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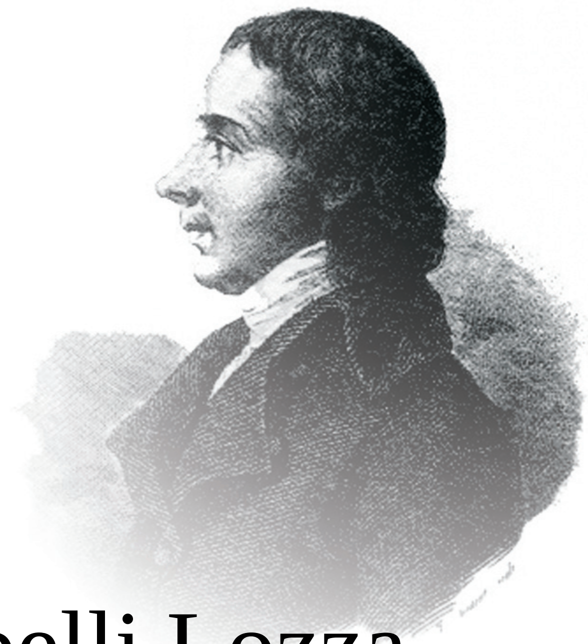

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Discrete Time Portfolio Selection with Lévy Processes

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Abstract. This paper analyzes discrete time portfolio selection models with Lévy processes. We first implement portfolio models under the hypotheses the vector of log-returns follow or a multivariate Variance Gamma model or a Multivariate Normal Inverse Gaussian model or a Brownian Motion. In particular, we propose an ex-ante and an ex-post empirical comparisons by the point of view of different investors. Thus, we compare portfolio strategies considering different term structure scenarios and different distributional assumptions when unlimited short sales are allowed.

Keywords: Subordinated Lévy models, term structure, expected utility, portfolio strategies.

1 Introduction

In this paper, we model the returns as a multidimensional time-changed Brownian motion where the subordinator follows or an Inverse Gaussian process or a Gamma process. Under these different distributional hypotheses we compare the portfolio strategies with the assumption that the log-returns follow a Brownian motion.
The literature in the multi-period portfolio selection has been dominated by the results of maximizing expected utility functions of terminal wealth and/or multi-period consumption. Differently from classic multi-period portfolio selection approaches, we consider mean-variance analysis alternative to that proposed by Li and Ng's (2000) by giving a mean-dispersion formulation of the optimal dynamic strategies. Moreover we also discuss a mean, variance, skewness and kurtosis extension of the original multi-period portfolio selection problem. These alternative multi-period approaches analyze portfolio selection, taking into consideration the admissible optimal portfolio choices when the log-returns follow a Lévy process. This analysis differs from other studies that assume Lévy processes with very heavy tails (see Rachev and Mittnik (2000), Ortobelli et al. (2004)), since we consider Lévy processes with semi-heavy tails. In order to compare the dynamic strategies under the different distributional assumptions, we analyze two investment allocation problems. The primary contribution of this empirical comparison is the analysis of the impact of distributional assumptions and different term structures on the multi-period asset allocation decisions. Thus, we propose a performance comparison among different multi-period mean-variance approach based on different Lévy processes and taking into consideration three different implicit term structures. For this purpose we discuss

the optimal allocation obtained by different risk averse investors with different risk aversion coefficients. We determine the multi-period optimal choices given by the minimization of the variance for different levels of final wealth average. Each investor, characterized by his/her utility function, will prefer the mean-variance model which maximizes his/her expected utility on the efficient frontier. Thus the portfolio policies obtained with this methodology represent the optimal investors' choices of the different approaches.

In Section 2, we introduce dynamic portfolio selection under the different multivariate distributional hypotheses. In Section 3 we propose a comparison of optimal portfolio strategies. Section 4, briefly summarizes the results.

2. Discrete time portfolio selection with subordinated Lévy processes

In this section we deal the dynamic portfolio selection problem among N+1 assets: N are risky assets and the (N+1)-th is risk free. We introduce portfolio selection models based on different assumptions of log-return distributions. In particular, we consider the Normal Inverse Gaussian process (NIG) and the Variance-Gamma one (VG) which are Lévy processes with semi heavy tails as suggested by Staino et al. (2007). These processes can be seen as subordinated Lévy processes where the subordinators are respectively the Inverse Gaussian process and the Gamma process.

Suppose that in the market the vector of risky assets has log-returns $\mathbf{X}_t = [X_t^{(1)},...,X_t^{(N)}]'$ distributed as:

$$\mathbf{X}_t = \mathbf{s}t + \boldsymbol{\mu} Z_t + \mathbf{Q}^{1/2} W_{Z_t}^{(N)}, \tag{1}$$

where $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_N]'$, $\mathbf{s} = [s_1, s_2, \ldots, s_N]'$, Z_t is the positive Lévy process subordinator, $\mathbf{Q} = \left[\sigma_{ij}^2\right]$ is a fixed definite positive variance-covariance matrix (i.e., $\sigma_{ij}^2 = \sigma_{ii}\sigma_{jj}\rho_{ij}$ where ρ_{ij} is the correlation between the conditional i-th component of $\mathbf{X}_t / Z_t$ and its conditional j-th component) and $W_t^{(N)}$ is a N-dimensional standard Brownian motion (i.e., $\mathbf{Q}^{1/2} W_{Z_t}^{(N)} = \sqrt{Z_t}\mathbf{Q}^{1/2}\mathbf{Y}$ where $\mathbf{Y}$ is a standard N-dimensional Gaussian independent of Z_t). Under the above distributional hypotheses we approximate the log-return of the portfolio with the portfolio of log-returns, that is the convex combination of the log-returns:

$$X_t^{(w)} = \mathbf{w}\mathbf{X}_t = \mathbf{w}\mathbf{s}t + (\mathbf{w}\boldsymbol{\mu})Z_t + \sqrt{\mathbf{w}\mathbf{Q}\mathbf{w}'} W_{Z_t}, \tag{2}$$

where $\mathbf{w} = [w_1, \ldots, w_N]$ is the vector of the weights invested in the risky assets, W_t is a 1-dimensional standard Brownian motion. At this point we will assume, that the subordinator Z_t is modeled either as an Inverse Gaussian process $Z_1 \sim \mathbf{IG}(1,b)$ or as a Gamma process $Z_1 \sim \mathbf{Gamma}\left(\frac{1}{\nu},\frac{1}{\nu}\right)$. An Inverse Gaussian process $X^{(IG)} = \left\{X_t^{(IG)}\right\}_{t\geq 0}$ assumes that any random variable $X_t^{(IG)}$ admits the following density function:

$$f_{IG}(x;ta,b)=\frac{ta}{\sqrt{2\pi}}\exp(tab)x^{-3/2}\exp\left(-\frac{1}{2}(t^2a^2x^{-1}+b^2x)\right)1_{[x>0]},$$

that is defined as Inverse Gaussian distribution $IG(ta,b)$ where *a, b* are positive. A Gamma process $X^{(Gamma)}=\left\{X_t^{(Gamma)}\right\}_{t\geq 0}$ assumes that any random variable $X_t^{(Gamma)}$ admits the following density function:

$$f_{Gamma}(x;t\,a,b)=\frac{b^{ta}}{\Gamma(ta)}x^{ta-1}\exp(-xb)1_{[x>0]},$$

that is defined as Gamma distribution $Gamma(ta,b)$ where *a, b* are positive.

NIG Processes. When Z_t follows an Inverse Gaussian process (i.e., $Z_1\sim\mathbf{IG}(1,b)$), then, the *i*-th log-return at time t=1 follows a NIG process $\mathbf{NIG}(\alpha_i,\beta_i,\delta_i,s_i)$ where the parameters are given by: $\alpha_i=\sqrt{(b/\delta_i)^2+\beta_i^2}$, $\beta_i=\mu_i/\delta_i^2\in(-\alpha_i,\alpha_i)$, $s_i\in R$ and $\delta_i=\sigma_{ii}>0$. Thus, the portfolio (2) follows a $\mathbf{NIG}(\alpha_w,\beta_w,\delta_w,s_w)$ process whose parameters are:

$$s_w=\mathbf{ws},\ \alpha_w=\sqrt{\left(\tfrac{b}{\delta_w}\right)^2+\beta_w^2},\quad \beta_w=\tfrac{\mathbf{w\mu}}{\mathbf{wQw'}},\quad \delta_w=\sqrt{\mathbf{wQw'}}\ .$$

In order to estimate all these parameters, we follow the MLE procedure suggested by Staino et al. (2007).

Variance-Gamma Processes. When Z_t follows a Gamma process (i.e., at time t=1 $Z_1\sim\mathbf{Gamma}\left(\frac{1}{v},\frac{1}{v}\right)$), then the log-return of the *i*-th asset at time t=1 follows a Variance-Gamma process with parameters $s_i,\mu_i\in R$, and $v,\sigma_{ii}>0$ (i.e., $X_1^{(i)}\sim\mathbf{VG}(\mu_i,\sigma_i,v,s_i)$). Analogously, the portfolio (2) follows a Variance-Gamma process with parameters $s_w=\mathbf{ws}$, $\mu_w=\mathbf{w\mu}$, $\sigma_w=\sqrt{\mathbf{wQw'}}$ and v. As for the NIG process we estimate all these parameters following the MLE procedure suggested by Staino et al. (2007). In portfolio theory, it has been widely used a Brownian Motion to model the vector of log-returns distribution. Under this assumption the portfolio follows a Brownian Motion $X^{(BM)w}=\left\{X_t^{(BM)w}\right\}_{t\geq 0}$ process, that is the portfolio of log returns at time *t* is normal distributed with mean $\mathbf{w\mu}t$ and standard deviation $\sqrt{t\mathbf{wQw'}}$. In the next subsection we describe the portfolio selection problem under the different distributional assumptions.

2.1 The discrete time portfolio selection problem

Suppose an investor has a temporal horizon t_T and he recalibrates its portfolio *T* times at some intermediate date, say $t=t_0,\ldots,t_{T-1}$ (where $t_0=0$). Since Lévy processes have independent and stationary increments the distribution of the random vector of log-returns on the period $(t_j,t_{j+1}]$ (i.e., $\mathbf{X}_{t_{j+1}}-\mathbf{X}_{t_j}$) is the same of $\mathbf{X}_{t_{j+1}-t_j}=[X_{t_{j+1}-t_j}^{(1)},...,X_{t_{j+1}-t_j}^{(N)}]'$. Considering that log-returns represent a good approximation of returns when $t_{j+1}-t_j$ is little enough, we assume that $\max_{j=0,\dots,T-1}\left(t_{j+1}-t_j\right)$ is less or equal than one month and we use $\mathbf{Y}_{t_j} := \mathbf{X}_{t_{j+1}} - \mathbf{X}_{t_j} = [Y_{1,t_j},...,Y_{N,t_j}]'$ to estimate the vector of returns on the period $(t_j, t_{j+1}]$. Suppose the deterministic variable r_{0,t_j} represents the return on the period $(t_j, t_{j+1}]$ of the risky-free asset, x_{i,t_j} the amount invested at time t_j in the *i*-th risky asset, and x_{0,t_j} the amount invested at time t_j in the risky-free asset. Then the investor's wealth at time t_{k+1} is given by:

$$\mathrm{W}_{t_{k+1}} = \sum_{i=0}^{N} x_{i,t_k}(1+Y_{i,t_k}) = (1+r_{0,t_k})\mathrm{W}_{t_k} + \mathbf{x}_{t_k}\mathbf{P}_{t_k}, \tag{3}$$

where $\mathbf{x}_{t_k} = [x_{1,t_k},\ldots,x_{N,t_k}]$, $\mathbf{P}_{t_k} = [P_{1,t_k},\ldots,P_{N,t_k}]'$ is the vector of excess returns $P_{i,t_k} = Y_{i,t_k} - r_{0,t_k}$. Thus, the final wealth is given by:

$$\mathrm{W}_{t_T} = \mathrm{W}_0 \prod_{k=0}^{T-1}(1+r_{0,t_k}) + \sum_{i=0}^{T-2}\mathbf{x}_{t_i}\mathbf{P}_{t_i}\prod_{k=i+1}^{T-1}(1+r_{0,t_k}) + \mathbf{x}_{t_{T-1}}\mathbf{P}_{T-1}, \tag{4}$$

where the initial wealth $\mathrm{W}_0 = \sum_{i=0}^{N} x_{i,0}$ is known. Assume that the amounts $\mathbf{x}_{t_j} = [x_{1,t_j},\ldots,x_{N,t_j}]$ are deterministic variables, whilst the amount invested in the risky-free asset is the random variable $x_{0,t_j} = W_{t_j} - \mathbf{x}_{t_j}\mathbf{e}$, where $\mathbf{e} = [1,\ldots,1]'$. Under these assumptions the mean, the variance the skewness and kurtosis of the final wealth are respectively

$$E(\mathrm{W}_{t_T}) = \mathrm{W}_0 B_0 + \sum_{i=0}^{T-1} E(\mathbf{x}_{t_i}\mathbf{P}_{t_i})B_{i+1} \tag{5a}$$

$$\mathtt{Variance}\left(W_T\right) = \sigma^2(W_{t_T}) = \sum_{i=0}^{T-1}\left(\mathbf{x}_{\mathbf{t_i}}\mathbf{Q}_{\mathbf{t_i}}\mathbf{x'}_{\mathbf{t_i}}\right)B_{i+1}^2 \tag{5b}$$

$$\mathtt{Sk}\left(\mathrm{W}_{t_T}\right) = \frac{\sum_{i=0}^{T-1} E\left((\mathbf{x}_{t_i}\mathbf{P}_{t_i} - E(\mathbf{x}_{t_i}\mathbf{P}_{t_i}))^3\right)B_{i+1}^3}{\sigma^3(W_{t_T})} \tag{5c}$$

$$\mathtt{Ku}\left(\mathrm{W}_{t_T}\right) = \frac{6\sum_{i=0}^{T-1}\sum_{j=i+1}^{T-1} B_{j+1}^2 B_{i+1}^2 \mathbf{x}_{\mathbf{t_i}}\mathbf{Q}_{\mathbf{t_i}}\mathbf{x'}_{\mathbf{t_i}}\left(\mathbf{x}_{t_j}\mathbf{Q}_{t_j}\mathbf{x'}_{t_j}\right) + \sum_{i=0}^{T-1} E\left((\mathbf{x}_{t_i}\mathbf{P}_{t_i} - E(\mathbf{x}_{t_i}\mathbf{P}_{t_i}))^4\right)B_{i+1}^4}{\sigma^4(W_{t_T})} \tag{5d}$$

where $B_T = 1$, $B_k = \prod_{j=k}^{T-1}(1+r_{0,t_j})$. Therefore, if we want to select the optimal portfolio strategies that solve the mean-variance problem:

$$\begin{cases} \min\limits_{\mathbf{x}_{t_0},\ldots,\mathbf{x}_{t_{T-1}}} \mathtt{Variance}[\mathrm{W}_{t_T}] \\ \mathtt{s.t.} \\ E[\mathrm{W}_{t_T}] = m \end{cases},$$

we can use the closed form solutions determined by Ortobelli et al. (2004). These solutions for Lévy subordinated processes are given by:

$$\mathbf{x}'_{t_k} = \frac{m - W_0 B_0}{B_{k+1}\sum_{j=0}^{T-1} E(\mathbf{P}_{t_j})'\mathbf{Q}_{t_j}^{-1} E(\mathbf{P}_{t_j})} \mathbf{Q}_{t_k}^{-1} E(\mathbf{P}_{t_k}) \quad k = 0,\dots,T-1 \tag{6}$$

where the components of the matrix $\mathbf{Q}_{t_k} = [q_{ij,t_k}]$, ($k$=0,…,$T$-1), are $q_{ij,t_k} = \mathbf{Cov}(X^{(i)}_{t_{k+1}-t_k}, X^{(j)}_{t_{k+1}-t_k})$. The optimal wealth invested in the riskless asset at time $t_0 = 0$ is the deterministic quantity $x_{0,0} = W_0 - \mathbf{x}_{t_0}\mathbf{e}$, while at time t_j it is given by the random variable $x_{0,t_j} = W_{t_j} - \mathbf{x}_{t_j}\mathbf{e}$, where W_{t_j} is formulated in equation (3). Observe that the covariance q_{ij,t_k} among components of the vector $\mathbf{X}_{t_{j+1}-t_j} = \mathbf{s}\left(t_{j+1} - t_j\right) + \boldsymbol{\mu} Z_{t_{j+1}-t_j} + \mathbf{Q}^{1/2} W^{(N)}_{Z_{t_{j+1}-t_j}}$ is given by

$$q_{ij,t_k} = \sigma_{ij}^2 E(Z_{t_{k+1}-t_k}) + \mu_i \mu_j \mathbf{Variance}[Z_{t_{k+1}-t_k}],$$

where $\sigma_{ij}^2 = \sigma_{ii}\sigma_{jj}\rho_{ij}$ are the components of matrix $\mathbf{Q} = \left[\sigma_{ij}^2\right]$ (see, among others, Cont and Tankov (2004)). So, for example, in the case the vector of log-returns $\mathbf{X}_t$ follows a NIG process we can rewrite the formulas (5) of mean, variance, skewness and kurtosis of final wealth:

$$E\left(\mathbf{x}_{t_i}\mathbf{P}_{t_i}\right) = \left(t_{i+1} - t_i\right)\left(b^{-1}\mathbf{x}_{t_i}\boldsymbol{\mu} + \mathbf{x}_{t_i}\mathbf{s}\right) - \mathbf{x}_{t_i}\mathbf{e} r_{0,t_i}$$

$$q_{ij,t_k} = \mathbf{Cov}(X^{(i)}_{t_{k+1}-t_k}, X^{(j)}_{t_{k+1}-t_k}) = \delta_i\delta_j\rho_{ij}E(I_{t_{k+1}-t_k}) +$$

$$+\beta_i\beta_j\delta_i^2\delta_j^2\mathbf{Variance}[I_{t_{k+1}-t_k}] = \frac{\delta_i\delta_j\rho_{ij}}{b}(t_{k+1}-t_k) + \frac{\beta_i\beta_j\delta_i^2\delta_j^2}{b^3}(t_{k+1}-t_k)$$

$$\mathbf{Sk}\left(W_{t_T}\right) = \frac{3\sum_{i=0}^{T-1} B_{i+1}^3\left(t_{i+1}-t_i\right)\mathbf{x}_{t_i}\boldsymbol{\mu}\left(b^2\mathbf{x}_{t_i}\mathbf{Q}_{t_i}\mathbf{x}'_{t_i} + \left(\mathbf{x}_{t_i}\boldsymbol{\mu}\right)^2\right)}{\sigma^3(W_{t_T})b^5}$$

$$\mathbf{Ku}\left(W_{t_T}\right) = \frac{6\sum_{i=0}^{T-1}\sum_{j=i+1}^{T-1} B_{j+1}^2 B_{i+1}^2 \mathbf{x}_{\mathbf{t_i}}\mathbf{Q}_{\mathbf{t_i}}\mathbf{x}'_{\mathbf{t_i}}\left(\mathbf{x}_{\mathbf{t}_j}\mathbf{Q}_{\mathbf{t}_j}\mathbf{x}'_{\mathbf{t}_j}\right)}{\sigma^4(W_{t_T})} +$$

$$+\frac{3\sum_{i=0}^{T-1} B_{i+1}^4\left(t_{i+1}-t_i\right)^2\left(b^2\mathbf{x}_{t_i}\mathbf{Q}_{t_i}\mathbf{x}'_{t_i} + \left(\mathbf{x}_{t_i}\boldsymbol{\mu}\right)^2\right)^2}{\sigma^4(W_{t_T})b^6} +$$

$$+\frac{3\sum_{i=0}^{T-1} B_{i+1}^4\left(t_{i+1}-t_i\right)\left(b^2\mathbf{x}_{t_i}\mathbf{Q}_{t_i}\mathbf{x}'_{t_i} + 5\left(\mathbf{x}_{t_i}\boldsymbol{\mu}\right)^2\right)\left(b^2\mathbf{x}_{t_i}\mathbf{Q}_{t_i}\mathbf{x}'_{t_i} + \left(\mathbf{x}_{t_i}\boldsymbol{\mu}\right)^2\right)}{\sigma^4(W_{t_T})b^7}$$

Instead, if $\mathbf{X}_t$ follows a Variance-Gamma process these formulas become:

$$E\left(\mathbf{x}_{t_i}\mathbf{P}_{t_i}\right) = \left(t_{i+1} - t_i\right)\left(\mathbf{x}_{t_i}\boldsymbol{\mu} + \mathbf{x}_{t_i}\mathbf{s}\right) - \mathbf{x}_{t_i}\mathbf{e} r_{0,t_i}$$

$$q_{ij,t_k} = \mathbf{Cov}(X^{(i)}_{t_{k+1}-t_k}, X^{(j)}_{t_{k+1}-t_k}) = \sigma_{ii}\sigma_{jj}\rho_{ij}E(Z_{t_{k+1}-t_k}) + \mu_i\mu_j \mathbf{Variance}[Z_{t_{k+1}-t_k}] =$$
$$= \sigma_{ii}\sigma_{jj}\rho_{ij}(t_{k+1}-t_k) + v\mu_i\mu_j(t_{k+1}-t_k).$$

$$\mathbf{Sk}\left(W_{t_T}\right) = \frac{\sum_{i=0}^{T-1}\left(\left(t_{i+1}-t_i\right)v\mathbf{x}_{t_i}\boldsymbol{\mu}\left(3\mathbf{x}_{t_i}\mathbf{Q}_{t_i}\mathbf{x}'_{t_i} + 2v\left(\mathbf{x}_{t_i}\boldsymbol{\mu}\right)^2\right)\right)B^3_{i+1}}{\sigma^3(W_{t_T})},$$

$$\mathbf{Ku}\left(W_{t_T}\right) = \frac{6\sum_{i=0}^{T-1}\sum_{j=i+1}^{T-1}B^2_{j+1}B^2_{i+1}\mathbf{x}_{t_i}\mathbf{Q}_{t_i}\mathbf{x}'_{t_i}\left(\mathbf{x}_{t_j}\mathbf{Q}_{t_j}\mathbf{x}'_{t_j}\right)}{\sigma^4(W_{t_T})} - \frac{\sum_{i=0}^{T-1}3B^4_{i+1}\left(v\left(\mathbf{x}_{t_i}\mathbf{Q}_{t_i}\mathbf{x}'_{t_i}\right)^2\left(t_{i+1}-t_i\right)\right)}{\sigma^4(W_{t_T})} +$$
$$+ \frac{\sum_{i=0}^{T-1}3B^4_{i+1}\left(\left(1+2v/\left(t_{i+1}-t_i\right)\right)\left(\mathbf{x}_{t_i}\mathbf{Q}_{t_i}\mathbf{x}'_{t_i}\left(t_{i+1}-t_i\right) + v\left(\mathbf{x}_{t_i}\boldsymbol{\mu}\right)^2\left(t_{i+1}-t_i\right)\right)^2\right)}{\sigma^4(W_{t_T})}.$$

Clearly, a more realistic portfolio selection problem should consider the investor's preference for skewness (see, among others, Ortobelli (2001)). Thus under the above distributional assumptions and under institutional restrictions of the market (such as no short sales and limited liability), all risk-averse investors optimize their portfolio choosing the solutions of the following constrained optimization problem:

$$\min_{\mathbf{x}_{t_0},\ldots,\mathbf{x}_{t_{T-1}}} \mathbf{Variance}[W_{t_T}]$$
$$\text{s.t.}$$
$$E[W_{t_T}] \geq m;\ \mathbf{Sk}\left(W_{t_T}\right) \geq q_1;\ \mathbf{Ku}\left(W_{t_T}\right) \leq q_2;$$
$$x_{i,t_j} \geq 0, i = 1,\ldots,N,\ j = 0,\ldots,T-1$$

for some mean m skewness q_1 and kurtosis q_2. This problem has not generally closed form solution. However using arguments similar to those proposed by Athayde, and Flôres (2004) based on a tensorial notation for the higher moments we can give an implicit analytical solution when unlimited short sales are allowed.

3. A comparison among Lévy dynamic strategies

In this section we examine the performances of Lévy processes approaches and we compare the Gaussian and Lévy non-Gaussian dynamic portfolio choice strategies when short sales are allowed. First, we analyze the optimal dynamic strategies during a period of five months, among the riskless return and 5 index-monthly returns from 04/10/1992 - 12/31/2005 (Down Jones Composite 65, Down Jones Industrials, Down Jones Utilities, S&P 500 and S&P 100). We start with a riskless of 0.3884% and we examine the different allocation considering three different implicit term structures. Table 1 describes the implicit term structures that we will use in this comparison.

Table 1. Term structures

	t_0	t_1	t_2	t_3	t_4
term 1	0.3884%	0.3984%	0.4084%	0.4184%	0.4284%
term 2	0.3884%	0.3884%	0.3884%	0.3884%	0.3884%
term 3	0.3884%	0.3784%	0.3684%	0.3584%	0.3484%

In particular, we approximate optimal solutions to the utility functional:

$$\max_{\{x_{i_j}\}_{j=0,1,\ldots,T-1}} E\left(1-\exp\left(-\frac{1}{a}W_T\right)\right) \tag{7}$$

where a (we use $a = 0.5, 1, 1.5, 2$) is an indicator of the risk tolerance and W_T is defined by formula (4). Secondly, we consider the utility function

$$u(x) = \begin{cases} cx - \frac{1}{c}x^2 & \text{if } x < \frac{c^2}{2} \\ \ln(x) + \left(\frac{c^3}{4} - \ln\left(\frac{c^2}{2}\right)\right) & \text{if } x \geq \frac{c^2}{2} \end{cases} \tag{8}$$

where for $x < c^2/2$ we have a quadratic utility function and for $x \geq c^2/2$ a logarithm utility function (we use $c = 1, 2, 3, 4, 5$). Thus, we are interested in finding optimal solutions to the functional

$$\max_{\{x_{i_j}\}_{j=0,1,\ldots,T-1}} E\left(u\left(W_T\right)\right). \tag{9}$$

Clearly we could obtain close form solutions to optimization problem (7) and (9) using arguments on the moments and on the Laplace transform. However, since we want to value the impact of different distributional assumptions in a mean-variance framework we will approximate formulas (7) and (9) using the historical observations of the final wealth valued for the optimal mean variance portfolios. In particular, we use the same algorithm proposed by Ortobelli et al. (2004) in order to compare the different models. Thus, first we consider the optimal strategies (6) for different levels of the mean. Then, we select the optimal portfolio strategies on the efficient frontiers which are solutions of problem (7) and (9) for different coefficients a and c. Therefore starting by an initial wealth $W_0 = 1$ we compute for every multi-period efficient frontier:

$$\max_{\{x_j\}_{j=0,1,\ldots,4}} \frac{1}{N}\sum_{i=1}^{N} u\left(W_5^{(i)}\right)$$

$$\text{subject to}$$

$$\{x_j\}_{j=0,1,\ldots,4} \text{ are optimal portfolio strategies (6)}$$

where $W_5^{(i)} = B_0 + \sum_{j=0}^{4} x_j p_j^{(i)} B_j$ is the $i-th$ observation of the final wealth and $p_t^{(i)} = [p_{1,t}^{(i)},..., p_{n,t}^{(i)}]'$ is the $i-th$ observation of the vector of excess returns $p_{k,t}^{(i)} = r_{k,t}^{(i)} - r_{0,t}$ relative to the $t-th$ period.

Table 2 Maximum expected utility $\max_{\{x_{t_j}\}_{j=0,1,...,T-1}} E\left(1\text{-exp}\left(-\frac{1}{a}W_T\right)\right)$ for $a = 0.5, 1, 1.5, 2$ under three different distributional hypotheses: BM, VG, and NIG and three different term structures

	TERM1			TERM2			TERM3		
	BM	VG	NIG	BM	VG	NIG	BM	VG	NIG
a=0.5	0.8727	**0.8731**	0.8728	0.8727	**0.8731**	0.8728	0.8726	**0.8732**	0.8730
a=1	0.6468	**0.6479**	0.6473	0.6471	**0.6485**	0.6479	0.6477	**0.6491**	0.6484
a=1.5	0.5037	**0.5053**	0.5045	0.5044	**0.5062**	0.5053	0.5052	**0.5073**	0.5063
a=2	0.4117	**0.4136**	0.4126	0.4127	**0.4148**	0.4137	0.4138	**0.4161**	0.4150

Table 3 Maximum expected utility $\max_{\{x_{t_j}\}_{j=0,1,...,T-1}} E\left(u\left(W_T\right)\right)$, where

$$u(x) = \begin{cases} cx - \frac{1}{c}x^2 & \text{if } x < \frac{c^2}{2} \\ \ln(x) + \left(\frac{c^3}{4} - \ln\left(\frac{c^2}{2}\right)\right) & \text{if } x \geq \frac{c^2}{2} \end{cases},$$

for $c = 1, 2, 3, 4, 5$ under three different distributional hypotheses: BM, VG, and NIG and three different term structures

	TERM1			TERM2			TERM3		
	BM	VG	NIG	BM	VG	NIG	BM	VG	NIG
c=1	0.9942	**0.9973**	0.9964	0.9983	**1.0025**	1.0012	1.0031	**1.0083**	1.0065
c=2	1.5396	**1.5422**	1.5407	1.5404	**1.5436**	1.5422	1.5419	**1.5453**	1.5436
c=3	2.8763	**2.8994**	2.8880	2.8910	**2.9168**	2.9043	2.9073	**2.9361**	2.9225
c=4	4.3106	**4.3799**	4.3454	4.3578	**4.4359**	4.3980	4.4106	**4.4974**	4.4565
c=5	5.9522	**6.1025**	6.0276	6.0572	**6.2264**	6.1443	6.1745	**6.3623**	6.2738

Finally, we obtain Tables 2 and 3 with the approximated maximum expected utility considering the three implicit term structures. In fact, we implicitly assume the approximation:

$$\frac{1}{N}\sum_{i=1}^{N} u\left(W_5^{(i)}\right) \approx E\left(u\left(W_5^{(i)}\right)\right).$$

Tables 2 and 3 show a superior performance of Lévy non Gaussian models with respect to the Gaussian one by the point of view of investors that maximize expected utility (7) and (9). In particular, the Variance Gamma model presents the best performance for different utility functions and term structures. Thus, from an ex-ante comparison among Variance Gamma, Normal Inverse Gaussian and Brownian motion models, investors characterized by the utility functions (7) and (8) should select portfolios assuming a Variance Gamma distribution.

The term structure determines the biggest differences in the portfolio weights of the same strategy and different periods. When the interest rates of the implicit term structure are growing (decreasing) we obtain that the investors are more (less) attracted to invest in the riskless in the sequent periods. Generally it does not exist a common factor between portfolio weights of different periods and of the same strategy. However, when we consider the flat term structure (2-nd term structure), the portfolio weights change over the time with the same capitalization factor.

Table 4 Ex-post final wealth obtained by the optimal strategies solutions of the problem $\max_{\{x_{t_j}\}_{j=0,1,\dots,T-1}} E\left(1-\exp\left(-\frac{1}{a}W_T\right)\right)$ for a=0.5, a=1, a=1.5, a=2 under three different distributional hypotheses: BM, VG, and NIG and three different term structures

	TERM1			TERM2			TERM3		
	BM	VG	NIG	BM	VG	NIG	BM	VG	NIG
a=0.5	1.0762	**1.0805**	1.0711	1.0716	**1.0755**	1.0670	1.0672	**1.0966**	1.0851
a=1	1.1319	**1.1404**	1.1470	1.1498	**1.1594**	1.1381	1.1401	**1.1487**	1.1295
a=1.5	1.1876	**1.2304**	1.1976	1.2019	**1.2154**	1.1856	1.1888	**1.2269**	1.1961
a=2	1.2433	**1.2903**	1.2482	1.2540	**1.2993**	1.2568	1.2617	**1.3050**	1.2627

Table 5 Ex-post final wealth obtained by the optimal strategies solutions of the problem $\max_{\{x_{t_j}\}_{j=0,1,\dots,T-1}} E\left(u\left(W_T\right)\right)$, where

$$u\left(x\right)=\begin{cases} cx-\frac{1}{c}x^2 & \text{if } x<\frac{c^2}{2} \\ \ln(x)+\left(\frac{c^3}{4}-\ln\left(\frac{c^2}{2}\right)\right) & \text{if } x\geq\frac{c^2}{2} \end{cases},$$

for $c=1,2,3,4,5$ under three different distributional hypotheses: BM, VG, and NIG and three different term structures

	TERM1			TERM2			TERM3		
	BM	VG	NIG	BM	VG	NIG	BM	VG	NIG
c=1	1.3269	**1.3503**	1.3241	1.3322	**1.3553**	1.3279	1.3347	**1.3571**	1.3294
c=2	1.1319	**1.1404**	1.1217	1.1237	**1.1594**	1.1381	1.1401	**1.1487**	1.1295
c=3	1.4383	**1.4702**	1.4252	1.4364	**1.4951**	1.4228	1.4320	**1.4873**	1.4404
c=4	1.8281	**1.9198**	1.8299	1.8532	**1.9428**	1.8261	1.8698	**1.9561**	1.8622
c=5	2.3573	**2.5194**	2.3358	2.3743	**2.5583**	2.3717	2.4049	**2.5812**	2.3951

In Table 4 and 5 we show the ex-post final wealth under the three term structures for the three distributional assumptions and for the two utility functions. The results confirm the better performance of the Variance Gamma approach with respect to the Normal Inverse Gaussian and Brownian motion ones. Moreover in this ex-post comparison we observe a better performance of the Brownian motion with respect to the NIG model.

4 Concluding remarks

This paper proposes an empirical comparison among three distributional hypotheses based on Lévy processes. We discuss the portfolio optimization problem by the point of view of investors that maximize either exponential utility functions or quadratic-logarithm utility functions. Therefore, we propose two models that take into account the heavier behavior of log-return distribution tails. The ex-ante empirical comparison shows a greater performance of two alternative subordinated Lévy processes. Instead, the ex-post comparison, even though it confirms the better behavior of the Variance Gamma model, shows a better performance of the Brownian motion model with respect to the Normal Inverse Gaussian one. However, several further empirical analysis should be necessary to validate the multi-period models here presented based on the mean, the variance of the final wealth maybe considering even its skewness and kurtosis as suggested by the first part of the paper.

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