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Quantum Computational Fuzzy Logics

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Abstract

The theory of logical gates in quantum computation has suggested new forms of quantum logic, called *quantum computational logics*. The basic semantic idea is the following: the meaning of a sentence is identified with a *quregister* (a system of *qubits* in a pure state) or, more generally, with a *mixture* of quregisters (called *qumix*). Following an approach proposed by Domenech and Freytes, we apply residuated structures associated with fuzzy logic to develop certain aspects of information processing in quantum computing from a logical perspective. For this purpose, we introduce an axiomatic system whose natural interpretation is the irreversible quantum Poincaré algebra. Such a system allows to establish a completeness theorem for the treatment of quantum information.

Key words: quantum computation; quantum logic; quantum algebra; product Łukasiewicz logic; PMV algebra.
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1 Introduction

The theory of logical gates in quantum computation has suggested new forms of quantum logic that have been called *quantum computational logics* (6). The main difference between orthodox quantum logic (first proposed by Birkhoff and von Neumann (1)) and quantum computational logics concerns a basic semantic question: how to represent the *meanings* of the sentences of a given language? The answer given by Birkhoff and von Neumann was the following: the meanings of the elementary experimental sentences of quantum theory

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have to be regarded as determined by convenient sets of *states* of quantum objects. Since these sets should satisfy some special closure conditions, it turns out that, in the framework of orthodox quantum logic, sentences can be adequately interpreted as *closed subspaces* of the Hilbert space associated to the physical systems under investigation (1). Interesting applications of orthodox quantum logic (and of its weaker variant, *orthologic*) have been recently investigated (19; 20; 22; 23; 18). Quantum computational logics give a different answer to our basic semantic question. The *meaning* of a sentence is identified with a quantum information quantity: a *quregister* or, more generally, a *mixture* of quregisters (briefly, a *qumix*) (6). Following an approach proposed by Domenech and Freytes (7) we study the logical formalization of the treatment of quantum information during the computational process. More precisely, we want to establish a lowest bound that allows to appreciate the relevance of inputs that are known with certainty with respect to the possible outputs. In order to accomplish this purpose, we use an approximate reasoning framework, which is a crucial theme studied within fuzzy logic and we introduce an axiomatic system for quantum computational logics. We show the relation between quantum computational logics and fuzzy logics in a rigorous manner, more precisely, the relation of the axiomatic system with the infinite valued product Łukasiewicz calculus. Such a system allows to establish a completeness theorem for the treatment of quantum information.

In this paper we come to a completion of the partial results obtained by Domenech and Freytes taking the square root of the identity into account. In the simplest case this connective corresponds to the Walsh-Hadamard gate. In the Domenech and Freytes approach, one can even shift down by one dimension and replace qumixes by points of the closed disc whereas in our approach we really make use of the Poincaré sphere.

The paper is organized as follows. In sections 2-5 we include some background material needed for what follows. In section 6 we introduce a quasi product many-valued algebra and a quantum product many-valued algebra. These structures have a reduct which is a generalization of the concept of MV algebra, whence an interesting connection arises with mainstream fuzzy logic. In section 7 an irreversible quantum computational logic with a Łukasiewicz fragment is also introduced and a completeness theorem is proved in section 8.

2 Quregisters and qumixes

We will first sum up some basic concepts of quantum computation that are used in the framework of quantum computational logics. Consider the two-dimensional Hilbert space $\mathbb{C}^2$ (where any vector $|\psi\rangle$ is represented by a pair of complex numbers). Let $\mathcal{B}^{(1)} = \{|0\rangle, |1\rangle\}$ be the canonical orthonormal basis for $\mathbb{C}^2$, where $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Recalling the Born rule, any $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ (with $|c_0|^2 + |c_1|^2 = 1$) can be regarded as an *uncertain piece of information*, where the answer *NO* has probability $|c_0|^2$, while the answer *YES* has probability $|c_1|^2$. The two basis-elements $|0\rangle$ and $|1\rangle$ are usually taken as encoding the classical bit-values 0 and 1, respectively. From a semantic point of view, they can be also regarded as the classical truth-values *Falsity* and *Truth*.

Definition 1 *Quregister.*
An n-quregister *is represented by a unit vector in the n-fold tensor product Hilbert space* $\otimes^n\mathbb{C}^2 := \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-times}$.

We will use $x, y, \ldots$ as variables ranging over the set $\{0, 1\}$. At the same time, $|x\rangle, |y\rangle, \ldots$ will range over the basis $\mathcal{B}^{(1)}$. Any factorized unit vector $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ of the space $\otimes^n\mathbb{C}^2$ will be called an *n-configuration* (which can be regarded as a quantum realization of a classical bit sequence of length n). Instead of $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$ we will also write $|x_1, \ldots, x_n\rangle$. Recall that the dimension of $\otimes^n\mathbb{C}^2$ is 2^n, while the set of all n-configurations $\mathcal{B}^{(n)} = \{|x_1, \ldots, x_n\rangle : x_1, \ldots, x_n \in \{0, 1\}\}$ is an orthonormal basis for the space $\otimes^n\mathbb{C}^2$. We will call this set a *computational basis* for the n-quregisters. Since any element of the computational basis can be labeled by a binary string which represents a natural number $j \in [0, 2^n - 1]$ in binary notation (where $j = 2^{n-1}x_1 + 2^{n-2}x_2 + \ldots + x_n$), any quregister can be briefly expressed as a superposition having the following form: $\sum_{j=0}^{2^n-1} c_j|j\rangle$, where $c_j \in \mathbb{C}$, $|j\rangle$ is the n-configuration corresponding to the number j and $\sum_{j=0}^{2^n-1} |c_j|^2 = 1$.

For semantic aims, it is useful to distinguish the *true* from the *false* in any space $\otimes^n\mathbb{C}^2$. We assume the following convention (which is a natural generalization of classical semantics): any n-configuration corresponds to a classical truth-value that is determined by its last element (i.e. $x_n = 1 :=$ *true* and $x_n = 0 := false$ or, in other words, by the parity of j, i.e. odd:=true and even:=false). Let us now decompose the Hilbert space $\otimes^n\mathbb{C}^2$ into its true and false subspaces $\otimes^n\mathbb{C}^2_0$ and $\otimes^n\mathbb{C}^2_1$ respectively, i.e. $\otimes^n\mathbb{C}^2 = \otimes^n\mathbb{C}^2_0 \oplus \otimes^n\mathbb{C}^2_1$, and denote by $P_1^{(n)}$ and $P_0^{(n)}$ the pertaining orthogonal projectors, $P_1^{(n)} + P_0^{(n)} = I^{(n)}$, where $I^{(n)}$ is the identity operator of $\otimes^n\mathbb{C}^2$. Therefore, the projectors $P_1^{(n)}$ and $P_0^{(n)}$ represent the *Truth-property* and the *Falsity-property* in $\otimes^n\mathbb{C}^2$, respectively. Let $\mathfrak{D}(\otimes^n\mathbb{C}^2)$ be the set of all positive trace class operators of $\otimes^n\mathbb{C}^2$ and let $\mathfrak{D} := \bigcup_{n=1}^{\infty} \mathfrak{D}(\otimes^n\mathbb{C}^2)$.

Definition 2 *Qumix.*
A qumix *is a density operator in* $\mathfrak{D}$.

Needless to say, quregisters correspond to particular qumixes that are *pure states* (i.e. projections onto one-dimensional closed subspaces of $\otimes^n \mathbb{C}^2$). Recalling the Born rule, we can now define the *probability-value* of any qumix.

Definition 3 *Probability of a qumix.*
For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$*:* $\mathtt{p}(\rho) = \mathtt{tr}(\rho P_1^{(n)})$.

$\mathtt{p}(\rho)$ is the probability that the information stocked by the qumix ρ is true. In the particular case where ρ corresponds to the 1-quregister

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

we obtain that $\mathtt{p}(\rho) = |c_1|^2$.

For any quregister $|\psi\rangle$, we will write $\mathtt{p}(|\psi\rangle)$ instead of $\mathtt{p}(P_{|\psi\rangle})$, where $P_{|\psi\rangle}$ (also indicated by $|\psi\rangle\langle\psi|$) is the density operator represented by the projection onto the one-dimensional subspace spanned by the vector $|\psi\rangle$. In particular, we have the matching of notions $P_0^{(1)} \equiv P_{|0\rangle}$ and $P_1^{(1)} \equiv P_{|1\rangle}$, with the projector also representing a pure state.

An interesting relation connects qumixes with the real numbers in the interval $[0, 1]$. For any $n \in \mathbb{N}^+$, any real number $\lambda \in [0, 1]$ uniquely determines a qumix $\rho_\lambda^{(n)}$:

$$\rho_\lambda^{(n)} := (1 - \lambda) k_n P_0^{(n)} + \lambda k_n P_1^{(n)}$$

(where $k_n = \frac{1}{2^{n-1}}$ is a normalization coefficient). From an intuitive point of view, $\rho_\lambda^{(n)}$ represents a *mixture of pieces of information* that might correspond to the *Truth* with probability λ. We will also write ρ_λ instead of $\rho_\lambda^{(1)}$.

3 Quantum Gates

In quantum computation, *quantum logical gates* (briefly, *gates*) are unitary operators that transform quregisters into quregisters. Being unitary, gates represent characteristic *reversible transformations*. The canonical gates (which are studied in the literature) can be naturally generalized to qumixes. Generally, gates correspond to some basic *logical operations* that admit a reversible behaviour. We will consider here the following gates: the *not*, the *Petri-Toffoli's* (17; 21) (also called *controlled-controlled-not*), the *controlled-not*, the *square root of the not*, the *square root of the identity*, the *Lukasiewicz's*.

Let us first describe our gates in the framework of quregisters.

Definition 4 *The not gate.*
For any $n \geq 1$*, the not gate on* $\otimes^n \mathbb{C}^2$ *is the linear operator* $\mathtt{Not}^{(n)}$ *such that*

for every element $|x_1,\ldots,x_n\rangle$ *of the computational basis* $\mathcal{B}^{(n)}$*:*

$$\texttt{Not}^{(n)}(|x_1,\ldots,x_{n-1},x_n\rangle) = |x_1,\ldots,x_{n-1},1-x_n\rangle.$$

In other words, $\texttt{Not}^{(n)}$ inverts the value of the last element of basis-vector of $\otimes^n\mathbb{C}^2$.

Clearly, $\texttt{Not}^{(n)} = I^{(n-1)} \otimes X$, where X is the Pauli matrix, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 5 *The Petri-Toffoli gate.*
For any $n \geq 1$ *and any* $m \geq 1$ *the Petri-Toffoli gate is the linear operator* $T^{(n,m,1)}$ *defined on* $\otimes^{n+m+1}\mathbb{C}^2$ *such that for every element* $|x_1,\ldots,x_n\rangle \otimes |y_1,\ldots,y_m\rangle \otimes |z\rangle$ *of the computational basis* $\mathcal{B}^{(n+m+1)}$*:*

$$T^{(n,m,1)}(|x_1,\ldots,x_n\rangle\otimes|y_1,\ldots,y_m\rangle\otimes|z\rangle) = |x_1,\ldots,x_n\rangle\otimes|y_1,\ldots,y_m\rangle\otimes|x_n y_m \boxplus z\rangle,$$

where $\boxplus$ *represents the sum modulo* 2.

Clearly, $T^{(n,m,1)} = (I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)}) \otimes I^{(1)} + P_1^{(n)} \otimes P_1^{(m)} \otimes X$.

One can easily show that both $\texttt{Not}^{(n)}$ and $T^{(n,m,1)}$ are unitary operators.

Consider now the set $\mathfrak{R} = \bigcup_{n=1}^{\infty} \otimes^n\mathbb{C}^2$ (which contains all quregisters $|\psi\rangle$ "living" in $\otimes^n\mathbb{C}^2$, for an $n \geq 1$). The gates Not and T can be uniformly defined on this set in the expected way:

$$\texttt{Not}(|\psi\rangle) := \texttt{Not}^{(n)}(|\psi\rangle), \qquad \textit{if } |\psi\rangle \in \otimes^n\mathbb{C}^2$$

$$T(|\psi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle) := T^{(n,m,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |\chi\rangle), \ \textit{if } |\psi\rangle \in \otimes^n\mathbb{C}^2, |\varphi\rangle \in \otimes^m\mathbb{C}^2 \textit{ and } |\chi\rangle \in \otimes^1\mathbb{C}^2.$$

On this basis, a conjunction And, a disjunction Or can be defined for any pair of quregisters $|\psi\rangle$ and $|\varphi\rangle$:

$$\texttt{And}(|\psi\rangle, |\varphi\rangle) := T(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle);$$

$$\texttt{Or}(|\psi\rangle, |\varphi\rangle) := \texttt{Not}(\texttt{And}(\texttt{Not}(|\psi\rangle), \texttt{Not}(|\varphi\rangle))).$$

Notice that our definition of And is reversible and, as such, needs a third ancillary system. Indeed, in this framework, $\texttt{And}(|\psi\rangle, |\varphi\rangle)$ should be regarded as a metalinguistic abbreviation for $T(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle)$. A similar observation holds for Or.

One can easily verify that, when applied to classical bits, Not, And and Or behave as the standard Boolean truth-functions.

An exclusive disjunction Xor can be defined by using the *controlled-not gate.*

Definition 6 *The controlled-not gate.*
For any $n \geq 1$ and any $m \geq 1$ the controlled-not gate is the linear operator $\mathtt{Xor}^{(n,m)}$ defined on $\otimes^{n+m}\mathbb{C}^2$ such that for every element $|x_1,\ldots,x_n\rangle \otimes |y_1,\ldots,y_m\rangle$ of the computational basis $\mathcal{B}^{(n+m)}$:

$$\mathtt{Xor}^{(n,m)}(|x_1,\ldots,x_n\rangle \otimes |y_1,\ldots,y_m\rangle) = |x_1,\ldots,x_n\rangle \otimes |y_1,\ldots,y_{m-1},x_n \boxplus y_m\rangle,$$

where $\boxplus$ represents the sum modulo 2.

Clearly, $\mathtt{Xor}^{(n,m)} = P_0^{(n)} \otimes I^{(m)} + P_1^{(n)} \otimes \mathtt{Not}^{(m)}$.

The gate Xor can be uniformly defined in the expected way:

$$\mathtt{Xor}(|\psi\rangle \otimes |\varphi\rangle) := \mathtt{Xor}^{(n,m)}(|\psi\rangle \otimes |\varphi\rangle) \;\; \textit{if}\, |\psi\rangle \in \otimes^n\mathbb{C}^2 \,\textit{and}\, |\varphi\rangle \in \otimes^m\mathbb{C}^2.$$

The quantum logical gates we have considered so far are, in a sense, "semi-classical". A quantum logical behaviour only emerges in the case where our gates are applied to superpositions. When restricted to classical registers, such operators turn out to behave as classical (reversible) truth-functions. We will now consider two important *genuine quantum gates* that transform classical registers (elements of $\mathcal{B}^{(n)}$) into quregisters that are superpositions: the *square root of the not* and the *square root of the identity.*

Definition 7 *The m-th root of the not.*
For any $n \geq 1$, the m-th root of the not on $\otimes^n\mathbb{C}^2$ is the linear operator $\sqrt[m]{\mathtt{Not}}^{(n)}$ such that for every element $|x_1,\ldots,x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\sqrt[m]{\mathtt{Not}}^{(n)}(|x_1,\ldots,x_n\rangle) = |x_1,\ldots,x_{n-1}\rangle \otimes \frac{1}{2}((1+e^{i\frac{\pi}{m}})|x_n\rangle + (1-e^{i\frac{\pi}{m}})|1-x_n\rangle),$$

where $i := \sqrt{-1}$.

One can easily show that $\sqrt[m]{\mathtt{Not}}^{(n)}$ is a unitary operator. The basic property of $\sqrt[m]{\mathtt{Not}}^{(n)}$ is the following:

$$\textit{for any}\ |\psi\rangle \in \otimes^n\mathbb{C}^2,\ \underbrace{\sqrt[m]{\mathtt{Not}}^{(n)}(\ldots\ \sqrt[m]{\mathtt{Not}}^{(n)}}_{m}(|\psi\rangle)\ldots) = \mathtt{Not}^{(n)}(|\psi\rangle).$$

In other words, applying m times the m-th root of the not means negating.

Clearly, $\sqrt[m]{\mathtt{Not}}^{(n)} = I^{(n-1)} \otimes M$, where $M := \frac{1}{2}\begin{pmatrix} 1+e^{i\frac{\pi}{m}} & 1-e^{i\frac{\pi}{m}} \\ 1-e^{i\frac{\pi}{m}} & 1+e^{i\frac{\pi}{m}} \end{pmatrix}$.

From a logical point of view, $\sqrt{\texttt{Not}}^{(n)}$ can be regarded as a "tentative partial negation" (a kind of "half negation") that transforms *precise pieces of information* into *maximally uncertain* ones. For, we have:

$$\texttt{p}(\sqrt{\texttt{Not}}^{(1)}(|1\rangle)) = \frac{1}{2} = \texttt{p}(\sqrt{\texttt{Not}}^{(1)}(|0\rangle)).$$

As expected, the square root of the not has no Boolean counterpart. Clearly, there exists no function $f : \{0,1\} \to \{0,1\}$ such that for any $x \in \{0,1\}$: $f(f(x)) = 1 - x$, since such a function is none of the possible four.

Interestingly enough, $\sqrt{\texttt{Not}}$ also does not have a continuous fuzzy counterpart.

Lemma 8 *There is no continuous function $f : [0,1] \to [0,1]$ such that for any $x \in [0,1]$: $f(f(x)) = 1 - x$ (6).*

Definition 9 *The m-th root of the identity.*
For any $n \geq 1$, the m-th root of the identity on $\otimes^n \mathbb{C}^2$ is the linear operator $\sqrt[m]{\texttt{I}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:
$\sqrt[m]{\texttt{I}}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle$
$\otimes \frac{1}{2\sqrt{2}}([(-1)^{1-x_n} + \sqrt{2} + ((-1)^{x_n} + \sqrt{2})e^{i\frac{2\pi}{m}}]|x_n\rangle + [1 - e^{i\frac{2\pi}{m}}]|1 - x_n\rangle)$.

The basic property of $\sqrt[m]{\texttt{I}}^{(n)}$ is the following:

$$\textit{for any } |\psi\rangle \in \otimes^n \mathbb{C}^2,\ \underbrace{\sqrt[m]{\texttt{I}}^{(n)}(\ldots \sqrt[m]{\texttt{I}}^{(n)}}_{m}(|\psi\rangle)\ldots) = |\psi\rangle.$$

Clearly, $\sqrt{\texttt{I}}^{(n)} = I^{(n-1)} \otimes H$, where H is the Walsh-Hadamard matrix $H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

As happens in the case of $\sqrt{\texttt{Not}}^{(n)}$, also $\sqrt{\texttt{I}}^{(n)}$ can be regarded as a "tentative partial assertion" (a kind of "half assertion") that transforms *precise pieces of information* into *maximally uncertain* ones. Apparently, one application of $\sqrt{\texttt{I}}^{(n)}$ to a precise information produces a *maximal disorder*, while two applications of $\sqrt{\texttt{I}}^{(n)}$ lead back to the initial information.

The gates considered so far can be naturally generalized to qumixes.

Definition 10 *Gates.*
For any qumix $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$,

$$\mathrm{G}^{(\mathrm{n})}(\rho) = \texttt{G}^{(n)} \rho\, \texttt{G}^{(n)\dagger},$$

where $\mathtt{G}^{(n)\dagger}$ *is the adjoint of* $\mathtt{G}^{(n)}$*.*

When our gates will be applied to density operators, we will use capital letters. Like in the quregister-case, the gates $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$, $\sqrt{\mathbb{I}}$, $\mathtt{AND}$, can be uniformly defined on the set $\mathfrak{D}$ of all qumixes.

The following theorems describe some basic properties of our gates.

Theorem 11 *(10)*

(1) $\forall n \in \mathbb{N}^+$*:* $\mathtt{NOT}(k_n\, P_0^{(n)}) = k_n\, P_1^{(n)}$*;*
(2) $\forall n \in \mathbb{N}^+$*:* $\mathtt{NOT}(k_n\, P_1^{(n)}) = k_n\, P_0^{(n)}$*;*
(3) $\mathtt{p}(\mathtt{NOT}(\rho)) = 1 - \mathtt{p}(\rho)$*.*

Theorem 12 *(10)*

(1) $\sqrt{\mathtt{NOT}}(\sqrt{\mathtt{NOT}}(\rho)) = \mathtt{NOT}(\rho)$*;*
(2) $\sqrt{\mathtt{NOT}}(\mathtt{NOT}(\rho)) = \mathtt{NOT}(\sqrt{\mathtt{NOT}}(\rho))$*;*
(3) $\forall n \in \mathbb{N}^+$*:* $\mathtt{p}(\sqrt{\mathtt{NOT}}(k_n P_1^{(n)})) = \mathtt{p}(\sqrt{\mathtt{NOT}}(k_n P_0^{(n)})) = \frac{1}{2}$*.*

Theorem 13 *(2; 10)*

(1) $\mathtt{p}(\mathtt{AND}(\rho, \sigma)) = \mathtt{p}(\rho)\mathtt{p}(\sigma)$*;*
(2) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\mathtt{AND}(\rho, \sigma))) = \frac{1}{2}$*.*

Theorem 14 *(6)*

(1) $\sqrt{\mathbb{I}}(\sqrt{\mathbb{I}}(\rho)) = \rho$*;*

(2) $\forall n \in \mathbb{N}^+$*:* $\mathtt{p}(\sqrt{\mathbb{I}}(k_n P_1^{(n)})) = \mathtt{p}(\sqrt{\mathbb{I}}(k_n P_0^{(n)})) = \frac{1}{2}$*;*

(3) $\forall n \in \mathbb{N}^+$*:* $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(k_n P_1^{(n)}))) = \mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(k_n P_0^{(n)}))) = \frac{1}{2}$*;*

(4) $\forall n \in \mathbb{N}^+$*:* $\mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(k_n P_1^{(n)}))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(k_n P_0^{(n)}))) = \frac{1}{2}$*;*

(5) $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(\rho))) = \mathtt{p}(\sqrt{\mathbb{I}}(\rho))$*;*

(6) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(\rho))) = 1 - \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$*;*

(7) $\mathtt{p}(\sqrt{\mathbb{I}}(\mathtt{AND}(\rho, \sigma))) = \frac{1}{2}$*;*

(8) $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(\mathtt{AND}(\rho, \sigma)))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(\mathtt{AND}(\rho, \sigma)))) = \frac{1}{2}$*.*

The gates we have considered so far represent typical *reversible* logical operations. Are there other interesting irreversible operations that might be considered in quantum computation? Some natural candidates are represented, for instance, by a Łukasiewicz-like conjunction and a Łukasiewicz-like disjunction (6).

Definition 15 *The Łukasiewicz disjunction.*
Let $\tau \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$ *and* $\sigma \in \mathfrak{D}(\otimes^m \mathbb{C}^2)$.

$$\tau \oplus \sigma := \rho^{(1)}_{\mathtt{p}(\tau)\oplus\mathtt{p}(\sigma)},$$

where $\oplus$ *in* $\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma)$ *is the Łukasiewicz "truncated sum" defined on the real interval* $[0,1]$ *(i.e.* $\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma) = \min\{1, \mathtt{p}(\tau) + \mathtt{p}(\sigma)\}$*) (24).*

The following theorems sum up some basic properties of the Łukasiewicz disjunction:

Theorem 16 *Let* $\tau \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$ *and* $\sigma \in \mathfrak{D}(\otimes^m \mathbb{C}^2)$ *(6).*

(1) $\tau \oplus \sigma = \begin{cases} \rho^{(1)}_{\mathtt{p}(\tau)+\mathtt{p}(\sigma)}, & \text{if } \mathtt{p}(\tau) + \mathtt{p}(\sigma) \leq 1; \\ P^{(1)}_1, & \text{otherwise}; \end{cases}$
(2) $\mathtt{p}(\tau \oplus \sigma) = \mathtt{p}(\tau) \oplus \mathtt{p}(\sigma)$;
(3) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\tau \oplus \sigma)) = \frac{1}{2}$;
(4) $\mathtt{p}(\sqrt{\mathbb{I}}(\tau \oplus \sigma)) = \frac{1}{2}$;
(5) $\mathtt{p}(\sqrt{\mathbb{I}}(\sqrt{\mathtt{NOT}}(\tau \oplus \sigma))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\sqrt{\mathbb{I}}(\tau \oplus \sigma))) = \frac{1}{2}$.

Lemma 17 *Let* $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$.

(1) $\forall n \in \mathbb{N}^+$: $\rho \oplus k_n P^{(n)}_1 = P^{(1)}_1$;
(2) $\forall n \in \mathbb{N}^+$: $\rho \oplus k_n P^{(n)}_0 = \rho^{(1)}_{\mathtt{p}(\rho)}$;
(3) $\rho \oplus \mathtt{NOT}(\rho) = P^{(1)}_1$.

PROOF. Straightforward.

From Lemma 17 it follows that $\mathtt{p}(\rho \oplus k_n P^{(n)}_1) = 1$, $\mathtt{p}(\rho \oplus k_n P^{(n)}_0) = \mathtt{p}(\rho)$ and $\mathtt{p}(\rho \oplus \mathtt{NOT}(\rho)) = 1$.

An interesting preorder relation can be defined on the set $\mathfrak{D}$ of all qumixes.

Definition 18 *Preorder.*
$\rho \preccurlyeq \sigma$ *iff the following conditions hold:*

(1) $\mathtt{p}(\rho) \leq \mathtt{p}(\sigma)$;
(2) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\sigma)) \leq \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))$;
(3) $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) \leq \mathtt{p}(\sqrt{\mathbb{I}}(\sigma))$.

One immediately shows that $\preccurlyeq$ is reflexive and transitive, but not antisymmetric (take, for example, $\rho = \frac{1}{2}I$ and $\sigma = P_{\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)}$). From an intuitive point of view, $\rho \preccurlyeq \sigma$ means that the information σ is "closer to the truth" than the information ρ.

The preorder $\preccurlyeq$ permits us to define on the set of all qumixes an equivalence relation $\equiv$ in the expected way.

Definition 19 *Equivalence.*

$$\rho \equiv \sigma \text{ iff } \rho \preccurlyeq \sigma \text{ and } \sigma \preccurlyeq \rho.$$

Clearly, $\equiv$ is an equivalence relation. Let

$$[\mathfrak{D}]_\equiv := \{[\rho]_\equiv \,:\, \rho \in \mathfrak{D}\}.$$

Unlike the qumixes (which are only preordered by $\preccurlyeq$), the equivalence-classes of $[\mathfrak{D}]_\equiv$ can be partially ordered in a natural way.

Definition 20 *Partial order.*

$$[\rho]_\equiv \preccurlyeq [\sigma]_\equiv \text{ iff } \rho \preccurlyeq \sigma.$$

The relation $\preccurlyeq$ (which is well defined) is a partial order.

Lemma 21

(1) $\forall n \in \mathbb{N}^+$*:* $[P_1]_\equiv = \left[k_n P_1^{(n)}\right]_\equiv$*;*
(2) $\forall n \in \mathbb{N}^+$*:* $[P_0]_\equiv = \left[k_n P_0^{(n)}\right]_\equiv$*;*
(3) $\forall n \in \mathbb{N}^+ \forall \lambda \in [0,1]$*:* $\left[\rho_\lambda^{(1)}\right]_\equiv = \left[\rho_\lambda^{(n)}\right]_\equiv$*.*

PROOF. Straightforward.

One can prove that $\equiv$ is a congruence relation with respect to the operations $\mathtt{AND}$, $\oplus$, $\mathtt{NOT}$, $\sqrt{\mathtt{NOT}}$, $\sqrt{\mathbb{I}}$.

In this framework, we can define, in the expected way, the operations:

Definition 22 *Operations.*
Let $\rho \in \mathfrak{D}(\otimes^n \mathbb{C}^2)$ *and* $\sigma \in \mathfrak{D}(\otimes^m \mathbb{C}^2)$*.*

(1) $[\rho]_\equiv \mathtt{AND} [\sigma]_\equiv = [\mathtt{AND}(\rho,\sigma)]_\equiv$*;*

(2) $[\rho]_\equiv \oplus [\sigma]_\equiv = [\rho \oplus \sigma]_\equiv$;
(3) $\mathtt{NOT}([\rho]_\equiv) = [\mathtt{NOT}(\rho)]_\equiv$;
(4) $\sqrt{\mathtt{NOT}}([\rho]_\equiv) = [\sqrt{\mathtt{NOT}}(\rho)]_\equiv$;
(5) $\sqrt{\mathbb{I}}([\rho]_\equiv) = [\sqrt{\mathbb{I}}(\rho)]_\equiv$.

Lemma 23

(1) The operation AND *is associative and commutative;*
(2) The operation $\oplus$ *is associative and commutative;*
(3) $\mathtt{NOT}(\mathtt{NOT}([\rho]_\equiv)) = [\rho]_\equiv$;
(4) $\sqrt{\mathtt{NOT}}(\sqrt{\mathtt{NOT}}([\rho]_\equiv)) = \mathtt{NOT}([\rho]_\equiv)$;
(5) $\sqrt{\mathbb{I}}(\sqrt{\mathbb{I}}([\rho]_\equiv)) = [\rho]_\equiv$.

PROOF. Straightforward.

On this basis, we can define the following quotient-structure:

Definition 24 *The standard irreversible quantum computational algebra. The structure*

$$([\mathfrak{D}]_\equiv, \mathtt{AND}, \oplus, \mathtt{NOT}, \sqrt{\mathtt{NOT}}, \sqrt{\mathbb{I}}, [P_0^{(1)}]_\equiv, [P_1^{(1)}]_\equiv, [\rho^{(1)}_{\frac{1}{2}}]_\equiv),$$

is called the standard irreversible quantum computational algebra.

4 The Poincaré quantum computational structures

We will now restrict our analysis to the qumixes living in the two-dimensional space $\mathbb{C}^2$. As is well known, every density operator of $\mathfrak{D}(\mathbb{C}^2)$ has the following matrix representation:

$$\frac{1}{2}\left(I + r_1 X + r_2 Y + r_3 Z\right),$$

where r_1, r_2, r_3 are real numbers such that $r_1^2 + r_2^2 + r_3^2 \leq 1$ and X, Y, Z are the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It turns out that a density operator $\frac{1}{2}(I + r_1 X + r_2 Y + r_3 Z)$ is pure iff $r_1^2 + r_2^2 + r_3^2 = 1$. Consequently,

- Pure density operators are in 1 : 1 correspondence with the points of the surface of the Poincaré sphere;
- Proper mixtures are in 1 : 1 correspondence with the inner points of the Poincaré sphere.

Let ρ be a density operator of $\mathfrak{D}(\mathbb{C}^2)$. We will denote by $\bar{\rho}$ the point of the Poincaré sphere that is univocally associated to ρ.

Let (r_1, r_2, r_3) be a point of the Poincaré sphere. We will denote by $\widehat{(r_1, r_2, r_3)}$ the density operator univocally associated to (r_1, r_2, r_3).

In the Domenech and Freytes approach one disregards the r_1 component of the Poincaré sphere whereas taking the square root of the identity gate into account one does not.

Lemma 25 *Let $\rho \in \mathfrak{D}(\mathbb{C}^2)$ such that $\bar{\rho} = (r_1, r_2, r_3)$. The following conditions hold:*

(1) $\mathtt{p}(\rho) = \frac{1-r_3}{2}$, $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \frac{1-r_2}{2}$, $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) = \frac{1-r_1}{2}$;
(2) $0 < \mathtt{p}(\rho) < 1$, $0 < \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) < 1$ *and* $0 < \mathtt{p}(\sqrt{\mathbb{I}}(\rho)) < 1$, *whenever ρ is a proper mixture.*
(3) $\frac{\mathtt{p}(\rho)}{4} + \frac{\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))}{4} + \frac{\mathtt{p}(\sqrt{\mathbb{I}}(\rho))}{4} \leq \frac{3+\sqrt{3}}{8}$

PROOF. (i) Easy computation;
(ii) Since proper mixtures are in 1:1 correspondence with inner points of the Poincaré sphere, we have: $r_1^2 + r_2^2 + r_3^2 < 1$. Hence: $r_1^2, r_2^2, r_3^2 < 1$ and $-1 < r_1, r_2, r_3 < 1$. Consequently: $0 < \mathtt{p}(\rho) = \frac{1-r_3}{2} < 1$, $0 < \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = \frac{1-r_2}{2} < 1$ and $0 < \mathtt{p}(\sqrt{\mathbb{I}}(\rho)) = \frac{1-r_1}{2} < 1$.
(iii) By (i), $\frac{\mathtt{p}(\rho)}{4} + \frac{\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho))}{4} + \frac{\mathtt{p}(\sqrt{\mathbb{I}}(\rho))}{4} = \frac{1-r_3}{8} + \frac{1-r_2}{8} + \frac{1-r_1}{8}$ and any point of the Poincaré sphere fulfills $r_1 + r_2 + r_3 \geq -\sqrt{3}$. This bound is given by a simple problem of minimization with conditional extremes. The factor $\frac{1}{4}$ is used to make the left-hand member of the inequality less than 1.

An irreversible conjunction can be now naturally defined on the set of all qumixes of $\mathfrak{D}(\mathbb{C}^2)$.

Definition 26 *The irreversible conjunction.*
Let $\tau, \sigma \in \mathfrak{D}(\mathbb{C}^2)$.

$$\tau \bullet \sigma := \rho^{(1)}_{\mathtt{p}(\tau)\mathtt{p}(\sigma)}$$

Interestingly enough, the density operator $\tau \bullet \sigma$ can be described as a *reduced state* of $\mathtt{AND}(\tau, \sigma)$. Suppose we have a compound physical system consisting of

r (possibly compound) subsystems, and let

$$\mathcal{H} = \mathcal{H}_1^{n_1} \otimes \ldots \otimes \mathcal{H}_r^{n_r}$$

be the Hilbert space associated to the total system (where $\mathcal{H}_j^{n_j} = \otimes^{n_j}\mathbb{C}^2$).

Let $\rho \in \mathfrak{D}(\mathcal{H})$ and $1 \leq j \leq r$. The *reduced state* of ρ with respect to the j-th subsystem is the unique density operator $red^j(\rho)$ that satisfies the following condition, for any self-adjoint operator A^j of $\mathcal{H}_j^{n_j}$:

$$\mathtt{tr}(A^j\, red^j(\rho)) = \mathtt{tr}((I^{(n_1)} \otimes \ldots \otimes I^{(n_{j-1})} \otimes A^j \otimes I^{(n_{j+1})} \otimes \ldots \otimes I^{(n_r)})\rho),$$

(where $I^{(n_h)}$ is the identity operator of $\mathcal{H}_h^{n_h}$).

Clearly, ρ and $red^j(\rho)$ turn out to be statistically equivalent with respect to the j-th subsystem of the total system.

One can prove that:

$$\tau \bullet \sigma = red^3(\mathtt{AND}(\tau, \sigma)).$$

In other words, $\tau \bullet \sigma$ represents the reduced state of $\mathtt{AND}(\tau, \sigma)$ on the third subsystem.

An interesting situation arises when both τ and σ are pure states. For instance, suppose that:

$$\tau = P_{|\psi\rangle} \; and \, \sigma = P_{|\varphi\rangle},$$

where $|\psi\rangle$ and $|\varphi\rangle$ are proper qubits. Then,

$$\mathtt{AND}(\tau, \sigma) = P_{T^{(1,1,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle)},$$

which is a pure state. At the same time, we have:

$$\tau \bullet \sigma = red^3(P_{T^{(1,1,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle)}),$$

which is a proper mixture. Apparently, when considering only the properties of the third subsystem, we lose some information. As a consequence, we obtain a final state that does not represent a maximal knowledge. As is well known, situations where the state of a compound system represents a maximal knowledge, while the states of the subsystems are proper mixtures, play an important role in the framework of entanglement-phenomena.

Lemma 27

(1) $\bullet$ *is associative and commutative;*
(2) $\rho \bullet P_0 = P_0$*;*
(3) $\rho \bullet P_1 = \rho_{\mathtt{p}(\rho)}$*;*
(4) $\mathtt{p}(\rho \bullet \sigma)) = \mathtt{p}(\rho)\mathtt{p}(\sigma)$*;*

(5) $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho \bullet \sigma)) = \frac{1}{2}$;
(6) $\mathtt{p}(\sqrt{\mathbb{I}}(\rho \bullet \sigma)) = \frac{1}{2}$.

PROOF. Easy.

Consider now the structure

$$\left(\mathfrak{D}(\mathbb{C}^2)\,, \bullet\,, \oplus\,, \mathtt{NOT}\,, \sqrt{\mathtt{NOT}}\,, \sqrt{\mathbb{I}}\,, P_0, P_1\,, \rho_{\frac{1}{2}}\right).$$

We will call such a structure *the Poincaré irreversible quantum computational algebra* (shortly, *the Poincaré IQC-algebra*).

Theorem 28 *The Poincaré algebra is isomorphic to the standard irreversible quantum computational algebra, via the map*
$g : \mathfrak{D}(\mathbb{C}^2) \to [\mathfrak{D}]_\equiv$ *such that* $\forall \rho \in \mathfrak{D}(\mathbb{C}^2)$*:*

$$g(\rho) = [\rho]_\equiv.$$

Moreover, for any $\rho\,, \sigma \in \mathfrak{D}(\mathbb{C}^2)$*:* $\rho \preccurlyeq \sigma$ *iff* $g(\rho) \preccurlyeq g(\sigma)$ *(6).*

5 The quantum computational algebra

An interesting algebraic property of the Poincaré IQC-structure is the following: our structure turns out to be isomorphic to a structure based on a particular subset of the set $\mathbb{R}^3$. Let

$$\mathbb{S} := \left\{(a, b, c) \mid a, b, c \in \mathbb{R} \text{ and } (1 - 2a)^2 + (1 - 2b)^2 + (1 - 2c)^2 \leq 1\right\}.$$

Note that for all triples $(a, b, c) \in \mathbb{S}$, the elements a, b, c belong to the real interval $[0, 1]$.

Let $\underline{0} := \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\underline{1} := \left(1, \frac{1}{2}, \frac{1}{2}\right)$, $\underline{\frac{1}{2}} := \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

The following operations ($\mathtt{NOT}^{\mathbb{S}}$, $\sqrt{\mathtt{NOT}}^{\mathbb{S}}$, $\sqrt{\mathbb{I}}^{\mathbb{S}}$, $\bullet^{\mathbb{S}}$, $\oplus^{\mathbb{S}}$) can be defined on $\mathbb{S}$.

Definition 29 *Operations.*

(1) $\mathtt{NOT}^{\mathbb{S}}(a_1, a_2, a_3) = (1 - a_1, 1 - a_2, a_3)$;
(2) $\sqrt{\mathtt{NOT}}^{\mathbb{S}}(a_1, a_2, a_3) = (a_2, 1 - a_1, a_3)$;
(3) $\sqrt{\mathbb{I}}^{\mathbb{S}}(a_1, a_2, a_3) = (a_3, 1 - a_2, a_1)$;
(4) $(a_1, a_2, a_3)\bullet^{\mathbb{S}}(b_1, b_2, b_3) = \left(a_1 b_1, \frac{1}{2}, \frac{1}{2}\right)$;

(5) $(a_1, a_2, a_3) \oplus^{\mathbb{S}} (b_1, b_2, b_3) = \begin{cases} (a_1 + b_1, \frac{1}{2}, \frac{1}{2}), & \textit{if } a_1 + b_1 \leq 1; \\ \underline{1}, & \textit{otherwise.} \end{cases}$

One can easily see that $\mathbb{S}$ is closed under the operations of Definition 29.

Lemma 30

(1) The operations $\bullet^{\mathbb{S}}$ and $\oplus^{\mathbb{S}}$ are commutative and associative;
(2) $(a_1, a_2, a_3)\bullet^{\mathbb{S}}\underline{0} = \underline{0}$;
(3) $(a_1, a_2, a_3) \oplus^{\mathbb{S}} \underline{0} = \left(a_1, \frac{1}{2}, \frac{1}{2}\right)$;
(4) $(a_1, a_2, a_3)\bullet^{\mathbb{S}}\underline{1} = \left(a_1 \frac{1}{2}, \frac{1}{2}\right)$;
(5) $(a_1, a_2, a_3) \oplus^{\mathbb{S}} \underline{1} = \underline{1}$;
(6) $\mathtt{NOT}^{\mathbb{S}}\mathtt{NOT}^{\mathbb{S}}(a_1, a_2, a_3) = (a_1, a_2, a_3)$;
(7) $\sqrt{\mathtt{NOT}}^{\mathbb{S}}\mathtt{NOT}^{\mathbb{S}}(a_1, a_2, a_3) = \mathtt{NOT}^{\mathbb{S}}\sqrt{\mathtt{NOT}}^{\mathbb{S}}(a_1, a_2, a_3)$;
(8) $\sqrt{\mathtt{NOT}}^{\mathbb{S}}\sqrt{\mathtt{NOT}}^{\mathbb{S}}(a_1, a_2, a_3) = \mathtt{NOT}^{\mathbb{S}}(a_1, a_2, a_3)$;
(9) (a_1, a_2, a_3) *is a fixed point of* $\mathtt{NOT}^{\mathbb{S}}$ *iff* (a_1, a_2, a_3) *is a fixed point of* $\sqrt{\mathtt{NOT}}^{\mathbb{S}}$ *iff* $(a_1, a_2, a_3) = \underline{\frac{1}{2}}$;
(10) $\sqrt{\mathbb{I}}^{\mathbb{S}}\sqrt{\mathbb{I}}^{\mathbb{S}}(a_1, a_2, a_3) = (a_1, a_2, a_3)$;
(11) $\underline{\frac{1}{2}}$ *is a fixed point of* $\sqrt{\mathbb{I}}^{\mathbb{S}}$.

PROOF. Easy computation.

Definition 31

$$(a_1, a_2, a_3) \preceq (b_1, b_2, b_3) \textit{ iff } a_1 \leq b_1,\ b_2 \leq a_2 \textit{ and } a_3 \leq b_3.$$

Consider now the structure $\left(\mathbb{S}, \bullet^{\mathbb{S}}, \oplus^{\mathbb{S}}, \mathtt{NOT}^{\mathbb{S}}, \sqrt{\mathtt{NOT}}^{\mathbb{S}}, \sqrt{\mathbb{I}}^{\mathbb{S}}, \underline{0}, \underline{1}, \underline{\frac{1}{2}}\right)$. We will call such a structure the $\mathbb{S}$ *quantum computational algebra* (shortly *the* $\mathbb{S}$*QC-algebra*).

We will prove that the Poincaré IQC-algebra and the $\mathbb{S}$QC-algebra are isomorphic.

Let $(a, b, c) \in \mathbb{S}$ and let $\rho(a, b, c)$ be the density operator associated to the triple $(1 - 2c, 1 - 2b, 1 - 2a)$. Thus,

$$\rho(a, b, c) := (1 - 2c, \widehat{1 - 2b}, 1 - 2a).$$

Hence:

$$\rho(a, b, c) = \begin{pmatrix} 1 - a & (\frac{1}{2} - c) - i(\frac{1}{2} - b) \\ (\frac{1}{2} - c) + i(\frac{1}{2} - b) & a \end{pmatrix}$$

Lemma 32

(1) $\rho(\mathtt{NOT}^{\mathbb{S}}(a_1, a_2, a_3)) = \mathtt{NOT}(\rho(a_1, a_2, a_3))$*;*
(2) $\rho(\sqrt{\mathtt{NOT}}^{\mathbb{S}}(a_1, a_2, a_3)) = \sqrt{\mathtt{NOT}}(\rho(a_1, a_2, a_3))$*;*
(3) $\rho(\sqrt{\mathbb{I}}^{\mathbb{S}}(a_1, a_2, a_3)) = \sqrt{\mathbb{I}}(\rho(a_1, a_2, a_3))$*;*
(4) $\rho((a_1, a_2, a_3) \bullet^{\mathbb{S}} (b_1, b_2, b_3)) = \rho(a_1, a_2, a_3) \bullet \rho(b_1, b_2, b_3)$*;*
(5) $\rho((a_1, a_2, a_3) \oplus^{\mathbb{S}} (b_1, b_2, b_3)) = \rho(a_1, a_2, a_3) \oplus \rho(b_1, b_2, b_3)$*.*

PROOF. Easy computation.

Theorem 33 *The* $\mathbb{S}QC$*-algebra*

$$\left(\mathbb{S}\,, \bullet^{\mathbb{S}}\,, \oplus^{\mathbb{S}}\,, \sqrt{\mathtt{NOT}}^{\mathbb{S}}\,, \sqrt{\mathbb{I}}^{\mathbb{S}}\,, \underline{0}\,, \underline{1}\,, \underline{\frac{1}{2}}\right)$$

is isomorphic to the Poincaré IQC-algebra

$$\left(\mathfrak{D}(\mathbb{C}^2)\,, \bullet\,, \oplus\,, \sqrt{\mathtt{NOT}}\,, \sqrt{\mathbb{I}}\,, P_0\,, P_1\,, \rho_{\frac{1}{2}}\right).$$

PROOF. Let h be the map of $\mathbb{S}$ into $\mathfrak{D}(\mathbb{C}^2)$ such that $\forall (a_1, a_2, a_3) \in \mathbb{S}$:

$$h((a_1, a_2, a_3)) := \rho(a_1, a_2, a_3).$$

That h is a homomorphism follows from Lemma 32. We now prove that h is injective. Suppose $(a_1, a_2, a_3) \neq (b_1, b_2, b_3)$. Suppose, by contradiction, that $h((a_1, a_2, a_3)) = h((b_1, b_2, b_3))$. Then, $\rho(a_1, a_2, a_3) = \rho(b_1, b_2, b_3)$. Thus, $\mathtt{p}(\rho(a_1, a_2, a_3)) = \mathtt{p}(\rho(b_1, b_2, b_3))$, $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho(a_1, a_2, a_3))) = \mathtt{p}(\sqrt{\mathtt{NOT}}(\rho(b_1, b_2, b_3)))$ and $\mathtt{p}(\sqrt{\mathbb{I}}(\rho(a_1, a_2, a_3))) = \mathtt{p}(\sqrt{\mathbb{I}}(\rho(b_1, b_2, b_3)))$. By Lemma 25, we obtain

$$\mathtt{p}(\rho(a_1, a_2, a_3)) = a_1 = b_1 = \mathtt{p}(\rho(b_1, b_2, b_3)),$$

$$\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho(a_1, a_2, a_3))) = a_2 = b_2 = \mathtt{p}(\sqrt{\mathtt{NOT}}\,\rho(b_1, b_2, b_3)),$$
$$\mathtt{p}(\sqrt{\mathbb{I}}(\rho(a_1, a_2, a_3))) = a_3 = b_3 = \mathtt{p}(\sqrt{\mathbb{I}}(\rho(b_1, b_2, b_3))).$$

Hence: $(a_1, a_2, a_3) = (b_1, b_2, b_3)$, contradiction. We now prove that h is surjective. Let ρ be a density operator of $\mathfrak{D}(\mathbb{C}^2)$ and let (a_1, a_2, a_3) be the point of the Poincaré sphere associated to ρ. Thus, $(a_1, a_2, a_3) = \bar{\rho}$.
Take $\left(\frac{1-a_3}{2}, \frac{1-a_2}{2}, \frac{1-a_1}{2}\right) \in \mathbb{S}$. By Lemma 25, $\rho\left(\frac{1-a_3}{2}, \frac{1-a_2}{2}, \frac{1-a_1}{2}\right) = \rho$. Consequently, $\rho = h\left(\left(\frac{1-a_3}{2}, \frac{1-a_2}{2}, \frac{1-a_1}{2}\right)\right)$.

As a consequence of Theorem 28 and of Theorem 33, we obtain that the IQC-algebra and the $\mathbb{S}$QC-algebra are isomorphic.

6 Quantum Many-valued algebra

Definition 34 *Many-valued algebra.*
A many-valued algebra (MV-algebra for short) is a structure $\mathbf{A} = (A, \oplus, \neg, 0)$ *of type* $(2, 1, 0)$ *such that the following conditions are satisfied:*

(1) $(A, \oplus, 0)$ *is a commutative monoid*
(2) $x \oplus \neg 0 = \neg 0$
(3) $\neg\neg x = x$
(4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

In each MV-algebra, one can define the following additional constant and connectives:

- $1 = \neg 0$
- $x \ominus y = \neg(\neg x \oplus y)$
- $x \odot y = \neg(\neg x \oplus \neg y)$
- $x \to y = \neg x \oplus y$
- $x \vee y = (x \ominus y) \oplus y$
- $x \wedge y = \neg(\neg x \vee \neg y)$

One can also define the relation $\leq$ $(\forall x, y \quad x \leq y \texttt{ iff } x \to y = 1)$ as a natural lattice order with top element 1 and bottom element 0 w.r.t. the lattice operations $\vee$ and $\wedge$ (because $(A, \vee, \wedge, 0, 1)$ forms a lattice).

Definition 35 *Quasi many-valued algebra (14).*
A quasi many-valued algebra is a structure $\mathbf{A} = (A, \oplus, \neg, 0)$ *of type* $(2, 1, 0)$ *such that, letting* $1 = \neg 0$, $x \ominus y = \neg(\neg x \oplus y)$*, the following conditions are satisfied:*

(1) $x \oplus (y \oplus z) = (x \oplus z) \oplus y$
(2) $x \oplus 1 = 1$
(3) $\neg\neg x = x$
(4) $(x \ominus y) \oplus y = (y \ominus x) \oplus x$
(5) $\neg(x \oplus 0) = \neg x \oplus 0$
(6) $(x \oplus y) \oplus 0 = x \oplus y$

Of course, a quasi-MV algebra is an MV algebra if and only if it satisfies the additional equation $x \oplus 0 = x$. Axioms (5) and (6) are there to enforce some regularity on the otherwise too wild behaviour of truncated sum: the operation of adding a zero should leave sums unaltered and should commute with inversion. In this setting, the relation $\leq$ turns out to be a preordering, but not necessarily a partial ordering of A.

Definition 36 *Product many-valued algebra (15).*

A product many-valued algebra (PMV-algebra for short) is a structure $(A, \oplus, \bullet, \neg, 0)$ *of type* $(2, 2, 1, 0)$ *such that the following conditions are satisfied:*

(1) $(A, \oplus, \neg, 0)$ *is an MV-algebra*
(2) $(A, \bullet, 1)$ *is a commutative monoid*
(3) $x \bullet (y \ominus z) = (x \bullet y) \ominus (x \bullet z)$

An important example of PMV-algebra is $[0,1]_{\mathtt{PMV}} = ([0,1], \bullet, \oplus, \neg, 0)$ such that $[0,1]$ is the real unit segment and $\bullet, \oplus, \neg$ are defined as follows: $x \bullet y = x\,y$, $x \oplus y = \min\{1, x+y\}$, $\neg x = 1 - x$. Note that every Boolean algebra becomes a PMV-algebra by letting the product operation coincide with the infimum operation. The following is almost immediate consequence of the definition.

We can now refer to the following equivalence relation:

$$\rho \restriction\!\approxeq \sigma \text{ iff } \mathtt{p}(\rho) = \mathtt{p}(\sigma).$$

For any $\rho \in \mathfrak{D}(\mathbb{C}^2)$, let

$$[\rho]_{\restriction\approxeq} := \left\{\sigma \in \mathfrak{D}(\mathbb{C}^2) \mid \rho \approxeq \sigma\right\}. \tag{1}$$

Further define

$$[\mathfrak{D}(\mathbb{C}^2)]_{\restriction\approxeq} := \left\{[\rho]_{\restriction\approxeq} \mid \rho \in \mathfrak{D}(\mathbb{C}^2)\right\}. \tag{2}$$

The operations $\bullet\,, \oplus\,, \mathtt{NOT}$ and the relation $\preceq$ can be defined on $[D(\mathbb{C}^2)]_{\restriction\approxeq}$ in the expected way.

On this basis we obtain the following quotient-structure

$$\left([\mathfrak{D}(\mathbb{C}^2)]_{\restriction\approxeq}\,, \bullet\,, \oplus\,, \mathtt{NOT}\,, [P_0]_{\restriction\approxeq}, [P_1]_{\restriction\approxeq}\right).$$

One can easily show:

Proposition 37 $([\mathfrak{D}(\mathbb{C}^2)]_{\restriction\approxeq}\,, \bullet\,, \oplus\,, \mathtt{NOT}\,, [P_0]_{\restriction\approxeq}, [P_1]_{\restriction\approxeq})$ *is a PMV-algebra isomorphic to* $[0,1]_{\mathtt{PMV}}$.

The isomorphism is given by $\rho_\lambda \to \lambda$. The operations $\wedge, \vee$ give the PMV lattice structure in $[\mathfrak{D}(\mathbb{C}^2)]_{\restriction\approxeq}$. This proposition shows that a logic associated to gates would admit an algebraic counterpart associated with PMV-algebras.

Lemma 38 *In each PMV-algebra we have*

(1) $0 \bullet x = 0$
(2) If $a \leq b$ *then* $a \bullet x \leq b \bullet x$
(3) $x \odot y \leq x \bullet y \leq x \wedge y$

Consider the subalgebra $\mathbf{S}$ of $[0,1]_{\texttt{PMV}}$ generated by $\frac{1}{2}$ which will play an important role in the logical treatment. The set S is contained in the rationals of $[0,1]$ and $(S,\leq)$ is a dense order in $[0,1]$.

Definition 39 *Quasi product many-valued algebra.*
A quasi product many-valued algebra is a structure $\mathbf{A}=(A,\oplus,\bullet,\neg,0)$ *of type* $(2,2,1,0)$ *such that, letting* $1=\neg 0$, $x\ominus y=\neg(\neg x\oplus y)$, *the following conditions are satisfied:*

(1) $(A,\oplus,\neg,0)$ *is a quasi-MV algebra*
(2) $x\bullet(y\bullet z)=(x\bullet z)\bullet y$
(3) $x\bullet(y\ominus z)=(x\bullet y)\ominus(x\bullet z)$
(4) $\neg(x\bullet 1)=\neg x\bullet 1$
(5) $(x\bullet y)\bullet 1=x\bullet y$
(6) $(x\bullet y)\oplus 0=x\bullet y$
(7) $(x\oplus y)\bullet 1=x\oplus y$

We can think of a quasi-PMV algebra as identical to an PMV algebra, except for the fact that 0 need not be a neutral element for the truncated sum and 1 need not be a neutral element for the product. Of course, a quasi-PMV algebra is an PMV algebra if and only if it satisfies the additional equations $x\oplus 0=x$ and $x\bullet 1=x$. Axioms (4)–(7) are there to enforce some regularity on the otherwise too wild behaviour of product.

Definition 40 *Quantum product many-valued algebra.*
A quantum product many-valued algebra (QPMV-algebra for short) is a structure $\mathbf{A}=\left(A,\oplus,\bullet,\sqrt{\neg},\sqrt{\mathtt{i}},0\right)$ *of type* $(2,2,1,1,0)$ *such that, letting* $1=\neg 0$, $x\ominus y=\neg(\neg x\oplus y)$ *and* $\neg x=\sqrt{\neg}\sqrt{\neg}x$, *the following conditions are satisfied:*

(1) $(A,\oplus,\neg,0)$ *is a quasi-PMV algebra*
(2) $\frac{1}{2}=\sqrt{\neg}\,\frac{1}{2}$
(3) $\sqrt{\neg}(x\oplus y)\oplus 0=\frac{1}{2}$
(4) $\sqrt{\mathtt{i}}\sqrt{\mathtt{i}}\,x=x$
(5) $\frac{1}{2}=\sqrt{\mathtt{i}}\,\frac{1}{2}$
(6) $\sqrt{\mathtt{i}}(x\oplus y)\oplus 0=\frac{1}{2}$
(7) $\sqrt{\mathtt{i}}\sqrt{\neg}\,x\oplus 0=\sqrt{\mathtt{i}}\,x\oplus 0$
(8) $\sqrt{\neg}\sqrt{\mathtt{i}}\,x\oplus 0=\neg\sqrt{\neg}x\oplus 0$
(9) $\sqrt{\mathtt{i}}\sqrt{\neg}\sqrt{\mathtt{i}}\sqrt{\neg}\sqrt{\mathtt{i}}\sqrt{\neg}\,x=x$

Examples of infinite QPMV algebras are given by the next two structures $\mathbf{C}$ (for cube) and $\mathcal{S}$ (for sphere).
Example Standard QPMV algebras.
$\mathbf{C}$ is the algebra $([0,1]\times[0,1]\times[0,1],\bullet^C,\oplus^C,\sqrt{\neg}^C,\sqrt{\mathtt{i}}^C,0^C,1^C)$, where:

$$\sqrt{\neg}^C\langle a_1,a_2,a_3\rangle=\langle a_2,1-a_1,a_3\rangle$$

$\sqrt{\mathtt{i}}^{C}\langle a_1,a_2,a_3\rangle = \langle a_3, 1-a_2, a_1\rangle$
$\langle a_1,a_2,a_3\rangle \oplus^{C} \langle b_1,b_2,b_3\rangle = \langle \min(a_1+b_1,1), \frac{1}{2}, \frac{1}{2}\rangle$
$\langle a_1,a_2,a_3\rangle \bullet^{C} \langle b_1,b_2,b_3\rangle = \langle a_1 b_1, \frac{1}{2}, \frac{1}{2}\rangle$

Note that $\langle a_1,a_2,a_3\rangle \oplus^{C} \langle 0, \frac{1}{2}, \frac{1}{2}\rangle \neq \langle a_1,a_2,a_3\rangle$.

$\mathcal{S}$ is the subalgebra of $\mathbf{C}$ whose universe is the set
$\mathbb{S} = \{(a,b,c) \mid a,b,c \in \mathbb{R} \text{ and } (1-2a)^2 + (1-2b)^2 + (1-2c)^2 \leq 1\}$.

Proposition 41 *Let* $\mathbf{n}\,x$ *be* $\underbrace{x \oplus \ldots \oplus x}_{\mathbf{n}}$,
$\mathbf{r}(x) := \neg\mathbf{4}(\neg\, x \bullet x) \oplus \neg\mathbf{4}(\neg\sqrt{\neg}\, x \bullet \sqrt{\neg}\, x) \oplus \neg\mathbf{4}(\neg\sqrt{\mathtt{i}}\, x \bullet \sqrt{\mathtt{i}}\, x)$.
In any QPMV *algebra the following conditions hold:*

(1) $x \oplus 0 = x \ominus 0$
(2) $x \bullet y = x \bullet (y \oplus 0)$
(3) $\sqrt{\mathtt{i}}\sqrt{\neg}\, x \bullet 1 = \sqrt{\mathtt{i}}\, x \bullet 1$
(4) $\sqrt{\neg}\sqrt{\mathtt{i}}\, x \bullet 1 = \neg\sqrt{\neg}\, x \bullet 1$
(5) $\mathbf{r}(x) = \mathbf{r}(\sqrt{\neg}\, x)$
(6) $\mathbf{r}(x) = \mathbf{r}(\sqrt{\mathtt{i}}\, x)$

PROOF.

(1) $x \oplus 0 = \neg(\neg x \oplus 0) = \neg\neg(x \oplus 0) = x \ominus 0$
(2) $x \bullet y = (x \bullet y) \oplus 0 = (x \bullet y) \ominus 0 = (x \bullet y) \ominus (x \bullet 0) = x \bullet (y \ominus 0) = x \bullet (y \oplus 0)$
(3) $\sqrt{\mathtt{i}}\sqrt{\neg}\, x \bullet 1 = (\sqrt{\mathtt{i}}\sqrt{\neg}\, x \oplus 0) \bullet 1 = (\sqrt{\mathtt{i}}\, x \oplus 0) \bullet 1 = \sqrt{\mathtt{i}}\, x \bullet 1$
(4) Similarly.
(5) $\mathbf{r}(x) = \neg\mathbf{4}(\neg\, x \bullet x) \oplus \neg\mathbf{4}(\neg\sqrt{\neg}\, x \bullet \sqrt{\neg}\, x) \oplus \neg\mathbf{4}(\neg\sqrt{\mathtt{i}}\, x \bullet \sqrt{\mathtt{i}}\, x)$
$= \neg\mathbf{4}(\neg\sqrt{\neg}\, x \bullet \sqrt{\neg}\, x) \oplus \neg\mathbf{4}(x \bullet \neg x) \oplus \neg\mathbf{4}((\neg\sqrt{\mathtt{i}}\, x \oplus 0) \bullet (\sqrt{\mathtt{i}}\, x \oplus 0))$
$= \neg\mathbf{4}(\neg\sqrt{\neg}\, x \bullet \sqrt{\neg}\, x) \oplus \neg\mathbf{4}(\neg\sqrt{\neg}\sqrt{\neg}\, x \bullet \sqrt{\neg}\sqrt{\neg}\, x) \oplus \neg\mathbf{4}(\neg\sqrt{\mathtt{i}}\sqrt{\neg}\, x \bullet \sqrt{\mathtt{i}}\sqrt{\neg}\, x)$
$= \mathbf{r}(\sqrt{\neg}\, x)$
(6) Similarly.

In a forecoming paper we will study some properties of quasi-PMV and QPMV algebra.

7 Irreversible quantum computational logics

The quantum computational structures we have investigated suggest a natural semantics, based on the following intuitive idea: any sentence α of the language is interpreted as a convenient qumix; at the same time, the logical connectives are interpreted as operations that either are *gates* or can be conveniently defined in terms of gates. We will consider a *propositional quantum computational language* $\mathcal{L}$ that contains privileged atomic formula $\perp$ (whose intended interpretation is the truth-value *Falsity*) and the following primitive connectives: a *square root of the negation* $\sqrt{\neg}$, a *square root of the identity* $\sqrt{\mathtt{i}}$, a *conjunction* $\bullet$ (which corresponds to the Petri-Toffoli gate), a *Łukasiewicz conjunction* $\odot$, a *Łukasiewicz disjunction* $\oplus$ (which corresponds to the truncated sum gate), a *Łukasiewicz implication* $\rightarrow$. When we omit parentheses, we assume these connectives bind from strongest to weakest in this order. Let $Form^{\mathcal{L}}$ be the set of all formulas of $\mathcal{L}$. In this framework, $\neg\alpha$ is dealt with a metalinguistic abbreviation for $\sqrt{\neg}\sqrt{\neg}\alpha$, $\sqrt{\neg}^n\alpha := \underbrace{\sqrt{\neg}\ldots\sqrt{\neg}}_{n}\alpha$, $\sqrt{\mathtt{i}}^n\alpha := \underbrace{\sqrt{\mathtt{i}}\ldots\sqrt{\mathtt{i}}}_{n}\alpha$, $\alpha^n := \underbrace{\alpha\odot\ldots\odot\alpha}_{n}$, $\alpha\equiv\beta$ is an abbreviation for $(\alpha\rightarrow\beta)\odot(\beta\rightarrow\alpha)$. We will use the following metavariables: $\mathbf{q},\mathbf{q_1},\mathbf{q_2},\ldots$ for atomic formulas and $\alpha,\beta,\ldots$ for formulas. The privileged formula $\top$ representing the *Truth* is defined as the negation of $\perp$ ($\top := \neg\perp$) while the constant $\bar{s}$ represent the elements ρ_s of the Poincaré IQC-algebra. In particular, $\perp,\top\in\bar{S}$ represent P_0 and P_1 respectively. This minimal quantum computational language can be extended to richer languages containing other primitive connectives that we will not consider here. We will deal with the usual notion of complexity of formulas (i.e. $Comp : Form^{\mathcal{L}}\rightarrow\mathbb{N}$, which associates to any formula α of the language a non-negative integer: $Comp(\alpha) = 0$ if α is an atomic formula; $Comp(\beta)+1$ if $\alpha = \sqrt{\neg}\beta$ or $\alpha = \sqrt{\mathtt{i}}\,\beta$; $Comp(\beta)+Comp(\gamma)+1$ if $\alpha = \beta * \gamma$ for each binary connective $*$).

Let us now introduce the concept of *irreversible quantum computational model* (briefly, *IQC-model*), where the "quasi-intensional" character of reversible models is lost. In fact, the interpretation of a formula in an irreversible model does not generally reflect the logical form of our formula: the meaning of the *whole* does not include the meanings of its *parts* (that are trace out).

Definition 42 *IQC-model.*
An IQC-model *of* $\mathcal{L}$ *is a function* $\mathtt{Qum} : Form^{\mathcal{L}}\rightarrow\mathfrak{D}(\mathbb{C}^2)$ *(which associates to any formula* α *of the language a qumix of* $\mathbb{C}^2$*):*

$$\mathtt{Qum}(\alpha) := \begin{cases} \rho & \textit{if } \alpha \textit{ is an atomic formula} \\ \rho_s & \textit{if } \alpha = \bar{s}; \\ \sqrt{\mathtt{NOT}}(\mathtt{Qum}(\beta)) & \textit{if } \alpha = \sqrt{\neg}\beta; \\ \sqrt{\mathbb{I}}(\mathtt{Qum}(\beta)) & \textit{if } \alpha = \sqrt{\mathtt{i}}\,\beta; \\ \mathtt{Qum}(\beta)\bullet\mathtt{Qum}(\gamma) & \textit{if } \alpha = \beta\bullet\gamma; \\ \mathtt{Qum}(\beta)\oplus\mathtt{Qum}(\gamma) & \textit{if } \alpha = \beta\oplus\gamma. \end{cases}$$

Clearly, if two models $\mathtt{Qum}, \mathtt{Qum}'$ coincide over atomic formulas, then $\mathtt{Qum} = \mathtt{Qum}'$.

Given an IQC-model $\mathtt{Qum}$, any formula α has a natural probability-value.

Definition 43 *The probability-value of α in a model $\mathtt{Qum}$.*

$$\mathtt{Qum}_{\mathtt{p}}(\alpha) := \mathtt{p}(\mathtt{Qum}(\alpha)).$$

Note that the probability-value of $\bar{s}$ is independent of the model. As we already know, qumixes are naturally preordered. This suggests to introduce the following consequence relation.

Definition 44 *Consequence and truth in a model $\mathtt{Qum}$.*
A formula β is a consequence in a model $\mathtt{Qum}$ *of a formula α ($\alpha \models_{\mathtt{Qum}} \beta$) iff* $\mathtt{Qum}(\alpha) \preccurlyeq \mathtt{Qum}(\beta)$.
A formula α is true in a model $\mathtt{Qum}$ *iff* $\top \models_{\mathtt{Qum}} \alpha$.

The notions of logical consequence and truth can be now defined in the expected way.

Definition 45 *Logical consequence and logical truth.*
A formula β is a logical consequence of a formula α *($\alpha \models \beta$) iff for any model* $\mathtt{Qum}$, $\alpha \models_{\mathtt{Qum}} \beta$.
A formula α is a logical truth *iff for any model $\mathtt{Qum}$, α is true in $\mathtt{Qum}$.*

We will indicate by IQCL, the logic that is semantically characterized by the logical consequence relation. In other words, we have: β is a logical consequence of α in the logic IQCL ($\alpha \models_{\mathtt{IQCL}} \beta$) iff β is a logical consequence of α.

The logical consequence is syntactically strongly related to the implication.

Proposition 46 *Let α, β be formulas. Then we have*

$$\alpha \models \beta \text{ iff } \alpha \to \beta \text{ is a logical truth.}$$

The following formulas are axioms of the IQCL.

Definition 47 • ***Łukasiewicz axioms:***
W1 $\alpha \to (\beta \to \alpha)$
W2 $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
W3 $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$
W4 $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$
• ***Equivalence axioms***
E1 $\top \equiv \neg\bot$

E2 $\neg\alpha \equiv \alpha \to \bot$
E3 $(\alpha \odot \beta) \equiv \neg(\neg\alpha \oplus \neg\beta)$
E4 $(\alpha \to \beta) \equiv (\neg\alpha \oplus \beta)$
• ***Product axioms***
P1 $\alpha \bullet \beta \to \beta \bullet \alpha$
P2 $\top \bullet \beta \equiv \alpha$
P3 $\alpha \bullet \beta \to \beta$
P4 $(\alpha \bullet \beta) \bullet \gamma \equiv \alpha \bullet (\beta \bullet \gamma)$
P5 $\alpha \bullet (\beta \odot \neg\gamma) \equiv (\alpha \bullet \beta) \odot \neg(\alpha \bullet \gamma)$
• ***S axioms*** *For each* $\bar{s}, \bar{t} \in \bar{S}$
S1 $\bar{s} \odot \bar{t} \equiv \overline{s \odot t}$
S2 $\bar{s} \to \bar{t} \equiv \overline{s \to t}$
S3 $\bar{s} \bullet \bar{t} \equiv \overline{s \bullet t}$
• ***Square root axioms***
Q1 $\sqrt{\neg}\, \bar{s} \equiv \bar{\frac{1}{2}}$, $\sqrt{\mathtt{i}}\, \bar{s} \equiv \bar{\frac{1}{2}}$, *for each* $\bar{s} \in \bar{S}$
Q2 $\sqrt{\neg}(\alpha * \beta) \equiv \bar{\frac{1}{2}}$, $\sqrt{\mathtt{i}}(\alpha * \beta) \equiv \bar{\frac{1}{2}}$, *for any binary connective* $*$
Q3 $\sqrt{\neg}\sqrt{\mathtt{i}}\sqrt{\neg}\sqrt{\mathtt{i}}\sqrt{\neg}\sqrt{\mathtt{i}}\, \alpha \equiv \alpha$
Q4
$$\Big\{\bar{\tfrac{1}{4}} \bullet \sqrt{\neg}^{n_1}\alpha \oplus \bar{\tfrac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}^{n_1}\alpha \oplus \bar{\tfrac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^{n_1}\alpha \to \bar{s},$$
$$\bar{\tfrac{1}{4}} \bullet \sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\alpha \oplus \bar{\tfrac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\alpha \oplus \bar{\tfrac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\alpha \to \bar{s},$$
$$\bar{\tfrac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^{2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\alpha \oplus \bar{\tfrac{1}{4}} \bullet \sqrt{\neg}\sqrt{\mathtt{i}}\sqrt{\neg}^{2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\alpha \oplus \bar{\tfrac{1}{4}} \bullet \sqrt{\neg}^{2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\alpha \to \bar{s} :$$
$$n_1, n_2, n_3, n_4 \in \{0,1,2,3\}, s \geq \tfrac{3+\sqrt{3}}{8}\Big\}$$

The unique deduction rule of IQCL is the modus ponens.

Note that axioms W1-W4 and E1-E4 conform the same propositional deductive system as the infinite valued Łukasiewicz calculus. Equivalence axioms are introduced in order to make the notion of model well defined. In fact, if we had defined $\neg\alpha$ as $\alpha \to \bot$, then we would have had to require that $\mathtt{Qum}(\sqrt{\neg}\neg\alpha) = \mathtt{Qum}(\sqrt{\neg}(\alpha \to \bot)) = \rho_{\frac{1}{2}}$. But this is not in general true except when $\mathtt{Qum}(\alpha) = \rho_s$ for some s. Moreover, the axioms Q4 refer to the relation between density operators ρ, $\sqrt{\mathtt{NOT}}(\rho)$, $\sqrt{\mathbb{I}}(\rho)$ with respect to the probability-values $\mathtt{p}(\sqrt{\mathbb{I}}(\rho)) = r_1$, $\mathtt{p}(\sqrt{\mathtt{NOT}}(\rho)) = r_2$, $\mathtt{p}(\rho) = r_3$ (see Lemma 25).

Definition 48 *Proof and theorem.*
A formula β is provable *in a set of formulas T ($T \vdash \beta$) iff β is the last formula of a sequence $\beta_1, \ldots, \beta_n$ such that each member is either an axiom of IQCL or follows from some preceding members of the sequence using modus ponens.*
α is a theorem of IQCL iff $\emptyset \vdash \alpha$ ($\vdash \alpha$, for short).

A model $\mathtt{Qum}$ of IQCL is a model of a theory T iff $\mathtt{p}(\mathtt{Qum}(\alpha)) = 1$ for each $\alpha \in T$. We will use $T \models \alpha$ when $\mathtt{p}(\mathtt{Qum}(\alpha)) = 1$ for each model $\mathtt{Qum}$ of a theory T. A theory T is inconsistent iff $T \vdash \bot$, otherwise it is consistent.

Proposition 49 *Let $\alpha, \beta, \gamma \in$IQCL. Then we have (7)*

(1) $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\beta \to \gamma))$
(2) $(\alpha \odot \beta) \to \alpha$
(3) $(\alpha \odot \beta) \to (\beta \to \alpha)$
(4) $(\alpha \odot (\alpha \to \beta)) \to (\beta \odot (\beta \to \alpha))$
(5) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \odot \beta) \to \gamma)$
(6) $((\alpha \odot \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$
(7) $((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma)$
(8) $\bot \to \alpha$

Proposition 50 *Let $\alpha, \beta, \gamma \in$IQCL and T be a theory. Then we have (7)*

(1) $\vdash \alpha \to (\beta \to \alpha)$
(2) $\vdash \alpha \to (\beta \to \alpha \odot \beta)$
(3) $\vdash (\alpha \to \beta) \to (\alpha \odot \gamma \to \beta \odot \gamma)$
(4) $T \vdash \alpha \odot \beta$ *iff* $T \vdash \alpha$ *and* $T \vdash \beta$
(5) $T \vdash \alpha \equiv \beta$ *iff* $T \vdash \alpha \to \beta$ *and* $T \vdash \beta \to \alpha$
(6) $T \vdash \alpha \to \beta$ *and* $T \vdash \beta \to \gamma$ *then* $T \vdash \alpha \to \gamma$
(7) $\vdash T$
(8) $\vdash \alpha \to (T \odot \alpha)$
(9) $\vdash (T \to \alpha) \to \alpha$
(10) $\vdash \neg\neg\alpha \to \alpha$
(11) $\vdash (\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$
(12) $\vdash (\alpha \to \beta) \to ((\alpha \oplus \gamma) \to (\beta \oplus \gamma))$
(13) $\vdash ((\alpha \equiv \beta) \odot (\beta \equiv \gamma)) \to (\alpha \equiv \gamma)$
(14) $\vdash ((\alpha \equiv \beta) \odot (\beta \to \gamma)) \equiv (\beta \to \gamma)$
(15) $\vdash ((\alpha \equiv \beta) \odot (\gamma \to \alpha)) \equiv (\gamma \to \beta)$
(16) $\vdash (\alpha \to \beta) \to (\gamma \bullet \alpha \to \alpha \bullet \beta)$
(17) $\vdash (\alpha \to \beta) \to (\alpha \bullet \gamma \to \beta \bullet \gamma)$
(18) $\vdash \alpha \odot \beta \to \alpha \bullet \beta$

The following theorem establishes a kind of deduction theorem for the IQCL calculus.

Theorem 51 *Deduction theorem.*
Let T be a theory and α, β be formulas. Then we have that if $T \bigcup \{\alpha\} \vdash \beta$ then there exists $n \in \mathbb{N}$ such that $T \vdash \alpha^n \to \beta$ (11).

Definition 52 *Relevance degree and proof degree.*
Let T be a theory over IQCL and α be a formula.
The relevance degree of T over α is $||\alpha||_T = \inf\{\mathtt{p}(\mathtt{Qum}(\alpha)) : \mathtt{p}(\mathtt{Qum}(T)) = 1\}$.
The proof degree of α is $|\alpha|_T = \sup\{s \in S : T \vdash \bar{s} \to \alpha\}$.

The truth degree in the Pavelka style logics represents the semantic relevance of a theory T with respect to a formula α, that is, the greatest lower bound of the probability-values that α may take if all probability-values of the formulas of T are known to be 1. Following Domenech and Freytes approach we show the Pavelka style strong completeness theorem. The equality between relevance degree and proof degree will give a syntactic notion of the relevance degree, that is, the lowest upper bound $s \in S$ such that $T \vdash \bar{s} \to \alpha$.

8 Completeness

In order to establish the formal connection between IQCL and fuzzy logic via a Pavelka style strong completeness theorem, we will deal with a fragment of IQCL whose algebraic counterpart is a PMV-algebra. We consider the subsystem of IQCL given by the axioms W1-W4, E1-E4, P1-P5, S1-S3 as the axiomatic system for the PMV-fragment $\text{IQCL}_{\texttt{PMV}}$. We must take formulas that do not contain $\bullet, \oplus$ (such as $\sqrt{\neg \mathbf{q}}$, $\sqrt{\mathbf{iq}}$) as atomic formulas of the fragment. We denote such formulas by $\mathbf{p}, \mathbf{p_0}, \mathbf{p_1}, \ldots$ and by $\vdash_{\texttt{PMV}}$ deductions on the PMV-fragment. A valuation over $\text{IQCL}_{\texttt{PMV}}$ is a function $v : \text{IQCL} \to [0,1]_{\texttt{PMV}}$ such that $v(\alpha * \beta) = v(\alpha) * v(\beta)$ for any binary connective $*$, $v(\neg\alpha) = \neg v(\alpha)$ and $v(\bar{s}) = s$ for each $\bar{s} \in \bar{S}$.

Let T be a theory and α be a formula, both in $\text{IQCL}_{\texttt{PMV}}$. A formula α is called $\text{IQCL}_{\texttt{PMV}}$ logical truth if and only if for each valuation v, $v(\alpha) = 1$. The relevance degree in $\text{IQCL}_{\texttt{PMV}}$ $||\alpha||_T^{PMV}$ is defined as in IQCL but in terms of the valuation over $\text{IQCL}_{\texttt{PMV}}$. Similarly, the proof degree $|\alpha|_T^{PMV}$ is defined in the PMV-fragment taking the axiomatic system of $\text{IQCL}_{\texttt{PMV}}$.

Definition 53 *Completeness.*
Let T be a theory in $IQCL_{\texttt{PMV}}$. Then T is complete iff for each pair of formulas α, β, we have: $T \vdash_{\texttt{PMV}} \alpha \to \beta$ or $T \vdash_{\texttt{PMV}} \beta \to \alpha$.

Lemma 54 *Let T be a theory and α be a formula, both in $IQCL_{\texttt{PMV}}$. Suppose that T does not prove α in the PMV-fragment. Then there exists a consistent complete theory T' in $IQCL_{\texttt{PMV}}$ such that $T \subseteq T'$ and T' does not prove α in the PMV-fragment (11).*

Theorem 55 *Let T be theory over $IQCL_{\texttt{PMV}}$. For each formula α, we define $[\alpha] = \{\beta : T \vdash_{\texttt{PMV}} \alpha \equiv \beta\}$. Let $L_T = \{[\alpha] : \alpha \in \text{IQCL}_{\texttt{PMV}}\}$. If we define, for each binary connective $*$, the following operations in L_T: $0 = [\bot]$, $1 = [\top]$, $\neg[\alpha] = [\neg\alpha]$, $[\alpha] * [\beta] = [\alpha * \beta]$, then we have $\langle L_T, \bullet, \odot, \oplus, \to, \neg, 0, 1\rangle$ is a PMV-algebra and $([\bar{s}])_{s \in S}$ is a subalgebra isomorphic to the algebra $\mathbf{S}$.*
If T is a complete theory then L_T is totally ordered (7).

We will refer to L_T as the Lindenbaum algebra associated to the theory T. Now we will establish a Pavelka style strong completeness theorem for the

PMV-fragment:

Theorem 56 *Let T be a theory and α be a formula, both over $IQCL_{\texttt{PMV}}$. Then we have (7)*

$$|\alpha|_T = ||\alpha||_T$$

The completeness of IQCL is obtained from the strong completeness of the PMV-fragment using the following translation of formulas.

Definition 57 *PMV-translation.*
A PMV-translation is a function $\alpha \mapsto \alpha_t$ associating to any IQCL formula an $IQCL_{\texttt{PMV}}$ formula, such that:

(1) $\mathbf{q} \mapsto \mathbf{q}$, $\sqrt{\neg}\mathbf{q} \mapsto \sqrt{\neg}\mathbf{q}$ *and* $\sqrt{\mathtt{i}}\mathbf{q} \mapsto \sqrt{\mathtt{i}}\mathbf{q}$ *for each atomic formula;*
(2) $\sqrt{\neg}\sqrt{\neg}\alpha \mapsto \sqrt{\neg}\sqrt{\neg}\alpha_t$*;*
(3) $\alpha * \beta \mapsto \alpha_t * \beta_t$ *for each binary connective* $*$*;*
(4) $\sqrt{\neg}(\alpha * \beta) \mapsto \overline{\tfrac{1}{2}}$ *for each binary connective* $*$*;*
(5) $\sqrt{\mathtt{i}}(\alpha * \beta) \mapsto \overline{\tfrac{1}{2}}$ *for each binary connective* $*$*.*

If T is a theory in IQCL, we define the PMV-translation over the theory as the set $T_t = \{\alpha_t : \alpha \in T\}$. From the definition of the PMV-translation, we can immediately establish the following lemma.

Lemma 58 *Let α be a formula in IQCL. Then, (7)*

$$\vdash_{\texttt{IQCL}} \alpha \equiv \alpha_t$$

Consider the following theory in $\text{IQCL}_{\texttt{PMV}}$ which plays an important role to deductions on IQCL with respect to deductions in $\text{IQCL}_{\texttt{PMV}}$.

Definition 59 *Theory $T_{\texttt{Q4}}$.*

$$T_{\texttt{Q4}} = \Big\{ \Big(\overline{\tfrac{1}{4}} \bullet \sqrt{\neg}^{n_1}\mathbf{p} \oplus \overline{\tfrac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}^{n_1}\mathbf{p} \oplus \overline{\tfrac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^{n_1}\mathbf{p}\Big) \to \bar{s},$$
$$\Big(\overline{\tfrac{1}{4}} \bullet \sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\mathbf{p} \oplus \overline{\tfrac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\mathbf{p} \oplus \overline{\tfrac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\mathbf{p}\Big) \to \bar{s},$$
$$\Big(\overline{\tfrac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^{2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\mathbf{p} \oplus \overline{\tfrac{1}{4}} \bullet \sqrt{\neg}\sqrt{\mathtt{i}}\sqrt{\neg}^{2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\mathbf{p} \oplus \overline{\tfrac{1}{4}} \bullet \sqrt{\neg}^{2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\mathbf{p}\Big) \to \bar{s} :$$
$$n_1, n_2, n_3, n_4 \in \{0,1,2,3\}, s \geq \tfrac{3+\sqrt{3}}{8}\Big\} \cup \Big\{ \Big(\overline{\tfrac{1}{4}} \bullet \mathbf{p} \oplus \overline{\tfrac{1}{4}}\Big) \to \bar{s} : s \geq \tfrac{1}{2}\Big\}$$

Lemma 60 *Let $\alpha \in IQCL$, $n_1, n_2, n_3, n_4 \in \{0,1,2,3\}$ and $s \in S$.*
If $s \geq \frac{3+\sqrt{3}}{8}$ then

$$T_{\texttt{Q4}} \vdash_{\texttt{PMV}} \left(\overline{\frac{1}{4}} \bullet \sqrt{\neg}^{n_1}\alpha \oplus \overline{\frac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}^{n_1}\alpha \oplus \overline{\frac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^{n_1}\alpha\right) \to \bar{s} \text{ noted } T_{\texttt{Q4}} \vdash_{\texttt{PMV}} \alpha_t^{n_1}$$

$$T_{\texttt{Q4}} \vdash_{\texttt{PMV}} \left(\overline{\frac{1}{4}} \bullet \sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\alpha \oplus \overline{\frac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\alpha \oplus \overline{\frac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^{n_2}\sqrt{\mathtt{i}}\sqrt{\neg}^{n_3}\alpha\right) \to \bar{s}$$

$$T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} \left(\overline{\frac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}^2\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\alpha \oplus \overline{\frac{1}{4}} \bullet \sqrt{\neg}\sqrt{\mathtt{i}}\sqrt{\neg}^2\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\alpha \oplus \overline{\frac{1}{4}} \bullet \sqrt{\neg}^2\sqrt{\mathtt{i}}\sqrt{\neg}^{n_4}\alpha\right) \to \bar{s}$$

If $\mathtt{s} \geq \frac{1}{2}$ *then* $T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} \left(\overline{\frac{1}{4}} \bullet \alpha \oplus \overline{\frac{1}{4}}\right) \to \bar{s}$

PROOF. We use induction on complexity of α. If $Comp(\alpha) = 0$, we have the following two cases.

$\alpha = \bar{s}$. By axiom Q1, $\sqrt{\neg}\bar{s} \equiv \overline{\frac{1}{2}}$ and by axiom S2, $\neg\overline{\frac{1}{2}} \equiv \overline{\frac{1}{2}}$. The result follows using proposition 50.

$\alpha = \mathbf{p}$. The translation of the formulas is over $T_{\mathtt{Q4}}$.

Suppose this result is valid for $Comp(\alpha) < n$ and consider α such that $Comp(\alpha) = n$. We have the following cases.

$\alpha = \sqrt{\neg}\beta$. $\alpha_t^0 = \left(\left(\overline{\frac{1}{4}} \bullet \sqrt{\neg}\beta \oplus \overline{\frac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}\beta \oplus \overline{\frac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}\beta\right) \to \bar{s}\right)_t$

$T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} \left(\overline{\frac{1}{4}} \bullet (\sqrt{\neg}\beta)_t \oplus \overline{\frac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}\beta_t \oplus \overline{\frac{1}{4}} \bullet (\sqrt{\mathtt{i}}\sqrt{\neg}\beta)_t\right) \to \bar{s}$

$\equiv \left(\left(\overline{\frac{1}{4}} \bullet \sqrt{\neg}\sqrt{\neg}\beta \oplus \overline{\frac{1}{4}} \bullet \sqrt{\neg}\beta \oplus \overline{\frac{1}{4}} \bullet \sqrt{\mathtt{i}}\sqrt{\neg}\beta\right) \to \bar{s}\right)_t$

By inductive hypothesis we have $T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} \beta_t^1$. For the rest of this case, it follows in a similar manner by using proposition 50.

$\alpha = \sqrt{\mathtt{i}}\beta$. Similarly.

$\alpha = \beta * \gamma$ for a binary connective $*$. Similar to the atomic case for the elements of $\bar{S}$.

The following theorem establishes the relation between the deductive system of IQCL and the deductive system $\text{IQCL}_{\mathtt{PMV}}$.

Theorem 61 *Let* T *be a theory and* α *be a formula both in IQCL. Then, we have*

$$T \vdash_{\mathtt{IQCL}} \alpha \text{ iff } T_t \cup T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} \alpha_t$$

PROOF. Suppose that $T \vdash_{\mathtt{IQCL}} \alpha$. We use induction on the length of the proof of α (noted by $Length(\alpha)$). If $Length(\alpha) = 1$, then we have the following cases:

(1) α is one of the axioms W1-W4, E1-E4, P1-P5, S1-S3. In this case, α_t results an axiom of the $\text{IQCL}_{\mathtt{PMV}}$.
(2) α is one of the axioms Q1-Q3 and by using proposition 50 it is PMV-theorem.
(3) If α is the axiom Q4, then we use Lemma 60.
(4) If $\alpha \in T$, then $\alpha_t \in T_t$.

Suppose the theorem is valid for $Length(\alpha) < n$. We consider $Length(\alpha) = n$. Thus, we have an IQCL-proof α from T as follows:

$$\alpha_1, \ldots, \alpha_m \to \alpha, \ldots, \alpha_m, \ldots, \alpha_{n-1}, \alpha$$

obtaining α by modus ponens from $\alpha_m \to \alpha$ and α_m. Using inductive hypothesis we have $T_t \cup T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} (\alpha_m \to \alpha)_t$ and $T_t \cup T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} (\alpha_m)_t$. By using $(\alpha_m \to \alpha)_t \equiv (\alpha_m)_t \to \alpha_t$ and modus ponens we have $T_t \cup T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} \alpha_t$. The converse follows from Lemma 58 and from the fact that formulas in $T_{\mathtt{Q4}}$ are IQCL theorems.

Theorem 61 shows in a formal sense the syntactic relation between IQCL and fuzzy logic. More precisely, we recall that provable formulas from IQCL-theories are identifiable to provable formulas from PMV-theories obtained by translation plus $T_{\mathtt{Q4}}$. In particular, IQCL-theorems are PMV-theorems of $T_{\mathtt{Q4}}$. Another important result that arises from these theorems is that the connectives $\sqrt{\neg}$, $\sqrt{\mathtt{i}}$ have importance only when applied to atomic formulas and their peculiarity are captured by the PMV-theory $T_{\mathtt{Q4}}$.

Corollary 62

$$\vdash_{\mathtt{IQCL}} \alpha \text{ iff } \cup T_{\mathtt{Q4}} \vdash_{\mathtt{PMV}} \alpha_t$$

Now we introduce the following sets in order to study the relation between models of IQCL and valuations of the PMV-fragment:
$E = \{\mathtt{Qum} : \mathtt{Qum} \text{ is a model of IQCL}\}$
$V_{\mathtt{Q4}} = \{v : v \text{ is a valuation of } \mathrm{IQCL}_{\mathtt{PMV}} \text{ with } v(\alpha) = 1 \text{ for any } \alpha \in T_{\mathtt{Q4}}\}$

Proposition 63 *There exists a bijection $E \to V_{\mathtt{Q4}}$, such that $\mathtt{Qum}_{\mathtt{p}}(\alpha) = v_{\mathtt{Qum}}(\alpha_t)$.*

PROOF. Let $\mathtt{Qum} \in E$ and let $v_{\mathtt{Qum}} = \mathtt{Qum}_{p|\mathtt{IQCL}_{\mathtt{PMV}}}$. We will see that $\mathtt{Qum} \mapsto v_{\mathtt{Qum}}$ is well defined in the sense that $v_{\mathtt{Qum}} \in V_{\mathtt{Q4}}$. Let $\alpha \in T_{\mathtt{Q4}}$. By Lemma 25 we have $\mathtt{Qum}_{\mathtt{p}}(\alpha) = 1$. Thus the assignation is well defined. The injectivity follows from Lemma 25. Now we prove the surjectivity. Let $v \in V_{\mathtt{Q4}}$. For each atomic formula α let $\mathtt{Qum}(\alpha) = \rho(1 - 2v(\alpha), 1 - 2v(\sqrt{\neg}\alpha), 1 - 2v(\sqrt{\mathtt{i}}\alpha))$. Thus, $\mathtt{Qum}$ is well defined over atomic formulas since v satisfies $T_{\mathtt{Q4}}$. Then we can extend $\mathtt{Qum}$ to IQCL and it is clear that $\mathtt{Qum}_{\mathtt{p}} = v$ since $\mathtt{Qum}_{\mathtt{p}}$ and v coincide over atomic formulas of $\mathrm{IQCL}_{\mathtt{PMV}}$. Finally, for each IQCL formula α, $\mathtt{Qum}_{\mathtt{p}}(\alpha) = v_{\mathtt{Qum}}(\alpha_t)$ taking into account the inductive argument on the complexity of formulas and translation.

Using Theorem 61 and Proposition 63 one can prove the following proposition that shows the rigorous semantic connection between IQCL and fuzzy logic.

More precisely, models of IQCL are identifiable to PMV-valuations that satisfy $T_{\mathtt{Q4}}$.

Proposition 64 *Let T be a theory and α be a formula, both in IQCL. Then we have*

(1) $|\alpha|_T^{\mathtt{IQCL}} = |\alpha_t|_{T_t \bigcup T_{\mathtt{Q4}}}^{\mathtt{PMV}}$
(2) $||\alpha||_T^{\mathtt{IQCL}} = ||\alpha_t||_{T_t \bigcup T_{\mathtt{Q4}}}^{\mathtt{PMV}}$

PROOF.

(1) Follows from Theorem 61.
(2) Follows from Theorem 63.

Finally, we can establish a Pavelka style strong completeness theorem and a somehow compactness theorem for IQCL.

Theorem 65 *Let T be a theory and α be a formula, both in IQCL. Then we have (7)*

$$|\alpha|_T^{IQCL} = ||\alpha||_T^{IQCL}$$

PROOF. Follows from Proposition 64 and Theorem 56.

Corollary 66 *Let T be a theory and α be a formula, both in IQCL. Then we have If $|\alpha||_T^{IQCL} = 0$, then for each $s \in S$, $\bar{s} \to \alpha$ is not provable from T.*

Theorem 67 *Let T be a theory and α be a formula, both in IQCL. Then we have (7)*

$$\text{If } r \leq ||\alpha||_T^{IQCL} \text{ then } \exists T_0 \subseteq T \text{ finite such that } r \leq ||\alpha||_{T_0}^{IQCL}.$$

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