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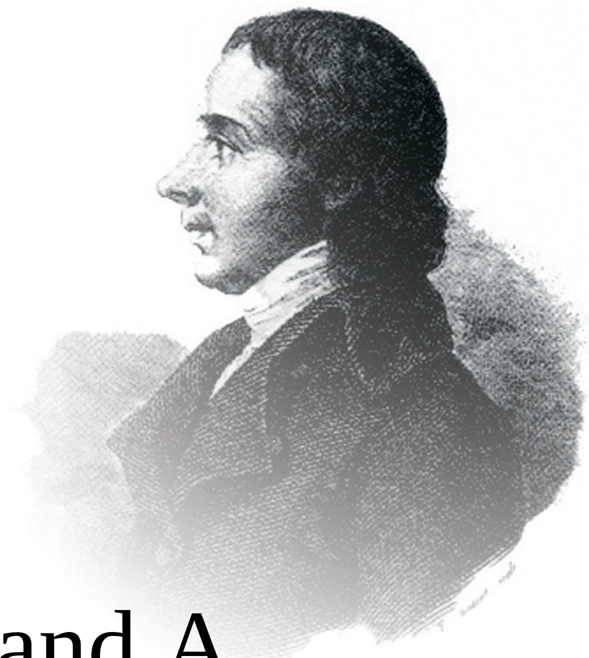

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Exotic Options with Lévy Processes: the Markovian Approach

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Exotic Options with Lévy Processes: the Markovian Approach

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Abstract: This paper proposes the markovian approach to price exotic options under Lévy processes. The markovian approach is simpler than the others proposed in literature for these processes and it allows to define hedging strategies. In particular, we consider three Lévy processes (Variance-Gamma, Meixner and Normal Inverse Gaussian) and we show how to compute American, barrier, compound and lookback option prices. We first discuss the use of an homogeneous Markov chain approximating the risk neutral log-return distribution. Then, we describe the methodology to price exotic contingent claims under the three different distributional assumptions and we compare the convergence results.
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1. Introduction

It is well known that log returns are not Gaussian distributed. As a matter of fact, several empirical investigations have shown that the log-returns present skew distributions with excess kurtosis and, for this reason, many alternative distributions have been proposed to characterize their distributional behavior. In particular Lévy processes have been widely used in the recent financial literature since they are a natural generalization of the Brownian motion, and they take into account the log-return skewness and kurtosis. Examples of such processes are the α Stable Lévy (see Mandelbrot and Taylor (1967) and Hurst, Platen and Rachev (1997)), the Tempered Stable (see Tweedie (1984)), the Normal Inverse Gaussian (see Barndorff-Nielsen (1995)), the Meixner (see Schoutens (2001)), the Variance Gamma (Madan and Seneta (1987, 1990)), the CGMY (see Carr *et al.* (2002)) and the Generalized Hyperbolic process (see, Eberlein *et al.* (1998), Prause (1999)).

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In this paper we face the problem to compute option prices assuming Lévy-exponential models. In this setting, we cannot always guarantee the classical predictable representation property that implies the completeness of the market. Thus, a more realistic market model based on a non-Brownian and a non-Poissonian Lévy process, will lead to incomplete markets. Clearly the complexity increases with respect to the usual framework of Black & Scholes model, since generally, there is not a unique risk neutral martingale measure and we have to solve a partial differential integral equation. European options, except for the few cases we know the risk neutral density distribution, can be priced using Fourier-based methods where we have to evaluate a Fourier transform numerically. These methods, like those due to Carr and Madan (1998) and Lewis (2001), can be easily applied using the FFT algorithm, but they cannot be easily used to price path dependent options. Generally, the methods proposed to price path dependent options when the log returns follow a Lévy process (such as those based on the Wiener-Hopf factorization) imply highly complex calculations with numerical integrations and numerical inversion methods, that do not always present stable results or that require long computational times, such as those based on Monte-Carlo simulations, (see, among others, Yor and Nguyen (2001), Boyarchenko and Levendorskii (2002), Schoutens (2003) and Cont and Tankov (2004) and the references therein).
In this paper we examine the markovian approach to price American, compound, barrier, and lookback options assuming Lévy-exponential models for the underlying. In particular, we compare option pricing results under the assumption the log return follows either a NIG process, or a Variance Gamma process or a Meixner process. The markovian approach to price contingent claims is as a logical extension of the well known binomial model (see Amin (1993)). The main idea is to exploit the possibility to build a sequence of Markov chains converging weakly to the Lévy process defined into the model. As suggested by Amin (1993), Krushner and Dupuis (2001), in many cases it is very simple to price contingent claims approximating the underlying markovian process with an homogeneous Markov chain. Moreover, with the markovian approach it is also possible to price contingent claims when the underlying follows a markovian non parametric process (see Iaquinta and Ortobelli (2006)). In the paper we obtain the arbitrage free prices, approximating the underlying risk neutral Markovian process. However, we can also obtain the absence of arbitrage by imposing some moment matching conditions that permit to estimate the risk neutral transition matrix (see Krushner and Dupuis (2001), Cont and Tankov (2003)). In particular, we adopt the methodology

proposed by Duan and Simonato (2001) and Duan et al. (2003). who have used the markovian approach to approximate Wiener processes and GARCH processes with Gaussian residuals in order to price American and barrier options. Therefore, the main contribution of this paper consists in showing the simplicity of the markovian approach to price exotic contingent claims when the underlying follows an exponential Lévy process. We first build a Markov chain that approximates the markovian behavior of three non Gaussian Lévy processes (Variance-Gamma, Meixner and Normal Inverse Gaussian) and the Brownian motion. This discretization process presents the same advantages of the binomial model since it permits us to price path dependent contingent claims. Then, we show the convergence of the compound option prices in the case analyzed by Geske (1979) for the Brownian motion and we extend the same analysis to the other three Lévy processes. For American and barrier options we just apply the markovian approach to Lévy processes as suggested by Duan et al (2003) for GARCH processes. For lookback options, instead, we show how to extend Cheuk and Vorst's algorithm in a Markov chain framework (see Babbs (2000) and Cheuk and Vorst (1997)). Recall that in the Black and Scholes framework there is an analytical pricing formula for lookback options derived by Goldman *et al.* (1979) and extended by Conze and Viswanathan (1991). However, discretizing the continuous markovian models we can approximate much better the right prices of lookback contracts, since in these contracts the maximum and/or the minimum of the underlying asset price are computed over some prespecified dates only, such as daily, weekly or monthly (see Cheuk and Vorst (1997)).

The paper is organized as follows. Section 2 introduces Lévy-exponential models and their risk neutral valuation. Section 3 discusses the markovian approach and shows some convergence results for American, European options and their Greeks when the log return follows either a NIG process, or a Variance Gamma process or a Meixner process. In Section 4 we deal with the compound, barrier, and lookback options. Finally we briefly summarize the results.

2. Lévy Processes and Risk Neutral Valuation

In this section we discuss the risk neutral valuation of Lévy processes when in the market there are two assets: a riskless asset with price process $B_t = \exp\left(\int_0^t r(s)ds\right)$, where the right continuous with left-hand limits time-dependent function $r(t)$ defines the short term interest rate, and a risky asset that pays

no dividends with price process $S_t = S_0 \exp(X_t)$. Thus, assume that log-return process $X = (X_t)_{t\geq 0}$ (i.e., $X_t = \log(S_t/S_0)$) is an adapted RCLL process defined on a filtered probability space $\left(\Omega, \Im, (\Im_t)_{0\leq t\leq\infty}, P\right)$, that satisfies the usual conditions. In particular, let us assume that the log-return process follows a Lévy process.
This assumption takes into account the skewness and the heavy tails often observed in the log-return distribution. As a matter of fact, Lévy processes are all the stochastic processes with stationary, independent increments and stochastically continuous sample paths. Since they have infinitely divisible distributions, their characteristic function $\phi(u)$ is univocally determined by the triplet $[\gamma, \sigma^2, \nu]$ that identifies the so called Lévy-Khintchine characteristic exponent $\psi(u) = \log \phi(u)$ given by:

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux1_{\{|x|<1\}})\nu(dx),$$

where $\gamma \in R$, $\sigma^2 > 0$ and ν is a measure on $R\backslash\{0\}$ with $\int_{-\infty}^{+\infty}(1 \wedge x^2)\nu(dx) < \infty$. In particular the Lévy triplet $[\gamma, \sigma^2, \nu]$ identifies the three main components of any Lévy process: the deterministic component (γ), the Brownian component (σ^2) and the pure jump component (ν). For further details on the theoretical aspects we refer to Sato (1999). Next, we consider three Lévy processes alternative to the Brownian motion that present skewness and semi heavy tails: the Normal Inverse Gaussian process (NIG), the Variance-Gamma process (VG) and the Meixner one.
Normal Inverse Gaussian: Under the assumption that the log return follows a NIG process $NIG(\alpha, \beta, \delta, q)$, with parameters $\alpha > 0$, $\beta \in (-\alpha, \alpha)$, $\delta > 0$, $q \in R$, we have that the characteristic function of the process at time t is given by:

$$\phi_{NIG}(u; \alpha, \beta, t\delta, tq) = \exp\left(-(t\delta)\left(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\right) + iutq\right).$$

That is the density of the log return at time t is given by:

$$\begin{aligned} f_{NIG}(x; \alpha, \beta, t\delta, tq) &= \frac{t\delta\alpha}{\pi} \exp\left(t\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - tq)\right) \times \\ &\times \frac{\mathbf{K}_1(\alpha\sqrt{(t\delta)^2 + (x - tq)^2})}{\sqrt{(t\delta)^2 + (x - tq)^2}} \end{aligned} \tag{1}$$

where $\mathbf{K}_\lambda(x)$ denotes the modified Bessel function of the third kind with index λ.
Variance Gamma: The Variance-Gamma process can be also defined as the

difference between two independent Gamma processes. Under the assumption that the log return follows a VG process $VG(\sigma,\nu,\theta,q)$ with parameters $\sigma>0$, $\nu>0$ and $q,\theta\in R$, the characteristic function of the process at time t is given by:

$$\phi_{VG}(u;\sigma\sqrt{t},\nu/t,t\theta,tq)=\left(1-iu\theta\nu+\frac{1}{2}\sigma^2\nu u^2\right)^{t/\nu}e^{iuqt}$$

That is the density of the log return at time t is given by:

$$\begin{aligned} f_{VG}(x;\sigma\sqrt{t},\nu/t,t\theta,qt) &= \frac{2e^{\frac{\theta(x-qt)}{\sigma^2}}\left(\frac{(x-qt)^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}}}{\nu^{t/\nu}\sqrt{2\pi}\sigma\Gamma(t/\nu)}\times \\ &\times\mathbf{K}_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(x-qt)^2(2\sigma^2/\nu+\theta^2)}\right) \end{aligned} \tag{2}$$

where $\mathbf{K}_{\frac{t}{\nu}-\frac{1}{2}}(x)$ is the modified Bessel function of the third kind with index $\dfrac{t}{v}-\dfrac{1}{2}$.
Meixner: Under the assumption that the log return follows a Meixner process $\mathbf{Meixner}(\alpha,\beta,\delta,q)$ with parameters $\alpha>0$, $\beta\in(-\pi,\pi)$, $\delta>0$, $m\in R$ the characteristic function of the process at time t is given by:

$$\phi_{\mathbf{Meixner}}(u;\alpha,\beta,\delta t,qt)=\left(\frac{\cos(\beta/2)}{\cosh((\alpha u-i\beta)/2)}\right)^{2\delta t}e^{iuqt}$$

That is the density of the log return at time t is given by:

$$\begin{aligned} f_{\mathbf{Meixner}}(x;\alpha,\beta,\delta t,qt) &= \frac{(2\cos(\beta/2))^{2\delta t}}{2\pi\Gamma(2\delta t)\alpha}\exp\left(\frac{\beta(x-qt)}{\alpha}\right)\times \\ &\times\left|\Gamma\left(\delta t+\frac{i(x-qt)}{\alpha}\right)\right|^2 \end{aligned} \tag{3}$$

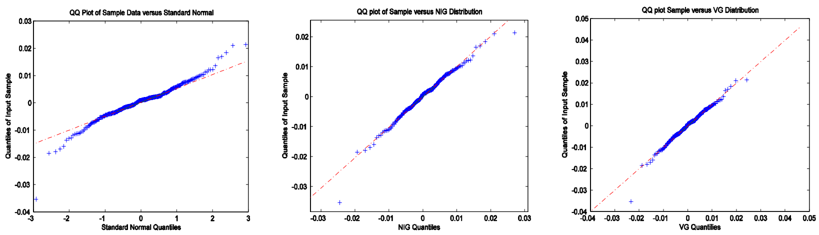
In order to value the best approximation of these distributions, we consider quotations of the index S&P500 from January 2006 to March 2007. Then we compute the parameters maximizing the likelihood function when the log-returns follow either a NIG process, or a Meixner process or a VG process (see Table 1). Finally we consider the Kolmogorov-Smirnov test

$$D=\sup_{x\in R}|F(x)-F_E(x)|,$$

Table 1. MLE of parameters and Kolmogorov-Smirnoff test of daily S&P500 log-returns assuming or a Normal Inverse Gaussian process, or a Variance-Gamma process or a Meixner process.

NIG	α=153.866	β=7.603	δ=1.562	q=-0.00029	D=0.0653
VG	θ=0.0756	σ=0.0984	ν=0.0024	q=0.00055	D=0.0667
Meixner	α=0.0146	β=0.1116	δ=94.676	q=-0.00026	D=0.0661

Figure 1. QQ plot among the sample and the Gaussian, NIG and VG distributions



where F_E is the empirical cumulative distribution and F the assumed distribution. Considering that the Brownian Motion hypothesis gives a value of the test $D = 0.0766$, then the other three distributional hypotheses present a better approximation. This empirical result is confirmed by the QQ-plot analysis of Figure 1.

Figure 1 reports a QQ-plot among the sample and the Gaussian, NIG and VG distributions (we get similar results with the Meixner distribution). Thus we can see how the empirical and theoretical distributions are closer on the whole real line when we use the NIG or VG distributions to model the log-returns. Under the assumption the log-return process follows a Lévy process whose trajectories are neither almost surely increasing nor almost surely decreasing we can always guarantee that there exists at least one equivalent martingale measure. Since the market is generally incomplete, then more than one equivalent martingale measure could exist. Given the risk neutral probability measure $\tilde{P}$, we can use it to determine the free-arbitrage price of any contingent claims with maturity T. That is, given the contingent claims function $\mathbf{H} : \Omega \to R$ ($\Im_T$-measurable function), then its price at time t is:

$$\Pi_t(\mathbf{H}) = \exp\left(-\int_t^T r(s)ds\right)\mathbf{E}^{\tilde{\mathbf{P}}}\left(\mathbf{H}|\Im_t\right). \tag{4}$$

There exist several techniques to determine a risk neutral martingale measure. A two steps methodology commonly used is:
1) determine a class of equivalent martingale measures;
2) determine the risk neutral measure, among the equivalent martingale measures, that minimizes a distance with respect to some historical contingent claim prices.

Typically, in order to determine the optimal parameters that better approximate the risk neutral distribution, we minimize the root mean squared prediction error (RMSE) with respect to the observed prices. Therefore, we consider N historical contingent claim prices cc_i (i=1,...,N) and we determine the risk neutral Lévy process parameters $\varpi \in \Theta$ that minimize

$$\text{RMSE}=\min_{\varpi\in\Theta}\sum_{i=1}^{N}\left(cc_i - \overline{Lp}_i(\varpi)\right)^2,$$

where $\overline{Lp}_i(\varpi)$ is the price of the i-th contingent claim obtained using the relation (4) under the equivalent martingale Lévy density with the parameters $\varpi \in \Theta$. Here in the following we briefly recall two classes of equivalent martingale measures used in Lévy processes literature: the mean-correcting equivalent martingale measure and the Esscher transform one.

Mean-correcting (see, among others, Schoutens (2003)): Mimicking the Black and Scholes model, the discounted price process $\tilde{S}_t = \exp\left(-\int_0^t r(s)ds\right) S_t$ becomes a martingale if we change the price process $S_t = S_0 \exp(X_t)$ with $S_t = S_0 \exp(\int_0^t \mu(s)ds + X_t)$, where $\mu(s) = q + r(s) - \log\phi(-i)$ and q is the translation parameter previously introduced for the three distributions. Therefore, we have to define a new equivalent probability measure $\tilde{P}$ on $(\Omega, \Im)$ under which the log-returns follow the Lévy process $\{\int_0^t \mu(s)ds + X_t\}$. In the three processes defined above, we have

$$\mu^{(\mathbf{NIG})}(s) = r(s) + \delta\left(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}\right), \tag{5a}$$

$$\mu^{(\mathbf{VG})}(s) = r(s) + \frac{1}{\nu}\log\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right), \tag{5b}$$

$$\mu^{(\mathbf{Meixner})}(s) = r(s) - 2\delta(\log(\cos(\beta/2)) - \log(\cos((\alpha+\beta)/2))). \tag{5c}$$

Esscher Transform (see Gerber, Shiu (1994, 1996)): Assume for simplicity that $r(t) = r$ constant. We observe that the discounted price process $\tilde{S}_t = S_0 \exp(-rt + X_t)$ becomes a martingale if we assume the new equivalent martingale density distribution

$$f_{\tilde{X}_t}(u) = \frac{f_{X_t}(u)\exp(\theta^* u)}{\int_{-\infty}^{+\infty} f_{X_t}(q)\exp(\theta^* q)dq}$$

where $\theta^*$is obtained as solution of the equation

$$\int_{-\infty}^{+\infty} f_{X_1}(u)\exp(\theta u)du = \int_{-\infty}^{+\infty} f_{X_1}(u)\exp\left((\theta+1)u - r\right)du.$$

That is, we define a new equivalent probability measure $\tilde{P}$ that has Radon-Nikodym derivative with respect to P given by $\dfrac{d\tilde{P}_{/\Im_t}}{dP_{/\Im_t}} = \dfrac{\exp(\theta X_t)}{E\left(\exp(\theta X_t)\right)}$. Then the three processes above admit the equivalent martingale density given by:

$$\begin{aligned} \tilde{f}_{NIG}(x;\alpha,\beta,t\delta,tq) &= \frac{t\delta\alpha}{\pi}\exp\Bigg(\delta(t-1)\sqrt{\alpha^2-\beta^2} + (\beta+\theta^*)(x-tq) + \\ &\quad +\delta\sqrt{\alpha^2-(\beta+\theta^*)^2}\Bigg)\frac{\mathbf{K}_1(\alpha\sqrt{(t\delta)^2+(x-tq)^2})}{\sqrt{(t\delta)^2+(x-tq)^2}}, \end{aligned}$$

$$\tilde{f}_{VG}(x;\sigma\sqrt{t},\nu/t,t\theta,tq) = \left(1+\theta^*\theta\nu - \frac{1}{2}\left(\sigma\theta^*\right)^2\nu\right)^{-1/\nu} \times$$

$$\times \frac{2e^{\frac{\left(\theta+\sigma^2\theta^*\right)(x-tq)}{\sigma^2}}\left(\frac{(x-tq)^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}}}{\nu^{t/\nu}\sqrt{2\pi}\sigma\Gamma(t/\nu)}\mathbf{K}_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(x-tq)^2(2\sigma^2/\nu+\theta^2)}\right),$$

$$\begin{aligned} \tilde{f}_{\mathbf{Meixner}}(x;\alpha,\beta,\delta t,tq) &= \left(\frac{\cos(\beta/2)}{\cosh(-i(\alpha\theta^*+\beta)/2)}\right)^{-2\delta}\frac{\left(2\cos\left(\beta/2\right)\right)^{2\delta t}}{2\pi\Gamma(2\delta t)\alpha} \times \\ &\quad \times \exp\left(\frac{\left(\beta+\alpha\theta^*\right)(x-tq)}{\alpha}\right)\left|\Gamma\left(\delta t + \frac{i(x-tq)}{\alpha}\right)\right|^2. \end{aligned}$$

3. Pricing and hedging American and European options with Lévy processes

In this section, we opportunely adapt to Lévy processes the markovian methodology proposed by Duan et al.(2003). Since Lévy processes are particular Markov processes we suggest to use an approximating Markov chain in order to price exotic options when the log return follows a Lévy process. This discretization process provides the same ductility of the binomial model and for this reason it is possible to price almost every path dependent contingent claim once we know the risk neutral distribution of the underlying Markov process.

3.1. The markovian approach

Assume the maturity of the contingent claim is T. Our task is to approximate, under the risk neutral probability P, the log price process $\{\ln(S_t)\}_{0\leq t\leq T}$ at times $\{0, \Delta t, 2\Delta t, \ldots, s\Delta t = T\}$ by a sequence of Markov chains $\{\tilde{Y}^{(m)}_{n\Delta t}, n = 0, 1, 2, \ldots, s\}_{m=2i+1, i\in N}$ with state space $\{p_1, p_2, \ldots, p_m\}$ and transition probability matrix $Q_{(m)} = [q_{ij}]_{1\leq i,j\leq m}$, where m is an odd integer and $p_{(m+1)/2} = \ln(S_0)$. In order to fix the ideas, we adopt the mean correcting risk neutral valuation considering the riskless rate $r(t) = r$ constant. Thus, we build a sequence of Markov chains $\{\tilde{Y}^{(m)}_{n\Delta t}, n = 0, 1, 2, \ldots, s\}_{m=2i+1, i\in N}$ with state space $\{p_1, p_2, \ldots, p_m\}$, converging weakly to the risk neutral Lévy process $\{\ln(S_0) + \mu t + X_t, \quad t = 0, \Delta t, 2\Delta t, \ldots, T\}$ (here $X = (X_t)_{t\geq 0}$ is the log-return process) as the state number m tends to infinite, where μ is defined (for the three processes introduced in the previous section) by formulas (5). Therefore, given the current price S_0, we define an interval centered in $\ln(S_0)$ such that the probability that $\ln(S_T) + \mu T$ belongs to the interval is almost equal to 1, i.e.,

$$P\left(\ln(S_T) + \mu T \in [\ln(S_0) - I(m), \ln(S_0) + I(m)]\right) \approx 1.$$

The m states of the Markov chain are defined as $p_i = \ln(S_0) + \frac{2i - m - 1}{m - 1} I(m)$, $i = 1, \ldots, m$. Note that $p_1 = \ln(S_0) - I(m)$, $p_m = \ln(S_0) + I(m)$ and $p_{(m+1)/2} = \ln(S_0)$. Fixed the m values p_i, we can always determine other m values starting by any other state $p^i_k = p_i + \frac{2k - m - 1}{m - 1} I(m)$. In particular, $p^i_k = p_j$ if and only if $k = j - i + \frac{m+1}{2}$, that is

$$p^i_k = p_i + \frac{2k - m - 1}{m - 1} I(m) = \ln(S_0) + \frac{2(i + k - \frac{m+1}{2}) - m - 1}{m - 1} I(m).$$

In order to get the convergence, we have to guarantee that $I(m) \to \infty$ and $I(m)/m \to 0$ as the number of the states converges to infinity ($m \to \infty$), see, among others, Pringent (2002). For example, when the Markov process $Y = \{\ln(S_t)\}_{0\le t\le T}$ admits finite mean (i.e., $E(|\ln(S_{\Delta t})|) < \infty$), we can use $I(m) = \max(|z_{1/m}|, |z_{1-1/m}|)$, where z_k is the $k\%$ quantile of $\ln(S_T) + \mu T$. Since $I(m) \to \infty$ and $I(m)/m \to 0$, we can guarantee the convergence of the Markov chain sequence. However, the speed of convergence is strictly linked to the choice of $I(m)$. Thus, we have to choose opportunely $I(m)$. Duan et al. (2003) suggest to use $I(m) = (2 + \ln(\ln(m)))\,\sigma\sqrt{T}$ for the Brownian Motion. When we assume the mean correcting risk neutral valuation for the three processes introduced in the previous section, we observe an higher speed of convergence using $I(m) = z + \dfrac{\log(\log(m))}{2}$,where with log we mean logarithm with base 10, $z = \max(|z_{0.01}|, |z_{0.99}|)$, $z_{0.01}$ and $z_{0.99}$ are respectively the 1% and 99% quantiles of the $\ln(S_T) + \mu T$ distribution. The transition probability between the i-th state and the k-th state is given by

$$q_{ik} = P\left(\ln(S_{\Delta t}) + \mu\Delta t \in (c_k^i, c_{k+1}^i]\right),$$

where $c_1^i = p_1^i - \dfrac{\log(\log(m))}{2}$, $c_k^i = (p_k^i + p_{k-1}^i)/2$, $k = 2, \ldots, m$ and $c_{m+1}^i = p_m^i + \dfrac{\log(\log(m))}{2}$. Then we deduce the convergence of the sequence of Markov chains $\{\tilde{Y}_{n\Delta t}^{(m)}, n = 0, 1, 2, \ldots, s\}_{m=2i+1, i\in N}$ with state space $\{p_1, p_2, \ldots, p_m\}$, to the risk neutral Lévy process $\{\mu t + \ln(S_t), t = 0, \Delta t, 2\Delta t, \ldots, T\}$ because

$$c_2^i - c_1^i = c_{m+1}^i - c_m^i = \frac{I(m)}{m-1} + \frac{\log(\log(m))}{2m} \to 0, \ \text{as } m \to \infty$$

and

$$c_{k+1}^i - c_k^i = 2\left(\frac{I(m)}{m-1}\right) \to 0, \ \text{as } m \to \infty, \ k = 2, \ldots, m-1.$$

Since $p_k^i = p_j$ if and only if $k = j - i + \frac{m+1}{2}$, then we have not to compute all the entries q_{ij} of the transition matrix $Q_{(m)}$. As a matter of fact, if we define $k(j) = j - i + \frac{m+1}{2}$, $j = 1, \ldots, m$, then the entries of the transition matrix $\mathrm{Q}_{(m)}$ are given by:
if $i < \frac{m+1}{2}$:

$$q_{ij} = \begin{cases} \sum_{k=1}^{1+\frac{m+1}{2}-i} \int_{c_k^i - p_i - \mu\Delta t}^{c_{k+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx \quad \text{if} \quad j = 1 \\ \int_{c_{k(j)}^i - p_i - \mu\Delta t}^{c_{k(j)+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx \text{ if } j = 2, \dots, i + \dfrac{m-1}{2} \\ 0 \text{ if } \quad j = i + \dfrac{m+1}{2}, \dots, m; \end{cases}$$

if $i > \frac{m+1}{2}$:

$$q_{ij} = \begin{cases} 0, \quad \text{if } j = 1, \dots, i - \dfrac{m+1}{2} \\ \int_{c_{k(j)}^i - p_i - \mu\Delta t}^{c_{k(j)+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx, \text{ if } j = i - \dfrac{m-1}{2}, \dots, m-1 \\ \sum_{k=m-i+\frac{m+1}{2}}^{m} \int_{c_k^i - p_i - \mu\Delta t}^{c_{k+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx, \text{ if } j = m; \end{cases}$$

if $i = \frac{m+1}{2}$:

$$q_{ij} = \int_{c_j^i - p_i - \mu\Delta t}^{c_{j+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx, \quad j = 1, \dots, m,$$

where $f_{X_{\Delta t}}(\cdot)$ is the density function of the log-return Lévy process. When m increases the intervals $(c_k^i, c_{k+1}^i]$ become so small that we can well approximate any integral with the area of only one rectangle, i.e.,

$$\int_{c_k^i - p_i - \mu\Delta t}^{c_{k+1}^i - p_i - \mu\Delta t} f_{X_{\Delta t}}(x)dx \approx f_{X_{\Delta t}}\left(\frac{c_k + c_{k+1}}{2}\right)(c_{k+1} - c_k).$$

3.2. Pricing of European contingent claims

When the maturity of an European contingent claim is T and we consider s steps (i.e., $s\Delta t = T$), then the price of the contingent claims is given by the $((m+1)/2)$-th component of the price vector

$$V(p, 0) = \left(Q_{(m)}\right)^s Z, \tag{6}$$

where Z is the m-dimensional vector of payoff at the maturity correspondent to the vector of log prices $p = [p_1, p_2, \dots, p_m]$.
So we can assume that the payoff vector is given by $Z = [g_{w,1}, ..., g_{w,m}]'$ where $g_{w,i} = \max\{w[\exp(p_i) - K], 0\}$, w is equal to 1 for a call and -1 for a put. Analogously, to the example reported by Duan and Simonato (2001) with the Black

Table 2. European put option prices under NIG, VG, and Meixner processes.

STATES	NIG process		VG process		Meixner process	
	Weekly	daily	weekly	daily	weekly	daily
m=101	1.7428	1.7984	1.6795	1.7489	1.7343	1.8022
m=501	1.7442	1.7442	1.6809	1.6852	1.7357	1.7357
m=1001	1.7442	1.7442	1.6810	1.6840	1.7358	1.7358
m=1501	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358
m=2001	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358
m=2501	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358
m=3001	1.7442	1.7442	1.6810	1.6810	1.7358	1.7358

and Scholes model, in Table 2 we show the convergence of this methodology under the three different distributional assumptions. In order to determine some prices which refer to the same underline stock process, for this table and all the following ones we use the mean correcting risk neutral measure applied to the parameters estimated in Table 1. Table 2 reports European put option prices at the money under NIG, VG, and Meixner processes on a stock price with current value $S_0 = 100$ euro, maturity $T = 0.5$ years, short interest rate $r = 5\%$ a.r.. Moreover, we consider that the temporal horizon is shared either in 24 periods or in 126 periods (i.e. Δt is equal respectively either to one week or to one day). In both cases we observe the convergence of the option prices when the number of the states m increases. The convergence price is the same we obtain approximating the integral that defines the risk neutral put option price $\exp(-rT)100\int_{-\infty}^{0}(1-e^x)f_{\tilde{X}_T}(x)dx$.

3.3. Pricing and hedging American contingent claims

Let us consider an American option with maturity T and strike price K. We assume that the contract may be exercised at times $\{0, \Delta t, 2\Delta t, \dots, s\Delta t\}$, where $T = s\Delta t$. Fixed the number of states m we build the vector of the state values $p = [p_1, p_2, \dots, p_m]$ of an approximating Markov chain $\{\tilde{Y}_{n\Delta t}^{(m)}, n = 0, 1, 2, \dots, s\}_{m=2i+1, i\in N}$, with risk neutral transition matrix $\mathrm{Q}_{(m)}$. Since the states remain the same for all the time steps, then at each time $\{0, \Delta t, 2\Delta t, \dots, s\Delta t\}$ there is an unique payoff vector

$$g_w(p, K) = [g_{w,1}, ..., g_{w,m}]'$$

where $g_{w,i} = \max\{w[\exp(p_i) - K], 0\}$, w is equal to 1 for a call and -1 for a put. For every couple of vectors $a = [a_1, ..., a_m]'$, $b = [b_1, ..., b_m]'$ we assume the vectorial notation $\max[a, b] := [\max(a_1, b_1), \max(a_2, b_2), ..., \max(a_m, b_m)]'$. Therefore, the price of the American option can be computed using the recursive vectorial formula:

$$\begin{gathered} V_w(p, T) = g_w(p, K), \\ V_w(p, t_i) = \max\left[g_w(p, K), e^{-r\Delta t} Q_{(m)} V_w(p, t_{i+1})\right], \\ i = 0, \ldots, s-1, \quad t_i = i\Delta t, \quad s\Delta t = T. \end{gathered} \tag{7}$$

The option price at time 0 is given by the $((m+1)/2)$-th element of $V_w(p, 0)$. When we price a contingent claim with the markovian approach we get the vector $V_w(p, 0)$ whose elements are option prices corresponding to discrete values of the stock price. Thus we can compute the Greeks in a way very similar to the finite-difference approach using the option prices adjacent to the $((m+1)/2)$-th element of $V_w(p, 0)$. However, as suggested by Duan *et al.*, in order to obtain higher quality Greeks it is advisable to have adjacent prices very close to the initial stock price. This approximation problem can be easily solved considering the states $p_{\frac{m+1}{2}} + \varepsilon$, and $p_{\frac{m+1}{2}} - \varepsilon$ in the Markov chain with ε opportunely small. In this way we can use the following approximation of delta and gamma values:

$$\Delta = \frac{\partial V_w}{\partial \ln S_0} \frac{1}{S_0} \approx \frac{V_w\left(p_{\frac{m+1}{2}} + \varepsilon, 0\right) - V_w\left(p_{\frac{m+1}{2}} - \varepsilon, 0\right)}{2\varepsilon} \frac{1}{S_0},$$

$$\begin{aligned} \Gamma &= \frac{\partial}{\partial S_0}\left(\frac{\partial V_w}{\partial \ln S_0} \frac{1}{S_0}\right) \approx \left(\frac{V_w\left(p_{\frac{m+1}{2}} - \varepsilon, 0\right) - V_w\left(p_{\frac{m+1}{2}} + \varepsilon, 0\right)}{2\varepsilon} + \right. \\ &\left. + \frac{V_w\left(p_{\frac{m+1}{2}} + \varepsilon, 0\right) + V_w\left(p_{\frac{m+1}{2}} - \varepsilon, 0\right) - 2V_w\left(p_{\frac{m+1}{2}}, 0\right)}{\varepsilon^2}\right) \frac{1}{S_0^2}. \end{aligned}$$

Consider American put options with exercise prices K=98 euro or K=102 euro under the assumption the log returns follow either a NIG, or a VG, or a Meixner process. We use the mean correcting risk neutral measure applied to the parameters estimated in Table 1 for puts on a stock price with current value $S_0 = 100$ euro, maturity $T = 0.5$ years, short interest rate $r = 5\%$ a.r.. In Table 3 we

Table 3. Delta, Gamma and American put option prices under NIG, VG, and Meixner processes.

	NIG process		VG process		Meixner process	
	K=98	**K=102**	**K=98**	**K=102**	**K=98**	**K=102**
m=501	1.2419	3.0101	1.2067	2.9527	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2914	-0.5739	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0572	0.0829	0.0561	0.0820
m=1001	1.2419	3.0101	1.1882	2.9529	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2881	-0.5732	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0571	0.0847	0.0561	0.0820
m=1501	1.2419	3.0101	1.1869	2.9509	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2879	-0.5732	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820
m=2001	1.2419	3.0101	1.1868	2.9507	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2879	-0.5732	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820
m=2501	1.2419	3.0101	1.1868	2.9508	1.2349	3.0025
delta	-0.2919	-0.5686	-0.2879	-0.5732	-0.2914	-0.5692
gamma	0.0560	0.0816	0.0571	0.0848	0.0561	0.0820

report the option prices and the values of delta and gamma when we assume $\varepsilon = 10^{-6}$. Even in this case we observe the convergence of these values for a number of states m greater than 500. We compare the results using Montecarlo simulations. However we observe that with Montecarlo simulations we need more than 10 millions simulations to get the same results we get with the Markovian approach approximated at 10^{-3}. Moreover, we observe that the results obtained with Montecarlo simulations are very unstable in comparison with those obtained from the Markovian procedure.

4. Compound, barrier, and lookback option prices with Lévy processes

In this section we propose to value exotic option prices assuming that a sequence of Markov chains $\{\tilde{Y}^{(m)}_{n\Delta t}, n = 0, 1, 2, \ldots, s\}_{m=2i+1, i\in N}$ describes the risk neutral behavior of $\ln(S_t)$ at times $\{0, \Delta t, 2\Delta t, \ldots, s\Delta t = T\}$. We compute compound, barrier, and lookback option prices under the three distributional assumptions. In particular, the methodology proposed is innovative for compound, and lookback options that have not been dealt by Duan and Simonato (2001) and Duan *et al.* (2003).

Compound options. Compound options are options written on options and can be of four types: a call on call, a put on call, a call on put, and a put on put. Consider a call on call. At the first maturity T_1 the compound option holder has the right to pay the first exercise price K_1 and get a call. Then, the call gives to the compound option holder the right to buy the underlying asset at the second maturity T_2 paying the second exercise price K_2. The markovian approach allows to price easily compound options. Using the recursive system to price an option with maturity $T_2 - T_1$ and exercise price K_2, we find a vector which represents the possible prices at time T_1 of the American (or European) option on which the first option is written. Denote this vector as

$$\tilde{V}_{w_1}(p, T_1) = [\tilde{V}_{w_1,1}, ..., \tilde{V}_{w_1,m}]'$$

where w_1 is equal to 1 for a call and -1 for a put. The payoff at time $T_1$of the compound option is given by the vector

$$V_{w_2}(p, T_1) = \max\{w_2[\tilde{V}_{w_1}(p, T_1) - K_1\overline{1}], \overline{0}\},$$

where $\overline{1}$and $\overline{0}$ are respectively vectors of ones and zeros, w_2 is equal to 1 for a call and -1 for a put. Thus, using again the recursive system with s steps (i.e., $s\Delta t = T_1$), the price at time 0 of an European option on an American (or European) option is given by the $((m+1)/2)$-th element of the vector $V_{w_2}(p,0) = e^{-rT_1}Q^s_{(m)}V_{w_2}(p,T_1)$.

Table 4 exhibits the prices of compound options obtained under Brownian motion, NIG, VG, and Meixner processes (considering different number of states m). In particular we compare the results we get under the Brownian Motion and those given by Geske′s closed formula (see Geske (1979)). These prices concern European calls on European calls, where the current asset price is $S = 100$, the first call has strike price K_1 and maturity $T_1 = 0.25$ years, and the second call has strike price K_2 and maturity $T_2 = 0.25$ years. We consider two possible strike prices K_1 (K_1=1.5, 2) and three possible strike prices K_2 (K_2=98,100,102). Moreover, the short interest rate is $r = 5\%$, the annual volatility of the Brownian motion is $\sigma = 10.14\%$, and the parameters of the NIG, Meixner and VG processes are always those ones of Table 1.

Barrier options Barrier options may be of two types, knock-out and knock-in. We proceed explaining how to use the markovian approach to price knock-out options and we refer to Duan et al. (2003) for knock-in options. An option is said knock-out when it becomes worthless if the underlying asset touches or crosses a constant barrier H at any monitoring time. The barrier can be lower or upper (i.e., H or H^*). A barrier option is double when there are two barriers and the underlying asset must remain between these two barriers at the monitoring days. Following Duan et al. (2003), we introduce an auxiliary variable a_t which takes the value 1 if the barrier condition is triggered before or at time t, and the value 0 otherwise. If we denote with $v(p_i, t; a_t)$ the option price at time t, for a knock-out option we have:

1) for every time

$$v_w(p_i, t_k; a_{t_k} = 1) = 0,$$

2) for $t_s = s\Delta t = T$,

$$v_w(p_i, T; a_T = 0) = \max\{w[\exp(p_i) - K], 0\},$$

3) $t_k = k\Delta t$, k=0,. . . ,s-1,

$$v_w(p_i, t_k; a_{t_k} = 0) = \max\left[g_w(p_i, K, a_{t_k} = 0),\ \ e^{-r\Delta t}\times \right.$$
$$\left. \times \sum_{j=1}^{m} \tilde{P}\left(X_{t_{k+1}} = p_j, a_{t_{k+1}} = 0 | X_{t_k} = p_i, a_{t_k} = 0\right) v(p_j, t_{k+1}; a_{t_{k+1}} = 0)\right],$$

where w is equal to 1 for a call and -1 for a put and

$$g_w(p_i, K, a_{t_k} = 0) = \begin{cases} \max\{w[\exp(p_i) - K], 0\} & \text{if American} \\ 0 & \text{if European.} \end{cases}$$

Table 4. Compound option prices under Brownian motion, NIG, VG, and Meixner processes. We consider European calls on European calls, where the current asset price is $S=100$, the first call has strike price K_1 and maturity $T_1=0.25$ years, and the second call has strike price K_2 and maturity $T_2=0.25$ years.

	Brownian motion				**Brownian motion**		
K_1=2	**K_2=98**	**K_2=100**	**K_2=102**	**K_1=1.5**	**K_2=98**	**K_2=100**	**K_2=102**
m=101	3.7530	2.5803	1.6764	**m=101**	4.1629	2.9332	1.9609
m=501	3.7540	2.5851	1.6747	**m=501**	4.1637	2.9381	1.9598
m=1001	3.7542	2.5851	1.6747	**m=1001**	4.1637	2.9385	1.9598
m=1501	3.7542	2.5852	1.6746	**m=1501**	4.1637	2.9386	1.9598
m=2001	3.7542	2.5852	1.6747	**m=2001**	4.1637	2.9386	1.9597
Geske	3.7542	2.5852	1.6747	**Geske**	4.1637	2.9386	1.9597
	NIG process				**NIG process**		
K_1=2	**K_2=98**	**K_2=100**	**K_2=102**	**K_1=1.5**	**K_2=98**	**K_2=100**	**K_2=102**
m=101	3.7380	2.5584	1.6607	**m=101**	4.1479	2.9127	1.9438
m=501	3.7360	2.5655	1.6577	**m=501**	4.1459	2.9189	1.9415
m=1001	3.7359	2.5660	1.6574	**m=1001**	4.1459	2.9190	1.9413
m=1501	3.7359	2.5660	1.6575	**m=1501**	4.1459	2.9191	1.9414
m=2001	3.7359	2.5660	1.6575	**m=2001**	4.1458	2.9191	1.9414
	Meixner process				**Meixner process**		
K_1=2	**K_2=98**	**K_2=100**	**K_2=102**	**K_1=1.5**	**K_2=98**	**K_2=100**	**K_2=102**
m=101	3.7304	2.5519	1.6552	**m=101**	4.1394	2.9065	1.9365
m=501	3.7289	2.5578	1.6494	**m=501**	4.1389	2.9107	1.9330
m=1001	3.7288	2.5580	1.6496	**m=1001**	4.1388	2.9108	1.9329
m=1501	3.7287	2.5580	1.6495	**m=1501**	4.1388	2.9110	1.9329
m=2001	3.7287	2.5580	1.6495	**m=2001**	4.1387	2.9110	1.9330
	VG process				**VG process**		
K_1=2	**K_2=98**	**K_2=100**	**K_2=102**	**K_1=1.5**	**K_2=98**	**K_2=100**	**K_2=102**
m=101	3.6634	2.4874	1.5795	**m=101**	4.0738	2.8397	1.8610
m=501	3.6800	2.5043	1.5965	**m=501**	4.0904	2.8564	1.8776
m=1001	3.6805	2.5048	1.5971	**m=1001**	4.0909	2.8570	1.8781
m=1501	3.6806	2.5049	1.5971	**m=1501**	4.0910	2.8571	1.8782
m=2001	3.6807	2.5050	1.5972	**m=2001**	4.0911	2.8571	1.8783

To compute the transition probability, we define the set of the states for which the option is knocked out and becomes worthless:

$$\Lambda = \begin{cases} \{i \in \{1,\ldots,m\} : \exp(p_i) \le H\} \text{ down-and-out option} \\ \{i \in \{1,\ldots,m\} : \exp(p_i) \ge H^*\} \text{ up-and-out option} \\ \{i \in \{1,\ldots,m\} : \exp(p_i) \le H \text{ or } \exp(p_i) \ge H^*\} \text{ double option} \end{cases}$$

When the states p_i and p_j do not belong to Λ, the conditional probabilities are the same of the matrix $Q_{(m)}$=$[q_{ij}]$ as described in the previous section, otherwise they are equal to zero. Therefore, the probability to transit from state p_i to state p_j are given by:

$$\pi_{ij} = \tilde{P}\{X_{t+1} = p_j, a_{t+1} = 0 | X_t = p_i, a_t = 0\} = \begin{cases} q_{ij} \text{ if } i \in \Lambda^c \text{ and } j \in \Lambda^c \\ 0 \text{ otherwise} \end{cases}$$

where Λ^c is the complement of Λ. Therefore the matrixes that define the conditional probabilities (that we call quasi-transition probabilities matrices) for the down-and-out, up-and-out, and double barrier-out options are respectively given by:

$$\Pi_{DO} = \begin{bmatrix} \mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,m-k+1} \\ \mathbf{0}_{m-k+1,k-1} & Q(k,m;k,m) \end{bmatrix},$$

$$\Pi_{UO} = \begin{bmatrix} Q(1,l;1,l) & \mathbf{0}_{l,m-l} \\ \mathbf{0}_{m-l,l} & \mathbf{0}_{m-l,m-l} \end{bmatrix},$$

$$\Pi_{DBO} = \begin{bmatrix} \mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,l-k+1} & \mathbf{0}_{k-1,m-l} \\ \mathbf{0}_{l-k+1,k-1} & Q(k,l;k,l) & \mathbf{0}_{l-k+1,m-l} \\ \mathbf{0}_{m-l,k-1} & \mathbf{0}_{m-l,l-k+1} & \mathbf{0}_{m-l,m-l} \end{bmatrix},$$

where k is the index number of the log price located immediately above the lower barrier H, l is the index number of the price located immediately below the upper barrier H^*, $\mathbf{0}_{i,j}$ is an $i \times j$ matrix of zeros, and $Q(i,j;k,l)$ is the sub-matrix of $Q_{(m)}$ taken from rows i to j and from columns k to l inclusively. Thus the knock-out option price with maturity T and strike price K can be computed using the recursive vectorial formula:

$$V_w(p,T;a_T = 0) = [v_w(p_1,T;a_T = 0), ..., v_w(p_m,T;a_T = 0)]'$$

and for $t_k = k\Delta t$, k=0,...,s-1,

$$\begin{aligned} V_w(p,t_k,a_{t_k} &= 0) = [v_w(p_1,t_k,a_{t_k} = 0), ..., v_w(p_m,t_k,a_{t_k} = 0)]' = \\ &= \max[g_w(p,K,a_{t_k} = 0), e^{-r\Delta t}\Pi V_w(p,t_{k+1};a_{t_{k+1}} = 0)], \end{aligned} \quad (8)$$

Table 5. European barrier option prices under NIG, VG, and Meixner processes. The current asset price, the short interest rate and the maturity are respectively *S*=100, *r*=5% and *T*=0.5.

	European down-out call options under NIG process				European down-out call options under VG process				European down-out call options under Meixner process			
Strike price	**Weekly**		**Daily**		**Weekly**		**Daily**		**Weekly**		**Daily**	
K=100	**H=94**	**H=98**	**H=94**	**H=98**	**H=94**	**H=98**	**H=94**	**H=98**	**H=94**	**H=98**	**H=94**	**H=98**
m=501	4.1358	3.1026	4.1059	2.8162	4.0826	3.0813	4.0536	2.7955	4.1288	3.0986	4.0993	2.8123
m=1001	4.1359	3.1033	4.1059	2.8183	4.0825	3.0820	4.0625	2.8071	4.1288	3.0993	4.0993	2.8145
m=1501	4.1359	3.1031	4.1058	2.8177	4.0825	3.0812	4.0553	2.7991	4.1288	3.0991	4.0991	2.8139
m=2001	4.1359	3.1029	4.1059	2.8171	4.0825	3.0815	4.0544	2.7996	4.1288	3.0989	4.0990	2.8132
m=2501	4.1359	3.1028	4.1059	2.8168	4.0825	3.0813	4.0546	2.7991	4.1288	3.0988	4.0991	2.8129
	European up-out call options under NIG process				**European up-out call options under VG process**				**European up-out call options under Meixner process**			
Strike price	**Weekly**		**Daily**		**Weekly**		**Daily**		**Weekly**		**Daily**	
K=90	**H=102**	**H=106**	**H=102**	**H=106**	**H=102**	**H=106**	**H=102**	**H=106**	**H=102**	**H=106**	**H=102**	**H=106**
m=501	1.1594	4.1648	0.9289	3.8133	1.1844	4.2817	0.9680	4.0230	1.1610	4.1780	0.9301	3.8265
m=1001	1.1563	4.1616	0.9203	3.8025	1.1847	4.2820	0.9439	3.9200	1.1579	4.1730	0.9210	3.8096
m=1501	1.1568	4.1607	0.9217	3.7997	1.1847	4.2818	0.9420	3.9126	1.1579	4.1735	0.9210	3.8115
m=2001	1.1565	4.1607	0.9206	3.7996	1.1849	4.2820	0.9425	3.9123	1.1580	4.1730	0.9214	3.8099
m=2501	1.1564	4.1604	0.9204	3.7995	1.1847	4.2818	0.9420	3.9119	1.1579	4.1732	0.9211	3.8103

where

$$g_w(p, K, a_{t_k} = 0) = [g_w(p_1, K, a_{t_k} = 0), ..., g_w(p_m, K, a_{t_k} = 0)]'$$

and Π is either Π_{DO}, or Π_{UO}, or Π_{DBO}, depending on the nature of the knock-out option. The knock-out option price at time 0 is given by the $((m+1)/2)$-th element of $V_w(p, 0; a_0 = 0)$. Barrier option prices are very sensitive to the position between discrete asset prices and barrier value. Thus, to reduce this effect it is important to define the cells of the markovian approach so that the barrier value correspond exactly to a cell's border.

Table 5 exhibits European barrier option prices. We consider two possible strike prices K=100 and K=90 for different fixed barriers and different distributional assumptions (NIG, VG, and Meixner). Even for this table we assume that the temporal horizon is shared either in 24 periods or in 126 periods (i.e., Δt is equal respectively either to one week or to one day). These prices refer to European down-out and up-out call options on a stock price with current value $S_0 = 100$ euro, maturity $T = 0.5$ years, short interest rate $r = 5\%$ a.r.. We also compare some of these results with those obtained with Montecarlo simulations. Since even in this case we get unstable results with more than five millions simulations, we did not report these partial results. Similarly, Table 6 displays American barrier option prices on a stock with the same current asset price, short interest rate

Table 6. American down-out and up-out put option prices, where both early exercise and monitoring are on daily basis under NIG, VG, and Meixner processes. The current asset price, the short interest rate and the maturity are respectively *S*=100, *r*=5% and *T*=0.5.

	American down-out put with daily monitoring		American down-out put with daily monitoring		American down-out put with daily monitoring	
Strike price K=101	**NIG process**		**VG process**		**Meixner process**	
	H=96	**H=99**	**H=96**	**H=99**	**H=96**	**H=99**
m=501	2.2453	1.1477	2.2568	1.1579	2.2496	1.1452
m=1001	2.2453	1.1462	2.2394	1.1540	2.2496	1.1438
m=1501	2.2454	1.1459	2.2382	1.1535	2.2497	1.1436
m=2001	2.2454	1.1458	2.2380	1.1534	2.2497	1.1434
m=2501	2.2454	1.1455	2.2380	1.1533	2.2498	1.1432
	American up-out put with daily monitoring		**American up-out put with daily monitoring**		**American up-out put with daily monitoring**	
Strike price K=101	**NIG process**		**VG process**		**Meixner process**	
	H=101	**H=104**	**H=101**	**H=104**	**H=101**	**H=104**
m=501	1.1425	2.0802	1.1174	2.0635	1.1302	2.0747
m=1001	1.1334	2.0800	1.1165	2.0417	1.1308	2.0744
m=1501	1.1335	2.0793	1.1164	2.0407	1.1309	2.0736
m=2001	1.1341	2.0793	1.1164	2.0405	1.1316	2.0736
m=2501	1.1337	2.0795	1.1165	2.0404	1.1312	2.0737

and maturity. In particular, we consider American down-out and up-out put option prices assuming a strike price K=101 and that the early exercise and the monitoring are on daily basis. As for American and European vanilla options Tables 5 and 6 show a good tendency towards a specific price when we increase the number of states of the Markov chain.

Lookback options An European lookback put option gives the right to sell the underlying asset at maturity for the maximum price monitored discretely during the time to maturity, while a call gives the right to buy the underlying asset for the minimum price. The option is American if the right is extended to the whole time to maturity. The pricing and hedging for a lookback option can be faced under the assumption that the asset follows a Markov chain. In the following, we consider an European lookback put option with maturity T and monitored at times $k = iT/n$, where n is the number of dates of monitoring and $i = 0, 1, \ldots, n$. In this setting it is implicitly assumed that the asset is monitored at constant time intervals where $\Delta t = T/n$. Clearly, we can easily extend these considerations to the case of lookback call options. Since we adapt Cheuk and Vorst's technique to the markovian approach, we express the final payoff in units of the asset price (see Babbs (2000) and Cheuk and Vorst (1997)). The payoff at the maturity T is equal to

$$\max\{S(iT/n) : i = 0, 1, \dots, n\} - S(T). \tag{9}$$

Dividing the payoff (9) by the asset price $S(T)$ we obtain the payoff expressed in asset price units:

$$Y(T) - 1,$$

where $Y(k) = \max\{S(iT/n) : i = 0, 1, \dots, nk/T\}/S(k)$. The evolution of the asset price S at times $k = iT/n$, $i = 0, 1, \dots, n$, is described by the Markov chain $\{X(i) := S(iT/n)\ | i = 0, 1, \dots n\}$ with state number m and risk neutral transition matrix $Q_{(m)} = [q_{ij}]_{1 \le i,j \le m}$. The random variables $X(i)$, $i = 1, \dots, n$, can assume the ordered values $x(j)$, $j = 1, \dots m$ (with $x(j) < x(j+1)$). Let us define the discrete stochastic process $Z(k; h, w)$, $h, w = 1, \dots m$, $k = iT/n$, $i = 0, 1, \dots, n$, where $Z(k; h, w)$ is the value at time k of a contingent claim with final payoff $Y(T) - 1$ when the current asset price is equal to $x(w)$ and the maximum asset price from time 0, to time $k - \Delta t$ is been $x(h)$. Therefore, at time T we consider the final payoff matrix:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ Z(T; 2, 1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Z(T; m, 1) & Z(T; m, 2) & \cdots & 0 \end{bmatrix}.$$

According to the risk-neutral pricing at time $T - \Delta t$, we have

$$Z(T - \Delta t; h, w) = \sum_{j=1}^{m} q_{wj} Z(T; h, j) \left(\frac{x(j)}{x(w)}\right) e^{-r\Delta t}, \text{ if } h > w, \tag{10}$$

$$Z(T - \Delta t; h, w) = \sum_{j=1}^{m} q_{wj} Z(T; w, j) \left(\frac{x(j)}{x(w)}\right) e^{-r\Delta t}, \text{ if } h \le w. \tag{11}$$

Formulas (10) and (11) have a quite immediate explanation. q_{wj} is just the probability to move from the state $x(w)$ to the state $x(j)$; in (10) we have $Z(T; h, j)$ because $x(h) > x(w)$ and thus the maximum at time $T - \Delta t$ is $x(h)$, while in (11) we have $Z(T; w, j)$ because $x(h) \le x(w)$ and the maximum is $x(w)$; the factor $x(j)/x(w)$ allows to express $Z(T - \Delta t; h, w)$ in units of $x(w)$; $e^{-r\Delta t}$ is the discount factor. Iterating the procedure, at time k we get

$$Z(k; h, w) = \sum_{j=1}^{m} q_{wj} Z(k + \Delta t; \max(h, w), j) \left(\frac{x(j)}{x(w)}\right) e^{-r\Delta t}.$$

Table 7. European and American lookback put option prices, where monitoring is on daily and weekly basis under NIG, VG and Meixner processes.

	European lookback put							
	Brownian Motion		NIG process		VG process		Meixner process	
	weekly	daily	weekly	daily	weekly	daily	weekly	daily
m=501	2.7121	3.1344	2.6680	3.0511	2.5998	3.0058	2.6605	3.0439
m=801	2.7125	3.1355	2.6683	3.0524	2.6000	2.9866	2.6609	3.0452
m=1001	2.7126	3.1358	2.6684	3.0528	2.6001	2.9843	2.6610	3.0456
m=1501	2.7127	3.1361	2.6685	3.0531	2.6002	2.9832	2.6611	3.0459
	American lookback put							
	Brownian Motion		NIG process		VG process		Meixner process	
	weekly	daily	weekly	daily	weekly	daily	weekly	daily
m=501	2.8587	3.2919	2.8176	3.2253	2.7528	3.1780	2.8113	3.2195
m=801	2.8695	3.3216	2.8180	3.2266	2.7532	3.1646	2.8117	3.2209
m=1001	2.8696	3.3218	2.8181	3.2269	2.7533	3.1630	2.8118	3.2212
m=1501	2.8697	3.3221	2.8182	3.2273	2.7534	3.1624	2.8119	3.2215

After n backward steps we obtain a matrix whose element $Z(0;h,w)$ is the value in units of $x(w)$ of the contingent claim with payoff $Y(T)-1$ when the current asset price is $x(w)$ and the maximum before time 0 is been $x(h)$. In our construction of the Markov chain $\{X(i) : i = 0,\dots,n\}$ the current asset price is $x(\frac{m+1}{2})$, thus the price of the European lookback put option is given by $Z(0;\frac{m+1}{2},\frac{m+1}{2})$ multiplied by $x(\frac{m+1}{2})$. American style options can be priced using the formula for $k = iT/n$, $i = 0,1,\dots,n-1$:

$$Z(k;h,w) = \max\left\{\sum_{j=1}^{m} q_{wj} Z(k+\Delta t;\max(h,w),j)\left(\frac{x(j)}{x(w)}\right)e^{-r\Delta t}, Y(k)-1\right\},$$

and then multiplying the element $Z(0;\frac{m+1}{2},\frac{m+1}{2})$ by $x(\frac{m+1}{2})$. In Table 7 we show the prices of European and American lookback put options, based on daily and weekly monitoring under the Brownian Motion, NIG, VG and Meixner processes. The current asset price, the short interest rate and the maturity are respectively S=100, r=5% and T=0.25. We compare the results for the European put with NIG and the VG processes with the prices obtained with 1000000 Montecarlo simulations and we obtain that the prices are respectively 2.6653 and 2.6025 with weekly monitoring and 3.0564 and 3.0011 with daily monitoring. Moreover we could observe that the results obtained with Montecarlo simulations are not very stable even when we simulate ten millions of values. While the prices obtained with the markovian approach are much more stable even with one thousand of states. As a matter of fact, for the European put with NIG and the VG processes we get 2.6691 and 2.6009 with weekly monitoring and 3.0534 and 2.9838 with daily monitoring. Thus even if these prices are much more near to those obtained with the Markovian approach they require much more computational time and present an higher level of instability.

5. Concluding remarks

The paper shows the simplicity of the markovian approach to price vanilla options and some types of exotic options when the log return follows a Lévy process. Clearly, we couldn't be exhaustive since this approach can be used to price many other markovian processes and exotic options. In particular, the discretization process with Markov chains permits to price path dependent options once we are able to approximate the risk neutral distribution of the underlying markovian log return process. Typically we can apply the markovian approach on GARCH type processes with markovian residuals, stochastic volatility Lévy processes and subordinated Lévy processes (see, among others, Shoutens (2003), DeGiovanni *et al.* (2007)).

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