that $T_L$ is an extension of $T$. Then, for each solution to the inclusion $0 \in T(x)$, the couple $(x, x)$ is a saddle point of $L$. If $T$ is maximal monotone, then there are no further saddle points of the form $(x, x)$ and $L$ is a Lagrangian saddle function to $T$.*

The two-way interplay between monotone operators and saddle functions accurately studied by Krauss was an important step for an explicit investigation of the links between monotonicity and convexity. Anyway, the use of saddle functions implies a level of sophistication which can be avoided by simply using convex functions, as showed by Simon Fitzpatrick a few years later.

## 2.2 Convex Representations of Monotone Operators

### 2.2.1 Seminal Contributions

Inspired by the work of Krauss, Fitzpatrick [34] proposed a far simpler way to represent maximal monotone operators by means of the subdifferentials of convex functions, instead of the subdifferentials of saddle functions.

To understand the main difference between the two approaches, we can consider the case when the maximal monotone operator $T : X \rightrightarrows X^*$ is the subdifferential of some proper lower semicontinuous convex function $g : X \to \mathbb{R} \cup \{+\infty\}$ defined on the Banach space $X$, i.e. $T = \partial g$. In this case (see Section 1.2), the graph of $T$ can be alternatively characterized as the set of points $(x, x^*) \in X \times X^*$ that satisfy the inequality

$$g(y) \geq g(x) + \langle y - x, x^* \rangle, \ \forall y \in X,$$

that is,

$$\langle y - x, x^* \rangle \leq g(y) - g(x), \ \forall y \in X,$$

or the equality

$$g(x) + g^*(x^*) = \langle x, x^* \rangle. \tag{2.1}$$

The theory developed by Krauss generalizes the former characterization, replacing the difference $g(y) - g(x)$ by a more general skew-symmetric saddle function defined on the product space $X \times X$. The procedure employed by Fitzpatrick, on the contrary, generalizes the latter characterization, since considers a convex function defined on the product space $X \times X^*$ of the Banach space with its dual, leading to the comparison between the values taken by this function and the duality product[2].

[2]The two characterizations of subdifferentials are equivalent and are linked by duality. An analogous relation holds for the two approaches of Krauss and Fitzpatrick that we are considering, as already observed in [34, Theorems 4.5, 4.6].

More precisely, to each convex function $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ one can associate the operator $T_f : X \rightrightarrows X^*$ such that, for all $x \in X$,

$$T_f(x) := \{x^* \in X^* : \; (x^*, x) \in \partial f(x, x^*)\},$$

which is monotone [34, Proposition 2.2]. On the other hand, for any monotone operator $T : X \rightrightarrows X^*$ with $\mathcal{D}(T) \neq \emptyset$, Fitzpatrick [34, Definition 3.1] introduces the function $\varphi_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\varphi_T(x, x^*) := \sup\{\langle y, x^* \rangle + \langle x - y, y^* \rangle : \; (y, y^*) \in \mathcal{G}(T)\} \tag{2.2}$$

for all $(x, x^*) \in X \times X^*$. This function is now called the *Fitzpatrick function* of $T$ and in the following we will always denote it by $\varphi_T$. By its very definition, $\varphi_T$ is convex and lower semicontinuous. The following theorem proves the analogy with (2.1) that we anticipated above.

**Theorem 2.2.1 ([34, Theorem 3.4])** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $(x, x^*) \in \mathcal{G}(T)$. Then $\varphi_T(x, x^*) = \langle x, x^* \rangle$ and $(x, x^*) \in \partial\varphi_T(x, x^*)$.*

As a consequence, if $T$ is a monotone operator, $T_{\varphi_T}$ is an extension of $T$ and, in particular, $T = T_{\varphi_T}$ whenever $T$ is maximal monotone [34, Corollary 3.5]. Anyway, the converse of the latter property is not true, as shown e.g. by the operator with graph $\mathcal{G}(T) := \{(0_X, 0_{X^*})\}$.
The Fitzpatrick function associated to a monotone operator $T$ enjoys an important minimality property[3] and, when $T$ is maximal monotone, it represents $T$ in the sense that it gives a complete characterization of its graph, as is specified in the last part of the following theorem.

**Theorem 2.2.2 ([34, Theorems 3.7, 3.8, Corollary 3.9])** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator.*

*(a) If $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ is a convex function such that $f(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ and if $f(y, y^*) = \langle y, y^* \rangle$ for all $(y, y^*) \in \mathcal{G}(T)$, then $\varphi_T \leq f$.*

*(b) $T$ is maximal monotone if and only if $\varphi_T(x, x^*) > \langle x, x^* \rangle$ for all $(x, x^*) \in (X \times X^*) \setminus \mathcal{G}(T)$.*

*(c) If $T$ is maximal monotone, then $\varphi_T(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ and $\varphi_T(x, x^*) = \langle x, x^* \rangle$ if and only if $(x, x^*) \in \mathcal{G}(T)$.*

[3]For this minimality property (stated in item $(a)$ of Theorem 2.2.2) a converse holds, when reasoning on maximal monotone operators defined on reflexive Banach spaces. Indeed, [62] proves that, in this setting, any minimal element in the family of convex functions bounded below by the duality product is the Fitzpatrick function of some maximal monotone operator.

The interest in convex functions for the study of monotone operators continued during the nineties, thanks in particular to the work of S. Simons and coauthors (see e.g. [32]), but definitely gained new momentum about a decade after the seminal paper of Fitzpatrick, when his approach was reintroduced, from different perspectives, in two papers. The first one was a brief article by Martínez-Legaz and Théra [63]. In this paper the authors introduce the family $\Phi(X)$ of all proper lower semicontinuous convex functions $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that

$$f(x, x^*) = (f + \delta_{b(f)})^*(x^*, x), \quad \forall (x, x^*) \in X \times X^*,$$

where

$$b(f) := \{(x, x^*) \in X \times X^* : \ f(x, x^*) \leq \langle x, x^* \rangle\}$$

and $X$ is a Banach space, as usual. Their main result establishes a one-to-one and onto correspondence between $\Phi(X)$ and the family $\mathfrak{M}(X)$ of all maximal monotone operators $T : X \rightrightarrows X^*$, by newly introducing the Fitzpatrick function, now written as

$$\varphi_T(x, x^*) := \langle x, x^* \rangle - \inf_{(y, y^*) \in \mathcal{G}(T)} \langle x - y, x^* - y^* \rangle. \tag{2.3}$$

**Theorem 2.2.3 ([63, Theorem 2])** *For any $T \in \mathfrak{M}(X)$, one has $\varphi_T \in \Phi(X)$. Moreover, the mapping*

$$\mathfrak{M}(X) \ni T \mapsto \varphi_T \in \Phi(X)$$

*is a bijection, with inverse*

$$\Phi(X) \ni f \mapsto T_f \in \mathfrak{M}(X),$$

*where $T_f(x) := \{x^* \in X^* : \ f(x, x^*) = \langle x, x^* \rangle\}$.*

The article of Burachik and Svaiter [27], on the other hand, studies the whole family of convex representations associated to a maximal monotone operator and its relations with the family of enlargements of the same operator (see Section 2.3 below for the precise definition of an enlargement). Indeed, considering a maximal monotone operator $T : X \rightrightarrows X^*$ defined on a real Banach space $X$, the authors introduce the family $\mathcal{H}_T$ of convex representations of $T$, defined as

$$\begin{aligned} \mathcal{H}_T \ := \ & \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} : \quad h \quad \text{is lower semicontinuous and convex,} \\ & h(x, x^*) \geq \langle x, x^* \rangle \ \forall (x, x^*) \in X \times X^*, \quad h(x, x^*) = \langle x, x^* \rangle \ \forall (x, x^*) \in \mathcal{G}(T)\}. \end{aligned} \tag{2.4}$$

We will present the relation of $\mathcal{H}_T$ with the family of enlargements of $T$ in Section 2.3 below, while we will now concentrate on the structure of $\mathcal{H}_T$.

Along with $\varphi_T$, another important representation of $T$ is introduced, that is, the function $\sigma_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\sigma_T := \text{cl conv}\,(\langle \cdot, \cdot \rangle + \delta_{\mathcal{G}(T)}).$$

Note that $\varphi_T(x, x^*) = \sigma_T^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$ [61], while $\varphi_T^*(x^*, x) \leq \sigma_T(x, x^*)$, with equality when $X$ is reflexive[4].

The following theorem collects the main results of [27, Sections 4, 5] concerning $\mathcal{H}_T$.

**Theorem 2.2.4 ([27, Sections 4, 5])** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator.*

*(a) If $h \in \mathcal{H}_T$, then $h(x, x^*) = \langle x, x^* \rangle$ if and only if $(x, x^*) \in \mathcal{G}(T)$.*

*(b) $\varphi_T$ and $\sigma_T$ are the minimum and the maximum of $\mathcal{H}_T$, respectively, so that*

$$\begin{aligned} \mathcal{H}_T \quad := \quad & \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} : \quad h \quad \text{is lower} \\ & \text{semicontinuous and convex, and} \quad \varphi_T \leq h \leq \sigma_T\}. \end{aligned} \tag{2.5}$$

*(c) If $h, k \in \mathcal{H}_T$ and $\lambda \in [0, 1]$, then $\lambda h + (1 - \lambda)k \in \mathcal{H}_T$.*

*(d) If $(h_i)_{i \in I}$ is a nonempty family in $\mathcal{H}_T$, then $\sup_{i \in I} h_i \in \mathcal{H}_T$.*

*(e) $\mathcal{H}_T$ is invariant under the operator $\mathscr{J}$ defined as $\mathscr{J} h(x, x^*) := h^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$. More precisely, the operator $\mathscr{J}$ maps $\mathcal{H}_T$ into itself and, if $X$ is reflexive, it is a bijection on $\mathcal{H}_T$.*

Notice that, in particular, $(h^*)^\top|_{X \times X^*} \in \mathcal{H}_T$ for all $h \in \mathcal{H}_T$, a fact that we will use extensively in the following.

The function $\sigma_T$ was also extensively studied and employed by Penot [72], in the reflexive setting. The first part of the paper proposes a systematic account of the main properties of $\sigma_T$ and its relations with other representations (including that of Krauss), while the second part concentrates on two main topics: existence of autoconjugate representations and maximality of sums and compositions of maximal monotone operators. An autoconjugate representation of a maximal monotone operator $T : X \rightrightarrows X^*$ is a proper lower semicontinuous convex function $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $f^*(x^*, x) = f(x, x^*)$ and $f(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$, with $f(x, x^*) = \langle x, x^* \rangle$ if and only if $(x, x^*) \in \mathcal{G}(T)$. [72, Theorem 10] proves that any maximal monotone operator admits an autoconjugate representation. Finally, concerning the composition

[4] The representation dual to $\varphi_T$ was already studied in [34].

and the sum of operators, [72] provides the following results using convex representations. In particular, the Attouch-Brézis conditions are obtained by means of convex representations.

**Theorem 2.2.5 ([72, Theorems 14, 15])** *Let $X$ be a reflexive Banach space.*

*(a) Let $A : X \to Y$ be a continuous linear map and $N : Y \rightrightarrows Y^*$ be a maximal monotone operator. Suppose that*

$$\bigcup_{\lambda>0} \lambda(\text{conv}\,\mathcal{D}(N) - \mathcal{R}(A)) = Y.$$

*Then $A^T NA$ is maximal monotone.*

*(b) Let $S, T : X \rightrightarrows X^*$ be maximal monotone operators such that*

$$\bigcup_{\lambda>0} \lambda(\text{conv}\,\mathcal{D}(S) - \text{conv}\,\mathcal{D}(T) = X.$$

*Then $S + T$ is maximal monotone.*

### 2.2.2 The Fitzpatrick Family and Representable Operators

In the remaining of this section we will restrict our consideration to those problems and articles that will be explicitly required to make the following chapters self-contained.
We first extend the definition of $\mathcal{H}_T$ introduced in the previous subsection. Indeed $\mathcal{H}_T$, as defined in (2.4), can be attached to any monotone operator, not necessarily maximal, due to the following result.

**Theorem 2.2.6 ([61, Theorem 5])** *Let $X$ be a Banach space. The operator $T : X \rightrightarrows X^*$ is monotone if and only if there exists a convex function $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $h(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ and $h(x, x^*) = \langle x, x^* \rangle$ for all $(x, x^*) \in \mathcal{G}(T)$. Moreover, $h$ can be taken to be lower semicontinuous.*

We will call $\mathcal{H}_T$ the *Fitzpatrick family* of $T$. According to [61, Proposition 9], the properties of $\mathcal{H}_T$ listed in items $(c)$ and $(d)$ of Theorem 2.2.4 still hold true.

Note that, in general, when $h \in \mathcal{H}_T$ and $T$ is not maximal monotone, there will be points for which $h(x, x^*) = \langle x, x^* \rangle$, even though $(x, x^*) \notin \mathcal{G}(T)$. This justifies the introduction of the following definition.

**Definition 2.2.7 ([61])** *Let $X$ be a Banach space. The monotone operator $T : X \rightrightarrows X^*$ is* representable *if there exists $h \in \mathcal{H}_T$ such that $h(x, x^*) = \langle x, x^* \rangle$ if and only if $(x, x^*) \in \mathcal{G}(T)$. The function $h$ is called a* (convex) representation *of $T$.*

Any maximal monotone operator is representable, but the converse is not true. Take for instance the monotone operator the graph of which coincides with the singleton $\{(0_X, 0_{X^*})\}$. It is representable by means of $\sigma_T = \delta_{\{(0_X,0_{X^*})\}}$, but it is certainly not maximal monotone.

From time to time, in the following we will use the name convex representation also in a broader sense, to make reference to any element of $\mathcal{H}_T$ for a non-representable operator $T$, or even of the bigger family $\mathcal{K}_T$, defined as

$$\begin{aligned} \mathcal{K}_T \quad := \quad & \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} : \quad h \quad \text{is lower} \\ & \text{semicontinuous and convex, and } \ \varphi_T \le h \le \sigma_T\}. \end{aligned} \tag{2.6}$$

**Remark 2.2.8** *(a)* In the case when $T$ is maximal monotone, Theorem 2.2.4 implies that $\mathcal{H}_T = \mathcal{K}_T$. If $T$ is monotone but not maximal, it remains true that $\mathcal{H}_T \subseteq \mathcal{K}_T$, as a consequence of Theorem 2.2.4 and of the definition of $\sigma_T$.
Furthermore, for all $h \in \mathcal{K}_T$ and $(x, x^*) \in \mathcal{G}(T)$, one has $h(x, x^*) = \langle x, x^*\rangle$, given that $\varphi_T(x, x^*) = \sigma_T(x, x^*) = \langle x, x^*\rangle$ for all $(x, x^*) \in \mathcal{G}(T)$.

*(b)* For all $h \in \mathcal{K}_T$, one has $(h^{*\top})|_{X\times X^*} \in \mathcal{K}_T$.

*Proof.* We only need to prove item $(b)$. Since $\varphi_T \le h \le \sigma_T$, then

$$\varphi_T = (\sigma_T^{*\top})|_{X\times X^*} \le (h^{*\top})|_{X\times X^*} \le (\varphi_T^{*\top})|_{X\times X^*}.$$

Since $\sigma_T$ is the greatest closed convex function majorized by the function $g : X \times X^* \to \overline{\mathbb{R}}$ such that $g(x, x^*) = \langle x, x^*\rangle + \delta_{\mathcal{G}(T)}(x, x^*)$ for all $(x, x^*) \in X \times X^*$ and given that $(\varphi_T^{*\top})|_{X\times X^*}$ is lower semicontinuous and convex, it suffices to prove that $(\varphi_T^{*\top})|_{X\times X^*}$ is majorized by the duality product on $\mathcal{G}(T)$. Indeed, for any $(x, x^*) \in \mathcal{G}(T)$, one has

$$\begin{aligned} (\varphi_T)^*(x^*, x) &= \sup_{(y,y^*)\in X\times X^*} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \varphi_T(y, y^*)\} \\ &= \sup_{(y,y^*)\in X\times X^*} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \langle x, x^*\rangle + \langle x, x^*\rangle - \varphi_T(y, y^*)\} \\ &\le \langle x, x^*\rangle + \sup_{(y,y^*)\in X\times X^*} \left\{ \sup_{(z,z^*)\in\mathcal{G}(T)} \{\langle z, y^*\rangle + \langle y, z^*\rangle - \langle z, z^*\rangle\} - \varphi_T(y, y^*) \right\} \\ &= \langle x, x^*\rangle + \sup_{(y,y^*)\in X\times X^*} \{\varphi_T(y, y^*) - \varphi_T(y, y^*)\} \\ &= \langle x, x^*\rangle \end{aligned}$$

and this completes the proof. □

In particular, the previous remark implies that, given a maximal monotone operator $T : X \rightrightarrows X^*$, for any $h \in \mathcal{H}_T$ also its Fenchel conjugate majorizes the duality product on $X \times X^*$, since $(h^{*\top})|_{X\times X^*} \in \mathcal{H}_T$ implies

$$h^*(x^*, x) \geq \varphi_T(x, x^*) \geq \langle x, x^* \rangle.$$

Thus, for any maximal monotone operator $T$ and $h \in \mathcal{H}_T$, one has $h(x, x^*), h^*(x^*, x) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ and

$$\mathcal{G}(T) = \{(x, x^*) \in X \times X^* : \quad h(x, x^*) = \langle x, x^* \rangle\}.$$

A natural and interesting question is whether the converse holds true as well. Burachik and Svaiter [28] show that the answer is in the positive if the Banach space $X$ is reflexive.

**Theorem 2.2.9 ([28, Theorem 3.1])** *Let $X$ be a reflexive Banach space and $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function such that, for all $(x, x^*) \in X \times X^*$,*

$$h(x, x^*) \geq \langle x, x^* \rangle \quad \textit{and} \quad h^*(x^*, x) \geq \langle x, x^* \rangle.$$

*Then the operator defined by*

$$\mathcal{G}(T) := \{(x, x^*) \in X \times X^* : \quad h(x, x^*) = \langle x, x^* \rangle\}$$

*is maximal monotone and $h, (h^{*\top})|_{X\times X^*} \in \mathcal{H}_T$.*

Two possible generalizations of this result to nonreflexive Banach spaces were obtained by Marques Alves and Svaiter by strengthening the hypotheses of the previous theorem in different ways.

**Theorem 2.2.10 ([53, Corollary 4.4],[54, Theorem 1.2] and [56, Theorem 3.1])** *Let $X$ be a Banach space, $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function and $T : X \rightrightarrows X^*$ be defined by*

$$\mathcal{G}(T) := \{(x, x^*) \in X \times X^* : \quad h^*(x^*, x) = \langle x, x^* \rangle\}.$$

*(a) If*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*,$$

$$h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle, \quad \forall (x^{**}, x^*) \in X^{**} \times X^*,$$

*then $T$ is maximal monotone of type (D) and $(h^{*\top})|_{X\times X^*} \in \mathcal{H}_T$. If, moreover, $h$ is lower semicontinuous, then $h \in \mathcal{H}_T$ as well.*

*(b) If*

$$h(x, x^*) \geq \langle x, x^* \rangle \quad \textit{and} \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*,$$

*and*

$$\bigcup_{\lambda > 0} \lambda \mathrm{Pr}_X(\mathrm{dom}\, h) \quad \textit{is a closed subspace of } X,$$

*then $T$ is maximal monotone and $(h^{*\top})|_{X \times X^*} \in \mathcal{H}_T$.*

### 2.2.3 Extensions to the Bidual and Operators of Type (D)

Theorem 2.2.10 provides a typical example of how simple and nice results that are true in reflexive Banach spaces can be generalized to the nonreflexive setting, either considering particularly well-behaved operators like monotone operators of type (D), or imposing additional qualification conditions. Another instance that fits this observation is the problem of maximality of the sum of two maximal monotone operators. In the general case of nonreflexive Banach spaces, the Attouch-Brézis condition in item $(b)$ of Theorem 2.2.5 is not sufficient in order to guarantee the maximality of the sum. Many papers (see for instance [3, 31, 99, 106, 107, 109] and the book [15, Chapter VI]) have investigated possible refinements of that condition and of the additional qualifications one has to impose. We only mention here one of the most recent contributions in this area. In the following theorem, given a subset $A$ of a Banach space $X$, ${}^{ic}A$ is the empty set if the affine hull aff $A$ (i.e., the intersection of all affine sets that contain $A$) is not closed, while, if aff $A$ is closed, it is the relative algebraic interior of $A$, that is to say, the set of points $a \in X$ for which, for all $x \in \mathrm{aff}\,(A - A)$, there exists $\delta > 0$ such that $a + \lambda x \in A$ for all $\lambda \in [0, \delta]$ (see [108]).

**Theorem 2.2.11 ([107, Corollary 4])** *Let $X$ be a Banach space and $M, N : X \rightrightarrows X^*$ be maximal monotone operators. If ${}^{ic}\mathcal{D}(M)$ and ${}^{ic}\mathcal{D}(N)$ are nonempty and*

$$0 \in {}^{ic}(\mathcal{D}(M) - \mathcal{D}(N)),$$

*then $M + N$ is maximal monotone.*

The Attouch-Brézis qualification is instead sufficient to ensure maximality if the two maximal monotone operators to be added are of type (D), according to the following result.

**Theorem 2.2.12 ([58, Lemma 3.5])** *Let $X$ be a Banach space and $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D). Given $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$, define $f : X \times X^* \to \overline{\mathbb{R}}$ as*

$$f(x, x^*) := \inf_{y^* \in X^*} \{h(x, y^*) + k(x, x^* - y^*)\}.$$

*If*

$$\bigcup_{\lambda>0} \lambda[\mathrm{Pr}_X(\mathrm{dom}\, h) - \mathrm{Pr}_X(\mathrm{dom}\, k)]$$

*is a closed subspace of $X$, then*

$$f(x,x^*) \geq \langle x, x^*\rangle, \quad \forall (x,x^*) \in X \times X^*,$$

$$f^*(x^*,x^{**}) \geq \langle x^{**}, x^*\rangle, \quad \forall (x^{**},x^*) \in X^{**} \times X^*,$$

$$\mathcal{G}(S+T) = \{(x,x^*) \in X \times X^* :\ f(x,x^*) = \langle x,x^*\rangle\} = \{(x,x^*) \in X \times X^* :\ f^*(x^*,x) = \langle x,x^*\rangle\}$$

*and $S+T$ is a maximal monotone operator of type (D), with* $\mathrm{cl}\, h, (h^{*\top})|_{X\times X^*} \in \mathcal{H}_T$.

In the previous theorem, as in item $(a)$ of Theorem 2.2.10, the condition that the Fenchel conjugate of a convex representation of an operator majorizes the duality product on $X^{**} \times X^*$ appears. A key result obtained and thoroughly investigated by Marques Alves and Svaiter in a series of very recent papers [53, 54, 57] states that the existence of a convex representation $h$ of the monotone operator $T$ such that $h^*(x^*,x^{**}) \geq \langle x^{**},x^*\rangle$ for all $(x^{**},x^*) \in X^{**} \times X^*$ is equivalent to $T$ being maximal monotone of type (D).

This characterization has been achieved in two main steps. First of all, Simons [92], in an attempt at defining a class of operators broader than that of type (D), but still preserving the same nice properties, introduced maximal monotone operators of type (NI) (from *negative infimum*).

**Definition 2.2.13** *Let $X$ be a Banach space. An operator $T : X \rightrightarrows X^*$ is of* type (NI) *if, for all $(x^{**},x^*) \in X^{**} \times X^*$,*

$$\inf_{(y,y^*)\in\mathcal{G}(T)} \langle x^{**} - y, x^* - y^*\rangle \leq 0.$$

Marques Alves and Svaiter [53, 54, 55, 58] proved several useful properties of maximal monotone operators of type (NI). Recall that $\widetilde{T}$ was defined in Section 1.3.3 by the relation

$$(x^{**},x^*) \in \mathcal{G}(\widetilde{T}) \quad \Longleftrightarrow \quad \langle x^{**} - y, x^* - y^*\rangle \geq 0,\ \ \forall (y,y^*) \in \mathcal{G}(T).$$

**Theorem 2.2.14 [55, Theorem 1.1]** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator of type (NI), which is equivalent to*

$$(\sigma_T)^*(x^*,x^{**}) \geq \langle x^{**},x^*\rangle, \quad \forall (x^*,x^{**}) \in X^* \times X^{**}.$$

*Then*

*(a)* $\widetilde{T} : X^{**} \rightrightarrows X^*$ *is the unique maximal monotone extension of* $T$ *to the bidual;*

*(b)* $(\sigma_T)^{*\top} = \varphi_{\widetilde{T}}$*;*

*(c) for all* $h \in \mathcal{H}_T$*,*

$$h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**},$$
$$h^{*\top} \in \mathcal{H}_{\widetilde{T}};$$

*(d)* $T$ *satisfies the strict Brønsted-Rockafellar property (see Definition 2.2.15 below).*

An immediate consequence of item $(b)$ above is that any maximal monotone operator $T$ of type (D) satisfies

$$\varphi_{\widetilde{T}}|_{X \times X^*} = \varphi_T. \tag{2.7}$$

Indeed, from the definition of $\varphi_T$ it follows that $((\sigma_T)^{*\top})|_{X \times X^*} = \varphi_T$.

Regarding item $(d)$, recall that the strict Brønsted-Rockafellar property is defined in [53] as follows.

**Definition 2.2.15** *Let* $X$ *be a Banach space and* $T : X \rightrightarrows X^*$ *be an operator. We say that* $T$ *satisfies the* strict Brønsted-Rockafellar property *when, for all* $\eta, \varepsilon$ *such that* $0 < \varepsilon < \eta$ *and for all* $(x, x^*) \in X \times X^*$*, if*

$$\inf_{(y,y^*) \in \mathcal{G}(T)} \langle x - y, x^* - y^* \rangle \geq -\varepsilon,$$

*then, for any* $\lambda > 0$ *there exists* $(x_\lambda, x^*_\lambda) \in \mathcal{G}(T)$ *such that*

$$\|x - x_\lambda\| < \lambda, \qquad \|x^* - x^*_\lambda\| < \frac{\eta}{\lambda}.$$

Following Simons [96, Definition 36.13], we will say that an operator satisfying the strict Brønsted-Rockafellar property is of type (BR).

Concerning the property of item $(c)$ of Theorem 2.2.14, the authors proved that, actually, it provides a characterization of maximal monotone operators of type (NI).

**Theorem 2.2.16 [54, Theorem 1.2]** *Let* $X$ *be a Banach space and* $T : X \rightrightarrows X^*$ *be a maximal monotone operator.* $T$ *is of type (NI) if and only if there exists* $h \in \mathcal{H}_T$ *such that* $h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle$ *for all* $(x^{**}, x^*) \in X^{**} \times X^*$*.*

The second step to obtain the equivalence between conjugate representations majorizing the duality product and the corresponding operator being of type (D) was the discovery that the families of type (D) and type (NI) operators coincide. As already observed in [92], any maximal monotone operator of type (D) is of type (NI). The converse was proved by Marques Alves and Svaiter [57]. Thus, taking into account Theorem 2.2.14, the following implications between classes of maximal monotone operators hold

$$(D) \quad \Longleftrightarrow \quad (NI) \quad \Longleftrightarrow \quad (MA) \quad \Longrightarrow \quad (BR),$$

where the class (MA), introduced in [54], consists of those maximal monotone operators $T$ that have a representation $h \in \mathcal{H}_T$ such that $h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle$ for all $(x^{**}, x^*) \in X^{**} \times X^*$.

To complete the previous chain of implications, we could add that the property of being of type (D) also implies that the maximal monotone operator admits a unique extension to the bidual which is maximal monotone with respect to the duality product on $X^{**} \times X^*$, as follows from item $(a)$ of Theorem 2.2.14. One could ask whether the opposite implication also holds, namely, whether the property of having a unique maximal monotone extension to the bidual implies in turn that the operator is of type (D). Marques Alves and Svaiter [55] proved that the answer is in the negative (we present the result according to the formulation given by the authors in a later paper).

**Theorem 2.2.17 [57, Theorem 4.5]** *Let $X$ be a Banach space. If $T : X \rightrightarrows X^*$ is maximal monotone and has a unique maximal monotone extension to the bidual, then one of the following conditions holds:*

*(a) $T$ is of type (D);*

*(b) $T$ is affine and non-enlargeable, that is $\varphi_T = \pi + \delta_{\mathcal{G}(T)}$ and $\mathcal{H}_T = \{\varphi_T\}$.*

### 2.2.4 Surjectivity Properties

As shown above, convex representations of maximal monotone operators can be employed to obtain new proofs of some key results, like those concerning the sum of two maximal monotone operators, and may be a valuable tool for achieving new refinements. We are now going to present another instance related to this point, concerning the surjectivity property stated in Theorem 1.3.12.

Simons and Zălinescu [98] used convex representations to obtain a new proof of that theorem, in its version based on the sum of the graphs [93, Theorem 10.6] (see Theorem 1.3.13 above). By means of a proof technique based on Fenchel duality, Martínez-Legaz [59] provided several generalizations of the surjectivity theorem and of its version with the sum of the graphs, replacing the duality mapping involved in those results by an arbitrary maximal monotone operator having finite-valued Fitzpatrick function. The main theorem of [59] is the following one.

**Theorem 2.2.18 [59, Theorem 2.1]** *Let $X$ be a reflexive Banach space. For every monotone operator $S : X \rightrightarrows X^*$, the following statements are equivalent:*

*(a) $S$ is maximal monotone;*

*(b) $\mathcal{G}(S) + \mathcal{G}(-T) = X \times X^*$ for every maximal monotone operator $T : X \times X^*$ such that $\varphi_T$ is finite-valued;*

*(c) there exists a maximal monotone operator $T : X \rightrightarrows X^*$ such that $\varphi_T$ is finite-valued, $\mathcal{G}(S) + \mathcal{G}(-T) = X \times X^*$, and there exists $(p, p^*) \in \mathcal{G}(T)$ such that $\langle p - y, p^* - y^* \rangle > 0$ for every $(y, y^*) \in \mathcal{G}(T) \backslash \{(p, p^*)\}$.*

On the other hand, the use of convex representations of maximal monotone operators also allowed Marques Alves and Svaiter [58] to generalize Rockafellar's surjectivity theorem to the case of nonreflexive Banach spaces, thus recovering the results of Gossez [37] using different techniques.

**Theorem 2.2.19 [58, Theorem 3.6]** *Let $X$ be a Banach space. If $T : X \rightrightarrows X^*$ is a monotone operator with $\mathcal{G}(T)$ closed in the norm topology of $X \times X^*$, then the conditions below are equivalent:*

*(a) $\overline{\mathcal{R}(T(\cdot + z_0) + J)} = X^*$ for all $z_0 \in X$;*

*(b) $\overline{\mathcal{R}(T(\cdot + z_0) + J_\varepsilon)} = X^*$ for all $\varepsilon > 0$, $z_0 \in X$;*

*(c) $\mathcal{R}(T(\cdot + z_0) + J_\varepsilon) = X^*$ for all $\varepsilon > 0$, $z_0 \in X$;*

*(d) $T$ is maximal monotone and of type (NI).*

## 2.3 Enlargements of Maximal Monotone Operators

In this final section, we will briefly recall the definition of an enlargement and its relations with convex representations of maximal monotone operators. A simple and important enlargement was considered by Veselý [105] under the name of $\varepsilon-$monotone operator, i.e. an operator $T : X \rightrightarrows X^*$ such that

$$\langle x - y, x^* - y^* \rangle \geq -\varepsilon \tag{2.8}$$

for all $(x, x^*), (y, y^*) \in \mathcal{G}(T)$ and for a fixed $\varepsilon \geq 0$. In this sense, $\varepsilon-$monotonicity is a generalization of monotonicity, since any monotone operator is $0-$monotone, by definition.

Anyway, inequality (2.8) can be also interpreted in another way. Given a maximal monotone operator $T : X \rightrightarrows X^*$, when trying to determine $\mathcal{G}(T)$ numerically, in general one will eventually obtain only approximate results, i.e. points $(x, x^*) \in X \times X^*$ that are not monotonically related to $\mathcal{G}(T)$ in an exact way, but only up to some small error $\varepsilon > 0$, as stated by inequality (2.8). Thus, for any arbitrary monotone operator $T$ and for all $\varepsilon > 0$, one can introduce the operator whose graph consists of those points $(x, x^*) \in X \times X^*$ satisfying inequality (2.8) for all $(y, y^*)\mathcal{G}(T)$. This operator is a particular instance of an enlargement and can be seen as a generalization to arbitrary monotone operators of the $\varepsilon-$subdifferential of Convex Analysis.

This notion of enlargement was introduced and thoroughly studied in a series of papers by R. S. Burachik, B. F. Svaiter and coauthors [21, 24, 25, 26, 27, 100]. The results of this research are also collected in the book by Burachik and Iusem [20]. The formal definition of an enlargement of a multifunction reads as follows.

**Definition 2.3.1 [100]** *Let $T : X \rightrightarrows X^*$ be a set valued function. We say that a point-to-set mapping $E : X \times \mathbb{R}_+ \rightrightarrows X^*$ is an* enlargement *of $T$ when the following hold.*

$(E_1)$ *$T(x) \subseteq E(x, \varepsilon)$ for all $\varepsilon \geq 0, x \in X$.*

$(E_2)$ *If $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $E(x, \varepsilon_1) \subseteq E(x, \varepsilon_2)$ for all $x \in X$.*

$(E_3)$ *The* transportation formula *holds for $E(\cdot, \cdot)$: Let $v^1 \in E(x^1, \varepsilon_1)$, $v^2 \in E(x^2, \varepsilon_2)$, and $\alpha \in [0, 1]$. Define*

$$\hat{x} := \alpha x^1 + (1 - \alpha)x^2,$$

$$\hat{v} := \alpha v^1 + (1 - \alpha)v^2,$$

$$\hat{\varepsilon} := \alpha\varepsilon_1 + (1 - \alpha)\varepsilon_2 + \alpha\langle x^1 - \hat{x}, v^1 - \hat{v}\rangle + (1 - \alpha)\langle x^2 - \hat{x}, v^2 - \hat{v}\rangle.$$

*Then $\hat{\varepsilon} \geq 0$ and $\hat{v} \in E(\hat{x}, \hat{\varepsilon})$.*

*When $E$ satisfies $(E_1) - (E_3)$, we write $E \in \mathbb{E}(T)$.*

Though the previous definition holds for general multifunctions, it is particularly important for maximal monotone operators. Thus, for the remaining of this section we will consider enlargements of a maximal monotone operator $T : X \rightrightarrows X^*$ on a real Banach space $X$. The prototypical example of an enlargement is the approximate subdifferential, defined as the multifunction that maps each $(x, \varepsilon) \in X \times \mathbb{R}_+$ to $\partial_\varepsilon f(x)$, in the case when $T$ is the subdifferential of a proper lower semicontinuos convex function $f : X \to \mathbb{R} \cup \{+\infty\}$. A second fundamental example is the one which guided our presentation up to now, i.e. the enlargement $T^e : X \times \mathbb{R}_+ \rightrightarrows X^*$ such that

$$T^e(x,\varepsilon) := \{x^* \in X^* : \ \langle x-y, x^*-y^* \rangle \geq -\varepsilon, \ \forall (y,y^*) \in \mathcal{G}(T)\}.$$

This enlargement was introduced in [21] for the finite-dimensional case, and extended first to Hilbert spaces in [24, 25] and then to Banach spaces in [26]. It is easy to see that the set $T^e(x,\varepsilon)$ is weak*-closed for every fixed $x$ and $\varepsilon$. If $x$ belongs to the interior of $\mathcal{D}(T)$, then the set $T^e(x,\varepsilon)$ is weak*-compact (see [26] and [20, Theorem 5.3.4]). On the other hand, the mapping $T^e$ is the biggest element in the family $\mathbb{E}(T)$ (see [100] and [20, Theorem 5.4.2]), meaning that $E \subseteq T^e$ for every $E \in \mathbb{E}(T)$.

We will denote by $\mathbb{E}_C(T)$ the subset of $\mathbb{E}(T)$ consisting of all $E \in \mathbb{E}(T)$ such that $E(x,\varepsilon)$ is weak*-closed for every $x \in X$ and every $\varepsilon \geq 0$. Then, in particular, for any $E \in \mathbb{E}_C(T)$ we have that $E(x,\varepsilon)$ is weak*-compact for any $\varepsilon \geq 0$ and any $x$ in the interior of $\mathcal{D}(T)$. An important property of this subfamily of enlargements is that every $E \in \mathbb{E}_C(T)$ fully characterizes $T$, as a consequence of [29, Corollary 3.6]. This means that, given $E_T \in \mathbb{E}_C(T)$ and $E_S \in \mathbb{E}_C(S)$, if $\mathcal{D}(T) = \mathcal{D}(S)$ and $E_T(x,\varepsilon) = E_S(x,\varepsilon)$ for every $x$ in the common domain and every $\varepsilon > 0$, then $S = T$.

With respect to the concerns of the present thesis, the most relevant property of the family of enlargements of a maximal monotone operator is its deep link with convex representations. As we anticipated in the previous section, Burachik and Svaiter [27] obtained their construction of the Fitzpatrick family $\mathcal{H}_T$ reasoning on enlargements. Indeed, up to a permutation one can identify the graph of an arbitrary $E \in \mathbb{E}(T)$ with a subset of $X \times X^* \times \mathbb{R}$, which we still denote $E$ for ease of notation. Considering then the lower envelope of the set $E$, $\lambda_E : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, defined as

$$\lambda_E(x,x^*) := \inf\{\varepsilon \geq 0 : \ x^* \in E(x,\varepsilon)\}, \quad \forall (x,x^*) \in X \times X^*,$$

and defining the function $\Lambda_E : X \times X^* \to \mathbb{R} \cup \{+\infty\}$

$$\Lambda_E(x,x^*) := \lambda_E(x,x^*) + \langle x, x^* \rangle, \quad \forall (x,x^*) \in X \times X^*,$$

the authors proved the following fundamental result.

**Theorem 2.3.2 [27, Theorem 3.6]** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then the map*

$$\mathbb{E}_C(T) \to \mathcal{H}_T, \quad E \mapsto \Lambda_E$$

*is a bijection, with inverse given by*

$$\mathcal{H}_T \to \mathbb{E}_C(T), \quad h \mapsto L^h,$$

*where* $L^h : X \times \mathbb{R}_+ \rightrightarrows X^*$ *is defined by*

$$L^h(x, \varepsilon) := \{x^* \in X^* : \ h(x, x^*) \leq \langle x, x^* \rangle + \varepsilon\}.$$

Thus, it is completely equivalent to characterize a maximal monotone operator $T$ either by enlargements of the subfamily $\mathbb{E}_C(T)$, or by convex representations taken from $\mathcal{H}_T$.

# Chapter 3

# Coincidence Results for Maximal Monotone Operators

In this chapter, which is based on [22], we establish minimal conditions under which two maximal monotone operators coincide. Our first result is inspired by an analogous result for subdifferentials of convex functions, namely, the fact that the difference of two convex functions is constant if and only if their subdifferentials intersect at every point of their common convex domain [46]. In particular, we prove that two maximal monotone operators $T$ and $S$ which share the same convex-like[1] domain $D$ coincide whenever $T(x) \cap S(x) \neq \emptyset$ for every $x \in D$. This is a consequence of a more general result for monotone operators (Theorem 3.1.3) that we prove using only simple algebraic techniques. As another consequence of the same theorem, we obtain a new easy proof of the well-known property according to which maximal monotone operators maintain their maximality when restricted to open subsets of their domain.

These results are presented in Section 3.1, while Section 3.2 extends them to the framework of enlargements of maximal monotone operators. More precisely, we prove that two operators coincide as long as their enlargements have nonempty intersection at each point of their common domain, assumed to be open. We then use this to obtain new facts for convex functions, showing that the difference of two proper lower semicontinuous and convex functions, the subdifferentials of which have a common open domain, is constant if and only if their $\varepsilon$-subdifferentials intersect at every point of that domain.

[1] This notion is defined in Section 3.1 below.

## 3.1 Coincidence Results

In this chapter we will work in the setting of real Banach spaces and no assumption of reflexivity is required. To state our results we need to recall Definition 1.3.2, according to which, given a monotone operator $T : X \rightrightarrows X^*$, $T^\mu$ denotes the multifunction whose graph consists of all points monotonically related to $\mathcal{G}(T)$, i.e.

$$\mathcal{G}(T^\mu) := \{(x, x^*) \in X \times X^* : \langle x - y, x^* - y^* \rangle \geq 0, \ \ \forall (y, y^*) \in \mathcal{G}(T)\}.$$

Moreover, we introduce some useful notation. If $Y$ is a vector space and $x, y \in Y$ with $x \neq y$, we denote by $[x, y]$, $]x, y[$ and $]y, x_{+\infty}[$ the sets of points $\lambda x + (1 - \lambda) y$, with $\lambda \in [0, 1]$, $\lambda \in\ ]0, 1[$ and $\lambda \in\ ]0, +\infty[$, respectively.

We will also consider the following notion, which suitably relaxes the concept of convexity.

**Definition 3.1.1** *Let $Y$ be a vector space and $A \subseteq Y$. We call $A$* convex-like *if, for any $x, y \in A$ with $x \neq y$, $]x, y[\ \cap\ A \neq \emptyset$.*

The class of convex-like sets contains that of nearly convex sets, which in turn includes all midpoint convex sets. Recall that a set $A$ in a vector space is called nearly convex [16] if there exists $\alpha \in ]0, 1[$ such that, for every $x, y \in A$, $\alpha x + (1 - \alpha) y \in A$. If $\alpha = \frac{1}{2}$, the set $A$ is called midpoint convex. It is easy to see that the intersection of a nearly convex set $A$ with any segment having its endpoints in $A$ is dense in the segment (with respect to the topology induced on the segment by its natural identification with an interval of the real line). On the contrary, convex-like sets do not necessarily enjoy this property; consider, e.g., the set of real numbers $]0, 1[\ \cup \{2\}$. Another example of a convex-like set which fails to be nearly convex is provided next.

**Example** Let $A := \{t(1, q) \in \mathbb{R}^2 \ : \ t \in \mathbb{R}, \ q \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers. In other words, $A$ is the union of all rays with rational slope.

The following lemma states a property of "monotonicity along lines" for maximal monotone operators, partially generalizing to the multidimensional setting a useful property of increasing functions in $\mathbb{R}$.

**Lemma 3.1.2** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $(y, y^*) \in \mathcal{G}(T)$. Then, for all $x \in X$ with $x \neq y$ and $]y, x_{+\infty}[\ \cap\ \mathcal{D}(T) \neq \emptyset$ and for all $z^* \in T(]y, x_{+\infty}[)$, $\langle x - y, z^* - y^* \rangle \geq 0$.*

*Proof.* By hypothesis, there exists $\lambda > 0$ such that $z := \lambda x + (1-\lambda)y \in \mathcal{D}(T)$ and $z^* \in T(z)$. Then, because of the monotonicity of $T$,

$$0 \leq \langle z - y, z^* - y^* \rangle = \langle \lambda x + (1-\lambda)y - y, z^* - y^* \rangle = \lambda \langle x - y, z^* - y^* \rangle.$$

Since $\lambda > 0$, the result follows. □

**Theorem 3.1.3** *Let $X$ be a Banach space and $S, T : X \rightrightarrows X^*$ be monotone operators such that, for all $x \in \mathcal{D}(S)$, $y \in \mathcal{D}(T)$, if $x \neq y$, then $S(]x, y_{+\infty}[) \cap T(]y, x_{+\infty}[) \neq \emptyset$. Then $\mathcal{G}(S) \subseteq \mathcal{G}(T^\mu)$ and, equivalently, $\mathcal{G}(T) \subseteq \mathcal{G}(S^\mu)$. In particular, if $T$ is maximal, then $\mathcal{G}(S) \subseteq \mathcal{G}(T)$, and, if both $S$ and $T$ are maximal, then $S = T$.*

*Proof.* Suppose by contradiction that there exist $(x, x^*) \in \mathcal{G}(S)$, $(y, y^*) \in \mathcal{G}(T)$ such that $\langle x - y, x^* - y^* \rangle < 0$. Then $x \neq y$ and, by hypothesis, there exists $z^* \in S(]x, y_{+\infty}[) \cap T(]y, x_{+\infty}[)$. By Lemma 3.1.2, $\langle x - y, x^* - z^* \rangle \geq 0$ and $\langle x - y, z^* - y^* \rangle \geq 0$, yielding

$$0 \leq \langle x - y, x^* - z^* \rangle + \langle x - y, z^* - y^* \rangle = \langle x - y, x^* - y^* \rangle < 0,$$

a contradiction.

Therefore, $\mathcal{G}(S) \subseteq \mathcal{G}(T^\mu)$ and $\mathcal{G}(T) \subseteq \mathcal{G}(S^\mu)$. The last assertions follow from the fact that an operator $S$ is maximal monotone if and only if $S = S^\mu$. □

**Remark 3.1.4** (a) The condition $S(]x, y_{+\infty}[) \cap T(]y, x_{+\infty}[) \neq \emptyset$, compared to the analogous condition in Corollary 3.1.5 below, allows the two operators to have different domains and, in principle, it does not imply a comparison of the values that the operators take at each point of their domains, but only in some proper subset of them.

(b) One can prove dual versions of Lemma 3.1.2 and Theorem 3.1.3, involving ranges instead of domains. For instance, the dual version of Theorem 3.1.3 reads:
*Let $X$ be a Banach space and $S, T : X \rightrightarrows X^*$ be monotone operators such that, for all $x^* \in \mathcal{R}(S)$, $y^* \in \mathcal{R}(T)$, if $x^* \neq y^*$, then $S^{-1}(]x^*, y^*_{+\infty}[) \cap T^{-1}(]y^*, x^*_{+\infty}[) \neq \emptyset$. Then $\mathcal{G}(S) \subseteq \mathcal{G}(T^\mu)$ and, equivalently, $\mathcal{G}(T) \subseteq \mathcal{G}(S^\mu)$. In particular, if $T$ is maximal, then $\mathcal{G}(S) \subseteq \mathcal{G}(T)$, and, if both $S$ and $T$ are maximal, then $S = T$.*
Since $S^{-1}$ and $T^{-1}$, considered as operators from $X^*$ to $X^{**}$, are monotone and their domains are the ranges of $S$ and $T$, respectively, one can apply Theorem 3.1.3 to conclude that $\mathcal{G}(S^{-1}) \subseteq \mathcal{G}((T^{-1})^\mu)$ and $\mathcal{G}(T^{-1}) \subseteq \mathcal{G}((S^{-1})^\mu)$. Given that $S^{-1}$ and $T^{-1}$ only take

values in $X$, the previous inclusions yield $\mathcal{G}(S) \subseteq \mathcal{G}(T^\mu)$ and $\mathcal{G}(T) \subseteq \mathcal{G}(S^\mu)$ as well. The last assertions follow as in the proof of Theorem 3.1.3.

The following corollary is a consequence of the previous theorem. We provide its proof to emphasize the convexity arguments on which it relies, arguments which motivated the proof of Theorem 3.1.3.

**Corollary 3.1.5** *Let $X$ be a Banach space and $T, S : X \rightrightarrows X^*$ be monotone operators such that $T$ is also maximal. Assume that $\mathcal{D}(T) = \mathcal{D}(S) =: D$, and this common set is convex-like. If $T(x) \cap S(x) \neq \emptyset$ for every $x \in D$, then $\mathcal{G}(S) \subseteq \mathcal{G}(T)$. In particular, if also $S$ is maximal, we will have $T = S$.*

*Proof.* The last assertion follows easily from the first assertion and the maximality of both operators. We proceed to prove the first assertion. Assume, on the contrary, that for some $x =: x_0$ we have $S(x_0) \not\subseteq T(x_0)$. Then, the maximality of $T$ implies that there exists $v_0 \in S(x_0)$ and $v_1 \in T(x_1)$ such that

$$0 > \langle x_0 - x_1, v_0 - v_1 \rangle. \tag{3.1}$$

Take now $z \in ]x_0, x_1[\ \cap\ D$. We can then take $w \in T(z) \cap S(z)$. Since $z = \lambda x_0 + (1-\lambda)\, x_1$ for some $\lambda \in ]0,1[$, we can write

$$\begin{array}{rcccl} 0 & \leq & \frac{1}{1-\lambda}\langle x_0 - z, v_0 - w\rangle & = & \langle x_0 - x_1, v_0 - w\rangle \\ 0 & \leq & \frac{1}{\lambda}\langle z - x_1, w - v_1\rangle & = & \langle x_0 - x_1, w - v_1\rangle, \end{array}$$

where the inequality in the first line of the expression above holds because $v_0 \in S(x_0)$, $w \in S(z)$ and $S$ is monotone, the inequality in the second line holds because $v_1 \in T(x_1)$, $w \in T(z)$ and $T$ is monotone, and the equalities follow from the definition of $z$. Adding up the right-hand sides we obtain

$$0 \quad \leq \quad \langle x_0 - x_1, v_0 - v_1\rangle,$$

contradicting (3.1). This proves the corollary. □

**Remark 3.1.6** The above corollary does not hold when none of the operators is maximal. Consider, e.g., two operators having a singleton as their common domain; they are monotone regardless their ranges. For another example in which the operators have full domain, consider $T(x) := \lfloor x \rfloor$ (where $\lfloor z \rfloor$ denotes the integer part of $z \in \mathbb{R}$) when $x \in \mathbb{R}\backslash\mathbb{Z}$ and $T(x) := [x-1, x-\frac{1}{2}]$

when $x \in \mathbb{Z}$, and $S(x) := [x - \frac{1}{2}, x]$ when $x \in \mathbb{Z}$, and $S(x) := \lfloor x \rfloor$ when $x \in \mathbb{R}\backslash\mathbb{Z}$. Then both operators coincide on the horizontal parts of their graphs, and intersect at the vertical parts, but they clearly don't coincide.

The following corollary, which is a consequence of Theorem 3.1.3, shows that the maximality of a monotone operator $T$ is preserved when taking its restriction to any nonempty open set $D$ contained in the interior of its domain. We thus provide a simple, essentially algebraic proof of [74, Corollary 7.8].

**Corollary 3.1.7** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a maximal monotone operator and $D \subseteq X$ be an open set such that $\emptyset \neq D \subseteq \operatorname{int}\mathcal{D}(T)$. For any monotone operator $S : X \rightrightarrows X^*$, if $T(x) \subseteq S(x)$ for all $x \in D$, then $T(x) = S(x)$ for all $x \in D$.*

*Proof.* Consider the operator $S_0 : X \rightrightarrows X^*$ with graph $\mathcal{G}(S_0) = \mathcal{G}(S) \cap (D \times X^*)$. The operators $T$ and $S_0$ satisfy the hypotheses of Theorem 3.1.3, given that, being $D$ open, for all $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(S_0) = D$ with $x \neq y$, there exists $z \in ]x, y[ \cap D$ and, by hypothesis, $\emptyset \neq T(z) \subseteq S(z) = S_0(z)$. Thus, we obtain $\mathcal{G}(S_0) \subseteq \mathcal{G}(T^\mu) = \mathcal{G}(T)$, i.e. $S(x) = S_0(x) \subseteq T(x)$ for all $x \in D$. Therefore, we conclude $T(x) = S(x)$ for all $x \in D$. $\square$

**Corollary 3.1.8** *Let $X$ be a Banach space and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Consider the following statements:*

*(i)* $\partial f(]x, y_{+\infty}[) \cap \partial g(]y, x_{+\infty}[) \neq \emptyset$ *for every* $x \in \mathcal{D}(\partial f)$ *and* $y \in \mathcal{D}(\partial g)$ *with* $x \neq y$*;*

*(ii) there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for every $x \in X$;*

*(iii)* $\mathcal{D}(\partial f) = \mathcal{D}(\partial g) =: D$ *and* $\partial f(x) \cap \partial g(x) \neq \emptyset$ *for every* $x \in D$*.*

*The implications (i) $\implies$ (ii) $\implies$ (iii) hold true. If $\mathcal{D}(\partial f)$ and $\mathcal{D}(\partial g)$ are convex-like, the three statements are equivalent.*

*Proof.* Assume that $(i)$ holds. Since $f$ and $g$ are lower semicontinuous and convex, we have that $\partial f$ and $\partial g$ are maximal monotone. This fact, together with $(i)$ and Theorem 3.1.3, implies that $\partial f = \partial g$. Therefore, there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in X$.

Part $(ii)$ directly yields $\mathcal{D}(\partial f) = \mathcal{D}(\partial g)$ and $\partial f(x) = \partial g(x)$ for every $x \in X$, and hence $(iii)$ holds.

If *(iii)* holds and $D$ is convex-like, for every $x, y \in D$ with $x \neq y$ and $z \in ]x, y[ \cap D$ one has $\partial f(]x, y_{+\infty}[) \cap \partial g(]y, x_{+\infty}[) \supseteq \partial f(z) \cap \partial g(z)$. Since the latter intersection is nonempty, we conclude that $\partial f(]x, y_{+\infty}[) \cap \partial g(]y, x_{+\infty}[)$ is nonempty too. This proves *(i)*.
□

## 3.2 Coincidence Results via Enlargements

We now make use of the concept of enlargements of maximal monotone operators, defined in Section 2.3, to establish another condition under which two maximal monotone operators coincide. Recall that, given a maximal monotone operator $T : X \rightrightarrows X^*$, we denote by $\mathbb{E}_C(T)$ the family of enlargements $E$ of $T$ such that $E(x,\varepsilon)$ is weak*-closed for every $x \in X$ and every $\varepsilon \geq 0$. Moreover, if $x \in \operatorname{int} \mathcal{D}(T)$, $E(x,\varepsilon)$ is weak*-compact. The biggest element of $\mathbb{E}_C(T)$ is $T^e : X \times \mathbb{R}_+ \rightrightarrows X^*$,

$$(x,\varepsilon) \mapsto T^e(x,\varepsilon) := \{x^* \in X^* : \langle x - y, x^* - y^* \rangle \geq -\varepsilon, \quad \forall (y, y^*) \in \mathcal{G}(T)\}.$$

When $T$ is the subdifferential of a proper lower semicontinuous convex function $f$, the approximate subdifferential of $f$ belongs to $\mathbb{E}_C(T)$ as well.

As a consequence of [29, Corollary 3.6], two maximal monotone operators $T$ and $S$ coincide whenever there exist enlargements $E_T \in \mathbb{E}_C(T)$ and $E_S \in \mathbb{E}_C(S)$, respectively, such that $D := \mathcal{D}(T) = \mathcal{D}(S)$ and $E_T(x,\varepsilon) = E_S(x,\varepsilon)$ for every $x \in D$ and every $\varepsilon > 0$. The result below uses Theorem 3.1.3 to relax the hypothesis $E_T(x,\varepsilon) = E_S(x,\varepsilon)$, so that we can simply require the intersection of both sets to be nonempty.

**Corollary 3.2.1** *Let $X$ be a Banach space, $T, S : X \rightrightarrows X^*$ be two maximal monotone operators and $D \subseteq X$ be an open set such that $\emptyset \neq D \subseteq \operatorname{int}\mathcal{D}(T)$. Let $E_T, E_S : X \times \mathbb{R}_+ \rightrightarrows X^*$ be such that $E_T \in \mathbb{E}_C(T)$ and $E_S \in \mathbb{E}_C(S)$. Then the following statements are equivalent:*

*(i)* *for any $x \in D$, $y \in \mathcal{D}(S)$ with $x \neq y$, there exist $u \in ]x, y_{+\infty}[ \cap D$ and $v \in ]y, x_{+\infty}[ \cap \mathcal{D}(S)$ such that $E_T(u,\varepsilon) \cap E_S(v,\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$;*

*(ii)* *$T(x) \subseteq S(x)$ for all $x \in D$;*

*(iii)* *$T(x) = S(x)$ for all $x \in D$;*

*(iv)* *$D \subseteq \operatorname{int}\mathcal{D}(S)$ and $E_T(x,\varepsilon) \cap E_S(x,\varepsilon) \neq \emptyset$ for every $x \in D$, $\varepsilon > 0$.*

*Proof.* $(i) \Longrightarrow (ii)$ Let $(x,x^*) \in \mathcal{G}(T) \cap (D \times X^*)$ and $(y,y^*) \in \mathcal{G}(S)$, with $x \neq y$. By $(i)$, there exist $u \in ]x, y_{+\infty}[ \cap D$ and $v \in ]y, x_{+\infty}[ \cap \mathcal{D}(S)$ such that $E_T(u,\varepsilon) \cap E_S(v,\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$. Therefore, the family $\{E_T(u,\varepsilon) \cap E_S(v,\varepsilon)\}_\varepsilon$ has the finite intersection property

$$\bigcap_{i=1}^{p} [E_T(u,\varepsilon_i) \cap E_S(v,\varepsilon_i)] = E_T(u,\bar{\varepsilon}) \cap E_S(v,\bar{\varepsilon}) \neq \emptyset,$$

where $\bar{\varepsilon} := \min\{\varepsilon_i : i = 1, \ldots, p\}$. Since, moreover, for all $\varepsilon > 0$, $E_S(v,\varepsilon)$ is weak$^*$-closed and $E_T(u,\varepsilon)$ is weak$^*$-compact, we conclude that the intersection of all elements of the family is nonempty. In other words,

$$\emptyset \neq \bigcap_{\varepsilon>0} [E_T(u,\varepsilon) \cap E_S(v,\varepsilon)] = \left[\bigcap_{\varepsilon>0} E_T(u,\varepsilon)\right] \cap \left[\bigcap_{\varepsilon>0} E_S(v,\varepsilon)\right] = T(u) \cap S(v),$$

where we used [20, Lemma 5.4.5(c)] in the last equality. Therefore, denoting by $T_0 : X \rightrightarrows X^*$ the operator such that $\mathcal{G}(T_0) = \mathcal{G}(T) \cap (D \times X^*)$, applying Theorem 3.1.3 to the operators $T_0$ and $S$ and taking the maximality assumption on $S$ into account, we obtain $\mathcal{G}(T_0) \subseteq \mathcal{G}(S^\mu) = \mathcal{G}(S)$, i.e. $T(x) \subseteq S(x)$ for all $x \in D$.

$(ii) \Longrightarrow (iii)$ It is an immediate consequence of Corollary 3.1.7.

$(iii) \Longrightarrow (iv)$ Because $D \subseteq \mathrm{int}\mathcal{D}(T)$, we have $D \subseteq \mathcal{D}(S)$; hence, as $D$ is open, $D \subseteq \mathrm{int}\mathcal{D}(S)$. Moreover,

$$E_T(x,\varepsilon) \cap E_S(x,\varepsilon) \supseteq T(x) \cap S(x) = T(x) = S(x) \neq \emptyset, \forall\, x \in D,\ \varepsilon > 0.$$

$(iv) \Longrightarrow (i)$ Since $D$ is open, for all $x \in D$ and $y \in \mathcal{D}(S)$, with $x \neq y$, there exists $z \in ]x,y[ \cap D$. Then $(iv)$ implies $E_T(z,\varepsilon) \cap E_S(z,\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. Thus, $(i)$ holds with $u = v = z$. □

**Remark 3.2.2** When the operators $T$ and $S$ are such that $T$ has an open domain, the above corollary yields a necessary and sufficient condition for $S$ and $T$ to coincide, a condition expressed in terms of enlargements.

**Corollary 3.2.3** *Let $X$ be a Banach space, $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two proper lower semicontinuous convex functions and $D \subseteq X$ be an open convex set such that $\emptyset \neq D \subseteq \mathrm{int}\mathcal{D}(\partial f) \cap \mathrm{int}\mathcal{D}(\partial g)$. Then the following are equivalent:*

*(i)* $(\partial f)^e(x,\varepsilon) \cap (\partial g)^e(x,\varepsilon) \neq \emptyset$ *for every* $x \in D$ *and every* $\varepsilon > 0$*;*

*(ii)* $\partial_\varepsilon f(x) \cap \partial_\varepsilon g(x) \neq \emptyset$ *for every* $x \in D$ *and every* $\varepsilon > 0$*;*

*(iii) there exists* $c \in \mathbb{R}$ *such that* $f(x) = g(x) + c$ *for every* $x \in D$*.*

*Proof.* By equivalence $(iii) \iff (iv)$ in Corollary 3.2.1, statements $(i)$ and $(ii)$ are equivalent to the equality $\partial f = \partial g$ on $D$, which is in turn equivalent to statement $(iii)$ of the present corollary [101, Theorem 2.1]. $\square$

# Chapter 4

# Autoconjugate Fitzpatrick Functions

This chapter presents the results on autoconjugate Fitzpatrick functions of subdifferentials contained in [79].

The study of autoconjugate elements of the Fitzpatrick family associated to a maximal monotone operator $T : X \rightrightarrows X^*$ defined on a Banach space $X$, i.e., those functions $h \in \mathcal{H}_T$ such that $h^*(x^*, x) = h(x, x^*)$ for all $(x, x^*) \in X \times X^*$, can be useful when studying the structure of $\mathcal{H}_T$. Indeed, for instance, an autoconjugate element of $\mathcal{H}_T$ is a fixed point for the map $\mathscr{J} : \mathcal{H}_T \to \mathcal{H}_T$ defined in [27] (see item $(e)$ of Theorem 2.2.4 above) by setting

$$\mathscr{J}(h)(x, x^*) = h^*(x^*, x), \quad \forall (x, x^*) \in X \times X^*.$$

Moreover, in the fundamental case where $T = \partial f$ for some proper lower semicontinuous convex function $f$, the function $f \oplus f^*$, defined as $(f \oplus f^*)(x, x^*) = f(x) + f^*(x^*)$ for all $(x, x^*) \in X \times X^*$, is an autoconjugate element of $\mathcal{H}_{\partial f}$.

The existence of autoconjugate representations of a maximal monotone operator was proved in [71], while explicit constructions were presented and studied in [9, 11, 73].

In particular, given the relevance of the Fitzpatrick function $\varphi_T$ among the elements of $\mathcal{H}_T$, it is of special interest to study those cases in which $\varphi_T$ is autoconjugate. It can be proved that, if the Banach space $X$ is reflexive, the property of autoconjugation of $\varphi_T$ is equivalent to the equality $\mathcal{H}_T = \{\varphi_T\}$, while, even when $X$ is nonreflexive, if $T = \partial f$, it is equivalent to $\varphi_{\partial f} = f \oplus f^*$ (see Remark 4.2.6 below). With respect to the latter case, two classes of functions, whose subdifferentials have autoconjugate Fitzpatrick functions, have been detected in the literature, namely indicator functions of nonempty closed convex sets and their conjugate functions, i.e. proper lower semicontinuous sublinear functions. The proof of the latter fact was given by Penot [72] under the hypothesis that $X$ be reflexive, by Burachick and Fitzpatrick [19] for the case of nonreflexive Banach spaces, but with the additional requirement that the function $f$ be everywhere finite, and finally by Bartz *et al.* [7] in the case of a general Banach space and a general proper lower semicontinuous sublinear function. Actually, this result for sublinear functions was partially anticipated also by the work of Carrasco-Olivera and Flores-Bazán [30, Corollary 3.7] (see Section 4.2 below for a precise discussion), studying enlargements instead of convex representations. Moreover, in [30, Remark 3.8] the same authors conjecture that the converse property holds. Restating the problem in our framework, it is a natural question to ask whether indicator and sublinear functions are the only ones the subdifferentials of which have autoconjugate Fitzpatrick functions.

The present chapter delivers new contributions to the above mentioned questions in a three-fold manner. First, it provides a necessary and sufficient condition for the equality $h = (\varphi_T)|_{\text{dom } h}$ to hold, where $h \in \mathcal{K}_T$ and $T$ is a monotone operator (not even necessarily maximal). As a second contribution, it applies this result to the case where $T$ is the subdifferential of a proper lower semicontinuous convex function, obtaining a new proof of the results of [7] for indicator and sublinear functions. In particular, Proposition 4.2.9 and Corollary 4.2.10 characterize a certain class of transformed indicator and sublinear functions as the unique family of functions satisfying in a peculiarly simple way the necessary and sufficient condition for the equality $\varphi_{\partial f} = f \oplus f^*$ to hold. As a consequence, those characterizations provide new insight concerning the reason why the subdifferentials of indicator and sublinear functions have autoconjugate Fitzpatrick functions (Section 4.2). This fact is exploited to tackle the inverse problem, proving that, in the one-dimensional setting ($X = \mathbb{R}$), the class of functions characterized by Proposition 4.2.9 and Corollary 4.2.10 is indeed the only one satisfying $\varphi_{\partial f} = f \oplus f^*$ (Section 4.3). This is no more true in multidimensional spaces and a very simple example in the case $X = \mathbb{R}^2$ is provided (Section 4.4).

## 4.1 Preliminary Results

Firstly, we recall some useful properties of subdifferentials. In particular, the relation between the subdifferential of a proper lower semicontinuous convex function $f : Y \to \mathbb{R} \cup \{+\infty\}$ on a Banach space $Y$ and the approximate subdifferential can be obtained specializing equality (1.3)

$$\forall y \in Y : \quad \partial f(y) = \bigcap_{\varepsilon > 0} \partial_\varepsilon f(y).$$

**Remark 4.1.1** Let $Y$ be a Banach space and $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function.

*(a)* For any $\varepsilon \geq 0$,

$$\mathcal{G}(\partial_\varepsilon f)^\top = \mathcal{G}(\partial_\varepsilon f^*) \cap (Y^* \times Y), \tag{4.1}$$

since, for any $(y, y^*) \in Y \times Y^*$,

$$\begin{aligned} (y, y^*) \in \mathcal{G}(\partial_\varepsilon f) &\iff f(y) + f^*(y^*) \leq \langle y, y^* \rangle + \varepsilon \\ &\iff f^{**}(y) + f^*(y^*) \leq \langle y, y^* \rangle + \varepsilon \\ &\iff (y^*, y) \in \mathcal{G}(\partial_\varepsilon f^*) \cap (Y^* \times Y); \end{aligned}$$

*(b)* $\varphi_{\partial f^*} = \sigma^*_{\partial f}$. The proof follows immediately combining [55, Theorem 1.1] and the well known fact that $(\partial f^*)^{-1}$ is the unique maximal monotone extension to the bidual of $\partial f$ (see for instance Lemma 5.1.2 below for a proof).

We now gather some results that we will need in the subsequent sections.

**Lemma 4.1.2 ([12, Lemma 3.1])** *Let $X$ be a Banach space and $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then for every lower semicontinuous function $f$ satisfying $f \geq g$ we have*

$$f|_{\mathcal{D}(\partial g)} = g|_{\mathcal{D}(\partial g)} \Longrightarrow f = g.$$

The following result is well known [108, Corollary 2.4.5].

**Lemma 4.1.3** *Let $X$ and $Y$ be Banach spaces and $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be proper functions. Then*

$$\forall (x, y) \in \operatorname{dom} f \times \operatorname{dom} g, \ \ \forall \varepsilon \geq 0: \qquad \partial_\varepsilon (f \oplus g)(x, y) = \bigcup_{\substack{\alpha, \beta \geq 0 \\ \alpha + \beta = \varepsilon}} (\partial_\alpha f(x) \times \partial_\beta g(y)).$$

The following lemma is used only in the proof of Lemma 4.1.5.

**Lemma 4.1.4** *Let $X$ be a Banach space and $\widetilde{K} \subseteq X^*$ be a convex set. Then $\left(\delta^*_{\widetilde{K}}|_X\right)^* = \delta_{\left(\mathrm{cl}^* \widetilde{K}\right)}$.*

*Proof.* Given an arbitrary $x^* \in X^*$, let

$$g(y) := \langle y, x^* \rangle - \delta^*_{\widetilde{K}}(y) = \langle y, x^* \rangle - \sup_{y^* \in \widetilde{K}} \langle y, y^* \rangle$$

for all $y \in X$.

If $x^* \in \mathrm{cl}^* \widetilde{K}$, there exists a net $(y^*_\alpha)$ in $\widetilde{K}$ that converges to $x^*$ in the weak$^*$ topology of $X^*$. Therefore, for all $y \in X$,

$$g(y) \leq \langle y, x^* \rangle - \lim_\alpha \langle y, y^*_\alpha \rangle = \langle y, x^* \rangle - \langle y, x^* \rangle = 0,$$

i.e., since $\left(\delta^*_{\widetilde{K}}|_X\right)^*(x^*) = \sup_{y \in X} g(y)$ and $g : X \to \overline{\mathbb{R}}$ is positive homogeneous, $\left(\delta^*_{\widetilde{K}}|_X\right)^*(x^*) = 0 = \delta_{\left(\mathrm{cl}^* \widetilde{K}\right)}(x^*)$.

On the other hand, if $x^* \notin \mathrm{cl}^* \widetilde{K}$, by the separation theorem there exists $\overline{y} \in X$ such that

$$\sup_{y^* \in \widetilde{K}} \langle \overline{y}, y^* \rangle \leq \sup_{y^* \in \mathrm{cl}^* \widetilde{K}} \langle \overline{y}, y^* \rangle < \langle \overline{y}, x^* \rangle,$$

implying $g(\overline{y}) > 0$. Thus, taking positive homogeneity of $g$ into account,

$$\left(\delta^*_{\widetilde{K}}|_X\right)^*(x^*) = \sup_{y \in X} g(y) = +\infty,$$

that is to say, $\left(\delta^*_{\widetilde{K}}|_X\right)^*(x^*) = \delta_{\left(\mathrm{cl}^* \widetilde{K}\right)}(x^*)$. □

We conclude this section with two lemmas that we will use in Proposition 4.2.9 and Corollary 4.2.10. The type of functions considered here will play an important role in Sections 4.2 and 4.3.

**Lemma 4.1.5** *Let $X$ be a Banach space, $f : X \to \overline{\mathbb{R}}$, $K \subseteq X$, $\widetilde{K} \subseteq X^*$ be a convex set, $z \in X$, $z^* \in X^*$ and $\alpha \in \mathbb{R}$.*

*(a) If $f = \delta_K + z^* + \alpha$, then $f^* = \left(\delta^*_K \circ \tau_{-z^*}\right) - \alpha$.*

*(b) If $f = \left(\delta^*_{\widetilde{K}} \circ \tau_z\right)|_X + \alpha$, then $f^* = \delta_{\left(\mathrm{cl}^* \widetilde{K}\right)} - z - \alpha$.*

*Proof.*

(a) For all $x^* \in X^*$,

$$\begin{aligned} f^*(x^*) &= \sup_{x \in X} \{\langle x, x^* \rangle - \delta_K(x) - \langle x, z^* \rangle - \alpha\} \\ &= \sup_{x \in K} \langle x, x^* - z^* \rangle - \alpha = (\delta^*_K \circ \tau_{-z^*})(x^*) - \alpha. \end{aligned}$$

(b) As a consequence of Lemma 4.1.4, for all $x^* \in X^*$, we obtain

$$\begin{aligned} f^*(x^*) &= \sup_{x \in X} \{\langle x, x^* \rangle - \delta^*_{\widetilde{K}}(x + z) - \alpha\} \\ &= \sup_{x \in X} \{\langle x + z, x^* \rangle - \delta^*_{\widetilde{K}}(x + z)\} - \langle z, x^* \rangle - \alpha \\ &= \sup_{x \in X} \{\langle x, x^* \rangle - \delta^*_{\widetilde{K}}(x)\} - \langle z, x^* \rangle - \alpha \\ &= \delta_{\left(\mathrm{cl}^* \widetilde{K}\right)}(x^*) - \langle z, x^* \rangle - \alpha. \end{aligned}$$

□

**Lemma 4.1.6** *Let $X$ be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function, $K \subseteq X$ be a nonempty closed convex set, $\widetilde{K} \subseteq X^*$ be a nonempty weak$^*$-closed convex set, $z \in X$, $z^* \in X^*$ and $\alpha \in \mathbb{R}$.*

*(a) If $f^* = \delta_{\widetilde{K}} + z + \alpha$, then $f^{**} = \left(\delta^*_{\widetilde{K}} \circ \tau_{-z}\right) - \alpha$. In particular, $f = \left(\delta^*_{\widetilde{K}} \circ \tau_{-z}\right)|_X - \alpha$.*

*(b) If $f^* = \left(\delta^*_K \circ \tau_{z^*}\right) + \alpha$, then $f = \delta_K - z^* - \alpha$.*

*Proof.*

(a) For all $x^{**} \in X$,

$$\begin{aligned} f^{**}(x^{**}) &= \sup_{x^* \in X^*} \{\langle x^{**}, x^* \rangle - \delta_{\widetilde{K}}(x^*) - \langle z, x^* \rangle - \alpha\} \\ &= \sup_{x^* \in \widetilde{K}} \langle x^{**} - z, x^* \rangle - \alpha = \left(\delta^*_{\widetilde{K}} \circ \tau_{-z}\right)(x^{**}) - \alpha. \end{aligned}$$

Since $f$ is proper lower semicontinuous and convex, $f = f^{**}$ on $X$.

(b) For all $x \in X$, since $f^{**} = f$ and $\delta^{**}_K = \delta_K$ on $X$,

$$\begin{aligned} f(x) &= \sup_{x^* \in X^*} \{\langle x, x^* \rangle - \delta^*_K(x^* + z^*) - \alpha\} \\ &= \sup_{x^* \in X^*} \{\langle x, x^* + z^* \rangle - \delta^*_K(x^* + z^*)\} - \langle x, z^* \rangle - \alpha \\ &= \delta^{**}_K(x) - \langle x, z^* \rangle - \alpha = \delta_K(x) - \langle x, z^* \rangle - \alpha. \end{aligned}$$

□

## 4.2 Autoconjugate Fitzpatrick Functions

The main results of this section are Theorem 4.2.3, Proposition 4.2.9 and Corollary 4.2.10. Theorem 4.2.3, in particular, provides a necessary and sufficient condition for an arbitrary element $h \in \mathcal{K}_T$, associated to a monotone (not necessarily maximal) operator $T$, to coincide with the Fitzpatrick function $\varphi_T$, under the hypothesis that their domains coincide. To make the proof clearer, we partition it in the following two lemmas.

**Lemma 4.2.1** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. Then, for all $(x, x^*) \in \operatorname{dom} \varphi_T$,*

$$\forall \varepsilon > 0 : \qquad \partial_\varepsilon \varphi_T(x, x^*) \cap \mathcal{G}(T)^\top \neq \emptyset.$$

*Proof.* Let $(x, x^*) \in \operatorname{dom} \varphi_T$. From the definition of the Fitzpatrick function of $T$ and the fact that, according to Remark 2.2.8, $(\varphi_T^{*\top})|_{X\times X^*} \in \mathcal{K}_T$, it follows that

$$\begin{aligned}\varphi_T(x, x^*) &= \sup_{(y,y^*)\in\mathcal{G}(T)} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle\} \\ &= \sup_{(y,y^*)\in\mathcal{G}(T)} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \varphi_T^{*\top}(y, y^*)\}.\end{aligned}$$

By the definition of a supremum, for any $\varepsilon > 0$, there exists $(\overline{y}, \overline{y}^*) \in \mathcal{G}(T)$, dependent on $\varepsilon$, such that

$$\varphi_T(x, x^*) - \varepsilon < \langle x, \overline{y}^*\rangle + \langle \overline{y}, x^*\rangle - \varphi_T^*(\overline{y}^*, \overline{y}),$$

i.e.

$$\varphi_T(x, x^*) + \varphi_T^*(\overline{y}^*, \overline{y}) < \langle x, \overline{y}^*\rangle + \langle \overline{y}, x^*\rangle + \varepsilon.$$

Therefore, $(\overline{y}^*, \overline{y}) \in \partial_\varepsilon \varphi_T(x, x^*)$. $\square$

**Lemma 4.2.2** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator, $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a proper function such that $h^*(y^*, y) \geq \langle y, y^*\rangle$ for all $(y, y^*) \in \mathcal{G}(T)$, and let $(x, x^*) \in \operatorname{dom} h$. Consider the following statements:*

*(a)* $\forall \varepsilon > 0 : \quad \partial_\varepsilon h(x, x^*) \cap \mathcal{G}(T)^\top \neq \emptyset$;

*(b)* $h(x, x^*) \leq \varphi_T(x, x^*)$.

*Then $(a) \Longrightarrow (b)$. If, in addition, $\varphi_T \leq h$, then $(b) \Longrightarrow (a)$, so that, in this case, $(a)$ is equivalent to the equality $h(x, x^*) = \varphi_T(x, x^*)$.*

*Proof.* $(a) \Longrightarrow (b)$ For any $\varepsilon > 0$ there exists $(\overline{y}, \overline{y}^*) \in \mathcal{G}(T)$, dependent on $\varepsilon$, such that

$$h(x, x^*) + h^*(\overline{y}^*, \overline{y}) \leq \langle x, \overline{y}^*\rangle + \langle \overline{y}, x^*\rangle + \varepsilon.$$

Since, by hypothesis, $h^*(\overline{y}^*, \overline{y}) \geq \langle \overline{y}, \overline{y}^*\rangle$, the previous inequality implies

$$\begin{aligned}h(x, x^*) - \varepsilon &\leq \langle x, \overline{y}^*\rangle + \langle \overline{y}, x^*\rangle - \langle \overline{y}, \overline{y}^*\rangle \\ &\leq \sup_{(y,y^*)\in\mathcal{G}(T)} \{\langle x, y^*\rangle + \langle y, x^*\rangle - \langle y, y^*\rangle\} \\ &= \varphi_T(x, x^*).\end{aligned}$$

Letting $\varepsilon \to 0^+$, one obtains $h(x, x^*) \leq \varphi_T(x, x^*)$.

$(b) \Longrightarrow (a)$ Suppose now that $\varphi_T \leq h$. As a consequence, $h^* \leq \varphi_T^*$ and, taking $(b)$ into account, $h(x,x^*) = \varphi_T(x,x^*)$. Thus,

$$h(x,x^*) + h^*(y^*,y) \leq \varphi_T(x,x^*) + \varphi_T^*(y^*,y)$$

for all $(y^*,y) \in X^* \times X$. Therefore, in particular, $\partial_\varepsilon \varphi_T(x,x^*) \subseteq \partial_\varepsilon h(x,x^*)$, for any $\varepsilon > 0$. Thus, since $\operatorname{dom} h \subseteq \operatorname{dom} \varphi_T$, by Lemma 4.2.1 we can conclude

$$\emptyset \neq \partial_\varepsilon \varphi_T(x,x^*) \cap \mathcal{G}(T)^\top \subseteq \partial_\varepsilon h(x,x^*) \cap \mathcal{G}(T)^\top,$$

for all $\varepsilon > 0$. □

The necessary and sufficient condition we announced at the beginning of this section is item $(b)$ of the following theorem. Item $(c)$ is simply a refinement of it that can save calculations when trying to prove that $(a)$ holds in some concrete application.

**Theorem 4.2.3** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. Then, for any $h \in \mathcal{K}_T$, the following are equivalent:*

*(a)* $h = \varphi_T$;

*(b)* $\operatorname{dom} \varphi_T \subseteq \operatorname{dom} h$ *and, for all* $(x,x^*) \in \operatorname{dom} \varphi_T$,

$$\forall \varepsilon > 0 : \qquad \partial_\varepsilon h(x,x^*) \cap \mathcal{G}(T)^\top \neq \emptyset; \tag{4.2}$$

*(c)* $\mathcal{D}(\partial \varphi_T) \subseteq \operatorname{dom} h$ *and, for all* $(x,x^*) \in \mathcal{D}(\partial \varphi_T)$, *condition* (4.2) *is satisfied.*

*Proof.* $(a) \Longrightarrow (b)$ Obviously $\operatorname{dom} \varphi_T = \operatorname{dom} h$ and, for all $(x,x^*) \in \operatorname{dom} \varphi_T$, (4.2) holds as a consequence of Lemma 4.2.1.
$(b) \Longrightarrow (c)$ Evident.
$(c) \Longrightarrow (a)$ Since $\varphi_T \leq h$, by Lemma 4.2.2 we obtain $h = \varphi_T$ on $\mathcal{D}(\partial \varphi_T)$. Hence, by Lemma 4.1.2, $h = \varphi_T$ on the whole of $X \times X^*$. □

**Remark 4.2.4** *(i)* Obviously, $\operatorname{dom} \varphi_T \subseteq \operatorname{dom} h$ implies $\operatorname{dom} \varphi_T = \operatorname{dom} h$, since $\varphi_T \leq h$. On the other hand, even if we don't assume this inclusion to hold, still we can prove the following equivalence

$$h = \varphi_T|_{\operatorname{dom} h} \iff \text{(4.2) holds for all } (x,x^*) \in \operatorname{dom} h.$$

*(ii)* Since the subdifferential of a proper lower semicontinuous convex function at a point is always a subset of the approximate subdifferential at the same point, if $(x, x^*) \in \mathcal{D}(\partial h)$, a sufficient condition for (4.2) to hold is

$$\partial h(x, x^*) \cap \mathcal{G}(T)^\top \neq \emptyset. \tag{4.3}$$

*(iii)* The previous theorem suggests two possible ways to prove that, given a monotone operator $T : X \rightrightarrows X^*$, the corresponding Fitzpatrick function is autoconjugate. A sufficient condition is that $\varphi_T$ be equal to $\sigma_T$ (since in this case we would have $\left(\varphi_T^{*\top}\right)|_{X\times X^*} \in \mathcal{K}_T = \{\varphi_T\}$), which can be verified by applying the conditions in the previous theorem with $h$ replaced by $\sigma_T$. On the other hand, a necessary and sufficient condition consists of proving that there is an autoconjugate $h \in \mathcal{K}_T$ with $\operatorname{dom}\varphi_T \subseteq \operatorname{dom} h$ and satisfying (4.2) for all $(x, x^*) \in \operatorname{dom}\varphi_T$. As we will see below, the latter approach is particularly useful when $T$ is the subdifferential of a proper lower semicontinuous convex function.

When the operator $T$ is maximal monotone, we can express condition (4.2) completely in terms of the (approximate) subdifferential of $h \in \mathcal{H}_T = \mathcal{K}_T$. To this end, given a function $g : X \times X^* \to \overline{\mathbb{R}}$, define

$$\operatorname{fix}(\partial g) := \{(x, x^*) \in X \times X^* : \quad (x^*, x) \in \partial g(x, x^*)\}.$$

**Proposition 4.2.5** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then*

$$\begin{aligned} \mathcal{H}_T \;=\; & \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} : \quad h \quad \text{is lower semicontinuous convex,} \\ & h(x, x^*), h^*(x^*, x) \geq \langle x, x^*\rangle \;\; \forall (x, x^*) \in X \times X^* \quad \text{and} \quad \operatorname{fix}(\partial h) = \mathcal{G}(T)\}. \end{aligned} \tag{4.4}$$

*Proof.* Denote by $\mathcal{L}_T$ the right-hand side of (4.4) and let $h \in \mathcal{L}_T$ and $(x, x^*) \in X \times X^*$. By definition, $(x, x^*) \in \mathcal{G}(T)$ if and only if $(x, x^*) \in \operatorname{fix}(\partial h)$, that is to say if and only if $h(x, x^*) + h^*(x^*, x) = 2\langle x, x^*\rangle$. Since $h \geq \langle\cdot,\cdot\rangle$ and $(h^{*\top})|_{X\times X^*} \geq \langle\cdot,\cdot\rangle$ on $X \times X^*$, the previous equality implies $h(x, x^*) = \langle x, x^*\rangle$ for all $(x, x^*) \in \mathcal{G}(T)$. Thus $h \in \mathcal{H}_T$.

Vice versa, suppose $h \in \mathcal{H}_T$. Then, according to Remark 2.2.8, $(h^{*\top})|_{X\times X^*} \in \mathcal{H}_T$. Since $h(x, x^*) = h^*(x^*, x) = \langle x, x^*\rangle$ for all $(x, x^*) \in \mathcal{G}(T)$, we obtain

$$h(x, x^*) + h^*(x^*, x) = 2\langle x, x^*\rangle, \quad \forall (x, x^*) \in \mathcal{G}(T), \tag{4.5}$$

i.e. $\mathcal{G}(T) \subseteq \mathrm{fix}(\partial h)$. On the other hand, $h \geq \langle \cdot, \cdot \rangle$ and $(h^{*\top})|_{X \times X^*} \geq \langle \cdot, \cdot \rangle$, so that (4.5) implies $h(x, x^*) = h^*(x^*, x) = \langle x, x^* \rangle$ for all $(x, x^*) \in \mathcal{G}(T)$, from which we conclude $\mathrm{fix}(\partial h) \subseteq \mathcal{G}(T)$. Therefore $h \in \mathcal{L}_T$. $\square$

As a consequence of the previous proposition, if $T$ is maximal monotone, (4.2) also reads

$$\forall \varepsilon > 0 : \quad \partial_\varepsilon h(x, x^*) \cap \mathrm{fix}(\partial h)^\top \neq \emptyset,$$

a condition which only involves the subdifferential and the $\varepsilon$-subdifferential of $h$, as anticipated.

In the particular case in which $T = \partial f$ with $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous convex function, (4.2) naturally reads as a condition on (approximate) subdifferentials and can give valuable information on the function $f$ itself. From now on we will focus exclusively on this important case.

**Remark 4.2.6** Note that, in this case,

$$\begin{aligned} \varphi_{\partial f} \text{ is autoconjugate} \quad &\iff \quad \varphi_{\partial f} = f \oplus f^* \\ &\iff \quad \forall x \in X, \forall \varepsilon \geq 0 : \quad (\partial f)^e(x, \varepsilon) = \partial_\varepsilon f(x). \end{aligned}$$

The second equivalence is a consequence of the correspondence between the Fitzpatrick family of any maximal monotone operator and the subfamily of enlargements $\mathbb{E}_C(T)$, studied in [27] (see Section 2.3). Anyway, we include here a direct proof for convenience of the reader.

*Proof.* Obviously, if $\varphi_{\partial f} = f \oplus f^*$, then $\varphi_{\partial f}$ is autoconjugate and $(\partial f)^e(x, \varepsilon) = \partial_\varepsilon f(x)$ for all $x \in X$, $\varepsilon \geq 0$.

Suppose now that $\varphi_{\partial f}$ is autoconjugate. Since $\varphi_{\partial f} \leq f \oplus f^*$, we have $f^* \oplus f^{**} \leq \varphi^*_{\partial f}$, from which

$$f \oplus f^* = (f^* \oplus f^{**})^\top|_{X \times X^*} \leq \left(\varphi^{*\top}_{\partial f}\right)|_{X \times X^*} = \varphi_{\partial f} \leq f \oplus f^*,$$

i.e. $\varphi_{\partial f} = f \oplus f^*$.

Finally, assume that $(\partial f)^e(x, \varepsilon) = \partial_\varepsilon f(x)$ for all $x \in X$, $\varepsilon \geq 0$ and suppose, by contradiction, that $\varphi_{\partial f}(y, y^*) \neq (f \oplus f^*)(y, y^*)$ for some $(y, y^*) \in X \times X^*$. In particular, since $\varphi_{\partial f} \leq f \oplus f^*$, $(y, y^*) \in \mathrm{dom}\, \varphi_{\partial f}$. Thus, there exists $\varepsilon \geq 0$ such that $\varphi_{\partial f}(y, y^*) \leq \langle y, y^* \rangle + \varepsilon$, so that, if $(y, y^*) \notin \mathrm{dom}\, f \times \mathrm{dom}\, f^*$, we obtain $y^* \in (\partial f)^e(y, \varepsilon)$, while $y^* \notin \partial_\varepsilon f(y)$, a contradiction to our assumption. If, on the contrary, $(y, y^*) \in \mathrm{dom}\, f \times \mathrm{dom}\, f^*$, since $\langle y, y^* \rangle \leq \varphi_{\partial f}(y, y^*) < (f \oplus f^*)(y, y^*)$, we can set $\varepsilon := (f \oplus f^*)(y, y^*) - \langle y, y^* \rangle > 0$, obtaining

$$\varphi_{\partial f}(y, y^*) < (f \oplus f^*)(y, y^*) = \langle y, y^* \rangle + \varepsilon.$$

Therefore, there exists $\bar{\varepsilon} \in ]0, \varepsilon[$ such that

$$\varphi_{\partial f}(y, y^*) \leq \langle y, y^* \rangle + \bar{\varepsilon} < (f \oplus f^*)(y, y^*).$$

Thus, $y^* \in (\partial f)^e(y, \bar{\varepsilon})$, while $y^* \notin \partial_{\bar{\varepsilon}} f(y)$, again a contradiction to our assumption. $\square$

The previous remark explains why the problem tackled in [30] could, in principle, be considered as equivalent to the problem of studying when $\partial f$ has an autoconjugate Fitzpatrick function. Anyway, the formulation given in [30] does not allow a perfect equivalence, since the condition

$$\forall x \in \operatorname{dom} f,\, \forall \varepsilon \geq 0: \quad (\partial f)^e(x, \varepsilon) = \partial_\varepsilon f(x)$$

is investigated, while the question whether $\varphi_{\partial f}$ can also take finite value at some point in $(X \backslash \operatorname{dom} f) \times X^*$ is not addressed.

In view of the previous remark, from now on we will confine ourselves to the consideration of the property $\varphi_{\partial f} = f \oplus f^*$, which can be studied by means of the necessary and sufficient condition provided by Theorem 4.2.3. We begin by explicitly stating how that theorem reads in the case when $T$ is the subdifferential of a proper lower semicontinuous and convex function. We skip the proof, since it is an immediate consequence of Theorem 4.2.3, along with Lemma 4.1.3 and the inclusion $f \oplus f^* \in \mathcal{H}_{\partial f}$.

**Corollary 4.2.7** *Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then the following are equivalent:*

*(a)* $\varphi_{\partial f} = f \oplus f^*$;

*(b)* $\operatorname{dom} \varphi_{\partial f} \subseteq \operatorname{dom} f \times \operatorname{dom} f^*$ *and, for all* $(x, x^*) \in \operatorname{dom} \varphi_{\partial f}$,

$$\forall \varepsilon > 0: \qquad \bigcup_{\substack{\alpha, \beta \geq 0 \\ \alpha + \beta = \varepsilon}} (\partial_\alpha f^*(x^*) \times \partial_\beta f(x)) \cap \mathcal{G}(\partial f) \neq \emptyset; \tag{4.6}$$

*(c)* $\mathcal{D}(\partial \varphi_{\partial f}) \subseteq \operatorname{dom} f \times \operatorname{dom} f^*$ *and, for all* $(x, x^*) \in \mathcal{D}(\partial \varphi_{\partial f})$, (4.6) *is satisfied.*

Notice that, if $f = \delta_K$ is the indicator function of a nonempty closed convex set $K \subseteq X$, then $0_{X^*} \in \partial f(x)$ for all $x \in K = \operatorname{dom} f$, so that

$$\emptyset \neq (\partial_\varepsilon f^*(x^*) \times \{0_{X^*}\}) \cap \mathcal{G}(\partial f) \subseteq \bigcup_{\substack{\alpha, \beta \geq 0 \\ \alpha + \beta = \varepsilon}} (\partial_\alpha f^*(x^*) \times \partial_\beta f(x)) \cap \mathcal{G}(\partial f)$$

for all $\varepsilon > 0$, $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$, whence (4.6) is satisfied at any $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$.

Analogously, if $f$ is a proper lower semicontinuous sublinear function, then

$$\emptyset \neq (\{0_X\} \times \partial_\varepsilon f(x)) \cap \mathcal{G}(\partial f) \subseteq \bigcup_{\substack{\alpha,\beta\geq 0\\ \alpha+\beta=\varepsilon}} (\partial_\alpha f^*(x^*) \times \partial_\beta f(x)) \cap \mathcal{G}(\partial f)$$

for all $\varepsilon > 0$, $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$. Thus, (4.6) is satisfied again at any $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$. In both cases, with few additional computation one can prove that $\operatorname{dom} \varphi_{\partial f} \subseteq \operatorname{dom} f \times \operatorname{dom} f^*$, yielding that the Fitzpatrick function of $\partial f$ is autoconjugate, i.e. the results proved in [7]. Actually, [7] proves more, since it shows that, in these cases, not only $\varphi_{\partial f}$ is autoconjugate, but in fact $\mathcal{H}_{\partial f} = \{\varphi_{\partial f}\}$. We will prove this point as well in Corollary 4.2.11.

A natural question is whether appropriately modified indicator and sublinear functions are the only families of functions which satisfy condition (4.6) in such a peculiar way, that is to say, with $\partial f(x)$ ($\partial f^*(x^*)$, respectively) containing a given $z^* \in X^*$ ($z \in X$) for any $x \in \mathcal{D}(\partial f)$ ($x^* \in \mathcal{D}(\partial f^*)$). The answer, which is given in Proposition 4.2.9 and Corollary 4.2.10, is in the positive, provided that we generalize indicator and sublinear functions according to the following definition, in which we introduce the two families of functions required.

**Definition 4.2.8** *Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function.*

*(a) We say that $f$ is* restricted-affine *if it is the sum of an indicator and an affine function, i.e. if there exist $K \subseteq X$ nonempty closed and convex, $z^* \in X^*$ and $\alpha \in \mathbb{R}$ such that $f = \delta_K + z^* + \alpha$.*

*(b) We call $f$* translated-sublinear *if it can be obtained from a sublinear function by translations (either of the domain or of the range), i.e. if there exist a nonempty subset $\widetilde{K} \subseteq X^*$, $z \in X$ and $\alpha \in \mathbb{R}$ such that $f = \big(\delta^*_{\widetilde{K}} \circ \tau_{-z}\big)|_X + \alpha$.*

**Proposition 4.2.9** *Let $X$ be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and $z^* \in X^*$. Then the following statements are equivalent:*

*(a) for all $(x, x^*) \in \mathcal{D}(\partial f) \times \operatorname{dom} f^*$ and for all $\varepsilon > 0$*

$$(\partial_\varepsilon f^*(x^*) \cap X) \times \{z^*\} \subseteq (\partial_\varepsilon f^*(x^*) \times \partial f(x)) \cap \mathcal{G}(\partial f); \tag{4.7}$$

*(b) for all $(x, x^*) \in \mathcal{D}(\partial f) \times \mathcal{D}(\partial f^*)$*

$$(\partial f^*(x^*) \cap X) \times \{z^*\} \subseteq (\partial f^*(x^*) \times \partial f(x)) \cap \mathcal{G}(\partial f); \tag{4.8}$$

*(c)* $z^* \in \bigcap_{x \in \mathcal{D}(\partial f)} \partial f(x)$;

*(d)* *there exist* $\alpha \in \mathbb{R}$ *and a nonempty, closed and convex set* $K \subseteq X$ *such that, for all* $x \in X$, $f(x) = \delta_K(x) + \langle x, z^* \rangle + \alpha$.

*Moreover, if any of the previous items holds, then*

$$\varphi_{\partial f} = f \oplus f^* = (\delta_K + z^*) \oplus (\delta_K^* \circ \tau_{-z^*}). \tag{4.9}$$

*Proof.* $(a) \Longrightarrow (b)$ Since $\mathcal{G}(\partial f) \neq \emptyset$ ($\mathcal{D}(\partial f)$ is dense in $\operatorname{dom} f \neq \emptyset$, being $f$ proper lower semicontinuous and convex), then, by Remark 4.1.1, $\mathcal{G}(\partial f^*) \cap (X^* \times X) \neq \emptyset$. Therefore, there exists $\overline{x}^* \in \mathcal{D}(\partial f^*)$ such that $\emptyset \neq \partial f^*(\overline{x}^*) \cap X \subseteq \partial_\varepsilon f^*(\overline{x}^*) \cap X$ for all $\varepsilon > 0$, so that (4.7) applied to $(x, \overline{x}^*)$ implies $z^* \in \partial f(x)$ for all $x \in \mathcal{D}(\partial f)$. Thus,

$$(\partial f^*(x^*) \cap X) \times \{z^*\} \subseteq \partial f^*(x^*) \times \partial f(x)$$

and

$$(\partial f^*(x^*) \cap X) \times \{z^*\} \subseteq \mathcal{D}(\partial f) \times \{z^*\} \subseteq \mathcal{G}(\partial f),$$

for all $(x, x^*) \in \mathcal{D}(\partial f) \times \mathcal{D}(\partial f^*)$.

$(b) \Longrightarrow (c)$ Reasoning as in the previous implication, since $\emptyset \neq \mathcal{G}(\partial f)^\top \subseteq \mathcal{G}(\partial f^*)$, there exists $x^* \in \mathcal{D}(\partial f^*)$ such that $\partial f^*(x^*) \cap X \neq \emptyset$. Therefore, (4.8) implies

$$\emptyset \neq (\partial f^*(x^*) \cap X) \times \{z^*\} \subseteq \partial f^*(x^*) \times \partial f(x)$$

for all $x \in \mathcal{D}(\partial f)$, i.e. $z^* \in \partial f(x)$ for all $x \in \mathcal{D}(\partial f)$.

$(c) \Longrightarrow (d)$ If $z^* \in \partial f(x)$ for all $x \in \mathcal{D}(\partial f)$, then $f(x) = \langle x, z^* \rangle - f^*(z^*)$ on $\mathcal{D}(\partial f)$ and, in fact, on $\operatorname{dom} f$, being $f$ proper lower semicontinuous and convex, by hypothesis. Thus, setting $K := \operatorname{dom} f$ and $\alpha := -f^*(z^*)$, we obtain $f(x) = \delta_K(x) + \langle x, z^* \rangle + \alpha$ for all $x \in X$. Moreover, $K = \operatorname{dom} f = \mathcal{D}(\partial f)$ is nonempty, convex (since $f$ is proper convex) and closed (being $f$ lower semicontinuous, by hypothesis, and being its affine part continuous).

$(d) \Longrightarrow (a)$ Since $f(x) = \delta_K(x) + \langle x, z^* \rangle + \alpha$, then $z^* \in \partial f(x)$ for all $x \in K$, while, taking Remark 4.1.1 into account, we obtain $\mathcal{R}(\partial_\varepsilon f^*) \cap X = \mathcal{D}(\partial_\varepsilon f) = K$ for all $\varepsilon \geq 0$. Thus, for all $x \in \mathcal{D}(\partial f)$, $x^* \in \operatorname{dom} f^*$,

$$(\partial_\varepsilon f^*(x^*) \cap X) \times \{z^*\} \subseteq \partial_\varepsilon f^*(x^*) \times \partial f(x)$$

and

$$(\partial_\varepsilon f^*(x^*) \cap X) \times \{z^*\} \subseteq K \times \{z^*\} \subseteq \mathcal{G}(\partial f),$$

so that $(a)$ holds.

If $(a) - (d)$ hold, we claim first that $\operatorname{dom} \varphi_{\partial f} = \operatorname{dom} f \times \operatorname{dom} f^*$. Indeed, [8, Theorem 2.6] states that

$$\operatorname{dom} f \times \operatorname{dom} f^* \subseteq \operatorname{dom} \varphi_{\partial f} \subseteq \operatorname{cl}(\operatorname{dom} f) \times \operatorname{cl}(\operatorname{dom} f^*).$$

Since, in the present case, $\operatorname{dom} f = K$ is a closed set, we only need to prove that $\operatorname{Pr}_{X^*}(\operatorname{dom} \varphi_{\partial f}) \subseteq \operatorname{dom} f^*$. Indeed, let $x \in X$ and $x^* \in X^* \backslash (\operatorname{dom} f^*)$. Because $K \times \{z^*\} \subseteq \mathcal{G}(\partial f)$ and, by Lemma 4.1.5, $f^* = \left(\delta_K^* \circ \tau_{-z^*}\right) - \alpha$, implying $\operatorname{dom} f^* = B_K + z^*$ (where $B_K$ is the barrier cone of $K$, i.e., the domain of $\delta_K^*$), then, for any $x \in X$, we have

$$\begin{aligned} \varphi_{\partial f}(x, x^*) &= \sup_{(y,y^*) \in \mathcal{G}(\partial f)} \{\langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle\} \\ &\geq \sup_{y \in K} \{\langle x, z^* \rangle + \langle y, x^* \rangle - \langle y, z^* \rangle\} = \langle x, z^* \rangle + \sup_{y \in K} \langle y, x^* - z^* \rangle = +\infty, \end{aligned}$$

i.e., $x^* \notin \operatorname{Pr}_{X^*}(\operatorname{dom} \varphi_{\partial f})$. Thus, $\operatorname{Pr}_{X^*}(\operatorname{dom} \varphi_{\partial f}) \subseteq \operatorname{dom} f^*$ and, actually, $\operatorname{dom} \varphi_{\partial f} = \operatorname{dom} f \times \operatorname{dom} f^*$.

We claim now that, for all $x^* \in \operatorname{dom} f^*$ and for all $\varepsilon > 0$,

$$\partial_\varepsilon f^*(x^*) \cap X \neq \emptyset.$$

Indeed, by definition of $\delta_K^*$, for any $\varepsilon > 0$ there exists $y \in K$ (dependent on $\varepsilon$) such that $(\delta_K^* \circ \tau_{-z^*})(x^*) - \varepsilon < \langle y, x^* - z^* \rangle$, implying

$$\begin{aligned} f^{**}(y) + f^*(x^*) &= f(y) + f^*(x^*) = \delta_K(y) + \langle y, z^* \rangle + \alpha + \delta_K^*(x^* - z^*) - \alpha \\ &= \langle y, z^* \rangle + \delta_K^*(x^* - z^*) < \langle y, x^* \rangle + \varepsilon, \end{aligned}$$

so that $y \in \partial_\varepsilon f^*(x^*) \cap X$.
Thus, as a consequence of $(a)$, condition (4.6) is satisfied for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^* = \mathcal{D}(\partial f) \times \operatorname{dom} f^*$. Therefore, by Corollary 4.2.7, equality (4.9) is satisfied as well. □

**Corollary 4.2.10** *Let $X$ be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and $z \in X$. The following statements are equivalent:*

*(a) for all $(x, x^*) \in \operatorname{dom} f \times \mathcal{D}(\partial f^*)$ and for all $\varepsilon > 0$*

$$\{z\} \times \partial_\varepsilon f(x) \subseteq (\partial f^*(x^*) \times \partial_\varepsilon f(x)) \cap \mathcal{G}(\partial f); \tag{4.10}$$

*(b) for all* $(x, x^*) \in \mathcal{D}(\partial f) \times \mathcal{D}(\partial f^*)$

$$\{z\} \times \partial f(x) \subseteq (\partial f^*(x^*) \times \partial f(x)) \cap \mathcal{G}(\partial f); \tag{4.11}$$

*(c)* $z \in \bigcap_{x^* \in \mathcal{D}(\partial f^*)} \partial f^*(x^*)$*;*

*(d) there exist* $\alpha \in \mathbb{R}$ *and a nonempty, weak*$^*$*-closed and convex set* $\widetilde{K} \subseteq X^*$ *such that, for all* $x \in X$, $f(x) = \delta^*_{\widetilde{K}}(x - z) + \alpha$.

*Moreover, if any of the previous items holds, then*

$$\varphi_{\partial f} = f \oplus f^* = \big(\delta^*_{\widetilde{K}} \circ \tau_{-z}\big)|_X \oplus (\delta_{\widetilde{K}} + z). \tag{4.12}$$

*Proof.* Applying Proposition 4.2.9 to $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ and $z \in X^{**}$ (recall that we identify $X$ with its image in the bidual), we obtain that the following statements are equivalent:

$(a')$ for all $(x^*, x^{**}) \in \mathcal{D}(\partial f^*) \times \operatorname{dom} f^{**}$ and for all $\varepsilon > 0$

$$(\partial_\varepsilon f^{**}(x^{**}) \cap X^*) \times \{z\} \subseteq (\partial_\varepsilon f^{**}(x^{**}) \times \partial f^*(x^*)) \cap \mathcal{G}(\partial f^*);$$

$(b')$ for all $(x^*, x^{**}) \in \mathcal{D}(\partial f^*) \times \mathcal{D}(\partial f^{**})$

$$(\partial f^{**}(x^{**}) \cap X^*) \times \{z\} \subseteq (\partial f^{**}(x^{**}) \times \partial f^*(x^*)) \cap \mathcal{G}(\partial f^*);$$

$(c')$ $z \in \bigcap_{x^* \in \mathcal{D}(\partial f^*)} \partial f^*(x^*)$;

$(d')$ there exist $\alpha \in \mathbb{R}$ and a nonempty, weak$^*$-closed and convex set $\widetilde{K} \subseteq X^*$ such that, for all $x^* \in X^*$, $f^*(x^*) = \delta_{\widetilde{K}}(x^*) + \langle x^*, z\rangle - \alpha$,

where in $(d')$ we can write weak$^*$−closed instead of closed, since $f^*$ is weak$^*$-lower semicontinuous. Moreover, if any of $(a') - (d')$ holds, then

$$\varphi_{\partial f^*} = f^* \oplus f^{**}. \tag{4.13}$$

Taking into account Lemmas 4.1.5 and 4.1.6 and the fact that, as a consequence of Remark 4.1.1, $\partial_\varepsilon f^{**}(x) \cap X^* = \partial_\varepsilon f(x)$ for all $\varepsilon \geq 0$ and for all $x \in \operatorname{dom} f$, then

$$(a') \Longrightarrow (a), \quad (b') \Longrightarrow (b), \quad (c') \Longleftrightarrow (c), \quad (d') \Longleftrightarrow (d).$$

Thus,

$$(c) \Longleftrightarrow (d) \Longrightarrow (a), (b).$$

In order to prove $(b) \Longrightarrow (c)$, simply reason as in Proposition 4.2.9; namely, consider that, since $\mathcal{G}(\partial f) \neq \emptyset$, there exists $x \in \mathcal{D}(\partial f)$, so that, by (4.11), $\emptyset \neq \{z\} \times \partial f(x) \subseteq \partial f^*(x^*) \times \partial f(x)$, for all $x^* \in \mathcal{D}(\partial f^*) \neq \emptyset$. Therefore, $z \in \partial f^*(x^*)$ for all $x^* \in \mathcal{D}(\partial f^*)$. The proof of the implication $(a) \Longrightarrow (c)$ is similar.

Finally, if any of items $(a) - (d)$ holds, then $(a') - (d')$ hold as well. Then, by (4.13), taking into account Remark 4.1.1,

$$\varphi_{\partial f} = (\sigma^{*\top}_{\partial f})|_{X \times X^*} = (\varphi^{\top}_{\partial f^*})|_{X \times X^*} = (f^{**} \oplus f^*)|_{X \times X^*} = f \oplus f^*,$$

which yields (4.12). □

**Corollary 4.2.11** *Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. If $f$ is restricted-affine or tranlated-sublinear, then the Fitzpatrick family of the subdifferential of $f$ is a singleton, i.e. $\mathcal{H}_{\partial f} = \{f \oplus f^*\}$.*

*Proof.* By Lemma 4.1.5, if $f$ is restricted-affine (tranlated-sublinear), then $f^*$ is tranlated-sublinear (restricted-affine, respectively). Therefore, by Corollary 4.2.10 (Proposition 4.2.9) and Remark 4.1.1, $\sigma^*_{\partial f} = \varphi_{\partial f^*} = f^* \oplus f^{**}$, from which

$$\sigma_{\partial f} = (\sigma^{**}_{\partial f})|_{X \times X^*} = (f^{**} \oplus f^{***})|_{X \times X^*} = f \oplus f^*.$$

On the other hand, by Proposition 4.2.9 (Corollary 4.2.10, respectively), $\varphi_{\partial f} = f \oplus f^*$. Hence, $\mathcal{H}_{\partial f} = \{f \oplus f^*\}$. □

**Remark 4.2.12** The previous results show that restricted-affine and translated-sublinear functions:

*(i)* satisfy condition (4.6) in a specially simple way (i.e. one of the two approximate subdifferentials is replaced by a singleton, which is independent of the points $x, x^*$ considered);

*(ii)* have subdifferentials whose Fitzpatrick functions are autoconjugate (and, actually, their Fitzpatrick families are singletons).

Moreover, since they are the only functions for which $(i)$ holds, we have that $(i) \Longrightarrow (ii)$. An interesting question to consider is whether the converse implication holds as well (this is a generalization of the conjecture [30, Remark 3.8] and a reformulation of it in our setting). The next section proves that it holds in the simple case $X = \mathbb{R}$, while Section 4.4 provides a counterexample showing that the implication already fails if we take $X = \mathbb{R}^2$.

## 4.3 The One-Dimensional Case

In this section we will consider the elementary case of a proper lower semicontinuous convex function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and prove that, in this framework, restricted-affine and translated-sublinear functions are the only ones whose subdifferentials have autoconjugate Fitzpatrick functions.

We will employ the following notation:

$$i := \inf \operatorname{dom} f, \qquad s := \sup \operatorname{dom} f, \qquad i^* := \inf \operatorname{dom} f^*, \qquad s^* := \sup \operatorname{dom} f^*.$$

Recall that, when $X = \mathbb{R}$, $I := \operatorname{dom} f$ is an interval and $f$ is continuous on $\operatorname{cl} I$ [108, Proposition 2.1.6]. Moreover, for any $\varepsilon \geq 0$ and for any $x \in I$, $\partial_\varepsilon f(x)$ is a closed interval in $\mathbb{R}$, possibly unbounded, or empty. It can be $\partial_\varepsilon f(x) = \emptyset$ only if $\varepsilon = 0$ and $x \in \{i, s\}$.

**Remark 4.3.1** Notice that, in the one-variable framework, the following useful property holds:

$$\forall x, y \in \mathcal{D}(\partial f) : \qquad x < y \qquad \Longrightarrow \qquad \max \partial f(x) \leq \min \partial f(y). \tag{4.14}$$

Indeed, for all $x^* \in \partial f(x)$, $y^* \in \partial f(y)$, by the monotonicity of $\partial f : \mathbb{R} \rightrightarrows \mathbb{R}$, one has $(y - x)(y^* - x^*) \geq 0$, yielding $y^* \geq x^*$, i.e.

$$\sup \partial f(x) \leq \inf \partial f(y).$$

Thus $\partial f(x)$ has an upper bound and, being a nonempty closed interval, it has a maximum. Similarly, $\partial f(y)$ has a lower bound and, actually, a minimum.

The following lemma recalls a well-known property of subdifferentials, stating it in the one-dimensional setting we are now considering. In this case, for all $E \subseteq \mathbb{R}$, the set $\operatorname{cl}\operatorname{conv} E$ is the smallest closed interval containing $E$.

**Lemma 4.3.2** *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. If there exist a nonempty set $E \subseteq \mathbb{R}$ and $x^* \in \mathbb{R}$ such that $x^* \in \partial f(x)$ for all $x \in E$, then $f|_{\operatorname{cl}\operatorname{conv} E}(x) = x^* x + c$, with $c \in \mathbb{R}$.*

*Proof.* It is obvious that $f|_{\partial f^*(x^*)}(x) = x^* x + c$, where $c := -f^*(x^*)$. On the other hand, it follows from Remark 4.1.1 and the properties of closure and convexity of $\partial f^*(x^*)$ that $E \subseteq \operatorname{cl}\operatorname{conv} E \subseteq \partial f^*(x^*)$. □

The following lemma will be at the heart of our analysis in this section. Loosely speaking, it states that, if relation (4.6) is satisfied, then $f$ cannot change its slope "too often". This property will be made explicit in Proposition 4.3.4 below.

**Lemma 4.3.3** *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function satisfying condition* (4.6) *for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$ and such that $i^* < s^*$. Moreover, let $x^* \in ]i^*, s^*[$ and $\overline{\alpha} > 0$.*

*(a) If $n := \max \partial_{\overline{\alpha}} f^*(x^*) < s$, then, for all $x \in \operatorname{dom} f$ such that $x \geq n$, $\max \partial f(n) \in \partial f(x)$.*

*(b) If $m := \min \partial_{\overline{\alpha}} f^*(x^*) > i$, then, for all $x \in \operatorname{dom} f$ such that $x \leq m$, $\min \partial f(m) \in \partial f(x)$.*

*Proof.* We will only prove the first item, since the proof of the second one is similar. Notice first that $\partial f(n) \neq \emptyset$ and that its maximum is well defined. Indeed, since $x^* \in ]i^*, s^*[$, there exists $y \in \partial f^*(x^*) \subseteq \partial_{\overline{\alpha}} f^*(x^*)$. Therefore $i \leq y \leq n < s$, so that either $n$ belongs to the interior of $\operatorname{dom} f$, or $n = y = i$. In both cases $\partial f(n) \neq \emptyset$. Moreover, there exists $x' \in ]n, s[ \subseteq \mathcal{D}(\partial f)$. Then it follows from Remark 4.3.1 that $\sup \partial f(n)$ is attained. For ease of notation, we will set $a^* := \max \partial f(n)$.

Let $x \in [n, s[$. When $x = n$, by definition $\max \partial f(n) \in \partial f(n) = \partial f(x)$. Thus, suppose $n < x$. A necessary condition for (4.6) to be satisfied at $(x, x^*)$, with $x^* \in \operatorname{dom} f^*$, is that, for any $0 < \varepsilon < \overline{\alpha}$, there exist $0 \leq \beta \leq \varepsilon$ such that $(\partial_{\overline{\alpha}} f^*(x^*) \times \partial_\beta f(x)) \cap \mathcal{G}(\partial f) \neq \emptyset$, since

$$\bigcup_{\substack{\alpha, \beta \geq 0 \\ \alpha + \beta = \varepsilon}} (\partial_\alpha f^*(x^*) \times \partial_\beta f(x)) \subseteq \partial_{\overline{\alpha}} f^*(x^*) \times \bigcup_{0 \leq \beta \leq \varepsilon} \partial_\beta f(x).$$

That is to say, a necessary condition is that there exist $0 \leq \beta \leq \varepsilon$ and $\overline{y} \in \partial_{\overline{\alpha}} f^*(x^*)$ such that

$$\partial f(\overline{y}) \cap \partial_\beta f(x) \neq \emptyset.$$

On the other hand, as a consequence of (4.14), for all $y \in \partial_{\overline{\alpha}} f^*(x^*)$ and $y^* \in \partial f(y)$, since $y \leq n < x$,

$$y^* \leq a^* \leq \min \partial f(x) \leq \sup \partial_\beta f(x). \tag{4.15}$$

In particular, taking $\overline{y}^* \in \partial f(\overline{y}) \cap \partial_\beta f(x)$, since $\partial_\beta f(x)$ is an interval and $\sup \partial_\beta f(x)$ is attained whenever finite, condition (4.15) implies $a^* \in \partial_\beta f(x)$, i.e.

$$f(x) + f^*(a^*) \leq x a^* + \beta \leq x a^* + \varepsilon.$$

Let $\varepsilon \to 0^+$ to conclude that $a^* \in \partial f(x)$. $\square$

In the remaining of this section, we will adopt the following notation:

$$[-\infty, m] := \,]-\infty, m], \quad \text{and} \quad [m, +\infty] := [m, +\infty[,$$

for all $m \in \mathbb{R}$, and $[-\infty, +\infty] := \,]-\infty, +\infty[$, while, as usual, $[x, x] = \{x\}$, for all $x \in \mathbb{R}$.

The following proposition and the subsequent lemma provide the analytical expressions of a proper lower semicontinuous convex function $f$ and its conjugate function $f^*$, respectively, under the hypothesis that $f$ satisfies condition (4.6) for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$.

**Proposition 4.3.4** *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function satisfying condition (4.6) for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$. Then $f$ is continuous on $[i, s]$ and there exist $m, n \in \operatorname{dom} f$, with $m \leq n$, and $a^*, b, c^*, d, e^*, g \in \mathbb{R}$ such that*

$$f(x) = \begin{cases} a^*x + b, & x \in [i, m] \\ c^*x + d, & x \in [m, n] \\ e^*x + g, & x \in [n, s]. \end{cases} \tag{4.16}$$

*Proof.* The continuity of $f$ on $[i, s]$ follows from $f$ being a proper lower semicontinuous convex function. Thus, we only have to prove that $f$ fits the scheme given in (4.16). If $i = s$, the result is trivial, so we can assume $i \neq s$. To prove the result, we will distinguish two main cases.

(a) Suppose that there exist $x^* \in \,]i^*, s^*[$ and $\overline{\beta} > 0$ such that $\max \partial_{\overline{\beta}} f^*(x^*) < s$. Then, in particular,

$$\max \partial_\beta f^*(x^*) < s$$

for all $\beta \in \,]0, \overline{\beta}]$. As a consequence of Lemma 4.3.3, there exists $e^*_\beta \in \mathbb{R}$, namely $e^*_\beta = \max \partial f(n_\beta)$ with $n_\beta := \max \partial_\beta f^*(x^*)$, such that $e^*_\beta \in \partial f(y)$ for all $y \in \operatorname{dom} f$ with $y \geq \max \partial_\beta f^*(x^*)$. Therefore, by Lemma 4.3.2, for all $\beta \in \,]0, \overline{\beta}]$ there exists $g_\beta \in \mathbb{R}$ such that

$$f|_{[\max \partial_\beta f^*(x^*), s]}(x) = e^*_\beta x + g_\beta,$$

where, actually, $e^*_\beta$ and $g_\beta$ do not depend on $\beta$, for $f$ to be uniquely defined on the nondegenerate interval $[\max \partial_{\overline{\beta}} f^*(x^*), s]$, so that we will simply write $e^*$ and $g$ respectively, dropping the index $\beta$. By (1.3), $\max \partial_\beta f^*(x^*) \to \max \partial f^*(x^*)$ as $\beta \to 0^+$, so that the previous equality implies $f|_{]\max \partial f^*(x^*), s]}(x) = e^*x + g$ and, by continuity,

$$f|_{[n, s]}(x) = e^*x + g,$$

where $n := \max \partial f^*(x^*)$.

Finally, consider two subcases. If there exists $\overline{\alpha} > 0$ such that $i < \min \partial_{\overline{\alpha}} f^*(x^*)$, then, reasoning in a similar way, we conclude that

$$f|_{[i,m]}(x) = a^* x + b, \tag{4.17}$$

where $m := \min \partial f^*(x^*)$ and $a^*, b \in \mathbb{R}$. Therefore, since, by Lemma 4.3.2,

$$f|_{[m,n]}(x) = c^* x + d,$$

with $c^* := x^*$ and $d \in \mathbb{R}$, the function $f$ is of the type described by (4.16). If, on the contrary, $\inf \partial_\alpha f^*(x^*) = i$ for all $\alpha > 0$, then $\inf \partial f^*(x^*) = i$ and (4.17) holds with $m = n$. Therefore $f$ has again the structure displayed in (4.16).

(b) Suppose now that, for all $x^* \in ]i^*, s^*[$ and for all $\beta > 0$, $\sup \partial_\beta f^*(x^*) = s$, i.e.

$$\sup \partial f^*(x^*) = s. \tag{4.18}$$

If $i^* = s^*$, then $\operatorname{dom} f^* = \{s^*\}$ and $f = xs^* - f^*(s^*)$ for all $x \in \mathbb{R}$, which is an instance of (4.16).

If, on the contrary, $i^* < s^*$, notice that $s < +\infty$. Indeed, if it were $s = +\infty$, under hypothesis (4.18) we would have $\sup \partial f^*(x^*) = +\infty$, for all $x^* \in ]i^*, s^*[$. As a consequence of Remark 4.3.1, this would imply $\operatorname{dom} f^* = \{s^*\}$, a contradiction. Then $s < +\infty$, so that, by (4.18), $s = \max \partial f^*(x^*)$ for all $x^* \in ]i^*, s^*[$. By Lemma 4.3.2, this yields $f^*(x^*) = sx^* + k$ on $[i^*, s^*]$, for some $k \in \mathbb{R}$. If $i^* = -\infty$, then it is easy to check that $f(x) = \delta_{\{s\}}(x) - k$, for all $x \in \mathbb{R}$. If $i^* > -\infty$, then $f(x) = i^* x - (i^* s + k)$, for all $x \in [i, s]$. In both cases, the analytic expression of $f$ fits again (4.16). □

**Remark 4.3.5** Since $f$ is convex, $a^* \leq c^* \leq e^*$. Moreover, the continuity of $f$ implies that

$$b = m(c^* - a^*) + d \quad \text{and} \quad g = -n(e^* - c^*) + d.$$

**Lemma 4.3.6** *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function defined as in* (4.16)*. Then $f^*$ is continuous on $[i^*, s^*]$ and there exist $p, q, r, t \in \mathbb{R}$ such that*

$$f^*(x^*) = \begin{cases} ix^* + p, & x^* \in [i^*, a^*[ \\ mx^* + q, & x^* \in [a^*, c^*] \\ nx^* + r, & x^* \in [c^*, e^*] \\ sx^* + t, & x^* \in ]e^*, s^*], \end{cases} \tag{4.19}$$

*where:*

*(i) if* $i = -\infty$, *then* $i^* = a^* \in \mathbb{R}$; *otherwise,* $i^* = -\infty$;

*(ii) if* $s = +\infty$, *then* $s^* = e^* \in \mathbb{R}$; *otherwise,* $s^* = +\infty$.

*Proof.* Since $f^*$ is proper lower semicontinuous and convex, it is continuous on $[i^*, s^*]$.

If $i = s$, then there exists $t \in \mathbb{R}$ such that $f^*(x^*) = sx^* + t$ for all $x^* \in X^*$, i.e. $f^*$ fits (4.19) and $(i)$ and $(ii)$ are satisfied.

Therefore, we will concentrate on the case when $i < s$. Hence, we can always rewrite $f$ as in (4.16), with $m, n \in \mathbb{R}$ and $i < m < n < s$ (possibly, $a^* = c^*$ or $c^* = e^*$). Then $\partial f(m) = [a^*, c^*]$ and $\partial f(n) = [c^*, e^*]$. Thus, by Lemma 4.3.2, there exist $q, r \in \mathbb{R}$ such that $f^*|_{[a^*,c^*]}(x^*) = mx^* + q$ and $f^*|_{[c^*,e^*]}(x^*) = nx^* + r$.

It follows from (4.16) that, if $i > -\infty$, then $\partial f(i) = \,]-\infty, a^*]$, i.e. $i \in \partial f^*(x^*)$ for all $x^* \in \,]-\infty, a^*]$. Therefore, by Lemma 4.3.2, there exists $p \in \mathbb{R}$ such that $f^*|_{]-\infty,a^*]}(x^*) = ix^* + p$ and we conclude that $i^* = -\infty$. On the other hand, if $i = -\infty$, we necessarily have $i^* = a^* \in \mathbb{R}$, since for all $x^* < a^*$ one has $x(x^* - a^*) \to +\infty$ as $x \to -\infty$, implying $f^*(x^*) = +\infty$ (note that we cannot have $i^* > a^*$, since $a^* \in \mathcal{R}(\partial f) = \mathcal{D}(\partial f^*) \subseteq \operatorname{dom} f^*$).

Analogously, it is easy to prove that, when $s < +\infty$, there exists $t \in \mathbb{R}$ such that $f^*|_{[e^*,+\infty[}(x^*) = sx^* + t$ and $s^* = +\infty$, while, if $s = +\infty$, $s^* = e^* \in \mathbb{R}$.

Thus $f^*$ corresponds to the scheme (4.19) and $(i)$ and $(ii)$ hold. $\square$

**Remark 4.3.7** By direct computation, one proves that:

$$\begin{aligned} p &= (m-i)a^* - (mc^* + d), & q &= -(mc^* + d), \\ r &= -(nc^* + d), & t &= -(s-n)e^* - (nc^* + d), \end{aligned}$$

where $p$ and $t$ are defined if $i > -\infty$ and $s < +\infty$, respectively (otherwise, $[i^*, a^*[\, = \emptyset$ and $]c^*, s^*] = \emptyset$, respectively).

Finally, the following theorem combines the previous two results to put restrictions on the admissible analytical expressions for $f$. To this end, notice that, if $f$ satisfies condition (4.6), then the same condition holds for $f^*$ as well, since, by Remark 4.1.1, one has

$$\bigcup_{\substack{\alpha,\beta \geq 0 \\ \alpha+\beta=\varepsilon}} (\partial_\alpha f^*(x^*) \times \partial_\beta f(x)) \cap \mathcal{G}(\partial f) \neq \emptyset \iff \bigcup_{\substack{\alpha,\beta \geq 0 \\ \alpha+\beta=\varepsilon}} (\partial_\alpha f(x) \times \partial_\beta f^*(x^*)) \cap \mathcal{G}(\partial f^*) \neq \emptyset,$$

for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$, $\varepsilon > 0$.

**Theorem 4.3.8** *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function satisfying condition (4.6) for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$. Then $f$ is a restricted-affine or a translated-sublinear function.*

*Proof.* Since $f$ satisfies condition (4.6) for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$, then, by Proposition 4.3.4, the analytic form of $f$ is given by (4.16), for some $m, n \in \operatorname{dom} f$ and $a^*, c^*, e^*, b, d, g \in \mathbb{R}$. If $i = s$, $f$ is trivially restricted-affine. Thus, as in the proof of the previous lemma, we can assume $i < m < n < s$, with possibly $a^* = c^*$ or $c^* = e^*$. Suppose by contradiction that $f$ is neither a restricted-affine nor a translated-sublinear function. This implies that (at least) one of the following cases holds:

(a) $i = -\infty$, $s = +\infty$ and $|\{a^*, c^*, e^*\}| = 3$;

(b) $i > -\infty$ and $|\{a^*, c^*, e^*\}| \geq 2$;

(c) $s < +\infty$ and $|\{a^*, c^*, e^*\}| \geq 2$.

Obviously item $(c)$ can be treated similarly to $(b)$; hence, we will concentrate only on the first two cases.

(a) By Lemma 4.3.6, we have $i^* = a^* \in \mathbb{R}$, $s^* = e^* \in \mathbb{R}$ and

$$f^*(x^*) = \begin{cases} mx^* + q, & x^* \in [i^*, c^*] \\ nx^* + r, & x^* \in [c^*, s^*]. \end{cases}$$

By definition, for any $\alpha \geq 0$, the set $\partial_\alpha f^*(i^*)$ consists exactly of those $y \in \operatorname{dom} f$ such that $f(y) + f^*(i^*) \leq yi^* + \alpha$, i.e., by Remark 4.3.7,

$$f(y) + mi^* - mc^* - d \leq yi^* + \alpha. \tag{4.20}$$

Taking into account that $f$ is described by (4.16) and that $\max \partial_\alpha f^*(i^*) < n$ if and only if $n \notin \partial_\alpha f^*(i^*)$ (since $n > m \in \partial f^*(i^*)$, it cannot be $\min \partial_\alpha f^*(i^*) > n$), we conclude from (4.20) that a sufficient condition for $\max \partial_\alpha f^*(i^*) < n$ to be satisfied is

$$c^*n + d + mi^* - mc^* - d > ni^* + \alpha,$$

i.e., $\alpha < (n - m)(c^* - i^*)$. Therefore, if $\alpha < (n - m)(c^* - i^*)$, $\max \partial_\alpha f^*(i^*) < n$, so that, for all $y \in \partial_\alpha f^*(i^*)$,

$$\max \partial f(y) \leq \min \partial f(n) = c^*,$$

according to Remark 4.3.1.

It follows from analogous computations that, if $\beta < s^* - c^*$, then, for all $z^* \in \partial_\beta f(n+1)$, $z^* > c^*$.

Therefore condition (4.6) cannot hold with $(x, x^*) = (n+1, i^*)$, considering

$$\varepsilon < \min\{(n-m)(c^* - i^*), s^* - c^*\}.$$

Thus, case $(a)$ is not a viable alternative.

(b) Notice first that we can suppose $s = +\infty$, since, otherwise: if $|\{a^*, c^*, e^*\}| = 2$, then it is easily checked that $f^*$ would be of the kind considered for $f$ in the previous item (and we could repeat the same proof, reasoning on $f^*$ instead of $f$); while, if $|\{a^*, c^*, e^*\}| = 3$, then $f^*$ would not comply with the necessary condition prescribed by Proposition 4.3.4 for any proper lower semicontinuous convex function satisfying condition (4.6) for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$.

Therefore, we are left with the case $i > -\infty$ and $s = +\infty$, i.e. $i^* = -\infty$ and $s^* = e^* \in \mathbb{R}$. Since, in the case we are now considering, $|\{a^*, c^*, s^*\}| \geq 2$, then $a^* \neq s^*$. Without loss of generality, just to fix notation, we can suppose $a^* < c^* = s^*$ when $|\{a^*, c^*, s^*\}| = 2$. By direct computation, similar to the previous point, we obtain $\max \partial_\alpha f^*(a^* - 1) < m$ for all $\alpha < m - i$. Hence, for all $y \in \partial_\alpha f^*(a^* - 1)$, one has $\max \partial f(y) \leq \min \partial f(m) = a^*$, by Remark 4.3.1. Similarly, setting $k := (m+n)/2$, one can calculate that, if $\beta < (k-m)(c^* - a^*)$, then $a^* \notin \partial_\beta f(k)$, implying $z^* > a^*$, for all $z^* \in \partial_\beta f(k)$. Thus, condition (4.6) cannot be satisfied at $(x, x^*) = (k, a^* - 1)$, considering

$$\varepsilon < \min\{m - i, (k-m)(c^* - a^*)\},$$

a contradiction to the hypothesis of the present theorem. $\square$

**Corollary 4.3.9** *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then $\mathcal{H}_{\partial f} = \{f \oplus f^*\}$ if and only if $f$ is a restricted-affine or a translated-sublinear function.*

*Proof.* If $f$ is a restricted-affine or a translated-sublinear function, it follows from Corollary 4.2.11 that $\mathcal{H}_{\partial f} = \{f \oplus f^*\}$. If, on the contrary, this equality holds, we have $\varphi_{\partial f} = f \oplus f^*$ and, by Corollary 4.2.7, condition (4.6) holds for all $(x, x^*) \in \operatorname{dom} f \times \operatorname{dom} f^*$. Thus, the previous theorem guarantees that the function $f$ is either restricted-affine or translated-sublinear. $\square$

## 4.4 A Counterexample in Two Dimensions

The arguments employed in the previous section for the one-dimensional case can be extended to multidimensional spaces only to a limited extent. In particular, the main result provided by Corollary 4.3.9 does not hold any more, as shown by the following simple example in $\mathbb{R}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined as

$$f(x,y) = \begin{cases} -x, & \text{if } (x,y) \in \ ]-\infty,0] \times [0,1] \\ 0, & \text{if } (x,y) \in [0,+\infty[ \times [0,1] \\ +\infty, & \text{if } (x,y) \in \mathbb{R} \times (\mathbb{R}\backslash[0,1]). \end{cases} \tag{4.21}$$

The function $f$ is proper lower semicontinuous and convex and has a closed domain $\operatorname{dom} f = \mathbb{R} \times [0,1]$. Its conjugate function $f^* : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f^*(x^*,y^*) = \begin{cases} 0, & \text{if } (x^*,y^*) \in [-1,0] \times \ ]-\infty,0] \\ y^*, & \text{if } (x^*,y^*) \in [-1,0] \times [0,+\infty[ \\ +\infty, & \text{if } (x^*,y^*) \in (\mathbb{R}\backslash[-1,0]) \times \mathbb{R}. \end{cases} \tag{4.22}$$

As we will prove in a moment, $f$ and $f^*$ are subdifferentiable on the whole of their respective domains. Thus, according to Remark 4.2.4 and Corollary 4.2.7, a sufficient condition for the equality $\varphi_{\partial f} = f \oplus f^*$ to hold is that $\operatorname{dom} \varphi_{\partial f} = \operatorname{dom} f \times \operatorname{dom} f^*$ and

$$(\partial f^*(x^*,y^*) \times \partial f(x,y)) \cap \mathcal{G}(\partial f) \neq \emptyset \tag{4.23}$$

for all $((x,y),(x^*,y^*)) \in \operatorname{dom} f \times \operatorname{dom} f^*$. The condition $\operatorname{dom} \varphi_{\partial f} = \operatorname{dom} f \times \operatorname{dom} f^*$ is true since, by [8, Theorem 2.6]

$$\operatorname{dom} f \times \operatorname{dom} f^* \subseteq \operatorname{dom} \varphi_{\partial f} \subseteq \operatorname{cl} (\operatorname{dom} f \times \operatorname{dom} f^*)$$

and, in the present example, $\operatorname{dom} f \times \operatorname{dom} f^*$ is a closed set. To prove that (4.23) is satisfied for all elements of $\operatorname{dom} f \times \operatorname{dom} f^*$, we explicitly calculate the subdifferentials of $f$ and $f^*$ (see Tables 4.1 and 4.2).

We deduce from the calculations in Tables 4.1 and 4.2 that, for all $(x^*,y^*) \in \operatorname{dom} f^*$,

$$\{(0,0),(0,1)\} \cap \partial f^*(x^*,y^*) \neq \emptyset.$$

On the other hand, since $\partial f(0,0) = [-1,0] \times \ ]-\infty,0]$ and $\partial f(0,1) = [-1,0] \times [0,+\infty[$, it is easy to check that

$$\partial f(0,0) \cap \partial f(x,y) \neq \emptyset \quad \text{and} \quad \partial f(0,1) \cap \partial f(x,y) \neq \emptyset$$

| $(x,y)$ | $\in$ | | $\partial f(x,y)$ | $=$ | |
|---|---|---|---|---|---|
| $]0,+\infty[$ | $\times$ | $\{1\}$ | $\{0\}$ | $\times$ | $[0,+\infty[$ |
| $]0,+\infty[$ | $\times$ | $]0,1[$ | $\{0\}$ | $\times$ | $\{0\}$ |
| $]0,+\infty[$ | $\times$ | $\{0\}$ | $\{0\}$ | $\times$ | $]-\infty,0]$ |
| $\{0\}$ | $\times$ | $\{1\}$ | $[-1,0]$ | $\times$ | $[0,+\infty[$ |
| $\{0\}$ | $\times$ | $]0,1[$ | $[-1,0]$ | $\times$ | $\{0\}$ |
| $\{0\}$ | $\times$ | $\{0\}$ | $[-1,0]$ | $\times$ | $]-\infty,0]$ |
| $]-\infty,0[$ | $\times$ | $\{1\}$ | $\{-1\}$ | $\times$ | $[0,+\infty[$ |
| $]-\infty,0[$ | $\times$ | $]0,1[$ | $\{-1\}$ | $\times$ | $\{0\}$ |
| $]-\infty,0[$ | $\times$ | $\{0\}$ | $\{-1\}$ | $\times$ | $]-\infty,0]$ |

Table 4.1: Subdifferential of $f$.

| $(x^*,y^*)$ | $\in$ | | $\partial f^*(x^*,y^*)$ | $=$ | |
|---|---|---|---|---|---|
| $\{0\}$ | $\times$ | $]0,+\infty[$ | $[0,+\infty[$ | $\times$ | $\{1\}$ |
| $]-1,0[$ | $\times$ | $]0,+\infty[$ | $\{0\}$ | $\times$ | $\{1\}$ |
| $\{-1\}$ | $\times$ | $]0,+\infty[$ | $]-\infty,0]$ | $\times$ | $\{1\}$ |
| $\{0\}$ | $\times$ | $\{0\}$ | $[0,+\infty[$ | $\times$ | $[0,1]$ |
| $]-1,0[$ | $\times$ | $\{0\}$ | $\{0\}$ | $\times$ | $[0,1]$ |
| $\{-1\}$ | $\times$ | $\{0\}$ | $]-\infty,0]$ | $\times$ | $[0,1]$ |
| $\{0\}$ | $\times$ | $]-\infty,0[$ | $[0,+\infty[$ | $\times$ | $\{0\}$ |
| $]-1,0[$ | $\times$ | $]-\infty,0[$ | $\{0\}$ | $\times$ | $\{0\}$ |
| $\{-1\}$ | $\times$ | $]-\infty,0[$ | $]-\infty,0]$ | $\times$ | $\{0\}$ |

Table 4.2: Subdifferential of $f^*$

for all $(x,y) \in \operatorname{dom} f$. Thus condition (4.23) is satisfied on $\operatorname{dom} f \times \operatorname{dom} f^*$ and we conclude that $\varphi_{\partial f} = f \oplus f^*$, though $f$ is clearly neither a restricted-affine nor a translated-sublinear function.

# Chapter 5

# Surjectivity Properties of Maximal Monotone Operators of Type (D)

As we have seen in Section 2.2.4, in the setting of reflexive Banach spaces, Martínez-Legaz [59] provided an interesting generalization of Rockafellar's surjectivity theorem, replacing the duality mapping by any maximal monotone operator having finite-valued Fitzpatrick function.

The aim of the present chapter, which is based on [80], is to further investigate in the domain of the convex analytical proofs contained in [59], especially with respect to their relevance for surjectivity results and applications of them. In this sense, we mainly generalize [59] along two directions.

First, by considering the case of a (possibly) nonreflexive Banach space with maximal monotone operators of type (D) defined on it. We mainly provide surjectivity properties that are stated in a natural way in terms of the unique extensions of the operators to the bidual, but we also consider a couple of results concerning density properties for the operators themselves, on the lines of [58].

Second, even for those results that hinge upon the hypothesis of reflexivity, we provide some generalizations with respect to [59] by refining the constraint qualifications and analyzing in full detail the structure and the scope of the proof techniques employed in that paper. Namely, we weaken the requirement of finite-valued Fitzpatrick functions typically used in [59], replacing it by conditions on the sum of the domains of convex representations, and characterize surjectivity properties in terms of the existence of Fenchel functionals (see Definition 5.1.3 below). This characterization, moreover, makes explicit the equivalent role played in our duality based proofs by any member of the Fitzpatrick family. The symmetry is such that they essentially have the same Fenchel functionals, if any.

The chapter is organized as follows. In the first section we set notation and recall basic definitions. Moreover, we collect some important results from [96], which we will need later on and we prove some simple preliminary lemmas. In the second section, we prove the surjectivity theorems in their form related to the sum of the graphs. In the third section, we prove the surjectivity result for the range of the sum of two maximal monotone operators of type (D) (satisfying appropriate conditions) and derive some corollaries (in particular an existence theorem for variational inequalities on reflexive Banach spaces) that refine the corresponding results in [59]. Finally, the last section provides, as an application of the previous results, a new convex analytical proof of the relations between the range of a maximal monotone operator of type (D) and the projections of the domains of its convex representations on the dual space, yielding as a consequence the convexity of the closure of the range.

## 5.1 Preliminary Results

Recall from Section 1.3 that, for any nonempty closed convex set $K \subseteq X$, the normal cone operator to $K$ is defined as $N_K = \partial\delta_K$, that is

$$N_K(x) := \begin{cases} \{x^* \in X^* : \langle y - x, x^* \rangle \le 0, \quad \forall y \in K\}, & x \in K \\ \emptyset, & x \notin K. \end{cases}$$

Moreover, we will denote by $B_K$ the *barrier cone* of $K$, i.e. the domain of the support function $\delta_K^*$.

In this chapter we will also need enlargements. For ease of notation, we will write $T^\varepsilon(x)$ instead of $T^e(x, \varepsilon)$. Recall that this means

$$\begin{aligned} \mathcal{G}(T^\varepsilon) &= \{(x, x^*) \in X \times X^* : \varphi_T(x, x^*) \le \langle x, x^* \rangle + \varepsilon\} \\ &= \{(x, x^*) \in X \times X^* : \langle x - y, x^* - y^* \rangle \ge -\varepsilon, \quad \forall (y, y^*) \in \mathcal{G}(T)\}. \end{aligned}$$

Another useful enlargement is the $\varepsilon-$subdifferential corresponding to the duality mapping, that we will denote by

$$\begin{aligned} J_\varepsilon : \quad & X \rightrightarrows X^* \\ & x \mapsto \{x^* \in X^* : \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2 \le \langle x, x^* \rangle + \varepsilon\}. \end{aligned}$$

**Lemma 5.1.1** *Let $X$ be a Banach space, $\alpha > 0$ and $|\cdot| : X \to \mathbb{R}$ be the norm on $X$ defined by $|\cdot| = \alpha\|\cdot\|$. Then, for all $\varepsilon \geq 0$,*

$$\left(J_X^{|\cdot|}\right)_\varepsilon = \alpha^2 \left(J_X^{\|\cdot\|}\right)_{\varepsilon/\alpha^2}.$$

*Proof.* Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be defined by $g(x) = 1/2|x|^2$ for all $x \in X$, so that $\left(J_X^{|\cdot|}\right)_\varepsilon = \partial_\varepsilon g$. For all $\varepsilon \geq 0$ and $(x, x^*) \in X \times X^*$, the inclusion $x^* \in \left(J_X^{|\cdot|}\right)_\varepsilon(x)$ is equivalent to $g(y) \geq g(x) + \langle y - x, x^* \rangle - \varepsilon$ for all $y \in X$, i.e.

$$\frac{1}{2}\alpha^2\|y\|^2 \geq \frac{1}{2}\alpha^2\|x\|^2 + \langle y - x, x^* \rangle - \varepsilon$$

and, dividing both sides by $\alpha^2$,

$$\frac{1}{2}\|y\|^2 \geq \frac{1}{2}\|x\|^2 + \left\langle y - x, \frac{1}{\alpha^2}x^* \right\rangle - \frac{\varepsilon}{\alpha^2},$$

which is in turn equivalent to $x^* \in \alpha^2 \left(J_X^{\|\cdot\|}\right)_{\varepsilon/\alpha^2}(x)$. Thus, $\left(J_X^{|\cdot|}\right)_\varepsilon = \alpha^2 \left(J_X^{\|\cdot\|}\right)_{\varepsilon/\alpha^2}$. □

For the sake of completeness, we provide here a proof of equality (1.7), that will be used in this chapter.

**Lemma 5.1.2** *Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then $\mathcal{G}(\widetilde{\partial f}) = \mathcal{G}(\partial f^*)^\top$.*

*Proof.* Let $(y^{**}, y^*) \in \mathcal{G}(\widetilde{\partial f})$. Since $\partial f$ is a maximal monotone operator of type (D) and $f \oplus f^* \in \mathcal{H}_{\partial f}$, then, by Theorem 2.2.14 (c), $f^{**} \oplus f^* = (f \oplus f^*)^{*\top} \in \mathcal{H}_{\widetilde{\partial f}}$, yielding

$$f^{**}(y^{**}) + f^*(y^*) = \langle y^{**}, y^* \rangle,$$

which, in turn, is satisfied if and only if $(y^*, y^{**}) \in \mathcal{G}(\partial f^*)$, since $f^* \oplus f^{**} \in \mathcal{H}_{\partial f^*}$. □

We now collect some important theorems of [96] that will be crucial to prove the results in the following sections. First, we adopt the terminology of [96], as specified in the definition below.

**Definition 5.1.3** *Let $X$ be a normed space and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions. We call $z^* \in X^*$ a* Fenchel functional *for $f$ and $g$ if*

$$f^*(z^*) + g^*(-z^*) \leq 0.$$

**Theorem 5.1.4 [96, Theorem 7.4]** *Let $X$ be a normed space and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions. Then:*

*(a) $f$ and $g$ have a Fenchel functional if, and only if, there exists $M \geq 0$ such that, for all $x, y \in X$,*

$$f(x) + g(y) + M\|x - y\| \geq 0;$$

*(b) if $z^* \in X^*$ is a Fenchel functional for $f$ and $g$, then*

$$\sup_{x,y \in X,\ x \neq y} \frac{-f(x) - g(y)}{\|x - y\|} \leq \|z^*\|;$$

*(c) if $f + g \geq 0$ on $X$ and*

$$\sup_{x,y \in X,\ x \neq y} \frac{-f(x) - g(y)}{\|x - y\|} < +\infty,$$

*then*

$$\min\{\|z^*\| : z^* \ is\ a\ Fenchel\ functional\ for\ f\ and\ g\} =$$

$$= \max\left\{ \sup_{x,y \in X,\ x \neq y} \frac{-f(x) - g(y)}{\|x - y\|}, 0 \right\}.$$

If $X = \{0\}$, the conditions on the supremum in $(b)$ and $(c)$ hold trivially with the usual convention $\sup \emptyset = -\infty$.

**Theorem 5.1.5 [96, Theorem 15.1]** *Let $X$ be a Banach space, $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions,*

$$\bigcup_{\lambda > 0} \lambda[\operatorname{dom} f - \operatorname{dom} g] \ be\ a\ closed\ subspace\ of\ X$$

*and*

$$f + g \geq 0 \ on\ X.$$

*Then there exists a Fenchel functional for $f$ and $g$.*

As a consequence of the previous theorem, one can obtain Attouch-Brézis theorem.

**Theorem 5.1.6 [96, Remark 15.2]** *Let $X$ be a Banach space, $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions and*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} f - \operatorname{dom} g] \text{ be a closed subspace of } X.$$

*Then, for all $x^* \in X^*$,*

$$(f+g)^*(x^*) = \min_{z^* \in X^*} \{f^*(x^* - z^*) + g^*(z^*)\}.$$

Taking $x^* = 0_{X^*}$ in the previous theorem, one obtains

$$\inf_{x \in X} (f+g)(x) = \max_{z^* \in X^*} \{-f^*(-z^*) - g^*(z^*)\}.$$

**Remark 5.1.7** In the following sections, we will frequently deal with translations of maximal monotone operators. In this connection, it can be useful to note that, given a maximal monotone operator $T : X \rightrightarrows X^*$, for all $(w, w^*) \in X \times X^*$,

$$\mathcal{G}(\tau_{-w^*} \circ T \circ \tau_w) = \mathcal{G}(T) - (w, w^*)$$

and an order preserving bijection between $\mathcal{H}_T$ and $\mathcal{H}_{\tau_{-w^*} \circ T \circ \tau_w}$ can be established as in [61], by means of the operator $\mathcal{T}_{(w,w^*)} : \mathcal{H}_T \to \mathcal{H}_{\tau_{-w^*} \circ T \circ \tau_w}$, such that $(\mathcal{T}_{(w,w^*)} h)(x, x^*) = h(x + w, x^* + w^*) - (\langle x, w^* \rangle + \langle w, x^* \rangle + \langle w, w^* \rangle)$ for any $h \in \mathcal{H}_T$, $(x, x^*) \in X \times X^*$. Therefore, it is equivalent to consider a convex representation of $\tau_{-w^*} \circ T \circ \tau_w$, or a convex representation of $T$ to which apply the bijection $\mathcal{T}_{(w,w^*)}$. Though we will usually work with the first representation, the equivalence of the two will sometimes be used. Note that the translation of a maximal monotone operator of type (D) is still maximal monotone of type (D) and that the unique maximal monotone extension of $\tau_{-w^*} \circ T \circ \tau_w$ to the bidual coincides with $\tau_{-w^*} \circ \widetilde{T} \circ \tau_w$.

We will also be interested in the effects of the composition of elements of $\mathcal{H}_T$ with reflections in the first or in the second component of points of $X \times X^*$. Such compositions will be essential for the duality proofs to work with elements of the Fitzpatrick family. The following lemma will then be useful.

**Lemma 5.1.8** *Let $X$ be a normed space and $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then, for all $(x^{**}, x^*) \in X^{**} \times X^*$:*

*(i)* $(f \circ \varrho_1)^*(x^*, x^{**}) = (f^* \circ \varrho_1)(x^*, x^{**}) = f^*(-x^*, x^{**})$;

*(ii)* $(f \circ \varrho_2)^*(x^*, x^{**}) = (f^* \circ \varrho_2)(x^*, x^{**}) = f^*(x^*, -x^{**})$.

*Proof.* We will only prove item $(i)$, since the proof of $(ii)$ is similar.

$$\begin{aligned}
(f \circ \varrho_1)^*(x^*, x^{**}) &= \sup_{(y,y^*) \in X\times X^*} \{\langle x^{**}, y^*\rangle + \langle y, x^*\rangle - (f \circ \varrho_1)(y, y^*)\} \\
&= \sup_{(y,y^*) \in X\times X^*} \{\langle x^{**}, y^*\rangle + \langle y, x^*\rangle - f(-y, y^*)\} \\
&= \sup_{(y,y^*) \in X\times X^*} \{\langle x^{**}, y^*\rangle + \langle -y, -x^*\rangle - f(-y, y^*)\} \\
&= \sup_{(y,y^*) \in X\times X^*} \{\langle x^{**}, y^*\rangle + \langle y, -x^*\rangle - f(y, y^*)\} \\
&= f^*(-x^*, x^{**}).
\end{aligned}$$

□

The following three simple algebraic lemmas will help us to manipulate Attouch-Brézis type conditions.

**Lemma 5.1.9** *Let $Y$ be a normed space, $A, B \subseteq Y$ and*

$$\bigcup_{\lambda>0} \lambda[A - B]$$

*be closed in $Y$.*
*Then*

$$\bigcup_{\lambda>0} \lambda[A - B] = \bigcup_{\lambda>0} \lambda[\text{cl } A - B].$$

*Proof.* The inclusion $\subseteq$ follows from $A \subseteq \text{cl } A$, implying $A - B \subseteq \text{cl } A - B$. In order to prove the opposite inclusion, let $p \in \bigcup_{\lambda>0} \lambda[\text{cl } A - B]$ (the inclusion is obvious if either $A$ or $B$ are empty). Then there exist $\mu > 0$, $x \in \text{cl } A$ and $y \in B$ such that $p = \mu(x - y)$. Moreover, since $x \in \text{cl } A$, there exists a sequence $(x_n)$ in $A$ such that $x_n \to x$. Therefore

$$p = \mu(x - y) = \mu\left(\lim_n x_n - y\right) = \lim_n [\mu(x_n - y)] \in \bigcup_{\lambda>0} \lambda[A - B]$$

because, for every $n \in \mathbb{N}$, $\mu(x_n - y) \in \bigcup_{\lambda>0} \lambda[A - B]$, which is a closed set by hypothesis. □

**Lemma 5.1.10** *Let $Y$ and $Z$ be normed spaces, $A \subseteq Y \times Z$ and $B \subseteq Y$. Let $S$ be any subspace of $Z$ containing $\mathrm{Pr}_Z A$. Then*

$$A - (B \times S) = (\mathrm{Pr}_Y A - B) \times S.$$

*Proof.* If either $A$ or $B$ are empty, the equality is trivial. If $A, B \neq \emptyset$, obviously, by definition of $S$, $A - (B \times S) \subseteq (\mathrm{Pr}_Y A - B) \times S$. Let $w \in (\mathrm{Pr}_Y A - B) \times S$. Then there exist $(a_1, a_2) \in A$, $b \in B$ and $c \in S$ such that $w = (a_1 - b, c)$. Therefore, letting $d := a_2 - c \in S$,

$$w = (a_1 - b, c) = (a_1 - b, a_2 - d) = (a_1, a_2) - (b, d) \in A - (B \times S).$$

$\square$

**Lemma 5.1.11** *Let $Y$ and $Z$ be normed spaces, $B \subseteq Y$, $C \subseteq Z$ and*

$$L := \bigcup_{\lambda > 0} \lambda (B \times C), \qquad M := \bigcup_{\lambda > 0} \lambda B, \qquad N := \bigcup_{\lambda > 0} \lambda C.$$

*Then:*

*(a) if $L$ is a closed subspace of $Y \times Z$, then $M$ and $N$ are closed subspaces of $Y$ and $Z$, respectively;*

*(b) if $C$ is a cone, then $L = M \times C$; in particular, if $M$ and $C$ are closed subspaces of $Y$ and $Z$, respectively, then $L$ is a closed subspace of $Y \times Z$. Analogously, if $B$ is a cone, then $L = B \times N$; if $B$ and $N$ are closed subspaces of $Y$ and $Z$, respectively, then $L$ is a closed subspace of $Y \times Z$.*

*Proof.*

(a) Since $L$ is a subspace of $Y \times Z$, then $(0_Y, 0_Z) \in L$, so that $0_Y \in B$ and $0_Z \in C$. Let $\lambda y_1, \mu y_2 \in M$, where $\lambda, \mu > 0$ and $y_1, y_2 \in B$. Then, for all $\alpha, \beta \in \mathbb{R}$, since $0_Z \in C$ and $L$ is a subspace, there exist $\tau > 0$ and $y \in B$ such that

$$\begin{aligned} \alpha(\lambda y_1) + \beta(\mu y_2) &= \alpha \mathrm{Pr}_Y(\lambda y_1, 0_Z) + \beta \mathrm{Pr}_Y(\mu y_2, 0_Z) = \\ &= \mathrm{Pr}_Y(\alpha\lambda(y_1, 0_Z) + \beta\mu(y_2, 0_Z)) = \mathrm{Pr}_Y(\tau y, 0_Z) = \tau y. \end{aligned}$$

Therefore $M$ is a subspace of $Y$. Moreover, if $(\lambda_n y_n)$ is a sequence in $M$ ($\lambda_n > 0$ and $y_n \in B$) converging to a given $x \in Y$, being $L$ closed, there exist $\varrho > 0$, $y \in B$ such that

$$(x, 0_Z) = \lim_n (\lambda_n y_n, 0_Z) = \varrho(y, 0_Z) \in L,$$

which yields $x = \varrho y \in M$. Thus $M$ is closed in $Y$.

In a similar way it can be proved that $N$ is a closed subspace of $Z$.

(b) If $B$ or $C$ are empty, the result is trivial. Thus, suppose that $B, C \neq \emptyset$ and, for instance, that $C$ is a cone (if $B$ is a cone, the proof is similar). Obviously $L \subseteq M \times N = M \times C$. On the other hand, given $x \in M$, $z \in C$, there exist $\lambda > 0$, $y \in B$ such that $x = \lambda y$, so that

$$(x, z) = (\lambda y, z) = \lambda \left( y, \frac{1}{\lambda} z \right) \in L,$$

since $C$ is a cone. In particular, if $M$ is a closed subspace of $Y$ and $C$ is a closed subspace of $Z$, then $L$ is a closed subspace of $Y \times Z$, since it is the cartesian product of two closed subspaces. □

## 5.2 The Sum of the Graphs

We begin with a lemma which takes on much of the burden needed to prove Theorem 5.2.2. Though, the purpose of stating a lemma on its own doesn't restrict to issues of ease, but it also enables us to underline the fact that the points $(x^{**}, x^*) \in X^{**} \times X^*$ satisfying the properties that are listed below are the same for any couple of representations $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, a fact which is not stressed in the statement of Theorem 5.2.2.

**Lemma 5.2.1** *Let $X$ be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D), $(x^{**}, x^*) \in X^{**} \times X^*$ and $(u, u^*), (v, v^*) \in X \times X^*$. Then the following facts are equivalent:*

*(a) $(u + x^{**}, u^* + x^*) \in \mathcal{G}(\widetilde{S})$ and $(v - x^{**}, x^* - v^*) \in \mathcal{G}(\widetilde{T})$;*

*(b) for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, the point $(x^*, x^{**})$ is a Fenchel functional for $h$ and $k \circ \varrho_1$;*

*(c) there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that $(x^*, x^{**})$ is a Fenchel functional for $h$ and $k \circ \varrho_1$.*

*If $X$ is reflexive, the previous statements are also equivalent to:*

*(d) for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$,*

$$(h + k \circ \varrho_1)(x^{**}, x^*) = 0; \tag{5.1}$$

*(e) there exist $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*}\circ T\circ\tau_v}$ such that (5.1) holds.*

*Proof.* $(a) \Longrightarrow (b)$ By hypothesis,

$$(x^{**}+u, x^*+u^*) \in \mathcal{G}(\widetilde{S}) \quad \text{and} \quad (v-x^{**}, x^*-v^*) \in \mathcal{G}(\widetilde{T}),$$

that is to say

$$(x^{**}, x^*) \in \mathcal{G}(\tau_{-u^*}\circ\widetilde{S}\circ\tau_u) \quad \text{and} \quad (-x^{**}, x^*) \in \mathcal{G}(\tau_{v^*}\circ\widetilde{T}\circ\tau_v). \tag{5.2}$$

Let $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*}\circ T\circ\tau_v}$. By Theorem 2.2.14,

$$h^{*\top} \in \mathcal{H}_{\tau_{-u^*}\circ\widetilde{S}\circ\tau_u} \quad \text{and} \quad k^{*\top} \in \mathcal{H}_{\tau_{v^*}\circ\widetilde{T}\circ\tau_v}.$$

Thus, by (5.2), we have

$$h^{*\top}(x^{**}, x^*) = \langle x^{**}, x^*\rangle \quad \text{and} \quad k^{*\top}(-x^{**}, x^*) = \langle -x^{**}, x^*\rangle,$$

which implies, by Lemma 5.1.8,

$$\begin{aligned} h^*(x^*, x^{**}) + (k\circ\varrho_1)^*(-x^*, -x^{**}) &= h^*(x^*, x^{**}) + k^*(x^*, -x^{**}) \\ &= h^{*\top}(x^{**}, x^*) + k^{*\top}(-x^{**}, x^*) \\ &= \langle x^{**}, x^*\rangle + \langle -x^{**}, x^*\rangle = 0, \end{aligned}$$

i.e. $(x^*, x^{**})$ is a Fenchel functional for $h$ and $k\circ\varrho_1$.

$(b) \Longrightarrow (c)$ Obvious.

$(c) \Longrightarrow (a)$ Suppose we are given $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*}\circ T\circ\tau_v}$ such that $(x^*, x^{**})$ is a Fenchel functional for $h$ and $k\circ\varrho_1$. Therefore

$$h^*(x^*, x^{**}) + (k\circ\varrho_1)^*(-x^*, -x^{**}) \le 0.$$

On the other hand, by Lemma 5.1.8 and Theorem 2.2.14, we obtain the opposite inequality as well, i.e.

$$\begin{aligned} h^*(x^*, x^{**}) + (k\circ\varrho_1)^*(-x^*, -x^{**}) &= h^{*\top}(x^{**}, x^*) + k^{*\top}(-x^{**}, x^*) \\ &\ge \langle x^{**}, x^*\rangle + \langle -x^{**}, x^*\rangle = 0. \end{aligned}$$

Thus, since

$$h^{*\top}(x^{**}, x^*) \ge \langle x^{**}, x^*\rangle \quad \text{and} \quad k^{*\top}(-x^{**}, x^*) \ge \langle -x^{**}, x^*\rangle,$$

then

$$h^{*\top}(x^{**}, x^*) = \langle x^{**}, x^* \rangle \quad \text{and} \quad k^{*\top}(-x^{**}, x^*) = \langle -x^{**}, x^* \rangle.$$

Hence, by the maximality of the operators $\tau_{-u^*} \circ \widetilde{S} \circ \tau_u$ and $\tau_{v^*} \circ \widetilde{T} \circ \tau_v$,

$$(x^{**}, x^*) \in \mathcal{G}(\tau_{-u^*} \circ \widetilde{S} \circ \tau_u) = \mathcal{G}(\widetilde{S}) - (u, u^*)$$

and

$$(-x^{**}, x^*) \in \mathcal{G}(\tau_{v^*} \circ \widetilde{T} \circ \tau_v) = \mathcal{G}(\widetilde{T}) - (v, -v^*),$$

yielding $(a)$.

Suppose now that $X$ is reflexive.

$(b) \Longrightarrow (d)$ Let $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$. We also have $h^{*\top} \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k^{*\top} \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$. Since $(b)$ holds, the point $(x^*, x^{**})$ is a Fenchel functional for $h^{*\top}$ and $k^{*\top} \circ \varrho_1$, so that

$$\begin{aligned} 0 = \langle x^{**}, x^* \rangle + \langle -x^{**}, x^* \rangle &\leq (h + k \circ \varrho_1)(x^{**}, x^*) \\ &= (h^{*\top})^*(x^*, x^{**}) + (k^{*\top} \circ \varrho_1)^*(-x^*, -x^{**}) \leq 0, \end{aligned}$$

implying (5.1).

$(d) \Longrightarrow (e)$ Obvious.

$(e) \Longrightarrow (c)$ By $(e)$, there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that

$$0 = (h + k \circ \varrho_1)(x^*, x^{**}) = (h^{*\top})^*(x^*, x^{**}) + (k^{*\top} \circ \varrho_1)^*(-x^*, -x^{**}).$$

Thus $(x^*, x^{**})$ is a Fenchel functional for $h^{*\top}$ and $k^{*\top} \circ \varrho_1$, where $h^{*\top} \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k^{*\top} \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$. □

As already announced, Lemma 5.2.1 makes the proof of Theorem 5.2.2 immediate. This theorem, along with its version for the range (Theorem 5.3.2), can be regarded as the basis of this chapter, since it provides a characterization of each point of the set $\mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$ in terms of Fenchel functionals of arbitrary convex representations of $\widetilde{S}$ and $\widetilde{T}$ or, equivalently, of their translations. Upon this duality characterization eventually hinge all the results that follow.

Note that condition (5.3) in Theorem 5.2.2 below and the analogous conditions in the results that follow are simply the necessary and sufficient condition for the existence of Fenchel functionals given by Simons (see Theorem 5.1.4 above). Although they are not new results, we include them in our statements for the sake of completeness, in order to give a thorough understanding of the correspondences involved.

**Theorem 5.2.2** *Let $X$ be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D) and $(u,u^*), (v,v^*), (w,w^*) \in X \times X^*$ such that $u+v=w$ and $u^*+v^*=w^*$. The following statements are equivalent:*

*(a)* $(w,w^*) \in \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$;

*(b) there exists* $(x^{**},x^*) \in X^{**} \times X^*$ *such that, for all* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$*, the point* $(x^*,x^{**})$ *is a Fenchel functional for* $h$ *and* $k \circ \varrho_1$*;*

*(c) there exist* $(x^{**},x^*) \in X^{**} \times X^*$*,* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ *such that the point* $(x^*,x^{**})$ *is a Fenchel functional for* $h$ *and* $k \circ \varrho_1$*;*

*(d)*

$$\inf_{\substack{(y,y^*) \,\in\, \mathrm{dom}\, \varphi_{\tau_{-u^*} \circ S \circ \tau_u} \\ (z,z^*) \,\in\, \varrho_1(\mathrm{dom}\, \varphi_{\tau_{v^*} \circ T \circ \tau_v}) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_{\tau_{-u^*} \circ S \circ \tau_u}(y,y^*) + (\varphi_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(z,z^*)}{\|(y,y^*) - (z,z^*)\|} > -\infty; \tag{5.3}$$

*(e) relation (5.3) holds with* $\varphi_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $\varphi_{\tau_{v^*} \circ T \circ \tau_v}$ *replaced by* $\sigma_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $\sigma_{\tau_{v^*} \circ T \circ \tau_v}$*, respectively.*

*Moreover, if $X$ is reflexive, the previous items are also equivalent to:*

*(f) there exists* $(x,x^*) \in X \times X^*$ *such that, for all* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$*,*

$$(h + k \circ \varrho_1)(x,x^*) = 0; \tag{5.4}$$

*(g) there exist* $(x,x^*) \in X \times X^*$*,* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ *such that (5.4) holds.*

*A sufficient condition for $(a) - (e)$ to hold is the existence of $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda > 0} \lambda[\mathrm{dom}\, h - \varrho_1(\mathrm{dom}\, k) - (w,w^*)] \quad \textit{is a closed subspace of } X \times X^*. \tag{5.5}$$

*Proof.* The equivalence $(a) \Longleftrightarrow (b) \Longleftrightarrow (c)$ ($\Longleftrightarrow (f) \Longleftrightarrow (g)$, if $X$ is reflexive) is an immediate consequence of Lemma 5.2.1.

$(b) \Longrightarrow (d)$ It follows from assertion $(b)$ of Theorem 5.1.4, since

$$\begin{aligned}
& \inf_{\substack{(y,y^*) \in \operatorname{dom} \varphi_{\tau_{-u^*} \circ S \circ \tau_u} \\ (z,z^*) \in \varrho_1(\operatorname{dom} \varphi_{\tau_{v^*} \circ T \circ \tau_v}) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_{\tau_{-u^*} \circ S \circ \tau_u}(y,y^*) + (\varphi_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(z,z^*)}{\|(y,y^*) - (z,z^*)\|} \\
= & -\sup_{\substack{(y,y^*),(z,z^*) \in X \times X^* \\ (y,y^*) \neq (z,z^*)}} \frac{-\varphi_{\tau_{-u^*} \circ S \circ \tau_u}(y,y^*) - (\varphi_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(z,z^*)}{\|(y,y^*) - (z,z^*)\|} \\
\geq & -\|(x^*, x^{**})\| > -\infty.
\end{aligned} \tag{5.6}$$

$(d) \Longrightarrow (e)$ Obvious.

$(e) \Longrightarrow (c)$ It follows from assertion $(a)$ of Theorem 5.1.4. Indeed, by setting

$$M := \max \left\{ \sup_{\substack{(y,y^*) \in \operatorname{dom} \sigma_{\tau_{-u^*} \circ S \circ \tau_u} \\ (z,z^*) \in \varrho_1(\operatorname{dom} \sigma_{\tau_{v^*} \circ T \circ \tau_v}) \\ (y,y^*) \neq (z,z^*)}} \frac{-\sigma_{\tau_{-u^*} \circ S \circ \tau_u}(y,y^*) - (\sigma_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(z,z^*)}{\|(y,y^*) - (z,z^*)\|}, 0 \right\}$$

and recalling (5.6), we have $0 \leq M < +\infty$ and

$$\sigma_{\tau_{-u^*} \circ S \circ \tau_u}(y,y^*) + (\sigma_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(z,z^*) + M\|(y,y^*) - (z,z^*)\| \geq 0$$

for any $(y,y^*),(z,z^*) \in X \times X^*$ with $(y,y^*) \neq (z,z^*)$. On the other hand, when $(y,y^*) = (z,z^*)$,

$$\begin{aligned}
& \sigma_{\tau_{-u^*} \circ S \circ \tau_u}(y,y^*) + (\sigma_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(y,y^*) + M\|(y,y^*) - (y,y^*)\| \\
= & \sigma_{\tau_{-u^*} \circ S \circ \tau_u}(y,y^*) + (\sigma_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(y,y^*) \\
\geq & \langle y, y^* \rangle + \langle -y, y^* \rangle = 0
\end{aligned}$$

for all $(y,y^*) \in X \times X^*$.

Finally, given $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ satisfying (5.5), we have $\mathcal{T}_{(u,u^*)}h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$, $\mathcal{T}_{(v,-v^*)}k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ and

$$\bigcup_{\lambda > 0} \lambda[\operatorname{dom} h - \varrho_1(\operatorname{dom} k) - (w,w^*)] = \bigcup_{\lambda > 0} \lambda[\operatorname{dom} \mathcal{T}_{(u,u^*)}h - \operatorname{dom}((\mathcal{T}_{(v,-v^*)}k) \circ \varrho_1)].$$

Moreover, $(\mathcal{T}_{(u,u^*)}h)(x,x^*) + ((\mathcal{T}_{(v,-v^*)}k) \circ \varrho_1)(x,x^*) \geq \langle x, x^* \rangle + \langle -x, x^* \rangle = 0$ for all $(x,x^*) \in X \times X^*$. Therefore, by Theorem 5.1.5, $(c)$ is satisfied. Consequently, $(a) - (e)$ hold. $\square$

**Remark 5.2.3** *(a)* Note that assertion $(d)$ can be restated by expressing the set over which the infimum in condition (5.3) is taken by means of the graphs of $S$ and $T$ instead of the domains of $\varphi_{\tau_{-u^*}\circ S\circ\tau_u}$ and $\varphi_{\tau_{v^*}\circ T\circ\tau_v}$, i.e.

$$\inf_{\substack{(y,y^*)\,\in\,\mathrm{cl\ conv}\ \mathcal{G}(S)\,-\,(u,u^*)\\ (z,z^*)\,\in\,\varrho_1(\mathrm{cl\ conv}\ \mathcal{G}(T))\,+\,(v,v^*)\\ (y,y^*)\neq(z,z^*)}} \frac{\varphi_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\varphi_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|} > -\infty. \tag{5.7}$$

Denote by $(d')$ this new statement. Obviously $(d)$ implies $(d')$. Vice versa, if $(d')$ holds, since for any maximal monotone operator $A : X \rightrightarrows X^*$ we have $\mathrm{conv}\ \mathcal{G}(A) \subseteq \mathrm{dom}\ \sigma_A \subseteq \mathrm{cl\ conv}\ \mathcal{G}(A)$, then

$$\begin{aligned}
&\inf_{\substack{(y,y^*)\,\in\,\mathrm{dom}\ \sigma_{\tau_{-u^*}\circ S\circ\tau_u}\\ (z,z^*)\,\in\,\varrho_1(\mathrm{dom}\ \sigma_{\tau_{v^*}\circ T\circ\tau_v})\\ (y,y^*)\neq(z,z^*)}} \frac{\sigma_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\sigma_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|}\\
=\ &\inf_{\substack{(y,y^*)\,\in\,\mathrm{cl\ conv}\ \mathcal{G}(S)\,-\,(u,u^*)\\ (z,z^*)\,\in\,\varrho_1(\mathrm{cl\ conv}\ \mathcal{G}(T))\,+\,(v,v^*)\\ (y,y^*)\neq(z,z^*)}} \frac{\sigma_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\sigma_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|}\\
\geq\ &\inf_{\substack{(y,y^*)\,\in\,\mathrm{cl\ conv}\ \mathcal{G}(S)\,-\,(u,u^*)\\ (z,z^*)\,\in\,\varrho_1(\mathrm{cl\ conv}\ \mathcal{G}(T))\,+\,(v,v^*)\\ (y,y^*)\neq(z,z^*)}} \frac{\varphi_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*)+(\varphi_{\tau_{v^*}\circ T\circ\tau_v}\circ\varrho_1)(z,z^*)}{\|(y,y^*)-(z,z^*)\|} > -\infty,
\end{aligned}$$

so that $(e)$ is satisfied and, consequently, $(d)$ holds as well.

*(b)* The sufficient condition (5.5) in the case $h = \sigma_S$ and $k = \sigma_T$ can be stated analogously in terms of the graphs of $S$ and $T$ as

$$\bigcup_{\lambda>0} \lambda[\mathrm{conv}\ \mathcal{G}(S) - \varrho_1(\mathrm{conv}\ \mathcal{G}(T)) - (w,w^*)] \quad \text{is a closed subspace of } X\times X^*. \tag{5.8}$$

Indeed, since for any $A \subseteq X\times X^*$, $\mathrm{conv}\ (\varrho_1 A) = \varrho_1(\mathrm{conv}\ A)$, $\mathrm{cl\ conv}\ (\varrho_1 A) = \varrho_1(\mathrm{cl\ conv}\ A)$ and $\mathrm{conv}\ [A-(w,w^*)] = \mathrm{conv}\ A - (w,w^*)$, we have

$$\begin{aligned}
&\bigcup_{\lambda>0} \lambda[\mathrm{conv}\ \mathcal{G}(S) - \varrho_1(\mathrm{conv}\ \mathcal{G}(T)) - (w,w^*)]\\
\subseteq\ &\bigcup_{\lambda>0} \lambda[\mathrm{dom}\ \sigma_S - \mathrm{dom}\ (\sigma_T\circ\varrho_1) - (w,w^*)]\\
\subseteq\ &\bigcup_{\lambda>0} \lambda[\mathrm{cl\ conv}\ \mathcal{G}(S) - \varrho_1(\mathrm{cl\ conv}\ \mathcal{G}(T)) - (w,w^*)].
\end{aligned}$$

Therefore, by Lemma 5.1.9, the sufficient condition (5.5) is satisfied with $h = \sigma_S$ and $k = \sigma_T$.

*(c)* Conditions (5.7) and (5.8) simplify whenever $\mathcal{G}(S)$ or $\mathcal{G}(T)$ are convex. By [55, Lemma 1.2] and [10, Theorem 4.2], this is the case if and only if $S$ or $T$ are translates of monotone linear relations.

*(d)* Since we will need it to prove Theorem 5.3.2, we observe that statement $(a)$ of Theorem 5.2.2 could be formulated in a less concise way by saying that there exists $(x^{**}, x^*) \in X^{**} \times X^*$ such that $(u + x^{**}, u^* + x^*) \in \mathcal{G}(\widetilde{S})$ and $(v - x^{**}, x^* - v^*) \in \mathcal{G}(\widetilde{T})$.

The following corollary extends to the nonreflexive setting, for maximal monotone operators of type (D), the surjectivity property in its version related to the sum of the graphs introduced in [93] (note that, strictly speaking, this version could not be called a surjectivity property, for it deals with the graphs, not with the ranges; anyway, as we will see in the next section, it is in some sense equivalent to surjectivity). On the basis of this corollary we will provide two possible reformulations of [59, Theorem 2.1] (the main result of that paper) in the nonreflexive setting for maximal monotone operators of type (D) (see Remark 5.2.5 and Corollary 5.2.6 below).

**Corollary 5.2.4** *Let $X$ be a Banach space and $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D). Then the following statements are equivalent:*

*(a)* $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$;

*(b) for all $(u, u^*), (v, v^*) \in X \times X^*$, there exists $(x^{**}, x^*) \in X^{**} \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, $(x^*, x^{**})$ is a Fenchel functional for $h$ and $k \circ \varrho_1$;*

*(c) for all $(u, u^*), (v, v^*) \in X \times X^*$, there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that $h$ and $k \circ \varrho_1$ have a Fenchel functional;*

*(d) for all $(u, u^*), (v, v^*) \in X \times X^*$,*

$$\inf_{\substack{(y,y^*) \in \text{cl conv } \mathcal{G}(S) - (u,u^*) \\ (z,z^*) \in \varrho_1(\text{cl conv } \mathcal{G}(T)) + (v,v^*) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_{\tau_{-u^*} \circ S \circ \tau_u}(y, y^*) + (\varphi_{\tau_{v^*} \circ T \circ \tau_v} \circ \varrho_1)(z, z^*)}{\|(y, y^*) - (z, z^*)\|} > -\infty; \qquad (5.9)$$

*(e) for all $(u, u^*), (v, v^*) \in X \times X^*$, relation (5.9) holds with $\varphi_{\tau_{-u^*} \circ S \circ \tau_u}$ and $\varphi_{\tau_{v^*} \circ T \circ \tau_v}$ replaced by $\sigma_{\tau_{-u^*} \circ S \circ \tau_u}$ and $\sigma_{\tau_{v^*} \circ T \circ \tau_v}$, respectively.*

*If $X$ is reflexive, they are also equivalent to:*

*(f) for all $(u, u^*), (v, v^*) \in X \times X^*$, there exists $(x, x^*) \in X \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$, $(h + k \circ \varrho_1)(x, x^*) = 0$;*

*(g) for all $(u, u^*), (v, v^*) \in X \times X^*$, there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{v^*} \circ T \circ \tau_v}$ such that $0 \in \mathrm{Im}(h + k \circ \varrho_1)$.*

*Thus, if for all* $(w,w^*) \in X \times X^*$ *there exist* $h \in \mathcal{H}_S$ *and* $k \in \mathcal{H}_T$ *such that*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_1(\operatorname{dom} k) - (w,w^*)] \quad \textit{is a closed subspace of } X \times X^*,$$

*then* $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$. *In particular, this is true whenever*

$$\operatorname{dom} \varphi_S - \varrho_1(\operatorname{dom} \varphi_T) = X \times X^*.$$

*Proof.* It follows immediately from Theorem 5.2.2. □

**Remark 5.2.5** *(i)* As we anticipated, the previous corollary provides a generalization of [59, Theorem 2.1], which reads as follows:
*Let* $X$ *be a Banach space and* $S : X \rightrightarrows X^*$ *be a monotone operator.*

*(a) If* $S$ *is maximal monotone of type (D), then, for any maximal monotone operator* $T : X \rightrightarrows X^*$ *of type (D) such that* $\operatorname{dom} \varphi_S - \varrho_1(\operatorname{dom} \varphi_T) = X \times X^*$, *one has* $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$.

*(b) If there exist a multifunction* $T : X \rightrightarrows X^*$, *such that* $\mathcal{G}(S) + \mathcal{G}(-T) = X \times X^*$, *and a point* $(p,p^*) \in X \times X^*$, *such that* $\langle p-y, p^*-y^* \rangle > 0$ *for any* $(y,y^*) \in \mathcal{G}(T)\setminus\{(p,p^*)\}$, *then* $S$ *is maximal monotone.*

Assertion (a) is a consequence of Corollary 5.2.4 that extends implication $(a) \Longrightarrow (b)$ of [59, Theorem 2.1] to a nonreflexive setting, for operators of type (D), and substitutes a constraint on the sum of the domains of the Fitzpatrick functions for the original condition requiring the second of these domains to be the whole of $X \times X^*$. Statement (b) is instead a refined version of implication $(c) \Longrightarrow (a)$ of the same theorem, taking into account that this implication already worked in a nonreflexive setting and that some hypotheses (namely, $T$ being a maximal monotone operator having finite-valued Fitzpatrick function) can be dropped.

*(ii)* In the particular case when $T$ is a subdifferential, we obtain an analogous generalization of [59, Corollary 2.5]. As in the previous case, the complete characterization given in the original result is recovered if $X$ is a reflexive Banach space.
*Let* $X$ *be a Banach space,* $S : X \rightrightarrows X^*$ *be a monotone operator and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous convex function.*

*(a) If $S$ is maximal monotone of type (D) and* $\mathrm{dom}\,\varphi_S + \mathrm{dom}\,f \times (-\mathrm{dom}\,f^*) = X \times X^*$, *then* $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{\partial f})$.

*(b) If $f$ is Gâteaux differentiable at some $p \in X$ and $\mathcal{G}(S) + \mathcal{G}(-\partial f) = X \times X^*$, then $S$ is maximal monotone.*

Recalling that subdifferentials of proper lower semicontinuous convex functions are maximal monotone operators of type (D), statement $(a)$ is a consequence of the previous result, setting $T = \partial f$ and taking into account that $f \oplus f^* \in \mathcal{H}_{\partial f}$ and $\mathrm{dom}\,(f \oplus f^*) = \mathrm{dom}\,f \times \mathrm{dom}\,f^*$, so that

$$\begin{aligned} \mathrm{dom}\,\varphi_S - \varrho_1(\mathrm{dom}\,\varphi_T) &\supseteq \mathrm{dom}\,\varphi_S - \varrho_1(\mathrm{dom}\,f \times \mathrm{dom}\,f^*) \\ &= \mathrm{dom}\,\varphi_S + \mathrm{dom}\,f \times (-\mathrm{dom}\,f^*) = X \times X^*. \end{aligned}$$

Statement $(b)$ is easily derived from [59, Corollary 2.5].

A closer similarity to the structure of [59, Theorem 2.1] can be obtained with a bit more involved version of statement $(c)$ of that theorem. We prove this fact in the following corollary, where we denote by $\mathrm{cl}_{\,(w,n)}(A)$ the closure of a set $A \subseteq X \times X^*$ in the $\sigma(X, X^*) \otimes$ norm topology of $X \times X^*$.

**Corollary 5.2.6** *Let $X$ be a Banach space and $S : X \rightrightarrows X^*$ be a monotone operator of type (D), whose graph is closed in the $\sigma(X, X^*) \otimes$ norm topology of $X \times X^*$. Then the following facts are equivalent:*

*(a) $S$ is maximal;*

*(b) for every maximal monotone operator $T : X \rightrightarrows X^*$ of type (D) such that* $\mathrm{dom}\,\varphi_S - \varrho_1(\mathrm{dom}\,\varphi_T) = X \times X^*$, *it holds $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$, so that, in particular,* $\mathrm{cl}_{\,(w,n)}(\mathcal{G}(S) + \mathcal{G}(-T)) = X \times X^*$;

*(c) there exist a monotone operator $T : X \rightrightarrows X^*$ of type (D) such that $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$ and a point $(p, p^*) \in \mathcal{G}(T)$ such that, for every net $(x_\alpha, x^*_\alpha)$ in $\mathcal{G}(T)$, if $\lim_\alpha \langle p - x_\alpha, p^* - x^*_\alpha \rangle = 0$, then $(x_\alpha)$ converges to $p$ in the $\sigma(X, X^*)$ topology of $X$ and $(x^*_\alpha)$ converges to $p^*$ in the norm topology of $X^*$.*

*Proof.* $(a) \Longrightarrow (b)$ It is a consequence of Corollary 5.2.4. To prove the density result, let $(w, w^*) \in X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$. Thus, there exist $(x^{**}, x^*) \in \mathcal{G}(\widetilde{S})$ and $(y^{**}, y^*) \in \mathcal{G}(\widetilde{T})$ such that $(x^{**}+y^{**}, x^*-y^*) = (w, w^*)$. Since $S$ and $T$ are both of type (D), there exist two nets $(x_\alpha, x^*_\alpha)_{\alpha\in A}$ in $\mathcal{G}(S)$ and $(y_\beta, y^*_\beta)_{\beta\in B}$ in $\mathcal{G}(T)$ converging to $(x^{**}, x^*)$ and to $(y^{**}, y^*)$, respectively, in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**}\times X^*$. The net $(z_\gamma, z^*_\gamma)_{\gamma\in\Gamma}$, with $\Gamma = A\times B$ and $z_\gamma := x_\alpha + y_\beta$, $z^*_\gamma := x^*_\alpha - y^*_\beta$ for every $\gamma = (\alpha,\beta)$, converges then to $(w, w^*)$ in the same topology of $X^{**}\times X^*$ and therefore in the $\sigma(X, X^*) \otimes$ norm topology of $X\times X^*$.

$(b) \implies (c)$ The duality mapping $J : X \rightrightarrows X^*$ is maximal monotone of type (D) and $\operatorname{dom}\varphi_J = X\times X^*$. Therefore, by hypothesis, $X\times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})$. Set then $(p, p^*) = (0_X, 0_{X^*})$ and consider a net $(x_\alpha, x^*_\alpha)$ in $\mathcal{G}(J)$, such that $\lim_\alpha \langle x_\alpha, x^*_\alpha\rangle = 0$. By definition of $J$, this implies $\lim_\alpha \left(\frac{1}{2}\|x_\alpha\|^2 + \frac{1}{2}\|x^*_\alpha\|^2\right) = 0$. Thus $(x_\alpha, x^*_\alpha)$ converges to $(0_X, 0_{X^*})$ in the norm topology of $X\times X^*$ and, consequently, in the $\sigma(X, X^*) \otimes$ norm topology.

$(c) \implies (a)$ Let $(x, x^*) \in X\times X^*$ be monotonically related to every point in $\mathcal{G}(S)$. Since $(x+p, x^*-p^*) \in X\times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$, then there exist two nets $(x_\alpha, x^*_\alpha) \in \mathcal{G}(S)$ and $(y_\beta, -y^*_\beta) \in \mathcal{G}(-T)$ converging in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**}\times X^*$ and the sum of which converges to $(x+p, x^*-p^*)$ in the same topology. Hence $(x - x_\alpha, x^* - x^*_\alpha)$ and $(y_\beta - p, -y^*_\beta + p^*)$ have the same limit. Since $(x, x^*)$ is monotonically related to $\mathcal{G}(S)$, then $\langle x - x_\alpha, x^* - x^*_\alpha\rangle \geq 0$ and, taking the limit, we obtain

$$\lim_\beta \langle y_\beta - p, -y^*_\beta + p^*\rangle = \lim_\alpha \langle x - x_\alpha, x^* - x^*_\alpha\rangle \geq 0,$$

which, taking into account the monotonicity of $T$, implies

$$\lim_\beta \langle y_\beta - p, y^*_\beta - p^*\rangle = 0.$$

Therefore, by the hypothesis on $(p, p^*)$, we obtain that $(y_\beta, y^*_\beta)$ converges to $(p, p^*)$ in the $\sigma(X, X^*) \otimes$ norm topology of $X\times X^*$. As a consequence, $(x_\alpha, x^*_\alpha)$ converges to $(x, x^*)$ in the same topology. Thus, by the hypothesis that $S$ has a closed graph in this topology, we conclude that $(x, x^*)$ belongs to $\mathcal{G}(S)$. Therefore, being $(x, x^*)$ an arbitrary point of $X\times X^*$ monotonically related to $\mathcal{G}(S)$, $S$ is maximal monotone. $\square$

Note that, in the previous proof, the hypothesis on the closure of the graph is needed only to prove the last implication.

A natural question to address at this point is whether the density property mentioned in statement $(b)$ of the previous corollary can be strengthened, introducing the closure in the norm topology of the product space $X\times X^*$. The answer is in the positive, but to obtain it we have to use, along with Fenchel duality, the strict Brønsted-Rockafellar property of maximal monotone operators of type (D).

**Theorem 5.2.7** *Let $X$ be a Banach space and $S, T : X \rightrightarrows X^*$ be monotone operators of type (D). If, for all $(w, w^*) \in X \times X^*$, there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda>0} \lambda[\mathrm{dom}\, h - \varrho_1(\mathrm{dom}\, k) - (w, w^*)] \quad \textit{is a closed subspace of } X \times X^*, \tag{5.10}$$

*then:*

*(a) for all $\varepsilon > 0$, $\mathcal{G}(S^\varepsilon) + \mathcal{G}(-T^\varepsilon) = X \times X^*$;*

*(b) if $S$ and $T$ are maximal monotone, $\mathrm{cl}\,(\mathcal{G}(S) + \mathcal{G}(-T)) = X \times X^*$.*

*Proof.*

(a) Let $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ satisfy condition (5.10), $(w, w^*) \in X \times X^*$ and $\varepsilon > 0$. Then, by Remark 5.1.7, $\mathcal{T}_{(w,w^*)}h \in \mathcal{H}_{\tau_{-w^*} \circ S \circ \tau_w}$ and, by hypothesis,

$$\bigcup_{\lambda>0} \lambda[\mathrm{dom}\, \mathcal{T}_{(w,w^*)}h - \varrho_1(\mathrm{dom}\, k)]$$

is a closed subspace of $X \times X^*$. Therefore, as a consequence of Theorem 5.1.6, there exists $(z^*, z^{**}) \in X^* \times X^{**}$ such that

$$\begin{aligned} 0 &\le \inf_{X\times X^*} \{\mathcal{T}_{(w,w^*)}h + (k \circ \varrho_1)\} \\ &= -((\mathcal{T}_{(w,w^*)}h)^*(z^*, z^{**}) - (k \circ \varrho_1)^*(-z^*, -z^{**}) \\ &= -\mathcal{T}_{(w^*,w)}h^*(z^*, z^{**}) - k^*(z^*, -z^{**}) \\ &\le -\langle z^{**}, z^*\rangle - \langle -z^{**}, z^*\rangle = 0, \end{aligned} \tag{5.11}$$

where the property $(\mathcal{T}_{(w,w^*)}h)^* = \mathcal{T}_{(w^*,w)}h^*$ and Lemma 5.1.8 have been used.
Thus the infimum in (5.11) is equal to zero and, for all $\varepsilon > 0$, there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$\mathcal{T}_{(w,w^*)}h(x_\varepsilon, x_\varepsilon^*) + k(-x_\varepsilon, x_\varepsilon^*) \le \varepsilon,$$

yielding

$$\varphi_{\tau_{-w^*} \circ S \circ \tau_w}(x_\varepsilon, x_\varepsilon^*) \le \mathcal{T}_{(w,w^*)}h(x_\varepsilon, x_\varepsilon^*) \le -k(-x_\varepsilon, x_\varepsilon^*) + \varepsilon \le \langle x_\varepsilon, x_\varepsilon^*\rangle + \varepsilon,$$

$$\varphi_T(-x_\varepsilon, x_\varepsilon^*) \le k(-x_\varepsilon, x_\varepsilon^*) \le -\mathcal{T}_{(w,w^*)}h(x_\varepsilon, x_\varepsilon^*) + \varepsilon \le \langle -x_\varepsilon, x_\varepsilon^*\rangle + \varepsilon,$$

that is, $(x_\varepsilon, x_\varepsilon^*) \in \mathcal{G}((\tau_{-w^*} \circ S \circ \tau_w)^\varepsilon) = \mathcal{G}(\tau_{-w^*} \circ S^\varepsilon \circ \tau_w)$ and $(-x_\varepsilon, x_\varepsilon^*) \in \mathcal{G}(T^\varepsilon)$. Hence,

$$(w, w^*) = (w + x_\varepsilon, w^* + x_\varepsilon^*) + (-x_\varepsilon, -x_\varepsilon^*) \in \mathcal{G}(S^\varepsilon) + \mathcal{G}(-T^\varepsilon).$$

(b) Let $(w, w^*) \in X \times X^*$. It follows from $(a)$ that, for all $n \in \mathbb{N}\backslash\{0\}$, there exists $(x_n, x_n^*) \in \mathcal{G}((\tau_{-w^*} \circ S \circ \tau_w)^{1/(9n^2)})$ such that $(-x_n, x_n^*) \in \mathcal{G}(T^{1/(9n^2)})$. By the strict Brønsted-Rockafellar property, there exist $(\overline{x}_n, \overline{x}_n^*) \in \mathcal{G}(\tau_{-w^*} \circ S \circ \tau_w)$ and $(-\overline{y}_n, \overline{y}_n^*) \in \mathcal{G}(T)$ such that

$$\|\overline{x}_n - x_n\| < \frac{1}{2\sqrt{2}n}, \quad \|\overline{x}_n^* - x_n^*\| < \frac{1}{2\sqrt{2}n}, \quad \|\overline{y}_n - x_n\| < \frac{1}{2\sqrt{2}n}, \quad \|\overline{y}_n^* - x_n^*\| < \frac{1}{2\sqrt{2}n},$$

implying

$$\|\overline{x}_n - \overline{y}_n\| < \frac{1}{\sqrt{2}n}, \quad \|\overline{x}_n^* - \overline{y}_n^*\| < \frac{1}{\sqrt{2}n}.$$

Therefore $(w + \overline{x}_n - \overline{y}_n, w^* + \overline{x}_n^* - \overline{y}_n^*)$ is a sequence in $\mathcal{G}(S) + \mathcal{G}(-T)$ such that

$$\|(w + \overline{x}_n - \overline{y}_n, w^* + \overline{x}_n^* - \overline{y}_n^*) - (w, w^*)\| = \left(\|\overline{x}_n - \overline{y}_n\|^2 + \|\overline{x}_n^* - \overline{y}_n^*\|^2\right)^{1/2} < \frac{1}{n},$$

i.e. $(w + \overline{x}_n - \overline{y}_n, w^* + \overline{x}_n^* - \overline{y}_n^*)$ converges in norm to $(w, w^*)$. □

Corollary 5.2.4 yields a sort of "extended Brønsted-Rockafellar property" (in the sense that it involves the extension of the operator to the bidual), that we can state as follows.

**Definition 5.2.8** *Let $X$ be a Banach space and $S : X \rightrightarrows X^*$ be an operator. We say that $S$ satisfies the* extended Brønsted-Rockafellar property *if, for all $\lambda, \varepsilon > 0$ and $(x, x^*) \in S^\varepsilon$, there exists $(\overline{x}^{**}, \overline{x}^*) \in \overline{S}$ such that $\|x - \overline{x}^{**}\| \leq \lambda$ and $\|x^* - \overline{x}^*\| \leq \varepsilon/\lambda$.*

**Proposition 5.2.9** *Let $X$ be a Banach space and $S : X \rightrightarrows X^*$ be a maximal monotone operator of type (D). Then $S$ satisfies the extended Brønsted-Rockafellar property.*

*Proof.* Let $\lambda, \varepsilon > 0$ and define the norm $|\cdot| = \sqrt{\varepsilon}/\lambda \|\cdot\|$. By Lemma 5.1.1,

$$J' := J_X^{|\cdot|} = \frac{\varepsilon}{\lambda^2} J_X^{\|\cdot\|}.$$

In particular, $J' : X \rightrightarrows X^*$ is then a maximal monotone operator of type (D), with finite-valued Fitzpatrick function. Therefore, by Corollary 5.2.4, $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J'})$. Hence, for any

$(x, x^*) \in S^\varepsilon$, there exists $(\overline{x}^{**}, \overline{x}^*) \in \widetilde{S} = \overline{S}$ such that $(x - \overline{x}^{**}, \overline{x}^* - x^*) \in \mathcal{G}(\widetilde{J}')$. Thus, by Lemma 5.1.2, item $(b)$ of Theorem 2.2.14 and (2.7),

$$\begin{aligned} |x - \overline{x}^{**}|^2 = |\overline{x}^* - x^*|^2 &= -\langle x - \overline{x}^{**}, x^* - \overline{x}^* \rangle \\ &\leq \varphi_{\widetilde{S}}(x, x^*) - \langle x, x^* \rangle \\ &= \varphi_S(x, x^*) - \langle x, x^* \rangle \leq \varepsilon, \end{aligned}$$

i.e.

$$|x - \overline{x}^{**}| \leq \sqrt{\varepsilon} \quad \text{and} \quad |\overline{x}^* - x^*| \leq \sqrt{\varepsilon}.$$

Since the norm that makes $X^*$ dual to $(X, |\cdot|)$ is $|\cdot| = \lambda/\sqrt{\varepsilon}\|\cdot\|$, this implies

$$\|x - \overline{x}^{**}\| \leq \lambda \quad \text{and} \quad \|\overline{x}^* - x^*\| \leq \frac{\varepsilon}{\lambda}.$$

□

Since in the reflexive case $\widetilde{S} = S$ and all maximal monotone operators are of type (D), then, in this setting, Proposition 5.2.9 yields Torralba's Theorem [102]. In order to recover the usual strict Brønsted-Rockafellar property, in the nonreflexive case one should invoke a different surjectivity result [58, Corollary 3.7], stating that, for a maximal monotone operator $S$ of type (D), one has $\mathcal{R}(S(\cdot + w) + \mu J_\eta) = X^*$ for all $w \in X$, $\mu, \eta > 0$. This result, which for monotone operators with closed graph is equivalent to the property of $S$ being maximal monotone of type (D), is obtained in [58] by means of the strict Brønsted-Rockafellar property. The opposite implication can be proved as well, as a consequence of the following statement (since it is easily verified that $\mathcal{R}(S(\cdot + w) + \mu J_\eta) = X^*$ for all $w \in X$ and $\mu, \eta > 0$ is equivalent to $\mathcal{G}(S) + \mathcal{G}(-\mu J_\eta) = X \times X^*$ for all $\mu, \eta > 0$).

**Proposition 5.2.10** *Let $X$ be a Banach space and $S : X \rightrightarrows X^*$ be a monotone operator. If $\mathcal{G}(S) + \mathcal{G}(-\mu J_\eta) = X \times X^*$ for all $\mu, \eta > 0$, then $S$ satisfies the strict Brønsted-Rockafellar property.*

*Proof.* Let $\lambda, \varepsilon, \widetilde{\varepsilon} > 0$, with $\varepsilon < \widetilde{\varepsilon}$, and $(w, w^*) \in S^\varepsilon$. Consider the norm

$$|\cdot| = \frac{\sqrt{\widetilde{\varepsilon}}}{\lambda}\|\cdot\|.$$

By Lemma 5.1.1, the hypothesis implies that, for any $\eta > 0$, there exists $(x_\eta, x_\eta^*) \in \mathcal{G}(S)$ such that $(w - x_\eta, x_\eta^* - w^*) \in \mathcal{G}(J_\eta^{|\cdot|})$, that is

$$\frac{1}{2}|w - x_\eta|^2 + \frac{1}{2}|x_\eta^* - w^*|^2 \leq -\langle w - x_\eta, w^* - x_\eta^* \rangle + \eta \leq \varepsilon + \eta. \tag{5.12}$$

Recall that, for any $(z, z^*) \in \mathcal{G}(J_\eta^{|\,\cdot\,|})$,

$$\begin{aligned}\frac{1}{2}(|z| - |z^*|)^2 &= \frac{1}{2}|z|^2 - |z|\,|z^*| + \frac{1}{2}|z^*|^2 \\ &\leq \frac{1}{2}|z|^2 + \frac{1}{2}|z^*|^2 - \langle z, z^* \rangle \leq \eta,\end{aligned}$$

implying $|z| \leq |z^*| + \sqrt{2\eta}$. Thus,

$$|z|^2 = \frac{1}{2}|z|^2 + \frac{1}{2}|z|^2 \leq \frac{1}{2}|z|^2 + \frac{1}{2}|z^*|^2 + |z^*|\,\sqrt{2\eta} + \eta.$$

In our case, setting $z = w - x_\eta$ and $z^* = x_\eta^* - w^*$ and taking into account (5.12), we obtain that $|x_\eta^* - w^*|^2 \leq 2(\varepsilon + \eta)$ and

$$\begin{aligned}|w - x_\eta|^2 &\leq \frac{1}{2}|w - x_\eta|^2 + \frac{1}{2}|x_\eta^* - w^*|^2 + |x_\eta^* - w^*|\,\sqrt{2\eta} + \eta \\ &\leq \varepsilon + \eta + \sqrt{2(\varepsilon + \eta)}\,\sqrt{2\eta} + \eta = \varepsilon + 2\eta + 2\sqrt{(\varepsilon + \eta)\eta}.\end{aligned}$$

Since $\lim_{\eta \to 0^+}(2\eta + 2\sqrt{(\varepsilon + \eta)\eta}) = 0$, there exists $\eta_1 > 0$ such that, for all $\eta \in \,]0, \eta_1[$,

$$\varepsilon + 2\eta + 2\sqrt{(\varepsilon + \eta)\eta} < \widetilde{\varepsilon},$$

i.e. $|w - x_\eta| < \sqrt{\widetilde{\varepsilon}}$.

Analogously, one can show that, for all $\eta \in \,]0, \eta_1[$, $|x_\eta^* - w^*| < \sqrt{\widetilde{\varepsilon}}$.

Finally, by definition of $|\,\cdot\,|$, we obtain

$$\|w - x_\eta\| < \lambda \quad \text{and} \quad \|x_\eta^* - w^*\| < \frac{\widetilde{\varepsilon}}{\lambda}.$$

$\square$

We end this section observing that Corollary 5.2.4 enables us to answer to a problem addressed at the end of [98], where the authors wonder if it is possible to prove with a technique similar to that employed in their paper the fact that, given a maximal monotone operator $S : X \rightrightarrows X^*$ of type (D), there exists $(x^{**}, x^*) \in \mathcal{G}(\widetilde{S}) \cap \mathcal{G}(-J_{X^*})^\top$, a fact already proved in [93] by means of a more traditional approach. The answer is in the positive.

**Corollary 5.2.11** *Let $X$ be a Banach space and $S : X \rightrightarrows X^*$ be a maximal monotone operator of type (D). Then $\mathcal{G}(\widetilde{S}) \cap \mathcal{G}(-J_{X^*})^\top \neq \emptyset$.*

*Proof.* Since $\operatorname{dom}\varphi_J = X \times X^*$ and the duality mapping $J$ is maximal monotone of type (D) (as the subdifferential of a proper lower semicontinuous convex function), then, by Corollary 5.2.4, $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})$. In particular, $(0_X, 0_X^*) \in \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})$, so that there exists $(x^{**}, x^*) \in \mathcal{G}(\widetilde{S})$ such that $(-x^{**}, -x^*) \in \mathcal{G}(-\widetilde{J})$, i.e., by Lemma 5.1.2, $(x^*, -x^{**}) \in \mathcal{G}(J_{X^*})$ and finally $(x^*, x^{**}) \in \mathcal{G}(-J_{X^*})$. □

## 5.3 The Range of the Sum

As in the previous section, we begin with a main theorem and then develop some of its consequences. In this case, the main result can be obtained directly from Theorem 5.2.2 by means of an appropriate transformation. Anyway, we state explicitly the analogous of Lemma 5.2.1, which is obtained exploiting the same transformation, since we will invoke it in the proof of Corollary 5.3.7.

**Lemma 5.3.1** *Let $X$ be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D), $(x^{**}, x^*) \in X^{**} \times X^*$ and $(u, u^*), (v, v^*) \in X \times X^*$. Then the following facts are equivalent:*

*(a)* $(u + x^{**}, u^* + x^*) \in \mathcal{G}(\widetilde{S})$ *and* $(v + x^{**}, v^* - x^*) \in \mathcal{G}(\widetilde{T})$*;*

*(b) for all* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$*, the point* $(x^*, x^{**})$ *is a Fenchel functional for* $h$ *and* $k \circ \varrho_2$*;*

*(c) there exist* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$ *such that* $(x^*, x^{**})$ *is a Fenchel functional for* $h$ *and* $k \circ \varrho_2$*.*

*If $X$ is reflexive, the previous statements are also equivalent to:*

*(d) for all* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$*,*

$$(h + k \circ \varrho_2)(x^{**}, x^*) = 0; \tag{5.13}$$

*(e) there exist* $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$ *such that (5.13) holds.*

*Proof.* Define $T' : X \rightrightarrows X^*$ by

$$x^* \in T'(x) \iff x^* \in -T(-x)$$

for all $(x,x^*) \in X \times X^*$. Then $T'$ is maximal monotone of type (D) and

$$(v + x^{**}, v^* - x^*) \in \mathcal{G}(\widetilde{T}) \quad \iff \quad (-v - x^{**}, -v^* + x^*) \in \mathcal{G}(\widetilde{T'}).$$

The result follows then from Lemma 5.2.1 with $v$ replaced by $-v$, considering the bijection

$$\begin{array}{rccc} \mathcal{R}: & \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v} & \to & \mathcal{H}_{\tau_{v^*}\circ T'\circ\tau_{-v}} \\ & k & \mapsto & k \circ \varrho_2 \circ \varrho_1. \end{array}$$

□

**Theorem 5.3.2** *Let $X$ be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D) and $w^* \in X^*$, $(u, u^*), (v, v^*) \in X \times X^*$ such that $u^* + v^* = w^*$. The following statements are equivalent:*

*(a)* $w^* \in \mathcal{R}(\widetilde{S}(\,\cdot + u) + \widetilde{T}(\,\cdot + v))$;

*(b) there exists* $(x^{**}, x^*) \in X^{**} \times X^*$ *such that, for all* $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$*, the point* $(x^*, x^{**})$ *is a Fenchel functional for* $h$ *and* $k \circ \varrho_2$*;*

*(c) there exist* $(x^{**}, x^*) \in X^{**} \times X^*$*,* $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$ *such that* $(x^*, x^{**})$ *is a Fenchel functional for* $h$ *and* $k \circ \varrho_2$*;*

*(d)*

$$\inf_{\substack{(y,y^*) \in \operatorname{dom} \varphi_{\tau_{-u^*}\circ S\circ\tau_u} \\ (z,z^*) \in \varrho_2(\operatorname{dom} \varphi_{\tau_{-v^*}\circ T\circ\tau_v}) \\ (y,y^*)\neq(z,z^*)}} \frac{\varphi_{\tau_{-u^*}\circ S\circ\tau_u}(y,y^*) + (\varphi_{\tau_{-v^*}\circ T\circ\tau_v} \circ \varrho_2)(z,z^*)}{\|(y,y^*) - (z,z^*)\|} > -\infty; \tag{5.14}$$

*(e) relation (5.14) holds with* $\varphi_{\tau_{-u^*}\circ S\circ\tau_u}$ *and* $\varphi_{\tau_{-v^*}\circ T\circ\tau_v}$ *replaced by* $\sigma_{\tau_{-u^*}\circ S\circ\tau_u}$ *and* $\sigma_{\tau_{-v^*}\circ T\circ\tau_v}$*, respectively.*

*If $X$ is reflexive, the previous items are also equivalent to:*

*(f) there exists* $(x, x^*) \in X \times X^*$ *such that, for all* $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$*,* $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$*,*

$$(h + k \circ \varrho_2)(x, x^*) = 0; \tag{5.15}$$

*(g) there exist* $(x, x^*) \in X \times X^*$*,* $h \in \mathcal{H}_{\tau_{-u^*}\circ S\circ\tau_u}$ *and* $k \in \mathcal{H}_{\tau_{-v^*}\circ T\circ\tau_v}$ *such that (5.15) holds.*

*A sufficient condition for (a) − (e) to hold is the existence of* $h \in \mathcal{H}_S$ *and* $k \in \mathcal{H}_T$ *such that*

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u - v, w^*)] \quad \textit{is a closed subspace of } X \times X^*. \tag{5.16}$$

*Proof.* The theorem follows either from Theorem 5.2.2 by means of the bijection $\mathcal{R}$ used in the proof of Lemma 5.3.1 (taking into account observation $(d)$ of Remark 5.2.3, replacing $v$ by $-v$ and setting $w = u - v$), or directly from the same lemma with a proof similar to that of Theorem 5.2.2. $\square$

**Corollary 5.3.3** *Let $X$ be a Banach space, $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type (D) and $u, v \in X$. Then the following statements are equivalent:*

*(a)* $\mathcal{R}(\widetilde{S}(\,\cdot + u) + \widetilde{T}(\cdot + v)) = X^*$;

*(b) for all $u^*, v^* \in X^*$, there exists $(x^{**}, x^*) \in X^{**} \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$, $(x^*, x^{**})$ is a Fenchel functional for $h$ and $k \circ \varrho_2$;*

*(c) for all $u^*, v^* \in X^*$, there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$ such that $h$ and $k \circ \varrho_2$ have a Fenchel functional;*

*(d) for all $u^*, v^* \in X^*$,*

$$\inf_{\substack{(y,y^*) \in \operatorname{cl}\operatorname{conv}\mathcal{G}(S) - (u,u^*) \\ (z,z^*) \in \varrho_2(\operatorname{cl}\operatorname{conv}\mathcal{G}(T)) + (-v,v^*) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_{\tau_{-u^*} \circ S \circ \tau_u}(y, y^*) + (\varphi_{\tau_{-v^*} \circ T \circ \tau_v} \circ \varrho_2)(z, z^*)}{\|(y, y^*) - (z, z^*)\|} > -\infty; \tag{5.17}$$

*(e) for all $u^*, v^* \in X^*$, relation (5.17) holds with $\varphi_{\tau_{-u^*} \circ S \circ \tau_u}$ and $\varphi_{\tau_{-v^*} \circ T \circ \tau_v}$ replaced by $\sigma_{\tau_{-u^*} \circ S \circ \tau_u}$ and $\sigma_{\tau_{-v^*} \circ T \circ \tau_v}$, respectively.*

*If $X$ is reflexive, they are also equivalent to:*

*(f) for all $u^*, v^* \in X^*$, there exists $(x, x^*) \in X \times X^*$ such that, for all $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$, $(h + k \circ \varrho_2)(x, x^*) = 0$;*

*(g) for all $u^*, v^* \in X^*$, there exist $h \in \mathcal{H}_{\tau_{-u^*} \circ S \circ \tau_u}$ and $k \in \mathcal{H}_{\tau_{-v^*} \circ T \circ \tau_v}$ such that*

$$0 \in \operatorname{Im}(h + k \circ \varrho_2).$$

*A sufficient condition for $\widetilde{S}(\,\cdot + u) + \widetilde{T}(\,\cdot + v)$ to be surjective is that, for all $w^* \in X^*$, there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda > 0} \lambda[\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u - v, w^*)] \quad \text{is a closed subspace of } X \times X^*. \tag{5.18}$$

*In particular, the previous condition is satisfied whenever there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u - v, 0) = A \times X^*, \quad \text{where } \bigcup_{\lambda > 0} \lambda A \text{ is a closed subspace of } X. \tag{5.19}$$

*Proof.* The equivalence of $(a)-(e)$ (and $(f)-(g)$, when $X$ is reflexive) is an immediate consequence of Theorem 5.3.2 (taking into account an observation similar to Remark 5.2.3, as $(d)$ and $(e)$ are concerned).

Condition (5.16) in the same theorem guarantees that the validity of (5.18) for any $w^* \in X^*$ is a sufficient condition for the surjectivity of $\widetilde{S}(\;\cdot + u) + \widetilde{T}(\;\cdot + v)$.
Finally, condition (5.19) yields

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - \varrho_2(\operatorname{dom} k) - (u+v, w^*)] = \bigcup_{\lambda>0} \lambda[A \times X^* - (0, w^*)] = \bigcup_{\lambda>0} \lambda(A \times X^*),$$

so that, by Lemma 5.1.11, (5.18) is satisfied for any $w^* \in X^*$. □

Condition (5.19) slightly refines the analogous condition given in [96, Theorem 30.2].

As a consequence of the previous corollary, one can provide generalizations of Corollary 2.7, Theorem 2.8 and Proposition 2.9 of [59]. We state for instance a possible improvement of Proposition 2.9 of that paper.

**Corollary 5.3.4** *Let $X$ be a Banach space, $S : X \rightrightarrows X^*$ be a monotone operator and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function.*

*(a) If $S$ is maximal monotone of type (D) and* $\operatorname{dom} \varphi_S + (-\operatorname{dom} f) \times \operatorname{dom} f^* = X \times X^*$, *then $\mathcal{R}(\widetilde{S}(\;\cdot + w) + \partial f) = X^*$ for all $w \in X$.*

*(b) If $f$ is Gâteaux differentiable at some $p \in X$ and $\mathcal{R}(S(\;\cdot + w) + \partial f) = X^*$ for all $w \in X$, then $S$ is maximal monotone.*

*(c) If $\mathcal{R}(S(\;\cdot + w) + \partial f) = X^*$ for all $w \in X$, $f$ admits a unique global minimizer $p$ and is Gâteaux differentiable at $p$, then $S$ is maximal monotone.*

*Proof.*

(a) It is a consequence of Corollary 5.3.3, setting $T = \partial f$, $u = w$, $v = 0$ and, in condition (5.19), $h = \varphi_S$ and $k = f \oplus f^*$.

(b) Note that the function $g : X \to \mathbb{R} \cup \{+\infty\}$ defined by $g(x) = f(-x)$ for any $x \in X$ is still proper lower semicontinuous and convex, and is Gâteaux differentiable at $-p$. Moreover, for any $x \in X$, $\partial g(x) = -\partial f(-x)$. Since the condition $\mathcal{R}(S(\;\cdot + w) + \partial f) = X^*$ for all $w \in X$ implies $\mathcal{G}(S) + \mathcal{G}(-\partial g) = X \times X^*$, the result is then an immediate consequence of $(ii)$ of Remark 5.2.5.

(c) Let $(x, x^*) \in X \times X^*$ be monotonically related to every point in $\mathcal{G}(S)$. Since $x^* \in X^* = \mathcal{R}(S(\,\cdot + x - p) + \partial f)$, we have $x^* \in S(a + x - p) + \partial f(a)$ for some $a \in X$. We can therefore write $x^* = a^* + s^*$ for some $a^* \in S(a + x - p)$ and $s^* \in \partial f(a)$. Using that $a^* - x^* = -s^*$, we obtain

$$\begin{aligned} 0 \leq \langle (a + x - p) - x, a^* - x^* \rangle &= \langle a - p, a^* - x^* \rangle = -\langle a - p, s^* \rangle \\ &= -\langle a, s^* \rangle + \langle p, s^* \rangle = -f(a) - f^*(s^*) + \langle p, s^* \rangle \\ &\leq -f(p) - f^*(s^*) + \langle p, s^* \rangle \leq 0, \end{aligned}$$

hence $f(a) = f(p)$ and $-f(p) - f^*(s^*) + \langle p, s^* \rangle = 0$, that is, $s^* \in \partial f(p) = \{0\}$. We deduce that $s^* = 0$ and, by the assumption on $f$, that $a = p$. We then conclude $x^* = a^* \in S(x)$, thus proving the maximality of $S$. $\square$

**Remark 5.3.5** As a consequence of Lemma 5.1.10, when $f$ is finite-valued the condition $\mathrm{dom}\,\varphi_S + (-\mathrm{dom}\,f) \times \mathrm{dom}\,f^* = X \times X^*$ is equivalent to $\mathrm{Pr}_{X^*}(\mathrm{dom}\,\varphi_S) + \mathrm{dom}\,f^* = X^*$. Analogously, if $f$ is cofinite, it is equivalent to $\mathrm{Pr}_X(\mathrm{dom}\,\varphi_S) - \mathrm{dom}\,f = X$.

As in the previous section, one could be interested in obtaining a surjectivity property for the range of the sum of the operators themselves, instead of their extensions. As in Theorem 5.2.7, one can prove a density result, rather than one of surjectivity. Item (*b*) of the following theorem is a generalization of implication 4. $\Longrightarrow$ 1. in [58, Theorem 3.6]; our proof is along the same lines.

**Theorem 5.3.6** *Let $X$ be a Banach space, $S, T : X \rightrightarrows X^*$ be monotone operators of type (D) and $u, v \in X$. If, for all $w^* \in X^*$, there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda > 0} \lambda[\mathrm{dom}\,h - \varrho_2(\mathrm{dom}\,k) - (u - v, w^*)] \quad \textit{is a closed subspace of } X \times X^*, \tag{5.20}$$

*then:*

*(a) for all $\varepsilon > 0$, $\mathcal{R}(S(\,\cdot + u)^\varepsilon + T(\,\cdot + v)^\varepsilon) = X^*$;*

*(b) if $S$ and $T$ are maximal monotone and*

$$\bigcup_{\lambda > 0} \lambda[\mathrm{Pr}_X \mathrm{dom}\,\mathcal{T}_{(u,w^*)}h - \mathrm{Pr}_X \mathrm{dom}\,\mathcal{T}_{(v,0_{X^*})}k] \quad \textit{is a closed subspace of } X, \tag{5.21}$$

*then* $\mathrm{cl}\,(\mathcal{R}(S(\,\cdot + u) + T(\,\cdot + v))) = X^*$.

*Proof.* Let $w^* \in X^*$ and $\varepsilon > 0$ be given and $h \in \mathcal{H}_S$, $k \in \mathcal{H}_T$ satisfy condition (5.20).

(a) With a reasoning analogous to the proof of item $(a)$ of Theorem 5.2.7, one can prove that

$$\inf_{(y,y^*)\in X\times X^*}\{\mathcal{T}_{(u,w^*)}h(y,y^*) + (\mathcal{T}_{(v,0_{X^*})}k \circ \varrho_2)(y,y^*)\} = 0. \tag{5.22}$$

Then there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$\mathcal{T}_{(u,w^*)}h(x_\varepsilon, x_\varepsilon^*) + \mathcal{T}_{(v,0_{X^*})}k(x_\varepsilon, -x_\varepsilon^*) \leq \varepsilon,$$

which implies

$$\varphi_{\tau_{-w^*}\circ S\circ\tau_u}(x_\varepsilon, x_\varepsilon^*) \leq \mathcal{T}_{(u,w^*)}h(x_\varepsilon, x_\varepsilon^*) \leq -\mathcal{T}_{(v,0_{X^*})}k(x_\varepsilon, -x_\varepsilon^*) + \varepsilon \leq \langle x_\varepsilon, x_\varepsilon^*\rangle + \varepsilon$$

and

$$\varphi_{T\circ\tau_v}(x_\varepsilon, -x_\varepsilon^*) \leq \mathcal{T}_{(v,0_{X^*})}k(x_\varepsilon, -x_\varepsilon^*) \leq -\mathcal{T}_{(u,w^*)}h(x_\varepsilon, x_\varepsilon^*) + \varepsilon \leq \langle x_\varepsilon, -x_\varepsilon^*\rangle + \varepsilon,$$

i.e. $(x_\varepsilon, x_\varepsilon^*) \in \mathcal{G}((\tau_{-w^*} \circ S \circ \tau_u)^\varepsilon)$ and $(x_\varepsilon, -x_\varepsilon^*) \in \mathcal{G}((T \circ \tau_v)^\varepsilon)$, yielding

$$w^* = (w^* + x_\varepsilon^*) - x_\varepsilon^* \in S(\ \cdot\ + u)^\varepsilon(x_\varepsilon) + T(\ \cdot\ + v)^\varepsilon(x_\varepsilon).$$

(b) Defining

$$H(x,x^*) := \inf_{y^*\in X^*}\{\mathcal{T}_{(u,w^*)}h(x,y^*) + \mathcal{T}_{(v,0_{X^*})}k(x, x^* - y^*)\},$$

by (5.21) and Lemma 2.2.12, one has that $\tau_{-w^*} \circ S \circ \tau_u + T \circ \tau_v$ is maximal monotone of type (D) and $\operatorname{cl} H \in \mathcal{H}_{\tau_{-w^*}\circ S\circ\tau_u + T\circ\tau_v}$.
By (5.22), there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that

$$\begin{aligned}
\varphi_{\tau_{-w^*}\circ S\circ\tau_u + T\circ\tau_v}(x_\varepsilon, 0_{X^*}) &\leq \operatorname{cl} H(x_\varepsilon, 0_{X^*}) \\
&\leq \mathcal{T}_{(u,w^*)}h(x_\varepsilon, x_\varepsilon^*) + (\mathcal{T}_{(v,0_{X^*})}k \circ \varrho_2)(x_\varepsilon, x_\varepsilon^*) \\
&< \varepsilon^2 = \langle x_\varepsilon, 0_{X^*}\rangle + \varepsilon^2.
\end{aligned}$$

Then, since $\tau_{-w^*} \circ S \circ \tau_u + T \circ \tau_v$, being of type (D), is of type (BR), for all $\eta > \varepsilon$ there exists $(\overline{x}, \overline{x}^*) \in \mathcal{G}(\tau_{-w^*} \circ S \circ \tau_u + T \circ \tau_v)$ such that $\|\overline{x} - x_\varepsilon\| < \eta$ and $\|\overline{x}^* - 0_{X^*}\| < \eta$, i.e.

$$w^* + \overline{x}^* \in \mathcal{R}(S(\ \cdot + u) + T(\ \cdot + v))$$

and $\|(w^* + \overline{x}^*) - w^*\| = \|\overline{x}^*\| < \eta$. The result follows from the arbitrariness of $\varepsilon$ and $\eta$. $\square$

We now present an application of the surjectivity results considered up to this point. A consequence of Theorem 5.3.2 is the possibility to provide a characterization of the solutions of variational inequalities written for maximal monotone operators in reflexive Banach spaces. This is accomplished by Corollary 5.3.7 below, which generalizes [59, Corollary 2.3].

Note that necessary and sufficient conditions for the existence of solutions to the variational inequality on $T$ and $K$ (where $K$ is a nonempty closed convex subset of $X$) in principle do not require $T + N_K$ to be maximal monotone, unlike standard sufficient conditions (see [85, 110]).

Recall from Section 5.1 that, given a cone $K$ in $X$, we denote by $B_K$ the barrier cone of $K$.

**Corollary 5.3.7** *Let $X$ be a reflexive Banach space, $S : X \rightrightarrows X^*$ be a maximal monotone operator, $K$ be a nonempty closed convex subset of $X$ and $(x, x^*) \in X \times X^*$. Consider the following statements:*

*(a)* *$(x, x^*)$ is a solution to the variational inequality on $S$ and $K$, i.e. $x \in K \cap \mathcal{D}(S)$, $x^* \in S(x)$ and*

$$\forall y \in K : \quad \langle y - x, x^* \rangle \geq 0; \tag{5.23}$$

*(b)* *for all $h \in \mathcal{H}_S$, the point $(x^*, x)$ is a Fenchel functional for $h$ and $(\delta_K \oplus \delta_K^*) \circ \varrho_2$;*

*(c)* *there exists $h \in \mathcal{H}_S$ such that $(x^*, x)$ is a Fenchel functional for $h$ and $(\delta_K \oplus \delta_K^*) \circ \varrho_2$;*

*(d)* *for all $h \in \mathcal{H}_S$,*

$$\left(h + (\delta_K \oplus \delta_K^*) \circ \varrho_2\right)(x, x^*) = 0; \tag{5.24}$$

*(e)* *there exists $h \in \mathcal{H}_S$ satisfying relation (5.24);*

*(f)*

$$\inf_{\substack{(y,y^*) \in \operatorname{cl\,conv} \mathcal{G}(S) \\ (z,z^*) \in K \times (-B_K) \\ (y,y^*) \neq (z,z^*)}} \frac{\varphi_S(y, y^*) + \delta_K^*(-z^*)}{\|(y, y^*) - (z, z^*)\|} > -\infty; \tag{5.25}$$

*(g)* *relation (5.25) holds with $\varphi_S$ replaced by $\sigma_S$;*

*(h)* *there exists $h \in \mathcal{H}_S$ such that*

$$\bigcup_{\lambda > 0} \lambda[\operatorname{dom} h - (K \times (-B_K))] \quad \textit{is a closed subspace of } X \times X^*. \tag{5.26}$$

*Statements $(a) - (e)$ are equivalent; $(f) - (g)$ are necessary and sufficient conditions for the existence of solutions to the variational inequality (5.23), while $(h)$ provides a sufficient condition.*

*Proof.* Note that $(a)$ is equivalent to the inclusions $(x, x^*) \in \mathcal{G}(S)$ and $(x, -x^*) \in \mathcal{G}(N_K)$. Since $N_K = \partial\delta_K$ and $\delta_K \oplus \delta_K^* = \varphi_{N_K}$, the equivalence of $(a) - (e)$ is a consequence of Lemma 5.3.1, with $x^{**} = x$, $u = v = 0_X$ and $u^* = v^* = 0_{X^*}$.

Moreover, since the existence of a solution to the variational inequality on $S$ and $K$ is equivalent to the inclusion $0 \in \mathcal{R}(S + N_K)$, then, by Theorem 5.3.2 with $u = v = 0_X$ and $w^* = 0_{X^*}$ (taking into account an observation similar to Remark 5.2.3), the relations between $(a) - (e)$ and $(f) - (h)$ follow as well. In particular, condition (5.26) is an instance of (5.16) with $k = \delta_K \oplus \delta_K^*$, because

$$\varrho_2(\mathrm{dom}\,(\delta_K \oplus \delta_K^*)) = \varrho_2(\mathrm{dom}\,\delta_K \times \mathrm{dom}\,\delta_K^*) = K \times (-B_K).$$

□

**Remark 5.3.8** *(a)* Similarly to Remark 5.2.3, a particular case of condition (5.26), namely when $h = \sigma_S$, can be stated as

$$\bigcup_{\lambda>0} \lambda[\mathrm{conv}\mathcal{G}(S) - (K \times (-B_K))] \quad \text{is a closed subspace of } X \times X^*. \tag{5.27}$$

*(b)* If $B_K$ is a closed subspace of $X^*$ containing $\mathrm{Pr}_{X^*}\mathrm{dom}\,h$, then, by Lemma 5.1.10 and Lemma 5.1.11, condition (5.26) simplifies to

$$\bigcup_{\lambda>0} \lambda(\mathrm{Pr}_X\mathrm{dom}\,h - K) \quad \text{is a closed subspace of } X.$$

Analogously, if $B_K$ is a closed subspace of $X^*$ containing $\mathrm{Pr}_{X^*}\mathrm{conv}\mathcal{G}(S)$, then condition (5.27) reduces to

$$\bigcup_{\lambda>0} \lambda(\mathrm{Pr}_X\mathrm{conv}\,\mathcal{G}(S) - K) \quad \text{is a closed subspace of } X.$$

In particular, this is the case whenever $K$ is bounded, since then $B_K = X^*$.

The previous corollary can be restated, with obvious changes, for the more general variational inequality (considered in [110, Proposition 32.36])

$$\forall y \in X : \qquad \langle y - x, x^* \rangle + \vartheta(y) \geq \vartheta(x),$$

where $\vartheta : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function, replacing $\delta_K \oplus \delta_K^*$ by $\vartheta \oplus \vartheta^*$ in the proof.

On the other hand, it is worthwhile explicitly stating how Corollary 5.3.7 specializes in the particular case of convex constrained optimization problems.

**Corollary 5.3.9** *Let $X$ be a reflexive Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and $K$ be a nonempty closed convex subset of $X$. If*

$$\bigcup_{\lambda>0} \lambda[(\text{dom } f - K) \times (\text{dom } f^* + B_K)] \quad \textit{is a closed subspace of} \quad X \times X^*, \tag{5.28}$$

*then $f$ has a global minimum on $K$.*

*Proof.* Condition (5.28) is derived from condition (5.26) of Corollary 5.3.7, taking into account that

$$\text{dom } (f \oplus f^*) - K \times (-B_K) = \text{dom } f \times \text{dom } f^* - K \times (-B_K) = (\text{dom } f - K) \times (\text{dom } f^* + B_K).$$

□

**Remark 5.3.10** By Lemma 5.1.11, condition (5.28) is implied by any of the two following conditions:

(a) $\bigcup_{\lambda>0} \lambda(\text{dom } f - K)$ is a closed subspace of $X$ and $\text{dom } f^* + B_K$ is a closed subspace of $X^*$;

(b) $\text{dom } f - K$ is a closed subspace of $X$ and $\bigcup_{\lambda>0} \lambda(\text{dom } f^* + B_K)$ is a closed subspace of $X^*$.

For instance, if $K$ is bounded, then $B_K = X^*$ and condition (5.28) can be replaced by

$$\bigcup_{\lambda>0} \lambda(\text{dom } f - K) \quad \text{is a closed subspace of} \quad X.$$

## 5.4 The Closure of the Range of a Maximal Monotone Operator

The duality methods used in the previous sections can provide geometrical insight in the theoretical framework of maximal monotone operators. An example of this fact is given by the following proposition, which provides a convex analytical proof of the well-known relations [96, Theorem 43.1 and Lemma 31.1] between the range of a maximal monotone operator of type (D) and the projection of the domains of its convex representations on $X^*$, implying as an immediate consequence the convexity of the closure of the former.

**Proposition 5.4.1** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator of type (D). Then, for any $h \in \mathcal{H}_T$:*

*(a)* $\mathrm{cl}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\, h) = \mathrm{cl}\,\mathcal{R}(T)$;

*(b)* $\mathrm{int}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\, h) \subseteq \mathrm{int}\,\mathcal{R}(\widetilde{T})$.

*Proof.*

(a) Since $\mathcal{G}(T) \subseteq \mathrm{dom}\, h \subseteq \mathrm{dom}\,\varphi_T$, it suffices to prove that $\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T \subseteq \mathrm{cl}\,\mathcal{R}(T)$. Actually, since $T$ is a maximal monotone operator of type (D), $\mathrm{cl}\,\mathcal{R}(\widetilde{T}) = \mathrm{cl}\,\mathcal{R}(T)$ (see e.g. [75, proof of Theorem 3.8]) and we will prove that $\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T \subseteq \mathrm{cl}\,\mathcal{R}(\widetilde{T})$.
Suppose, by contradiction, that there exists $(x_0, x_0^*) \in \mathrm{dom}\,\varphi_T$ such that $x_0^* \notin \mathrm{cl}\,\mathcal{R}(\widetilde{T})$. Without loss of generality, we can suppose that $(x_0, x_0^*) = (0_X, 0_{X^*})$.
Take $\eta \in\ ]0, d_{X^*}(0, \mathrm{cl}\,\mathcal{R}(\widetilde{T}))[$ and set $K := \mathrm{cl}\, B_{X^*}(0_{X^*}, \eta)$ (the closed ball of $X^*$ centered at $0_{X^*}$ and with radius $\eta$) and $g := (\delta_K^*)_{|X}$. Then we have $g^* = \delta_K$ (by Lemma 4.1.4) and $(0_X, 0_{X^*}) \in X \times B_{X^*}(0_{X^*}, \eta) = \mathrm{int}\ \mathrm{dom}\,(g \oplus g^*)$. Hence

$$(0_X, 0_{X^*}) \in \mathrm{dom}\,\varphi_T\ \cap\ \mathrm{int}\ \mathrm{dom}\,((g \oplus g^*) \circ \varrho_1).$$

Clearly this condition implies that

$$\bigcup_{\lambda > 0} \lambda[\mathrm{dom}\,\varphi_T - \varrho_1 \mathrm{dom}\,(g \oplus g^*)] = X \times X^*.$$

Therefore, by Theorem 5.1.5, there exists a Fenchel functional $(y^*, y^{**}) \in X^* \times X^{**}$ for $\varphi_T$ and $(g \oplus g^*) \circ \varrho_1$. By Lemma 5.2.1, this implies $y^* \in \mathcal{R}(\widetilde{T}) \cap \mathcal{R}(\widetilde{\partial g})$, which is absurd, since, by Lemma 5.1.2, $\mathcal{R}(\widetilde{\partial g}) = \mathcal{D}(\partial g^*) = K$ and $\mathcal{R}(\widetilde{T}) \cap K = \emptyset$ by construction. Then $\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T \subseteq \mathrm{cl}\,\mathcal{R}(\widetilde{T})$.

(b) Similarly to item $(a)$, we only have to prove that $\mathrm{int}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T) \subseteq \mathrm{int}\,\mathcal{R}(\widetilde{T})$.
The result is obvious when $\mathrm{int}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T) = \emptyset$. Suppose then that there exists $(x_0, x_0^*) \in \mathrm{dom}\,\varphi_T$, such that $x_0^* \in \mathrm{int}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T)$. This means that there exists $\varrho > 0$ such that $B_{X^*}(x_0^*, \varrho) \subseteq \mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T$. Let $y^* \in B_{X^*}(x_0^*, \varrho)$ and define $A : X \rightrightarrows X^*$ by $\mathcal{G}(A) = X \times \{y^*\}$. It is easy to check that $A$ is maximal monotone of type (D) and that $\mathrm{dom}\,\varphi_A = \mathcal{G}(A)$.
Since $y^* \in \mathrm{int}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T)$, we have $0_{X^*} \in \mathrm{int}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T) - \{y^*\} = \mathrm{int}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T - \{y^*\})$, which implies that the set $\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T - \{y^*\}$ is absorbing in $X^*$, i.e.

$$\bigcup_{\lambda > 0} \lambda(\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T - \{y^*\}) = X^*.$$

Therefore, by Lemma 5.1.10 (with the roles of $Y$ and $Z$ interchanged) and Lemma 5.1.11 $(b)$, we obtain

$$\begin{aligned}\bigcup_{\lambda>0}\lambda[\mathrm{dom}\,\varphi_T-\varrho_1(\mathrm{dom}\,\varphi_A)] &= \bigcup_{\lambda>0}\lambda(\mathrm{dom}\,\varphi_T - X\times\{y^*\})\\ &= \bigcup_{\lambda>0}\lambda[X\times(\mathrm{Pr}_{X^*}\mathrm{dom}\,\varphi_T-\{y^*\})] = X\times X^*.\end{aligned}$$

Thus, by Theorem 5.2.2, $(0_X, 0_{X^*}) \in \mathcal{G}(\widetilde{T}) + \mathcal{G}(-\widetilde{A})$. Therefore, there exists $(x, x^*) \in X \times X^*$ such that $(x, x^*) \in \mathcal{G}(\widetilde{T})$ and $(-x, x^*) \in \mathcal{G}(\widetilde{A})$, so that $\emptyset \neq \mathcal{R}(\widetilde{T}) \cap \mathcal{R}(\widetilde{A}) = \mathcal{R}(\widetilde{T}) \cap \mathcal{R}(A) \subseteq \{y^*\}$. Thus $y^* \in \mathcal{R}(\widetilde{T})$. Since $y^*$ was arbitrarily chosen in $B_{X^*}(x_0^*, \varrho)$, we have $B_{X^*}(x_0^*, \varrho) \subseteq \mathcal{R}(\widetilde{T})$, i.e. $x_0^* \in \mathrm{int}\,\mathcal{R}(\widetilde{T})$. □

**Remark 5.4.2** When $X$ is a reflexive space, Proposition 5.4.1 $(b)$ yields the relation

$$\mathrm{int}\,(\mathrm{Pr}_{X^*}\mathrm{dom}\,h) = \mathrm{int}\,\mathcal{R}(T)$$

as a particular case, which is part of [96, Lemma 31.1].

# Chapter 6

# A Family of Representable Extensions to the Bidual

As we have seen in Section 1.3.3, a deep study of maximal monotone operators defined on nonreflexive Banach spaces can be traced back to the pioneering work of J.-P. Gossez [37, 38, 39, 40], in the seventies. In particular, he defined a well-behaved class of operators, called of type (D), which admit a unique maximal monotone extension to the bidual. More classes were defined in the nineties by S. Simons [93, 94], S. Fitzpatrick and R. R. Phelps [35], and A. Verona and M. E. Verona [103].

In the present chapter, which is based on the results of [78], we provide some contributions to a general study of monotone extensions of monotone operators to the bidual, employing some of the main results of [57]. Indeed, the literature concerning extensions to the bidual has mostly focused on maximal monotone extensions, especially in the cases when they are unique. Here, on the other hand, we consider some remarkable monotone extensions to the bidual that are neither necessarily maximal nor premaximal monotone. These extensions can be somehow studied by means of convex representations of the given operator.

The chapter is organized as follows. Section 6.1 gathers some important facts that we are using in the following. Section 6.2 contains the main results concerning monotone extensions of monotone operators to the bidual, focusing in particular on the problem of representability of these extensions, on their reciprocal relations and on how they are affected by different properties of the operators under consideration. Finally, section 6.3 considers the implications of asking for two of these properties (strict Brønsted-Rockafellar property and being of type (D)) to be satisfied by the extension of an operator given by its closure in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, rather than simply by the operator itself.

## 6.1 Preliminary Results

In this chapter we will work in the setting of a nonreflexive real Banach space $X$. Recall from Section 2.2.2 that, to any monotone operator $T : X \rightrightarrows X^*$ one can associate the two families

$$\begin{aligned}\mathcal{H}_T \quad := \quad & \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} : \quad h \quad \text{is lower semicontinuous and convex,} \\ & h(x, x^*) \geq \langle x, x^* \rangle \ \forall (x, x^*) \in X \times X^*, \quad h(x, x^*) = \langle x, x^* \rangle \ \forall (x, x^*) \in \mathcal{G}(T)\}.\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_T \quad := \quad & \{h : X \times X^* \to \mathbb{R} \cup \{+\infty\} : \quad h \quad \text{is lower} \\ & \text{semicontinuous and convex, and } \ \varphi_T \leq h \leq \sigma_T\}.\end{aligned}$$

Note that, while $\sigma_T \in \mathcal{H}_T$, unless $T$ is maximal monotone $\varphi_T$ will not majorize the duality product on $X \times X^*$ and consequently $\varphi_T \notin \mathcal{H}_T$. Moreover, in principle, none of the two functions will characterize the operator (while this is the case when $T$ is maximal monotone), since they can be equal to the duality product also at points which do not belong to $\mathcal{G}(T)$. Anyway, a monotone operator can be representable (without being maximal monotone), in the sense that there exists $h \in \mathcal{H}_T$ such that $h(x, x^*) = \langle x, x^* \rangle$ implies $(x, x^*) \in \mathcal{G}(T)$ (Definition 2.2.7).

**Remark 6.1.1** As a consequence of [61, Theorem 5], given a lower semicontinuous convex function $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ such that $f(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$, the set

$$\{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x, x^* \rangle\}$$

is the graph of a representable monotone operator (with $f$ as a convex representation of it).

In this chapter we will be interested in considering extensions of a monotone operator $T : X \rightrightarrows X^*$ to the bidual, i.e. operators $S : X^{**} \rightrightarrows X^*$ such that $\mathcal{G}(T) \subseteq \mathcal{G}(S)$, via the natural inclusion of $X$ in its bidual. In particular, recall (Section 1.3.3) that $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$ if and only if there exists a bounded net $(x_\alpha, x_\alpha^*)$ in $\mathcal{G}(T)$ that converges to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, while

$$\mathcal{G}(\widetilde{T}) = \{(x^{**}, x^*) \in X^{**} \times X^* : \quad \langle x^{**} - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in \mathcal{G}(T)\}.$$

The operator $T$ is of type (D) if and only if $\overline{T} = \widetilde{T}$, while it is of type (BR) if it satisfies the strict Brønsted-Rockafellar property (Definition 2.2.15). As we observed in Section 2.2.3, any maximal monotone operator of type (D) is of type (BR) as well.

In the following we will use some basic properties of the extensions $\overline{T}$ and $\widetilde{T}$.

**Proposition 6.1.2** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator.*

*(a) $\overline{T}$ and $\widetilde{T}$ are extensions of $T$.*

*(b) $\overline{T}$ is a monotone operator and $\mathcal{G}(\overline{T})$ is contained in the graph of any maximal monotone extension of $T$ to the bidual.*

*(c) $\mathcal{G}(\widetilde{T})$ contains the graph of any monotone extension of $T$ to the bidual. Therefore, in particular, $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widetilde{T})$.*

*Proof.*

(a) Given an arbitrary point $(x, x^*) \in \mathcal{G}(T)$, taking $(x_\alpha, x^*_\alpha) = (x, x^*)$ for any $\alpha$ in an ordered set $A$, we trivially obtain that $(x, x^*) \in \mathcal{G}(\overline{T})$.
On the other hand, the monotonicity of $T$ implies that $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(y, y^*) \in \mathcal{G}(T)$. Thus $(x, x^*) \in \mathcal{G}(\widetilde{T})$.

(b) Notice that, for any $(x^{**}, x^*) \in X^{**} \times X^*$ and any bounded net $(x_\alpha, x^*_\alpha)$ converging to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, one has

$$\lim_\alpha \langle x_\alpha, x^* \rangle = \langle x^{**}, x^* \rangle$$

and

$$0 \leq \lim_\alpha |\langle x_\alpha, x^*_\alpha - x^* \rangle| \leq \lim_\alpha \|x_\alpha\| \|x^*_\alpha - x^*\| = 0.$$

Hence,

$$\lim_\alpha \langle x_\alpha, x^*_\alpha \rangle = \lim_\alpha (\langle x_\alpha, x^* \rangle + \langle x_\alpha, x^*_\alpha - x^* \rangle) = \langle x^{**}, x^* \rangle. \tag{6.1}$$

To show that $\overline{T}$ is monotone, consider first an arbitrary $(u^{**}, u^*) \in \mathcal{G}(\overline{T})$. Given a bounded net $(u_\beta, u^*_\beta)$ in $\mathcal{G}(T)$ converging to $(u^{**}, u^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, for all $(v, v^*) \in \mathcal{G}(T)$, taking (6.1) into account, one has

$$\langle u^{**} - v, u^* - v^* \rangle = \lim_\beta \langle u_\beta - v, u^*_\beta - v^* \rangle \geq 0,$$

since any point $(u_\beta, u^*_\beta)$ belongs to $\mathcal{G}(T)$ and $T$ is a monotone operator.
Therefore, given any $(x^{**}, x^*), (y^{**}, y^*) \in \mathcal{G}(\overline{T})$ and a bounded net $(x_\alpha, x^*_\alpha)$ in $\mathcal{G}(T)$ converging to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, one also obtains

$$\langle x^{**} - y^{**}, x^* - y^* \rangle = \lim_\alpha \langle x_\alpha - y^{**}, x^*_\alpha - y^* \rangle \geq 0,$$

so that $\overline{T}$ is monotone.

Finally, let $S : X^{**} \rightrightarrows X^*$ be a maximal monotone extension of $T$ to the bidual. Then, for all $(y^{**}, y^*) \in \mathcal{G}(S)$ and $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$, given a net $(x_\alpha, x_\alpha^*)$ in the graph of $T$ converging to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, one has

$$\langle x^{**} - y^{**}, x^* - y^* \rangle = \lim_{\alpha} \langle x_\alpha - y^{**}, x_\alpha^* - y^* \rangle \geq 0,$$

since $S$ is monotone and, being $S$ an extension of $T$, $(x_\alpha, x_\alpha^*) \in \mathcal{G}(S)$ for all $\alpha$. From the previous inequality, because of the maximality of $S$, it follows $(x^{**}, x^*) \in \mathcal{G}(S)$.

(c) Let $S : X^{**} \rightrightarrows X^*$ be a monotone extension of $T$ to the bidual. Then, for all $(x^{**}, x^*) \in \mathcal{G}(S)$ and for all $(y, y^*) \in \mathcal{G}(T)$, it holds $\langle x^{**} - y, x^* - y^* \rangle \geq 0$, since $\mathcal{G}(T) \subseteq \mathcal{G}(S)$ and $S$ is monotone. Then, by definition of $\widetilde{T}$, $\mathcal{G}(S) \subseteq \mathcal{G}(\widetilde{T})$.

The fact that $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widetilde{T})$ follows from $(b)$.

□

**Remark 6.1.3** $\widetilde{T}$ is not necessarily monotone, even when $T$ is maximal monotone [40].

We will also employ the main results of [57].

**Theorem 6.1.4 [57, Lemma 4.1]** *Let $X$ be a Banach space and $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$f^{**}(x^{**}, x^*) = \liminf_{(y,y^*) \to (x^{**},x^*)} f(y, y^*), \qquad \forall (x^{**}, x^*) \in X^{**} \times X^*,$$

*where the* lim inf *is taken over all nets in $X \times X^*$ converging to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$.*

**Theorem 6.1.5 [57, Theorem 4.2]** *Let $X$ be a Banach space and $f : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, for any $(x^{**}, x^*) \in X^{**} \times X^*$ there exists a bounded net $(z_i, z_i^*)_{i \in I}$ in $X \times X^*$ which converges to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$ and*

$$f^{**}(x^{**}, x^*) = \lim_{i \in I} f(z_i, z_i^*).$$

The third result we will refer to is not stated explicitly, but is embedded in the proof of [57, Theorem 4.4].

**Theorem 6.1.6 [57, proof of Theorem 4.4]** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator of type (BR). For any $h \in \mathcal{H}_T$ and $(x^{**}, x^*) \in X^{**} \times X^*$, if $h^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle$, then there exists a bounded net $(x_\alpha, x^*_\alpha)_\alpha$ in $\mathcal{G}(T)$ converging to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$.*

## 6.2 Monotone Extensions to the Bidual

For any monotone operator $T : X \rightrightarrows X^*$, we introduce a family of extensions of $T$ to the bidual which is generated by the Fitzpatrick family of $T$.

**Definition 6.2.1** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $h \in \mathcal{H}_T$. Then we denote by $\widehat{T}_h : X^{**} \rightrightarrows X^*$ the operator with graph*

$$\mathcal{G}(\widehat{T}_h) := \{(x^{**}, x^*) \in X^{**} \times X^* : h^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle\}.$$

The following theorem states that $\widehat{T}_h$ is a representable monotone extension of $T$ to the bidual and considers the relations holding between $\overline{T}$, $\widehat{T}_h$ and $\widetilde{T}$, both in the sense of graph inclusion and with respect to the Fitzpatrick functions (as $\overline{T}$ and $\widehat{T}_{\sigma_T}$ are concerned).

**Theorem 6.2.2** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $h \in \mathcal{H}_T$. Then:*

*(a) for all $(x^{**}, x^*) \in X^{**} \times X^*$, one has $h^{**}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle$;*

*(b) $\widehat{T}_h : X^{**} \rightrightarrows X^*$ is a representable monotone operator, with $(h^{**})|_{X^{**} \times X^*}$ as a convex representation and such that $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widehat{T}_h) \subseteq \mathcal{G}(\widetilde{T})$;*

*(c) $\varphi_{\overline{T}^{-1}} = \varphi_{\widehat{T}^{-1}_{\sigma_T}} = \sigma^*_T$.*

*Proof.*

(a) Let $(x^{**}, x^*) \in X^{**} \times X^*$. Given a bounded net $(x_\alpha, x^*_\alpha)$ in $X \times X^*$ converging to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology and such that $h(x_\alpha, x^*_\alpha)$ converges to $h^{**}(x^{**}, x^*)$ (the existence of such a net is guaranteed by Theorem 6.1.5), one has

$$h^{**}(x^{**}, x^*) = \lim_\alpha h(x_\alpha, x^*_\alpha) \geq \lim_\alpha \langle x_\alpha, x^*_\alpha \rangle = \langle x^{**}, x^* \rangle,$$

where the inequality follows from $h \geq \langle \cdot, \cdot \rangle$ on $X \times X^*$ (being $h \in \mathcal{H}_T$), while the latter equality is given by (6.1).

(b) Since $(h^{**})_{|X^{**} \times X^*}$ is a lower semicontinuous convex function and, by $(a)$, $h^{**}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle$ for all $(x^{**}, x^*) \in X^{**} \times X^*$, then, according to Remark 6.1.1, $\widehat{T}_h$ is a representable monotone operator, with the function $(h^{**})_{|X^{**} \times X^*}$ as a convex representation.
Moreover, since $h^{**} = h$ on $X \times X^*$ and $h \in \mathcal{H}_T$, then, for any $(x, x^*) \in \mathcal{G}(T)$, $h^{**}(x, x^*) = \langle x, x^* \rangle$, i.e. $(x, x^*) \in \widehat{T}_h$. Thus $\widehat{T}_h$ is an extension of $T$ to the bidual.
The inclusion $\mathcal{G}(\widehat{T}_h) \subseteq \mathcal{G}(\widetilde{T})$ is obvious, given that $\widetilde{T}$ contains the graph of any monotone extension of $T$ to the bidual (Proposition 6.1.2 (c)).
On the other hand, for any $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$, given a bounded net $(x_\alpha, x^*_\alpha)$ in $\mathcal{G}(T)$ converging to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, we have

$$\langle x^{**}, x^* \rangle = \lim_\alpha \langle x_\alpha, x^*_\alpha \rangle = \lim_\alpha h(x_\alpha, x^*_\alpha) = \lim_\alpha h^{**}(x_\alpha, x^*_\alpha) \geq h^{**}(x^{**}, x^*) \geq \langle x^{**}, x^* \rangle,$$

where the first inequality is a consequence of Theorem 6.1.4, while the second one follows from item $(a)$. Thus $h^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle$ and $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widehat{T}_h)$.

(c) Since $\overline{T}$ is an extension of $T$ and given that, by $(b)$ (recalling that $\sigma_T \in \mathcal{H}_T$), $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widehat{T}_{\sigma_T})$, then, for all $(x^*, x^{**}) \in X^* \times X^{**}$,

$$\begin{aligned} \sigma_T^*(x^*, x^{**}) &= \sup_{(y, y^*) \in \mathcal{G}(T)} \{\langle x^{**}, y^* \rangle + \langle y, x^* \rangle - \langle y, y^* \rangle\} \\ &\leq \sup_{(y^{**}, y^*) \in \mathcal{G}(\overline{T})} \{\langle x^{**}, y^* \rangle + \langle y^{**}, x^* \rangle - \langle y^{**}, y^* \rangle\} \\ &= \sup_{(y^{**}, y^*) \in \mathcal{G}(\overline{T})} \{\langle x^{**}, y^* \rangle + \langle y^{**}, x^* \rangle - \sigma_T^{**}(y^{**}, y^*)\} \\ &\leq \sigma_T^{***}(x^*, x^{**}) = \sigma_T^*(x^*, x^{**}). \end{aligned}$$

Thus, as

$$\sup_{(y^{**}, y^*) \in \mathcal{G}(\overline{T})} \{\langle x^{**}, y^* \rangle + \langle y^{**}, x^* \rangle - \langle y^{**}, y^* \rangle\} = \varphi_{\overline{T}^{-1}}(x^*, x^{**}),$$

it holds $\varphi_{\overline{T}^{-1}} = \sigma_T^*$.
Substituting $\mathcal{G}(\widehat{T}_{\sigma_T})$ for $\mathcal{G}(\overline{T})$ in the previous inequalities, one proves that $\varphi_{\widehat{T}^{-1}_{\sigma_T}} = \sigma_T^*$ as well. □

With respect to item $(c)$ of the previous theorem, we can draw the following consequence. In principle, one could conceive a generalization of the class of maximal monotone operators of type (D) given by the family of maximal monotone operators $T : X \rightrightarrows X^*$ such that $\overline{T}$ is maximal monotone, though not necessarily equal to $\widetilde{T}$. Anyway, this actually turns out not to be a broader class, according to the following corollary.

**Corollary 6.2.3** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then $\overline{T}$ is maximal monotone if and only if $T$ is of type (D).*

*Proof.* If $T$ is of type (D), then $\overline{T}$ coincides with $\widetilde{T}$, by definition. Therefore, as a consequence of Proposition 6.1.2, $\overline{T}$ is maximal monotone.

Vice versa, if $\overline{T}$ is a maximal monotone operator, i.e. if $\overline{T}^{-1}$ is maximal monotone, then the Fitzpatrick function $\varphi_{\overline{T}^{-1}} = \sigma_T^*$ (Theorem 6.2.2 (c)) majorizes the duality product on $X^* \times X^{**}$, implying that $T$ is of type (D), according to Theorem 2.2.14. □

With respect to item $(b)$ of Theorem 6.2.2, a natural question is to determine when the inclusions considered either hold as equalities, or are strict. The following corollary provides a partial answer to this question.

**Corollary 6.2.4** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator of type (BR). Then, for all $h \in \mathcal{H}_T$, $\overline{T} = \widehat{T}_h$. Thus, $\overline{T}$ is a representable monotone operator and $\{(h^{**})_{|X^{**}\times X^*} : h \in \mathcal{H}_T\}$ is a collection of convex representations of $\overline{T}$.*

*Proof.* It is an immediate consequence of item $(b)$ of Theorem 6.2.2 and of Theorem 6.1.6. □

As a consequence of the previous corollary, we have the following characterization of the property of being of type (D) for maximal monotone operators.

**Proposition 6.2.5** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. Then the following statements are equivalent:*

*(a) $T$ is of type (D);*

*(b) for all $h \in \mathcal{H}_T$, it holds $\mathcal{G}(\widehat{T}_h) = \mathcal{G}(\widetilde{T})$;*

*(c) there exists $h \in \mathcal{H}_T$ such that $\mathcal{G}(\widehat{T}_h) = \mathcal{G}(\widetilde{T})$.*

*Proof.* $(a) \Longrightarrow (b)$ Obvious, as a consequence of Theorem 6.2.2 and of the definition of type (D).

$(b) \Longrightarrow (c)$ Obvious.

$(c) \Longrightarrow (a)$ Let $h \in \mathcal{H}_T$ be such that $\mathcal{G}(\widehat{T}_h) = \mathcal{G}(\widetilde{T})$. As a consequence of Theorem 6.2.2 $(b)$, $\widetilde{T}$ is monotone, hence maximal monotone, by Proposition 6.1.2 $(c)$. Thus $T$ has a unique maximal monotone extension to the bidual and, by Theorem 2.2.17, it is either of type (D) or non enlargeable. In both cases, $T$ is of type (BR). Indeed, maximal monotone operators of type (D) are of type (BR) according to Theorem 2.2.14, while non enlargeable operators are trivially of type (BR), since, for any $\varepsilon > 0$,

$$\inf_{(y,y^*)\in\mathcal{G}(T)} \langle x - y, x^* - y^*\rangle \geq -\varepsilon \quad \Leftrightarrow \quad \varphi_T(x, x^*) \leq \langle x, x^*\rangle + \varepsilon \quad \Leftrightarrow \quad (x, x^*) \in \mathcal{G}(T).$$

Then, by Corollary 6.2.4, it holds $\mathcal{G}(\overline{T}) = \mathcal{G}(\widehat{T}_h)$, implying $\mathcal{G}(\overline{T}) = \mathcal{G}(\widetilde{T})$, i.e. $T$ is of type (D). □

Notice that we need maximality of $T$ only to prove the implication $(c) \Longrightarrow (a)$.

**Remark 6.2.6** Corollary 6.2.4 and Proposition 6.2.5 imply that, for any maximal monotone operator $T$ and any $h \in \mathcal{H}_T$:

*(a)* if $T$ admits a unique maximal monotone extension to the bidual but it is not of type (D), then $\mathcal{G}(\overline{T}) = \mathcal{G}(\widehat{T}_h) \subsetneq \mathcal{G}(\widetilde{T})$;

*(b)* the relations $\mathcal{G}(\overline{T}) \subsetneq \mathcal{G}(\widehat{T}_h)$ and $\mathcal{G}(\widehat{T}_h) = \mathcal{G}(\widetilde{T})$ cannot hold simultaneously.

When $T$ is a monotone operator that is not of type (BR), we don't know if the inclusion $\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(\widehat{T}_h)$ is proper or not. If the former case is true, we cannot approximate arbitrary points of $\mathcal{G}(\widehat{T}_h)$ by means of bounded nets in $\mathcal{G}(T)$ converging in the $\sigma(X^{**}, X^*) \otimes$ norm topology. Anyway, there is a family of monotone extensions to the bidual for which we can recover a weaker approximation result by means of Proposition 6.2.8 below, as a consequence of Theorem 6.1.5, based on a proof similar to that of Theorem 6.1.6 with the strict Brønsted-Rockafellar property replaced by the usual Brønsted-Rockafellar property of subdifferentials. Specifically, Proposition 6.2.8 will give an approximation result for the set

$$\operatorname{fix}\partial h^* := \{(x^*, x^{**}) \in X^* \times X^{**} : \; h^*(x^*, x^{**}) + h^{**}(x^{**}, x^*) = 2\langle x^{**}, x^*\rangle\}$$

by means of $\mathcal{G}(\partial h)^\top$, for all $h \in \mathcal{K}_T$.

The interest of this set of fixed points can be motivated if we refer to the case of maximal monotone operators defined on reflexive Banach spaces, since, in this particular setting, $\mathcal{K}_T = \mathcal{H}_T$ and $\operatorname{fix} \partial h^* = \operatorname{fix} \partial h = \mathcal{G}(T)$, where

$$\operatorname{fix} \partial h := \{(x, x^*) \in X \times X^* : \ \ h(x, x^*) + h^*(x^*, x) = 2\langle x, x^* \rangle\}$$

and the equality $\operatorname{fix} \partial h = \mathcal{G}(T)$ follows from the fact that, in this case, $h, h^{*\top} \geq \langle \cdot, \cdot \rangle$ on $X \times X^*$ and $\mathcal{G}(T) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x, x^* \rangle\} = \{(x, x^*) \in X \times X^* : h^*(x^*, x) = \langle x, x^* \rangle\}$ (see for instance [27]).

Consider first the following properties of the operator $F_h : X^{**} \rightrightarrows X^*$, defined by

$$\mathcal{G}(F_h) = (\operatorname{fix} \partial h^*)^\top.$$

**Proposition 6.2.7** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator. Then, for all $h \in \mathcal{K}_T$, $F_h$ is an extension of $T$ to the bidual and it is a representable monotone operator.*

*Proof.* Let $h \in \mathcal{K}_T$. Since $\varphi_T \leq h \leq \sigma_T$, one has

$$\sigma_T^* \leq h^* \leq \varphi_T^*.$$

Notice that, for any $(x, x^*) \in \mathcal{G}(T)$,

$$\begin{aligned}
\varphi_T^*(x^*, x) &= \sup_{(y,y^*) \in X \times X^*} \{\langle x, y^* \rangle + \langle y, x^* \rangle - \varphi_T(y, y^*)\} \\
&= \sup_{(y,y^*) \in X \times X^*} \{\langle x, y^* \rangle + \langle y, x^* \rangle - \sup_{(z,z^*) \in \mathcal{G}(T)} \{\langle y, z^* \rangle + \langle z, y^* \rangle - \langle z, z^* \rangle\}\} \\
&= \sup_{(y,y^*) \in X \times X^*} \{\langle x, y^* \rangle + \langle y, x^* \rangle + \inf_{(z,z^*) \in \mathcal{G}(T)} \{-\langle y, z^* \rangle - \langle z, y^* \rangle + \langle z, z^* \rangle\}\} \\
&\leq \sup_{(y,y^*) \in X \times X^*} \{\langle x, y^* \rangle + \langle y, x^* \rangle - \langle y, x^* \rangle - \langle x, y^* \rangle + \langle x, x^* \rangle\} \\
&= \langle x, x^* \rangle.
\end{aligned}$$

Thus,

$$\langle x, x^* \rangle = \varphi_T(x, x^*) = \sigma_T^*(x^*, x) \leq h^*(x^*, x) \leq \varphi_T^*(x^*, x) \leq \langle x, x^* \rangle,$$

implying $h^*(x^*, x) = \langle x, x^* \rangle$. Since, moreover, $h^{**}(x, x^*) = h(x, x^*) = \langle x, x^* \rangle$, then $F_h$ is an extension of $T$ to the bidual.

Finally, by Fenchel inequality, the lower semicontinuous convex function

$$(x^*, x^{**}) \mapsto \frac{1}{2}(h^*(x^*, x^{**}) + h^{**}(x^{**}, x^*))$$

majorizes the duality product. Therefore, according to Remark 6.1.1, $F_h^{-1}$, and hence $F_h$, is a representable monotone operator. $\square$

As a consequence, for all $h \in \mathcal{K}_T$, it holds

$$\mathcal{G}(T) \subseteq \mathcal{G}(F_h) \subseteq \mathcal{G}(\widetilde{T}).$$

When $h = \sigma_T$, the first inclusion can be refined, yielding

$$\mathcal{G}(\overline{T}) \subseteq \mathcal{G}(F_{\sigma_T}) \subseteq \mathcal{G}(\widetilde{T}),$$

since, for all $(x^{**}, x^*) \in \mathcal{G}(\overline{T})$, one has $\sigma_T^*(x^*, x^{**}) = \varphi_{\overline{T}^{-1}}(x^*, x^{**}) = \langle x^{**}, x^* \rangle$ and $\sigma_T^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle$, by items $(c)$ and $(b)$, respectively, of Theorem 6.2.2.

**Proposition 6.2.8** *Let $X$ be a Banach space, $T : X \rightrightarrows X^*$ be a monotone operator and $h \in \mathcal{K}_T$. For any $(x^{**}, x^*) \in \mathcal{G}(F_h)$, there exists a bounded net $((x_\alpha, x_\alpha^*), (y_\alpha^*, y_\alpha^{**}))$ in $\mathcal{G}(\partial h)$ such that $(x_\alpha, x_\alpha^*)$ converges to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$ and $(y_\alpha^*, y_\alpha^{**})$ converges to $(x^*, x^{**})$ in the norm topology of $X^* \times X^{**}$.*

*Proof.* By Theorem 6.1.5, there exists a bounded net $(z_\alpha, z_\alpha^*)$ in $X \times X^*$ converging to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$ and such that $\lim_\alpha h(z_\alpha, z_\alpha^*) = h^{**}(x^{**}, x^*)$. Since $h^{**}(x^{**}, x^*)$ is finite, we can choose $(z_\alpha, z_\alpha^*)$ such that the net $(h(z_\alpha, z_\alpha^*))_\alpha$ is bounded. Therefore, taking Fenchel inequality into account, one can define

$$\varepsilon_\alpha^2 := h(z_\alpha, z_\alpha^*) + h^*(x^*, x^{**}) - \langle z_\alpha, x^* \rangle - \langle x^{**}, z_\alpha^* \rangle \geq 0,$$

where $\varepsilon_\alpha$ is bounded and, since $(x^{**}, x^*) \in \mathcal{G}(F_h) = (\operatorname{fix} \partial h^*)^\top$, $\lim_\alpha \varepsilon_\alpha^2 = 0$.

By the Brønsted-Rockafellar property of subdifferentials, for any $\alpha$ there exists $((x_\alpha, x_\alpha^*), (y_\alpha^*, y_\alpha^{**})) \in \mathcal{G}(\partial h)$ such that

$$\|(x_\alpha, x_\alpha^*) - (z_\alpha, z_\alpha^*)\| \leq \varepsilon_\alpha \quad \text{and} \quad \|(y_\alpha^*, y_\alpha^{**}) - (x^*, x^{**})\| \leq \varepsilon_\alpha,$$

implying

$$\|x_\alpha - z_\alpha\| \leq \varepsilon_\alpha, \qquad \|x_\alpha^* - z_\alpha^*\| \leq \varepsilon_\alpha.$$

Thus $(x_\alpha, x_\alpha^*)$ is bounded and converges to $(x^{**}, x^*)$ in the $\sigma(X^{**}, X^*) \otimes$ norm topology of $X^{**} \times X^*$, while $(y_\alpha^*, y_\alpha^{**})$ is bounded and converges to $(x^*, x^{**})$ in the norm topology of $X^* \times X^{**}$. $\square$

## 6.3 Adding Properties to the Extensions

Until now, we have only considered properties of $T$. Now we are going to study what happens when we require $\overline{T}$ to satisfy some particular property as well. While the properties of $T$ considered in the previous section were typically weaker than (or equivalent to) that of being of type (D), endowing $\overline{T}$ with the property of being of type (BR) will imply that $T$ is of type (D).

**Lemma 6.3.1** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator of type (BR). Then $T$ is premaximal monotone and the graph of its unique maximal monotone extension is equal to* $\mathrm{cl}\,\mathcal{G}(T)$*, the closure of $\mathcal{G}(T)$ in the norm topology of $X \times X^*$.*

*Proof.* Let $(x_0, x_0^*) \in X \times X^*$ be monotonically related to $\mathcal{G}(T)$. Then, for all $n \in \mathbb{N}\backslash\{0\}$,

$$\inf_{(y,y^*)\in\mathcal{G}(T)} \langle x_0 - y, x_0^* - y^* \rangle \geq 0 > -\frac{1}{n+1}.$$

Thus, since $T$ is of type (BR), there exists $(x_n, x_n^*) \in \mathcal{G}(T)$ such that $\|x_n - x_0\| < 1/\sqrt{n}$ and $\|x_n^* - x_0^*\| < 1/\sqrt{n}$. Thus $(x_0, x_0^*)$ belongs to the closure of $\mathcal{G}(T)$ in the norm topology of $X \times X^*$. Hence, the graph of any monotone extension $S : X \rightrightarrows X^*$ of $T$ is contained in $\mathrm{cl}\,\mathcal{G}(T)$.

On the other hand, if $S$ is maximal monotone, the opposite inclusion holds as well, being

$$\mathcal{G}(T) \subseteq \mathcal{G}(S) \Rightarrow \mathrm{cl}\,\mathcal{G}(T) \subseteq \mathrm{cl}\,\mathcal{G}(S)$$

and $\mathrm{cl}\,\mathcal{G}(S) = \mathcal{G}(S)$, since any maximal monotone operator has a closed graph. In particular, then, $T$ admits only one maximal monotone extension (in $X \times X^*$), defined by $\mathrm{cl}\,\mathcal{G}(T)$, i.e. it is premaximal monotone. □

**Remark 6.3.2** As a consequence of the previous lemma, any representable monotone operator of type (BR) is maximal monotone, since its graph is closed, according to [61, Proposition 8].

**Definition 6.3.3** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. We call $T$ of* type (BR*) *if $\overline{T}^{-1} : X^* \rightrightarrows X^{**}$ is of type (BR).*

**Proposition 6.3.4** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. If $T$ is of type (BR*), then $T$ is of type (D).*

*Proof.* If $\overline{T}^{-1}$ is of type (BR), then, by Lemma 6.3.1, $\overline{T}^{-1}$ is premaximal monotone. Thus, by Proposition 6.1.2 (*b*), $T$ admits a unique maximal monotone extension to the bidual. As a consequence, by Theorems 2.2.17 and 2.2.14, $T$ is of type (BR), implying that $\overline{T} = \widehat{T}_h$ for all $h \in \mathcal{H}_T$, by Corollary 6.2.4. Thus $\widehat{T}_h^{-1}$ is of type (BR) for all $h \in \mathcal{H}_T$. Moreover, by Theorem 6.2.2, $\widehat{T}_h^{-1}$ is a representable monotone operator. Therefore, according to Remark 6.3.2, $\widehat{T}_h^{-1}$ is maximal monotone, i.e. $\overline{T}^{-1}$ is maximal monotone. Finally, this fact is equivalent to $T$ being of type (D), as stated in Corollary 6.2.3. $\square$

In the end, we consider for $\overline{T}^{-1}$ a condition which, in principle, is stronger than that of being of type (BR), i.e. the condition of being of type (D).

**Definition 6.3.5** *Let $X$ be a Banach space and $T : X \rightrightarrows X^*$ be a maximal monotone operator. We call $T$ of* type (D$^*$) *if $\overline{T}^{-1} : X^* \rightrightarrows X^{**}$ is maximal monotone of type (D).*

Note that any maximal monotone operator of type (D$^*$) is of type (D) as well, since, by Corollary 6.2.3, the maximality of $\overline{T}$ implies $T$ being of type (D) (the same conclusion can also be drawn as an immediate consequence of Proposition 6.3.4). Therefore, $\overline{T} = \widetilde{T}$.

Moreover, the subdifferential of a proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ is of type (D$^*$), given that, as a consequence of (1.7) and the fact that subdifferentials of proper lower semicontinuous convex functions are of type (D), $\overline{\partial f} = \widetilde{\partial f} = (\partial f^*)^{-1}$ and $\partial f^*$, being again the subdifferential of a lower semicontinuous proper convex function, is a maximal monotone operator of type (D).

Summarizing, for any maximal monotone operator $T : X \rightrightarrows X^*$, the following relations hold:

$T$ is a subdifferential $\Longrightarrow$ $T$ is of type (D$^*$) $\Longrightarrow$ $T$ is of type (BR$^*$) $\Longrightarrow$ $T$ is of type (D).

It is an open question whether the converse of any of the previous implications holds as well. Finally, we know from Corollary 5.2.4 that two maximal monotone operators $S, T : X \rightrightarrows X^*$ of type (D), under suitable conditions, satisfy the inclusion $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$. Operators of type (D$^*$), by means of Theorem 5.3.6, enable us to extend this property, in terms of a density result, to the whole of $X^{**} \times X^*$.

**Theorem 6.3.6** *Let $X$ be a Banach space and $S, T : X \rightrightarrows X^*$ be maximal monotone operators of type ($D^*$). For all $(w^{**}, w^*) \in X^{**} \times X^*$, if there exist $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$ such that*

$$\bigcup_{\lambda>0} \lambda[\text{dom}\,(h^{**}|_{X^{**}\times X^*}) - \varrho_1(\text{dom}\,(k^{**}|_{X^{**}\times X^*})) - (w^{**}, w^*)] \text{ is a closed subspace of } X^{**} \times X^*, \tag{6.2}$$

*and*

$$\bigcup_{\lambda>0} \lambda[\text{Pr}_{X^*}\text{dom}\, \mathcal{T}_{(w^{**},w^*)}(h^{**}|_{X^{**}\times X^*}) - \text{Pr}_{X^*}\text{dom}\,(k^{**}|_{X^{**}\times X^*})] \text{ is a closed subspace of } X^*, \tag{6.3}$$

*then there exists a sequence $(w_n^{**})$ in $X^{**}$ converging to $w^{**}$ and such that $(w_n^{**}, w^*) \in \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})$. Therefore, in particular,* $\text{cl}\,(\mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{T})) = X^{**} \times X^*$.

*Proof.* Since $S$ and $T$ are maximal monotone operators of type (D), then, for all $h \in \mathcal{H}_S$ and $k \in \mathcal{H}_T$, setting $\tilde{h} := (h^{**\top})_{|X^*\times X^{**}}$ and $\tilde{k} := (k^{**\top})_{|X^*\times X^{**}}$, as a consequence of Theorem 6.2.2 we have $\tilde{h} \in \mathcal{H}_{\widetilde{S}^{-1}}$ and $\tilde{k} \in \mathcal{H}_{\widetilde{T}^{-1}}$. Thus, by (6.2), for any $(w^*, w^{**}) \in X^* \times X^{**}$, there exist $\tilde{h} \in \mathcal{H}_{\widetilde{S}^{-1}}$ and $\tilde{k} \in \mathcal{H}_{\widetilde{T}^{-1}}$ such that

$$\bigcup_{\lambda>0} \lambda[\text{dom}\,\tilde{h} - \varrho_2(\text{dom}\,\tilde{k}) - (w^*, w^{**})]$$

is a closed subspace of $X^* \times X^{**}$ and, by (6.3),

$$\bigcup_{\lambda>0} \lambda[\text{Pr}_{X^*}\text{dom}\, \mathcal{T}_{(w^*,w^{**})}\tilde{h} - \text{Pr}_{X^*}\text{dom}\,\tilde{k}]$$

is a closed subspace of $X^*$.

Therefore, by Theorem 5.3.6, one has

$$\text{cl}\,\mathcal{R}(\widetilde{S}^{-1}(\;\cdot + w^*) + \widetilde{T}^{-1}) = X^{**}$$

for all $w^* \in X^*$.

Thus, there exist a sequence $(x_n^{**}, x_n^*)$ in $X^{**} \times X^*$ and a sequence $(-y_n^{**})$ in $X^{**}$ such that

$$(w^* + x_n^*, w^{**} + x_n^{**}) \in \mathcal{G}(\tilde{S}^{-1}), \qquad (x_n^*, -y_n^{**}) \in \mathcal{G}(\tilde{T}^{-1})$$

and $x_n^{**} - y_n^{**}$ converges to $0_{X^{**}}$ in the norm topology of $X^{**}$. As a consequence, one has

$$(w^{**} + x_n^{**} - y_n^{**}, w^*) = (w^{**} + x_n^{**}, w^* + x_n^*) + (-y_n^{**}, -x_n^*) \in \mathcal{G}(\tilde{S}) + \mathcal{G}(-\tilde{T})$$

and $w_n^{**} := w^{**} + x_n^{**} - y_n^{**}$ converges to $w^{**}$ in the norm topology of $X^{**}$.

Since $(w^{**}, w^*) \in X^{**} \times X^*$ was chosen arbitrarily and $(w_n^{**}, w^*)$ converges to $(w^{**}, w^*)$ in the norm topology of $X^{**} \times X^*$, we conclude that $\text{cl}\,(\mathcal{G}(\tilde{S}) + \mathcal{G}(-\tilde{T})) = X^{**} \times X^*$. $\square$

A relevant example illustrating the previous theorem is given by the case when $T$ is the duality mapping $J$, i.e. the subdifferential of the function $j : X \to \mathbb{R} \cup \{+\infty\}$ defined by $j(x) = 1/2\|x\|^2$ for all $x \in X$. As the subdifferential of a proper lower semicontinuous convex function, $J$ is a maximal monotone operator of type (D) and the function $j \oplus j^* : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, defined by $(j \oplus j^*)(x, x^*) = j(x) + j^*(x^*)$ for all $(x, x^*) \in X \times X^*$, belongs to $\mathcal{H}_J$. Moreover,

$$\text{dom}\,(j \oplus j^*) = X \times X^* \quad \text{and} \quad \text{dom}\,(j \oplus j^*)^{**}|_{X^{**} \times X^*} = \text{dom}\,(j^{**} \oplus j^{***})|_{X^{**} \times X^*} = X^{**} \times X^*.$$

Therefore, for every maximal monotone operator $S : X \rightrightarrows X^*$ of type (D$^*$) and $T = J$, the hypotheses of Theorem 6.3.6 are satisfied for any given $h \in \mathcal{H}_S$, by setting $k = j \oplus j^*$. Thus, not only (by Corollary 5.2.4) we have $X \times X^* \subseteq \mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})$, but it also holds

$$\text{cl}\,(\mathcal{G}(\widetilde{S}) + \mathcal{G}(-\widetilde{J})) = X^{**} \times X^*.$$

# Chapter 7

# Surjectivity and Abstract Monotonicity

This chapter presents a surjectivity theorem in the context of abstract monotonicity, and is based on [69], to which we refer the reader for more results and examples. In particular, a theory of monotone operators can be developed in the framework of abstract convexity (on the same lines of [33, 68]) and we provide in this general setting a surjectivity result for abstract monotone operators satisfying a given qualification condition.

Abstract convexity has found many applications in the study of problems of Mathematical Analysis and Optimization, generalizing classical results of Convex Analysis. It is well-known that every proper lower semicontinuous convex function is the upper envelope of a set of affine functions. In abstract convexity, the role of the set of affine functions is taken by an alternative set $H$ of functions, in the sense that their upper envelopes constitute the set of abstract convex functions. Different choices of the set $H$ have been studied in the literature, yielding important applications [87, 88, 89, 90].

Abstract convexity has mainly been used for the study of point-to-point functions. Examples of its use in the analysis of multifunctions can be found in [23, 50, 51, 71]. Recently, a theory of monotone operators has been developed in the framework of abstract convexity [33, 68].

The structure of the chapter is as follows. In Section 7.1, we provide some preliminary definitions and results related to abstract convexity and abstract monotonicity, while Section 7.2 presents the surjectivity result we mentioned (Theorem 7.2.4), which is a partial extension of Corollary 5.3.3 to the setting of abstract monotonicity.

## 7.1 Preliminary Notions

Let $X$ be an arbitrary set and $L$ be a set of real-valued functions $l : X \to \mathbb{R}$ defined on $X$. For each $l \in L$ and $c \in \mathbb{R}$, consider the shift $h_{l,c}$ of $l$ on the constant $c$

$$h_{l,c}(x) := l(x) - c, \quad \forall x \in X.$$

The function $h_{l,c}$ is called $L$-affine. Recall [86] that the set $L$ is called a set of *abstract linear functions* if $h_{l,c} \notin L$ for all $l \in L$ and all $c \in \mathbb{R} \setminus \{0\}$. The set of all $L$-affine functions will be denoted by $H_L$. If $L$ is the set of abstract linear functions, then $h_{l,c} = h_{l_0,c_0}$ if and only if $l = l_0$ and $c = c_0$.

If $L$ is a set of abstract linear functions, then the mapping $(l, c) \to h_{l,c}$ is a one-to-one correspondence. In this case, we identify $h_{l,c}$ with $(l, c)$, in other words, we consider an element $(l, c) \in L \times \mathbb{R}$ as a function defined on $X$ by $x \mapsto l(x) - c$.

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called proper if $\operatorname{dom} f \neq \emptyset$, where $\operatorname{dom} f$ is defined by

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\}.$$

Let $\mathcal{F}(X)$ be the set of all functions $f : X \to \mathbb{R} \cup \{+\infty\}$ and the function $-\infty$.

Recall [86] that a function $f \in \mathcal{F}(X)$ is called $H$-convex ($H = L$, or $H = H_L$) if

$$f(x) = \sup\{h(x) : h \in \operatorname{supp}\,(f, H)\}, \quad \forall\, x \in X,$$

where

$$\operatorname{supp}\,(f, H) := \{h \in H : h \leq f\}$$

is the *support set* of the function $f$.

Let $\mathcal{P}(H)$ be the set of all $H$-convex functions $f : X \to \mathbb{R} \cup \{+\infty\}$. We say that (see [44]) the set-valued mapping $\operatorname{supp}\,(\cdot, H) : \mathcal{P}(H) \rightrightarrows H$ is additive in $f$ and $g$ if

$$\operatorname{supp}\,(f + g, H) = \operatorname{supp}\,(f, H) + \operatorname{supp}\,(g, H).$$

Note that if $X$ is a locally convex Hausdorff topological vector space and $L$ is the set of all real-valued continuous linear functionals defined on $X$, then $f : X \to \mathbb{R} \cup \{+\infty\}$ is an $L$-convex function if and only if $f$ is lower semicontinuous and sublinear. Also, $f$ is an $H_L$-convex function if and only if $f$ is lower semicontinuous and convex.

In the framework of abstract convexity, the coupling function can be defined as

$$\langle \cdot, \cdot \rangle : X \times L \to \mathbb{R}, \qquad (x, l) \mapsto \langle x, l \rangle := l(x).$$

For a function $f \in \mathcal{F}(X)$, define the Fenchel-Moreau $L$-conjugate $f_L^*$ of $f$ [86] by

$$f_L^*(l) := \sup_{x \in X} \{l(x) - f(x)\}, \quad \forall\, l \in L.$$

Similarly, the Fenchel-Moreau $X$-conjugate $g_X^*$ of an extended real valued function $g$ defined on $L$ is given by

$$g_X^*\,(x) := \sup_{l \in L} \{l(x) - g(l)\}, \quad \forall\, x \in X.$$

The function $f_{L,X}^{**} := (f_L^*)_X^*$ is called the second conjugate (or biconjugate) of $f$, and by definition we have

$$f_{L,X}^{**}(x) := \sup_{l \in L} \{l(x) - f_L^*(l)\}, \quad \forall\, x \in X.$$

For the second conjugate, the following result holds.

**Theorem 7.1.1 ([86, Theorem 7.1])** *Let $f \in \mathcal{F}(X)$. Then, $f = f_{L,X}^{**}$ if and only if $f$ is an $H_L$-convex function.*

The following properties of the conjugate function follow directly from the definition.

(i) Fenchel-Young's inequality: if $f \in \mathcal{F}(X)$, then

$$f(x) + f_L^*(l) \geq l(x), \quad \forall\, x \in X, l \in L.$$

(ii) For $f_1$ and $f_2 \in \mathcal{F}(X)$, we have

$$f_1 \leq f_2 \implies f_2^* \leq f_1^*.$$

A set $C \subset \mathcal{F}(X)$ is called additive if, for all $f_1,\ f_2 \in C$, one has $f_1 + f_2 \in C$.

If $X$ is a set on which an addition $+$ is defined, then we say that a function $f \in \mathcal{F}(X)$ is additive if

$$f(x + y) = f(x) + f(y), \quad \forall\, x,\ y \in X.$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function and $x_0 \in \text{dom}\, f$. Recall [86] that an element $l \in L$ is called an $L$-subgradient of $f$ at $x_0$ if

$$f(x) \geq f(x_0) + l(x) - l(x_0), \quad \forall\, x \in X.$$

The set $\partial_L f(x_0)$ of all $L$-subgradients of $f$ at $x_0$ is called $L$-subdifferential of $f$ at $x_0$. The subdifferential $\partial_L f(x_0)$ (see [86, Proposition 1.2]) is nonempty if and only if $x_0 \in \text{dom}\, f$ and

$$f(x_0) = \max\{h(x_0) : h \in \text{supp}\ (f, H_L)\}.$$

Recall [44] that for proper functions $f,\ g \in \mathcal{F}(X)$, the infimal convolution of $f$ with $g$ is denoted by $f \boxplus g : X \to \mathbb{R} \cup \{+\infty\}$ and is defined by

$$(f \boxplus g)(x) := \inf_{x_1+x_2=x} \{f(x_1) + g(x_2)\}, \quad \forall\, x \in X.$$

The infimal convolution of $f$ with $g$ is said to be exact provided the above infimum is achieved for every $x \in X$ [44].

**Theorem 7.1.2 ([44, Theorem 7.1])** *Let $L$ be an additive set of abstract linear functions and $f,\ g : X \to \mathbb{R} \cup \{+\infty\}$ be $H_L$-convex functions such that $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \emptyset$. Then the following assertions are equivalent:*

*(i) the mapping $\operatorname{supp}(\cdot, H_L)$ is additive in $f$ and $g$;*

*(ii) $(f+g)^*_L = f^*_L \boxplus g^*_L$ with exact infimal convolution.*

Now, assume as above that $X$ is an arbitrary set and $L$ is a set of real-valued abstract linear functions. In the following, we present some definitions and properties of abstract monotone operators [33, 50, 68, 71].

$(i)$ A set-valued mapping $T : X \rightrightarrows L$ is called an *$L$-monotone operator* (or *abstract monotone operator*) if

$$l(x) - l(x') - l'(x) + l'(x') \geq 0 \tag{7.1}$$

for all $(x, l)$, $(x', l') \in \mathcal{G}(T)$.

It is worth noting that, if $X$ is a Banach space with dual space $X^*$ and $L := X^*$, then $T$ is a monotone operator in the classical sense.

$(ii)$ A set-valued mapping $T : X \rightrightarrows L$ is called *maximal $L$-monotone* (or *maximal abstract monotone*) if $T$ is $L$-monotone and $T = T'$ for any $L$-monotone operator $T' : X \rightrightarrows L$ such that $\mathcal{G}(T) \subseteq \mathcal{G}(T')$.

There exist examples of abstract convex functions such that their $L$-subdifferentials are maximal $L$-monotone operators [68, 70].

$(iii)$ Definitions analogous to $(i)$ and $(ii)$ can be given for a subset $S \subseteq X \times L$, instead of an operator.

(*iv*) Let $T : X \rightrightarrows L$ be a set-valued mapping. Corresponding to the mapping $T$ define the *L-Fitzpatrick function* (or, *abstract Fitzpatrick function*) $\varphi_T : X \times L \to \mathbb{R} \cup \{+\infty\}$ by

$$\varphi_T(x,l) := \sup_{(x',l') \in \mathcal{G}(T)} \{l(x') + l'(x) - l'(x')\} \tag{7.2}$$

for all $(x,l) \in X \times L$.

Similarly to the case of ordinary maximal monotone operators, the following result relates the $L-$Fitzpatrick function with the duality product.

**Theorem 7.1.3 ([68])** *Let $T : X \rightrightarrows L$ be a maximal L-monotone operator. Then*

$$\varphi_T(x,l) \geq l(x), \quad \forall x \in X,\ l \in L, \tag{7.3}$$

*with equality holding if and only if $l \in T(x)$.*

## 7.2 A Surjectivity Result

Let $U$ be an arbitrary set and $L$ be an additive group of abstract linear functions on $U$. We define the coupling between $U \times L$ and $L \times U$ as

$$< (u,l), (m,v) > = m(u) + l(v),$$

for all $(u,l) \in U \times L$ and $(m,v) \in L \times U$. Let $X \subseteq U$. We will say that $A : X \rightrightarrows L$ is $L$-monotone if so is its extension to $U$ obtained by assigning empty images to the elements in $U \setminus X$. Similarly, a function $h : X \times L \to \mathbb{R} \cup \{+\infty\}$ will be called $H_{L\times U}$-convex if it is the restriction of an $H_{L\times U}$-convex function on $U \times L$.

Given an $L-$monotone operator $A : X \rightrightarrows L$, consider the Fitzpatrick family of abstract convex representations of $A$

$$\begin{aligned} \mathcal{H}_A \quad = \quad & \{h : X \times L \to \mathbb{R} \cup \{+\infty\} : \quad h \text{ is } H_{L\times U} - \text{convex}, \\ & h(x,l) \geq l(x) \ \ \forall (x,l) \in X \times L, \quad h(x,l) = l(x) \ \ \forall (x,l) \in \mathcal{G}(A)\}. \end{aligned}$$

Moreover, for all $l_0 \in L$, denote by $A_{l_0} : X \rightrightarrows L$ the multifunction such that $A_{l_0}(x) = A(x) - l_0$, for all $x \in X$. It is easy to check that, for any $h \in \mathcal{H}_A$, the function $h_{l_0} : X \times L \to \mathbb{R} \cup \{+\infty\}$, defined by

$$h_{l_0}(x,l) := h(x, l + l_0) - l_0(x), \ \ \forall (x,l) \in X \times L,$$

belongs to $\mathcal{H}_{A_{l_0}}$. Notice that, for any $(m,u) \in L \times U$,

$$\begin{aligned}
(h_{l_0})^*_{L\times U}(m,u) &= \sup_{(x,l)\in X\times L} \{m(x) + l(u) - h(x, l+l_0) + l_0(x)\} \\
&= \sup_{(x,l)\in X\times L} \{(m+l_0)(x) + (l-l_0)(u) - h(x,l)\} \\
&= \sup_{(x,l)\in X\times L} \{(m+l_0)(x) + l(u) - h(x,l)\} - l_0(u) \\
&= h^*_{L\times U}(m+l_0, u) - l_0(u).
\end{aligned}$$

If $A : X \rightrightarrows L$ is an $L-$monotone operator and $h \in \mathcal{H}_A$, denote by $\widetilde{A}_h : U \rightrightarrows L$ the operator defined by

$$G(\widetilde{A}_h) = \{(u,l) \in U \times L : \quad h^*_{L\times U}(l,u) = l(u)\}.$$

In particular, when $h = \varphi_A$, we will simply write $\widetilde{A}$, instead of $\widetilde{A}_{\varphi_A}$, for ease of notation. According to the following proposition, $\widetilde{A}$ is an extension of $A$, i.e. $\mathcal{G}(A) \subseteq \mathcal{G}(\widetilde{A})$.

**Proposition 7.2.1** *Let $X \subseteq U$ and $A : X \rightrightarrows L$ be an $L-$monotone operator. Then $\widetilde{A}$ is an extension of $A$.*

*Proof.* Notice first that, for any $(m,x) \in L \times X$, one has $(\varphi_A)^*_{L\times U}(m,x) \geq \varphi_A(x,m)$. Indeed, since $\varphi_A(y,l) = l(y)$ for all $(y,l) \in \mathcal{G}(A)$,

$$\begin{aligned}
(\varphi_A)^*_{L\times U}(m,x) &= \sup_{(y,l)\in X\times L} \{m(y) + l(x) - \varphi_A(y,l)\} \\
&\geq \sup_{(y,l)\in \mathcal{G}(A)} \{m(y) + l(x) - \varphi_A(y,l)\} \\
&= \sup_{(y,l)\in \mathcal{G}(A)} \{m(y) + l(x) - l(y)\} \\
&= \varphi_A(x,m).
\end{aligned}$$

Moreover, for all $(x,m) \in \mathcal{G}(A)$, one has $(\varphi_A)^*_{L\times U}(m,x) \leq m(x)$, since

$$\begin{aligned}
(\varphi_A)^*_{L\times U}(m,x) &= \sup_{(y,l)\in X\times L} \{m(y) + l(x) - \varphi_A(y,l)\} \\
&= \sup_{(y,l)\in X\times L} \{m(y) + l(x) - m(x) + m(x) - \varphi_A(y,l)\} \\
&\leq m(x) + \sup_{(y,l)\in X\times L} \left\{ \sup_{(z,n)\in\mathcal{G}(A)} \{l(z) + n(y) - n(z)\} - \varphi_A(y,l) \right\} \\
&= m(x) + \sup_{(y,l)\in X\times L} \{\varphi_A(y,l) - \varphi_A(y,l)\} \\
&= m(x).
\end{aligned}$$

Therefore, for all $(x,m) \in \mathcal{G}(A)$, one obtains

$$m(x) = \varphi_A(x,m) \leq (\varphi_A)^*_{L\times U}(m,x) \leq m(x),$$

i.e. $(x,m) \in \mathcal{G}(\widetilde{A})$. Thus, $\widetilde{A}$ is an extension of $A$. □

**Definition 7.2.2** *Let $f, g : X \times L \to \mathbb{R} \cup \{+\infty\}$ be $H_{L\times U}$−convex functions. We call an* abstract skewed Fenchel functional *for $f$ and $g$ any $(m,u) \in L \times U$ such that*

$$f^*_{L\times U}(m,u) + g^*_{L\times U}(-m,u) \leq 0.$$

**Remark 7.2.3** If $U$ is an additive set and the elements of $L$ are odd functions, then, defining the function $\varrho_2 : X \times L \to X \times L$ by $\varrho_2(x,l) = (x,-l)$ for all $(x,l) \in X \times L$, the existence of an abstract skewed Fenchel functional for $f$ and $g$ is equivalent to the existence of an abstract Fenchel functional for $f$ and $g \circ \varrho_2$, i.e. an element $(m,u) \in L \times U$ such that

$$f^*_{L\times U}(m,u) + (g \circ \varrho_2)^*_{L\times U}(-m,-u) \leq 0.$$

The proof of this fact is immediate, given that, for all $(m,u) \in L \times U$,

$$\begin{aligned}
(k \circ \varrho_2)^*_{L\times U}(-m,-u) &= \sup_{(x,l)\in X\times L} \{-m(x) + l(-u) - (k \circ \varrho_2)(x,l)\} \\
&= \sup_{(x,l)\in X\times L} \{-m(x) + l(-u) - k(x,-l)\} \\
&= \sup_{(x,l)\in X\times L} \{-m(x) - l(-u) - k(x,l)\} \\
&= \sup_{(x,l)\in X\times L} \{-m(x) + l(u) - k(x,l)\} \\
&= k^*_{L\times U}(-m,u).
\end{aligned}$$

**Theorem 7.2.4** *Let $X \subseteq U$ and $A, B : X \rightrightarrows L$ be $L$−monotone operators. If there exist $h \in \mathcal{H}_A$ and $k \in \mathcal{H}_B$ such that $h^*_{L\times U}(m,u) \geq m(u)$ and $k^*_{L\times U}(m,u) \geq m(u)$, for all $(m,u) \in L \times U$, and such that, for any $l_0 \in L$, the functions $h_{l_0}$ and $k$ admit an abstract skewed Fenchel functional, then $\mathcal{R}(\widetilde{A}_h + \widetilde{B}_k) = L$.*

*Proof.* By hypothesis, there exists an abstract skewed Fenchel functional $(\overline{m},\overline{u})$ for $h_{l_0}$ and $k$, i.e.

$$(h_{l_0})^*_{L\times U}(\overline{m},\overline{u}) + k^*_{L\times U}(-\overline{m},\overline{u}) \leq 0.$$

Moreover, since $h^*_{L\times U}(m,u) \geq m(u)$ and $k^*_{L\times U}(m,u) \geq m(u)$ for all $(m,u) \in L \times U$, by hypothesis

$$\begin{aligned}(h_{l_0})^*_{L\times U}(m,u) + k^*_{L\times U}(-m,u) &= h^*_{L\times U}(m+l_0,u) - l_0(u) + k^*_{L\times U}(-m,u) \\ &\geq (m+l_0)(u) - l_0(u) - m(u) \\ &= m(u) + l_0(u) - l_0(u) - m(u) \\ &= 0.\end{aligned}$$

Then one concludes

$$(h_{l_0})^*_{L\times U}(\overline{m},\overline{u}) + k^*_{L\times U}(-\overline{m},\overline{u}) = 0,$$

from which

$$h^*_{L\times U}(\overline{m}+l_0,\overline{u}) = (\overline{m}+l_0)(\overline{u}) \quad \text{and} \quad k^*_{L\times U}(-\overline{m},\overline{u}) = -\overline{m}(\overline{u}),$$

so that

$$(\overline{u},\overline{m}+l_0) \in \mathcal{G}\big(\widetilde{A}_h\big) \quad \text{and} \quad (\overline{u},-\overline{m}) \in \mathcal{G}\big(\widetilde{B}_k\big).$$

Thus,

$$l_0 = l_0 + \overline{m} - \overline{m} \in \widetilde{A}_h(\overline{u}) + \widetilde{B}_k(\overline{u}),$$

i.e., as a consequence of the arbitrariness of $l_0 \in L$,

$$\mathcal{R}\big(\widetilde{A}_h + \widetilde{B}_k\big) = L. \tag{7.4}$$

□

**Remark 7.2.5** *(a)* The hypotheses of the previous theorem hold whenever $A$ and $B$ are maximal monotone operators of type (D) defined on a Banach space $X$ and there exist $h \in \mathcal{H}_A$ and $k \in \mathcal{H}_B$ such that

$$\operatorname{dom} h - \varrho_2(\operatorname{dom} k) = F \times X^*,$$

where $\bigcup_{\lambda>0} \lambda F$ is a closed subspace of $X$. Indeed, in this case Corollary 5.3.3 guarantees the existence of a Fenchel functional for $h_{w^*}$ and $k \circ \varrho_2$, for all $w^* \in X^*$. Then, identifying $X$ with its image through the canonical inclusion in $X^{**}$, setting $L := X^*$, $U := X^{**}$ and taking Remark 7.2.3 into account, the previous theorem applies.

*(b)* Let $X, Y$ be reflexive Banach spaces and $t : X \to Y$ be an injective and continuous function. Define

$$L := \{f : X \to \mathbb{R} : \quad \exists y^* \in Y^*, f = y^* \circ t\}$$

and, for all $l \in L$, set

$$\|l\|_L := \sup \left\{ \left| \frac{l(x)}{\|t(x)\|_Y} \right| : \quad x \in X, \quad t(x) \neq 0_Y \right\}.$$

It is easy to check that the definition of $\|\cdot\|_L$ does not depend on the choice of $y^*$ and that $(L, \|\cdot\|_L)$ is a normed space. Setting $U := L^*$, then $(t, \mathrm{Id}) : X \times L \to Y \times L$ is a continuous and injective function, $L \times L^*$ can be taken as a set of abstract linear functions on $X \times L$ and the $H_{L\times L^*}-$convex functions will be called *hidden convex functions* [91]. Moreover, one can prove that the function $\zeta : X \to L^*$ defined by

$$\zeta(x)(l) = l(x), \qquad \forall l \in L,$$

for any $x \in X$, is injective. It does indeed take values in $L^*$, given that $\zeta(x)$ is linear and

$$|\zeta(x)(l)| = |l(x)| \leq \|l\|_L \|t(x)\|_Y$$

for all $x \in X$ and $l \in L$, and its injectivity is a direct consequence of that of $t$.

As a consequence of [44, Corollary 5.4], if $A, B : X \rightrightarrows L$ are maximal $L-$monotone operators and the abstract Fitzpatrick function of $B$, $\varphi_B : X \times L \to \mathbb{R} \cup \{+\infty\}$, is continuous on $X \times L$, then, for all $l_0 \in L$, there exists a Fenchel functional $(\overline{m}, \overline{m}^*) \in L \times L^*$ for $(\varphi_A)_{l_0}$ and $\varphi_B \circ \varrho_2$. Therefore, if the functions in $L$ are odd, identifying $X$ with $\zeta(X)$ and taking Remark 7.2.3 into account, then the surjectivity condition (7.4) holds for the extensions $\widetilde{A}$ and $\widetilde{B}$.

# Conclusions and Applications

The property of monotonicity has been deeply studied in the mathematical literature on operators during the last decades, from both a theoretical and an applied point of view. The links of this notion with convexity were soon revealed, though their thorough exploitation only began a decade ago, when convex functions representing maximal monotone operators were extensively employed as a powerful tool of investigation. Having a proper lower semicontinuous convex function that characterizes the graph of a multifunction allows one to translate problems concerning monotone operators into problems that can be dealt with in the realm of Convex Analysis, which considerably expanded and deepened in the recent past and is nowadays a valuable instrument of common use in many applications, including economic theory. This translation of the original problem in terms of convexity puts several classical results on maximal monotone operators in a new perspective, yielding surprising simplifications and elegant revisits of their proofs. In particular, Convex Analysis provides a very powerful duality theory, which plays a crucial role in the literature on convex representations of maximal monotone operators.

In the present thesis we have built once more on this deep relation between monotonicity, convexity and duality. For instance, the importance of duality theory was particularly emphasized in Chapter 5, where, on the lines of [59], the surjectivity property of the sum of the extensions of two maximal monotone operators to the bidual was characterized in terms of Fenchel-Rockafellar duality theorem [81, Corollary 9] and its generalizations, via the notion of a Fenchel functional.

This way of intermingling monotonicity, convexity and duality, and making them interact, is not only meant as a theoretical exercise, but also brings along some possible applications. We mention here two of them. The first one concerns Optimization and Variational Analysis, while the second is about an economic property, namely, the monotonicity of the demand correspondence. While a contribution to the former application was already provided in Chapter 5, the second application is presented as an indication for future research.

A valuable generalization of optimization problems to which much attention has been devoted in the last decades is represented by variational inequalities (see [45]). In particular, variational inequalities involving monotone operators can be considered, that is, given a monotone operator $S : X \rightrightarrows X^*$ and a closed convex set $K$, the problem of finding $(x, x^*) \in X \times X^*$ such that $x \in K \cap \mathcal{D}(S)$, $x^* \in S(x)$ and

$$\langle y - x, x^* \rangle \geq 0, \quad \forall y \in K.$$

The following classical result regarding the application of the theory of monotone operators to variational inequalities is due to Rockafellar.

**Theorem 7.2.6 ([85, Theorem 5])** *Let $X$ be a reflexive Banach space, $K \subseteq X$ be a closed convex set and $A : X \rightrightarrows X^*$ be a maximal monotone operator. Suppose there exist an $a \in K$ and an $\alpha > 0$ such that*

$$\langle x - a, x^* \rangle \geq 0, \quad \forall (x, x^*) \in \mathcal{G}(A) : \; x \in K, \; \|x\| > \alpha. \tag{7.5}$$

*Suppose also that one of the following conditions is satisfied:*

*(a)* $K \cap \operatorname{int} \mathcal{D}(A) \neq \emptyset$;

*(b)* $\mathcal{D}(A) \cap \operatorname{int} K \neq \emptyset$.

*Then the variational inequality for $A$ and $K$ has a solution, i.e. there exists at least one $x \in \mathcal{D}(A) \cap K$ such that, for some $x^* \in A(x)$, $\langle y - x, x^* \rangle \geq 0$ for all $y \in K$.*

In the previous result, the possible lack of compactness of the set $K$ is compensated by a coercivity-type assumption (7.5) on the operator $A$. In addition to it, a qualification condition is required, $(a)$ or $(b)$, to guarantee that the sum of $A$ and the normal cone operator to $K$ be maximal.

Convex representations can also be used to obtain an existence result for variational inequalities, as shown in [59].

**Theorem 7.2.7 ([59, Corollary 2.3])** *Let $X$ be a reflexive Banach space and $S : X \rightrightarrows X^*$ be a maximal monotone operator. If $\varphi_S$ is finite-valued, then for every closed convex set $K \subseteq X$ there exist $x \in K$ and $x^* \in S(x)$ such that*

$$\langle y - x, x^* \rangle \geq 0, \quad \forall y \in K.$$

In the perspective of convex representations, the coercivity assumption and the qualification conditions $(a) - (b)$ are replaced by a qualification condition on the Fitzpatrick function of $T$.

Anyway, this condition can be restrictive. A refinement of it was provided by means of the qualification (5.26) in Corollary 5.3.7 above, i.e.

$$\bigcup_{\lambda>0} \lambda[\operatorname{dom} h - (K \times (-B_K))] \quad \text{is a closed subspace of} \quad X \times X^*,$$

for some $h \in \mathcal{H}_S$. Actually, Corollary 5.3.7 extends the use of convex representations further, providing a complete characterization of the solutions of a variational inequality in terms of Fenchel functionals. That is to say, the set of solutions of a variational inequality for a maximal monotone operator $S$ and a closed convex cone $K$ can be univocally determined considering duality properties of the convex representations of the operator $S$ itself ($h \in \mathcal{H}_S$) and of the convex representation of the normal cone operator to $K$ ($\delta_K \oplus \delta_K^*$).

This application was already studied in the present thesis and rather deals with the methods than with the contents of economic theory. As a complementary example, we finally present an application of the theory of monotone operators to Economics, indicating at the same time a possible path for future investigation.

In the context of Microeconomics, monotonicity may be worthwhile considering in connection with demand correspondences. Recall (see for instance [64]) that the consumer faces a decision problem in which, having a given wealth $w$, he has to choose his consumption over a certain number $L$ of commodities ($L \in \mathbb{N}\backslash\{0\}$), with given prices. Suppose that the consumption set, i.e. the set of consumption bundles that the individual can conceivably choose, is $\mathbb{R}^L_+$, the nonnegative orthant of $\mathbb{R}^L$. Moreover, suppose that to each commodity a strictly positive price is associated, so that any vector $p$ of prices belongs to the strictly positive orthant $\mathbb{R}^L_{++}$, and, for simplicity, normalize the wealth level to 1. We will model consumer's choice behavior by means of a preference relation $\succeq$ that is reflexive ($x \succeq x$, for all $x \in \mathbb{R}^L_+$), transitive (if $x \succeq y$ and $y \succeq z$, then $x \succeq z$, for all $x, y, z \in \mathbb{R}^L_+$), complete (for any $x, y \in \mathbb{R}^L_+$, at least one of $x \succeq y$ and $y \succeq x$ holds) and locally nonsatiated, i.e. for every $x \in \mathbb{R}^L_+$ and $\varepsilon > 0$ there exists $y \in \mathbb{R}^L_+$ such that $\|x - y\| \leq \varepsilon$ and $y \succ x$ (this last notation means that $y \succeq x$ but not $x \succeq y$, as usual). Moreover, we will assume that $\succeq$ is representable by a utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$, i.e. a function such that, for all $x, y \in \mathbb{R}^L_+$, $u(x) \geq u(y)$ if and only if $x \succeq y$. The consumer's utility maximization problem is the problem of maximizing his utility, choosing a consumption bundle belonging to the consumption set, subject to a budget constraint

$$\begin{aligned} \max_{x \in \mathbb{R}^L_+} \quad & u(x) \\ \text{s.t.} \quad & \langle x, p \rangle \leq 1. \end{aligned} \tag{7.6}$$

The optimal solution mapping of the previous problem, that is to say the operator $X : \mathbb{R}^L_{++} \rightrightarrows \mathbb{R}^L_+$ which associates to each $p \in \mathbb{R}^L_{++}$ the set of elements $x \in \mathbb{R}^L_+$ that are solutions of problem (7.6), is called the (*Walrasian*, or *market*) *demand correspondence*, while the optimal value function $v : \mathbb{R}^L_{++} \to \mathbb{R} \cup \{+\infty\}$ is called the *indirect utility function*.

In this connection, the notion of monotonicity allows a formal treatment of the law of demand. Indeed, if the operator $-X$ is monotone, then, for all $p, p' \in \mathbb{R}^L_{++}$ and for all $x \in X(p), x' \in X(p')$,

$$\langle p - p', x - x' \rangle \leq 0. \tag{7.7}$$

In particular, if $-X$ is a strictly monotone operator[1], then the previous inequality is strict whenever $p \neq p'$, that is, the *uncompensated law of demand* [64, Definition 4.C.2] holds, which is an important property in the study of the aggregate demand function.

Concerning the monotonicity of $-X$, it has by now become a classical result the following theorem of Mitjushin and Polterovich [67], published in Russian in 1978.

**Theorem 7.2.8 ([41, Theorem 6.24])** *Suppose that consumer's preferences are represented by a utility function $u$ of class $C^2$ and such that:*

*(a)* $\nabla u(x) \in \mathbb{R}^L_{++}$ *for all* $x \in \mathbb{R}^L_{++}$*;*

*(b)* $u$ *is concave;*

*(c)* $u$ *induces a demand function (i.e., $X$ is single-valued) of class $C^1$.*

*Suppose in addition that*

$$-\frac{\langle x, \nabla^2 u(x) x \rangle}{\langle x, \nabla u(x) \rangle} < 4, \tag{7.8}$$

*for all* $x \in \mathbb{R}^L_{++}$*. Then $-X$ is strictly monotone.*

Another result concerning monotonicity was obtained a few years before by Milleron [65] and essentially states that, if the utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ is concave and has no maximum, while its associated indirect utility function $v : \mathbb{R}^L_+ \to \mathbb{R} \cup \{+\infty\}$ is convex, then the negative of the demand function, $-X$, is monotone (see [41, Corollary 6.12]).

This result can be obtained as a corollary of the following theorem, the statement of which can be found in [41, Theorem 6.34], while the proof will appear in a forthcoming paper by Martínez-Legaz and Quah. Notice that in this case no utility representation is required for the

[1] An operator $T : X \rightrightarrows X^*$ is strictly monotone if $\langle x - y, x^* - y^* \rangle > 0$ for all $(x, x^*), (y, y^*) \in \mathcal{G}(T)$ with $x \neq y$.

preference relation. A demand correspondence can be defined also in this case, associating to each $p \in \mathbb{R}^L_{++}$ the set

$$X(p) := \{x \in B(p) : \quad x \succeq y, \quad \forall y \in B(p)\},$$

where $B(p) := \{y \in \mathbb{R}^L_+ : \ \langle y, p\rangle \leq 1\}$.

**Theorem 7.2.9 ([41, Theorem 6.34])** *Let $\succeq$ be a reflexive, transitive, complete and locally nonsatiated preference relation on $\mathbb{R}^L_+$ and let $X$ be its associated demand correspondence. If the set*

$$C := \{(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_+ : \quad x \succeq y, \quad \forall y \in B(p)\}$$

*is convex, then $-X$ is monotone.*

*Proof.* Let $p, p' \in \mathbb{R}^L_{++}$ and $x \in X(p), x' \in X(p')$. Then $(p, x), (p', x') \in C$ and, by hypothesis,

$$\left(\frac{1}{2}(p + p'), \frac{1}{2}(x + x')\right) = \frac{1}{2}(p, x) + \frac{1}{2}(p', x') \in C$$

as well. As a consequence,

$$\frac{1}{2}(x + x') \succeq y, \quad \text{for all } y \text{ such that } \left\langle y, \frac{1}{2}(p + p')\right\rangle \leq 1. \tag{7.9}$$

Now, suppose by contradiction that

$$\left\langle \frac{1}{2}(x + x'), \frac{1}{2}(p + p')\right\rangle < 1.$$

Then there would be $x_0 \in \left\{y \in \mathbb{R}^L_+ : \ \langle y, \frac{1}{2}(p + p')\rangle < 1\right\}$ such that $x_0 \succ \frac{1}{2}(x+x')$, a contradiction to (7.9). Therefore,

$$\left\langle \frac{1}{2}(x + x'), \frac{1}{2}(p + p')\right\rangle \geq 1,$$

from which, taking into account that

$$1 = \frac{1}{2}(\langle x, p\rangle + \langle x', p'\rangle),$$

we obtain $\langle x - x', p - p'\rangle \leq 0$. $\square$

As we anticipated, the result of Milleron [65] can be obtained as a consequence of the previous theorem. Indeed, if $\succeq$ can be represented by a utility function $u$, the set $C$ can be rewritten as

$$C = \{(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_+ : \quad u(x) \geq u(y), \quad \forall y \in B(p)\},$$

implying

$$C = \left\{(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_+ : \quad u(x) \geq \sup\{u(y) : \ y \in B(p)\}\right\}.$$

It then follows from the definition of the indirect utility function that

$$C = \{(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{+} : \quad u(x) - v(p) \geq 0\}.$$

The hypotheses of Milleron's theorem imply that the function $(p, x) \mapsto u(x) - v(p)$ is concave, so that $C$ is convex and Theorem 7.2.9 can be applied (since $u$ has no maximum, $\succeq$ is locally nonsatiated), yielding the monotonicity of $-X$.

Future research in this area could follow two main directions. On the one hand, convex representations could be introduced to study $-X$ and, hence, the demand correspondence $X$. In particular, rather than looking for exact representability, it would be advisable to find conditions under which the family $\mathcal{H}_{-X}$ is nonempty. Indeed, the existence of a lower semicontinuous convex function majorizing the duality product and being equal to it on the graph of $-X$, though possibly not only there, would imply that $-X$ is monotone. Therefore, on these lines one could obtain new conditions for the monotonicity of the demand function.

On the other hand, when preferences are locally nonsatiated, the negative $-X$ of the demand correspondence is cyclically quasimonotone [41, Theorem 6.22], i.e., for all $p_k \in \mathbb{R}^L_{++}$, $x_k \in X(p_k)$, $k = 1, \ldots, n$ $(n \geq 2)$, one has

$$\min_{k=1,\ldots,n} \langle x_k - x_{k+1}, p_k \rangle \leq 0,$$

with $x_{k+1} = x_1$. In this connection, a natural extension of the current research on convex representations of monotone operators would consist of obtaining analogous representations for quasimonotone operators, for instance using notions from generalized convexity.

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