



**UNIVERSITA' DEGLI STUDI DI BERGAMO**  
**DIPARTIMENTO DI SCIENZE ECONOMICHE**

**“Hyman P. Minsky”**

**Via dei Caniana 2, I-24127 Bergamo, Italy**

**Tel. +39-035-2052501; Fax: +39-035-2052549**

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**Maria Rosa Battaggion, Piero Tedeschi**

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# Do Process Innovations Induce Product Ones?

Maria Rosa Battaggion<sup>#</sup> and Piero Tedeschi<sup>§</sup>

<sup>#</sup> Dipartimento di Scienze Economiche  
Università di Bergamo  
Via dei Caniana 2  
24127 Bergamo, Italy.

E-Mail: maria-rosa.battaggion@unibg.it.

<sup>§</sup> Dipartimento di Statistica  
Università di Milano Bicocca  
Via Bicocca degli Arcimboldi 8  
20126 Milan, Italy.

E-Mail: piero.tedeschi@unimib.it

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### **Abstract**

We study the relationship between process and product innovations in vertically differentiated duopolies. A process innovation can lead two competing firms to improve the quality of their goods introducing a product innovation. In fact, a cost reducing innovation has two effects: it spurs production and it enhances price competition. The former effect induces both firms to increase quality. The latter encourages differentiation, inducing low quality firm to decrease it. Therefore, high quality firm always improves its quality, while the other may or may not. The prevailing effect depends on the nature of quality costs (fixed or variable).

## 1 Introduction

In the present paper we study the relationship between process and product innovation, which we judge of the utmost relevance for the understanding of technological dynamics. The theoretical literature represents process innovation by cost reductions and product innovation by increases of the demand schedule. However, only few contributions deal with both kinds of innovation (e.g. Bonanno, Haworth - 1988, Lamberini, Orsini - 2000) and even fewer investigate the relationship between the two. Remarkable examples are the papers due to Athey, Schmutzler (1995) and Eswaran, Gallini (1996).

Athey and Schmutzler (1995) prove that process innovation (cost-reducing) and product innovation (demand-enhancing) are complementary. In the short run, an increase in firm's net revenue of one type of innovation induces the firm to implement also innovation of the other kind. Intuitively, product innovation shifts the demand curve incentivating the firm to increase output. The higher the quantity, the bigger is the return to lowering unit costs (process innovation). Therefore, the firm will tend to implement process as well as product innovation. For an analogous reasoning, long run variables such as investments in product design flexibility and process flexibility show the same kind of complementarity. Lin, Saggi (2002) in a framework of horizontal product differentiation find similar results to Athey and Schmutzler (1995). Moreover, in their model firms invest more in product R&D if they are allowed to invest also in process one, rather than in the case where process R&D is not available.

Eswaran, Gallini (1996), instead, study the relationship between process and product innovation in order to describe the effects of different patent policies. They present a model of horizontally differentiated products, where a competitor can eventually challenge the incumbent firm, called pioneer. The former enters the market with an horizontally differentiated product. In this model, the degree of differentiation corresponds to the intensity of product innovation. However, the more the entrant differentiates and the softer price competition is, hence the lower is the incentive to introduce process innovations. Thus, in their framework, the two kinds of innovation are substitute, since the incentive to adopt a process innovation is lower when product innovation is larger.<sup>1</sup>

In our paper we consider a model of vertical product differentiation,

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<sup>1</sup>The approach of Eswaran Gallini was further developed in a recent paper by *Rosenkranz* (2003). The author shows that antitrust policy towards R&D cooperation affects innovative decisions and moreover the direction of technological change, in a model where consumer has a preference over variety and firms simultaneously choose between process *or* product innovation.

which we think as more suitable to represent product innovations (i.e., quality improvements), than horizontal ones. Furthermore, we assume imperfect competition (a duopoly) even before the innovation is adopted and the adoption of a process innovation at the outset of the game. In summary, our setup is more similar to that of Eswaran, Gallini (1996), but our results are closer to those of Athey Schmutzler (1995). In fact, we prove that a process innovation (cost reducing) can lead firms to invest and improve the quality of their goods and thus to adopt a product innovation. Therefore the two kinds of innovation are complementary. However, we model the opposite causal effect between product and process innovation with respect to the previous contributions, namely, that from process to product one.

The choice of a vertical differentiation model seems to us a natural one, even though it implies a non trivial choice among different models. The two main choices concern the market coverage and the cost of quality. As for the former, we assumed a partially covered market, since we wish to better capture the demand enhancing character of product innovation. This assumption is consistent with most of the literature on the subject. More crucial is the assumption on the latter choice, that concerning costs of quality. The literature of vertical product differentiation can be divided into two classes of models. The first one assumes that a higher quality level implies a higher fixed cost (Shaked, Sutton, 1983; Bonanno, 1986), that is, the cost of quality does not interact with the cost of quantity. While the second one assumes that the choice of a better quality induces a higher marginal production cost (Mussa, Rosen, 1978; Gal-Or, 1983). Usually the former assumption is considered suitable to describe a firm investing in R&D. While the assumption of variable costs of quality is compatible with the adoption of a new, quality improving, technology, involving the choice of inputs of better quality. For instance, in the pharmaceutical industry, the production of a better drug often entails the introduction of a higher percentage of active ingredient for each unit of production, which implies higher marginal production costs. Therefore, in this specific case a product innovation entails higher marginal costs. In our analysis we consider separately both fixed and variable costs of quality.

The assumptions on the quality cost affects important details also of our results. In a duopoly with variable costs of quality, firms improve the quality levels of their products, but also product differentiation increases, if a process innovation is available to both firms. That is, the quality of the high quality firm increases more than that of the low quality one. Differently, in a duopoly with fixed costs of quality, the results depend on the initial production costs. A process innovation makes the leader

firm increase its quality level and the follower one decrease it, if production cost are low also before adoption. Conversely, if production costs before innovation are high, both firms will improve their quality level, but in this case (and differently from variable costs) reducing vertical differentiation.

The economic intuition of the results is the following. A process innovation has two different effects. A process innovation induces lower costs and therefore makes it convenient to increase production. Therefore any strategy which allows to increase the price becomes more profitable, for given cost function. Hence firms have higher incentives to improve the quality of their goods. However there is a second effect of process innovation. Since products are less costly price competition becomes tougher. Firms will react adopting strategies based on non-price-factor, the most important of which is an increase in product differentiation. In our setup firms have different incentives to differentiate. Therefore the high quality firm wishes to increase the quality of its product, while the low quality one to decrease it. Hence this latter effect works in the same direction of the former for the high quality firm, while it works in the opposite one for the low quality one. The implication of the two combined effects are that the high quality firm will always improve the quality of its product, while the low quality one will improve it only if the former effect prevails on the latter. When the latter prevails firm 2 reduces quality. This happens only with fixed costs when production costs are low.

Recently, Bandyopadadhyay, Acharyya, (2004) proposed a model which has some similarities to ours. They examine the complementarity between process and product innovations in a monopoly with vertical differentiation. Complementarity arises when a process innovation makes product innovation profitable, thus inducing the monopolist to adopt it. Also their results depend on the nature of the innovation costs (fix or variable): when quality costs are variable complementarity holds only with partial coverage, but when they are fixed the complementarity results holds unambiguously. However, since they study a monopoly, they completely neglect any strategic issue.

Some examples of the interdependence between product and process technology are, in the field of laser printers and of automobile industries; in both cases of process innovations for building established products induce new and better quality versions of the goods. For instance, in the Seventies, Bob Metcalfe developed Ethernet in the Xerox Parc Laboratories. Ethernet technology was able to send a larger amount of data to the laser printers, therefore it can be considered a process innovation. Simultaneously, Xerox has been forced to produce a new hardware, namely

a faster printer to exploit the new technology (Varian, Shapiro 1999). Our model offers a theoretical explanation to this process. Initially a firm introduces a cost reducing innovation; then it is forced to invest in R&D and to introduce a new, higher quality hardware, namely a product innovation. The final level of market differentiation depends upon the competitors' reactions.

Analogously, the use of plastic in the car industry was initially meant to reduce production costs and certainly not car quality; in this case we have a process innovation. Later on, research on modern materials helped car manufacturers to use plastic also to improve product quality and now luxury brands too make use of plastic parts for the body and the interior of cars. Again, our model is suitable to describe this equilibrium path. At the beginning, plastic parts (for instance bumpers) were a cost-reducing innovation. Then, according to an increase in market competition, firms try to differentiate their products from the competitors' ones and improve the quality of bumpers (and hence of cars).

In the following sections of the paper we will introduce the model and our analytical results. Section 2 introduces the formal model of vertical differentiation and the assumptions. In the Section 3, we fully develop the model with variable costs of quality. First, we analyze the comparative statics, uniqueness and existence of the equilibrium in the model without innovation. Then we introduce the game with innovation, both in the symmetric and asymmetric cases. Analogously, Section 4 solves the model with fixed costs. In the Section 5, some concluding remarks will close the paper.

## 2 Description of the Model

In this section we present the model, starting from the description of the timing of the game. Then the demand will be described. Finally we will introduce the technology, distinguishing between the cases of variable and fixed costs of quality.

### 2.1 Timing

The timing of the game is as follows. In the first stage, the two firms simultaneously decide whether to adopt a freely available process innovation. In the second stage firms set the quality  $\theta_i, i = 1, 2$ . Then in the final stage of the game, firms simultaneously set prices. Therefore, referring to the standard model of quality choice, we add a stage at the beginning of the game. Namely, in the first stage firms decide whether to adopt or not a cost reducing innovation.

## 2.2 The Demand

As in standard models of vertical differentiation, we assume that each consumer has the following indirect utility function:

$$u_h = h\theta_i - p_i \text{ with } i = 1, 2$$

where the parameter  $h$  is uniformly distributed on  $[0, 1]$ , and represents the consumers' marginal rate of substitution between money and quality.  $\theta_i$  measures the quality level of the final good  $y_i$ .  $p_i$  is the unit price for the good produced by firm  $i$ . We impose that consumers can purchase at most one unit of the good, as it is standard in this class of models.

Without loss of generality, we assume that  $\theta_1 \geq \theta_2$ . Hence, we have two marginal consumers, the former is indifferent between buying the low quality good and not buying at all, characterized by a parameter value:  $h_{02} = \frac{p_2}{\theta_2}$ ; the latter is indifferent between buying good 1 or good 2, characterized by a parameter value:  $h_{21} = \frac{p_1 - p_2}{\theta_1 - \theta_2}$ . Hence, the demand functions for the high quality and the low quality firms are the standard ones in this class of models with partial coverage:

$$y_1 = 1 - h_{21} = 1 - \frac{p_1 - p_2}{\theta_1 - \theta_2} \quad (1)$$

$$y_2 = h_{21} - h_{02} = \frac{p_1 - p_2}{\theta_1 - \theta_2} - \frac{p_2}{\theta_2} \quad (2)$$

## 2.3 The Technology

Empirically, cost of quality might have both a variable component and a fixed one. However, following the literature, we will solve separately the two models. We start introducing the variable costs model, because its results are more clear-cut than those of the fixed costs one.

### 2.3.1 Variable Costs

For computation simplicity and in agreement with most of the literature, we assume that the cost function of the firm  $i$  is:

$$c_i \frac{(\theta_i)^2}{2} y_i \quad \text{with } i = 1, 2 \quad (3)$$

with  $c_i < 1$ .

Using the respective demand functions (1) and (2), the profit functions become:

$$\pi_1(p_1, p_2, \theta_1, \theta_2) = \left(1 - \frac{p_1 - p_2}{\theta_1 - \theta_2}\right) \left(p_1 - c_1 \frac{\theta_1^2}{2}\right) \quad (4)$$

$$\pi_2(p_1, p_2, \theta_1, \theta_2) = \left(\frac{p_1 - p_2}{\theta_1 - \theta_2} - \frac{p_2}{\theta_2}\right) \left(p_2 - c_2 \frac{\theta_2^2}{2}\right) \quad (5)$$



### 2.3.2 Fixed Costs

Analogously, assuming fixed costs of quality, cost function of firm  $i$  is as follows:

$$k \frac{(\theta_i)^2}{2} + a_i y_i \quad \text{with} \quad i = 1, 2$$

with  $k < 1$ .

Therefore, profit functions for Firms 1 and 2 are respectively:

$$\pi_1(p_1, p_2, \theta_1, \theta_2) = \left(1 - \frac{p_1 - p_2}{\theta_1 - \theta_2}\right) (p_1 - a_1) - \frac{k}{2} \theta_1^2 \quad (6)$$

$$\pi_2(p_1, p_2, \theta_1, \theta_2) = \left(\frac{p_1 - p_2}{\theta_1 - \theta_2} - \frac{p_2}{\theta_2}\right) (p_2 - a_2) - \frac{k}{2} \theta_2^2 \quad (7)$$

## 3 Solution of the Model: Variable Costs

Given the profit functions, (4) and (5), we solve the game backwards, starting from the price stage. First, we will analyze the existence and the uniqueness of the equilibrium of the model without innovation. Second, we perform the comparative statics of the same model. Finally, we will show that the comparative statics is useful in characterizing the game with innovation.

Solving the system of first order conditions for prices, we obtain the following equilibrium prices as a function of the two quality levels:

$$p_1(\theta_1, \theta_2) = \frac{1}{2} \theta_1 \frac{4(\theta_1 - \theta_2) + 2c_1\theta_1^2 + c_2\theta_2^2}{4\theta_1 - \theta_2} \quad (8)$$

$$p_2(\theta_1, \theta_2) = \frac{1}{2} \theta_2 \frac{2(\theta_1 - \theta_2) + c_1\theta_1^2 + 2c_2\theta_1\theta_2}{4\theta_1 - \theta_2} \quad (9)$$

Substituting (8) and (9) in the profit function of Firm 1 and simplify, we obtain the profit as a function only of the two quality levels. The profit function of Firm 1 and 2 are respectively:

$$\begin{aligned} \Pi_1(\theta_1, \theta_2) &= \pi_1(p_1(\theta_1, \theta_2), p_2(\theta_1, \theta_2), \theta_1, \theta_2) = \\ &= \frac{1}{4} \theta_1^2 \frac{(4(\theta_1 - \theta_2) - (2\theta_1 - \theta_2)c_1\theta_1 + c_2\theta_2^2)^2}{(4\theta_1 - \theta_2)^2(\theta_1 - \theta_2)} \end{aligned} \quad (10)$$

$$\begin{aligned} \Pi_2(\theta_1, \theta_2) &= \pi_2(p_1(\theta_1, \theta_2), p_2(\theta_1, \theta_2), \theta_1, \theta_2) = \\ &= \frac{1}{4} \theta_1 \theta_2^2 \frac{(2(\theta_1 - \theta_2) - (2\theta_1 - \theta_2)c_2\theta_2 + c_1\theta_1^2)^2}{(4\theta_1 - \theta_2)^2(\theta_1 - \theta_2)} \end{aligned} \quad (11)$$

Now, we are ready to solve the second stage of the game where firms choose the products quality. The following proposition proves that the equilibrium does exist. In equilibrium, firms will choose to partially differentiate their products when there is a cost in improving quality. Moreover, we are able to find a set of parameter values (the cost parameter of each firm  $c_i$ , as defined in (3)) for which the equilibrium is “unique”. Later, we are able to fully characterize the unique equilibrium of our model.

**Proposition 1** *There exists a non empty open set of parameter values -  $\frac{c_2}{c_1} \in (0.94, 1.06)$  - for which the model has an equilibrium. The equilibrium where  $\theta_1 > \theta_2$  is unique and it implies that firms will differentiate the quality level of their products.*

**Proof.** See Appendix A. ■

Notice that the set of parameter values specified in the previous proposition is rather large. In fact, it implies a difference of  $\pm 6\%$  in the marginal costs, a difference which seems to us quite substantial. There are still some difficulties in performing comparative statics. It is obvious that when the model is symmetric there exist two equilibria: one where Firm 1 produces at a higher quality level (the one we studied) and a second (symmetric with respect to the former) where Firm 2 produces at higher quality level. In the case of a symmetric model the choice between the two equilibria is irrelevant, because they are equivalent up to a name permutation. However, if we set:

$$\begin{aligned} c &= c_2 \\ c_1 &= mc, \end{aligned} \tag{12}$$

thus  $m = c_1/c_2$ , we can find a set of equilibria (parameterized to  $m$ ) which converges to that of the symmetric equilibrium as  $m \rightarrow 1$ . This, together with the existence of two equilibria in the symmetric case, implies that in the relevant range of parameter values there are two equilibria, which are substantially different if  $m \neq 1$ . In order to simplify the analysis and to maintain uniqueness, we assume that Firm 1 produces the highest quality good. With this restriction in mind, Proposition 1 guarantees a unique equilibrium, which allows to prove a simple comparative statics result, summarized in the next Proposition. Before stating the following Proposition notice that  $m$  is an inverse measure of the technological advantage of Firm 1; therefore when it decreases Firm’s 1 technology improves compared to that of Firm 2. Namely, when  $m < 1$  Firm 1 is more productive than Firm 2, while the opposite occurs when  $m > 1$ .

**Proposition 2** (a) *Product quality and product differentiation increase if both firms production costs reduce proportionally, i.e. if  $c$  decreases, namely*

$$\theta_1 = \frac{\theta_1(m)}{c}$$

$$\theta_2 = \frac{\theta_2(m)}{c}$$

with  $\theta_1(m) > \theta_2(m)$ .

(b) *Product quality of Firm 1 and product differentiation increase also if Firm 1 increases its technological advantage, i.e.  $m$  decreases. Product quality of Firm 2 first decreases then increases.*

**Proof.** See Appendix A. ■

We derived all the needed results for the vertical differentiation model without innovation. We now have to find their implications for the innovation model, starting from the solution of the first stage of the game, where the two firms simultaneously decide whether to innovate or not. Afterwards, we have to characterize how innovation affects vertical products differentiation. Process innovation is represented by a decrease in  $c_i$ , namely  $c = c_2$  and  $c_1 = mc$ , which can be induced, for instance, by the use of a new production input or process. Obviously, firms innovate only if it is profitable to do so. Therefore, first we have to ascertain what happens to profits. The next Proposition solves this problem in a rather simple way.

**Proposition 3** *Adopting an innovation is an equilibrium strategy for both firms.*

**Proof.** See Appendix A. ■

In the comparative statics analysis we will consider two different kinds of process innovations. We define the first one *equiproportional*, because it leaves the ratio between the two marginal costs,  $m$ , unchanged. This kind of innovation can be parameterized through a decrease in  $c$ . The second type of innovation increases the technological advantage of Firm 1 with respect to Firm 2 and can be represented through a decrease in  $m$ , for given  $c$ . We call this type of process innovation, *disproportional*. We will deal first with the equiproportional innovation.

In Proposition 2 we found that the solution has the form:  $\theta_i = \frac{\theta_i(m)}{c}$ ,  $i = 1, 2$ , which is very simply parameterized in  $c$  and makes it easy to perform comparative statics when the innovation is introduced. Therefore a direct consequence of the mentioned Proposition part (a) is that

a proportional process innovation makes  $\theta_1$ ,  $\theta_2$  and  $(\theta_1 - \theta_2)$  increase. If  $m$  decreases, provided that  $m \leq 1.3$ , part (b) of Proposition 2 implies that differentiation increases, as part (b) of the following Proposition states. We summarize the above discussion in the following:

**Proposition 4** *If the parameter values are those described in Proposition 1:*

- (a) *an equiproportional process innovation increases qualities of both firms and vertical product differentiation,*
- (b) *a disproportional process innovation at the advantage of Firm 1 increases quality of Firm 1 and vertical product differentiation, while product quality of Firm 2 first decreases and then increases.*

The result of Proposition 4, in the case of equiproportional process innovation, has a very compelling economic intuition. In fact, as we said in the Introduction, a process innovation makes price competition among firms tougher and therefore induces them to adopt defensive strategies, the most important of which is an increase in product differentiation. Given the result on differentiation, it is not surprising that Firm 1 exploits the cost reduction in order to increase the quality. Slightly more surprising is that also the second Firm quality increases, since it faces a trade-off. On the one hand, it finds it more convenient to increase quality, given the cost reduction. On the other hand, it has an incentives to lower quality in order to increase product differentiation. However, since Firm 1 is increasing significantly its quality, the latter effect is dominated by the former.

By the same reasoning, in the case of disproportional process innovation, Firm 1 has an incentive to differentiate its output from that of the opponent firm, increasing quality differentiation. By so doing, it also increases its advantage in producing higher quantities of the good, since it benefits of lower marginal costs. Firm 2 faces a trade-off, as already said. It has an incentive to increase differentiation and therefore decrease quality, in order to lower price competition. On the other hand, an increase of Firm 1 quality allows to exploit the higher willingness to pay of the consumers. The two forces act differently for different ranges of parameter values, since the intensity of the second effect varies with its technological advantage. In fact, the quality of Firm 1 increases in a much faster way as it becomes more and more efficient. Therefore if the asymmetries between firms are significant also Firm 2 can increase its quality. As a matter of facts, for small differences between firms an increase of Firm 1 efficiency induces a lower quality of the product of Firm 2, if Firm 1 is much more efficient the opposite is true.

## 4 Solution of the Model: Fixed Costs

In the present section we assume that there are fixed costs of quality. While with variable costs we were able to characterize also the asymmetric model, with fixed cost we are able to solve the symmetric model only. For this reason and in order to simplify the exposition, we present the results of the fixed costs model in a more condensed way, with respect to the variable costs ones.

We solve the game backwards, starting from the prices stage, as usual. Solving the system of first order conditions, from (6) and (7), with respect to prices we obtain the equilibrium prices as a function of the two quality level:

$$p_1(\theta_1, \theta_2, a_1, a_2) = \theta_1 \frac{2(\theta_1 - \theta_2) + 2a_1 + a_2}{4\theta_1 - \theta_2} \quad (13)$$

$$p_2(\theta_1, \theta_2, a_1, a_2) = \frac{(\theta_1 - \theta_2)\theta_2 + 2\theta_1 a_2 + a_1 \theta_2}{4\theta_1 - \theta_2} \quad (14)$$

Substituting the equilibrium prices in the profit functions we obtain:

$$\Pi_1(\theta_1, \theta_2, a_1, a_2, k) = \frac{(2\theta_1^2 - 2\theta_1\theta_2 - 2\theta_1 a_1 + \theta_1 a_2 + a_1 \theta_2)^2}{(4\theta_1 - \theta_2)^2 (\theta_1 - \theta_2)} - \frac{1}{2} k \theta_1^2 \quad (15)$$

$$\Pi_2(\theta_1, \theta_2, a_1, a_2, k) = \frac{\theta_1 (\theta_1 \theta_2 - \theta_2^2 + a_1 \theta_2 + \theta_2 a_2 - 2\theta_1 a_2)^2}{\theta_2 (4\theta_1 - \theta_2)^2 (\theta_1 - \theta_2)} - \frac{1}{2} k \theta_2^2 \quad (16)$$

as a function of the two quality levels. Now, we can solve the second stage of the game where firms choose the products quality. As we already stated, we can solve only the symmetric model, i.e.  $a_1 = a_2 = a$ . First we will prove the existence of the equilibrium in the model without innovation. Then we will characterize the game with innovation by means of comparative statics.

**Proposition 5** *There exists a unique equilibrium of the symmetric game (i.e., with  $a_1 = a_2 = a$ ) with fixed costs for  $a \cdot k \in A \cup B$ , where:*

$$A = [0, 0.3042E^{-2}] \quad \text{and} \quad B = [0.59466E^{-2}, 0.82724E^{-2}] .$$

**Proof.** See Appendix B. ■

In equilibrium firms differentiate the quality level of their products, as it is standard in the literature on vertical differentiation. We are able to characterize the range of parameter values where an equilibrium exists

and it is unique. Now we can solve the first stage of the game, where the two firms simultaneously decide whether to innovate or not, and eventually how innovation affects quality and vertical products differentiation. Firms introduce the innovation only if it is profitable. This is proved in the following proposition.

**Proposition 6** *For the equilibrium strategy profile of the symmetric game, the adoption of a process innovation is always profitable for one of the firms.*

**Proof.** See Appendix B ■

We cannot prove whether to adopt the innovation is an equilibrium, since we did not solve the asymmetric model. However, in the appendix we can prove that a small reduction in production costs is always profitable for one of the firms. Moreover, we proved graphically that the adoption of a process innovation which allows a discrete reduction in production costs is an equilibrium.

Notice that in the proof of Proposition 5, we come up with a unique solution, for specific ranges of parameters values. In the model there are two driving forces induced by a process innovation and already explained in the introduction of the paper. The former induces all firms to improve the quality, while the latter enhances differentiation. For the high quality firm, these two forces both contribute in improving quality, while for the low quality firm the attempt to increase product differentiation can induce a lowering of the quality level. Moreover, these two forces have different intensities, according to the parameter values. When we consider low production costs, price competition is necessarily tough and the incentive to improve the quality for Firm 2 are low, since its market share is relatively small and decreases with the innovation. Therefore in equilibrium the innovations will induce more product differentiation. At the opposite, when we start with high production costs, price competition is milder and both firms will increase quality. However, the low quality firm will have an higher incentive to increase quality than the high quality one and this will induce a lower vertical differentiation. This economic intuition is properly stated in the following proposition.

**Proposition 7** (a) *If  $ak \in A$ , process innovation increases Firm's 1 product quality,  $\theta_1$ , while it decreases that of Firm 2,  $\theta_2$ . Therefore, vertical product differentiation increases.*

(b) *If  $ak \in B$ , process innovation increases the quality of both products,  $\theta_1, \theta_2$ . Moreover, vertical product differentiation decreases.*

**Proof.** See Appendix B. ■

In the equilibrium of case (a), firms produce at such low costs that make price competition very tough. Therefore, when process innovation is introduced, both firms pursue a quality differentiation policy, to relax price-competition. This strategy induces the quality leader, Firm 1, to increase product quality. Differently Firm 2 can only decrease its quality. In summary, process innovation induces product innovation, as an improvement of product quality, for the high-quality firm and induces a lower quality level for the low-quality firm. If we interpret fixed cost of quality as R&D investment necessary to obtain higher quality, product innovation becomes complementary to the original process innovation.

This statement is even more convincing in case (b). Here the equilibrium is characterized by high production costs. To recover this costs, firms are forced to compete on non-price factors, such as quality. After the introduction of process innovation and the related fall down in production costs, both firms improve the quality level of their goods, reducing vertical differentiation. In summary, for both the firms process innovation induces a quality improvement which is a product innovation, under specific interpretation of the model, e.g., if fixed costs are generated by R&D investments. In both cases, a process innovation allows the two firms to lower the price and increase the quantities sold, as it can be proved graphically.

## 5 Concluding Remarks

The adoption of a process innovation always induces Firm 1, the quality leader, to choose a better quality; in both cases of variable and fixed costs of quality. Firm 2, instead, with variable costs of quality, will certainly choose a higher quality level, while with fixed costs the quality level can increase or decrease according to the cost level. Product differentiation increases with variable costs, while it increases with fixed costs, only in the second case.

The economic reason for the results is that a process innovation has two different effects. It induces lower costs boosting production, which makes it more convenient to increase quality. It also makes price competition tougher and therefore spurs vertical differentiation. The two effects work in the same direction for the high quality firm and in the opposite one for the low quality one. Hence the high quality firm will always improve the quality, while the low quality one will improve it only if the former effect prevails on the latter.

The more natural interpretation of our result is that an adoption of a process innovation induces always product innovation for the high

quality firm and in important regions of the parameter values also for the low quality one, since the quality of their products improves. Therefore we can say that in our model process innovation induces product innovation. For this aspect, our model provides a further motivation for considering product and process innovations as complementary (Athey and Schmutzler, 1995).

Moreover our results provide a new logical example of the difficulty in distinguishing between product and process innovations, both theoretically and empirically, even though we deserve the distinction between process and product innovation as one of the most relevant in the whole theory of technical change.



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## Appendix A

### *Proof of Proposition 1*

In order to prove the existence and the uniqueness of the solution in the asymmetric model for the specified set of parameters values, we start by maximizing the firms' profit functions and computing the first order conditions. Then we show that the solution is an internal one and that the second order conditions are satisfied. Finally we prove that the equilibrium is unique.

### *First order conditions*

The two profit functions can be written in terms of  $c$  and  $m$  setting  $c = c_2$  and  $c_1 = mc$ , as follows

$$\begin{aligned}\Pi_1(\theta_1, \theta_2, c, m) &= \frac{1}{4}\theta_1^2 \frac{(4(\theta_1 - \theta_2) + c\theta_2^2 - 2mc\theta_1^2 + mc\theta_1\theta_2)^2}{(4\theta_1 - \theta_2)^2(\theta_1 - \theta_2)} \\ \Pi_2(\theta_1, \theta_2, c, m) &= \frac{1}{4}\theta_1\theta_2 \frac{(2(\theta_1 - \theta_2) + mc\theta_1^2 + c\theta_2^2 - 2\theta_1c\theta_2)^2}{(4\theta_1 - \theta_2)^2(\theta_1 - \theta_2)}\end{aligned}$$

and their first order conditions respectively for  $\theta_1$  and  $\theta_2$  are:

$$\begin{aligned}&\frac{1}{4}\theta_1(-4\theta_1 + 4\theta_2 - c\theta_2^2 + 2mc\theta_1^2 - mc\theta_1\theta_2) \\ &(-20\theta_1\theta_2^2 + 8\theta_2^3 + 28\theta_1^2\theta_2 + 24mc\theta_1^4 - 46mc\theta_1^3\theta_2 + 23mc\theta_1^2\theta_2^2 - \\ &2c\theta_2^4 - 4mc\theta_1\theta_2^3 + 4\theta_1^2c\theta_2^2 + c\theta_2^3\theta_1 - 16\theta_1^3)(4\theta_1 - \theta_2)^{-3}(\theta_1 - \theta_2)^{-2} = 0\end{aligned}\quad (17)$$

and

$$\begin{aligned}&\frac{1}{4}\theta_1(2\theta_1 - 2\theta_2 + mc\theta_1^2 + c\theta_2^2 - 2\theta_1c\theta_2) \\ &(-19c\theta_2^3\theta_1 + 14\theta_1\theta_2^2 - 22\theta_1^2\theta_2 - 2mc\theta_1^2\theta_2^2 + mc\theta_1^3\theta_2 + 38\theta_1^2c\theta_2^2) \\ &(-24c\theta_1^3\theta_2 + 8\theta_1^3 + 4mc\theta_1^4 + 2c\theta_2^4)(4\theta_1 - \theta_2)^{-3}(\theta_1 - \theta_2)^{-2} = 0\end{aligned}\quad (18)$$

Substituting (8) and (9) in (1) and (2) we obtain:

$$y_1 = \frac{1}{2}\theta_1 \frac{4\theta_1 - 4\theta_2 - 2mc\theta_1^2 + c\theta_2^2 + mc\theta_1\theta_2}{(4\theta_1 - \theta_2)(\theta_1 - \theta_2)}\quad (19)$$

$$y_2 = \frac{1}{2}\theta_1 \frac{2\theta_1 - 2\theta_2 + mc\theta_1^2 + c\theta_2^2 - 2\theta_1c\theta_2}{(4\theta_1 - \theta_2)(\theta_1 - \theta_2)}\quad (20)$$

Notice that the first terms of (17) and (18) are the numerator of (19) and (20) respectively. So in our internal solution only the second terms

in (17) and (18) matter. Therefore the first order conditions are satisfied in internal maxima for the firm only if the numerator of the fractions in (17) and (18) are nought. Given that, it is not possible to find simply the zeros of those terms, we introduce the following transformations:

$$\theta_1 = s/c \quad (21)$$

$$\theta_2 = xs/c$$

with  $0 \leq x \leq 1$ , since  $\theta_2 \leq \theta_1$ . Therefore, we obtain the following equivalent conditions:

$$f_1(s/c, xs/c, m) = -s^3 [(2x^4 + 4mx^3 - x^3 - 23mx^2 - 4x^2 + 46mx - 24m) s - 8x^3 - 20x^2 - 28x + 16] = 0 \quad (22)$$

$$f_2(s/c, xs/c, m) = s^3 [(+2x^4 - 19x^3 + 38x^2 - 2mx^2 + mx - 24x + 4m) s + 14x^2 - 22x + 8] = 0 \quad (23)$$

The system (22) – (23) has a trivial solution, which is  $\theta_1 = \theta_2 = s = 0$ . But  $\theta_1 = \theta_2 = 0$  cannot be an equilibrium. In fact:

$$\Pi_1(\theta_1, 0, c, m) = \frac{1}{16}\theta_1(2 - mc\theta_1)^2$$

is increasing in  $\theta_1$  in a right neighborhood of 0. Setting  $s > 0$ , the first order conditions are satisfied only if the two expressions in square brackets are nought. We can easily solve (23) for  $s$  and obtain:

$$s = -\frac{14x^2 - 22x + 8}{(+2x^3 - 19x^2 + 38x - 24)x + (-2x^2 + x + 4)m} \quad (24)$$

Substituting then (24) in (22) we have:

$$2 \frac{(x-1)(4-x)(8x^5 - 42x^4 + 20mx^3 + 99x^3 - 104x^2 - 81mx^2 + 48x + 84mx - 32m)}{(-24m + 46mx - 23mx^2 + 2x^4 + 4mx^3 - 4x^2 - x^3)(4m + 38x^2 + mx + 2x^4 - 19x^3 - 24x - 2mx^2)} = 0$$

which is zero only if the numerator is nought, that is:

$$(x-1)(4-x) \cdot [(8x^4 - 42x^3 + 99x^2 - 104x + 48)x + (20x^3 - 81x^2 + 84x - 32)m] = 0 \quad (25)$$

This equation has two obvious solutions:  $x = 4$  and  $x = 1$ . The first one can be discarded because we have  $x \leq 1$ . It is easy to check that the second one is not an optimum for Firm 1. In fact,  $x = 1$  implies  $\theta_1 = \theta_2$  and the limit of the rhs of (17) for  $\theta_1 \rightarrow \theta_2$  is  $-\infty$ , since the

numerator equals  $-\frac{3}{4}(1-m)^2 c^2 \theta^7$ , while the denominator tends to 0. We have finally that the first order condition for internal maxima are satisfied only if (24) holds and if:

$$(8x^4 - 42x^3 + 99x^2 - 104x + 48)x + (20x^3 - 81x^2 + 84x - 32)m = 0$$

It is rather difficult to solve this expression for  $x$ . However, we can solve it for  $m$  with the interpretation that we find the ratio between the two marginal costs,  $m$ , that induces a specific equilibrium ratio among qualities,  $x$ . This solution is:

$$m = -\frac{8x^4 - 42x^3 + 99x^2 - 104x + 48}{20x^3 - 81x^2 + 84x - 32}x \quad (26)$$

and substituting again in (24) we obtain:

$$s = 2\frac{20x^3 - 81x^2 + 84x - 32}{(4-x)(8x^3 - 46x^2 + 71x - 36)}x \quad (27)$$

#### *Internal solutions*

If we consider again the expression for  $y_1$ , substitute (21) and then the last two expressions for  $m$  and  $s$ , we obtain the following expression in  $x$ :

$$y_1 = -2\frac{4x^2 - 11x + 6}{8x^3 - 46x^2 + 71x - 36}$$

The numerator has two solutions:  $x = 2$  and  $x = \frac{3}{4}$ , while the denominator has only one real solution, whose approximate value is  $x \simeq 3.662$ . Therefore the above expression changes sign in the relevant region  $x \in [0, 1]$  only once, for  $x = \frac{3}{4}$ . Moreover, if we evaluate it for  $x = 0$ , we obtain  $y_1 = 1/3 > 0$ . Therefore the relevant range for an internal solution becomes  $x \in [0, 0.75]$ . If we do the same for  $y_2$  we obtain:

$$y_2 = 2\frac{4x^3 - 17x^2 + 26x - 16}{(4-x)(8x^3 - 46x^2 + 71x - 36)}$$

We already proved that the denominator does not change sign and hence it is negative. The numerator has only one real root:  $x = 2$  and therefore it is also negative in the relevant range. Hence  $y_2 > 0$ . Finally consider the expression for  $s$ , (27). We already proved that the denominator is negative and the numerator has only one real root:

$$x = \frac{1}{20}\sqrt[3]{3403 + 120\sqrt{469}} + \frac{169}{20\sqrt[3]{3403 + 120\sqrt{469}}} + \frac{27}{20} \approx 2.7236$$

and it is therefore negative in the relevant range. Hence  $s$  is positive and, for positive  $x$ , both  $\theta_1$  and  $\theta_2$  are positive. Therefore we have to prove for which values of  $m$  we have a positive  $x$ . Consider (26) and notice that the denominator is equal to the numerator of (27). Therefore the denominator is always negative. The numerator is always decreasing. In fact, differentiation yields:

$$32x^3 - 126x^2 + 198x - 104$$

which has only one real solution  $x = 1$ . Notice that the numerator for  $x = 1$  is equal to 9 and therefore it is always positive. Therefore  $m$  is positive for any  $x \in [0, 0.75]$ . Since positive values of  $m$  are admissible, the first order conditions will characterize an internal solution, provided that the profits are non negative and the second order conditions are globally satisfied.

*Second order conditions*

Notice that the profit functions are continuous. Moreover, it is easy to check that  $\Pi_i(\theta_1/c, \theta_2/c, c, m) = c\Pi_i(\theta_1, \theta_2, 1, m)$ . Therefore we can set  $c = 1$ , without loss of generality. Thus the only relevant parameter is  $m$ . Using (26) and (27) we can plot  $\Pi_1(\theta_1, x \cdot s(x), 1, m(x))$  and  $\Pi_2(s(x), \theta_2, 1, m(x))$  in order to ascertain whether there exists a set of parameter values which satisfies globally the second order conditions. For instance, if we set  $m = 0.94$ , the corresponding value of  $x$  is that satisfying  $m(x) = 0.94$ , whose approximate value is:  $x = 0.46616$ . Substituting this value in  $\Pi_1(\theta_1, x \cdot s(x), 1, m(x))$  and in  $\Pi_2(s(x), \theta_2, 1, m(x))$ , and noticing that the relevant ranges are  $\theta_1 \geq \theta_2^* = x^*s(x^*) \simeq 0.39413$ ,  $\theta_1 \leq 2$ ,  $\theta_2 \geq 0$  and finally  $\theta_2 \leq \theta_1^* = s(x^*) \simeq 0.84549$ , we can plot the two profit functions in the relevant regions as in Figures 1. Again, we can set  $m = 1.06$ , the corresponding value of  $x$  is that satisfying  $m(x) = 1.06$ , whose approximate value is:  $x = 0.51188$ . Substituting in  $\Pi_1(\theta_1, x \cdot s(x), 1, m(x))$  and in  $\Pi_2(s(x), \theta_2, 1, m(x))$ , and noticing that the relevant ranges are  $\theta_1 > 0.41031$ , because otherwise  $y_1$  is non positive and also that  $0.40811 \geq x^*s(x^*) \simeq 0.40663$ ,  $\theta_1 \leq 2$ , the plot of Firm 1 profit function in the relevant regions is that in Figure 1. As for  $\theta_2$ , the relevant range is  $\theta_2 \geq 0$  and  $\theta_2 \leq \theta_1 = s(x) \simeq 0.79439$ . However, it is easy to check that for  $\theta_2 > 0.77062$ ,  $y_1$  is nought and for  $0 \leq \theta_2 \leq 0.77062$  the plot of Firm 2 profit function is that in Figure 1. In the interval  $\theta_2 \in (0.77062, 0.79439]$  Firm 1 does not sell any good. Therefore Firm 2 profit function is:

$$\Pi_2 = \left(1 - \frac{p_2}{\theta_2}\right) \left(p_2 - \frac{(\theta_2)^2}{2}\right)$$

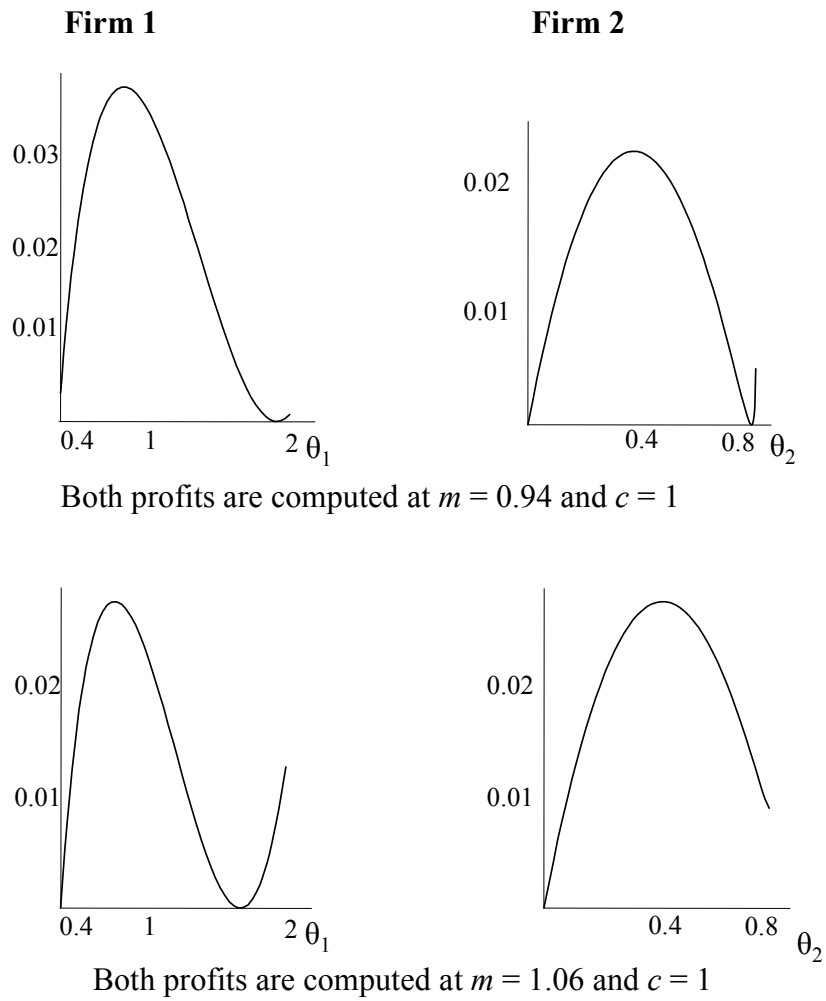


Figure 1: Profits as a function of own quality.  
 The other firm's quality is held constant at the equilibrium level

which reaches its maximum for:

$$p_2 = \frac{1}{4} (\theta_2 + 2) \theta_2$$

which substituted again in Firm 2 profit function yields:

$$\Pi_2 = \frac{1}{16} (2 - \theta_2)^2 \theta_2$$

which is decreasing in the relevant range. Moreover it is easy to check that  $\Pi_2(s(x), \theta_2, 1, m(x))$  evaluated at  $x = 0.51188$  and  $\theta_2 = 0.77062$  is bigger than this last profit function. Hence  $\Pi_2$  has a unique maximum, which is the one depicted in Figure 1.

We can perform the same exercise for various values of the whole range  $m \in [0.96, 1.06]$  reaching the same qualitative results. By continuity we can ascertain that the second order conditions are satisfied. Finally notice that the profit levels are always positive and this completes the proof that the solution are not corner ones. ■

*Proof of Proposition 2*

Part (a). Let us start with the comparative statics with respect to  $c$ . First notice that first order conditions, (22) and (23), they do not depend any more on  $c$ . Therefore if  $\widehat{s}$  and  $\widehat{x}$ , with  $\widehat{x} \leq 1$ , are a solution of the system, they do not depend on  $c$ . Moreover we have:  $c\theta_1 = \widehat{s}$  and  $c\theta_2 = \widehat{x}\widehat{s}$  and hence the difference between  $\theta_1$  and  $\theta_2$  increases when  $c$  decreases. Notice that a decrease in  $c$  represents an equiproportional decrease of  $c_1$  and  $c_2$ . Hence we have an increase in differentiation if both costs decrease in a proportional way and this proves the first part of the Proposition.

Part (b). Plotting  $m$  as in (26) in the range  $[0, 0.75]$  we obtain the upper left graph in Figure 2. Notice that the relation is monotonically increasing for  $x \in [0, 0.69]$ . At  $x = 0.69$ , (26) is almost at its maximum, whose approximate value is 1.3. Therefore the relation between  $m$  and  $x$  is monotonically increasing for  $m \in [0, 1.3]$ . Now, the plot of (27) in the range  $x \in [0, 0.69]$  shows that  $\theta_1/c$  is monotonically decreasing in  $x$  and hence in  $m$  (See lower left graph in Figure 2). Therefore, when the technological advantage of Firm 1 increases, its quality increases as well.

For (27), we have the following expression of the quality of Firm 2:

$$sx = -2 \frac{20x^3 - 81x^2 + 84x - 32}{8x^4 - 78x^3 + 255x^2 - 320x + 144} \quad (28)$$

Plotting (28) in the range  $x \in [0, 0.69]$  we have the lower right graph of Figure 2. Therefore, when Firm 1 increases its technological advantage,

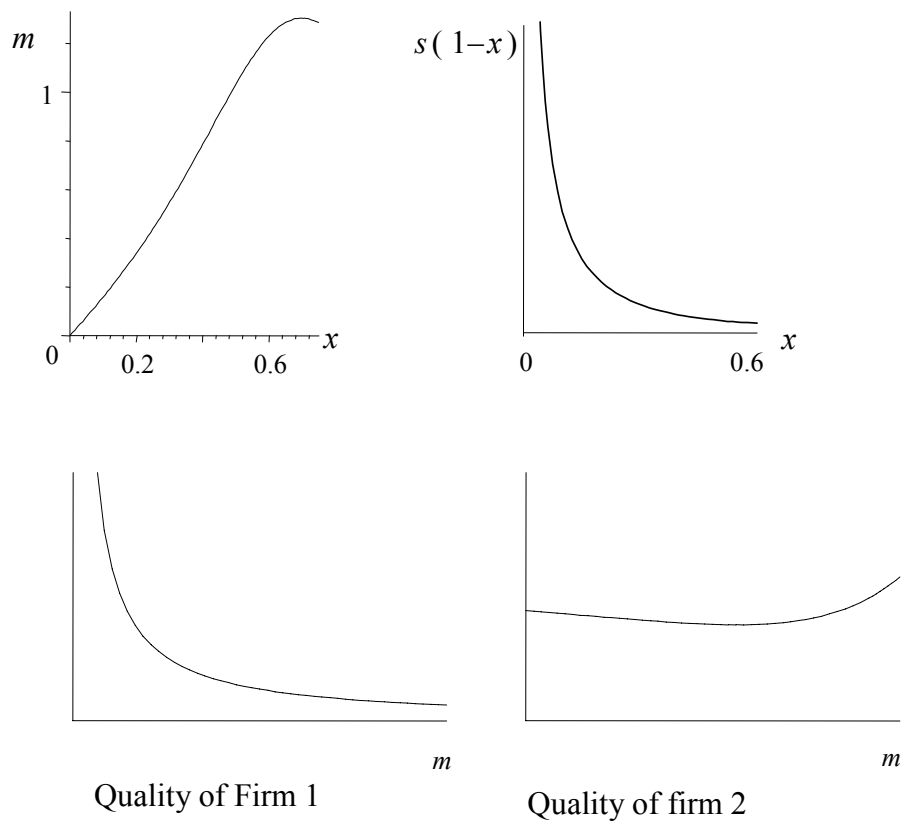


Figure 2: Comparative statics with a disproportional innovation



product quality of Firm 2 initially decreases, then increases slightly. Recall that, for given  $c$ , the differentiation increases if  $(\theta_1 - \theta_2) c = s(1 - x)$  increases. For (27) we have:

$$s(1 - x) = 2 \frac{(1 - x)(20x^3 - 81x^2 + 84x - 32)}{(4 - x)(8x^3 - 46x^2 + 71x - 36)x}$$

whose plot in the range  $x \in [0, 0.69]$  is the upper right graph in Figure 2 and which is monotonically decreasing in  $x$  and hence in  $m$ . Therefore, product differentiation increases as the technological advantage of Firm 1 increases, for given cost of Firm 2,  $c$ . ■

*Proof of Proposition 3*

Notice that a process innovation for Firm 1 is equivalent to a reduction of  $c_1$ , holding  $c_2$  constant that given our variable transformations is equivalent to a reduction in  $m$ , holding  $c$  constant. A process innovation for Firm 2, instead, is equivalent to a reduction in  $c_2$ , holding  $c_1$  constant, which after our transformation is equivalent to a decrease in  $c$  with a proportional increase in  $m$ , so that  $mc = c_1$  is constant. Using (26) and (27), Firm's 1 profits in equilibrium satisfy the following equality:

$$c\Pi_1(s(x)/c, xs(x)/c, m(x)c, c) = \frac{8(4x^2 - 11x + 6)(80x^6 - 624x^5 + 1891x^4 - 2885x^3 + 2394x^2 - 1048x + 192)}{x(-4 + x)(8x^3 - 46x^2 + 71x - 36)^3}$$

If we plot the vector:

$$\frac{c\Pi_1(s(x)/c, xs(x)/c, m(x)c, c)}{m(x)}$$

we obtain the equilibrium profit of Firm 1 as a function of  $m$  and see in Figure 3 that it is decreasing. Hence it is always profitable for Firm 1 to adopt a process innovation.

Let us move to the second Firm. Using again (26) and (27), its equilibrium profits satisfy:

$$\begin{aligned} mc\Pi_2(s(x)/c, xs(x)/c, m(x)c, c) &= \\ m(x)c\Pi_2(s(x)/c, xs(x)/c, m(x)c, c) &= \\ 8x \frac{(1-x)(x-2)^2(4x^2-9x+8)^2}{(4-x)^3} &= \\ \frac{(8x^4-42x^3+99x^2-104x+48)}{(-8x^3+46x^2-71x+36)^3} & \end{aligned}$$

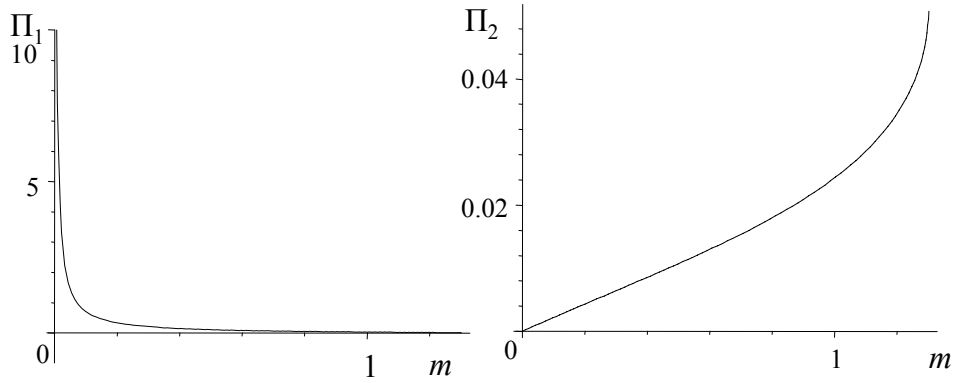


Figure 3: Profits of the two firms and asymmetries

In case of process innovation of Firm 2, the above expression is proportional to profits of Firm 2, since  $m(x)c$  is constant. Process innovation implies an increase in  $m$ . If we plot the vector:

$$m(x)c\Pi_2(s(x)/c, xs(x)/c, m(x)c, c)$$

$$\mu(x)$$

we obtain the equilibrium profit of Firm 2 as a function of  $m$  and see in Figure 3 that it is increasing. Hence it is always profitable for Firm 2 to adopt a process innovation. ■

## Appendix B

*Proof of Proposition 5.*

In order to prove the existence of a unique solution, given the set of parameters, we start maximizing the firms' profit functions, computing the first order conditions. Then we prove that the candidate equilibrium is an internal solution. Finally we show that second order conditions are satisfied.

*First order conditions*

Consider first that in the symmetric model the profit functions (15) and (16) rewrite as follows:

$$\Pi_1(a\theta_1, a\theta_2, a, a, k) / a = \frac{(2\theta_1^2 - 2\theta_1\theta_2 - \theta_1 + \theta_2)^2}{(4\theta_1 - \theta_2)^2 (\theta_1 - \theta_2)} - \frac{1}{2}ak\theta_1^2 \quad (29)$$

$$\Pi_2(a\theta_1, a\theta_2, a, a, k) / a = \frac{\theta_1 (\theta_1\theta_2 - \theta_2^2 + 2\theta_2 - 2\theta_1)^2}{\theta_2 (4\theta_1 - \theta_2)^2 (\theta_1 - \theta_2)} - \frac{1}{2}ak\theta_2^2 \quad (30)$$

Therefore maximizing (29) and (30) is equivalent to maximizing the following expressions:

$$\Pi_1(\theta_1, \theta_2, 1, 1, k) = \frac{(2\theta_1^2 - 2\theta_2\theta_1 - \theta_1 + \theta_2)^2}{(4\theta_1 - \theta_2)^2 (\theta_1 - \theta_2)} - \frac{1}{2}k\theta_1^2 \quad (31)$$

$$\Pi_2(\theta_1, \theta_2, 1, 1, k) = \theta_1 \frac{(\theta_2\theta_1 - \theta_2^2 + 2\theta_2 - 2\theta_1)^2}{(4\theta_1 - \theta_2)^2 (\theta_1 - \theta_2) \theta_2} - \frac{1}{2}k\theta_2^2 \quad (32)$$

Differentiating the above profit functions respectively for  $\theta_1$  and  $\theta_2$  we obtain the first order conditions for an internal solution:

$$F_1(\theta_1, \theta_2, k) = (2\theta_1 - 1) \frac{8\theta_1^2 - 6\theta_2\theta_1 + 4\theta_1 + 4\theta_2^2 - 7\theta_2}{(4\theta_1 - \theta_2)^3} - k\theta_1 = 0 \quad (33)$$

$$F_2(\theta_1, \theta_2, k) = (2\theta_1 - 1) \frac{8\theta_1^2 - 6\theta_2\theta_1 + 4\theta_1 + 4\theta_2^2 - 7\theta_2}{(4\theta_1 - \theta_2)^3} - k\theta_1 = 0 \quad (34)$$

In order to solve the system, we introduce the following transformations:  $\theta_1 = \theta$ ,  $\theta_2 = x\theta$ . Substituting in (33) and (34) we obtain the following equivalent conditions:

$$F_1(\theta, x\theta, k) = \theta \left( (2\theta - 1) \frac{8\theta - 6\theta x + 4 + 4x^2\theta - 7x}{(4-x)^3 \theta^3} - k \right) = 0 \quad (35)$$

$$F_2(\theta, x\theta, k) = (2 - x\theta) \frac{-4x^2 + 7\theta x^2 + 6x - 4\theta x - 8}{\theta^2 x^2 (4-x)^3} - kx\theta = 0 \quad (36)$$

It is trivial to solve (35) for  $k$ .

$$k = K(x, \theta) = (2\theta - 1) \frac{8\theta - 6\theta x + 4 + 4x^2\theta - 7x}{(4-x)^3 \theta^3} \quad (37)$$

Then substituting  $K(\cdot)$  in (36) we obtain the following equation:

$$(8x^3 - 12x^2 + 23x - 4) x^2 \theta^2 - 4(x^3 + 2x^2 + x + 2) x^2 \theta + 7x^4 - 4x^3 + 8x^2 - 12x + 16 = 0$$

which can be solved for  $x\theta$  and has the following two roots:

$$\Theta_A(x) = \frac{2x(x+2)(x^2+1) + \sqrt{(4x^6 - 8x^5 + 12x^4 - 17x^3 + 20x^2 - 24x + 4)(4-x)^2}}{(8x^3 - 12x^2 + 23x - 4)} \quad (38)$$

$$\Theta_B(x) = \frac{2x(x+2)(x^2+1) - \sqrt{(4x^6 - 8x^5 + 12x^4 - 17x^3 + 20x^2 - 24x + 4)(4-x)^2}}{(8x^3 - 12x^2 + 23x - 4)} \quad (39)$$

Notice that:  $2x(x+2)(x^2+1)$  is always positive, the discriminant is positive for  $0 \leq x < 0.193306$  and finally the denominator is positive for

$$x > \hat{x} = -\frac{1}{12} \left( 594 + 6\sqrt{39279} \right)^{\frac{1}{3}} + \frac{17}{2} \left( 594 + 6\sqrt{39279} \right)^{-\frac{1}{3}} + \frac{1}{2} \simeq 0.19043$$

Therefore the first root is positive in the interval  $\hat{x} \leq x \leq 0.193306$ , which is its relevant range and can be approximated by  $0.19043 \leq x \leq 0.193306$ . In this range  $\Theta_A(\cdot)$  can be represented as in Figure 4. The numerator of the second root is negative for  $0 \leq x \leq \hat{x}$ . Recalling that also the denominator is negative in the same region,  $0 \leq x \leq \hat{x}$  is the relevant area for the second root. The approximate values of the range of definition of the second root, whose plot is in Figure 4, are  $0 \leq x \leq 0.19043$ .

#### *Internal solutions*

Notice that prices  $p_i(\theta, x\theta, 1, 1)$  for  $i = 1, 2$ , defined in (13) and (14), are always positive for  $x < 1$ . Moreover, if we substitute (13) and (14) in (1) and (2), we obtain respectively:

$$y_1(\theta, x\theta) = \frac{2\theta_1 - 1}{4\theta_1 - \theta_2}$$

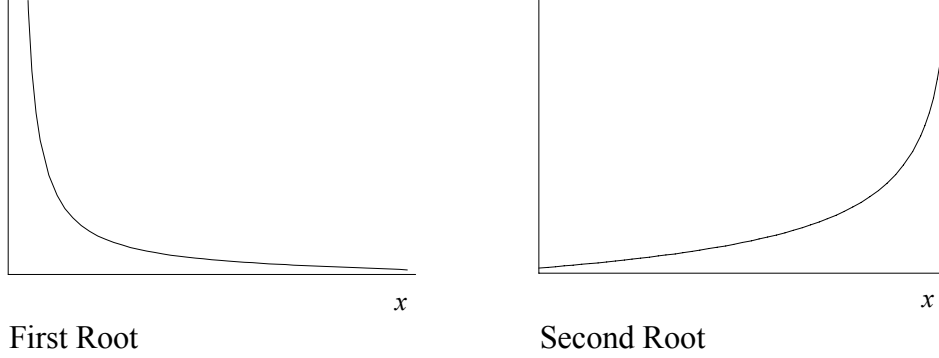


Figure 4:  $\Theta_A(\cdot)$  and  $\Theta_B(\cdot)$  respectively

$$y_2(\theta, x\theta, \cdot) = (\theta_2 - 2) \frac{\theta_1}{(4\theta_1 - \theta_2)\theta_2}$$

$y_1$  is positive if  $\theta \geq \frac{1}{2}$  and  $y_2$  if  $x\theta \geq 2$ . Therefore quantities are always positive if  $\theta_2 \geq 2$ .

Now we have to check whether profits are positive. Let us start considering the profit function of Firm 1, using  $K(x, \theta)$  as defined in (37):

$$\begin{aligned} \Pi_1(\theta, x\theta, 1, 1, K(x, \theta)) = \\ \frac{1}{2}(1 - 2\theta) \frac{14x\theta - 8\theta - 17x + 2x^2 + 12}{(4 - x)^3 \theta} \end{aligned} \quad (40)$$

is positive if and only if one of the two following conditions hold:

$$14\theta x - 8\theta + 12 + 2x^2 - 17x \geq 0 \text{ and } 1 - 2\theta \geq 0 \quad (41)$$

$$14\theta x - 8\theta + 12 + 2x^2 - 17x \leq 0 \text{ and } 1 - 2\theta \leq 0 \quad (42)$$

(41) is satisfied if  $\theta \leq \frac{1}{2}$ , which is not possible. (42) is satisfied if  $\theta \geq \frac{1}{2}$  and  $\theta \geq \frac{(2x^2 - 17x + 12)}{2(4 - 7x)}$ . Plotting  $\theta = \frac{(2x^2 - 17x + 12)}{2(4 - 7x)}$  in Figure 5, we can see that for  $\theta > 2$  the above condition is satisfied. Thus condition (42) is always satisfied.

Analogously for the second Firm:

$$\begin{aligned} \Pi_2(\theta, x\theta, 1, 1, K(x, \theta)) = \\ \frac{1}{2} \frac{-2(4x^3 - 7x^2 + 13x - 4)x^2\theta^2 + 4(x^4 + 2x^3 - 2x^2 + 10x - 8)x\theta - (x + 2)(7x^3 - 18x^2 + 28x - 16)}{\theta(4 - x)^3 x} \end{aligned} \quad (43)$$

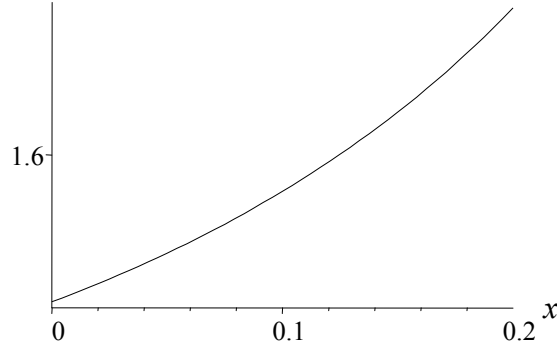


Figure 5: Condition for a positive profit of Firm 1

which is positive if:

$$G(x, \theta) = -2(4x^3 - 7x^2 + 13x - 4)x^2\theta^2 + \quad (44)$$

$$4(x^4 + 2x^3 - 2x^2 + 10x - 8)x\theta - (x + 2)(7x^3 - 18x^2 + 28x - 16) \geq 0$$

Let start substituting the first solution  $\theta = \theta_2/x = \Theta_A(x)/x$ , for  $0.19043 \leq x \leq 0.193306$  in  $G(x, \theta)$  and plot  $G(x, \Theta_A(x)/x)$  in Figure 6. Substituting the second solution  $\theta = \theta_2/x = \Theta_B(x)/x$ , for  $0 \leq x \leq 0.19043$  in  $G(x, \theta)$  and plot  $G(x, \Theta_B(x)/x)$ , we obtain the plot in Figure 6. The equation is zero for  $x = 0$  and  $x = 0.1821247$ . Therefore  $\Pi_2(\cdot)$  is positive if  $0.1821257 \leq x \leq 0.19043$ . Then we can plot  $K(x, \Theta_i(x)/x)$ , for  $i = A, B$ , in the range where both profits,  $\Pi_1$  and  $\Pi_2$ , are positive in Figure 7.

#### *Second order conditions*

Now we still have to check whether second order conditions are satisfied. Differentiating (33) and (34) with respect to  $\theta_1$  and  $\theta_2$ , and introducing the usual transformation  $\theta_1 = \theta$ ,  $\theta_2 = x\theta$ , we get the following expressions:

$$\tilde{F}_{11}(\theta, x, k) = -8(x\theta - 2) \frac{x^2\theta + 5\theta x - 5x + 2}{(4-x)^4\theta^3} - k \quad (45)$$

$$\tilde{F}_{22}(\theta, x, k) = -2 \frac{-64+64x-24x^2+12x^3-4(5+x)\theta x^3+(8+7x)x^3\theta^2}{x^3(-4+x)^4\theta^3} - k \quad (46)$$

Substituting the first root in (45) and (46) respectively, we get the graphs in Figures 6. From the graphs, we see that second order conditions are

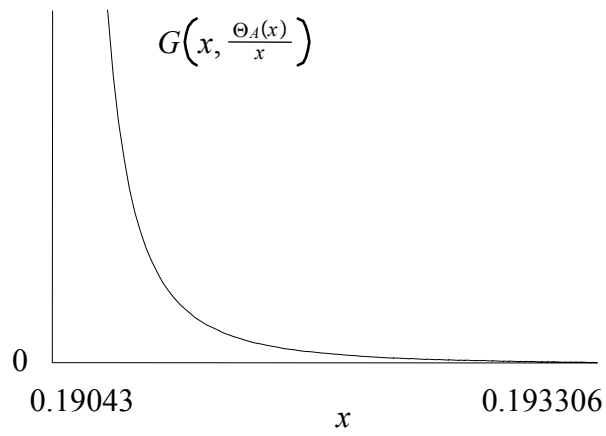
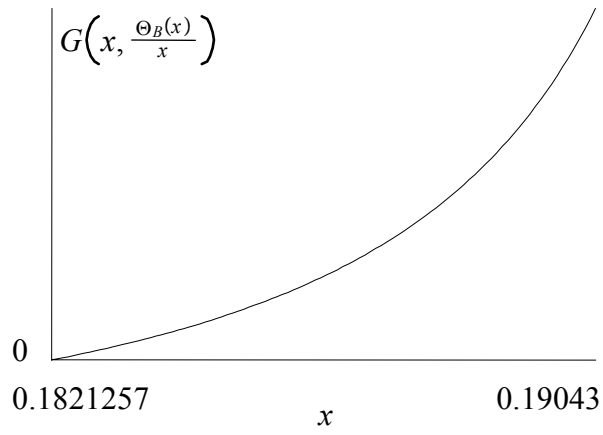


Figure 6: Positive profit for Firm 2

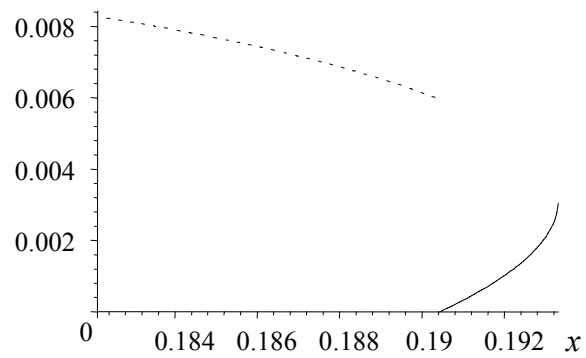


Figure 7: Relation between  $K$  and  $x$ .  $K(x, \Theta_A(x)/x)$  in solid line and  $K(x, \Theta_B(x)/x)$  in dotted line

always satisfied. Thus the relevant range for  $x$  is:  $0.1904315 \leq x \leq 0.193306$ . Therefore there exists a solution in the range:  $0 \leq k \leq 0.003042$ . Analogously, substituting the second root in (45) and (46) we reach similar results, represented in Figures 8. Second order condition for Firm 1 are always satisfied in the range of definition. Second order conditions for the second root are always satisfied in the range of positive profits, therefore  $0.182126 \leq x \leq 0.19043$ . Finally we compute the range of  $k$  where the solution exists:  $0.0059466 \leq k \leq 0.0082724$ . ■

*Proof of Proposition 6*

We have to prove that in the neighborhood of the symmetric equilibrium it is always convenient to innovate. Therefore we study the sign of the following derivatives:

$$\begin{aligned} \frac{d\Pi_i(\theta_i, \theta_j, a_i, a_j, k)}{da_i} = \\ \frac{\partial}{\partial \theta_j} \Pi_i(\theta_i, \theta_j, a_i, a_j, k) \cdot \frac{\partial \theta_j}{\partial a_i} + \frac{\partial}{\partial a_i} \Pi_i(\theta_i, \theta_j, a_i, a_j, k) \end{aligned} \quad (47)$$

with  $i, j = 1, 2$  and  $i \neq j$

Let us start with *Firm 1*. According to expression (47), we have to prove that:

$$\begin{aligned} \frac{d\Pi_1(\theta_1, \theta_2, a_1, a_2, k)}{da_1} = \\ \frac{\partial}{\partial \theta_2} \Pi_1(\theta_1, \theta_2, a_1, a_2, k) \cdot \frac{\partial \theta_2}{\partial a_1} + \frac{\partial}{\partial a_1} \Pi_1(\theta_1, \theta_2, a_1, a_2, k) \leq 0 \end{aligned} \quad (48)$$

Differentiating profit function (15) for  $\theta_1$ , we get:

$$\begin{aligned} \frac{d\Pi_1(\theta_1, \theta_2, a_1, a_2, 1)}{d\theta_1} = (2(\theta_1 - \theta_2)\theta_1 - (2\theta_1 - \theta_2)a_1 + \theta_1 a_2) \\ \frac{(2\theta_2^2 - \theta_1\theta_2 - 4\theta_1^2)a_2 + (8\theta_1^2 - 10\theta_1\theta_2 + 5\theta_2^2)a_1 - 4\theta_2^3 + 10\theta_1\theta_2^2 - 14\theta_1^2\theta_2 + 8\theta_1^3}{(4\theta_1 - \theta_2)^3(\theta_1 - \theta_2)^2} - \theta_1 \end{aligned}$$

which rewrites, in the symmetric case, as :

$$\begin{aligned} G_1(\theta_1, \theta_2, a, a, 1) = \\ (2\theta_1 - a) \frac{(4\theta_1 - 7\theta_2)a + 2(2\theta_2^2 - 3\theta_1\theta_2 + 4\theta_1^2)}{(4\theta_1 - \theta_2)^3} - \theta_1 \end{aligned} \quad (49)$$



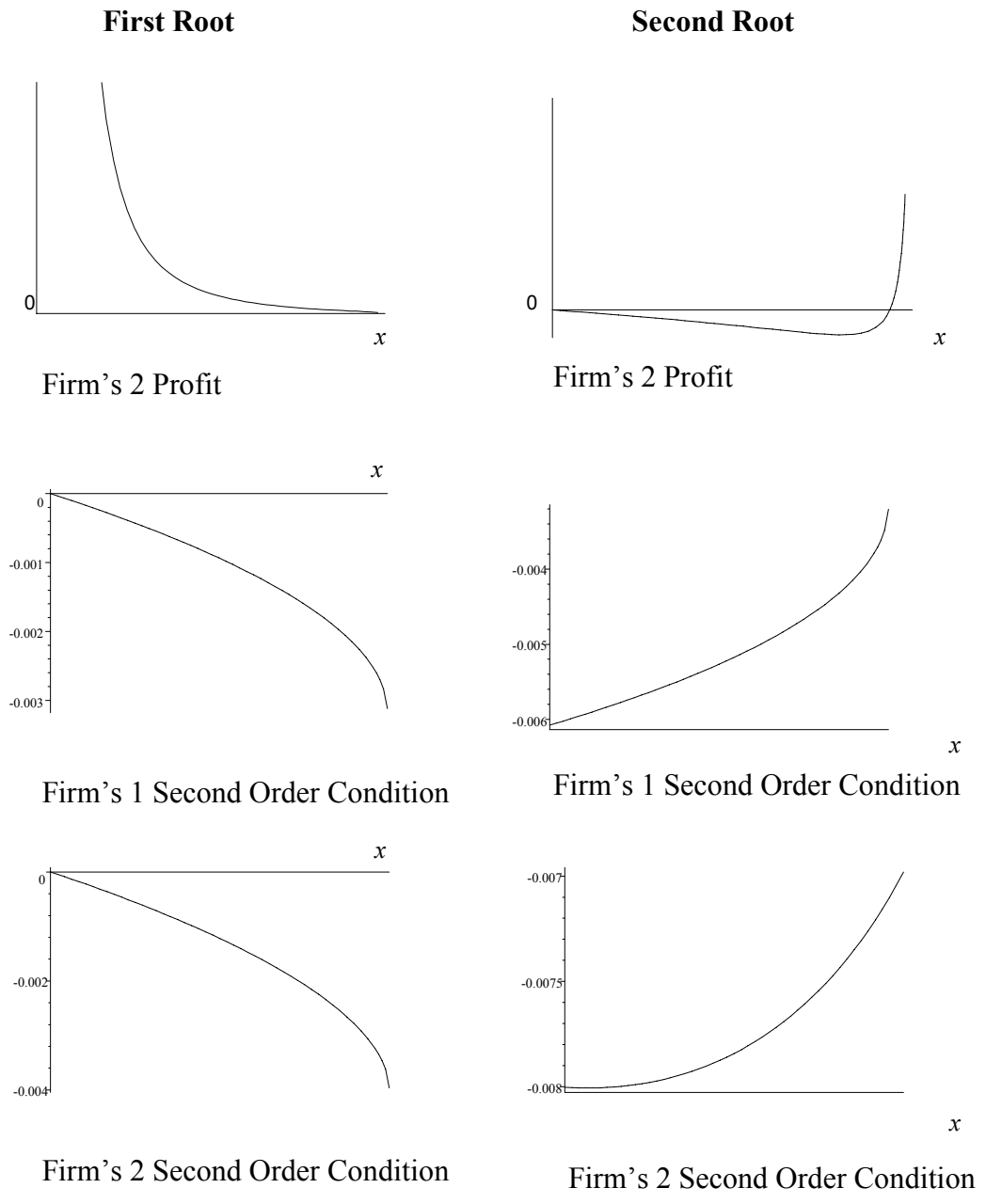


Figure 8: Positive profits and second order condition

Analogously, differentiating (15) for  $\theta_2$  we obtain:

$$\frac{\partial}{\partial \theta_2} \Pi_1(\theta_1, \theta_2, a, a, 1) = -\frac{(2\theta_1 + \theta_2)(2\theta_1 - a)^2}{(4\theta_1 - \theta_2)^3}$$

and also, differentiating (15) for  $a_1$ , we have:

$$\frac{\partial}{\partial a_1} \Pi_1(\theta_1, \theta_2, a_1, a_2, 1) = -2\frac{(2\theta_1 - \theta_2)(2\theta_1 - a)}{(4\theta_1 - \theta_2)^2}$$

By the implicit function theorem we have:

$$\frac{\partial \theta_2}{\partial a_1} = -\frac{-\frac{\partial}{\partial \theta_1} G_2(\theta_1, \theta_2, a_1, a_2) \cdot \frac{\partial}{\partial a_1} G_1(\theta_1, \theta_2, a_1, a_2) + \frac{\partial}{\partial \theta_1} G_1(\theta_1, \theta_2, a_1, a_2) \cdot \frac{\partial}{\partial a_1} G_2(\theta_1, \theta_2, a_1, a_2)}{\frac{\partial}{\partial \theta_1} G_1(\theta_1, \theta_2, a_1, a_2) \cdot \frac{\partial}{\partial \theta_2} G_2(\theta_1, \theta_2, a_1, a_2) - \frac{\partial}{\partial \theta_2} G_1(\theta_1, \theta_2, a_1, a_2) \cdot \frac{\partial}{\partial \theta_1} G_2(\theta_1, \theta_2, a_1, a_2)}{\rho_1(\theta_1, \theta_2, a_1, a_2)} \equiv$$

Finally, we get:

$$\begin{aligned} \frac{d\Pi_1(\theta_1, \theta_2, a_1, a_2, k)}{da_1} = \\ - \left( \frac{(2\theta_1 + \theta_2)(2\theta_1 - a)}{(4\theta_1 - \theta_2)} \cdot \rho_1(\theta_1, \theta_2, a_1, a_2) + 2(2\theta_1 - \theta_2) \right) \frac{(2\theta_1 - a)}{(4\theta_1 - \theta_2)^2} \end{aligned} \quad (50)$$

We should prove that  $\frac{d\Pi_1(\theta_1, \theta_2, a_1, a_2, k)}{da_1} > 0$ . Therefore, we have to show that  $(2\theta_1 - a)$  and the term in parenthesis have the same sign. Notice that in the symmetric model, where  $a_1 = a_2 = a$ , we defined  $K = ak$  and therefore  $a = K/k = \left[ K \left( x, \frac{\Theta_i(x)}{x} \right) \right] / k$ , for  $i = A, B$ , where  $\Theta_i(x)$  is defined in (38) and (39).

We start with the first root. In order to ascertain the sign of  $(2\theta_1 - a)$ , recall that:

$$\frac{1}{k}(2\theta_1 - a) = 2 \cdot \Theta_A(x) - xK \left( x, \frac{\Theta_A(x)}{x} \right)$$

and plotting the r.h.s. in the range  $0.1904315 \leq x \leq 0.193306$ , it has the slope of Figure 9 in the left. Analogously, considering the second root, the plot of  $2 \cdot \Theta_B(x) - xK \left( x, \frac{\Theta_B(x)}{x} \right)$  in the range  $0.1904315 \leq x \leq 0.193306$  becomes as in Figure 9 in the right. Given that the above expressions are both positive, we should prove that:

$$\left( \frac{(2\theta_1 + \theta_2)(2\theta_1 - a)}{(4\theta_1 - \theta_2)} \cdot \rho_1(\theta_1, \theta_2, a, a) + 2(2\theta_1 - \theta_2) \right) \geq 0$$

Introducing the usual transformation,  $\theta_1 = \theta$  and  $\theta_2 = x\theta$ , and using (37) and (38) (or (39)) the above expression becomes:

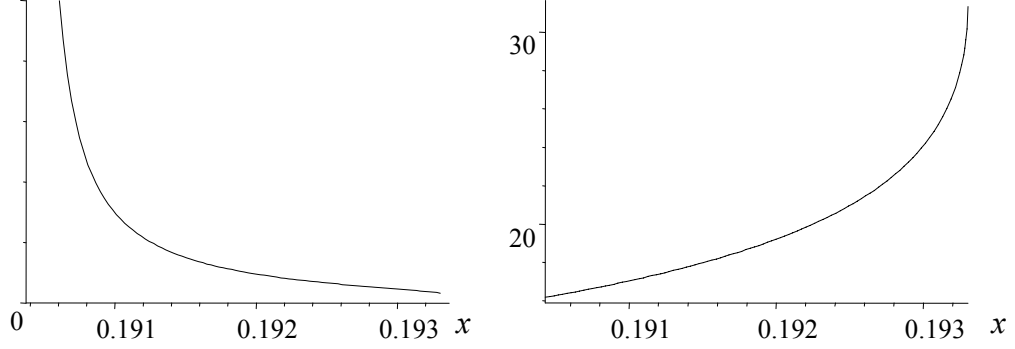


Figure 9:  $2 \cdot \Theta_i(x) - xK\left(x, \frac{\Theta_i(x)}{x}\right)$ , with  $i = A$  (left) and  $i = B$  (right)

$$\frac{(2+x) \left(2 \frac{\Theta_i(x)}{x} - K\left(x, \frac{\Theta_i(x)}{x}\right)\right)}{(4-x)} \cdot \rho_1\left(\frac{\Theta_i(x)}{x}, \Theta_i(x), K\left(x, \frac{\Theta_i(x)}{x}\right), K\left(x, \frac{\Theta_i(x)}{x}\right)\right) + \quad (51)$$

$$+ 2 \cdot \frac{\Theta_i(x)}{x} \cdot (2-x)$$

Substituting (38) in (51) we can plot the function in Figure 10. Analogously, using the second root (39) in (51) and we plot it in Figure 10. Let us consider, now, *Firm 2*. As in the previous case we should prove that:

$$\frac{d\Pi_2(\theta_1, \theta_2, a_1, a_2, k)}{da_2} = \frac{\partial}{\partial \theta_1} \Pi_2(\theta_1, \theta_2, a_1, a_2, k) \cdot \frac{\partial \theta_1}{\partial a_2} + \frac{\partial}{\partial a_2} \Pi_2(\theta_1, \theta_2, a_1, a_2, k) \leq 0$$

Differentiating profit function (16) for  $\theta_2$ , we obtain:

$$\frac{\partial \Pi_2(\theta_1, \theta_2, a_1, a_2, 1)}{\partial \theta_2} = \theta_1 \left( (\theta_1 - \theta_2) \theta_2 - (2\theta_1 - \theta_2) a_2 + a_1 \theta_2 \right) \cdot \quad (52)$$

$$\frac{(8\theta_1^3 - 18\theta_1^2\theta_2 + 9\theta_1\theta_2^2 - 2\theta_2^3)a_2 + \theta_2(4\theta_1^2 + \theta_1\theta_2 - 2\theta_2^2)a_1 + \theta_1\theta_2(\theta_1 - \theta_2)(4\theta_1 - 7\theta_2)}{\theta_2^2(4\theta_1 - \theta_2)^3(\theta_1 - \theta_2)^2} - \theta_2$$

In the symmetric case (52) rewrites:

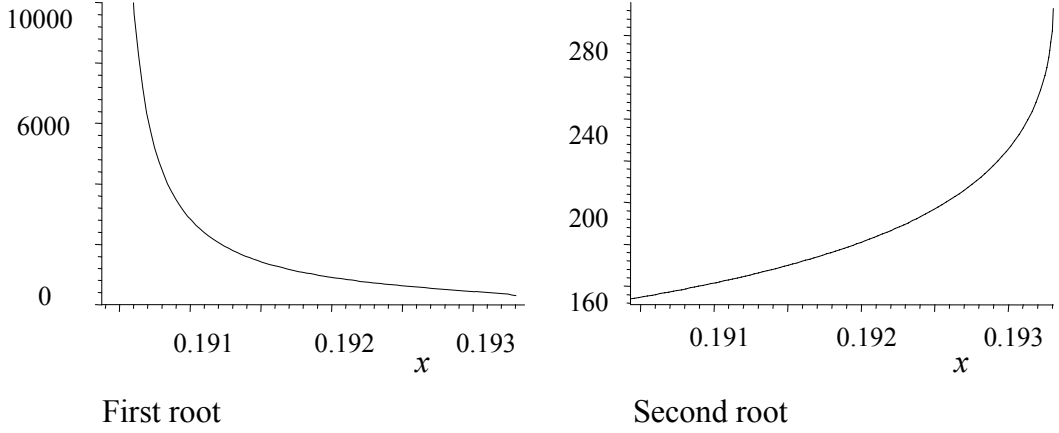


Figure 10: Condition of profitability of an infinitesimal innovation for Firm 1

$$G_2(\theta_1, \theta_2, a, a, 1) = \theta_1(\theta_2 - 2a) \quad (53)$$

$$\frac{2(2\theta_2^2 - 3\theta_1\theta_2 + 4\theta_1^2)a + \theta_1\theta_2(4\theta_1 - 7\theta_2)}{\theta_2^2(4\theta_1 - \theta_2)^3} - \theta_2$$

Moreover, differentiating (16) for  $\theta_1$  we get:

$$\frac{\partial}{\partial \theta_1} \Pi_2(\theta_1, \theta_2, a, a, 1) = (\theta_2 - 2a)^2 \frac{2\theta_1 + \theta_2}{(4\theta_1 - \theta_2)^3}$$

and differentiating (16) for  $a_2$ , we obtain:

$$\frac{\partial}{\partial a_2} \Pi_2(\theta_1, \theta_2, a_1, a_2, 1) = -2 \frac{(2\theta_1 - \theta_2)(\theta_2 - 2a)\theta_1}{(4\theta_1 - \theta_2)^2 \theta_2}$$

Finally, by the implicit function theorem:

$$\frac{\partial \theta_1}{\partial a_2} = \frac{\frac{\partial}{\partial \theta_2} F_2(\theta_1, \theta_2, a_1, a_2) \cdot \frac{\partial}{\partial a_2} F_1(\theta_1, \theta_2, a_1, a_2) - \frac{\partial}{\partial \theta_2} F_1(\theta_1, \theta_2, a_1, a_2) \cdot \frac{\partial}{\partial a_2} F_2(\theta_1, \theta_2, a_1, a_2)}{\frac{\partial}{\partial \theta_1} F_1(\theta_1, \theta_2, a_1, a_2) \cdot \frac{\partial}{\partial \theta_2} F_2(\theta_1, \theta_2, a_1, a_2) - \frac{\partial}{\partial \theta_2} F_1(\theta_1, \theta_2, a_1, a_2) \cdot \frac{\partial}{\partial \theta_1} F_2(\theta_1, \theta_2, a_1, a_2)} \equiv \rho_2(\theta_1, \theta_2, a_1, a_2)$$

Therefore, we have to prove that:

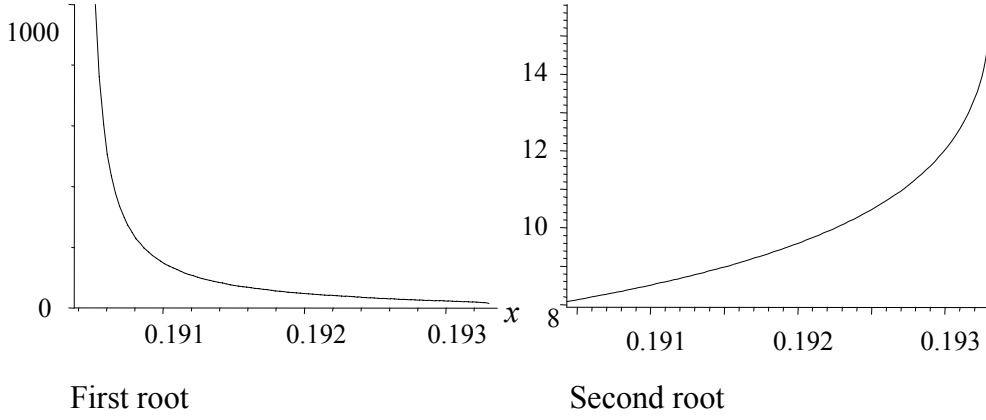


Figure 11:  $\Theta_i(x) - 2K\left(x, \frac{\Theta_i(x)}{x}\right)$ , with  $i = A$  (left) and  $i = B$  (right)

$$\frac{d\Pi_2(\theta_1, \theta_2, a_1, a_2, k)}{da_2} = \quad (54)$$

$$\left( \frac{\theta_2 - 2a}{(4\theta_1 - \theta_2)} \rho_2(\theta_1, \theta_2, a_1, a_2) - 2 \right) \frac{(2\theta_1 - \theta_2)(\theta_2 - 2a)\theta_1}{(4\theta_1 - \theta_2)^2 \theta_2} \geq 0$$

In order to ascertain the sign of  $(\theta_2 - 2a)$  recall that:

$$\frac{1}{k}(\theta_2 - 2a) = \Theta_i(x) - 2K\left(x, \frac{\Theta_i(x)}{x}\right), \quad i = A, B \quad (55)$$

Therefore we have to ascertain the sign of the r.h.s. If we use the first root (38) in order to obtain  $\Theta_i(x) - 2K\left(x, \frac{\Theta_i(x)}{x}\right)$  and plot it in the range  $0.1904315 \leq x \leq 0.193306$ , we obtain Figure 11 on the left, while if we substitute the second root we obtain the plot on the right of the same figure.

Therefore (55) is always positive. For those values (5) simplifies to:

$$\frac{x\theta - 2a}{\theta(4-x)} \rho_2(\theta, x\theta, a, a) - 2 \leq 0$$

or, introducing the usual transformation:

$$\frac{\Theta_i(x) - 2K\left(x, \frac{\Theta_i(x)}{x}\right)}{\left(4\frac{\Theta_i(x)}{x} - \Theta_i(x)\right)} \cdot \rho_2\left(\frac{\Theta_i(x)}{x}, \Theta_i(x), K\left(x, \frac{\Theta_i(x)}{x}\right), K\left(x, \frac{\Theta_i(x)}{x}\right)\right) - 2 \leq 0 \quad (56)$$

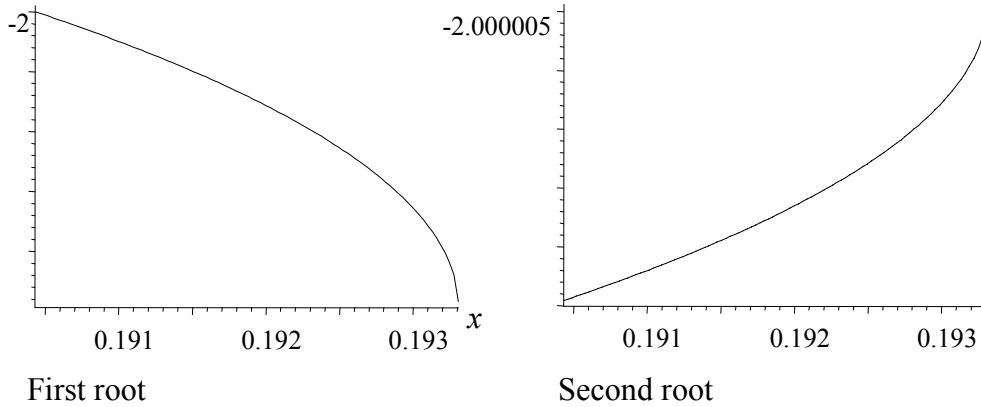


Figure 12: Condition of profitability of an infinitesimal innovation for Firm 2

whose plot in the range  $0.1904315 \leq x \leq 0.193306$  is in Figure 12 and shows that it is always verified, both for the first root and the second one. ■

**Remark 8** Notice however that if we set  $a_1 = a_2 = \frac{1}{100}$  and increase the efficiency of  $a_1$  and  $a_2$  separately of 50%, the reaction functions move as in figure 13 in the upper and lower graphs respectively. One could also show that the equilibrium moves in a higher isoprofit, even though we did not trace it in order not to complicate too much the graph.

*Proof of Proposition 7*

When the innovation is introduced we can perform comparative statics on  $a$ . Notice that in the symmetric model  $a_1 = a_2 = a$ . Let define  $\theta_i = a\tilde{\theta}_i$ , therefore  $\tilde{\theta}_1 = \Theta_i(x)/x$  with  $i = A, B$  and  $\tilde{\theta}_2 = \Theta_i(x)$  with  $i = A, B$ . We define also  $K = ak$  and therefore  $a = K/k$ . According to (38), (39) and (37), we can rewrite:

$$\begin{aligned}
 a &= \frac{K(x, \Theta_i(x)/x)}{k}, \quad \text{with } i = A, B \\
 \theta_2 &= a\Theta_i(x) = K \left( x, \frac{\Theta_i(x)}{x} \right) \frac{\Theta_i(x)}{k}, \quad \text{with } i = A, B \\
 \theta_1 &= a\Theta_i(x)/x = K \left( x, \frac{\Theta_i(x)}{x} \right) \frac{\Theta_i(x)}{kx}, \quad \text{with } i = A, B
 \end{aligned}$$

$a$ ,  $\theta_1$  and  $\theta_2$  are function of  $x$ , except for the multiplicative term  $\frac{1}{k}$ . Therefore, for comparative statics purposes, we can ignore that term. Notice also that  $\Theta_A$  is monotonically decreasing in  $x$ , as in previous Figure 4. While  $\Theta_B$  is monotonically increasing in  $x$ , as in the same Figure 4. Now we can plot  $\theta_2 = a\Theta_i(x)$  and  $\theta_1 = a\Theta_i(x)/x$  with  $i = A, B$  as a function of  $a$ . Let start with the first root: case a.

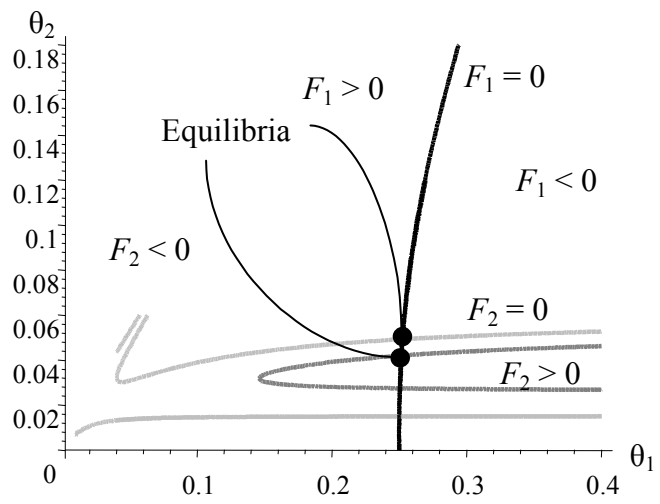
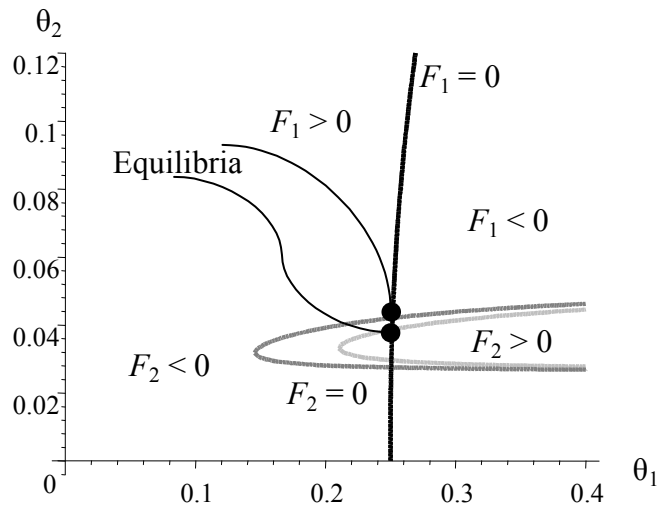


Figure 13: Discrete Changes in the Production Costs

Case a. In Figures 14, we can see  $\theta_2 = a\Theta_A(x)$  and  $\theta_1 = a\Theta_A(x)/x$  for  $0.1904315 \leq x \leq 0.193306$ .

As stated in the Proposition, the graphs show that a process innovation, decreasing  $a$ , increases  $\theta_1$  and decreases  $\theta_2$ . Considering, also, that the distance between  $\theta_1 = a\Theta_A(x)/x$  and  $\theta_2 = a\Theta_A(x)$  is a measure of differentiation, we can plot:

$$\theta_1 - \theta_2 = K \left( x, \frac{\Theta_A(x)}{x} \right) \Theta_A(x) \frac{1-x}{x} \quad (57)$$

always in Figure 14. The graph shows that the product differentiation increases. Considering a different measure of the differentiation level, such as the ratio  $x = \frac{\theta_2}{\theta_1}$ , we get the same result, as it is shown again in Figure 14.

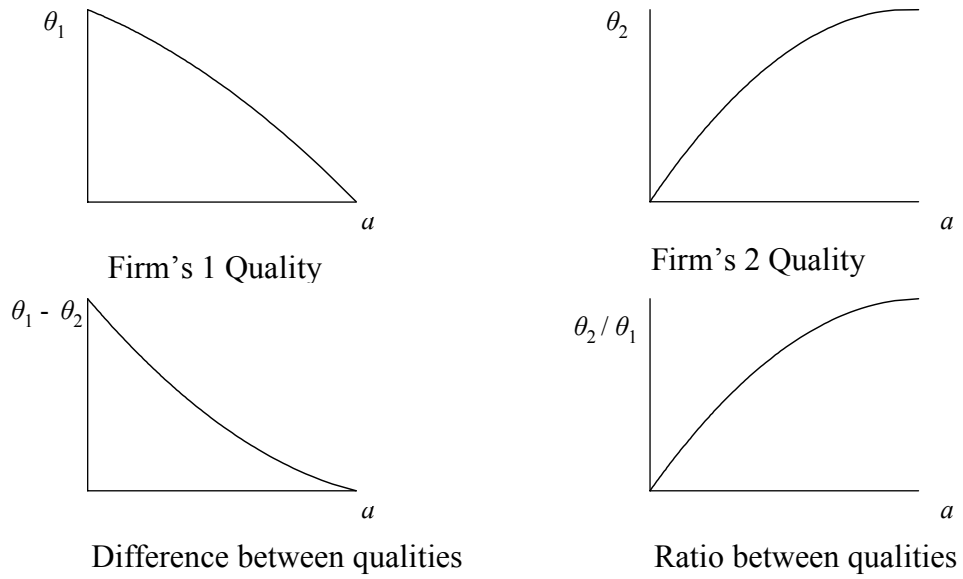
Case b. In Figure 14 we also plot  $\theta_2 = a\Theta_B(x)$  and  $\theta_1 = a\Theta_B(x)/x$  for  $0.1821257 \geq x \geq 0.19043$ . Differently from the previous case, here we have that both  $\theta_1$  and  $\theta_2$  increases with a process innovation decreasing  $a$ . Plotting the difference between  $\theta_1 = a\Theta_B(x)/x$  and  $\theta_2 = a\Theta_B(x)$  in the same Figure 14, we can see that the differentiation increases.

$$\theta_1 - \theta_2 = K \left( x, \frac{\Theta_B(x)}{x} \right) \Theta_B(x) \frac{1-x}{x} \quad (58)$$

Again, if we measure the degree of differentiation by means of the ratio  $x = \frac{\theta_1}{\theta_2}$ , we get the same result in Figure 14. As stated in the Proposition, when  $a$  decreases the differentiation grows up. ■



**First Root**



**Second Root**

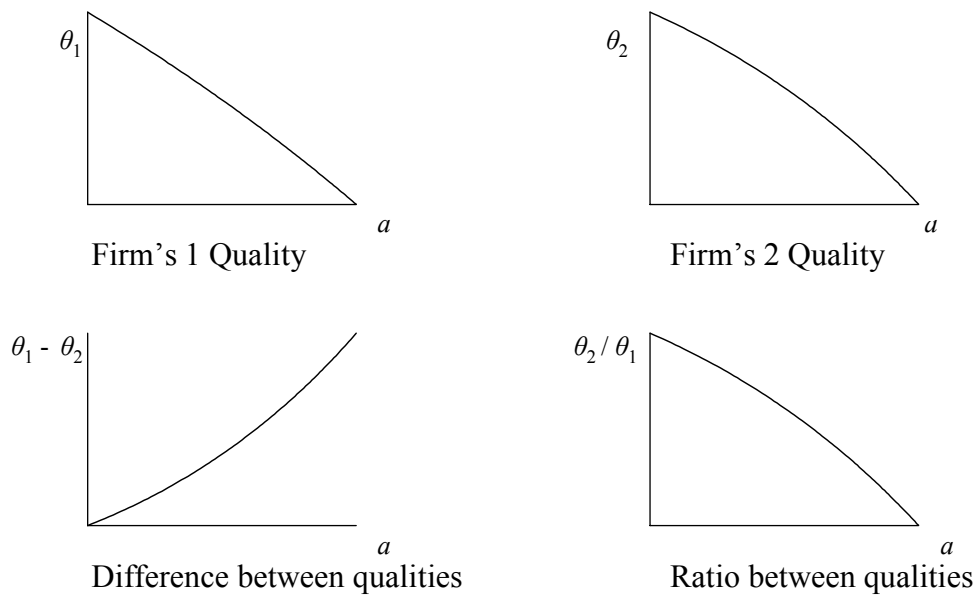


Figure 14: Comparative Statics with Fixed Costs of Quality